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A topological construction of families of Galois covers of the line

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We describe a new construction of families of Galois coverings of the line using basic properties of configuration spaces, covering theory, and the Grauert–Remmert extension theorem. Our construction provides an alternative to a previous construction due to González-Díez and Harvey (which uses Teichmüller theory and Fuchsian groups) and, in the case the Galois group is nonabelian, corrects an inaccuracy therein. In the opposite case where the Galois group has trivial center, we recover some results due to Fried and Völklein.

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1 Introduction

The object of this note are families of Galois coverings of the line.

Let G be a finite group and let C and C' be smooth projective curves over the complex numbers endowed with a G–action. We say that C and C' are *topologically equivalent* or have the same (unmarked) *topological type* if there is an $\eta \in \operatorname{Aut} G$ and an orientation-preserving homeomorphism $f : C \to C'$ such that $f(g \cdot x) = \eta(g) \cdot f(x)$ for $x \in C'$ and $g \in G$. We say that C and C' are (unmarkedly) *G–isomorphic* if moreover f is a biholomorphism.

Given a G–covering $C \to \mathbb{P}^1$, it has been proved by González-Díez and Harvey [1992] that there exists an algebraic family of curves with a G–action

$$\pi : \mathscr{C} \to \mathsf{B}$$

such that

(1) every curve C' in the family is *topologically equivalent* to C;

(2) every curve with an action of the given topological type is *G–isomorphic* to some fiber of the family and to at most a finite number of fibers.

This result has been subsequently used in several papers, eg [Conti et al. 2022; Frediani et al. 2015; Frediani and Neumann 2003; Penegini 2015; Perroni 2022], just to mention a few.

The construction in [González-Díez and Harvey 1992] uses Teichmüller theory. Other approaches to this construction include [Fried and Völklein 1991; Li 2018; Völklein 1994]. In this paper we describe an alternative, explicit and mostly topological construction of such families. We expect this to be useful to make explicit computations on the family. For example, we expect this to allow a better understanding of the monodromy and the generic Hodge group for the natural variation of Hodge structure associated with the family, generalizing the results of [Rohde 2009] carried out in the cyclic case. Our motivation comes from the fact that these families and their variation of the Hodge structure are important in the study of Shimura subvarieties of the moduli space A_g (of principally polarized abelian varieties of dimension g) in relation with the Coleman–Oort conjecture; see eg [Moonen 2010; Moonen and Oort 2013; Frediani et al. 2015; Tamborini 2022]. The results presented here are, nevertheless, of independent interest.

1.1 We give a quick glance at our construction. For $n \geq 3$ let $\mathsf{M}_{0,n}$ denote the set of n–tuples

$$X = (x_1, \dots, x_n) \in (\mathbb{P}^1)^n$$

such that $x_i \neq x_j$ for $i \neq j$, $x_{n-2} = 0$, $x_{n-1} = 1$ and $x_n = \infty$. Consider the group

$$\Gamma_n = \langle \gamma_1, \dots, \gamma_n \mid \gamma_1 \cdots \gamma_n = 1 \rangle.$$

Let G be a finite group and let $\theta\colon \Gamma_n \to G$ be an epimorphism. Fix $X \in \mathsf{M}_{0,n}$. After choosing a base point $x_0 \in \mathbb{P}^1 - X$ and an isomorphism $\chi\colon \Gamma_n \cong \pi_1(\mathbb{P}^1 - X, x_0)$, Riemann's existence theorem yields a G–covering $C_X \to \mathbb{P}^1$ with monodromy $\theta \circ \chi^{-1}$ and branch locus X. Nevertheless this covering depends on the choices. Our goal is to make this construction for all $X \in \mathsf{M}_{0,n}$ together, in order to get a family of curves parametrized by $\mathsf{M}_{0,n}$. Consider the map

$$p\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}, \quad p(x_0, x_1, \dots, x_n) = (x_1, \dots, x_n).$$

We have $p^{-1}(X) = \mathbb{P}^1 - X$. Hence p can be thought as the universal family of genus 0 curves with n marked points. The basic idea of our construction is that the total space of our family should be a suitable G–covering of $\mathsf{M}_{0,n+1}$. For the construction of this covering, choose

(i) an element $x = (x_0, X) \in \mathsf{M}_{0,n+1}$;

(ii) an isomorphism $\chi\colon \Gamma_n \to \pi_1(\mathbb{P}^1 - X, x_0)$.

The following sequence is exact and splits:

$$1 \to \pi_1(\mathbb{P}^1 - X, x_0) \to \pi_1(\mathsf{M}_{0,n+1}, x) \to \pi_1(\mathsf{M}_{0,n}, X) \to 1.$$

Set for simplicity $N_x := \pi_1(\mathbb{P}^1 - X, x_0)$, $H_X := \pi_1(\mathsf{M}_{0,n}, X)$ and $f := \chi^{-1} \circ \theta$. Assume that we can find an extension $\tilde{f}$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathbb{P}^1 - X, x_0) & \longrightarrow & \pi_1(\mathsf{M}_{0,n+1}, x) & \longrightarrow & H_X & \longrightarrow & 1 \\ & & & \overset{f}{\searrow} & \Big\downarrow \tilde{f} & & & & \\ & & & & G & & & & \end{array}$$

From $\tilde{f}$ we get a topological G–covering $\mathscr{C}^* \to \mathsf{M}_{0,n+1}$. By the Grauert–Remmert extension theorem (see Theorem 7.4 below) this compactifies to a branched covering $\mathscr{C} \to \mathbb{P}^1 \times \mathsf{M}_{0,n}$ of quasiprojective varieties. Composing with the projection to $\mathsf{M}_{0,n}$ we get a holomorphic family $\pi : \mathscr{C} \to \mathsf{M}_{0,n}$ satisfying properties (1) and (2).

1.2 Thus, if one is able to find the extension $\tilde{f}$, one can construct the families using only basic properties of configuration spaces, covering theory and the Grauert–Remmert extension theorem, avoiding Teichmüller theory and Fuchsian groups. In fact this strategy is not new, as it has already been used in exactly the same context in various papers by Michael D Fried and Helmut Völklein; see eg [Fried 1977; Fried and Völklein 1991; Völklein 1994].

If G is abelian, one is always able to find the extension $\tilde{f}$; see Section 10. In general however the extension $\tilde{f}$ does not exist, contrary to what is claimed in [González-Díez and Harvey 1992]. One can at least show that there are always finite-index subgroups $H_a \subset H_X$ such that f extends to a morphism $f_a : N_x \rtimes H_a \to G$. Geometrically passing from H_X to the subgroup H_a means that one builds a family satisfying (1) and (2) over a base which is no longer $\mathsf{M}_{0,n}$, but some finite cover Y_a of it. The pair (H_a, f_a) is far from unique, there are many of them and different choices yield families differing by finite étale pullback (see Section 7 for precise definitions.) So another problem arises: how is one supposed to choose the pair (H_a, f_a) in order to determine the family in a canonical way?

For a special class of groups, namely for groups G with trivial center, there is a canonical choice of (H_a, f_a), which allows to construct a canonical family of coverings. This case corresponds to the one studied in [Fried and Völklein 1991; Völklein 1994; 1996] where the condition that G be centerless plays a crucial role.

It is odd that for this problem the two special cases occur in opposite directions, namely for abelian and for centerless groups.

For general G one is not able to pick out a distinguished choice in a canonical way. This problem was already considered long ago in [Fried 1977, pages 57–58] where a cohomological interpretation of this difficulty is given.

Our approach instead is the following. Since we are stuck with a whole collection of pairs (H_a, f_a), each one giving rise to a family of coverings with base the cover Y_a of $\mathsf{M}_{0,n}$, we decide to consider the whole collection instead of the single families. This collection comes naturally with the structure of a directed set coming from the pullbacks among families. We are able to show that this collection with this structure is well defined and depends only on the topological data.

Summing up, our construction, which builds heavily on previous approaches, corrects an inaccuracy in [González-Díez and Harvey 1992], where it is erroneously claimed that one has always $\mathsf{Y}_a = \mathsf{M}_{0,n}$, confirms that $\mathsf{Y}_a = \mathsf{M}_{0,n}$ if G is abelian (Theorem 10.1), and allows to recover at least part of the results in the papers of Fried and Völklein quoted above, while generalizing them to arbitrary groups with nontrivial center.

1.3 The paper is organized as follows. In Section 2 we recall basic facts about the configuration spaces of $\mathbb{P}^1$. Section 3 deals with parallel transport for fiber bundles. This material is for sure known to the experts, but rather hard to locate in the literature. Since these arguments are quite useful and we like their geometric flavor, we prefer to expound them concisely. In Section 4 we recall some classical concepts of surface topology. After these preliminaries, in Section 5 we study the set $\mathcal{T}^n(G)$ of topological types of G–actions; the main result is Theorem 5.6, which gives a combinatorial description of the set of topological types. The proof of this well-known fact presents our ideas in a simple context. Section 6 is dedicated to the description of some technical tools for the construction of the families. In Section 7 we construct the collection of families $\{\mathscr{C}_a \to \mathsf{Y}_a\}_a$ as sketched above. In Section 8 we study the dependence of the collection on the choices (i) and (ii), and on the epimorphism $\theta\colon \Gamma_n \to G$. Also this point becomes quite neat using our approach. Section 9 is dedicated to the case where G has trivial center and Section 10 to case where G is abelian. Summing up our main theorem is the following:

Theorem 1.4 (1) *The topological types of G–curves C with $g(C) = g$, $g(C/G) = 0$ and n branch points are in bijection with the set $(\mathscr{D}^n(G)/\operatorname{Aut} G)/\operatorname{Out}^* \Gamma_n$ (see Corollary 5.7, Definitions 4.8 and 5.2, and (4-1) for notation).*

(2) *For any topological type there is a nonempty ordered set $(\mathscr{I}, \geq)$ and for any $a \in \mathscr{I}$ there is an algebraic family $\pi_a\colon \mathscr{C}_a \to \mathsf{Y}_a$ of genus g curves with a G–action. The following properties hold:*

- (a) *Every curve C in the family has the given topological type.*
- (b) *For any $a \in \mathscr{I}$ and for any G–curve C with $C/G \cong \mathbb{P}^1$, there is at least one fiber of $\pi_a\colon \mathscr{C}_a \to \mathsf{Y}_a$ which is G–isomorphic to C, and there are only finitely many such fibers.*
- (c) *Each Y_a is a finite étale cover of $\mathsf{M}_{0,n}$;*
- (d) *$(\mathscr{I}, \geq)$ is a directed set: for any $a, b \in \mathscr{I}$ there is a c with $c \geq a$ and $c \geq b$.*
- (e) *If $a \geq b$, there is an algebraic étale covering $v\colon \mathsf{Y}_a \to \mathsf{Y}_b$ such that $\mathscr{C}_a \cong v^*\mathscr{C}_b$.*
- (f) *All the families have the same moduli image.*
- (g) *If $Z(G) = \{1\}$, then $(\mathscr{I}, \geq)$ has a minimum; hence in this case we can associate to any topological type a single family instead of the whole collection.*
- (h) *If G is abelian, then there exists $a \in \mathscr{I}$ such that $Y_a = \mathsf{M}_{0,n}$.*

The precise statement can be found in Theorems 7.8 and 8.2. Roughly speaking one can say that for any topological type there is a "universal" family of G–curves with that topological type. Such a family is not unique, but only unique up to the equivalence relation generated by finite étale pullbacks.

1.5 The existence problem that we address in this paper can of course be generalized: instead of considering just Galois covers, one can ask for the construction of families satisfying (1) and (2) for all the coverings with a fixed Galois closure (equivalently with fixed monodromy). These kinds of problems have been studied a lot and they are extremely important also because of their relevance for the inverse Galois problem; see [Fried 1977; 2010; Fried and Jarden 1986; Fried and Völklein 1991; 1992; Völklein 1996]. In these cases it often happens that the "universal" family has more than one component. We stress that in this paper we restrict only to the Galois case and that in this case all families are connected. In fact, the base of each family is a (connected) cover of $\mathsf{M}_{0,n}$.

Another variant of the problem studied in this paper is obtained by letting G^* be a group such that $\operatorname{Inn} G \subset G^* \subset \operatorname{Aut} G$ and considering two data equivalent if and only if they belong to the same $G^* \times \operatorname{Out}^* \Gamma_n$–orbit. This also has attracted a lot of attention in the literature. Our case corresponds to the choice $G^* = \operatorname{Aut} G$. In this paper we restrict to this case since we are interested in the topological types.

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2 Configuration spaces

2.1 If M is a manifold, its configuration space is

$$\boldsymbol{F}_{0,n}\, M := \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

We use the following notation: $X = (x_1, \dots, x_n)$ is a point of $\boldsymbol{F}_{0,n}\, M$ and $x = (x_0, X) = (x_0, x_1, \dots, x_n)$ is a point of $\boldsymbol{F}_{0,n+1}\, M$. We set $M - X := M - \{x_1, \dots, x_n\}$. The group $\pi_1(\boldsymbol{F}_{0,n}\, M)$ is called the *pure braid group* with n strings of the manifold M.

2.2 If $n \geq 3$, then the group $\mathrm{PGL}(2, \mathbb{C})$ acts freely and holomorphically on $\boldsymbol{F}_{0,n}\, \mathbb{P}^1$. The quotient $\boldsymbol{F}_{0,n}\mathbb{P}^1/\mathrm{PGL}(2, \mathbb{C})$ is the moduli space of smooth curves of genus 0 with n marked points. Setting $\mathbb{C}^{**} := \mathbb{C} - \{0, 1\}$, the map

$$\boldsymbol{F}_{0,n-3}\, \mathbb{C}^{**} \to \boldsymbol{F}_{0,n}\mathbb{P}^1, \quad (z_1, \dots, z_{n-3}) \mapsto (z_1, \dots, z_{n-3}, 0, 1, \infty),$$

is a section for the action of $\mathrm{PGL}(2,\mathbb{C})$, ie its image intersects each orbit in exactly one point and it induces a biholomorphism of $\boldsymbol{F}_{0,n-3}\,\mathbb{C}^{**}$ onto the moduli space $\boldsymbol{F}_{0,n}\mathbb{P}^1/\mathrm{PGL}(2,\mathbb{C})$. We *define* $\mathsf{M}_{0,n}$ as the image of the section, ie we set

$$\mathsf{M}_{0,n} := \boldsymbol{F}_{0,n-3}\,\mathbb{C}^{**}\times\{(0,1,\infty)\} = \{X=(x_1,\dots,x_n)\in\boldsymbol{F}_{0,n}\mathbb{P}^1 \mid x_{n-2}=0,\ x_{n-1}=1,\ x_n=\infty\}.$$

Points of $\mathsf{M}_{0,n}$ will be denoted by $X=(x_1,\dots,x_n)$ with the understanding that $x_{n-2}=0$, $x_{n-1}=1$ and $x_n=\infty$. Similarly we set

$$\mathsf{M}_{0,n+1} := \{x=(x_0,\dots,x_{n+1})\in\boldsymbol{F}_{0,n+1}\,\mathbb{P}^1 \mid x_{n-2}=0,\ x_{n-1}=1,\ x_n=\infty\}.$$

It is often useful to compare the configuration space of $\mathbb{P}^1$ with that of the plane. Denote by $\mathrm{T}(2,\mathbb{C})$ the subset of elements of $\mathrm{PGL}(2,\mathbb{C})$ fixing ∞. The group $\mathrm{T}(2,\mathbb{C})$ acts on $\boldsymbol{F}_{0,n-1}\,\mathbb{C}$ and the map

$$\mathsf{M}_{0,n}\to\boldsymbol{F}_{0,n-1}\,\mathbb{C},\quad X\mapsto(x_1,\dots,x_{n-3},0,1), \tag{2-1}$$

is a section for this action; hence $\mathsf{M}_{0,n}\times\mathrm{T}(2,\mathbb{C})\cong\boldsymbol{F}_{0,n-1}\,\mathbb{C}$. In particular, $\pi_1(\mathsf{M}_{0,n})\subset\pi_1(\boldsymbol{F}_{0,n-1}\,\mathbb{C})$. Thus, when dealing with $\pi_1(\mathsf{M}_{0,n})$, we can work with the more classical braid group of the plane. The map

$$p\colon\mathsf{M}_{0,n+1}\to\mathsf{M}_{0,n},\quad p(x_0,X):=X \tag{2-2}$$

is a fiber bundle. In fact it is the restriction of the bundle $\boldsymbol{F}_{0,n}\,\mathbb{C}\to\boldsymbol{F}_{0,n-1}\,\mathbb{C}$; see [Birman 1974]. The fiber over X is $\mathbb{P}^1-X=\mathbb{C}^{**}-\{x_1,\dots,x_{n-3}\}$. Hence (2-2) is the universal family of genus 0 curves with n ordered marked points.

2.3 Fix $x=(x_0,X)\in\mathsf{M}_{0,n+1}$ and let $\tilde{x}=(x_0,\widetilde{X})\in\boldsymbol{F}_{0,n}\,\mathbb{C}$ be the corresponding point via (2-1). We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\mathbb{P}^1-X,x_0) & \longrightarrow & \pi_1(\mathsf{M}_{0,n+1},x) & \longrightarrow & \pi_1(\mathsf{M}_{0,n},X) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(\mathbb{C}-\widetilde{X},x_0) & \longrightarrow & \pi_1(\boldsymbol{F}_{0,n}\,\mathbb{C},\tilde{x}) & \longrightarrow & \pi_1(\boldsymbol{F}_{n-1}\,\mathbb{C},\widetilde{X}) & \longrightarrow & 1 \end{array} \tag{2-3}$$

The rows are the split exact sequence of the fibrations p and $\boldsymbol{F}_{0,n}\,\mathbb{C}\to\boldsymbol{F}_{0,n-1}\,\mathbb{C}$; see eg [Birman 1974, Corollary 1.8.1; Fadell 1962, Theorem 3.1]. A geometric way to exhibit the splitting is to produce a cross section as follows: given $x=(x_1,\dots,x_n)\in\mathsf{M}_{0,n}$ we set

$$f(x):=\tfrac{1}{2}\min\{1,|x_1|,\dots,|x_{n-3}|\}.$$

Then $s(x):=(f(x),x_1,\dots,x_n)$ is a section of $p\colon\mathsf{M}_{0,n+1}\to\mathsf{M}_{0,n}$. (A similar idea is used in [Fadell 1962, Theorem 3.1].) The morphism $s_*\colon\pi_1(\mathsf{M}_{0,n},X)\to\pi_1(\mathsf{M}_{0,n+1},x)$ is a splitting. Setting

$$\begin{gathered}\varepsilon\colon\pi_1(\mathsf{M}_{0,n},X)\to\mathrm{Aut}(\pi_1(\mathbb{P}^1-X,x_0)),\\ \varepsilon([\alpha])([\gamma]):=s_*[\alpha]\cdot[\gamma]\cdot s_*[\alpha]^{-1}=[s\circ\alpha*\gamma*s\circ i(\alpha)],\end{gathered} \tag{2-4}$$

we get

$$\pi_1(\mathsf{M}_{0,n+1},x)=\pi_1(\mathbb{P}^1-X,x_0))\rtimes_\varepsilon\pi_1(\mathsf{M}_{0,n},X).$$

3 Parallel transport

In this section we recall a notion of parallel transport up to homotopy on any fiber bundle. In the sequel, we will use it for the bundle $p\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$ to study the dependence of the construction of Section 7 from the choices made.

3.1 Given $b_0, b_1 \in B$ let $\Omega(B, b_0, b_1)$ denote the set of all paths α in B with $\alpha(0) = b_0$ and $\alpha(1) = b_1$. We write $\alpha \sim \beta$ if $\alpha \simeq \beta$ rel $\{0, 1\}$. Let $\Pi_1(B)$ denote the fundamental groupoid of B; this is the small category whose objects are the points of B and with morphisms from b_0 to b_1 equal to $\Omega(B, b_0, b_1)/{\sim}$, composition being given by $[\alpha] \cdot [\beta] = [\alpha * \beta]$.

3.2 Let $p\colon E \to B$ be a fiber bundle (in the sense of [Spanier 1966, page 90], ie a locally trivial bundle). Assume that the base B is Hausdorff and paracompact. Then p is a fibration [Spanier 1966, Corollary 14, page 96], ie it has the homotopy lifting property for every topological space Z: if $H\colon Z \times [0,1] \to B$ is any map and $f\colon Z \to E$ lifts $H(\cdot\,,0)$, then there is a lift $\widetilde{H}$ of H with $\widetilde{H}(\cdot\,,0) = f$; see eg [Spanier 1966, page 66]. For any fiber bundle $p\colon E \to B$ one can define a sort of parallel transport up to homotopy, which is a contravariant functor T from $\Pi_1(B)$ to the homotopy category of topological spaces, denoted by $\mathrm{h-TOP}$. For $b \in B$ set $T(b) := E_b = p^{-1}(b)$. Given $[\alpha] \in \Pi_1(B)(b_0, b_1)$ consider the map $H\colon E_{b_0} \times [0,1] \to B$ defined by $H(e,t) := \alpha(t)$. The inclusion $i\colon E_{b_0} \hookrightarrow E$ is a lift of $H(\cdot\,,0)$. By the homotopy lifting property there is $\widetilde{H}\colon E_{b_0} \times [0,1] \to E$ with $p\widetilde{H} = H$ and $\widetilde{H}(\cdot\,,0) = i$. Moreover the homotopy class of $\widetilde{H}(\cdot\,,1)$ is well defined. We call $T([\alpha]) = [\widetilde{H}(\cdot\,,1)] \in [E_{b_0}, E_{b_1}]$ the homotopy parallel transport along α; see eg [Spanier 1966, page 100f; May 1999, page 54].

3.3 If $p\colon E \to B$ is a differentiable fiber bundle one can say more. Recall the following basic fact from differential topology. Let M and N be smooth manifolds. An *isotopy* of M in N is a smooth map $f\colon M \times [0,1] \to N$ such that $f(\cdot\,,t)$ is an embedding for any t. If $M = N$, $f(\cdot\,,t)$ is a diffeomorphism of M for any t and $f(\cdot\,,0) = \mathrm{id}_M$; we say that f is an *ambient isotopy*.

Theorem 3.4 *If M is a compact submanifold of N, any isotopy $f\colon M \times [0,1] \to N$ such that $f(\cdot\,,0)$ is the inclusion $M \hookrightarrow N$ extends to an ambient isotopy.*

For a proof, see eg [Hirsch 1976, Theorem 1.3, page 180].

Lemma 3.5 *Assume that $p\colon E \to B$ is a differentiable bundle. Let α be a path in B from b_0 to b_1. Let σ be a path in E with $p\sigma = \alpha$ and set $x_0 = \sigma(0) \in E_{b_0}$ and $x_0' = \sigma(1) \in E_{b_1}$. Then there is a map $\widetilde{H}\colon E_{b_0} \times [0,1] \to E$ such that*

(1) *$\widetilde{H}(\cdot\,,0)$ is the inclusion $E_{b_0} \hookrightarrow E$;*

(2) *$\widetilde{H}(\cdot\,,t)$ is a diffeomorphism of E_{b_0} onto $E_{\alpha(t)}$;*

(3) $\widetilde{H}(x_0, t) = \sigma(t)$.

In particular, the map $f^\alpha := \widetilde{H}(\,\cdot\,, 1)$ *is a diffeomorphism of* E_{b_0} *onto* E_{b_1} *such that* $f^\alpha(x_0) = x_0'$ *and* $T([\alpha]) = [f^\alpha]$. *Moreover if* G *is a finite group acting fiberwise on* E *and the fiber is compact, then* f^α *can be chosen to be* G*–equivariant.*

Proof If the fiber of E is compact, the argument is the usual proof of the Ehresmann theorem: pullback E to $[0, 1]$, choose a lift to E of the vector field d/dt and integrate it; see eg [Voisin 2002]. A G–invariant lift gives the last statement. But we also need the case of noncompact fibers. This can be treated as follows. Denote by $\tilde{\alpha}\colon \alpha^*E \to E$ the bundle map covering α. Since $[0, 1]$ is contractible, there is a (smooth) trivialization $\psi\colon E_{b_0} \times [0, 1] \to \alpha^*E$ such that $\psi(x, 0) = x$; see [Steenrod 1951, Corollary 11.6, page 53]. Given any such ψ the composition $\tilde{\alpha} \circ \psi\colon E_{b_0} \times [0, 1] \to E$ is a possible choice for the map $\widetilde{H}$ in 3.2. We now modify ψ so that it matches conditions (1)–(3). First notice that if $\{h_t\}_{t\in[0,1]}$ is any path in $\mathrm{Diff}(E_{b_0})$ starting at the identity, then $\psi_t' := \psi_t h_t$ is a new trivialization of α^*E. Next observe that $t \mapsto \psi_t^{-1}(\sigma(t))$ is a path in E_{b_0} from x_0 to $\psi_1^{-1}(x_0')$, ie an isotopy of $\{x_0\}$ in E_{b_0}. By Theorem 3.4 there is $\{h_t\}$ that extends this isotopy. Then $\psi_t' := \psi_t h_t$ is a trivialization and $\widetilde{H} := \tilde{\alpha} \circ \psi'$ satisfies (1)–(3). □

We now use this construction for the fiber bundle $\mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$ and give a geometric interpretation of the morphism (2-4) in terms of parallel transport.

Proposition 3.6 *Let* $x, x' \in \mathsf{M}_{0,n+1}$. *Let* $\beta\colon [0, 1] \to \mathsf{M}_{0,n}$ *be a path such that* $\beta(0) = X$ *and* $\beta(1) = X'$. *Let* $\widetilde{H}$, f^β *and* $T([\beta])$ *be as in Lemma 3.5. Assume that* $f^\beta(x_0) = x_0'$. *Set* $\tilde{\beta}(t) := \widetilde{H}(t, x_0)$. *Then for* $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$ *we have* $f_*^\beta([\gamma]) = \tilde{\beta}_\#([\gamma])$.

Proof Take $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. Consider the map

$$F\colon [0, 1] \times [0, 1] \to \mathsf{M}_{0,n+1}, \quad F(t, s) = \widetilde{H}(\gamma(s), t).$$

Then $F(0, s) = \widetilde{H}(\gamma(s), 0) = \gamma(s)$, $F(0, 1) = \widetilde{H}(\gamma(s), 1) = f^\beta \circ \gamma(s)$ and

$$F(t, 0) = F(t, 1) = \widetilde{H}(x_0, t) = \tilde{\beta}(t).$$

It follows that $i(\tilde{\beta}) * \gamma * \tilde{\beta} \simeq f^\beta \circ \gamma$ rel $\{0, 1\}$. Hence $f_*^\beta([\gamma]) = \tilde{\beta}_\#([\gamma])$ for any $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. □

Proposition 3.7 *Let* $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$ *and let* $\widetilde{H}$, f^α *and* $T([\alpha])$ *be as in Lemma 3.5. Assume that* $\sigma(t) := \widetilde{H}(t, x_0) = s \circ \alpha$. *Then* $\varepsilon([\alpha]) = f_*^\alpha$.

Proof By Proposition 3.6, we get $f_*^\alpha[\gamma] = \sigma_\#([\gamma]) = [\sigma * \gamma * i(\sigma)]$ for any $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. Hence f^α satisfies $f_*^\alpha[\gamma] = [s \circ \alpha * \gamma * s \circ i(\alpha)] = \varepsilon([\alpha])([\gamma])$ for every $[\gamma] \in \pi_1(\mathbb{P}^1 - X, x_0)$. □

4 Dehn–Nielsen theorems and consequences

We dedicate this section to fixing some notation and recalling some classical concepts of surface topology.

4.1 Let Σ be an oriented surface and set $\Sigma^* := \Sigma - \{y\}$ for some $y \in \Sigma$. Given $b_0, b_1 \in \Sigma$ let $\Omega(\Sigma, b_0, b_1)$ denote the set of all paths α in Σ with $\alpha(0) = b_0$ and $\alpha(1) = b_1$. Fix $x_0 \in \Sigma^*$. Let $\tilde{\alpha} \in \Omega(\Sigma, x_0, y)$ be such that $\tilde{\alpha}(t) = y$ only for $t = 1$ and let D be a small disk around y. Let α be the loop that starts at x_0, travels along $\tilde{\alpha}$ till it reaches ∂D, then makes a complete tour of ∂D counterclockwise and finally goes back to x_0 again along $\tilde{\alpha}$. An important observation is that the conjugacy class of $[\alpha]$ in $\pi_1(\Sigma^*, x_0)$ is well defined. Indeed the choice of the disk does not change $[\alpha]$, while if a different path $\tilde{\beta} \in \Omega(\Sigma, x_0, y)$ is chosen, then $[\beta]$ and $[\alpha]$ are conjugate by the class of a loop in Σ^* that starts at x_0 travels along $\tilde{\alpha}$ up to ∂D, then along a piece of ∂D and finally goes back along $\tilde{\beta}$.

4.2 Fix a point $(x_0, X) \in \boldsymbol{F}_{0,n}\, S^2$. Consider a smooth regular arc $\tilde{\alpha}_i$ joining x_0 to x_{σ_i} (for some permutation σ). Assume that the paths $\tilde{\alpha}_i$ intersect only at x_0 and that the tangent vectors at x_0 are all distinct and follow each other in counterclockwise order (we orient S^2 by the outer normal). Now consider the loops α_i constructed as in 4.1 and assume that the circles are pairwise disjoint and that the intersection of the interior of the i^{th} circle with X reduces to x_{σ_i}.

Definition 4.3 Let $x = (x_0, X) \in \boldsymbol{F}_{0,n+1}\, S^2$. We call a set of generators $\mathcal{B} = \{[\alpha_1], \dots, [\alpha_n]\}$ obtained as above a *geometric basis* of $\pi_1(S^2 - X, x_0)$. We say that a geometric basis $\mathcal{B} = \{[\alpha_i]\}_{i=1}^n$ is *adapted* to x if it respects the order of the points in X, that is α_i turns around x_i, ie the permutation $\sigma = \mathrm{id}$.

Notice that, thanks to the permutation, the definition of geometric basis depends only on the set $\{x_1, \dots, x_n\}$, not on the ordering of the points. On the other hand the classes $\{[\alpha_i]\}$ have a fixed order.

4.4 For $n \geq 3$, set $\Gamma_n := \langle \gamma_1, \dots, \gamma_n \mid \prod_{i=1}^n \gamma_i = 1 \rangle$. From a geometric basis $\mathcal{B} = \{[\alpha_i]\}_{i=1}^n$ we get an isomorphism

$$\chi \colon \Gamma_n \to \pi_1(S^2 - X, x_0)$$

such that $\chi(\gamma_i) = [\alpha_i]$. Assume that $\mathcal{B} = \{[\alpha_i]\}_{i=1}^n$ and $\overline{\mathcal{B}} = \{[\bar{\alpha}_i]\}_{i=1}^n$ are two geometric bases for $\pi_1(S^2 - X, x_0)$. It follows from 4.1 that every $[\alpha_i]$ is conjugate to some $[\bar{\alpha}_j]$. If we denote by

$$\chi, \bar{\chi} \colon \Gamma_n \to \pi_1(S^2 - X, x_0)$$

the isomorphisms corresponding to the two bases, then $\mu := \bar{\chi} \circ \chi^{-1} \in \operatorname{Aut} \pi_1(S^2 - X, x_0)$ has the following properties:

(1) for every $i = 1, \dots, n$, $\mu([\alpha_i])$ is conjugate to $[\alpha_j]$ for some j;

(2) the induced homomorphism on $H^2(\pi_1(S^2 - X, x_0), \mathbb{Z})$ is the identity.

Definition 4.5 We denote by $\operatorname{Aut}^* \pi_1(S^2 - X, x_0)$ the subgroup of elements of $\operatorname{Aut} \pi_1(S^2 - X, x_0)$ satisfying properties (1) and (2) above. By 4.4 this definition does not depend on the choice of the geometric basis $\mathscr{B}$.

4.6 Now assume that $\mathscr{B}$ and $\overline{\mathscr{B}}$ are adapted to X. In this case, for every $i = 1, \dots, n$, $[\alpha_i]$ is conjugate to $[\bar{\alpha}_i]$. As a consequence, the automorphism $\mu := \bar{\chi} \circ \chi^{-1}$ of $\pi_1(S^2 - X, x_0)$ belongs to the subgroup $\operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)$ defined as follows.

Definition 4.7 We denote by $\operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)$ the subgroup of $\operatorname{Aut}^* \pi_1(S^2 - X, x_0)$ of elements that map $[\alpha_i]$ to a conjugate of $[\alpha_i]$ for every $i = 1, \dots, n$. This definition does not depend on the choice of the geometric basis $\mathscr{B}$ adapted to x.

Definition 4.8 Similarly, we denote by $\operatorname{Aut}^* \Gamma_n \subset \operatorname{Aut} \Gamma_n$ the subgroup of automorphisms ν satisfying:

(1) For $i = 1, \dots, n$ the element $\nu(\gamma_i)$ is conjugate to γ_j for some j.

(2) The automorphism of $H^2(\Gamma_n, \mathbb{Z})$ induced by ν is the identity.

We denote by $\operatorname{Aut}^{**} \Gamma_n \subset \operatorname{Aut}^* \Gamma_n$ the subgroup of automorphisms ν such that:

(1′) For $i = 1, \dots, n$ the element $\nu(\gamma_i)$ is conjugate to γ_i.

If $\chi\colon \Gamma_n \to \pi_1(S^2 - X, x_0)$ is induced from a geometric basis (not necessarily adapted to x), then $\nu \in \operatorname{Aut}^* \Gamma_n$ (resp. $\operatorname{Aut}^{**} \Gamma_n$) if and only if $\chi\nu\chi^{-1} \in \operatorname{Aut}^* \pi_1(S^2 - X, x_0)$ (resp. $\operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)$).

4.9 If G is a group and $a \in G$, then $\operatorname{inn}_a : G \to G$ denotes conjugation by a, ie $\operatorname{inn}_a(x) = axa^{-1}$. Notice that if $f : G \to H$ is a morphism, then $f \circ \operatorname{inn}_a = \operatorname{inn}_{f(a)} \circ f$. The group of inner automorphisms of G is denoted $\operatorname{Inn} G$. It is a normal subgroup of $\operatorname{Aut} G$. We set $\operatorname{Out} G := \operatorname{Aut} G / \operatorname{Inn} G$. For $(x_0, X) \in \boldsymbol{F}_{0,n+1}\, S^2$, we observe that $\operatorname{Inn}(\pi_1(S^2 - X, x_0)) \subset \operatorname{Out}^{**}(\pi_1(S^2 - X, x_0))$ and $\operatorname{Inn} \Gamma_n \subset \operatorname{Aut}^{**} \Gamma_n$, and we define

$$
\begin{aligned}
\operatorname{Out}^* \pi_1(S^2 - X, x_0) &:= \frac{\operatorname{Aut}^* \pi_1(S^2 - X, x_0)}{\operatorname{Inn} \pi_1(S^2 - X, x_0)}, \\
\operatorname{Out}^{**} \pi_1(S^2 - X, x_0) &:= \frac{\operatorname{Aut}^{**} \pi_1(S^2 - X, x_0)}{\operatorname{Inn} \pi_1(S^2 - X, x_0)}, \qquad \text{(4-1)} \\
\operatorname{Out}^* \Gamma_n &:= \frac{\operatorname{Aut}^* \Gamma_n}{\operatorname{Inn} \Gamma_r}.
\end{aligned}
$$

Using a geometric basis we immediately get $\operatorname{Out}^* \Gamma_n \cong \operatorname{Out}^* \pi_1(S^2 - X, x_0)$.

4.10 If $S_{g,n}$ is a topological surface of genus g with n punctures, the mapping class group of $S_{g,n}$ is denoted by $\operatorname{Mod}(S_{g,n})$, while $\operatorname{PMod}(S_{g,n})$ denotes the pure mapping class group of $S_{g,n}$, which is defined to be the subgroup of $\operatorname{Mod}(S_{g,n})$ of elements that fix each puncture individually.

4.11 In the sequel we will need the following variants of the Dehn–Nielsen–Baer theorem, for which see [Farb and Margalit 2012, Theorem 8.8, page 234; Ivanov 2002, Section 2.9; Zieschang et al. 1980, Theorem 5.7.1, page 197, and Theorem 5.13.1, page 214].

Theorem 4.12 (Dehn–Nielsen–Baer) *Let $x = (x_0, X) \in \boldsymbol{F}_{0,n+1}\, S^2$. Then $\varphi \in \mathrm{Aut}^*\, \pi_1(S^2 - X, x_0)$ if and only if there exists $\sigma \in \mathrm{Inn}\, \pi_1(S^2 - X, x_0)$ and an orientation-preserving homeomorphism*

$$h\colon S^2 - X \to S^2 - X$$

such that $h(x_0) = x_0$ and $\varphi = \sigma \circ h_*$. In other words, $\mathrm{Mod}(S^2 - X) \cong \mathrm{Out}^*(\pi_1(S^2 - X, x_0))$.

Corollary 4.13 *Let $x, y \in \boldsymbol{F}_{0,n+1}\, S^2$ and $\varphi\colon \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - Y, y_0)$ be a homomorphism that sends geometric bases to geometric bases. Then there exists $\sigma \in \mathrm{Inn}(\pi_1(S^2 - Y, y_0))$ and an orientation-preserving homeomorphism $h\colon S^2 - X \to S^2 - Y$ such that $h(x_0) = y_0$ and $\varphi = \sigma \circ h_*$.*

Proof Fix an orientation-preserving homeomorphism $f : S^2 - Y \to S^2 - X$ such that $f(y_0) = x_0$ and apply the Dehn–Nielsen–Baer theorem to $f_* \circ \varphi$. □

Corollary 4.14 *Let $x = (x_0, X) \in \boldsymbol{F}_{0,n+1}\, S^2$. Then $\varphi \in \mathrm{Aut}^{**}\, \pi_1(S^2 - X, x_0)$ if and only if there exists $\sigma \in \mathrm{Inn}(\pi_1(S^2 - X, x_0))$ and an orientation-preserving self-homeomorphism h of S^2 such that $h(x_i) = x_i$ for $0 \le i \le n$ and $\varphi = \sigma \circ h_*$. In other words, $\mathrm{PMod}(S^2 - X) \cong \mathrm{Out}^{**}\, \pi_1(S^2 - X, x_0)$.*

Proof Applying the Dehn–Nielsen–Baer theorem we get the homeomorphism h of $S^2 - X$ and σ. It is elementary that h extends to a homeomorphism of S^2. Next assume $h(x_1) = x_j$ and fix a geometric basis $\mathscr{B} = \{[\alpha_i]\}$ adapted to x. Here α_i is a loop at x_0 that makes a counterclockwise turn around x_i as in 4.1. Hence $[h\alpha_1]$ is a loop making a turn around $h(x_1) = x_j$. But $[h\alpha_1]$ is conjugate to $\sigma h_*([\alpha_1]) = \varphi([\alpha_1])$ which is conjugate to $[\alpha_1]$ since $\varphi \in \mathrm{Aut}^{**}\, \pi_1(S^2 - X, x_0)$. Since α_1 makes a turn around x_1 it follows that $h(x_1) = x_j = x_1$. Similarly $h(x_i) = x_i$ for any i. □

4.15 We conclude this section by interpreting some classical constructions in the theory of braid groups using parallel transport. We consider the (pure version of the) generalized Birman exact sequence associated with $\mathbb{C}^{**} = \mathbb{P}^1 - \{0, 1, \infty\}$ (see [Farb and Margalit 2012, Theorem 9.1, page 245])

$$1 \to \pi_1(\mathsf{M}_{0,n}, X) \xrightarrow{\text{Push}} \mathrm{PMod}(\mathbb{P}^1 - X) \xrightarrow{\text{Forget}} \mathrm{PMod}(\mathbb{C}^{**}) \to 1. \tag{4-2}$$

The map Forget is the natural homeomorphism obtained by filling in the punctures, ie it is the map induced by the inclusion $\mathbb{P}^1 - X \hookrightarrow \mathbb{C}^{**}$. The map Push is defined as follows; see [Farb and Margalit 2012, Section 4.2.1]. Let $\alpha = (\alpha_1, \dots, \alpha_n)\colon [0, 1] \to \mathsf{M}_{0,n}$ be a pure braid in $\mathbb{P}^1$, with $\alpha(0) = \alpha(1) = X$. Thinking of α as an isotopy from X to X (sending each x_i to x_i) we get by Theorem 3.4 that it can be extended to an isotopy of the whole $\mathbb{P}^1$. Denoting by Φ_α the homeomorphism of $\mathbb{P}^1$ obtained at the end of the isotopy, we have that $\Phi_\alpha(x_i) = \alpha_i(1) = x_i$, and thus Φ_α can be regarded as an homeomorphism of $\mathbb{P}^1 - X$. Taking its isotopy class we get $\mathrm{Push}(\alpha) = [\Phi_\alpha] \in \mathrm{PMod}(\mathbb{P}^1 - X)$. This map is well defined, ie it does not depend on the choice of α within its homotopy class nor on the choice of the isotopy extension.

4.16 It is useful to reinterpret the morphism ε defined in (2-4) in this setting. In particular we note that $\operatorname{Im}\varepsilon \subset \operatorname{Aut}^{**}(\pi_1(\mathbb{P}^1 - X, x_0))$. Fix $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$.

Arguing as in Proposition 3.7 note that f^α extends to a homeomorphism $f^\alpha \colon \mathbb{P}^1 \to \mathbb{P}^1$ that fixes every x_i individually. Hence $[f^\alpha] \in \operatorname{PMod}(\mathbb{P}^1 - X)$. Since $\varepsilon([\alpha]) = f^\alpha_*$, we have $\varepsilon([\alpha]) \in \operatorname{Aut}^{**}(\pi_1(\mathbb{P}^1 - X))$.

Let $\tilde{\varepsilon} \colon \pi_1(\mathsf{M}_{0,n}, X) \to \operatorname{Out}^{**}(\pi_1(\mathbb{C}^{**} - X, x_0))$ denote the composition of ε with the natural projection $\operatorname{Aut}^{**} \to \operatorname{Out}^{**}$. Also, denote by $F \colon \operatorname{PMod}(\mathbb{C}^{**} - X) \to \operatorname{Out}^{**}(\pi_1(\mathbb{C}^{**} - X, x_0))$ the isomorphism $F \colon [h] \mapsto [h_*]$ coming from Corollary 4.14 of the Dehn–Nielsen–Baer theorem. The following proposition is the analogue of [Birman 1974, Theorem 1.10] for configurations of points in $\mathbb{C}^{**}$ (instead of $\mathbb{C}$).

Proposition 4.17 *For* $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$, *let* f^α *be the parallel transport as in Lemma 3.5. Then* $\operatorname{Push}([\alpha]) = [f^\alpha]$. *Moreover, the following diagram commutes*:

$$\begin{array}{ccc} & & \operatorname{PMod}(\mathbb{P}^1 - X) \\ & \overset{\operatorname{Push}}{\nearrow} & \big\downarrow F \\ \pi_1(\mathsf{M}_{0,n}, X) & \underset{\tilde{\varepsilon}}{\searrow} & \\ & & \operatorname{Out}^{**}(\pi_1(\mathbb{P}^1 - X, x_0)) \end{array}$$

Proof Let $\alpha \colon [0,1] \to \mathsf{M}_{0,n}$ be a pure braid in $\mathbb{P}^1$, with $\alpha(0) = \alpha(1) = X$, that we think as an isotopy from X to X. Let $\widetilde{H} \colon (\mathbb{P}^1 - X) \times [0,1] \to \mathsf{M}_{0,n+1}$ and f^α be as in Lemma 3.5. Define a map $\psi \colon \mathbb{P}^1 \times [0,1] \to \mathbb{P}^1$ by $\psi(u,t) := \widetilde{H}(u,t)$ for $u \notin X$ and $\psi(x_i, t) := \alpha_i(t)$. So ψ is an ambient isotopy of $\mathbb{P}^1$ extending the isotopy α. This proves the result, since by Proposition 3.7 $\varepsilon([\alpha]) = f^\alpha_*$, so $\tilde{\varepsilon}([\alpha]) = f^\alpha_* \bmod \operatorname{Inn}\pi_1(\mathbb{P}^1 - X, x_0)$, while $\operatorname{Push}([\alpha]) = [f^\alpha]$. □

Remark 4.18 Considering configurations of points in $\mathbb{C}$ instead of $\mathbb{C}^{**}$, Proposition 4.17 corresponds to [Birman 1974, Theorem 1.10].

Proposition 4.19 *Let* $x = (x_0, X) \in \mathsf{M}_{0,n+1}$ *and let* $\nu \in \operatorname{Aut}^{**}\pi_1(\mathbb{P}^1 - X, x_0)$. *Then there is an* $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$, *a lift* $\tilde{\alpha}$ *of* α *with* $\tilde{\alpha}(0) = \tilde{\alpha}(1) = x_0$, *a parallel transport* f^α_t *such that* $f^\alpha_t(x_0) = \tilde{\alpha}(t)$ *and* $z \in \pi_1(\mathbb{P}^1 - X, x_0)$ *such that* $\nu = \operatorname{inn}_z \circ f^\alpha_*$.

Proof Since $\operatorname{PMod}(\mathbb{C}^{**})$ is trivial — see [Farb and Margalit 2012, Proposition 2.3] — it follows from (4-2) that Push (and thus $\tilde{\varepsilon}$) is an isomorphism. In particular, for every $\nu \in \operatorname{Aut}^{**}(\pi_1(\mathbb{P}^1 - X, x_0))$, there exists $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$ and $\sigma \in \operatorname{Inn}(\pi_1(\mathbb{P}^1 - X, x_0))$ such that $f^\alpha_* = \varepsilon([\alpha]) = \nu \circ \sigma$. Thus $\nu = \operatorname{inn}_z \circ f^\alpha_*$ for some $z \in \pi_1(\mathbb{P}^1 - X, x_0)$. □

5 Topological types of actions

Definition 5.1 Let G be a finite group and let Σ_1 and Σ_2 be oriented topological surfaces both endowed with an action of G. We say that the two actions are *topologically equivalent* if there is an $\eta \in \operatorname{Aut} G$ and an orientation-preserving homeomorphism $f : \Sigma_1 \cong \Sigma_2$ such that $f(g \cdot x) = \eta(g) \cdot f(x)$ for any $x \in \Sigma_1$ and any $g \in G$; see [González-Díez and Harvey 1992]. An equivalence class is called a *topological type* of G–action (sometimes this is called *unmarked topological type*).

Definition 5.2 Fix on S^2 the orientation by the outer normal. We let $\mathcal{T}^n(G)$ denote the set of topological types of G–actions on a topological surface Σ such that $\Sigma/G \cong S^2$ (as oriented surfaces) and the projection $\pi\colon \Sigma \to \Sigma/G$ has n branch points.

Definition 5.3 If G is a finite group an n–datum is an epimorphism $\theta\colon \Gamma_n \to G$ is such that $\theta(\gamma_i) \neq 1$ for $i = 1, \dots, n$. We let $\mathcal{D}^n(G)$ denote the set of all n–data associated with the group G.

5.4 Fix a point $x = (x_0, X) \in \boldsymbol{F}_{0,n+1}\, S^2$ and a geometric basis $\mathcal{B} = \{[\alpha_i]\}_{i=1}^n$ of $\pi_1(S^2 - X, x_0)$. Denote by $\chi\colon \Gamma_n \cong \pi_1(S^2 - X, x_0)$ the corresponding isomorphism. If $\theta\colon \Gamma_n \to G$ is a n–datum, the epimorphism $\theta \circ \chi^{-1}$ gives rise to a topological G–covering $p\colon \Sigma_0^\theta \to S^2 - X$. By the topological part of the Riemann existence theorem, this can be completed to a branched G–cover $p\colon \Sigma^\theta \to S^2$. By taking the equivalence class of Σ^θ we get a topological type of G–action. We get a map

$$\mathcal{F}_{x,\mathcal{B}}\colon \mathcal{D}^n(G) \to \mathcal{T}^n(G), \quad (\theta\colon \Gamma_n \to G) \mapsto [\Sigma^\theta].$$

5.5 We now introduce an action on the set of data that will be very important for the rest of the paper. By the Dehn–Nielsen–Baer theorem, $\operatorname{Out}^* \Gamma_n \cong \operatorname{Out}^*(\pi_1(S^2 - X, x_0)) \cong \operatorname{Mod}(S^2 - X)$. The latter group has a presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\begin{aligned} &\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for } |i-j| \geq 2, && (\sigma_1 \cdots \sigma_{n-1})^n = 1, \\ &\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i, && \sigma_1 \cdots \sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2} \cdots \sigma_1 = 1. \end{aligned} \tag{5-1}$$

See [Birman 1974, Theorem 4.5, page 164]. Consider the short exact sequence

$$1 \to \operatorname{Inn} \Gamma_n \hookrightarrow \operatorname{Aut}^* \Gamma_n \xrightarrow{\pi} \operatorname{Out}^* \Gamma_n \to 1.$$

Let $\tilde{\sigma}_i : \Gamma_r \to \Gamma_r$ be the automorphism defined by the rule

$$\tilde{\sigma}_i(\gamma_i) = \gamma_{i+1}, \qquad \tilde{\sigma}_i(\gamma_{i+1}) = \gamma_{i+1}^{-1}\gamma_i\gamma_{i+1}, \qquad \tilde{\sigma}_i(\gamma_j) = \gamma_j \quad \text{for } j \neq i, i+1.$$

These automorphisms belong to $\operatorname{Aut}^* \Gamma_n$ and satisfy the relations (5-1) up to inner automorphisms. Moreover, $\pi(\tilde{\sigma}_i) = \sigma_i$. The group $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$ acts on the set $\mathcal{D}^n(G)$ by the rule

$$(\eta, \nu) \cdot \theta := \eta \circ \theta \circ \nu^{-1},$$

where $(\eta, \nu) \in \operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$ and $\theta \in \mathscr{D}^n(G)$ is a datum. We can view this action as an iterated action: first $\operatorname{Aut} G$ acts on $\mathscr{D}^n(G)$, then $\operatorname{Aut}^* \Gamma_n$ acts on the quotient $\mathscr{D}^n(G)/\operatorname{Aut} G$. Observe also that for any $a \in \Gamma_n$,

$$\theta \circ (\mathrm{inn}_a)^{-1} = \mathrm{inn}_{\theta(a)^{-1}} \circ \theta.$$

So inner automorphisms of Γ_n can be absorbed in the action of $\operatorname{Aut} G$. Since the automorphisms $\tilde{\sigma}_i$ above satisfy the relations (5-1) up to inner automorphisms, it follows that they do satisfy them exactly when acting on $\mathscr{D}^n(G)/\operatorname{Aut} G$. In this way one gets an action of $\operatorname{Out}^* \Gamma_n$ on $\mathscr{D}^n(G)/\operatorname{Aut} G$. Finally we claim that the actions of $\operatorname{Aut}^* \Gamma_n$ and $\operatorname{Out}^* \Gamma_n$ on $\mathscr{D}^n(G)/\operatorname{Aut} G$ have the same orbits; hence

$$\mathscr{D}^n/(\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n) = (\mathscr{D}^n/\operatorname{Aut} G)/\operatorname{Aut}^* \Gamma_n = (\mathscr{D}^n/\operatorname{Aut} G)/\operatorname{Out}^* \Gamma_n.$$

The reason is the same as before: inner automorphisms of Γ_n can be absorbed in the action of $\operatorname{Aut} G$.

Theorem 5.6 *Let G be a finite group. Choose*

(1) *an element $x = (x_0, X) \in \boldsymbol{F}_{0,n+1} S^2$;*

(2) *a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ of $\pi_1(S^2 - X, x_0)$.*

Then the map $\mathscr{F}_{x,\mathscr{B}}$ induces a bijection between $\mathscr{D}^n(G)/(\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n)$ and the set $\mathscr{T}^n(G)$ of topological types of G–actions. The bijection does not depend on the choices of the point $x \in \boldsymbol{F}_{0,n+1} S^2$ and of the geometric basis $\mathscr{B}$.

From the discussion in 5.5 we immediately get the following.

Corollary 5.7 *The topological types of G–actions on curves of genus g are in bijection with*

$$(\mathscr{D}^n/\operatorname{Aut} G)/\operatorname{Out}^* \Gamma_n.$$

The proof of Theorem 5.6 is based on the following two propositions.

Proposition 5.8 *The map $\mathscr{F}_{x,\mathscr{B}}$ is constant on the orbits of the action of* $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$.

Proof Let $\theta\colon \Gamma_n \to G$ be a datum and $(\eta, \nu) \in \operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$. Let $\theta' = \eta \circ \theta \circ \nu^{-1}$. We want to show that Σ^θ and $\Sigma^{\theta'}$ have the same topological type of G–action. Set $\bar{\nu} = \chi \circ \nu \circ \chi^{-1}$. Observe that $\bar{\nu} \in \mathrm{Aut}^*(\pi_1(S^2 - X, x_0))$ since $\nu \in \mathrm{Aut}^* \Gamma_n$. By the Dehn–Nielsen–Baer Theorem 4.12, there is $\sigma \in \mathrm{Inn}(\pi_1(S^2 - X, x_0))$ and an orientation-preserving diffeomorphism $h\colon (S^2 - X, x_0) \to (S^2 - X, x_0)$ such that $h(x_0) = x_0$ and $\sigma \circ h_* = \bar{\nu}$. Let p and p' denote the projections

$$\begin{array}{ccc} \Sigma_0^{\theta} & & \Sigma_0^{\theta'} \\ \big\downarrow p & & \big\downarrow p' \\ (S^2 - X, x_0) & \xrightarrow{\ h\ } & (S^2 - X, x_0) \end{array}$$

Choose $\tilde{x}_0 \in \Sigma_0^{\theta}$ and $\tilde{x}'_0 \in \Sigma_0^{\theta'}$ both over x_0. We have that

$$h_*(p_*(\pi_1(\Sigma_0^{\theta}, \tilde{x}_0))) = (\sigma^{-1} \circ \bar{\nu})(\ker(\theta \circ \chi^{-1})) = \sigma^{-1}(\ker(\theta \circ \chi^{-1} \circ \bar{\nu}^{-1})) = \ker(\theta \circ \chi^{-1} \circ \bar{\nu}^{-1}),$$

where the last equality holds because σ is an inner automorphism. Moreover, since $\eta \in \mathrm{Aut}\, G$,

$$\ker(\theta \circ \chi^{-1} \circ \bar{\nu}^{-1}) = \ker(\eta \circ \theta \circ \chi^{-1} \circ \bar{\nu}^{-1}).$$

Thus $h_*(p_*(\pi_1(\Sigma_0^{\theta}, \tilde{x}_0))) = \ker(\eta \circ \theta \circ \chi^{-1} \circ \bar{\nu}^{-1}) = (p')_*(\pi_1(\Sigma_0^{\theta'}))$. By the lifting theorem we get an oriented homeomorphism $\tilde{h}\colon \Sigma_0^{\theta} \to \Sigma_0^{\theta'}$ such that the diagram commutes and which extends to the compactifications. Hence the G–actions on Σ^{θ} and $\Sigma^{\theta'}$ have the same topological type. □

Proposition 5.9 *If $\theta \in \mathscr{D}^n(G)$, then $\mathscr{F}_{x,\mathscr{B}}(\theta)$ does not depend on the choices of the point $x \in \boldsymbol{F}_{0,n+1}\, S^2$ and of the geometric basis $\mathscr{B}$.*

Proof First fix x and consider two geometric bases $\mathscr{B}$ and $\overline{\mathscr{B}}$. Let $\chi, \bar{\chi}\colon \Gamma_n \to \pi_1(S^2 - X, x_0)$ denote the corresponding isomorphisms. Then $\nu := \chi^{-1} \circ \bar{\chi} \in \mathrm{Aut}^* \Gamma_n$. For a datum θ, we have $\theta \circ \bar{\chi}^{-1} = \theta \circ \nu^{-1} \circ \chi^{-1}$. So $\mathscr{F}_{x,\overline{\mathscr{B}}}(\theta) = \mathscr{F}_{x,\mathscr{B}}(\theta \circ \nu^{-1})$. By Proposition 5.8, $\mathscr{F}_{x,\mathscr{B}}(\theta \circ \nu^{-1}) = \mathscr{F}_{x,\mathscr{B}}(\theta)$. Hence $\mathscr{F}_{x,\overline{\mathscr{B}}}(\theta) = \mathscr{F}_{x,\mathscr{B}}(\theta)$, as desired. Now suppose that $x, y \in \boldsymbol{F}_{0,n+1}\, S^2$. Let $\chi\colon \Gamma_n \to \pi_1(S^2 - X, x_0)$ and $\bar{\chi}\colon \Gamma_n \to \pi_1(S^2 - Y, y_0)$ be the isomorphisms associated with two geometric bases $\mathscr{B}$ and $\overline{\mathscr{B}}$. Then

$$\nu := \bar{\chi} \circ \chi^{-1}\colon \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - Y, y_0)$$

sends a geometric basis to a geometric basis. Hence, by Corollary 4.13, there is $\sigma \in \mathrm{Inn}(\pi_1(S^2 - Y, y_0))$ and an orientation-preserving homeomorphism $h\colon (S^2 - X, x_0) \to (S^2 - Y, y_0)$ such that $h(x_0) = y_0$ and $\sigma \circ h_* = \nu$. Given a datum θ, h_* maps the kernel of $\theta \circ \chi^{-1}$ to the kernel of $\theta \circ \bar{\chi}^{-1}$. By the lifting theorem there is an oriented diffeomorphism $\tilde{h}$ that extends to the compactifications. Hence the G–actions on Σ_x^{θ} and Σ_y^{θ} have the same topological type. □

We recall two basic facts about monodromy maps. Let $p\colon E \to B$ be a topological G–covering. For $b \in B$ and $e \in p^{-1}(b)$, we denote by $\mu_{p,e}$ the monodromy map $\mu_{p,e}\colon \pi_1(B, b) \to G$ such that $g = \mu_{p,e}[\alpha]$ maps e to $\alpha_e(1)$, where α_e is the lift of α with initial point e.

Lemma 5.10 *Let $p\colon E \to B$ be a topological G–covering. Fix $b_0, b_1 \in B$ and $e_i \in p^{-1}(b_i)$. Let δ be a path from e_0 to e_1 and $\gamma = p \circ \delta$. Then $\mu_{p,e_0} = \mu_{p,e_1} \circ \gamma_{\#}$. In particular, if $b_0 = b_1$ then μ_{p,e_0} and μ_{p,e_1} differ by an inner automorphism of $\pi_1(B, b_0)$ or — equivalently — of G.*

Lemma 5.11 *Let $p\colon E \to B$ and $p'\colon E' \to B'$ be G–coverings. Let $\tilde{h}\colon E \to E'$ be a G–equivariant homeomorphism and denote by $h\colon B \to B'$ the induced homeomorphism. Fix $e_0 \in E$. Then $\mu_{p,e_0} = \mu_{p',\tilde{h}(e_0)} \circ h_*$.*

Proof of Theorem 5.6 By Proposition 5.8, $\mathcal{F}_{x,\mathcal{B}}$ induces a map between $\mathcal{D}^n(G)/\ (\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n)$ and $\mathcal{T}^n(G)$. To prove the statement we have to check that

(1) if two epimorphisms $\theta, \theta'\colon \Gamma_r \to G$ give rise to the same topological type of G–action, then θ and θ' are in the same orbit for the action of $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$;

(2) every topological type of G–action with n branch points can be constructed from a datum in $\mathcal{D}^n(G)$.

To prove (1), consider the branched covers $p\colon \Sigma \to S^2$ and $p'\colon \Sigma' \to S^2$ associated with $\theta \circ \chi^{-1}$ and $\theta' \circ \chi^{-1}$ and suppose that there exists $\eta \in \operatorname{Aut} G$ and an orientation-preserving homeomorphism $\tilde{h}\colon \Sigma \to \Sigma'$ such that $\tilde{h}(g \cdot e) = \eta(g)\tilde{h}(e)$. We get an induced homeomorphism $h\colon \Sigma/G \to \Sigma'/G$ and an isomorphism $h_*\colon \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - X, h(x_0))$. Fix $e_0 \in p^{-1}(x_0)$. From Lemma 5.11 it follows that $\mu_{p,e_0} = \eta \circ \mu_{p',\tilde{h}(e_0)} \circ h_*$. Now fix $e'_0 \in (p')^{-1}(x_0)$ and a path in Σ' from $\tilde{h}(e_0)$ to e'_0. Finally let $\gamma = p' \circ \delta$. By Lemma 5.10 we get that $\mu_{p',\tilde{h}(e_0)} = \mu_{p',e'_0} \circ \gamma_\#$. Thus

$$\mu_{p,e_0} = \eta \circ \mu_{p',e'_0} \circ \gamma_\# \circ h_*. \tag{5-2}$$

Observe that, since $\tilde{h}$ preserves the orientation, so does h; hence $\gamma_\# \circ h_*\colon \pi_1(S^2 - X, x_0) \to \pi_1(S^2 - X, x_0)$ lies in $\operatorname{Aut}^*(\pi_1(S^2 - X, x_0))$. Let $\nu := \chi^{-1} \circ (\gamma_\# \circ h_*) \circ \chi \in \operatorname{Aut}^* \Gamma_n$ be the corresponding automorphism in $\operatorname{Aut}^* \Gamma_n$. (Again we are using that χ comes from a geometric basis.) Also, observe that $\theta \circ \chi^{-1}$ coincides with μ_{p,e_0} up to an inner automorphism of G, and the same holds for $\theta' \circ \chi^{-1}$ and for μ_{p',e'_0}. We get that there exists $\eta \in \operatorname{Aut} G$ such that (5-2) becomes

$$\theta \circ \chi^{-1} = \eta \circ \theta' \circ \chi^{-1} \circ (\gamma_\# \circ h_*) = \eta \circ \theta' \circ \nu^{-1} \circ \chi^{-1}.$$

Thus $(\eta, \nu).\theta' = \theta$; that is, they are in the same orbit for the action of $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$. To prove (2) assume that G acts effectively on a surface Σ in such a way that $\Sigma/G \cong S^2$. Up to diffeomorphism we can assume that the set of critical values of $p\colon \Sigma \to S^2$ coincides with X. Fix a point $\tilde{x}_0 \in p^{-1}(x_0)$. Let $\theta := \mu_{p,\tilde{x}_0} \circ \chi\colon \Gamma_n \to G$ be the monodromy of the unramified cover. Since Σ_0^θ is connected θ is surjective, and $\theta(\gamma_i) \neq 1$ since all the points of X are branch points. So it is an n–datum. By construction the associated cover coincides with Σ. Finally, it follows from Proposition 5.9 that the bijection induced by $\mathcal{F}_{x,\mathcal{B}}$ does not depend on x and $\mathcal{B}$. □

6 Tools for the construction

This section is dedicated to some tools that we will need in the following section for the construction of the families. We start with some considerations from group theory, that will be at the basis of the construction of the ordered set $(\mathcal{I}, \geq)$ of Theorem 1.4.

Consider an exact sequence of groups

$$(*) \qquad 1 \to N \xrightarrow{i} K \xrightarrow{p} H \to 1$$

and an epimorphism

$$f : N \twoheadrightarrow G$$

onto a *finite* group G.

Definition 6.1 An *extension* a of $(*, f)$ is a pair $a = (H_a, f_a)$, whose first element is a subgroup H_a of H of *finite index*, and whose second element is a morphism $f_a \colon p^{-1}(H_a) \to G$ such that $f_a i = f$. We denote by $\mathscr{I}(*, f)$ the set of all extensions.

If $a = (H_a, f_a)$ is an extension, we set

$$K_a := p^{-1}(H_a).$$

K_a is a subgroup of K and f_a is defined on K_a.

On the set $\mathscr{I}(*, f)$ we introduce the order relation

$$a \geq b \iff H_a \subset H_b \text{ and } f_a = f_b|_{K_a}.$$

Proposition 6.2 *$(\mathscr{I}(*, f), \geq)$ is a directed set.*

Proof Given $a, b \in \mathscr{I}(*, f)$, set $H_c := \{h \in H_a \cap H_b \mid f_a(h) = f_b(h)\}$. Then H_c has finite index in H since G is finite. Set $f_c := f_a|_{H_c}$. Then $c := (H_c, f_c) \in \mathscr{I}(*, f)$, and $c \geq a$ and $c \geq b$. □

In the following lemmas we describe two natural bijections between the sets $\mathscr{I}(*, f)$, when f and $(*)$ change under some specific rule.

Lemma 6.3 *Given $f : N \twoheadrightarrow G$ and $\eta \in \operatorname{Aut} G$, set $\bar{f} := \eta \circ f$. Then*

$$\text{(6-1)} \qquad \Phi \colon \mathscr{I}(*, f) \to \mathscr{I}(*, \bar{f}), \quad \Phi(H_a, f_a) := (H_a, \eta \circ f_a).$$

is an order-preserving bijection.

The proof is immediate.

Lemma 6.4 *Consider a commutative diagram of groups*

$$\begin{array}{ccccccccccc}
(*) & & 1 & \longrightarrow & N & \xrightarrow{i} & K & \xrightarrow{p} & H & \longrightarrow & 1 \\
\downarrow & & & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \\
(\bar{*}) & & 1 & \longrightarrow & \overline{N} & \xrightarrow{\bar{i}} & \overline{K} & \xrightarrow{\bar{p}} & \overline{H} & \longrightarrow & 1
\end{array}$$

with exact rows and α, β and γ isomorphisms. In other words, $(*)$ and $(\bar{*})$ are isomorphic short exact sequences. Given $\bar{f} : \overline{N} \twoheadrightarrow G$, set $f := \bar{f} \circ \alpha : N \twoheadrightarrow G$. Then the map

$$\text{(6-2)} \qquad \Phi \colon \mathscr{I}(*, f) \to \mathscr{I}(\bar{*}, \bar{f}), \quad \Phi(H_a, f_a) := (\beta(H_a), f_a \circ \gamma^{-1}|_{\gamma(K_a)}),$$

is an order-preserving bijection.

Proof If $a = (H_a, f_a)$, set $K_a = p^{-1}(H_a)$ as above. Set $\overline{H}_{\bar{a}} := \beta(H_a)$. Then

$$\overline{K}_{\bar{a}} := \bar{p}^{-1}(\beta(H_a)) = (\beta^{-1}\bar{p})^{-1}(H_a) = (p\gamma^{-1})^{-1}(H_a) = \gamma(K_a).$$

Set also $\bar{f_{\bar{a}}} := f_a \circ \gamma^{-1}|_{\overline{K}_{\bar{a}}}$. Then it is immediate to check that $\bar{a} := (\overline{H}_{\bar{a}}, \bar{f_{\bar{a}}}) = \Phi(a)$ belongs to $\mathscr{I}(\bar{*}, \bar{f})$ and that Φ is an order-preserving bijection. □

Lemma 6.5 *Let N, H and G be groups and let $\varepsilon\colon H \to \operatorname{Aut} N$, $h \mapsto \varepsilon_h$, be a morphism. Let $f : N \to G$ and $\varphi\colon H \to G$ be morphisms. There is a morphism $f'\colon N \rtimes_\varepsilon H \to G$ extending both f and φ (when N and H are included in $N \rtimes_\varepsilon H$ in the obvious way) if and only if for any $h \in H$*

$$\operatorname{inn}_{\varphi(h)} \circ f = f \circ \varepsilon_h. \tag{6-3}$$

The proof is elementary.

Lemma 6.6 *Let N, H, G, $\varepsilon\colon H \to \operatorname{Aut} N$ and f be as above. Assume that f is surjective, that N is finitely generated and that G is finite. Then*

(a) *$H'' := \{h \in H \mid \varepsilon_h(\ker\theta) = \ker\theta\}$ is a finite-index subgroup of H;*

(b) *there is a morphism $\tilde{\varepsilon}\colon H'' \to \operatorname{Aut} G$ such that the diagram*

$$\begin{array}{ccc} N & \xrightarrow{\varepsilon_h} & N \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ G & \xrightarrow{\tilde{\varepsilon}_h} & G \end{array}$$

commutes for $h \in H''$;

(c) *$H' := \ker\tilde{\varepsilon}$ is a finite-index subgroup of H;*

(d) *there is a unique morphism $f'\colon N \rtimes_\varepsilon H' \to G$ that extends f and such that $f'|_{H'} \equiv 1$.*

Proof The subgroup $\ker f$ has index $d := |G| < \infty$ in N. Since N is finitely generated, there are a finite number of index d subgroups of N; see eg [Hall 1950, page 128; Kurosh 1960, page 56]. The natural action of $\operatorname{Aut} N$ on the subgroups of N preserves the index. Therefore the orbit of $\operatorname{Aut} N$ through $\ker f$ is finite. Hence $(\operatorname{Aut} N)_{\ker f}$ has finite index in $\operatorname{Aut} N$. Since H/H'' injects in $\operatorname{Aut} N/(\operatorname{Aut} N)_{\ker f}$, H'' also has finite index in H. The existence of $\tilde{\varepsilon}_h$ follows immediately from the inclusion $\varepsilon_h(\ker f) \subset \ker f$ for $h \in H''$. Since $\operatorname{Aut} G$ is finite, H' has finite index in H'' and in H. By construction, for any $h \in H'$ we have $f = \tilde{\varepsilon}_h \circ f = f \circ \varepsilon_h$, ie (6-3) holds with $\varphi\colon H' \to G$ the trivial morphism. □

Theorem 6.7 *If the sequence $(*)$ splits, then $\mathscr{I}(*, f) \neq \varnothing$.*

Proof By Lemma 6.4 we can assume that the split exact sequence $(*)$ is a semidirect product. The result then follows from Lemma 6.6. □

6.8 We dedicate the second part of this section to some considerations on coverings and fiber bundles, which will be fundamental tools for our construction.

In the following we assume that all the spaces considered are semilocally 1–connected. Let X be a connected space and let $x \in X$. For every subgroup $H \subset \pi_1(X,x)$ there is a pointed covering $p\colon (E,e) \to (X,x)$ such that $\operatorname{Im} p_* = H$. Moreover p is unique up to pointed isomorphism. If $\beta \in \Omega(X,x,x')$ and $\beta_\#\colon \pi_1(X,x) \to \pi_1(X,x')$ is the induced isomorphism, then the pointed coverings of X associated with $H \subset \pi_1(X,x)$ and with $\beta_\# H \subset \pi_1(X,x')$ are isomorphic. Indeed if β_e denotes the lift with $\beta_e(0)=e$ and $e'=\beta_e(1)$, then $p_*\pi_1(E,e')=\beta_\# H$, so E is associated with both subgroups. If X is a complex manifold, any covering has a unique complex structure such that p is holomorphic and the coverings associated to $H \subset \pi_1(X,x)$ and with $\beta_\# H \subset \pi_1(X,x')$ are biholomorphic.

Lemma 6.9 *Let $\bar E$, $\bar B$ and B be connected and locally arcwise connected topological spaces. Let $p\colon \bar E \to \bar B$ be a fiber bundle and $q\colon B \to \bar B$ be a covering. Let $E := q^*\bar E$ be the pullback bundle. Then in the diagram*

$$\begin{array}{ccc} (E,e) & \xrightarrow{\bar q} & (\bar E,\bar e) \\ \psi\downarrow & & \downarrow p \\ (B,b) & \xrightarrow{q} & (\bar B,\bar b) \end{array} \tag{6-4}$$

$\bar q\colon E \to \bar E$ is also a covering. Moreover, if the fiber of p is arcwise connected, then

$$\bar q_*\pi_1(E,e) = p_*^{-1}(q_*\pi_1(B,b)).$$

Proof Fix $\bar e \in \bar E$, set $\bar b = p(\bar e)$ and let $V \subset \bar B$ be an evenly covered open subset of $\bar B$, ie $q^{-1}(V) = \bigsqcup U_i$ and $q|_{U_i}$ is a homeomorphism of U_i onto V. We claim that $p^{-1}(V)$ is an evenly covered neighborhood of e. Indeed $\bar q^{-1}(p^{-1}(V)) = \bigsqcup \psi^{-1}(U_i)$. Moreover $\psi^{-1}(U_i) = (q|_{U_i})^*\bar E$ is mapped homeomorphically on $p^{-1}(V)$ by $\bar q$ since $q|_{U_i}$ is a homeomorphism onto V. This proves the first assertion. Next choose $e \in \bar q^{-1}(\bar e)$ and set $b = \psi(e)$. Obviously $q(b) = \bar b$. Set $\bar F := p^{-1}(\bar b)$ and $F := \psi^{-1}(\bar b)$. The diagram (6-4) induces a morphism of the homotopy exact sequences of the bundles:

$$\begin{array}{ccccccccccc} \longrightarrow & \pi_2(B) & \longrightarrow & \pi_1(F,e) & \longrightarrow & \pi_1(E,e) & \xrightarrow{p_*} & \pi_1(B,b) & \longrightarrow & \pi_0(F)=1 \\ & \downarrow\cong & & \downarrow\cong & & \downarrow\bar q_* & & \downarrow q_* & & \downarrow\cong \\ \longrightarrow & \pi_2(\bar B) & \longrightarrow & \pi_1(\bar F,\bar e) & \longrightarrow & \pi_1(\bar E,\bar e) & \xrightarrow{p_*} & \pi_1(\bar B,\bar b) & \longrightarrow & \pi_0(F)=1 \end{array}$$

Set $H := q_*\pi_1(B,b) \subset \pi_1(\bar B,\bar b)$, and $K := p_*^{-1}(H) \subset \pi_1(\bar E,\bar e)$. In the lower row we can substitute $\pi_1(B,b)$ with H and $\pi_1(E,e)$ with K and the row remains exact. Clearly $\bar q_*$ maps into K since the diagram commutes. So we get the diagram

$$\begin{array}{ccccccccccc} \longrightarrow & \pi_2(B) & \longrightarrow & \pi_1(F,e) & \longrightarrow & \pi_1(E,e) & \xrightarrow{p_*} & \pi_1(B,b) & \longrightarrow & \pi_0(F)=1 \\ & \downarrow\cong & & \downarrow\cong & & \downarrow\bar q_* & & \downarrow q_* & & \downarrow\cong \\ \longrightarrow & \pi_2(\bar B) & \longrightarrow & \pi_1(\bar F,\bar e) & \longrightarrow & K & \xrightarrow{p_*} & H & \longrightarrow & \pi_0(F)=1 \end{array}$$

Now q_* is an isomorphism. Applying the short five lemma [Eilenberg and Steenrod 1952, page 16], we get that $K = \operatorname{Im}\bar q_*$, as desired. □

The following lemma is a sort of converse which will be needed later.

Lemma 6.10 *Let A, $\overline{E}$, E, B and $\overline{B}$ be connected and locally arcwise connected topological spaces. Consider the diagram*

(6-5)
$$\begin{array}{ccc} (A,a) & \xrightarrow{\tilde{q}} & (\overline{E},\bar{e}) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ (B,b) & \xrightarrow{q} & (\overline{B},\bar{b}) \end{array}$$

Assume that $\varphi\colon A \to B$ and $p\colon \overline{E} \to \overline{B}$ are fiber bundles with arcwise connected fibers, that q and $\tilde{q}$ are finite degree coverings, and that $\tilde{q}_*\pi_1(A,a) = p_*^{-1}(q_*\pi_1(B,b))$. Then A is isomorphic to $q^*\overline{E}$ as a fiber bundle over B.

Proof Apply Lemma 6.9. Using the same notation as in (6-4),

$$\bar{q}_*\pi_1(E,e) = p_*^{-1}(q_*\pi_1(B,b)) = \tilde{q}_*\pi_1(A,a).$$

Moreover $\bar{q}$ is also a covering. So there is $w\colon (A,a) \to (E,e)$ such that $\bar{q} \circ w = \tilde{q}$. It remains to show that $\psi \circ w = \varphi$. Combining (6-5) with (6-4) we get the commutative diagram

$$\begin{array}{ccccc} & & \overset{\tilde{q}}{\curvearrowright} & & \\ (A,a) & \xrightarrow{w} & (E,e) & \xrightarrow{\bar{q}} & (\overline{E},\bar{e}) \\ & {\scriptstyle\varphi}\searrow & {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ & & (B,b) & \xrightarrow{q} & (\overline{B},\bar{b}) \end{array}$$

From $\bar{q} \circ w = \tilde{q}$ we get $p \circ \bar{q} \circ w = p \circ \tilde{q}$; hence $q \circ \psi \circ w = q \circ \varphi$. So $\psi \circ w$ and φ lift the same map with respect to the covering q. Since $\psi \circ w(a) = \varphi(a)$, we conclude that $\psi \circ w = \varphi$ and the result follows. □

7 Construction of the families of G–curves

7.1 Fix an element $x = (x_0, X) \in \mathsf{M}_{0,n+1}$ and set

(7-1)
$$N_x := \pi_1(\mathbb{P}^1 - X, x_0), \quad K_x := \pi_1(\mathsf{M}_{0,n+1}, x), \quad H_X := \pi_1(\mathsf{M}_{0,n}, X).$$

Consider the split exact sequence in the top row of (2-3), namely

($*_x$)
$$1 \to N_x \xrightarrow{i_*} K_x \xrightarrow{p_*} H_X \to 1.$$

Here $i\colon \mathbb{P}^1 - X \hookrightarrow \mathsf{M}_{0,n+1}$ is the map $i(x') := (x', x_1, \dots, x_n)$ and $p\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$ is the fibration. Now let G be a finite group and let $\theta\colon \Gamma_n \to G$ be a datum. Choose a geometric basis $\mathscr{B} = \{[\alpha_i]\}_{i=1}^n$ of N_x. As in 4.4, let $\chi\colon \Gamma_n \to N_x$ be the isomorphism induced from the basis $\mathscr{B}$. We apply the group theoretical considerations of Section 6 to the exact sequence $(*_x)$ with $f := \theta \circ \chi^{-1}\colon N_x \twoheadrightarrow G$. We get a directed set $\mathcal{I}(*_x, f)$, which is nonempty since $(*_x)$ splits. To stress the dependence from the choices made, we will set

$$\mathcal{I}(x, \mathcal{B}, \theta) := \mathcal{I}(*_x, \theta \circ \chi^{-1}).$$

Indeed χ contains the same information as the basis $\mathcal{B}$.

Definition 7.2 A *collection of families* is an indexed set $\{\mathcal{C}_a \to \mathsf{Y}_a\}_{a \in \mathcal{I}}$ where

(1) $(\mathcal{I}, \geq)$ is a directed set;

(2) (Y_a, y_a) is a pointed smooth complex quasiprojective variety;

(3) $\mathcal{C}_a \to \mathsf{Y}_a$ is a family of curves;

(4) if $a, b \in \mathcal{I}$ and $a \geq b$, then there is an étale cover of finite degree $v_{ab}\colon (\mathsf{Y}_a, y_a) \to (\mathsf{Y}_b, y_b)$ such that $\mathcal{C}_a \cong v_{ab}^* \mathcal{C}_b$.

In this section we construct a collection of families indexed by $\mathcal{I}(x, \mathcal{B}, \theta)$.

7.3 Fix $a = (H_a, f_a) \in \mathcal{I}(x, \mathcal{B}, \theta)$. Let $q_a\colon (\mathsf{Y}_a, y_a) \to (\mathsf{M}_{0,n}, X)$ be the pointed covering with $q_{a*}\pi_1(\mathsf{Y}_a, x_a) = H_a$. Endow Y_a with the unique structure of a complex manifold making q_a an unramified analytic cover. Consider the diagram

$$\begin{array}{ccc} (E_a := q_a^* \mathsf{M}_{0,n+1}, e_a) & \xrightarrow{\bar{q}_a} & (\mathsf{M}_{0,n+1}, x) \\ \big\downarrow \psi_a & & \big\downarrow p \\ (\mathsf{Y}_a, y_a) & \xrightarrow{q_a} & (\mathsf{M}_{0,n}, X) \end{array} \tag{7-2}$$

with $e_a := (y_a, x)$. Notice that $p\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n}$ is the universal family of lines with n holes and hence $\psi_a\colon E_a \to \mathsf{Y}_a$ is also a holomorphic family of curves (lines with holes).

By Lemma 6.9 applied to the diagram (7-2), the map $\bar{q}_a\colon E_a \to \mathsf{M}_{0,n+1}$ is the covering such that $\bar{q}_{a*}\pi_1(E_a, e_a) = K_a := p_*^{-1}(H_a)$. Hence $f_a\colon K_a \to G$ gives a morphism $\pi_1(E_a, e_a) \to G$ and thus a pointed G–covering $u_a\colon (\mathcal{C}_a^*, z_a) \to (E_a, e_a)$ such that $\operatorname{Im} u_{a*} = (\bar{q}_{a*})^{-1}(\ker f_a)$. In other words, u_a is the covering such that

$$\operatorname{Im} u_{a*} = \bar{q}_{a*}^{-1}(\ker f_a). \tag{7-3}$$

Composing with ψ_a we finally get a holomorphic family of noncompact Riemann surfaces

$$\pi_a = \psi_a \circ u_a\colon \mathcal{C}_a^* \to \mathsf{Y}_a.$$

The following diagram describes the whole situation:

$$\begin{array}{ccccc} (\mathcal{C}_a^*, z_a) & \xrightarrow{u_a} & (E_a, e_a) & \xrightarrow{\bar{q}_a} & (\mathsf{M}_{0,n+1}, x) \\ & \searrow^{\pi_a} & \big\downarrow \psi_a & & \big\downarrow p \\ & & (\mathsf{Y}_a, y_a) & \xrightarrow{q_a} & (\mathsf{M}_{0,n}, X) \end{array}$$

It might help to compare this diagram with the corresponding diagram of groups:

$$\begin{array}{ccccc} \ker f_a & \hookrightarrow & K_a & \hookrightarrow & K_x \\ & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ & & H_a & \hookrightarrow & H_X \end{array}$$

Summing up: p is the universal family of lines with n holes, q_a is a covering used as a base change, ψ_a is the pullback family of lines with n holes, u_a is a Galois cover and π_a is a family of noncompact Riemann surfaces. Each fiber of π_a covers the corresponding fiber of ψ_a. More precisely, if $y \in \mathsf{Y}_a$ and $X = q_a(y) \in \mathsf{M}_{0,n}$, looking at the fibers over y we have the unramified G–covering

$$\mathscr{C}^*_{a,y} \to E_{a,y} = \mathbb{P}^1 - X. \tag{7-4}$$

The last step in the construction is the fiberwise compactification, which is an application of the Grauert–Remmert extension theorem; see [Grothendieck 1971, Chapter XII, Theorem 5.4, page 340].

Theorem 7.4 (Grauert–Remmert extension theorem) *Let Y be a connected complex manifold and $Z \subset Y$ a closed analytic subset such that $Y^\circ := Y - Z$ is dense in Y. Let $f^\circ \colon X^\circ \to Y^\circ$ be a finite unramified cover. Then up to isomorphism there exists a unique normal analytic space X and a unique analytic covering $f : X \to Y$ such that $X^\circ \subset X$ and $f^\circ = f|_{X^\circ}$.*

Corollary 7.5 *In the hypotheses above, if Z is a smooth divisor, then X is smooth.*

Proof Let D be the unit disc. Using a local chart $U \cong D^n$ of Y such that $U \cap Z = D^n \cap \{z_1 = 0\}$ we get a finite cover of $D^* \times D^{n-1}$. By the topological classification of coverings disc, it is of the form $(z_1, \dots, z_n) \mapsto (z_1^m, z_2, \dots, z_n)$ for some $m \geq 1$, hence extends to an analytic cover $D^n \to D^n$. So, by uniqueness, $f^{-1}(U) \cong D^n$. In particular, $f^{-1}(U)$ is smooth. □

Lemma 7.6 *The unramified covering $u_a \colon \mathscr{C}^*_a \to E_a$ extends uniquely to an algebraic ramified cover $u_a \colon \mathscr{C}_a \to \mathbb{P}^1 \times \mathsf{Y}_a$, with $\mathscr{C}_a$ and Y_a smooth and quasiprojective.*

Proof Consider $\mathbb{P}^1 \times \mathsf{M}_{0,n}$. Let $x_0 \in \mathbb{P}^1$ and $X = (x_1, \dots, x_n) \in \mathsf{M}_{0,n}$. Recall that this means that $x_{n-2} = 0$, $x_{n-1} = 1$, $x_n = \infty$ and $(x_1, \dots, x_{n-3}) \in \boldsymbol{F}_{0,n-3}\, \mathbb{C}^{**}$. Let $Z_i \subset \mathbb{P}^1 \times \mathsf{M}_{0,n}$ be the smooth divisor $Z_i := \{x_0 = x_i\}$ for $i = 1, \dots, n$. The divisors $Z_1, \dots, Z_n$ are pairwise disjoint, so their union, which we denote by Z, is a smooth divisor of $\mathbb{P}^1 \times \mathsf{M}_{0,n}$. The map $\bar{q}_a$ in (7-2) obviously extends to a map

$$\bar{q}_a \colon \mathbb{P}^1 \times \mathsf{Y}_a \to \mathbb{P}^1 \times \mathsf{M}_{0,n}.$$

Then $\bar{q}_a^* Z$ is a smooth divisor of $\mathbb{P}^1 \times \mathsf{Y}_a$. Since $\mathsf{M}_{0,n+1} = (\mathbb{P}^1 \times \mathsf{M}_{0,n}) - Z$, $E_a = (\mathbb{P}^1 \times \mathsf{Y}_a) - \bar{q}_a^* Z$. So we can apply the Grauert–Remmert extension theorem to the topological covering $u_a \colon \mathscr{C}^*_a \to E_a$, which can be thus completed to a ramified cover $u_a \colon \mathscr{C}_a \to \mathbb{P}^1 \times \mathsf{Y}_a$, with $\mathscr{C}_a$ smooth. To prove the quasiprojectivity one uses a similar argument. An étale analytic cover of a quasiprojective variety is quasiprojective and the covering map is algebraic; see eg [Grothendieck 1971, Chapter XII, Theorem 5.1, page 333]. Since $\mathsf{M}_{0,n}$ and $\mathsf{M}_{0,n+1}$ are quasiprojective, and q_a and $\bar{q}_a$ are étale, we get that Y_a and E_a are quasiprojective and q_a and $\bar{q}_a$ are algebraic morphisms. Let $\bar{\mathsf{Y}}_a$ be a projective manifold containing Y_a as an open subset. Then E_a is a Zariski open subset of $\mathbb{P}^1 \times \bar{\mathsf{Y}}_a$ and we can apply the Grauert–Remmert extension theorem to $u_a \colon \mathscr{C}_a^* \to E_a$, this time viewing E_a as an open subset of $\mathbb{P}^1 \times \overline{\mathsf{Y}}_a$. We obtain a ramified cover $\bar{u}_a \colon \overline{\mathscr{C}}_a \to \mathbb{P}^1 \times \overline{\mathsf{Y}}_a$. Since $\mathbb{P}^1 \times \overline{\mathsf{Y}}_a$ is projective, $\overline{\mathscr{C}}_a$ is also projective. By uniqueness, $\mathscr{C}_a = \bar{u}_a^{-1}(\mathbb{P}^1 \times \mathsf{Y}_a)$, so it is quasiprojective. □

7.7 Notice that the projection $\mathbb{P}^1 \times \mathsf{Y}_a \to \mathsf{Y}_a$ extends ψ_a in (7-2), while the composition

$$\mathscr{C}_a \xrightarrow{u_a} \mathbb{P}^1 \times \mathsf{Y}_a \to \mathsf{Y}_a$$

extends π_a. We denote the extensions by the same symbol. We claim that

$$\pi_a \colon \mathscr{C}_a \to \mathsf{Y}_a$$

is a submersion. Indeed, let $U \cong D^n$ be a local chart in $\mathbb{P}^1 \times \mathsf{Y}_a$ such that

$$U \cap \pi^* Z = U \cap \pi^* Z_i = \{x_0 - x_i = 0\}$$

for some $i = 1, \dots, n$ (with $x_{n-2} = 0$, $x_{n-1} = 1$ and $x_n = \infty$). Denoting $w = x_0 - x_i$, we get that $w, x_1, \dots, x_n$ are local coordinates on U and $\pi'|_{\pi'^{-1}(U)} \colon \pi'^{-1}(U) \to U$ is of the form

$$(w, x_1, \dots, x_n) \mapsto (w^m, x_1, \dots, x_n)$$

for some $m \geq 2$. We conclude that locally $\pi_a(w, x_1, \dots, x_n) = (x_1, \dots, x_n)$. Thus π_a is a submersion onto a smooth base and its fibers are smooth curves.

If $y \in \mathsf{Y}_a$, the fiber $\mathscr{C}_{a,y} \to \mathbb{P}^1$ of π_a over y is the unique smooth compactification of the unramified cover (7-4), ie the one given by Riemann's existence theorem.

We call

$$\begin{array}{ccc} \mathscr{C}_a & \xrightarrow{\quad u_a \quad} & \mathbb{P}^1 \times \mathsf{Y}_a \\ & {\scriptstyle \pi_a} \searrow \quad \swarrow {\scriptstyle \psi_a} & \\ & \mathsf{Y}_a & \end{array}$$

the family of G–coverings associated with the datum $\theta \in \mathscr{D}^n(G)$, the point $x = (x_0, X) \in \mathsf{M}_{0,n+1}$, the geometric basis $\mathscr{B}$ of $\pi_1(\mathbb{P}^1 - X, x_0)$ and the extension $a \in \mathscr{I}(x, \mathscr{B}, \theta)$.

Theorem 7.8 *If $x \in \mathsf{M}_{0,n+1}$, $\mathscr{B}$ is a basis of N_x and θ is an n–datum, then*

$$\mathfrak{K}(x, \mathscr{B}, \theta) := \{\mathscr{C}_a \to \mathsf{Y}_a\}_{a \in \mathscr{I}(x,\mathscr{B},\theta)} \tag{7-5}$$

is a collection of families in the sense of Definition 7.2.

Proof It remains only to prove property (4). We start with an observation. If $p_i \colon (E_i, e_i) \to (B, b)$ are coverings and $\operatorname{Im} p_{1*} \subset \operatorname{Im} p_{2*}$, the unique continuous map $f \colon (E_1, e_1) \to (E_2, e_2)$ such that $p_2 \circ f = p_1$ is a covering map. Indeed let $f \colon (X, x) \to (E_2, e_2)$ be the covering with $\operatorname{Im} f_* = p_{2*}^{-1}(\operatorname{Im} p_{1*})$. Then $p_2 \circ f$ is a covering isomorphic to p_1, so we can assume $p_1 = p_2 \circ f$.

Now, given $a = (H_a, f_a)$ and $b = (H_b, f_b)$, $a \geq b$ means that $H_a \subset H_b \subset \pi_1(\mathsf{M}_{0,n}, X)$; hence $K_a = p_*^{-1}(H_a) \subset K_b = p_*^{-1}(H_b)$ and $f_a\colon K_a \to G$ is the restriction of f_b. We have coverings $q_i : (\mathsf{Y}_i, y_i) \to (\mathsf{M}_{0,n}, X)$ with $H_i = \operatorname{Im} q_{i*}$ for $i = a, b$. By the observation at the beginning there is a unique covering map $v\colon (\mathsf{Y}_a, y_a) \to (\mathsf{Y}_b, y_b)$ such that $q_b \circ v = q_a$ and $\operatorname{Im} v_* = q_{b*}^{-1}(H_a)$. For the same reason, since $\operatorname{Im} \bar{q}_{i*} = K_i$ for $i = a, b$, there is a covering $\bar{v}\colon (E_a, e_a) \to (E_b, e_b)$ such that $\bar{q}_b \circ \bar{v} = \bar{q}_a$. We claim that

$$\psi_b \bar{v} = v \psi_a. \tag{7-6}$$

Indeed, $q_b \psi_b \bar{v} = p \bar{q}_b \bar{v} = p \bar{q}_a = q_a \psi_a = q_b v \psi_a$. Hence $\psi_b \bar{v}$ and $v \psi_a$ lift the same map with respect to the covering q_b. Since $\psi_b \bar{v}(e_a) = y_b = v \psi_a(e_a)$ we conclude that $\psi_b \bar{v} = v \psi_a$ as claimed.

Finally we have the coverings $u_i : \mathscr{C}_i^* \to E_i$ such that $\operatorname{Im} u_{i*} = \bar{q}_{i*}^{-1}(\ker f_i)$; see (7-3). Since $\bar{v}_* = \bar{q}_{b*}^{-1} \circ \bar{q}_{a*}$ and $\ker f_a \subset \ker f_b$ we have $\bar{v}_*(\bar{q}_{a*}^{-1}(\ker f_a)) = \bar{q}_{b*}^{-1}(\ker f_a) \subset \bar{q}_{b*}^{-1}(\ker f_b)$. This means that

$$\operatorname{Im}(\bar{v} \circ u_a)_* = \bar{v}_*(\bar{q}_{a*}^{-1}(\ker f_a)) \subset \operatorname{Im} u_{b*}. \tag{7-7}$$

So we can apply once more the observation at the beginning and we get a covering $\tilde{v}\colon \mathscr{C}_a^* \to \mathscr{C}_b^*$ such that

$$u_b \tilde{v} = \bar{v} u_a, \quad \operatorname{Im} \tilde{v}_* = u_{b*}^{-1}(\operatorname{Im}(\bar{v} \circ u_a)_*). \tag{7-8}$$

Composing with ψ_a and ψ_b and using (7-6) we get a commutative diagram

$$\begin{array}{ccc} \mathscr{C}_a^* & \xrightarrow{\tilde{v}} & \mathscr{C}_b^* \\ {\scriptstyle \pi_a}\downarrow & & \downarrow{\scriptstyle \pi_b} \\ \mathsf{Y}_a & \xrightarrow{v} & \mathsf{Y}_b \end{array} \tag{7-9}$$

with π_a and π_b bundles, and $\bar{v}$ and $\tilde{v}$ coverings. We claim that

$$\operatorname{Im} \tilde{v}_* = \pi_{b*}^{-1}(\operatorname{Im} v_*). \tag{7-10}$$

Indeed starting from (7-7) we compute

$$\operatorname{Im}(\bar{v} \circ u_a)_* = \bar{q}_{b*}^{-1}(\ker f_a) = \bar{q}_{b*}^{-1}(K_a \cap \ker f_b) = \bar{q}_{b*}^{-1}(K_a) \cap \bar{q}_{b*}^{-1}(\ker f_b),$$
$$\bar{q}_{b*}^{-1}(\ker f_b) = \operatorname{Im} u_{b*},$$
$$K_a = p_*^{-1}(H_a),$$
$$\bar{q}_{b*}^{-1}(K_a) = \bar{q}_{b*}^{-1} p_*^{-1}(H_a) = (p\bar{q}_{b*})_*^{-1}(H_a) = (q_{b*}\psi_{b*})^{-1}(H_a) = \psi_{b*}^{-1}(q_{b*}^{-1}(H_a)) = \psi_{b*}^{-1}(\operatorname{Im} v_*),$$
$$\operatorname{Im}(\bar{v} \circ u_a)_* = \psi_{b*}^{-1}(\operatorname{Im} v_*) \cap \operatorname{Im} u_{b*}.$$

So from (7-8) we get

$$\operatorname{Im} \tilde{v}_* = u_{b*}^{-1}(\operatorname{Im}(\bar{v} \circ u_a)_*) = u_{b*}^{-1}(\psi_{b*}^{-1}(\operatorname{Im} v_*) \cap \operatorname{Im} u_{b*}) = u_{b*}^{-1}(\psi_{b*}^{-1}(\operatorname{Im} v_*)) = \pi_{b*}^{-1}(\operatorname{Im} v_*).$$

This proves (7-10). Applying Lemma 6.10 to the diagram (7-9) we get that $\tilde{v}\colon \mathscr{C}_a^* \to v^*\mathscr{C}_b^*$ is an isomorphism of bundles over Y_a. The map $\tilde{v}$ is an isomorphism of the coverings $\mathscr{C}_a^* \to E_a$ and

$v^*\mathscr{C}_b^* \to E_a$. By the uniqueness statement in the Grauert–Remmert extension theorem, it extends to an isomorphism of the coverings $\mathscr{C}_a \to \mathbb{P}^1 \times \mathsf{Y}_a$ and $v^*\mathscr{C}_b \to \mathbb{P}^1 \times \mathsf{Y}_a$. This extension is an isomorphism of the families of curves, $\mathscr{C}_a \cong v^*\mathscr{C}_b$. □

8 Independence from the choices

In this section, we conclude the proof of our Theorem 1.4. We present two main arguments. The first one is Theorem 8.2, whose proof will take up most of the section. It states the independence of the collection $\mathfrak{K}(x, \mathscr{B}, \theta)$ on $x \in \mathsf{M}_{0,n+1}$, on the geometric basis $\mathscr{B}$, and on the $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$–orbit of θ. Secondly, we show (Theorem 8.6) that every curve in a family of the collection $\pi_a \colon \mathscr{C}_a \to \mathsf{Y}_a$ has the topological type associated with θ, and that, conversely, for any G–curve C with topological type $[\theta]$, there is at least one fiber of $\mathscr{C}_a \to \mathsf{Y}_a$ which is (unmarkedly) G–isomorphic to C (and there are only finitely many such fibers).

Definition 8.1 We say that two collections of families $\{\mathscr{C}_a \to \mathsf{Y}_a\}_{a \in \mathscr{I}}$ and $\{\overline{\mathscr{C}}_{\bar a} \to \overline{\mathsf{Y}}_{\bar a}\}_{\bar a \in \overline{\mathscr{I}}}$ are *equivalent* if there is an order-preserving bijection $a \mapsto \bar a$ of $\mathscr{I}$ onto $\overline{\mathscr{I}}$ and for every $a \in \mathscr{I}$ a biholomorphism $w_a \colon \mathsf{Y}_a \to \overline{\mathsf{Y}}_{\bar a}$ such that:

(1) $\mathscr{C}_a \cong w_a^* \overline{\mathscr{C}}_{\bar a}$.

(2) If $a, b \in \mathscr{I}$ and $a \ge b$, the following diagram commutes:

$$\begin{array}{ccc} \mathsf{Y}_a & \xrightarrow{w_a} & \overline{\mathsf{Y}}_{\bar a} \\ {\scriptstyle v_{ab}}\downarrow & & \downarrow{\scriptstyle \bar v_{\bar a \bar b}} \\ \mathsf{Y}_b & \xrightarrow{w_b} & \overline{\mathsf{Y}}_{\bar b} \end{array}$$

In the following we conclude the independence of our collection from the choices made; different choices yield equivalent collections.

Theorem 8.2 *Up to equivalence, the collection of families $\mathfrak{K}(x, \mathscr{B}, \theta)$ is independent of the choices of x and $\mathscr{B}$ and only depends on the $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$–orbit of θ. In particular, the collection $\mathfrak{K}(x, \mathscr{B}, \theta)$ only depends on the topological type $[\theta]$.*

The proof of Theorem 8.2 is organized as follows: We start by showing that the action of $\operatorname{Aut} G$ on θ does not change the collection (Lemma 8.3); and then we prove that changing x and $\mathscr{B}$ by parallel transport leads to equivalent collections (Lemma 8.4). The combination of these two results implies that, up to equivalence, the collection of families $\mathfrak{K}(x, \mathscr{B}, \theta)$ does not change under the action $\operatorname{Aut} G \times \operatorname{Aut}^{**} \Gamma_n$ on θ (Lemma 8.5). Finally, we combine these results and complete the proof of Theorem 8.2.

Lemma 8.3 *Let $\theta \in \mathscr{D}^n(G)$ and $\eta \in \operatorname{Aut} G$. Set $\bar\theta := \eta \circ \theta$. Let $\mathscr{I}(x, \mathscr{B}, \theta) \to \mathscr{I}(x, \mathscr{B}, \bar\theta)$, $a \mapsto \bar a$, be the bijection of Lemma 6.3. Then $\mathsf{Y}_{\bar a} = \mathsf{Y}_a$ and $\mathscr{C}_{\bar a} = \mathscr{C}_a$. So $\mathfrak{K}(x, \mathscr{B}, \theta) = \mathfrak{K}(x, \mathscr{B}, \bar\theta)$. In particular, for $z \in N_x$, $\mathfrak{K}(x, \mathscr{B}, \theta) = \mathfrak{K}(x, \mathscr{B}, \theta \circ \operatorname{inn}_z)$.*

Proof Let $\chi\colon \Gamma_n \to N_x$ be the isomorphism induced from the basis $\mathscr{B}$. Set $f := \theta \circ \chi^{-1}$ and

$$\bar{f} := \bar{\theta} \circ \chi^{-1} = \eta \circ f.$$

By Lemma 6.3 we get a bijective correspondence $I(x, \mathscr{B}, \theta) \to I(x, \mathscr{B}, \bar{\theta})$ which sends $a = (H_a, f_a)$ to $\bar{a} := (H_{\bar{a}}, f_{\bar{a}})$, where $H_{\bar{a}} = H_a$, and $f_{\bar{a}} = \eta \circ f_a$. It follows that $K_{\bar{a}} = K_a$ and $\ker f_{\bar{a}} = \ker f_a$. Therefore $\mathsf{Y}_{\bar{a}} = \mathsf{Y}_a$, $E_{\bar{a}} = E_a$, $\mathscr{C}^*_{\bar{a}} = \mathscr{C}^*_a$ and $\mathscr{C}_{\bar{a}} = \mathscr{C}_a$. For the last statement, observe that $\theta \circ \mathrm{inn}_z = \mathrm{inn}_{\theta(z)} \circ \theta$. □

Lemma 8.4 *Let $\theta \in \mathscr{D}^n(G)$ and $x, x' \in \mathsf{M}_{0,n+1}$. Let the notation be as in Proposition 3.7: β is a path in $\mathsf{M}_{0,n}$ from X to X', f^β represents the parallel transport along β, $f^\beta(x_0) = x'_0$ and $\tilde{\beta}(t) = \widetilde{H}(t, x_0)$. Then the collections $\mathfrak{K}(x, \mathscr{B}, \theta)$ and $\mathfrak{K}(x', f^\beta_*(\mathscr{B}), \theta)$ are equivalent.*

Proof Let $\chi\colon \Gamma_n \to N_x$ be the isomorphism induced from the basis $\mathscr{B}$. Set $f := \theta \circ \chi^{-1}$ and $\bar{f} := f \circ (f^\beta_*)^{-1}$. We show that if $a \in \mathscr{I}(x, \mathscr{B}, \theta)$ and $\bar{a} = \Phi(a)$, where Φ is the map in (6-2), then the families $\mathscr{C}_a \to \mathsf{Y}_a$ and $\mathscr{C}_{\bar{a}} \to \mathsf{Y}_{\bar{a}}$ are canonically isomorphic. Consider the diagram

$$\begin{array}{ccccccccccc}
(*_x) & & 1 & \longrightarrow & N_x & \xrightarrow{i_*} & K_x & \xrightarrow{p_*} & H_X & \longrightarrow & 1 \\
\downarrow & & & & \downarrow f^\beta_* & & \downarrow \tilde{\beta}_\# & & \downarrow \beta_\# & & \\
(*_{x'}) & & 1 & \longrightarrow & N_{x'} & \xrightarrow{i_*} & K_{x'} & \xrightarrow{p_*} & H_{X'} & \longrightarrow & 1
\end{array}$$

Assume $a = (H_a, f_a)$ and $\bar{a} = (H_{\bar{a}}, f_{\bar{a}})$. By the definition of Φ we have $H_{\bar{a}} = \beta_\#(H_a)$, $K_{\bar{a}} = \tilde{\beta}_\#(K_a)$, $f_{\bar{a}} = f_a \circ (\tilde{\beta}_\#)^{-1}$ and $\ker f_{\bar{a}} = \tilde{\beta}_\#(\ker f_a)$. It follows from 6.8 that there are canonical isomorphisms $\mathsf{Y}_{\bar{a}} \cong \mathsf{Y}_a$, $E_{\bar{a}} \cong E_a$ and $\mathscr{C}^*_{\bar{a}} \cong \mathscr{C}^*_a$. By compactifying we get that the families $\mathscr{C}_a \to \mathsf{Y}_a$ and $\mathscr{C}_{\bar{a}} \to \mathsf{Y}_{\bar{a}}$ are isomorphic. □

Lemma 8.5 *Let $(\eta, \nu) \in \mathrm{Aut}\, G \times \mathrm{Aut}^{**}\, \Gamma_n$. Then the collections $\mathfrak{K}(x, \mathscr{B}, \theta)$ and $\mathfrak{K}(x, \mathscr{B}, \eta \circ \theta \circ \nu^{-1})$ are equivalent.*

Proof We have $\bar{\nu} := \chi \circ \nu \circ \chi^{-1} \in \mathrm{Aut}^{**}\, N_x$. Set $\bar{\theta} := \eta \circ \theta \circ \nu^{-1}$, $f := \theta \circ \chi^{-1}\colon N_x \twoheadrightarrow G$ and $\bar{f} := \bar{\theta} \circ \chi^{-1} = \eta \circ f \circ \bar{\nu}^{-1}$. By Proposition 4.19, there is an $[\alpha] \in \pi_1(\mathsf{M}_{0,n}, X)$, a lift $\tilde{\alpha}$ of α with $\tilde{\alpha}(0) = \tilde{\alpha}(1) = x_0$, and a parallel transport f^α_t such that $f^\alpha_t(x_0) = \tilde{\alpha}(t)$ and $z \in \pi_1(\mathbb{P}^1 - X, x_0)$ such that $\bar{\nu} = \mathrm{inn}_z \circ f^\alpha_*$. Note that, in particular, $f^\alpha(x_0) = x_0$. We get $\bar{f} = \eta \circ f \circ (f^\alpha_*)^{-1} \circ \mathrm{inn}_{z^{-1}}$. The statement follows from the previous two lemmas. □

Proof of Theorem 8.2 Since changing geometric bases of N_x adapted to X corresponds to acting with $\mathrm{Aut}^{**}\, \Gamma_n$, by the previous lemma it follows that if the point x is fixed, changing the adapted basis does not matter. Next fix $x, \bar{x} \in \mathsf{M}_{0,n+1}$. Choose a path $\tilde{\beta}$ in $\mathsf{M}_{0,n+1}$ joining x to $\bar{x}$. Set $\beta := p \circ \tilde{\beta}$ and let f^β be a parallel transport such that $f^\beta(x_0) = \bar{x}_0$. Let $\mathscr{B}$ be an adapted basis at x. Then $f^\beta_* \mathscr{B}$ is an adapted basis at $\bar{x}$. By Lemma 8.4 we get that $\mathfrak{K}(x, \mathscr{B}, \theta)$ and $\mathfrak{K}(\bar{x}, f^\beta_* \mathscr{B}, \theta)$ are equivalent. In other words we have independence from x and $\mathscr{B}$ as long as $\mathscr{B}$ is adapted to x. We also have that θ only matters through its $\operatorname{Aut} G \times \operatorname{Aut}^{**} \Gamma_n$–orbit by Lemma 8.5. It remains to show independence from the $\operatorname{Aut}^{*} \Gamma_n$–orbit. It follows from the definitions in 4.4 that this is equivalent to showing that if $x \in \mathsf{M}_{0,n+1}$, $\mathscr{B}$ is a basis adapted to x and $\overline{\mathscr{B}}$ is an arbitrary basis of $\pi_1(\mathbb{P}^1 - X, x_0)$, then the collections $\mathfrak{K}(x, \mathscr{B}, \theta)$ and $\mathfrak{K}(x, \overline{\mathscr{B}}, \theta)$ are equivalent. Let us prove this statement. There is a permutation $\sigma \in S_n$ such that $\overline{\mathscr{B}}$ is adapted to $(x_0, x_{\sigma_1}, \dots, x_{\sigma_n})$. Define

$$\tau\colon \mathsf{M}_{0,n} \to \mathsf{M}_{0,n}, \qquad \tau(x_1, \dots, x_n) := (x_{\sigma_1}, \dots, x_{\sigma_n}),$$
$$\tilde{\tau}\colon \mathsf{M}_{0,n+1} \to \mathsf{M}_{0,n+1}, \quad \tilde{\tau}(x_0, x_1, \dots, x_n) := (x_0, x_{\sigma_1}, \dots, x_{\sigma_n}).$$

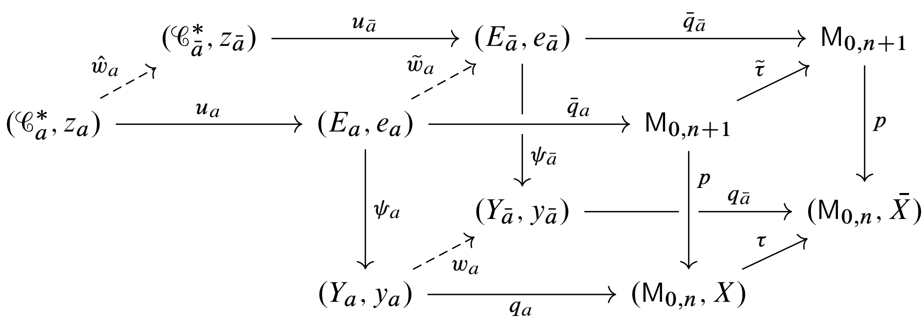
Set $\bar{x} = \tilde{\tau}(x)$ and $\overline{X} = \tau(X)$. By the previous results we know that $\mathfrak{K}(x, \mathscr{B}, \theta)$ and $\mathfrak{K}(\bar{x}, \overline{\mathscr{B}}, \theta)$ are equivalent. It remains to check that also $\mathfrak{K}(\bar{x}, \overline{\mathscr{B}}, \theta)$ and $\mathfrak{K}(x, \overline{\mathscr{B}}, \theta)$ are equivalent. Consider the diagram

$$\begin{array}{llcccccccc}
(*_x) & & 1 \longrightarrow & N_x & \xrightarrow{i_*} & K_x & \xrightarrow{p_*} & H_X & \longrightarrow 1 \\
\big\downarrow & & & \big\downarrow \mathrm{id}_{N_x} & & \big\downarrow \tilde{\tau}_* & & \big\downarrow \tau_* & \\
(*_{\bar{x}}) & & 1 \longrightarrow & N_{\bar{x}} & \xrightarrow{i_*} & K_{\bar{x}} & \xrightarrow{p_*} & H_{\overline{X}} & \longrightarrow 1
\end{array}$$

To check commutativity observe that $\tilde{\tau}$ sends the fiber over X to the fiber over $\overline{X} := (x_{\sigma_1}, \dots, x_{\sigma_n})$, ie $\tilde{\tau}(\mathbb{P}^1 - X) \times \{X\} = (\mathbb{P}^1 - X) \times \{\overline{X}\}$ and on the first factor it is the identity map. We use this diagram with $f = \bar{f} = \theta \circ \bar{\chi}^{-1}$. We get the usual correspondence $a \mapsto \bar{a}$, $\mathscr{I}(x, \overline{\mathscr{B}}, \theta) \to \mathscr{I}(\bar{x}, \overline{\mathscr{B}}, \theta)$, with

$$H_{\bar{a}} = \tau_*(H_a), \quad K_{\bar{a}} = \tilde{\tau}_*(K_a), \quad \ker f_{\bar{a}} = \tilde{\tau}_*(\ker f_a). \tag{8-1}$$

Consider the diagram



By a repeated use of the lifting theorem and using (8-1) we can show the existence of homeomorphisms w_a, $\tilde{w}_a$ and $\hat{w}_a$ making the diagram commute. Indeed $(\operatorname{Im}(\tau \circ q_a)_*) = \tau_*(H_a) = \operatorname{Im} q_{\bar{a}*}$ by the first equation in (8-1). So w_a is the isomorphism between the pointed coverings $\tau \circ q_a$ and $q_{\bar{a}}$. By the same argument, using the second equation in (8-1), we get the isomorphism $\tilde{w}_a$. Consider the cube on the right in the diagram. All its faces (except the left one) commute. But then

$$q_{\bar{a}} \psi_{\bar{a}} \tilde{w}_a = p \bar{q}_{\bar{a}} \tilde{w}_a = p \tilde{\tau} \bar{q}_a = \tau p \bar{q}_a = \tau q_a \psi_a = q_{\bar{a}} w_a \psi_a.$$

So $\psi_{\bar{a}} \tilde{w}_a$ and $w_a \psi_a$ lift the same map with respect to $q_{\bar{a}}$. Since $\psi_{\bar{a}} \tilde{w}_a(e_a) = y_{\bar{a}} = w_a \psi_a(e_a)$ we conclude that $\psi_{\bar{a}} \tilde{w}_a = w_a \psi_a$.

Finally consider the horizontal square on the left of the diagram. We want to show that

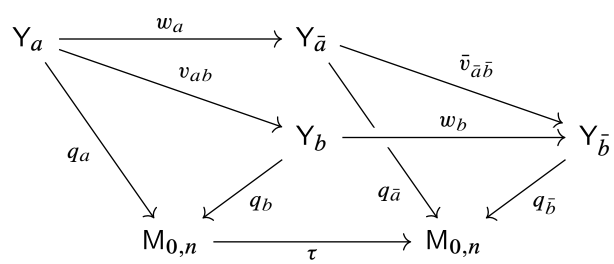
$$\operatorname{Im}(\tilde{w}_a \circ u_a)_* = \operatorname{Im} u_{\bar{a}*}.$$

We compose with the injective morphism $\bar{q}_{\bar{a}*}$ and compute

$$\bar{q}_{\bar{a}*}(\operatorname{Im}(\tilde{w}_a \circ u_a)_*) = \tilde{\tau}_* q_{a*}(\operatorname{Im} u_{a*}) = \tilde{\tau}_*(\ker f_a).$$

By the third equation in (8-1) this equals $\ker f_{\bar{a}} = \bar{q}_{\bar{a}*}(\operatorname{Im} u_{\bar{a}*})$. Thus $\bar{q}_{\bar{a}*}(\operatorname{Im}(\tilde{w}_a \circ u_a)_*) = \bar{q}_{\bar{a}*}(\operatorname{Im} u_{\bar{a}*})$ and $\operatorname{Im}(\tilde{w}_a \circ u_a)_* = \operatorname{Im} u_{\bar{a}*}$. So the lifting theorem again yields existence of an isomorphism $\hat{w}_a$ making everything commutative. The homeomorphisms w_a, $\tilde{w}_a$ and $\hat{w}_a$ are in fact biholomorphisms as observed in 6.8. It follows that $\pi_{\bar{a}} \hat{w}_a = w_a \pi_a$. By the uniqueness statement in the Grauert–Remmert extension theorem, $\tilde{w}_a$ extends to a biholomorphism between $\mathscr{C}_a$ and $\mathscr{C}_{\bar{a}}$. Thus $\mathscr{C}_a \cong \tilde{w}_a^* \mathscr{C}_{\bar{a}}$.

Property (2) in Definition 8.1 follows again by the lifting theorem:



We have $\bar{q}_{\bar{b}} \bar{v}_{\bar{a}\bar{b}} w_a = q_{\bar{a}} w_a = \tau q_a = \tau q_b v_{ab} = q_{\bar{b}} w_b v_{ab}$, so $\bar{v}_{ab} w_a$ and $w_a v_{ab}$ lift the same map. Moreover, $\bar{v}_{\bar{a}\bar{b}} w_a(y_a) = \bar{v}_{ab}(y_{\bar{a}}) = y_{\bar{b}} = w_b(y_b) = w_b v_{ab}(y_a)$, so the two maps coincide; $\bar{v}_{ab} w_a = w_a v_{ab}$. This proves (2). □

Theorem 8.6 *Let G be a finite group and $\theta \in \mathscr{D}^n(G)$. Choose a point $x \in \mathsf{M}_{0,n+1}$ and a geometric basis $\mathscr{B}$ of N_x. Let $\pi_a : \mathscr{C}_a \to \mathsf{Y}_a$ be any family in the collection $\mathfrak{K}(x, \mathscr{B}, \theta)$. Then every curve in the family has the topological type given by $[\theta] \in \mathscr{D}^n(G)/\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$. Conversely, every algebraic curve with a G–action of the topological type given by $[\theta]$ is (unmarkedly) G–isomorphic to some fiber. Moreover, there are only finitely many such fibers.*

Proof Consider $\pi_a : \mathscr{C}_a \to \mathsf{Y}_a$ and let $y, y' \in \mathsf{Y}_a$. Let β be a path in Y_a from y to y', and let f^β represent the parallel transport along β. By Lemma 3.5, we get a G–equivariant diffeomorphism $\mathscr{C}_{a,y} \to \mathscr{C}_{a,y'}$. Hence the G–actions on $\mathscr{C}_{a,y}$ and $\mathscr{C}_{a,y'}$ have the same topological type. This proves the first statement. Now let C be an algebraic curve such that G acts effectively on C in such a way that $C/G \cong \mathbb{P}^1$. We get the ramified covering $\pi : C \to \mathbb{P}^1$. By acting via $\mathrm{PGL}(2, \mathbb{C})$, one can move any three branch points of π to 0, 1 and ∞. We can thus assume that the set of critical values of $\pi : C \to \mathbb{P}^1$ coincides with $Y \in \mathsf{M}_{0,n}$. Set $C^* := \pi^{-1}(\mathbb{P}^1 - Y)$. Fix a point $y_0 \in \mathbb{P}^1 - Y$ and consider the monodromy $f : \pi_1(\mathbb{P}^1 - Y, y_0) \to G$ associated with $\pi|_{C^*} : C^* \to \mathbb{P}^1 - Y$. Finally fix a basis $\mathscr{B}'$ of $\pi_1(\mathbb{P}^1 - Y, y_0)$ to Y. Let $\chi : \Gamma_n \to \pi_1(\mathbb{P}^1 - Y, y_0)$ denote the associated isomorphism. Denote by $\theta' = f \circ \chi : \Gamma_n \to G$ the datum associated with C. We get a collection $\mathfrak{K}(y, \mathscr{B}', \theta')$. Assume that C has the same topological type of G–action as $[\theta]$, namely that $[\theta]=[\theta']\in\mathscr{D}^n(G)/\mathrm{Aut}^{**}\,\Gamma_n\times\mathrm{Aut}\,G$. By Theorem 8.2 the collections $\mathfrak{K}(x,\mathscr{B},\theta)$ and $\mathfrak{K}(y,\mathscr{B}',\theta')$ are equivalent. Thus there exist $\bar{a}\in\mathscr{I}(y,\mathscr{B}',\theta')$ and a biholomorphism $w_a\colon \mathsf{Y}_a\to\overline{\mathsf{Y}}_{\bar{a}}$ as in Definition 8.1. In particular, $\mathscr{C}_a\cong w_a^*\overline{\mathscr{C}}_{\bar{a}}$. It follows that C, which is the central fiber for $\pi_{\bar{a}}\colon\mathscr{C}_{\bar{a}}\to\mathsf{Y}_{\bar{a}}$, is G–isomorphic to some fiber of $\pi_a\colon\mathscr{C}_a\to\mathsf{Y}_a$. To check that only finitely many fibers can be G–isomorphic to C we argue as follows. For any $\sigma\in S_n$ there is a unique $g_\sigma\in\mathrm{Aut}\,\mathbb{P}^1$ such that $g_\sigma(y_{\sigma_{n-2}})=0$, $g_\sigma(y_{\sigma_{n-1}})=1$ and $g_\sigma(y_{\sigma_n})=\infty$. If $f\colon C\to\mathscr{C}_{a,y}$ is a G–isomorphism for some $y\in\mathsf{Y}_a$, then f descends to an isomorphism $\bar{f}\in\mathrm{Aut}\,\mathbb{P}^1$ that maps branch points to branch points. So if $X:=q_a(y)$, we have $\bar{f}(\{y_1,\dots,y_n\})=\{x_1,\dots,x_n\}$. Then there is a permutation σ such that $\bar{f}(y_{\sigma_i})=x_i$ for any $i=1,\dots,n$. So $\bar{f}=g_\sigma$ and $X=(g_\sigma(y_1),\dots,g_\sigma(y_n))$. This shows that there is a finite number of possibilities for X, so a finite number of possibilities for y since q_a is finite. □

9 The centerless case

If the group G has trivial center, the whole discussion in Sections 6, 7 and 8 is greatly simplified.

Indeed, let us go back to the setting at the beginning of Section 6 and let us consider again the sequence $(*)$.

Theorem 9.1 *If the sequence $(*)$ on page 1585 splits and $Z(G)=\{1\}$, then there exists a minimum $a_{\min}\in\mathscr{I}(*,f)$ and it is unique.*

Proof With the notation of Lemma 6.6, set $H''':=\{h\in H''\mid\tilde{\varepsilon}_h\in\mathrm{Inn}\,G\}$. Note that $H'\subset H'''\subset H''$ and that H''' has finite index in H'' and in H. By assumption the map $G\to\mathrm{Inn}\,G$ is bijective. So for every $h\in H'''$, there is a unique element of G, denoted by $\varphi(h)$, such that $\tilde{\varepsilon}_h=\mathrm{inn}_{\varphi(h)}$. We get a map $\varphi\colon H'''\to G$. Since $\tilde{\varepsilon}$ is a morphism, we have $\mathrm{inn}_{\varphi(hh')}=\mathrm{inn}_{\varphi(h)\varphi(h')}$ and, since $Z(G)=\{1\}$, this implies that φ is a morphism. Also, by construction, φ satisfies $\mathrm{inn}_{\varphi(h)}\circ f=f\circ\varepsilon_h$. Therefore, by Lemma 6.5, there exists a morphism $\tilde{f}\colon N\rtimes_\varepsilon H'''\to G$ extending f such that $\tilde{f}|_{H'''}=\varphi$. Thus $(H''',\tilde{f})\in\mathscr{I}(*,f)$. Moreover, since φ is unique, so is $\tilde{f}$. Now let $a=(H_a,f_a)\in\mathscr{I}(*,f)$ and observe that, by Lemma 6.5, every $h\in H_a$ satisfies (6-3). It follows that $H_a\subset H'''$ and $\varphi_a=\varphi|_{H_a}$ and thus we conclude that $a=(H_a,f_a)\geq(H''',\tilde{f})$. Uniqueness of the minimum in obvious in any ordered set. □

Next let N_x, K_x and H_X be as in (7-1) and consider the splitting exact sequence $(*_x)$. As usual, choose a geometric basis $\mathscr{B}=\{[\alpha_i]\}_{i=1}^n$ of N_x, let $\chi\colon\Gamma_n\to N_x$ be the isomorphism induced from the basis $\mathscr{B}$, and, for a datum $\theta\colon N_x\to G$, set $f:=\theta\circ\chi^{-1}\colon N_x\to G$. Theorem 9.1 applied to $(*_x)$ reads as follows:

Theorem 9.2 *If G has trivial center, then there exists a minimum $a_{\min}\in\mathscr{I}(x,\mathscr{B},\theta)$ and it is unique.*

Thus in this case by choosing the minimum we have a canonical choice of a family. Thus, if the center of G is trivial, the choice of a point $x\in\mathsf{M}_{0,n+1}$, a geometric basis $\mathscr{B}=\{[\alpha_i]\}_{i=1}^n$, and a datum $\theta\colon N_x\to G$ yields a well-defined minimum family

$$\pi_{(x,\mathscr{B},\theta)}\colon\mathscr{C}_{(x,\mathscr{B},\theta)}\to\mathsf{Y}_{(x,\mathscr{B},\theta)},$$

and we can forget about the whole collection. Moreover by Theorem 8.2 changing x or $\mathscr{B}$ or θ inside its $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$–orbit amounts to passing from a collection to an equivalent one. Since equivalence is order-preserving, it naturally maps the minimum to the minimum. This yields the following.

Theorem 9.3 *If G has trivial center, then up to isomorphism the family $\pi_{(x,\mathscr{B},\theta)}\colon \mathscr{C}_{(x,\mathscr{B},\theta)} \to \mathsf{Y}_{(x,\mathscr{B},\theta)}$ is independent of the choices of x and $\mathscr{B}$ and only depends on the $\operatorname{Aut} G \times \operatorname{Aut}^* \Gamma_n$–orbit of θ. In particular, the family $\pi_{(x,\mathscr{B},\theta)}\colon \mathscr{C}_{(x,\mathscr{B},\theta)} \to \mathsf{Y}_{(x,\mathscr{B},\theta)}$ only depends on the topological type $[\theta]$.*

10 The abelian case

We conclude looking at the special case where the group G is abelian, the opposite of G being centerless.

Theorem 10.1 *If G is abelian, then there exists $a \in \mathscr{I}(x,\mathscr{B},\theta)$ such that $\mathsf{Y}_a = \mathsf{M}_{0,n}$.*

Proof Let N_x, K_x and H_X be as in (7-1) and consider the splitting exact sequence $(*_x)$, ie the top row of (2-3). Let $\chi\colon \Gamma_n \to N_x$ be the isomorphism induced from the basis $\mathscr{B}$. Set $f := \theta \circ \chi^{-1}\colon N_x \to G$. Now let $\varphi\colon H_X \to G$ be any morphism. Let

$$\varepsilon\colon \pi_1(\mathsf{M}_{0,n}) \to \operatorname{Aut}(\pi_1(\mathbb{P}^1 - X, x_0))$$

denote the morphism giving the semidirect product in $(*_x)$. By the considerations in 2.3, ε is just the restriction to $\pi_1(\mathsf{M}_{0,n})$ of the morphism $\tilde{\varepsilon}$ giving the splitting of the exact sequence in the second row of (2-3). In [Birman 1974, Corollary 1.8.3] it is explicitly described the image via $\tilde{\varepsilon}$ of the generators of the pure braid group of $n-1$ strings of the plane. To be more precise, the notation in [Birman 1974] corresponds to identify

$$\mathsf{M}_{0,n} \cong \{(x_1,\dots,x_{n-1}) \in \boldsymbol{F}_{0,n-1}\,\mathbb{C} \mid x_1 = 0, x_2 = 1\}$$

instead of (2-1). By this description one sees that, for a generator h of $\pi_1(\mathsf{M}_{0,n})$, ε_h sends a generator γ_j of $\pi_1(\mathbb{P}^1 - X, x_0)$ to a conjugate of it. In the setting of Lemma 6.5 we have $f \circ \varepsilon_h(\gamma_j) = f(\gamma_j)$ since G is abelian. Similarly $\operatorname{inn}_{\varphi(h)}$ is the identity since G is abelian. It follows immediately that there exists $f_a\colon K_x \to G$ extending both f and φ. Thus $(H_x, f_a) \in \mathscr{I}(x,\mathscr{B},\theta)$. □

10.2 The proof of Theorem 10.1 shows that, when G is abelian, for every morphism $\varphi\colon H_X \to G$ we can build $f_\varphi\colon K_x \to G$ extending both f and φ. We point out that this is the opposite of the uniqueness result in Theorem 9.2. Of course, $(H_X, f_\varphi) \in \mathscr{I}(x,\mathscr{B},\theta)$ is a minimal element for $(\mathscr{I}(x,\mathscr{B},\theta), \geq)$ since H_X is as big as possible, ie if $b \in \mathscr{I}(x,\mathscr{B},\theta)$ and $(H_X, f_\varphi) \geq b$, then $H_X = H_b$, so $b = (H_x, f_\varphi)$. But different choices of φ yield elements in $\mathscr{I}(x,\mathscr{B},\theta)$ that are not comparable with respect to the order relation $\geq$.

10.3 An important point to stress is that, in the general case, $H_a \subsetneq H_X$ and $\mathsf{Y}_a \neq \mathsf{M}_{0,n}$ for every $a \in \mathscr{I}(x,\mathscr{B},\theta)$. We now show this via an easy example. As in the proof of Theorem 10.1, we use the description in [Birman 1974] of image via $\tilde{\varepsilon}$ of the generators of the pure braid group of the plane and we show that, in general, there may not exist any morphism $\tilde{f}\colon \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H_X \to G$ extending f. Thus, in this case $H_a \subsetneq H_X$ for any $a \in \mathcal{I}$. Let $\theta\colon \Gamma_4 \to S_3$ be given by $\theta(\gamma_1) = (12)$, $\theta(\gamma_2) = (23)$, $\theta(\gamma_3) = (23)$ and $\theta(\gamma_4) = (12)$. With the notation in [Birman 1974], $\pi_1(\mathsf{M}_{0,4})$ is free on the generators A_{12} and A_{13}. We have $\theta(\varepsilon(A_{12})\gamma_1) = (23)$, $\theta(\varepsilon(A_{12})\gamma_2) = (13)$, $\theta(\varepsilon(A_{12})\gamma_3) = (23)$ and $\theta(\varepsilon(A_{12})\gamma_4) = (12)$. Now note that, on one side, $\gamma_1\gamma_2\gamma_3\gamma_4 = 1$ and thus $\gamma_1\gamma_2\gamma_3\gamma_4 \in \ker\theta$, but on the other side $\theta(\varepsilon(A_{12})(\gamma_1\gamma_2\gamma_3\gamma_4)) = (23)(13) = (123) \neq 1$. With the notation of Lemma 6.6 it follows that $A_{12} \notin H''$, so $H'' \neq H_X$. It follows from Lemma 6.5 that for any $a \in \mathcal{I}$ we have $H_a \subset H''$. Thus in particular, $H_a \subset H'' \neq H_X$. Thus there is no morphism $\tilde{f}\colon \pi_1(\mathbb{P}^1 - X, x_0) \rtimes H_X \to G$ extending f. Geometrically, one can interpret this fact as follows. On $\mathsf{M}_{0,4} \cong \mathbb{C}^{**}$ there is the universal family of elliptic curves $\mathscr{E} \to \mathsf{M}_{0,4}$. We denote by E_λ the fiber of $\mathscr{E} \to \mathsf{M}_{0,4}$ over $\lambda \in \mathbb{C}^{**}$. The family corresponding to θ shows that every elliptic curve has an effective action of S_3, which is built as follows: $S_3 = \mathbb{Z}/3 \rtimes \mathbb{Z}/2$, where $\mathbb{Z}/2$ is the multiplication by -1 on E and $\mathbb{Z}/3$ is a subgroup of the translations $(E, +)$. So to build such an action one has to choose a line inside $E_\lambda[3]$. If an extension $\tilde{f}\colon \pi_1(\mathsf{M}_{0,5}) \to S_3$ exists, then there is a family of lines $l_\lambda \subset E_\lambda[3] \cong H_1(E_\lambda, \mathbb{Z}/3)$ defined over $\mathsf{M}_{0,4}$. Equivalently, fixing a base point $\lambda_0 \in \mathsf{M}_{0,4}$, there is a line $l_{\lambda_0} \subset E_{\lambda_0}$ which is stable under the action of the monodromy of the family $\mathscr{E}$. But the image of this monodromy is Γ_2, the congruence subgroup of level 2, which fixes no line in $H_1(E_{\lambda_0}, \mathbb{Z}/3)$.

10.4 It follows from the previous remarks that, in the general case, Y_a cannot be $\mathsf{M}_{0,n}$ itself, but is necessarily a finite cover of it. As pointed out in the introduction, this corrects an inaccuracy in [González-Díez and Harvey 1992]. There it is claimed that $\mathsf{Y} = \mathsf{M}_{0,n}$ always. As $\mathsf{M}_{0,n}$ is birational to projective space, the authors concluded that the image of the family in M_g is always a unirational variety. By Theorem 10.1 their proof works for abelian covers, hence the moduli image of a family of abelian covers is always unirational. In the general case this argument fails and in fact the result is false. Indeed, Michael D Fried informed us that he recently found examples of families for which the moduli image is not unirational. In his work in progress [Fried $\geq$ 2024], Fried considers the moduli space of Galois covers of the line with fixed datum and fixed Nielsen class. When a component of this moduli space is of general type (ie a multiple of its canonical class gives an embedding), then the component is not unirational. When the datum is for covers with 4 branch points, and the equivalences include reduction by the action of Möbius transformations, there is an explicit formula for the genus of the components — see [Bailey and Fried 2002] — which in this case are one-dimensional and covers of the j–line. When that genus exceeds 1, these spaces have general type. For the group A_n, $n \equiv 1 \bmod 4$, and the branching type of the covers having all four conjugacy classes $(n+1)/2$–cycles, Fried has computed the components and their genuses. For n large, the genus is a nonconstant multiple of n^2. When the equivalence comes from the degree n permutation representation of A_n, the base Y_a of any family in the collection $\{\mathscr{C}_a \to \mathsf{Y}_a\}_{a \in \mathcal{I}}$ associated with the datum, has a natural map to one of these components. Thus its moduli image cannot be unirational.

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