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Bounded subgroups of relatively finitely presented groups

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Let G be a finitely generated group that is relatively finitely presented with respect to a collection H_Λ of peripheral subgroups such that the corresponding relative Dehn function is well defined. We prove that every infinite subgroup H of G that is bounded in the relative Cayley graph of G with respect to H_Λ is conjugate into a peripheral subgroup. As an application, we obtain a trichotomy for subgroups of relatively hyperbolic groups. Moreover we prove the existence of the relative exponential growth rate for all subgroups of limit groups.

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1 Introduction

The notion of a group G that is hyperbolic relative to a finite set H_Λ of its subgroups was introduced by Gromov [10] as a generalization of a word hyperbolic group. In his definition, the groups $H \in H_\Lambda$ appear as stabilizers of points at infinity of a certain hyperbolic space X the group G acts on. Since then, the study of relatively hyperbolic groups has remained an active field of research, and several characterizations of relative hyperbolicity were introduced by Bowditch [2], Farb [8] and Osin [15]. In the last work, Osin uses the concept of relative presentations in order to define the relative hyperbolicity of a group G with respect to a set $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. To make this more precise, let $X \subseteq G$ be a symmetric subset such that G is generated by $\bigcup_{\lambda\in\Lambda} H_\lambda \cup X$. Then we obtain a canonical epimorphism

$$\varepsilon\colon F := \left(\mathop{\ast}_{\lambda\in\Lambda} \widetilde{H}_\lambda\right) * F(X) \to G,$$

where the groups $\widetilde{H}_\lambda$ are disjoint isomorphic copies of H_λ, and $F(X)$ denotes the free group over X. Consider a subset $\mathcal{R} \subseteq F$ whose normal closure is the kernel of ε. Then $\mathcal{R}$ gives rise to a so-called *relative presentation of* G *with respect to* H_Λ of the form

$$\left\langle X, \mathcal{H} \;\middle|\; S = 1, S \in \bigcup_{\lambda\in\Lambda} \mathcal{S}_\lambda, R = 1, R \in \mathcal{R} \right\rangle, \tag{1}$$

where $\mathcal{H} := \bigcup_{\lambda\in\Lambda}(\widetilde{H}_\lambda \setminus \{1\})$ and $\mathcal{S}_\lambda$ is the set of all relations over the alphabet $\widetilde{H}_\lambda$. In this framework, G is said to be *hyperbolic relative to* H_Λ if X and $\mathcal{R}$ can be chosen to be finite and (1) admits a *linear* *relative Dehn function*. That is, there is some $C > 0$ such that for every word w of length at most ℓ over $X \cup \mathcal{H}$ that represents the identity in G, there is an equality of the form

$$w =_F \prod_{i=1}^{k} f_i^{-1} R_i^{\pm 1} f_i \tag{2}$$

that holds in F, where $k \leq C\ell$, $f_i \in F$ and $R_i \in \mathcal{R}$. Note that in general, there is no reason to expect that for every $\ell \in \mathbb{N}$ and every relation w of length at most ℓ there is a uniform upper bound $n \in \mathbb{N}$ such that w can be written as in (2) with $k \leq n$. Even if X and $\mathcal{R}$ are finite, in which case we say that (1) is a *finite relative presentation for* G, there are easy examples where there is no such n; see [15, Example 1.3].

Here we study groups G that admit a finite relative presentation as in (1) whose relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. This means that for every $\ell \in \mathbb{N}$ there is a minimal number $\delta^{\mathrm{rel}}_{G,H_\Lambda}(\ell)$ such that for every relation w of length at most ℓ there is an expression of the form (2) with $k \leq \delta^{\mathrm{rel}}_{G,H_\Lambda}(\ell)$. Examples of relatively finitely presented groups that admit a well-defined nonlinear relative Dehn function were considered by Hughes, Martínez-Pedroza and Sánchez Saldaña [12]. The study of groups with a well-defined relative Dehn function typically involves considerations in the so-called *relative Cayley graph* $\Gamma(G, X \cup \mathcal{H})$ of G. Since $X \cup \mathcal{H}$ can be (and usually is) infinite, it is natural to ask the following:

Question 1.1 *Which subgroups of G have bounded diameter in $\Gamma(G, X \cup \mathcal{H})$?*

Note that, aside from the finite subgroups of G, every subgroup of G that can be conjugated into some of the groups H_λ has bounded diameter in $\Gamma(G, X \cup \mathcal{H})$. It turns out that for finitely generated G, the existence of a well-defined relative Dehn function is enough to deduce that there are no further examples of subgroups of G whose diameter in $\Gamma(G, X \cup \mathcal{H})$ is finite.

Theorem 1.2 *Let G be a finitely generated group. Suppose that G is relatively finitely presented with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Then every subgroup $K \leq G$ satisfies exactly one of the following conditions*:

(i) *K is finite.*

(ii) *K is infinite and conjugate to a subgroup of some H_λ.*

(iii) *K is unbounded in $\Gamma(G, X \cup \mathcal{H})$.*

Note for example that if one of the subgroups H_λ in Theorem 1.2 is infinite, then there is no subgroup $K \leq G$ that contains H_λ as a proper subgroup of finite index. This also follows from the fact that each H_λ is almost malnormal, which is shown in [15, Proposition 2.36].

Remark 1.3 The condition that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ in Theorem 1.2 is well defined cannot be removed. To see this, let G be the infinite cyclic group generated by an element a and let H be the subgroup of G that is generated by a^2. Then the relative Cayley graph of G with respect to $H_\Lambda = \{H\}$ is clearly bounded. In particular, G is a bounded subset of its relative Cayley graph while not being conjugate to a subgroup of H. On the other hand, it can be easily seen that G admits a finite relative presentation with respect to H.

If the group G in Theorem 1.2 is relatively hyperbolic with respect to H_Λ, then it is known that a subgroup $K \leq G$ with infinite diameter in $\Gamma(G, X \cup \mathcal{H})$ contains a loxodromic element; see Osin [16, Theorem 1.1 and Proposition 5.2]. Recall that an element $g \in G$ is called *loxodromic* if the map

$$\mathbb{Z} \to \Gamma(G, X \cup \mathcal{H}) \quad \text{given by } n \mapsto g^n$$

is a quasiisometric embedding. We therefore obtain the following classification of subgroups of relatively hyperbolic groups which, to the best of my knowledge, was not recorded before:

Corollary 1.4 *Let G be a finitely generated group. Suppose that G is relatively hyperbolic with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. Then every subgroup $K \leq G$ satisfies exactly one of the following conditions*:

(i) *K is finite.*

(ii) *K is infinite and conjugate to a subgroup of some H_λ.*

(iii) *K contains a loxodromic element.*

As an application of Corollary 1.4, we consider relative exponential growth rates in finitely generated groups. Recall that for a finitely generated group G and a finite generating set X of G, the *growth function* $\beta_G^X\colon \mathbb{N} \to \mathbb{N}$ of G with respect to X is defined by $\beta_G^X(n) = |B_G^X(n)|$, where $B_G^X(n)$ denotes the set of all elements of G that are represented by words of length at most n in the generators of X and X^{-1}. Using Fekete's lemma, it is easy to see that the limit $\lim_{n\to\infty} \sqrt[n]{\beta_G^X(n)}$, known as the exponential growth rate of G with respect to X, always exist; see for example Milnor [13]. Given a subgroup $H \leq G$, a relative analogue of the exponential growth rate is obtained by counting the elements in the relative balls $B_H^X(n) := B_G^X(n) \cap H$. The resulting function

$$\beta_H^X\colon \mathbb{N} \to \mathbb{N} \quad \text{given by } n \mapsto |B_H^X(n)|$$

is called the *relative growth function* of H with respect to X. In [14, Remark 3.1], Olshanskii pointed out that, unlike in the nonrelative case, the limit $\lim_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$ does not exist in general. As a consequence, the *relative exponential growth rate of H in G with respect to X* is typically defined as $\limsup_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$. Nevertheless, in many cases where the relative exponential growth rate is studied in the literature (see for example Cohen [3], Grigorchuk [9], Olshanskii [14], Sharp [19], Coulon, Dal'Bo and Sambusetti [5] and Dahmani, Futer and Wise [7] where G is free or hyperbolic) the limit $\lim_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$ is known to exist, in which case we say that the relative exponential growth rate of H in G exists with respect to X. In the case where G is a free group, the existence of the relative exponential growth rate was proven by Olshanskii in [14], extending prior results of Cohen [3] and Grigorchuk [9] who have independently proven the existence for normal subgroups of G. More recently, these existence results were generalized by the author to the case where G is a finitely generated acylindrically hyperbolic group and H is a subgroup that contains a generalized loxodromic element of G; see [17]. By combining this with Corollary 1.4, we will be able conclude the following:

Theorem 1.5 *Let G be a finitely generated group that is relatively hyperbolic with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. Suppose that each of the groups H_λ has subexponential growth. Then the relative exponential growth rate of every subgroup $H \leq G$ exists with respect to every finite generating set of G.*

By Osin [15, Theorem 1.1], each of the groups H_λ in Theorem 1.5 is finitely generated, so the assumption on subexponential growth indeed makes sense. Relatively hyperbolic groups G as in Theorem 1.5 include many naturally occurring examples of groups. A particularly interesting such class is given by limit groups, which were introduced by Sela in his solution of the Tarski problems [18], and naturally generalize the class of free groups. By work of Dahmani [6] and Alibegović [1], limit groups are known to be relatively hyperbolic with respect to a system of representatives for the conjugacy classes of their maximal abelian noncyclic subgroups. As a consequence, we obtain the following generalization of Olshanskii's existence result:

Corollary 1.6 *Let G be a limit group. Then the relative exponential growth rate of every subgroup $H \leq G$ exists with respect to every finite generating set of G.*

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2 Preliminaries

In this section we introduce some definitions and properties that will be relevant for our study of relatively finitely presented groups. More information about these groups can be found in [15].

2.1 Relative presentations

Let us fix a group G and a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of so-called *peripheral subgroups* of G. Let $X \subseteq G$ be a symmetric subset such that G is generated by $\bigcup_{\lambda\in\Lambda} H_\lambda \cup X$. Such an X will be referred to as a *relative generating* of G with respect to H_Λ. Note that this gives us a canonical epimorphism

$$\varepsilon\colon F := \Big(\mathop{\ast}_{\lambda\in\Lambda} \widetilde{H}_\lambda \Big) \ast F(X) \to G,$$

where the groups $\widetilde{H}_\lambda$ are pairwise disjoint isomorphic copies of H_λ and $F(X)$ denotes the free group over X. Let us also assume that $\widetilde{H}_\lambda \cap X = \varnothing$ for every $\lambda \in \Lambda$. Let N denote the kernel of ε and let $\mathcal{R} \subseteq N$ be a subset whose normal closure in F coincides with N. For each $\lambda \in \Lambda$ let $\mathcal{S}_\lambda$ be the set of words over $\widetilde{H}_\lambda \setminus \{1\}$ that represent the identity in G.

Definition 2.1 With the notation above, we say that a *relative presentation of G with respect to H_Λ* is a presentation of the form

$$\left\langle X, \mathcal{H} \;\middle|\; S = 1, S \in \bigcup_{\lambda \in \Lambda} \mathcal{S}_\lambda,\ R = 1,\ R \in \mathcal{R} \right\rangle, \tag{3}$$

where $\mathcal{H} := \bigcup_{\lambda\in\Lambda}(\widetilde{H}_\lambda \setminus \{1\})$. The relative presentation (3) is called finite if X and $\mathcal{R}$ are finite. In this case G is said to be *relatively finitely presented with respect to H_Λ*.

The following result will be crucial for us:

Theorem 2.2 [15, Theorem 1.1] *Let G be a finitely generated group and let $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ be a collection of its subgroups. Suppose that G is finitely presented with respect to H_Λ. Then the following conditions hold*:

- (i) *The collection H_Λ is finite*, ie $|\Lambda| < \infty$.
- (ii) *Each subgroup H_λ is finitely generated.*

2.2 Relative Dehn functions

Let G be a relatively finitely presented group with a finite relative presentation as in Definition 2.1. For each $\ell \in \mathbb{N}$, let N_ℓ denote the set of words of length at most ℓ over $X \cup \mathcal{H}$ that represent the identity in G. Given $w \in N_\ell$, let $\mathrm{vol}(w) \in \mathbb{N}$ be minimal with the property that there is an expression of the form

$$w =_F \prod_{i=1}^{\mathrm{vol}(w)} f_i^{-1} R_i^{\pm 1} f_i, \tag{4}$$

where the equality is taken in F, and $f_i \in F$ and $R_i \in \mathcal{R}$ for every $1 \le i \le \mathrm{vol}(w)$.

Definition 2.3 The *relative Dehn function* for the finite relative presentation (3) of G is defined by

$$\delta^{\mathrm{rel}}_{G,H_\Lambda} : \mathbb{N} \to \mathbb{N} \cup \{\infty\} \quad \text{given by } \ell \mapsto \sup\{\mathrm{vol}(w) \mid w \in N_\ell\}.$$

We say that $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined if $\delta^{\mathrm{rel}}_{G,H_\Lambda}(\ell) < \infty$ for every $\ell \in \mathbb{N}$.

An important class of relatively finitely presented groups with a well-defined Dehn function consists of relatively hyperbolic groups, which can be defined in terms of the relative Dehn function.

Definition 2.4 A relatively finitely presented group G with a relative presentation (3) is called *relatively hyperbolic with respect to H_Λ* if there is some $C > 0$ such that $\delta^{\mathrm{rel}}_{G,H_\Lambda}(\ell) \le C\ell$ for every $\ell \in \mathbb{N}$.

Of course, the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ depends on the finite relative presentation (3), and not just on H_Λ. But as for ordinary (nonrelative) Dehn functions of finitely presented groups, different finite relative presentations lead to asymptotically equivalent relative Dehn functions; see [15, Theorem 2.34]. In particular, the property of $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ being well defined or bounded above by a linear function does not depend on the choice of a finite relative presentation.

2.3 Geometry of the relative Cayley graph

Let us again consider a relatively finitely presented group G with a finite relative presentation as in Definition 2.1. The Cayley graph of G with respect to $X \cup \mathcal{H}$ is called *the relative Cayley graph* of G and will be denoted by $\Gamma(G, X \cup \mathcal{H})$. We will study the local geometry of $\Gamma(G, X \cup \mathcal{H})$. In order to do so, let us fix some terminology. Given an edge e of $\Gamma(G, X \cup \mathcal{H})$, we write $\partial_0(e)$ to denote the initial vertex of e and $\partial_1(e)$ to denote the terminal vertex of e. A sequence $p = (e_1, \dots, e_n)$ of edges in $\Gamma(G, X \cup \mathcal{H})$ is called a *path* if $\partial_1(e_i) = \partial_0(e_{i+1})$ for $1 \le i < n$. If moreover $\partial_0(e_1) = \partial_1(e_n)$, then p is said to be *cyclic*. The label of a path p will be denoted by $\mathrm{Lab}(p)$. Sometimes it is useful to forget about the initial vertex of a cyclic path $p = (e_1, \dots, e_n)$. To make this precise, we define the *loop* associated to p as the set $[p]$ of all paths of the form $(e_i, \dots, e_n, e_1, \dots, e_{i+1})$ for $1 \le i \le n$. A *subpath of a loop* $[p]$ is a subpath of some representative $p' \in [p]$. The algebraic counterpart of a loop is the set $[w]$ of all cyclic conjugates of a word w over $X \cup \mathcal{H}$, which will be referred to as a *cyclic word*. Accordingly, the label of a loop $[p]$ is defined as $\mathrm{Lab}([p]) := [\mathrm{Lab}(p)]$. Up to minor notational differences, the following definitions can be found in [15].

Definition 2.5 Let w be a word over $X \cup \mathcal{H}$. A subword v of w is a λ*–subword* if it consists of letters of $\widetilde{H}_\lambda$. If a λ–subword v of w is not properly contained in any other λ–subword of w, then v is called a λ*–syllable* of w. Similarly, we say that a word v over $X \cup \mathcal{H}$ is a λ–subword of a cyclic word $[w]$ if it is a λ–subword of some cyclic conjugate of w. If a λ–subword v of $[w]$ is not properly contained in any other λ–subword of $[w]$, then v is called a λ*–syllable* of $[w]$.

Let us now translate Definition 2.5 into conditions for paths in $\Gamma(G, X \cup \mathcal{H})$.

Definition 2.6 Let q be a path in $\Gamma(G, X \cup \mathcal{H})$. A subpath p of q is a λ*–subpath* if $\mathrm{Lab}(p)$ is a λ–subword of $\mathrm{Lab}(q)$. A λ–subpath p of q is called a λ*–component* of q if $\mathrm{Lab}(p)$ is a λ–syllable of $\mathrm{Lab}(q)$. Suppose now that q is cyclic, and consider the loop $[q]$ associated to q. We say that a subpath p of $[q]$ is a λ–subpath of $[q]$ if $\mathrm{Lab}(p)$ is a λ–subword of $\mathrm{Lab}([q])$. If moreover $\mathrm{Lab}(p)$ is a λ–syllable of $\mathrm{Lab}([q])$, then p is called a λ*–component* of $[q]$.

Definition 2.7 Let p_1 and p_2 be λ–components of a path p (resp. a loop $[q]$) in $\Gamma(G, X \cup \mathcal{H})$. We say that p_1 and p_2 are *connected*, if there is a path c in $\Gamma(G, X \cup \mathcal{H})$ that connects a vertex of p_1 with a vertex of p_2 and $\mathrm{Lab}(c)$ consists of letters of $\widetilde{H}_\lambda$. We say that p_1 is *isolated* in p (resp. $[q]$) if there are no further λ–components of p (resp. $[q]$) that are connected to p_1.

Let us now translate the notion of an isolated component of a path (loop) into a corresponding notion for syllables in (cyclic) words.

Definition 2.8 Let w be a word over $X \cup \mathcal{H}$ and let p be any path in $\Gamma(G, X \cup \mathcal{H})$ with $\mathrm{Lab}(p) = w$. We say that two λ–syllables v_1 and v_2 of w are *connected* (resp. *isolated*) if the corresponding λ–components p_1 and p_2 of p are connected (resp. isolated). If w represents the identity in G, and v_1 and v_2 are

λ–syllables of the cyclic word $[w]$, then v_1 and v_2 are *connected* (resp. *isolated*) if the corresponding λ–components p_1 and p_2 of the loop $[p]$ are connected (resp. isolated).

The following lemma is a direct consequence of [15, Lemma 2.27]. It will help us study the local structure of $\Gamma(G, X \cup \mathcal{H})$ and often lets us switch between the word metrics d_X and $d_{X \cup \mathcal{H}}$.

Lemma 2.9 *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups, and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Then for every $n \in \mathbb{N}$ there is a finite subset $\Omega_n \subseteq G$ with the property that for every cyclic path q in $\Gamma(G, X \cup \mathcal{H})$ of length at most n and every isolated component p of the loop $[q]$, the label $\mathrm{Lab}(p)$ represents an element in Ω_n.*

3 The alternating growth condition

In this section we introduce the alternating growth condition, which will play a central role in our proof of Theorem 1.2.

3.1 Regular neighborhoods

Let us start by defining a condition for paths in graphs that can be thought of as a strong form of having no self-intersections.

Definition 3.1 Let Γ be a graph and let p be a path in Γ that consecutively traverses the sequence $v_0, \dots, v_n$ of vertices in Γ. We say that p has a *regular neighborhood in* Γ if every two vertices v_i and v_j that can be joined by an edge in Γ satisfy $|i - j| \le 1$.

Example 3.2 If p is a geodesic path in a graph Γ then p has a regular neighborhood in Γ.

Example 3.3 If p is a nontrivial cyclic path in a graph Γ then p does not have a regular neighborhood in Γ.

Remark 3.4 Every path p that has a regular neighborhood in a graph Γ is locally 2–geodesic, ie the restriction of p to each subpath of length at most 2 is geodesic.

It will be useful for us to translate the concept of regular neighborhoods to words over some generating set of a group.

Definition 3.5 Let G be a group and let X be a generating set of G. A word w over X is called *regular* (with respect to X) if some path p in $\Gamma(G, X)$ with $\mathrm{Lab}(p) = w$ has a regular neighborhood in $\Gamma(G, X)$.

Remark 3.6 Let G be a group and let X be a generating set of G. From the definitions, it directly follows that a word w over X is regular if and only if every subword v of w of length at least 2 satisfies $|v|_X \ge 2$, where $|\cdot|_X$ denotes the word metric corresponding to X.

3.2 Sequences of alternating growth

We want to study sequences of regular words in the context of finitely generated relatively finitely presented groups. Let us therefore fix a finitely generated group G, a finite generating set X of G, and a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of peripheral subgroups of G. Suppose that G is relatively finitely presented with respect to H_Λ and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. As in Section 2 we write $\widetilde{H}_\lambda$ to denote pairwise disjoint isomorphic copies of H_λ that also intersect trivially with X. Let us fix some notation in order to avoid ambiguities concerning the length and the evaluation of a word over $X \cup \mathcal{H}$, where as always $\mathcal{H} = \bigcup_{\lambda\in\Lambda}(\widetilde{H}_\lambda \setminus \{1\})$.

Notation 3.7 Let $w = w_1 \dots w_\ell$ be a word over $X \cup \mathcal{H}$. We write $\|w\| = \ell$ for the word length of w. The image of w in G will be denoted by $\bar{w}$. For any subset $Y \subseteq G$ we write $|\bar{w}|_Y$ for the length of a shortest word over Y that represents $\bar{w}$. If there is no such word, then we set $|\bar{w}|_Y = \infty$.

Definition 3.8 A sequence of words $(w_1^{(n)} \dots w_\ell^{(n)})_{n\in\mathbb{N}}$ of fixed length $\ell \geq 2$ over $X \cup \mathcal{H}$ satisfies the *alternating growth condition* if the following conditions are satisfied:

(I) If $w_i^{(n)} = x$ for some $1 \leq i \leq \ell$, $n \in \mathbb{N}$ and $x \in X$, then $w_i^{(m)} = x$ for every $m \in \mathbb{N}$. In this case we say that i is an *index of type* X.

(II) If $w_i^{(n)} \in \widetilde{H}_\lambda$ for some $1 \leq i \leq \ell$, $n \in \mathbb{N}$ and $\lambda \in \Lambda$, then $w_i^{(m)} \in \widetilde{H}_\lambda$ for every $m \in \mathbb{N}$. In this case we say that i is an *index of type* λ.

(III) The index 1 is not of type X.

(IV) Two consecutive indices are never of the same type.

(V) If i is of type λ, then $\bar{w}_i^{(n)} \notin H_\mu$ and $|\bar{w}_i^{(n)}|_X \geq n$ for every $\mu \in \Lambda \setminus \{\lambda\}$ and every $n \in \mathbb{N}$.

(VI) Each word $w_1^{(n)} \dots w_\ell^{(n)}$ is regular with respect to $X \cup \mathcal{H}$.

The following observation will be used frequently:

Remark 3.9 Given a regular word w over $X \cup \mathcal{H}$, it directly follows from the definitions that every syllable v in w is isolated and consists of a single edge.

3.3 Concatenating sequences of alternating growth

In what follows, we need to construct certain sequences $(w_1^{(n)} \dots w_\ell^{(n)})_{n\in\mathbb{N}}$ of words over $X \cup \mathcal{H}$ that satisfy the alternating growth condition so that ℓ can be chosen arbitrarily large. In order to do so, we will use the following lemma, which allows us to "concatenate" two sequences of words that satisfy the alternating growth condition so that the resulting sequence also satisfies the alternating growth condition.

Lemma 3.10 *With the notation above, suppose that there are two sequences* $(v_1^{(n)} \dots v_M^{(n)})_{n\in\mathbb{N}}$ *and* $(w_1^{(n)} \dots w_N^{(n)})_{n\in\mathbb{N}}$ *of words over* $X \cup \mathcal{H}$ *that satisfy the alternating growth condition. Let* $\lambda \in \Lambda$ *be such that* $w_1^{(n)} \in \widetilde{H}_\lambda$ *for some* $n \in \mathbb{N}$.

(i) *Suppose that* $\bar{v}_M^{(n)} \notin H_\lambda$ *for every* $n \in \mathbb{N}$. *Then there is a strictly increasing sequences of natural numbers* $(s_n)_{n\in\mathbb{N}}$ *such that*

$$(v_1^{(s_n)} \dots v_M^{(s_n)} w_1^{(s_n)} \dots w_N^{(s_n)})_{n\in\mathbb{N}}$$

satisfies the alternating growth condition.

(ii) *Suppose that* $\bar{v}_M^{(n)} \in H_\lambda$ *for every* $n \in \mathbb{N}$. *Then there are strictly increasing sequences of natural numbers* $(s_n)_{n\in\mathbb{N}}$ *and* $(t_n)_{n\in\mathbb{N}}$ *such that the sequence*

$$(v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)})_{n\in\mathbb{N}},$$

where $z^{(n)} \in \widetilde{H}_\lambda$ *is the element representing* $\overline{v_M^{(s_n)} w_1^{(t_n)}} \in H_\lambda$, *satisfies the alternating growth condition.*

Proof Let us first prove (i). Suppose that there is no such sequence $(s_i)_{i\in\mathbb{N}}$. Then there are infinitely many $n \in \mathbb{N}$ such that $v_1^{(n)} \dots v_M^{(n)} w_1^{(n)} \dots w_N^{(n)}$ does not satisfy some of the conditions of Definition 3.8. Since (I)–(V) are clearly satisfied, it follows that $v_1^{(n)} \dots v_M^{(n)} w_1^{(n)} \dots w_N^{(n)}$ is not regular (with respect to $X \cup \mathcal{H}$) for infinitely many $n \in \mathbb{N}$. By restriction to a subsequence if necessary, we can assume that no word $v_1^{(n)} \dots v_M^{(n)} w_1^{(n)} \dots w_N^{(n)}$ is regular. Since $\bar{v}_M^{(n)} \notin H_\lambda$ for every $n \in \mathbb{N}$, none of the subwords $v_M^{(n)} w_1^{(n)}$ represent the trivial element in G. Along with the assumption that $v_1^{(n)} \dots v_M^{(n)}$ and $w_1^{(n)} \dots w_N^{(n)}$ are regular, it follows that there is a maximal index a_n such that

$$|v_{a_n}^{(n)} \dots v_M^{(n)} w_1^{(n)} \dots w_{b_n}^{(n)}|_{X\cup\mathcal{H}} = 1 \tag{5}$$

for some index b_n. Suppose that each b_n is chosen to be minimal with respect to a_n. Then there are generators $u_n \in X \cup \mathcal{H}$ such that

$$q_n = v_{a_n}^{(n)} \dots v_M^{(n)} w_1^{(n)} \dots w_{b_n}^{(n)} u_n$$

represents the identity in G for every $n \in \mathbb{N}$. We want to argue that $w_1^{(n)}$ is an isolated λ–syllable in the cyclic word $[q_n]$. Suppose that this is not the case. Then there are three cases to consider:

Case 1 $(\overline{v_i^{(n)} \dots v_M^{(n)} w_1^{(n)}} \in H_\lambda$ for some $a_n \le i \le N)$ Then $\overline{v_i^{(n)} \dots v_M^{(n)}} \in H_\lambda$, and since $v_1^{(n)} \dots v_M^{(n)}$ is regular, $i = M$. Thus $\bar{v}_M^{(n)} \in H_\lambda$, in contrast to our assumption $\bar{v}_M^{(n)} \notin H_\lambda$.

Case 2 $(\overline{w_1^{(n)} \dots w_i^{(n)}} \in H_\lambda$ for some $2 \le i \le b_n)$ This is a contradiction since $w_1^{(n)} \dots w_N^{(n)}$ is regular.

Case 3 $(\overline{w_1^{(n)} \dots w_{b_n}^{(n)} u_n} \in H_\lambda)$ In this case we also have $\overline{v_{a_n}^{(n)} \dots v_M^{(n)}} \in H_\lambda$. Using again the assumption that $v_1^{(n)} \dots v_M^{(n)}$ is regular, $a_n = M$ and $\bar{v}_M^{(n)} \in H_\lambda$, which contradicts our assumption that $\bar{v}_M^{(n)} \notin H_\lambda$.

Thus $w_1^{(n)}$ is indeed an isolated λ–syllable in $[q_n]$. Moreover, $\|q_n\| \le M+N+1$ for every $n \in \mathbb{N}$. From Lemma 2.9 it therefore follows that $\{\bar{w}_1^{(n)} \mid n \in \mathbb{N}\}$ is a finite subset of G. On the other hand, the alternating growth condition ensures that $|w_1^{(n)}|_X \ge n$ for every $n \in \mathbb{N}$. This finally gives us the contradiction that arose from our assumption that there is no sequence $(s_i)_{i\in\mathbb{N}}$ as in (i).

Let us now prove (ii). From the alternating growth condition, $|w_1^{(n)}|_X \ge n$ for every $n \in \mathbb{N}$. Thus we can choose strictly increasing sequences of natural numbers $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ such that $|v_M^{(s_n)} w_1^{(t_n)}|_X \ge n$ for every $n \in \mathbb{N}$. Note that Definition 3.8(I)–(V) are clearly satisfied for

$$(v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)})_{n\in\mathbb{N}},$$

where $z^{(n)} \in \widetilde{H}_\lambda$ is the element representing $\overline{v_M^{(s_n)} w_1^{(t_n)}}$. In order to prove the lemma it therefore suffices to show that $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ is regular for all but finitely many $n \in \mathbb{N}$. To see this, let us first consider the subwords

$$v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} \quad \text{and} \quad z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}.$$

Suppose that there is some $1 \le i \le M-1$ with $|v_i^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}|_{X\cup\mathcal{H}} \le 1$. Then there are two cases to consider:

Case 1 $(\overline{v_i^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}} \in H_\lambda)$ Then we also have $\overline{v_i^{(s_n)} \dots v_{M-1}^{(s_n)}} \in H_\lambda$, and since $v_1^{(s_n)} \dots v_M^{(s_n)}$ is regular, it follows that $M-1=1$. But then $v_{M-1}^{(s_n)}$ and $v_M^{(s_n)}$ both represent elements of H_λ, which in turn contradicts the regularity of $v_1^{(s_n)} \dots v_M^{(s_n)}$.

Case 2 $(\overline{v_i^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}} \notin H_\lambda)$ Then there is some $u_n \in X \cup \mathcal{H}$ that does not lie in $\widetilde{H}_\lambda$ such that $q_n := v_i^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} u_n$ represents the identity in G. We claim that $z^{(n)}$ is an isolated syllable in the cyclic word $[q_n]$. Otherwise there would be some $i \le j \le M-1$ with

$$\overline{v_j^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}} \in H_\lambda,$$

which is impossible as we have seen in Case 1. Moreover, $\|q_n\| \le M$. From Lemma 2.9 it therefore follows that $\{\bar{z}_1^{(n)} \mid n \in \mathbb{N}\}$ is a finite subset of G. Since $|z^{(n)}|_X \ge n$, there are only finitely many $n \in \mathbb{N}$ such that $|v_i^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}|_{X\cup\mathcal{H}} \le 1$ for some $1 \le i \le M-1$. Thus $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}$ is regular for all but finitely many $n \in \mathbb{N}$. Symmetric argument shows that $z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ is regular for all but finitely many $n \in \mathbb{N}$. By restriction to a subsequence if necessary, we can therefore assume that the words $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}$ and $w_2^{(t_n)} \dots w_N^{(t_n)}$ are regular for every n.

Now assume $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ is not regular, and choose $1 \le a_n \le M-1$ and $2 \le b_n \le N$ such that $v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}$ is a minimal subword of $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ with

$$|v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}|_{X\cup\mathcal{H}} \le 1.$$

Case 1 $(q_n := v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}$ represents the identity in $G)$ Since $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}$ and $z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ are regular, it follows that $z^{(n)}$ is an isolated syllable in the cyclic word $[q_n]$. In view of Lemma 2.9, there are only finitely many such n.

Case 2 ($v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}$ does not represent an element of H_λ) Then there is some $u_n \in \bigcup_{\mu \in \Lambda \setminus \{\lambda\}} (\widetilde{H}_\lambda \setminus \{1\}) \cup X$ such that

$$q_n := v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)} u_n$$

represents the trivial element in G. In particular, u_n is not part of a λ–syllable in the cyclic word $[q_n]$. Another application of Lemma 2.9 now reveals that there are only finitely many $n \in \mathbb{N}$ such that $v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}$ does not represent an element of H_λ.

Case 3 ($v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)}$ represents a nontrivial element in H_λ) Then there is some $u_n \in \widetilde{H}_\lambda$ such that

$$q_n := v_{a_n}^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)} u_n$$

represents the identity in G. Suppose $z^{(n)}$ is connected to some further λ–syllable in the cyclic word $[q_n]$. Since $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)}$ and $z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ are regular, $z^{(n)}$ has to be connected to u_n. Hence

$$\overline{z^{(n)} w_2^{(t_n)} \dots w_{b_n}^{(t_n)} u_n} \in H_\lambda,$$

which implies

$$\overline{w_2^{(t_n)} \dots w_{b_n}^{(t_n)}} \in H_\lambda.$$

From the regularity of $z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ it therefore follows that $N = 2$. But then $\bar{w}_2^{(t_n)} \in H_\lambda$, which contradicts the regularity of $w_1^{(t_n)} w_2^{(t_n)} \dots w_N^{(t_n)}$. Thus u_n is an isolated syllable in $[q_n]$ and a final application of Lemma 2.9 proves that Case 3 can only occur finitely many times.

Altogether we have shown that $v_1^{(s_n)} \dots v_{M-1}^{(s_n)} z^{(n)} w_2^{(t_n)} \dots w_N^{(t_n)}$ is regular for all but finitely many $n \in \mathbb{N}$, which proves the lemma. □

Corollary 3.11 *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of peripheral subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$, and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Let $(w^{(n)})_{n \in \mathbb{N}}$ be a sequence of words over $X \cup \mathcal{H}$ that satisfies the alternating growth condition and let K be the subgroup of G generated by $\{\bar{w}_n \mid n \in \mathbb{N}\}$. Then there is some $C \in \mathbb{N}$ that satisfies the following. For every $L \in \mathbb{N}$ there is a sequence of words $(v_n)_{n \in \mathbb{N}}$ over $X \cup \mathcal{H}$ such that*:

(i) *$(v_n)_{n \in \mathbb{N}}$ satisfies the alternating growth condition.*

(ii) *The length of every word v_n is bounded by $L \le \|v_n\| \le L + C$.*

(iii) *Every word v_n represents an element of K.*

Proof Let us write $w^{(n)} = w_1^{(n)} \dots w_\ell^{(n)}$ for every $n \in \mathbb{N}$. From properties (II) and (III) of the alternating growth condition there is some $\lambda \in \Lambda$ such that $w_1^{(n)} \in \widetilde{H}_\lambda$ for every $n \in \mathbb{N}$. By restriction to a subsequence if necessary, we may assume that $(w_n)_{n \in \mathbb{N}}$ satisfies one of the following two conditions:

(i) $\bar{w}_\ell^{(n)} \notin H_\lambda$ for every $n \in \mathbb{N}$.

(ii) $\bar{w}_\ell^{(n)} \in H_\lambda$ for every $n \in \mathbb{N}$.

Suppose the first and let $k \in \mathbb{N}$. Then an inductive application of Lemma 3.10(i) provides us with subsequences

$$(w_1^{(s_{i,n})} \dots w_\ell^{(s_{i,n})})_{n\in\mathbb{N}}$$

of $w^{(n)}$ for each $1 \le i \le k$ such that the sequence of concatenated words

$$v_n := (w_1^{(s_{1,n})} \dots w_\ell^{(s_{1,n})})(w_2^{(s_{2,n})} \dots w_\ell^{(s_{2,n})}) \dots (w_1^{(s_{k,n})} \dots w_\ell^{(s_{k,n})})$$

has length $k\ell$ and satisfies the alternating growth condition. Thus the corollary is clearly satisfied for $C = \ell$.

Let us now consider (ii), and let $k \in \mathbb{N}$. Then an inductive application of Lemma 3.10(ii) provides us with subsequences

$$(w_1^{(s_{i,n})} \dots w_\ell^{(s_{i,n})})_{n\in\mathbb{N}}$$

of $w^{(n)}$ for each $1 \le i \le k$ such that the sequence of words v_n given by

$$(w_1^{(s_{1,n})} \dots w_{\ell-1}^{(s_{1,n})}) z^{(t_{1,n})} (w_2^{(s_{2,n})} \dots w_{\ell-1}^{(s_{2,n})}) z^{(t_{2,n})} \dots z^{(t_{k-1,n})} (w_2^{(s_{k,n})} \dots w_\ell^{(s_{k,n})}),$$

where $z^{(t_{i,n})} \in \widetilde{H}_\lambda$ is the element representing $\overline{w_\ell^{(s_{i,n})} w_1^{(s_{i+1,n})}} \in H_\lambda$, satisfies the alternating growth condition. In this case v_n has length $k(\ell-1)+1$ and we see that the corollary is satisfied for $C = \ell$. □

4 Dichotomy of infinite subgroups

Endowed with Corollary 3.11, we are now ready to study the subgroup of a relatively finitely presented group G that is generated by all the elements $\bar{w}_n$, where $(w_n)_{n\in\mathbb{N}}$ is a sequence that satisfies the alternating growth condition.

Lemma 4.1 *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of peripheral subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Suppose that $(w_n)_{n\in\mathbb{N}}$ is a sequence of words over $X \cup \mathcal{H}$ that satisfies the alternating growth condition. Then the subgroup $K \le G$ generated by $\{\bar{w}_n \in G \mid n \in \mathbb{N}\}$ is unbounded with respect to $d_{X\cup\mathcal{H}}$.*

Proof Suppose that K is bounded with respect to $d_{X\cup\mathcal{H}}$, ie that there is some $N \in \mathbb{N}$ with $|k|_{X\cup\mathcal{H}} \le N$ for every $k \in K$. Due to Corollary 3.11 there is a number $L \ge 4N$ and a sequence $(v_n)_{n\in\mathbb{N}}$ of words $v_n = v_1^{(n)} \dots v_L^{(n)}$ over $X \cup \mathcal{H}$ that satisfies the alternating growth condition such that each v_n represents an element of K. By restriction to a subsequence, we can assume that there is some $M \in \mathbb{N}$ with $|v_n|_{X\cup\mathcal{H}} = M \le N$ for every $n \in \mathbb{N}$. Let $u_1^{(n)} \dots u_M^{(n)}$ be a shortest word over $X \cup \mathcal{H}$ representing $\bar{v}_n^{-1}$. Then each word $q_n := v_1^{(n)} \dots v_L^{(n)} u_1^{(n)} \dots u_M^{(n)}$ represents the identity in G. The alternating growth condition ensures $v_1^{(n)} \dots v_L^{(n)}$ is regular and that two consecutive letters of v_n do not lie in X. It therefore follows that at least every second of its letters is an isolated syllable in v_n. Thus there are at least $2N$ isolated syllables in $v_n = v_1^{(n)} \dots v_L^{(n)}$. Note that for every $\lambda \in \Lambda$ and every λ–syllable of $u_1^{(n)} \dots u_M^{(n)}$, which necessarily consists of a single letter $u_i^{(n)}$, there is at most one λ–syllable $v_j^{(n)}$ in $v_1^{(n)} \dots v_L^{(n)}$ that is connected to $u_i^{(n)}$ in the cyclic word $[q_n]$. Otherwise there would be a connection between two different isolated λ–syllables of $v_1^{(n)} \dots v_L^{(n)}$ by a λ–word. This implies that there are at least $2N - M \geq N$ isolated syllables in $[q_n]$ that become arbitrarily large with respect to X as n goes to ∞. But this contradicts Lemma 2.9 since $\|q_n\| \leq M + L$ for every $n \in \mathbb{N}$. Thus K is an unbounded subset of $\Gamma(G, X \cup \mathcal{H})$. $\square$

Lemma 4.2 *Let G be a finitely generated group with a finite generating set X. Suppose that G is relatively finitely presented with respect to a collection of peripheral subgroups $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Let $K \leq G$ be an infinite subgroup that is bounded with respect to $d_{X \cup \mathcal{H}}$. Then there is an element $g \in G$ and an index $\eta \in \Lambda$ such that $|gKg^{-1} \cap H_\eta| = \infty$.*

Proof Since K is bounded with respect to $d_{X \cup \mathcal{H}}$, each of its conjugates gKg^{-1} is a bounded subset of $\Gamma(G, X \cup \mathcal{H})$. Let $m \in \mathbb{N}$ be minimal with the following property:

($*$) There is a conjugate $H := gKg^{-1}$ of K, a finite relative generating set Y of G, and an infinite sequence $(k_n)_{n \in \mathbb{N}}$ of pairwise distinct elements $k_n \in H$ with $|k_n|_{Y \cup \mathcal{H}} = m$ for every $n \in \mathbb{N}$.

Let g, Y and $(k_n)_{n \in \mathbb{N}}$ be as in ($*$). For each n let $u^{(n)} = u_1^{(n)} \dots u_m^{(n)}$ be a (shortest) word over $Y \cup \mathcal{H}$ that represents k_n. Due to the minimality of m, we can extend Y to any finite relative generating set Y' of G such that ($*$) is still satisfied for an appropriate subsequence of $(k_n)_{n \in \mathbb{N}}$. Since G is finitely generated, we can therefore assume that Y is a symmetric generating set of G.

Suppose first that $m = 1$. Then $u_1^{(n)} \in \mathcal{H} = \bigcup_{\lambda \in \Lambda} (\widetilde{H}_\lambda \setminus \{1\})$ for all but finitely many $n \in \mathbb{N}$. Since Λ is finite by Theorem 2.2, there is some $\eta \in \Lambda$ such that infinitely many pairwise distinct letters $u_1^{(n)}$ lie in $\widetilde{H}_\eta$. It therefore follows that $|gKg^{-1} \cap H_\eta| = \infty$.

Let us now consider the case $m \geq 2$. We want to modify Y and $u^{(n)}$ in such a way that some subsequence of $(u^{(n)})_{n \in \mathbb{N}}$ satisfies the alternating growth condition. This will be done inductively by going through the letters $u_i^{(n)}$ of $u^{(n)}$.

Suppose that $u_1^{(n)} \in Y$ for infinitely many $n \in \mathbb{N}$. Then we can choose some $x_1 \in Y$ and a subsequence $(k_{j_n})_{n \in \mathbb{N}}$ of $(k_n)_{n \in \mathbb{N}}$ with $u_1^{(j_n)} = x_1$ for every n. In this case we replace $(k_n)_{n \in \mathbb{N}}$ by $(k_{j_n})_{n \in \mathbb{N}}$.

Suppose next that $u_1^{(n)} \in \mathcal{H}$ for all but finitely many $n \in \mathbb{N}$. Since Λ is finite, there is some $\lambda_1 \in \Lambda$ with $u_1^{(n)} \in \widetilde{H}_{\lambda_1}$ for infinitely many $n \in \mathbb{N}$. We have to consider 2 cases:

Case 1 (there is some $\tilde{h}_1 \in \widetilde{H}_{\lambda_1}$ with $u_1^{(n)} = \tilde{h}_1$ for infinitely many $n \in \mathbb{N}$) Restrict $(k_n)_{n \in \mathbb{N}}$ to a subsequence $(k_{j_n})_{n \in \mathbb{N}}$ such that $u_1^{(j_n)} = \tilde{h}_1$ for every $n \in \mathbb{N}$. Moreover we add h_1 and h_1^{-1} to Y and replace the letter $u_1^{(j_n)} = \tilde{h}_1 \in \widetilde{H}_{\lambda_1}$ in $u^{(j_n)}$ by $h_1 \in Y$ for every $n \in \mathbb{N}$. Next we replace the resulting sequence by a subsequence that satisfies ($*$), which is possible by the choice of m.

Case 2 (there is no $\tilde{h}_1 \in \widetilde{H}_{\lambda_1}$ with $u_1^{(n)} = \tilde{h}_1$ for infinitely many $n \in \mathbb{N}$) Replace $(u^{(n)})_{n \in \mathbb{N}}$ by a subsequence $(u^{(j_n)})_{n \in \mathbb{N}}$ such that $|\bar{u}_1^{(j_n)}|_Y > n$ for every $n \in \mathbb{N}$.

We proceed analogously with the other indices $i \in \{2, \dots, m\}$. The resulting sequence of words over $Y \cup \mathcal{H}$ will be denoted by $(v_1^{(n)} \dots v_m^{(n)})_{n \in \mathbb{N}}$. Let $g_n \in H$ be the element represented by $v_1^{(n)} \dots v_m^{(n)}$.

Suppose that either two consecutive letters $v_i^{(n)}$ and $v_{i+1}^{(n)}$ or $v_1^{(n)}$ and $v_m^{(n)}$ both lie in Y. Then we could add $v_i^{(n)}v_{i+1}^{(n)}$ and $(v_i^{(n)}v_{i+1}^{(n)})^{-1}$ (resp. $v_m^{(n)}v_1^{(n)}$ and $(v_m^{(n)}v_1^{(n)})^{-1}$) to Y in order to obtain a shorter sequence of infinitely many pairwise distinct elements of H (resp. of $v_{-1}^{(n)}Hv_1^{(n)}$) with respect to $d_{Y\cup\mathcal{H}}$. But this is a contradiction to the choice of m. Thus neither $v_i^{(n)}$ and $v_{i+1}^{(n)}$ nor $v_1^{(n)}$ and $v_m^{(n)}$ both lie in Y. In particular, we can replace $v_1^{(n)}\dots v_m^{(n)}$ by its inverse $(v_m^{(n)})^{-1}\dots(v_1^{(n)})^{-1}$ to ensure that the first letter does not lie in Y. Let us therefore assume that $v_1^{(n)}$ is never contained in Y. To prove that $(v_1^{(n)}\dots v_m^{(n)})_{n\in\mathbb{N}}$ satisfies the alternative growth condition, it remains to show that each $v_1^{(n)}\dots v_m^{(n)}$ is regular and that two consecutive letters $v_i^{(n)}$ and $v_{i+1}^{(n)}$ cannot lie in the same group $\widetilde{H}_\lambda$. But these properties are direct consequences of the condition $|g_n|_{Y\cup\mathcal{H}}=m$ from $(*)$, where k_n plays the role of g_n. Altogether we have shown that there is a conjugate H of K and a sequence $(g_n)_{n\in\mathbb{N}}$ of elements in H that can be represented by a sequence $(v_1^{(n)}\dots v_m^{(n)})_{n\in\mathbb{N}}$ of words over $Y\cup\mathcal{H}$ that satisfies the alternating growth condition. In this case, Lemma 4.1 tells us that H is an unbounded subset of $\Gamma(G,Y\cup\mathcal{H})$, which clearly contradicts our assumption that K is a bounded subset of $\Gamma(G,X\cup\mathcal{H})$. Hence $m=1$, in which case we have already proven the lemma. □

We are now ready to prove our main theorem:

Theorem 4.3 *Let G be a finitely generated group and let X be a finite generating set of G. Suppose that G is relatively finitely presented with respect to a collection of peripheral subgroups $H_\Lambda=\{H_\lambda\mid\lambda\in\Lambda\}$ and that the relative Dehn function $\delta^{\mathrm{rel}}_{G,H_\Lambda}$ is well defined. Then every subgroup $K\le G$ satisfies exactly one of the following conditions*:

(i) *K is finite.*

(ii) *K is infinite and conjugated to a subgroup of a peripheral subgroup.*

(iii) *K is unbounded in $\Gamma(G,X\cup\mathcal{H})$.*

Proof Suppose that K is infinite and bounded as a subset of $\Gamma(G,X\cup\mathcal{H})$. From Lemma 4.2 we know that there is an index $\eta\in\Lambda$ and an element $g\in G$ such that the $gKg^{-1}\cap H_\eta$ is infinite. We can therefore choose a sequence $(h_n)_{n\in\mathbb{N}}$ of distinct nontrivial elements $h_n\in gKg^{-1}\cap H_\eta$. Suppose that gKg^{-1} is not a subgroup of H_η and let $a\in gKg^{-1}\setminus H_\eta$. Let $\tilde{h}_n\in\widetilde{H}_\eta$ be the element representing h_n. Then, after adding $\{a,a^{-1}\}$ to X if necessary, we can consider the sequence of words $(\tilde{h}_na)_{n\in\mathbb{N}}$ over $X\cup\mathcal{H}$. We claim that $(\tilde{h}_na)_{n\in\mathbb{N}}$ has a subsequence that satisfies the alternating growth condition. The only property that is not directly evident is that $(\tilde{h}_na)_{n\in\mathbb{N}}$ has a subsequence consisting of regular words. Suppose that this is not the case. Since Λ is finite by Theorem 2.2, it then follows that there is some $\mu\in\Lambda$ such that $\tilde{h}_na$ represents an element in H_μ for infinitely many $n\in\mathbb{N}$. Then $\tilde{h}_ma(\tilde{h}_na)^{-1}=\tilde{h}_m\tilde{h}_n^{-1}$ represents an element in $H_\mu\cap H_\eta$ whenever $\tilde{h}_ma$ and $\tilde{h}_na$ both represent elements of H_μ. Since a was chosen outside of H_η, it moreover follows that $\tilde{h}_na$ can never represent an element of H_η. In particular, $\eta\ne\mu$. But this is a contradiction to [15, Proposition 2.36], which says that $H_\mu\cap H_\eta$ is finite for $\mu\ne\eta$. Thus $(\tilde{h}_na)_{n\in\mathbb{N}}$ has a subsequence that satisfies the alternating growth condition. In this case Lemma 4.1 tells us that the subgroup $\langle\{ah_n \mid n \in \mathbb{N}\}\rangle$ of gKg^{-1} is unbounded in $\Gamma(G, X \cup \mathcal{H})$, which contradicts our assumption that K is bounded in $\Gamma(G, X \cup \mathcal{H})$. Finally, this proves that gKg^{-1} is a subgroup of H_η. □

Let us now consider the important special case of Theorem 1.2 where G is relatively hyperbolic with respect to H_Λ. Recall that an element $g \in G$ is called *loxodromic* if the map

$$\mathbb{Z} \to \Gamma(G, X \cup \mathcal{H}) \quad \text{given by } n \mapsto g^n$$

is a quasiisometric embedding. It is known that a subgroup $K \leq G$ with infinite diameter in $\Gamma(G, X \cup \mathcal{H})$ contains a loxodromic element. This follows from a corresponding result for acylindrically hyperbolic groups [16, Theorem 1.1] and the fact that relatively hyperbolic groups act acylindrically on the (hyperbolic) graph $\Gamma(G, X \cup \mathcal{H})$ [16, Proposition 5.2].

Corollary 4.4 *Let G be a finitely generated group. Suppose that G is relatively hyperbolic with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. Then every subgroup $K \leq G$ satisfies exactly one of the following conditions*:

(i) *K is finite.*

(ii) *K is infinite and conjugate to a subgroup of some H_λ.*

(iii) *K contains a loxodromic element.*

4.1 A geometric proof of Corollary 4.4

As pointed out to the author by Jason Manning, there is a short and more geometric proof of Corollary 4.4 that uses the cusped space $\Omega = \Omega(G, H_\Lambda, X)$ associated to G, H_Λ and an appropriate finite generating set X of G (see [11, Definition 3.15], where cusped spaces for relatively hyperbolic groups were introduced). Indeed, according to [11, Remark 3.14 and Theorem 3.25], the space Ω is hyperbolic and admits a proper isometric action of G. Moreover it is evident from the construction of Ω that for each $x \in \Omega$ and every infinite subgroup $K \leq \Omega$, the orbit $K.x$ has infinite diameter in Ω. Thus $K.x$ has at least one limit point $\xi \in \partial\Omega$. If $K.x$ has another limit point $\eta \in \partial\Omega$, then we can choose $g, h \in H$ such that the distances of $d_\Omega(g.x, x)$ and $d_\Omega(h.x, x)$ are arbitrarily large while the Gromov product $(g.x, h.x)_x$ is bounded. In this case, a standard argument tells us that at least one of the elements $g, h, gh \in K$ is loxodromic; see [4, Lemme 2.3]. In the remaining case, ξ is a fixed point of H and it is a consequence of the construction of Ω that H is conjugate to a subgroup of some H_λ.

5 Applications

As an application of the classification of subgroups of relatively hyperbolic groups given in Corollary 4.4, we prove the existence of the relative exponential growth rate for all subgroups of a large variety of relatively hyperbolic groups.

Definition 5.1 Let G be a finitely generated group and let X be a finite generating set of G. Given a subgroup $H \leq G$, we define the *relative growth function* of H in G with respect to X by

$$\beta_H^X \colon \mathbb{N} \to \mathbb{N}, \quad n \mapsto |B_H^X(n)|,$$

where $B_H^X(n)$ denotes the set of elements in H that are represented by words of length at most n over $X \cup X^{-1}$. The relative exponential growth rate of H in G with respect to X is defined by $\limsup_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$.

It is natural to ask whether lim sup can be replaced by lim, ie whether the limit $\lim_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$ exists. Unlike in the important special case $H = G$, in which it is well known that this limit exists (see eg [13]), it does not exist in general; see [14, Remark 3.1]. In the case where the limit $\lim_{n\to\infty} \sqrt[n]{\beta_H^X(n)}$ does exist, we will say that the relative exponential growth rate of H in G exists with respect to X. The following result provides us with a large variety of finitely generated relatively hyperbolic groups G for which the relative exponential growth rate exists for each of its subgroups and generating sets.

Theorem 5.2 *Let G be a finitely generated group that is relatively hyperbolic with respect to a collection $H_\Lambda = \{H_\lambda \mid \lambda \in \Lambda\}$ of its subgroups. Suppose that each of the groups H_λ has subexponential growth. Then the relative exponential growth rate of every subgroup $K \leq G$ exists with respect to every finite generating set of G.*

Proof Let X be a finite generating set of G. We go through the three cases of Corollary 4.4.

Suppose first that K is finite. Then β_K^X is eventually constant and it trivially follows that $\lim_{n\to\infty} \sqrt[n]{\beta_K^X(n)}$ exists and is equal to 1.

Let us next consider the case where K contains a loxodromic element k. By [16, Proposition 5.2], G acts acylindrically on the (hyperbolic) graph $\Gamma(G, X \cup \mathcal{H})$. In this case, [16, Theorem 1.1] tells us that either G is virtually cyclic, in which case the claim follows trivially, or G is acylindrically hyperbolic, in which case the claim is covered by [17, Theorem 5.8].

Consider now the remaining case, where K is infinite and conjugated to a subgroup of some peripheral subgroup. Thus there is some $g \in G$ and some $\lambda \in \Lambda$ such that $K \leq gH_\lambda g^{-1}$. By Theorem 2.2 each H_λ, and hence $gH_\lambda g^{-1}$, is finitely generated. We can therefore choose a finite generating set Y of $gH_\lambda g^{-1}$. Moreover, it follows from [15, Lemma 5.4] that each peripheral subgroup, and hence $gH_\lambda g^{-1}$, is undistorted in G. We can therefore choose a constant $C > 0$ such that

$$\beta^X_{gH_\lambda g^{-1}}(n) \leq \beta^Y_{gH_\lambda g^{-1}}(Cn) \tag{6}$$

for every $n \in \mathbb{N}$. By assumption, each H_λ, and therefore $gH_\lambda g^{-1}$, has subexponential growth. Thus we have $\lim_{n\to\infty} \beta^Y_{gH_\lambda g^{-1}}(n)/a^n = 0$ for every $a > 1$. In view of (6), this implies that

$$\lim_{n\to\infty} \beta^X_{gH_\lambda g^{-1}}(n)/a^n = 0.$$

Then $\lim_{n\to\infty} \sqrt[n]{\beta_K^X(n)} = 1$ since $\beta_K^X(n) \leq \beta^X_{gH_\lambda g^{-1}}(n)$ for $n \in \mathbb{N}$, and in particular the limit exists. □

References

[1] **E Alibegović**, *A combination theorem for relatively hyperbolic groups*, Bull. Lond. Math. Soc. 37 (2005) 459–466 MR Zbl

[2] **B H Bowditch**, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. 22 (2012) art. id. 1250016 MR Zbl

[3] **J M Cohen**, *Cogrowth and amenability of discrete groups*, J. Funct. Anal. 48 (1982) 301–309 MR Zbl

[4] **M Coornaert**, **T Delzant**, **A Papadopoulos**, *Géométrie et théorie des groupes: les groupes hyperboliques de Gromov*, Lecture Notes in Math. 1441, Springer (1990) MR Zbl

[5] **R Coulon**, **F Dal'Bo**, **A Sambusetti**, *Growth gap in hyperbolic groups and amenability*, Geom. Funct. Anal. 28 (2018) 1260–1320 MR Zbl

[6] **F Dahmani**, *Combination of convergence groups*, Geom. Topol. 7 (2003) 933–963 MR Zbl

[7] **F Dahmani**, **D Futer**, **D T Wise**, *Growth of quasiconvex subgroups*, Math. Proc. Cambridge Philos. Soc. 167 (2019) 505–530 MR Zbl

[8] **B Farb**, *Relatively hyperbolic groups*, Geom. Funct. Anal. 8 (1998) 810–840 MR Zbl

[9] **R I Grigorchuk**, *Symmetrical random walks on discrete groups*, from "Multicomponent random systems" (R L Dobrushin, Y G Sinai, D Griffeath, editors), Adv. Probab. Related Topics 6, Dekker, New York (1980) 285–325 MR Zbl

[10] **M Gromov**, *Hyperbolic groups*, from "Essays in group theory" (S M Gersten, editor), Math. Sci. Res. Inst. Publ. 8, Springer (1987) 75–263 MR Zbl

[11] **D Groves**, **J F Manning**, *Dehn filling in relatively hyperbolic groups*, Israel J. Math. 168 (2008) 317–429 MR Zbl

[12] **S Hughes**, **E Martínez-Pedroza**, **L J Sánchez Saldaña**, *Quasi-isometry invariance of relative filling functions*, Groups Geom. Dyn. 17 (2023) 1483–1515 MR Zbl

[13] **J Milnor**, *A note on curvature and fundamental group*, J. Differential Geom. 2 (1968) 1–7 MR Zbl

[14] **A Y Olshanskii**, *Subnormal subgroups in free groups, their growth and cogrowth*, Math. Proc. Cambridge Philos. Soc. 163 (2017) 499–531 MR Zbl

[15] **D V Osin**, *Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems*, Mem. Amer. Math. Soc. 843, Amer. Math. Soc., Providence, RI (2006) MR Zbl

[16] **D Osin**, *Acylindrically hyperbolic groups*, Trans. Amer. Math. Soc. 368 (2016) 851–888 MR Zbl

[17] **E Schesler**, *The relative exponential growth rate of subgroups of acylindrically hyperbolic groups*, J. Group Theory 25 (2022) 293–326 MR Zbl

[18] **Z Sela**, *Diophantine geometry over groups, I: Makanin–Razborov diagrams*, Publ. Math. Inst. Hautes Études Sci. 93 (2001) 31–105 MR Zbl

[19] **R Sharp**, *Relative growth series in some hyperbolic groups*, Math. Ann. 312 (1998) 125–132 MR Zbl

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