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Manifolds with small topological complexity

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We study closed orientable manifolds whose topological complexity is at most 3 and determine their cohomology rings. For some of the admissible cohomology rings we are also able to identify corresponding manifolds up to a homeomorphism.

55M30; 57N65

1 Introduction

Topological complexity of a (path-connected) space X, denoted by $\mathrm{TC}(X)$, is a numerical homotopy invariant introduced by M Farber [8] as a quantitative measure for the complexity of motion planning in a configuration space X of some robot device. Although configuration spaces of robots can be quite general topological spaces (see Kapovich and Millson [16] and Pavešić [20]), of particular importance are those that have the structure of a manifold (eg ordered configuration spaces of manifolds, see Cohen [3]; configuration spaces of spidery linkages, see O'Hara [19]; and of general parallel mechanisms, see Shvalb, Shoham and Blanc [24]). It is thus of interest to determine which closed manifolds M have a given value of $\mathrm{TC}(M)$. The case $\mathrm{TC}(M) = 1$ is void, because a nontrivial closed manifold cannot be contractible. Grant, Lupton and Oprea [10, Corollary 1.2] showed that the only closed manifolds with topological complexity equal to 2 are the odd-dimensional spheres. In this paper we study closed oriented manifolds M with $\mathrm{TC}(M) = 3$. Some examples immediately spring to mind: even-dimensional spheres S^{2n} by [8, Theorem 8] and products of two odd-dimensional spheres, by [8, Theorems 8 and 11]. Are there any other examples? Our main result is Theorem 3.2 in which we give an exact description of admissible cohomology rings of manifolds whose topological complexity is at most 3.

Theorem 3.2 *If M is a closed, orientable manifold with $\mathrm{TC}(M) \le 3$, then $\pi_1(M)$ is either trivial or isomorphic to $\mathbb{Z}$, and one of the following alternatives holds*:

1. $H^*(M;\mathbb{Z}) \cong \bigwedge(x_m)$, *or*
2. $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$ *with k and l odd, $k \ge 1$, $l \ge 3$ and $k+l=m$, or*
3. $H_i(M;\mathbb{Z}) = 0$ *for $i \ne 0, k, m$ with $k \ge 2$ and $m = 2k+1$, and* $H^*(M;\mathbb{F}_2) \cong \bigwedge(x_k, x_{k+1}) \otimes \mathbb{F}_2$.

The conditions in the theorem are necessary but not sufficient to guarantee that $\mathrm{TC}(M) = 3$, as illustrated by the case of the symplectic group $\mathrm{Sp}(2)$ whose cohomology is of type (2), but $\mathrm{TC}(\mathrm{Sp}(2)) = 4$ (see Section 4).

In the next section we recall the definition and the main properties of the topological complexity. In Section 3 we state and prove our main result. Finally, in Section 4 we discuss specific manifolds whose cohomology ring is described in the mentioned theorem. We also obtain some specific results for closed orientable manifolds M that admit cellular decompositions with at most four cells: if $\mathrm{TC}(M) \le 3$, then certain Hopf invariants must vanish (Proposition 4.2); if in addition M is smooth and its dimension is even and smaller than 12, then M is the total space of an orthogonal sphere bundle over a sphere (Proposition 4.3).

2 Preliminaries on topological complexity

For a topological space X let X^I denote the space of continuous paths $\alpha: I \to X$, and let $\pi: X^I \to X \times X$ be the evaluation map $\pi(\alpha) := (\alpha(0), \alpha(1))$. *Topological complexity* of a path-connected topological space X is the least integer $\mathrm{TC}(X) = n$ for which there exists a covering $U_1, \dots, U_n$ of $X \times X$, where each U_i is open and admits a continuous section to the map $\pi: X^I \to X \times X$ [8, Definition 2]. Note that the topological complexity of X is not defined if X is not path-connected, because in that case the map π is not onto. We will thus assume throughout the paper that X (or M) is a path-connected space. Moreover, if X is a compact ANR space (which includes closed manifolds) then the requirement that the sets in the covering are open is superfluous, since by [21, Theorem 4.6], one can consider coverings of $X \times X$ by arbitrary subsets.

The main properties of topological complexity are listed in the following proposition, where the value of $\mathrm{TC}(X)$ is related to the Lusternik–Schnirelmann category $\mathrm{cat}(X)$ (for which we refer to the classical monograph [5]), and to the nilpotency of certain ideal in the cohomology ring of $X \times X$.

Note that in this work we use the nonnormalized versions of category and topological complexity for which $\mathrm{cat}(X) = \mathrm{TC}(X) = 1$ if X is a contractible space. Many authors use a normalized or reduced category and topological complexity, which is one less than in our definition, so that the category and the topological complexity of a contractible space are equal to 0. This holds in particular for the above mentioned monograph [5] and the article [10], so the reader should be careful when comparing results stated under different conventions.

Proposition 2.1 (1) $\mathrm{TC}(X) = 1$ *if and only if* X *is contractible.*

(2) *Homotopy invariance*:
$$X \simeq Y \implies \mathrm{TC}(X) = \mathrm{TC}(Y).$$

(3) *Category estimate*:
$$\mathrm{cat}(X) \le \mathrm{TC}(X) \le \mathrm{cat}(X \times X).$$

(4) *If* X *is a topological group, then* $\mathrm{TC}(X) = \mathrm{cat}(X)$.

(5) *Cohomological estimate*:
$$\mathrm{TC}(X) \ge \mathrm{nil}(\mathrm{Ker}\,\Delta^*),$$

where $\Delta^*\colon H^*(X\times X;R)\to H^*(X;R)$ *is the homomorphism induced by the diagonal map* $\Delta\colon X\to X\times X$ *on the cohomology with coefficients in a ring* R, *and* $\mathrm{nil}(\mathrm{Ker}\,\Delta^*)$ *is the minimal integer* k *for which all* k*–fold products in* $\mathrm{Ker}\,\Delta^*$ *are zero.*

(6) *Product formula*: *if* X *and* Y *are ANR spaces*, *then*

$$\mathrm{TC}(X\times Y)\le \mathrm{TC}(X)+\mathrm{TC}(Y)-1.$$

Recall that the value of Δ^* on the cross product $u\times v\in H^*(X\times X;R)$ of elements $u,v\in H^*(X;R)$ can be given in terms of their cup product as

$$\Delta^*(u\times v)=u\cdot v,$$

and the cup product of elements $u\times v$ and $u'\times v'$ is given as

$$(u\times v)\cdot(u'\times v')=(-1)^{|v|\cdot|u'|}(u\cdot u')\times(v\cdot v'),$$

where $|v|$ and $|u'|$ are the dimensions of cohomology classes v and u'; see [11, pages 215–216]. This explains why Farber [8, Definition 6] called $\mathrm{Ker}\,\Delta^*$ the *ideal of zero-divisors* of $H^*(X;R)$. For every $u\in H^*(X;R)$ we have

$$\Delta^*(u\times 1-1\times u)=u\cdot 1-1\cdot u=0,$$

therefore $(u\times 1-1\times u)\in\mathrm{Ker}\,\Delta^*$. Indeed, if $H^*(X;R)$ is a finitely generated free R–module (which implies that $H^*(X\times X;R)\cong H^*(X;R)\otimes H^*(X;R)$ by the Künneth theorem), then $\mathrm{Ker}\,\Delta^*$ is generated as an ideal by elements of the form $(u\times 1-1\times u)$ because $\Delta^*\big(\sum u_i\times v_i\big)=\sum u_i\cdot v_i=0$ implies

$$\sum u_i\times v_i=\sum(u_i\times v_i-1\times u_iv_i)=\sum(u_i\times 1-1\times u_i)\cdot(1\times v_i)\,.$$

3 Admissible cohomology rings

Computation of topological complexity of closed surfaces was completed in the orientable case by Farber [8, Theorem 9] and in the nonorientable case by Dranishnikov [6] and Cohen and Vandembroucq [4]. Thus we know that the only closed surfaces whose topological complexity is 3 are the sphere S^2 and the torus $S^1\times S^1$. To avoid making unnecessary exceptions, for the rest of this section let M denote a closed, orientable m–dimensional manifold with $m\ge 3$.

In this section we show that the condition $\mathrm{TC}(M)\le 3$ poses strong restrictions on the fundamental group and the cohomology ring of M. As a starting point we take the following consequence of a deep theorem proved by Dranishnikov, Katz and Rudyak [7].

Theorem 3.1 *If* $\mathrm{TC}(M)\le 3$, *then* $\pi_1(M)$ *is either trivial or isomorphic to* $\mathbb{Z}$.

Proof If $\mathrm{TC}(M)\le 3$, then $\mathrm{cat}(M)\le 3$ by Proposition 2.1(3), which by [7, Theorem 1.1] implies that $\pi_1(M)$ is a free group. Let us assume that the rank of $\pi_1(M)$ is at least 2 and consider the cup product pairing

$$H^1(M;\mathbb{Z})\times H^{m-1}(M;\mathbb{Z})\stackrel{\cdot}{\longrightarrow} H^m(M;\mathbb{Z})$$

which is nonsingular by [11, Proposition 3.38]. Indeed, by the Hurewicz theorem $H_1(M;\mathbb{Z})$ is free abelian; therefore $H^1(M;\mathbb{Z})$ and $H^{m-1}(M;\mathbb{Z})$ are also free by the universal coefficients theorem and by Poincaré duality, respectively. Since the rank of $H^1(M;\mathbb{Z})$ is at least 2, nonsingularity of the pairing implies that there exist linearly independent elements $u, v \in H^1(M;\mathbb{Z})$ and $u', v' \in H^{m-1}(M;\mathbb{Z})$ such that $u \cdot u' = v \cdot v' = g$, where g is a generator of $H^m(M;\mathbb{Z})$, and furthermore $u \cdot v' = v \cdot u' = 0$. Then we obtain by direct computation a nontrivial four-fold product of zero-divisors,

$$(u \times 1 - 1 \times u) \cdot (u' \times 1 - 1 \times u') \cdot (v \times 1 - 1 \times v) \cdot (v' \times 1 - 1 \times v') = 2(g \times g) \neq 0.$$

Therefore by Proposition 2.1(6), if $\operatorname{rank}(\pi_1(M)) \geq 2$, then $\mathrm{TC}(M) \geq 5$. Thus, if $\mathrm{TC}(M) \leq 3$, then $\pi_1(M)$ is a free group or rank 0 or 1, as claimed. □

In the following theorem we determine all admissible cohomology rings for a manifold whose topological complexity is at most 3.

Theorem 3.2 *Assume that M is a closed, orientable manifold with $\mathrm{TC}(M) \leq 3 \leq \dim(M)$. Then $\pi_1(M)$ is either trivial or isomorphic to $\mathbb{Z}$ and one of the following alternatives holds:*

(1) $H^*(M;\mathbb{Z}) \cong \bigwedge(x_m)$, *or*

(2) $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$ *with k and l odd, $k \geq 1$, $l \geq 3$ and $k + l = m$, or*

(3) $H_i(M;\mathbb{Z}) = 0$ *for $i \neq 0, k, m$ with $k \geq 2$ and $m = 2k+1$, and* $H^*(M;\mathbb{F}_2) \cong \bigwedge(x_k, x_{k+1}) \otimes \mathbb{F}_2$.

Proof In order to prove the theorem we need to consider several cases and subcases. Let g denote the generator of the top-dimensional cohomology $H^m(M;R)$ and for every $u \in H^*(M;R)$ let

$$\hat{u} := u \times 1 - 1 \times u \in H^*(M \times M; R)$$

be the shorthand for the corresponding zero-divisor. By Theorem 3.1 we must consider two possibilities, $\pi_1(M) \cong \mathbb{Z}$ or $\pi_1(M) = 0$.

(1) If $\pi_1(M) \cong \mathbb{Z}$, let u be a generator of $H^1(M;\mathbb{Z}) \cong \mathbb{Z}$ and let, as in the proof of Theorem 3.1, $v \in H^{m-1}(M;\mathbb{Z})$ be such that $u \cdot v = g$. If $m-1$ is even, then

$$\hat{v}^2 \cdot \hat{u} = -2(v \times v) \cdot \hat{u} = -2(g \times u - u \times g) \neq 0$$

(note that $v^2 = 0$ for dimensional reasons), and thus $\mathrm{TC}(M) \geq 4$ by Proposition 2.1(6). On the other hand, if $m-1$ is odd, and if there exists a nonzero element $w \in H^i(M;\mathbb{Z})$ for some $2 \leq i \leq m-2$, then

$$\hat{u} \cdot \hat{v} \cdot \hat{w} = w \times g - g \times w \pm uw \times v - v \times uw \neq 0,$$

so again $\mathrm{TC}(M) \geq 4$.

We conclude that if $\pi_1(M) \cong \mathbb{Z}$ and $\mathrm{TC}(M) = 3$, then $H^*(M;\mathbb{Z})$ is multiplicatively generated by two cohomology classes in dimensions 1 and $m-1$, which are Poincaré duals to each other, and furthermore $m-1$ must be odd. In other words, $H^*(M;\mathbb{Z}) \cong \bigwedge(x_1, x_k)$ for some odd integer $k > 1$.

(2) If M is simply connected, then we consider four subcases depending on the structure of the group

$$\widehat{H}(M;R) := \bigoplus_{i=2}^{m-2} H_i(M;R).$$

(2a) If $\widehat{H}(M;\mathbb{Q}) \neq 0$ we argue similarly as in case (1). First of all we note that $\widehat{H}(M;\mathbb{Z})$ is not all torsion, so by [11, Corollary 3.39] we may find homogeneous elements $u, v \in \widehat{H}(M;\mathbb{Z})$ of infinite order, such that $u \cdot v = g$. As in case (1), if either u or v is of even degree, then we can find a nontrivial product of three zero-divisors, and then $\mathrm{TC}(M) \geq 4$. Therefore, if $\mathrm{TC}(M) \leq 3$, then both u and v must be of odd degree, which as before implies that $H^*(M;\mathbb{Z})$ contains a subring of the form $\bigwedge(x_k, x_l)$ where k and l are odd integers and $1 < k \leq l < m-1$. Furthermore, if there exists an element $w \in H^*(M;\mathbb{Z})$ which is not contained in the mentioned subring, then $\hat{u} \cdot \hat{v} \cdot \hat{w} \neq 0$ similarly as in the second part of case (1). Thus, $\widehat{H}(M;\mathbb{Q}) \neq 0$ and $\mathrm{TC}(M) = 3$ imply $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$.

(2b) Let us now assume that $\widehat{H}(M;\mathbb{Q}) = 0$ but $\widehat{H}(M;\mathbb{F}_p) \neq 0$ for some odd prime p, and let k be the minimal $k \geq 2$ for which $H_k(M;\mathbb{Z})$ has p–torsion. By the universal coefficient theorem for cohomology (see [11, Theorem 3.2]) $H^i(M;\mathbb{F}_p) \neq 0$ for $i = k, k+1$. It then follows by Poincaré duality that $H^i(M;\mathbb{F}_p) \neq 0$ for $i = m-k-1, m-k$. Therefore, $H^i(M;\mathbb{F}_p) \neq 0$ in three different dimensions, unless $m = 2k+1$. In the first case, we may find (as in case (1)) three nontrivial cohomology classes u, v and w of different dimension (with $u \cdot v = g$ by [11, Corollary 3.39]), for which $\hat{u} \cdot \hat{v} \cdot \hat{w} \neq 0$ and thus $\mathrm{TC}(M) \geq 4$.

On the other hand, if $m = 2k+1$, then let $u \in H^k(M;\mathbb{F}_p)$ and $v \in H^{k+1}(M;\mathbb{F}_p)$ be such that $u \cdot v = g$. If k is even, then

$$\hat{u}^2 \cdot \hat{v} = 2(u \times g - g \times u) + v \times u^2 - u^2 \times v \neq 0.$$

Similarly, if k is odd, then $\hat{u} \cdot \hat{v}^2 \neq 0$, so in both cases $\mathrm{TC}(M) \geq 4$.

(2c) The next subcase arises if $\widehat{H}(M;\mathbb{Q}) = 0$ and $\widehat{H}(M;\mathbb{F}_p) = 0$ for p odd but $\widehat{H}(M;\mathbb{F}_2) \neq 0$. The argument is similar as in (2b), except if $m = 2k+1$, since in that case the proof that $\hat{u}^2 \cdot \hat{v} \neq 0$ for k even (or that $\hat{u} \cdot \hat{v}^2 \neq 0$ for k odd) breaks down because of 2–torsion. On the other hand, if $u \in H^k(M;\mathbb{F}_2)$ and $v \in H^{k+1}(M;\mathbb{F}_2)$ such that $u \cdot v = g$, and if additionally $u^2 \neq 0$, then

$$\hat{u}^2 \cdot \hat{v} = u^2 \times v + v \times u^2 \neq 0,$$

so $\mathrm{TC}(M) \geq 4$. Thus, under the assumptions of (2c), if $\mathrm{TC}(M) \leq 3$ then $H^*(M;\mathbb{F}_2) \cong \bigwedge(x_k, x_{k+1}) \otimes \mathbb{F}_2$.

(2d) The final possibility is that $\widehat{H}(M;R) = 0$ for all coefficient rings R, which clearly implies that $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k)$. □

4 Some manifolds with small TC

Theorem 3.2 shows that the condition $\mathrm{TC}(M) \leq 3$ is much more restrictive than the analogous condition $\mathrm{cat}(M) \leq 3$. Indeed the class of manifolds whose Lusternik–Schnirelmann category is at most 3 includes all surfaces, two-fold products of spheres, all $(n-1)$–connected $2n$–manifolds and a variety of other examples. In this section we study the actual manifolds M satisfying $\mathrm{TC}(M) \leq 3$ (without the restriction that $\dim(M) \geq 3$). For some admissible cohomology rings we describe exactly the corresponding manifolds, while in other cases we are only able to present suitable candidates and compute their Lusternik–Schnirelmann category.

(1) The simplest case to consider are manifolds whose cohomology ring is given by Theorem 3.2(1). In fact, since the fundamental group of M is free, M must be simply connected (except in the trivial case $M = S^1$). This fact, together with $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k)$ immediately yields that M is homotopy equivalent to S^k. Finally, the positive solution to the Poincaré conjecture implies that M is actually homeomorphic to S^k.

(2) If $H^*(M;\mathbb{Z}) \cong \bigwedge(x_1, x_k)$ as in Theorem 3.2(2), then we can use the fact that $S^1 \simeq K(\mathbb{Z},1)$ to find a map $f_1\colon M \to S^1$ which represents the cohomology class

$$x_1 \in H^1(M;\mathbb{Z}) \cong [M, S^1].$$

Similarly, there is a map $f_k\colon M \to K(\mathbb{Z},k)$ representing the cohomology class

$$x_k \in H^k(M;\mathbb{Z}) \cong [M, K(\mathbb{Z},k)].$$

It is well known that $K(\mathbb{Z},k)$ can be constructed by attaching cells of dimension bigger or equal to $k+2$ to the sphere S^k. Since the dimension of M is $m = k+1$, we may assume by cellular approximation theorem that the image of f_k is contained in S^k. Thus we obtain a map

$$(f_1, f_k)\colon M \to S^1 \times S^k,$$

which is clearly an isomorphism on the integral cohomology and is thus a homotopy equivalence, because $\pi_1(M) \cong \mathbb{Z}$. By a rigidity theorem of Kreck and Lück [17, Theorem 0.13(a)] we conclude that M is actually homeomorphic to $S^1 \times S^k$.

(3) If $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_k)$ with k odd, then M is a $(k-1)$–connected $2k$–dimensional manifold. Thus we may invoke C T C Wall's classification [27] by which $M \approx S^k \times S^k$ provided $k \equiv 3, 5, 7 \pmod 8$; see also [2, Theorem 3.1].

(4) The instances of Theorem 3.2(2) when $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$ for $1 < k < l$ with k and l odd are more complicated. First of all, they include products of odd spheres of the form $S^k \times S^l$ and we know that $\mathrm{TC}(S^k \times S^l) = 3$. Moreover, by the above-mentioned theorem of Kreck and Lück [17, Theorem 0.13(a)], a manifold that is homotopy equivalent to a product of odd spheres is actually homeomorphic to that product.

The first example that is not a product of spheres is the special unitary group $\mathrm{SU}(3)$ whose cohomology is $H^*(\mathrm{SU}(3);\mathbb{Z}) \cong \bigwedge(x_3, x_5)$. Singhof [25, Theorem 1(a)] proved that $\mathrm{cat}(\mathrm{SU}(3)) = 3$; therefore by Proposition 2.1(4), we conclude that $\mathrm{TC}(\mathrm{SU}(3)) = 3$, as well.

The cohomology ring of the symplectic group Sp(2) is $H^*(\mathrm{Sp}(2);\mathbb{Z}) \cong \bigwedge(x_3, x_7)$. However, Schweitzer [23] used secondary cohomology operations to prove that $\mathrm{cat}(\mathrm{Sp}(2)) = 4$, which in turn implies that $\mathrm{TC}(\mathrm{Sp}(2)) = 4$. Hilton and Roitberg [13] discovered three more examples of H–spaces whose cohomology is isomorphic to $\bigwedge(x_3, x_7)$, which are usually denoted by $E_{3\omega}$, $E_{4\omega}$ and $E_{5\omega}$ (and Sp(2) corresponds to E_ω). Their Lusternik–Schnirelmann category (and thus topological complexity) is equal to 4; see [5, Chapter 4].

In fact, we have a complete description of manifolds that admit H–space structure and whose topological complexity is equal to 3. First observe, that by the classification of H–spaces of low rank (Hilton and Roitberg [14]; see also [12, Section III.2]), the following list exhausts all (homotopy types of) H–spaces whose cohomology ring is isomorphic to one of the rings listed in Theorem 3.2: spheres S^k for $k \in \{1,3,7\}$, products $S^k \times S^l$ for $k, l \in \{1,3,7\}$, SU(3), $E_{k\omega}$ for $k = 1,3,4,5$ and $\mathbb{R}P^3$. By a cup-length argument, $\mathrm{TC}(\mathbb{R}P^3) = \mathrm{cat}(\mathbb{R}P^3) = 4$, which together with the above discussion yields:

Proposition 4.1 *Let M be a closed orientable manifold with $\mathrm{TC}(M) = 3$. If M admits an H–space structure, then M is either $\mathrm{SU}(3)$ or $S^k \times S^l$ for $k, l \in \{1,3,7\}$.*

More generally, let us consider fibre bundles $p\colon M \to S^l$ with fibre S^k for some odd integers $1 < k < l$. The cohomology of M is easily computed using Gysin sequence, so we obtain $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$ and the manifold itself admits a CW–decomposition of the form

$$M = S^k \cup_\alpha e^l \cup_\beta e^{k+l},$$

with attaching maps $\alpha\colon S^{l-1} \to S^k$ and $\beta\colon S^{k+l-1} \to S^k \cup_\alpha e^l$. If α is a suspension or more generally a coH–map (eg if $l < 2k-1$ so that $\pi_{l-1}(S^k)$ is in the stable range), then $S^k \cup_\alpha e^l$ is a coH–space and $\mathrm{cat}(S^k \cup_\alpha e^l) = 2$ (see [5]). Therefore, $\mathrm{cat}(M) \le 3$ but, since the cup length of M equals 2, we have that $\mathrm{cat}(M) = 3$. This yields many important examples like the complex and quaternionic Stiefel manifolds, $V_2(\mathbb{C}^n) = U(n)/U(n-2)$ whose cohomology ring is given as $H^*(V_2(\mathbb{C}^n);\mathbb{Z}) \cong \bigwedge(x_{2n-1}, x_{2n-3})$, and $V_2(\mathbb{H}^n) = \mathrm{Sp}(n)/\mathrm{Sp}(n-2)$ with $H^*(V_2(\mathbb{H}^n);\mathbb{Z}) \cong \bigwedge(x_{4n-1}, x_{2n-5})$. It is known (see [15]) that except for the case $V_2(\mathbb{C}^4) = S^5 \times S^7$, the spaces $V_2(\mathbb{C}^n)$ and $V_2(\mathbb{H}^n)$ do not split as products of spheres.

If the attaching map α is not a coH–map, then $\mathrm{cat}(S^k \cup_\alpha e^l) = 3$. In that case $\mathrm{cat}(M) = 3$ if and only if certain set of Hopf invariants $\mathcal{H}(\beta)$ contains the zero class (see [5, Chapter 6], in particular Theorem 6.19 therein).

As we have seen, there are many sphere bundles over spheres whose category is 3. Unfortunately, we are currently lacking a general method to determine their topological complexity, so this remains an interesting open problem. Some cases can be settled by applying a method that was recently developed by Gonzalez, Grant and Vandembroucq [9] and which uses higher Hopf invariants. They computed topological complexity of many two-cell complexes, but the technical details are quite formidable, and the full analysis of three-cell complexes seems to be beyond reach at this point. Nevertheless, we were able to combine some of their computations with results from Pavešić [22] that relate topological complexity of a space with topological complexity of its skeleta, to show that some sphere bundles over spheres have topological complexity at least 4. We will work in the so-called *metastable range* and assume that $2k < l < 3k-1$. Under this assumption one can associate to every map $\alpha\colon S^{l-1}\to S^k$ a *generalized Hopf invariant* $H_0(\alpha)\colon S^{l-1}\to S^{2k-1}$ (see [9, Section 5] for relevant definitions and results), which allows us to determine $\mathrm{TC}(S^k\cup_\alpha e^l)\geq 4$.

Proposition 4.2 *Let k be an odd integer and let $2k<l<3k-1$. Assume that M has a CW–decomposition of the form $M=S^k\cup_\alpha e^l\cup_\beta e^{k+l}$ with attaching maps $\alpha\colon S^{l-1}\to S^k$ and $\beta\colon S^{k+l-1}\to S^k\cup_\alpha e^l$ (this in particular applies if M is an S^l–bundle over S^k). If $H_0(\alpha)\neq 0$, then $\mathrm{TC}(M)\geq 4$.*

Proof Note that the inclusion $S^k\cup_\alpha e^l\hookrightarrow M$ is a $(k+l-1)$–equivalence because $S^k\cup_\alpha e^l$ is the $(k+l-1)$–skeleton of M. The topological complexity of $S^k\cup_\alpha e^l$ was bounded from below in [9, Theorem 5.6]: $\mathrm{TC}(S^k\cup_\alpha e^l)\geq 4$. On the other hand, [22, Theorem 3.6] implies that

$$\mathrm{cat}(M)\geq \mathrm{cat}(S^k\cup_\alpha e^l)=3.$$

Therefore $\mathrm{TC}(M)\geq 3$. Then we may apply [22, Theorem 3.1], which states that if

$$2\dim(S^k\cup_\alpha e^l)<k(\mathrm{TC}(M)-1)+(k+l-1)$$

(which is clearly satisfied if $l<3k-1$), then $\mathrm{TC}(M)\geq \mathrm{TC}(S^k\cup_\alpha e^l)\geq 4$. □

It turns out that up to dimension 10 the case of sphere bundles over spheres is generic for smooth, even-dimensional manifolds (that is quite relevant if one is mainly interested in configuration spaces of specific mechanical systems). In fact, we have the following result.

Proposition 4.3 *Let M be a smooth, orientable, closed manifold with $\mathrm{TC}(M)\leq 3$. If M is even-dimensional and $\dim(M)\leq 10$, then M is homotopy equivalent to the total space of an orthogonal sphere bundle over a sphere.*

Proof By the assumptions, the cohomology of M is given by cases (1) or (2) of Theorem 3.2. If M is not simply connected, then we already proved that M is homeomorphic to a product of spheres. If M is simply connected, then $\dim(M)\leq 10$ implies that its cohomology is isomorphic to either $\bigwedge(x_3,x_5)$ or to $\bigwedge(x_3,x_7)$. Thus we may apply [14, Theorem 6.1] to conclude that M is homotopy equivalent to the total space of an orthogonal S^3–bundle with base S^5 or S^7. □

For manifolds of dimension higher than 10 we may describe a convenient Morse decomposition of M. Smale [26, Theorem G] showed that if the dimension of M is at least 6, then it has a Morse decomposition with the minimal number of handles compatible with its homology. Therefore, if $H^*(M;\mathbb{Z})\cong\bigwedge(x_k,x_l)$, then M admits a decomposition with four handles whose indices are 0, k, l and $k+l$, respectively. The union of the 0– and k–handles depends on the framing which is given by an element of $\pi_{k-1}(O(l))$. This group is known to be trivial for $k\not\equiv 1 \pmod 8$, therefore the union of the first two handles is homeomorphic to $S^k \times D^l$. By the same argument, the union of the l– and $(k+l)$–handles is also homeomorphic to $S^k \times D^l$.

Proposition 4.4 *Let M be a smooth, orientable, closed manifold with $\dim(M) > 10$ and $\mathrm{TC}(M) = 3$. If $H^*(M;\mathbb{Z}) \cong \bigwedge(x_k, x_l)$ with $k \not\equiv 1 \pmod 8$, then M can be obtained by glueing together two copies of $S^k \times D^l$ along the common boundary $S^k \times S^{l-1}$.*

(5) Let us finally consider manifolds that satisfy condition (3) of Theorem 3.2. The lowest-dimensional case is a simply connected 5–dimensional manifold whose $\mathbb{F}_2$ cohomology is

$$H^*(M;\mathbb{F}_2) \cong \bigwedge(x_2, x_3) \otimes \mathbb{F}_2.$$

Barden [1] showed that every simply connected 5–dimensional manifolds can be decomposed as a connected sum of certain basic 5–manifolds. We are not dwelling into details but one can easily check that the only 5–manifold that satisfies the above condition is the famous Wu manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$. It admits a CW–decomposition $\mathrm{SU}(3)/\mathrm{SO}(3) = S^2 \cup e^3 \cup e^5$, where the 3–cell is attached by a degree 2 map; therefore the 3–skeleton of $\mathrm{SU}(3)/\mathrm{SO}(3)$ is the Moore space $M(\mathbb{Z}/2, 2)$. The category of a Moore space is 2; therefore the category of the Wu manifold is 3. However, we were not able to determine whether its topological complexity is also 3. One can construct higher analogues of the Wu manifold using handle decompositions, for example by gluing together two copies of a (twisted or untwisted, depending on the dimension) D^{k+1}–bundle over S^k along a suitable homeomorphism of the boundary. All of these spaces have a CW–decomposition with the top-cell attached to a suspension, so their category is equal to 3.

We should also mention an interesting result that was recently proved by S Mescher [18, Proposition 6.2]. He used weighted cohomology classes to show that a closed oriented manifold M with $\mathrm{TC}(M) \le 3$ is either a rational homology sphere or it admits a degree 1 map from a closed oriented manifold of the form $S^1 \times P$ (in other words, it is 1–*dominated* by a product of a $(\dim(M)-1)$–dimensional manifold with a circle).

Let us conclude with a brief discussion on two possible extensions of the presented results. Theorem 3.2 gives a precise description of cohomology rings of closed orientable manifolds whose topological complexity is at most 3, so it is natural to ask what can be said about nonorientable closed manifolds M with $\mathrm{TC}(M) \le 3$. As in the orientable case, the fundamental group $\pi_1(M)$ must be free. That rank of $\pi_1(M)$ cannot exceed 1 can be seen similarly as in Section 2. On the other hand, $\pi_1(M)$ cannot be trivial, because M is nonorientable. We thus conclude that $H^*(M;\mathbb{F}_2) \cong \bigwedge(x_1, x_{m-1}) \otimes \mathbb{F}_2$, and the corresponding manifolds are the generalized Klein-bottles (nonorientable S^{m-1}–bundles over S^1). Their category is 3 but we do not know whether their topological complexity can be, at least in some cases, also equal to 3.

Another extension that could be pursued is determination of manifolds whose topological complexity is at most 4. Although the general case seems to be beyond reach because we have very little information on manifolds whose category is 4, we believe that some reasonable progress could be achieved on closed manifolds M satisfying $\mathrm{TC}(M) \le 4$ and $\mathrm{cat}(M) \le 3$.

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