Algebraic & Geometric Topology

Volume 24 (2024)

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Algebraic & Geometric Topology **24**:3 (2024) 1725–1738
DOI: 10.2140/agt.2024.24.1725
Published: 28 June 2024

Steenrod problem and some graded Stanley–Reisner rings

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"What kind of ring can be represented as the singular cohomology ring of a space?" is a classic problem in algebraic topology, posed by Steenrod. We consider this problem when rings are the graded Stanley–Reisner rings, in other words the polynomial rings divided by an ideal generated by square-free monomials. We give a necessary and sufficient condition that a graded Stanley–Reisner ring is realizable when there is no pair of generators x, y such that $|x| = |y| = 2^n$ and $xy \neq 0$.

55N10; 55R35, 13F55

1 Introduction

A classical problem in algebraic topology posed by Steenrod in [14] asks which graded rings occur as the cohomology ring of a space. Especially when the graded ring is polynomial ring, this problem was studied by many researchers, for example Adams and Wilkerson [1], Aguadé [2], Andersen and Grodal [4], Clark and Ewing [6], Dwyer, Miller and Wilkerson [8], Dwyer and Wilkerson [9; 10], Hubbuck [11], Sugawara and Toda [15] and Thomas [16]. This polynomial ring case was finally solved by Andersen and Grodal [3].

On the other hand, when the graded ring is a monomial ideal ring, in other words a polynomial ring divided by an ideal generated by monomials, some researchers studied this problem. The realizability of Stanley–Reisner rings, square-free monomial ideal rings, generated by degree 2 elements is proved by Davis and Januszkiewicz in [7]. Trevisan [17] generalize their construction and prove the realizability of monomial ideal rings generated by degree 2 elements. By using polyhedral products, the realizability of Stanley–Reisner rings of a certain class is proved by Bahri, Bendersky, Cohen and Gitler in [5]. So and Stanley [13] prove the realizability of graded monomial ideal ring modulo torsion. Thus there are results about the realizability of monomial ideal rings, but there are few results about necessary conditions for monomial ideal rings to be realizable.

In this paper we obtain a necessary and sufficiently condition for a graded Stanley–Reisner ring to be realizable when there is no pair of generators x, y such that $|x| = |y| = 2^n$ and $xy \neq 0$. At first, we define the graded Stanley–Reisner ring. A simplicial complex with the vertex set V is a subset of the power set of V which closed under taking subsets. In this paper we allow for there to exist $x \in V$ such that $\{x\} \notin K$, and we assume that the empty set is always a face of the simplicial complex. Let K be a simplicial complex with the vertex set V, and $\phi\colon V \to 2\mathbb{Z}_{>0}$. Then the graded Stanley–Reisner ring $\mathrm{SR}(K,\phi)$ is defined by

$$\mathrm{SR}(K,\phi) \cong \mathbb{Z}[V]/I,$$

where $\mathbb{Z}[V]$ is the polynomial ring generated by $x \in V$ with $|x| = \phi(x)$ and I is the ideal generated by monomials $x_1x_2\cdots x_k$ with $\{x_1, x_2, \dots, x_k\} \notin K$ as a simplex. When $K = \{\varnothing\}$, there is an isomorphism $\mathrm{SR}(K,\phi) \cong \mathbb{Z}$.

To state the main theorem in this paper we set notation. A simplex of a simplicial complex is maximal when the simplex is not a face of a larger simplex in the simplicial complex. For a simplicial complex K with the vertex set V, we define a poset (not subcomplex) $P_{\max}(K) \subset K$, where we regard K as a subset of the power set of V. For $\sigma \in K$, $\sigma \in P_{\max}(K)$ if and only if there exist maximal simplices $\sigma_1, \dots, \sigma_n \in K$ such that $\bigcap \sigma_i = \sigma$. And for $\sigma, \tau \in P_{\max}$, we have $\sigma < \tau$ when σ is a face of τ in K.

Theorem 1.1 *Let $\mathrm{SR}(K,\phi)$ be the finitely generated graded Stanley–Reisner ring for a simplicial complex K with the vertex set V and $\phi\colon V \to 2\mathbb{Z}_{>0}$. Suppose that the graded Stanley–Reisner ring $\mathrm{SR}(K,\phi)$ satisfies the following*:

- *If generators $x, y \in V$ satisfy $\phi(x) = \phi(y) = 2^i$ for some $i \geq 2$, then $xy = 0$ in $\mathrm{SR}(K,\phi)$.*

Then there is a space X such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$ if and only if $\mathrm{SR}(K,\phi)$ satisfies the following condition:

- *For $\sigma \in P_{\max}(K)$ the set $\{\phi(x) \mid x \in \sigma\}$ is equal to $\{2,2,\dots,2\}$, $\{4,6,\dots,2n+2\} \cup \{2,2,\dots,2\}$ or $\{4,8,\dots,4n\} \cup \{2,2,\dots,2\}$ as a multiset for some n.*

This is the main theorem in this paper.

Remark 1.2 In the main theorem there is an artificial assumption:

- If generators $x, y \in V$ satisfy $\phi(x) = \phi(y) = 2^i$ for some $i \geq 2$, then $xy = 0$ in $\mathrm{SR}(K,\phi)$.

We believe that this assumption in the main theorem can be replaced by the following condition:

- If generators $x, y \in V$ satisfy $\phi(x) = \phi(y) = 4$, then $xy = 0$ in $\mathrm{SR}(K,\phi)$.

This condition is the case that $i = 2$ in the upper assumption. Andersen and Grodal proved that the degree of the generators of realizable polynomial is a union of copies of $\{2\}$, $\{4,6,\dots,2n+2\}$ or $\{4,8,\dots,4n\}$. Since in polynomial case there is one generator with degree 4 except in the case $\{2\}$, this condition implies that the tensor products of two of polynomial rings with the case $\{4,6,\dots,2n+2\}$ and $\{4,8,\dots,4n\}$ is not included. Therefore this condition seems natural.

But now we are not able to prove the theorem that replaces the artificial assumption with this condition. The reason why the artificial assumption is required is in the latter part of this paper.

We can generalize the construction of a space X with $H^*(X;\mathbb{Z})$ being isomorphic to the graded Stanley–Reisner ring to a wider classes. The following theorem is proved in Section 3.

Theorem 1.3 *Let* $\mathrm{SR}(K,\phi)$ *be the finitely generated graded Stanley–Reisner ring for a simplicial complex* K *with vertex set* V *and* $\phi\colon V \to 2\mathbb{Z}_{>0}$. *If* $\mathrm{SR}(K,\phi)$ *satisfies the following condition, we can construct a space* X *as a homotopy colimit such that* $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$:

- *There is a decomposition* $\coprod_i A_i = V$ *such that for all* i *and* $\sigma \in P_{\max}(K)$, *the set* $\{\phi(x) \mid x \in \sigma \cap A_i\}$ *is equal to* $\{2,2,\dots,2\}$, $\{4,6,\dots,2n\} \cup \{2,2,\dots,2\}$ *or* $\{4,8,\dots,4n\} \cup \{2,2,\dots,2\}$ *as a multiset for some* n.

In the first half of this paper, Sections 2, 3 and 4, we construct a space X with $H^*(X;\mathbb{Z})$ isomorphic to a graded Stanley–Reisner ring, and prove Theorem 1.3. In the latter half, Sections 5 and 6, we obtain the necessary condition that graded Stanley–Reisner rings occur as the cohomology ring of a space. At last, by combining these results, we prove the main theorem in Section 7.

Acknowledgements The author is grateful to Donald Stanley for suggesting this issue and for valuable advice. The author is supported by JSPS KAKENHI Grant 21J10117.

2 Homotopy colimit

In this section we recall a homotopy colimit and prove some lemmas we will use.

Let P be a finite poset. The order complex of P, $\Delta(P)$, is a simplicial complex whose faces are totally ordered subsets in P. We regard P as a category. For a functor $F\colon P \to \mathrm{Top}$, the homotopy colimit is defined as

$$\mathrm{hocolim}_P\, F = \coprod_{\sigma=(x_1<x_2<\cdots<x_k)\in\Delta(P)} |\sigma| \times F(x_k)/\sim,$$

where the equivalence is $(\iota(x), y) \sim (x, F(\iota)(y))$ for $\iota\colon \tau \hookrightarrow \sigma$ and $x \in |\tau|$, $y \in F(\max(\sigma))$.

We write $P_{<a} = \{p \in P \mid p < a\}$ and $P_{\le a} = \{p \in P \mid p \le a\}$ for some $a \in P$.

Lemma 2.1 *Let* $(P,<)$ *be a finite poset and* $F\colon P \to \mathrm{Top}$ *be a functor. Let* $a \in P$ *be a maximal element. Then there is a pushout diagram*

$$\begin{tikzcd} \mathrm{hocolim}_{P_{<a}} F \arrow[r] \arrow[d] & \mathrm{hocolim}_{P\setminus\{a\}} F \arrow[d] \\ \mathrm{hocolim}_{P_{\le a}} F \arrow[r] & \mathrm{hocolim}_P\, F \end{tikzcd}$$

where for a subset $P' \subset P$ $\mathrm{hocolim}_{P'} F$ *means the homotopy colimit of the functor* $F|_{P'}\colon P' \to Top$.

Proof By the definition of homotopy colimit, we obtain that

$$\mathrm{hocolim}_{P\setminus\{a\}} F \cup \mathrm{hocolim}_{P_{\le a}} F = \mathrm{hocolim}_P\, F,$$
$$\mathrm{hocolim}_{P\setminus\{a\}} F \cap \mathrm{hocolim}_{P_{\le a}} F = \mathrm{hocolim}_{P_{<a}} F.$$

The inclusions
$$\operatorname{hocolim}_{P_{<a}} F \hookrightarrow \operatorname{hocolim}_{P\setminus\{a\}} F,$$
$$\operatorname{hocolim}_{P_{<a}} F \hookrightarrow \operatorname{hocolim}_{P_{\leq\{a\}}} F$$
are cofibrations. By combining these we obtain this lemma. □

Next, we see the relation between the homotopy pushout and the graded Stanley–Reisner ring. For a subcomplex $K' \subset K$, let $V(K')$ be the vertex set of K'.

Lemma 2.2 *Let K be a simplicial complex with the vertex set V, and $\phi\colon V \to 2\mathbb{Z}_{>0}$. Let K_1 and K_2 be subcomplexes of K. We assume the following*:

- *There is a space X with $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K_1 \cap K_2, \phi)$.*
- *For $i = 1, 2$ there are spaces X_i with $H^*(X_i;\mathbb{Z}) \cong \mathrm{SR}(K_i, \phi)$.*
- *For $i = 1, 2$ there are maps $\pi_i\colon X \to X_i$ such that π_i^* is identified with the natural projection $\mathrm{SR}(K_i, \phi) \to \mathrm{SR}(K_1 \cap K_2, \phi)$ in cohomology.*

Then the cohomology ring of the homotopy pushout of the diagram
$$\begin{tikzcd} X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X_1 \\ X_2 & \end{tikzcd}$$
is isomorphic to $\mathrm{SR}(K_1 \cup K_2, \phi)$.

Proof Let $p_i\colon \mathrm{SR}(K_1 \cup K_2, \phi) \to \mathrm{SR}(K_i, \phi)$ be the natural projection for $i = 1, 2$. Then it is easy to see that the following sequence is a short exact sequence as a graded module
$$1 \to \mathrm{SR}(K_1 \cup K_2, \phi) \xrightarrow{p_1 \oplus p_2} \mathrm{SR}(K_1, \phi) \oplus \mathrm{SR}(K_2, \phi) \xrightarrow{\pi_1^* - \pi_2^*} \mathrm{SR}(K_1 \cap K_2, \phi) \to 1.$$
By the Mayer–Vietoris sequence for X_1 and X_2 of the pushout, we obtain that the cohomology of the homotopy pushout is isomorphic to $\mathrm{SR}(K_1 \cup K_2, \phi)$ as a graded module. Since the cohomology ring of the homotopy pushout is a graded subring of $H^*(X_1 \amalg X_2) \cong \mathrm{SR}(K_1, \phi) \oplus \mathrm{SR}(K_2, \phi)$, this isomorphism becomes an isomorphism as a graded ring. □

3 Construction of homotopy colimit

In this section we construct a homotopy colimit representation of a space X with $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K, \phi)$ for some graded Stanley–Reisner rings $\mathrm{SR}(K, \phi)$. This construction is an analogy to the construction in [7]. The Davis–Januszkiewicz space that first appeared in [7] is constructed by the union of the products of complex projective spaces. As far as looking at cohomology, our construction is like a graded version of their construction.

3.1 Maps between the classifying spaces of Lie groups

We define maps between the classifying spaces of Lie groups and see some properties. Consider the inclusions

$$\iota_1\colon \mathrm{SU}(n)\to \mathrm{SU}(n+1), \qquad \iota_1(A)=A\oplus 1 \quad \text{for } A\in \mathrm{SU}(n),$$
$$\iota_2\colon \mathrm{Sp}(n)\to \mathrm{Sp}(n+1), \qquad \iota_2(A)=A\oplus 1 \quad \text{for } A\in \mathrm{Sp}(n).$$

For the quaternion $\mathbb{H}$ and the set of complex 2×2–matrices $M(2,\mathbb{C})$, let $c\colon \mathbb{H}\to M(2,\mathbb{C})$ be the map

$$c(z+jw)=\begin{pmatrix} z & -\overline{w}\\ w & \overline{z}\end{pmatrix} \quad \text{for } z,w\in\mathbb{C}.$$

Let $\iota_3\colon \mathrm{Sp}(n)\to \mathrm{SU}(2n)$ be the map such that for $A=(a_{i,j})_{ij}\in \mathrm{Sp}(n)$,

$$\iota_3(A)=\begin{pmatrix} c(a_{1,1}) & c(a_{1,2}) & \dots\\ c(a_{2,1}) & c(a_{2,2}) & \\ \vdots & & \ddots\end{pmatrix}\in \mathrm{SU}(2n).$$

Since ι_i is a homomorphism, ι_i induces the map between classifying map. We denote these maps as same symbol ι_i. Since the diagram

$$\begin{tikzcd} \mathrm{Sp}(n) \arrow[r, "\iota_2"] \arrow[d, "\iota_3"] & \mathrm{Sp}(n+1) \arrow[d, "\iota_3"] \\ \mathrm{SU}(2n) \arrow[r, "\iota_1"] & \mathrm{SU}(2n+2) \end{tikzcd}$$

is commutative, there is a commutative diagram

$$\begin{tikzcd} B\mathrm{Sp}(n) \arrow[r, "\iota_2"] \arrow[d, "\iota_3"] & B\mathrm{Sp}(n+1) \arrow[d, "\iota_3"] \\ B\mathrm{SU}(2n) \arrow[r, "\iota_1"] & B\mathrm{SU}(2n+2) \end{tikzcd}$$

We recall the cohomology of these classifying spaces. There is an isomorphism

$$H^*(B\mathrm{SU}(n);\mathbb{Z})\cong \mathbb{Z}[c_2,c_3,\dots,c_n],$$

where c_i is the i^{th} Chern class. For degree reasons, we obtain $\iota_3^*(c_{2n+1})=0$, and the next lemma holds.

Lemma 3.1 (cf [12, Chapter III, Theorem 5.8]) *There is an isomorphism*

$$H^*(B\mathrm{Sp}(n);\mathbb{Z})\cong \mathbb{Z}[\iota_3^*(c_2),\iota_3^*(c_4),\dots,\iota_3^*(c_{2n})].$$

In this paper we take the generators of $H^*(B\mathrm{Sp}(n);\mathbb{Z})$ as in this lemma. Then there are equations for $\iota_1\colon B\mathrm{SU}(n)\to B\mathrm{SU}(n+1)$ and $\iota_2\colon B\mathrm{Sp}(n)\to B\mathrm{Sp}(n+1)$ (cf [12, Chapter III]):

$$\iota_1^*(c_i)=\begin{cases} c_i & \text{if } i\le n,\\ 0 & \text{if } i=n+1,\end{cases} \qquad \iota_2^*(\iota_3^*(c_{2i}))=\begin{cases} \iota_3^*(c_{2i}) & \text{if } i\le n,\\ 0 & \text{if } i=n+1.\end{cases}$$

In summary, ι_1, ι_2 and ι_3 are the maps that send each generator to its corresponding generator or 0 in cohomology.

3.2 Construction

We define a functor by using the maps ι_1, ι_2 and ι_3. Let K be a simplicial complex with the vertex set V, and $\phi\colon V \to 2\mathbb{Z}_{>0}$ satisfying the following condition:

- There is a decomposition $\coprod_i A_i = V$ such that for all i and $\sigma \in P_{\max}(K)$, the set $\{\phi(x) \mid x \in \sigma \cap A_i\}$ is equal to $\{2, 2, \dots, 2\}$, $\{4, 6, \dots 2n+2\} \cup \{2, \dots, 2\}$ or $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$ as a multiset for some n.

The simplicial complex K can be regarded as a poset by inclusions. We define a subposet $P \subset K$ satisfying

- $P_{\max}(K) \subset P$,
- for any $\sigma \in P$ and i, the set $\{\phi(x) \mid x \in \sigma \cap A_i\}$ is equal to $\{2, 2, \dots, 2\}$, $\{4, 6, \dots 2n+2\} \cup \{2, \dots, 2\}$ or $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$ as a multiset for some n.

Then we regard the poset P as a category and we define a functor $F\colon P \to \mathrm{Top}$. For $\sigma \in K$,

$$X_\sigma = \begin{cases} B\mathrm{Sp}(n) \times \prod_{\{x \in \sigma \mid \phi(x)=2\}} \mathbb{C}\mathrm{P}^\infty & \text{when } \{\phi(x) \mid x \in \sigma\} = \{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}, \\ B\mathrm{SU}(n+1) \times \prod_{\{x \in \sigma \mid \phi(x)=2\}} \mathbb{C}\mathrm{P}^\infty & \text{when } \{\phi(x) \mid x \in \sigma\} = \{4, 6, \dots, 2n+2\} \cup \{2, \dots, 2\}, \\ \text{point} & \text{when } \sigma \text{ is the empty set.} \end{cases}$$

For $\sigma \subset \tau \in K$, let

$$\iota\colon \prod_{\{x \in \sigma \mid \phi(x)=2\}} \mathbb{C}\mathrm{P}^\infty \to \prod_{\{x \in \tau \mid \phi(x)=2\}} \mathbb{C}\mathrm{P}^\infty$$

be the inclusion such that each vertex corresponds to the same vertex. Then let $f_{\sigma,\tau}\colon X_\sigma \to X_\tau$ be the map constructed by the product of the composition of ι_1, ι_2 and ι_3 between $B\mathrm{SU}(n)$ and $B\mathrm{Sp}(n)$, and ι between the products of $\mathbb{C}\mathrm{P}^\infty$. We define a functor $F\colon P \to \mathrm{Top}$ as follows:

- For $\sigma \in P$, put $F(\sigma) = \prod_i X_{\sigma \cap A_i}$.
- For $\sigma, \tau \in P$ with $\sigma \subset \tau$, the map between $F(\sigma) \to F(\tau)$ is defined by the product

$$\prod_i f_{\sigma \cap A_i, \tau \cap A_i}\colon \prod_i X_{\sigma \cap A_i} \to \prod_i X_{\tau \cap A_i}.$$

We define $X = \operatorname{hocolim}_P F$; then the following lemma holds.

Lemma 3.2 *Under the above notation, the cohomology ring of X is isomorphic to* $\mathrm{SR}(K, \phi)$.

Proof We prove this lemma by induction on $|P|$. Let σ be a maximal simplex in K. Let K' be the simplicial complex consisting of the faces of simplices in $P \setminus \{\sigma\}$. Then by the assumption of the induction,

$$\begin{aligned} H^*(\operatorname{hocolim}_{P\setminus\{\sigma\}} F; \mathbb{Z}) &\cong \mathrm{SR}(K', \phi), \\ H^*(\operatorname{hocolim}_{P_{\le\sigma}} F; \mathbb{Z}) &\cong \mathbb{Z}[\sigma], \\ H^*(\operatorname{hocolim}_{P_{<\sigma}} F; \mathbb{Z}) &\cong \mathrm{SR}(K', \phi)/(V \setminus \sigma) \cong \mathrm{SR}(K'', \phi), \end{aligned}$$

where K'' is the simplicial complex consisting of the simplices that a simplex in K' and a face of σ. By Lemma 2.1, X is represented by the following homotopy pushout diagrams

$$\begin{tikzcd} \operatorname{hocolim}_{P_{<\sigma}} F \arrow[r] \arrow[d] & \operatorname{hocolim}_{P\setminus\{\sigma\}} F \arrow[d] \\ \operatorname{hocolim}_{P_{\leq\sigma}} F \arrow[r] & X \end{tikzcd}$$

Since ι_1, ι_2 and ι_3 are the maps that send each generator to its corresponding generator or 0 in cohomology, the maps in the upper diagram satisfy the condition in Lemma 2.2. Therefore by Lemma 2.2, we obtain that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$. □

Proof of Theorem 1.3 By this discussion, we apply Lemma 3.2 to the case $P = P_{\max}(K)$, completing the proof. □

When the degree of generators of $\mathrm{SR}(K,\phi)$ are only 2 and 4, Theorem 1.3 becomes a well-known result. This corollary is directly proved by the result of Davis and Januszkiewicz [7], and a special case of [5, Theorem 2.34].

Corollary 3.3 *Let $\mathrm{SR}(K,\phi)$ be the finitely generated graded Stanley–Reisner ring for a simplicial complex K with the vertex set V and $\phi\colon V \to 2\mathbb{Z}_{>0}$. When the image of ϕ is in $\{2,4\}$, we can construct a space X such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$.*

When, in $\mathrm{SR}(K,\phi)$, there is no pair of generators $x, y \in V$ such that $|x| = |y| = 4$ and $xy \neq 0$, we don't have to take the decomposition of the vertex set. In this case, we can restate Theorem 1.3 as follows.

Corollary 3.4 *Let $\mathrm{SR}(K,\phi)$ be the finitely generated graded Stanley–Reisner ring for a simplicial complex K with the vertex set V and $\phi\colon V \to 2\mathbb{Z}_{>0}$. We assume that there is no pair of generators $x, y \in V$ such that $|x| = |y| = 4$ and $xy \neq 0$ in $\mathrm{SR}(K,\phi)$. Then if $\mathrm{SR}(K,\phi)$ satisfies the following condition, we can construct a space X such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$:*

- *For $\sigma \in P_{\max}(K)$, the set $\{\phi(x) \mid x \in \sigma\}$ is equal to $\{2,2,\dots,2\}$, $\{4,6,\dots,2n+2\} \cup \{2,2,\dots,2\}$ or $\{4,8,\dots,4n\} \cup \{2,2,\dots,2\}$ as a multiset.*

4 Examples

In this section we look at some examples about Corollary 3.4.

Let $\mathrm{SR}[K,\phi] \cong \mathbb{Z}[x_4,x_6,x_8]/(x_6x_8)$. Then the corresponding diagram is

$$B\mathrm{SU}(3) \leftarrow B\mathrm{Sp}(1) \rightarrow B\mathrm{Sp}(2).$$

Let $\mathrm{SR}[K,\phi] \cong \mathbb{Z}[x_4, x_{6,1}, x_{6,2}, \dots, x_{6,n}, x_8]/(x_{6,j}x_{6,k} \text{ for } j \neq k)$, where $|x_{i,j}| = i$. Then the corresponding diagram is

$$\begin{tikzcd} & B\mathrm{Sp}(2) \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ B\mathrm{SU}(4) & B\mathrm{SU}(4) & \cdots & B\mathrm{SU}(4) \end{tikzcd}$$

Let $\mathrm{SR}[K,\phi] \cong \mathbb{Z}[x_4, x_{6,1}, x_{6,2}, x_{8,1}, x_{8,2}]/(x_{6,1}x_{6,2}, x_{8,1}x_{8,2})$, where $|x_{i,j}| = i$. Then the corresponding diagram is

$$\begin{tikzcd} & & & B\mathrm{SU}(4) \\ & B\mathrm{SU}(3) \arrow[urr] \arrow[rr] & & B\mathrm{SU}(4) \\ & & B\mathrm{Sp}(2) \arrow[uur] \arrow[dddr] & \\ B\mathrm{Sp}(1) \arrow[uur] \arrow[urr] \arrow[drr] \arrow[ddr] & & & \\ & & B\mathrm{Sp}(2) \arrow[uuur] \arrow[ddr] & \\ & B\mathrm{SU}(3) \arrow[rr] \arrow[drr] & & B\mathrm{SU}(4) \\ & & & B\mathrm{SU}(4) \end{tikzcd}$$

5 Approach from algebra over the Steenrod algebra

This section discusses when a graded polynomial ring has an unstable algebra structure over mod p Steenrod algebra by using previous results. All of the properties in this section are similar to the properties used by Aguadé in [2]. There, Aguadé obtains which polynomial algebras over $\mathbb{Z}$ are realizable as the integral cohomology ring of a space when the orders of the generators are all different. To prove this, Aguadé observes which polynomial rings have an unstable algebra structure over the mod p Steenrod algebra by using the result of Adams and Wilkerson [1]. In this section, we consider which polynomial rings have an unstable algebra structure over the mod p Steenrod algebra under the condition that there is at most 1 generator with degree 4.

When a commutative graded algebra A^* over $\mathbb{Z}/p$ has an action of mod p Steenrod algebra with Cartan formula, we say A^* an algebra over the mod p Steenrod algebra. An algebra over the mod p Steenrod algebra A^* with $A^{2i+1} = 0$ for all i is unstable if and only if for all homogeneous elements $x \in A^{2d}$, there are equations

$$\mathcal{P}^k(x) = \begin{cases} x^p & \text{if } k = d, \\ 0 & \text{if } k > d \end{cases} \quad \text{when } p \geq 3, \qquad \text{or} \qquad \mathrm{Sq}^{2k}(x) = \begin{cases} x^2 & \text{if } k = d, \\ 0 & \text{if } k > d \end{cases} \quad \text{when } p = 2.$$

When the odd-degree part of A^* is equal to 0, the unstable condition can be defined by only these equations. Conversely, if the odd-degree part of A^* is not equal to 0, there are more equations needed to define the unstable condition.

The following theorem can be obtained by combining Theorems 1.1 and 1.2 in Adams and Wilkerson [1].

Theorem 5.1 (cf Adams and Wilkerson [1, Theorems 1.1 and 1.2]) *Let A^* be a graded polynomial algebra over $\mathbb{Z}/p$ for prime p. We assume that the following conditions hold*:

- *A^* is an unstable algebra over the mod p Steenrod algebra.*
- *A^* is evenly generated.*
- *A^* is finitely generated as ring.*
- *The degrees of generators of A^* are prime to p.*

Then there is an isomorphism

$$A^* \cong H^*(BT^n;\mathbb{Z}/p)^W$$

for some n and a group W generated by pseudoreflections.

By using this theorem, we can prove the next theorem.

Proposition 5.2 (cf Aguadé [2, Proposition 2]) *Let A^* be a graded polynomial algebra over $\mathbb{Z}$ satisfying the following condition*:

- *There is a number N such that for all prime numbers $p > N$, $A \otimes \mathbb{Z}/p$ has unstable algebra structure over the mod p Steenrod algebra.*

Then the degree of the generator of A^* is the union of the following list:

- $\{2\}$
- $\{4, 6, \dots, 2n\}$
- $\{4, 8, \dots, 4n\}$
- $\{4, 8, \dots, 4(n-1), 2n\}$ for $n \geq 4$
- $\{4, 12\}$
- $\{4, 12, 16, 24\}$
- $\{4, 10, 12, 16, 18, 24\}$
- $\{4, 12, 16, 20, 24, 28, 36\}$
- $\{4, 16, 24, 28, 36, 40, 48, 60\}$
- $\{4, 16\}$
- $\{4, 24\}$
- $\{4, 48\}$

We prove this proposition by the same method in the proof of [2, Proposition 2].

Proof Let $p_1, \dots, p_i$ be the primes larger than 7 which divide the degree of generators of A^*. Then by a theorem of Dirichlet we can take a prime number $p > N$ such that

$$p \equiv 7 \bmod 16, \quad p \equiv 2 \bmod 3, \quad p \equiv 3 \bmod 5, \quad p \equiv 3 \bmod 7, \quad p \equiv 2 \bmod p_i.$$

By Theorem 5.1, $A^* \otimes \mathbb{Z}/p$ is isomorphic to an invariant ring $H^*(BT^n;\mathbb{Z}/p)^W$ for some n and a group W generated by pseudoreflections. By the classification theorem of p–adic pseudoreflection groups (cf Clark and Ewing [6]), we obtain this proposition. □

For a graded algebra A^*, write $QA^* = A^*/(A^*_+)^2$. The following lemma is proved by Thomas.

Theorem 5.3 (Thomas [16, Theorem 1.4]) *Let A^* be a finitely generated polynomial algebra over $\mathbb{Z}/2$ and an unstable algebra over the mod 2 Steenrod algebra. Then for any number i and odd number $n \geq 3$, the map*

$$\mathrm{Sq}^{2^i} : QA^{2^i(n-1)} \to QA^{2^i n}$$

is a surjection.

Lemma 5.4 *Let A^* be a polynomial algebra over $\mathbb{Z}$ such that the degrees of generators are equal to one of the following list as a multiset*:

- $\{4, 8, \dots, 4(n-1), 2n\} \cup \{2, 2, \dots, 2\}$ *with $n \geq 4$ and n is not a power of* 2,
- $\{4, 12\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 12, 16, 24\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 10, 12, 16, 18, 24\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 12, 16, 20, 24, 28, 36\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 16, 24, 28, 36, 40, 48, 60\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 24\} \cup \{2, 2, \dots, 2\}$,
- $\{4, 48\} \cup \{2, 2, \dots, 2\}$.

Then $A^* \otimes \mathbb{Z}/2$ doesn't have an unstable algebra structure over the mod 2 Steenrod algebra.

Proof We assume that $A^* \otimes \mathbb{Z}/2$ has an unstable algebra over the mod 2 Steenrod algebra. By Theorem 5.3, if there is a generator x with $|x| = 12$, then there must be a generator y with $|y| = 8$. Therefore the second, third, fourth and fifth cases don't have an unstable algebra structure over the mod 2 Steenrod algebra.

Similarly, if there is a generator x such that $|x| = 60, 24, 48$, then there must be a generator y with $|y| = 56, 16, 32$, respectively. Therefore the sixth, seventh and eighth cases don't have an unstable algebra structure over the mod 2 Steenrod algebra.

It remains to show the first case. In this case we can denote $n = 2^i m$ for an integer i and an odd number $m \geq 3$. When $i = 0$, by [2, Proposition 3], A^* doesn't have an unstable algebra structure over the mod 2 Steenrod algebra. When $i \geq 1$, by Theorem 5.3 $\mathrm{Sq}^{2^{i+1}} : QA^{2^{i+1}(m-1)} \to QA^{2^{i+1}m}$ must be a surjection. But $\dim(QA^{2^{i+1}(m-1)}) = 1$ and $\dim(QA^{2^{i+1}m}) = 2$; a contradiction. Therefore the first case doesn't have an unstable algebra structure over the mod 2 Steenrod algebra.

Combining these discussions, the proof is complete. □

Lemma 5.5 *Let A^* be a polynomial algebra over $\mathbb{Z}$ such that the degrees of generators are equal to $\{4, 16\} \cup \{2, 2, \dots, 2\}$ as a multiset. Then $A^* \otimes \mathbb{Z}/3$ doesn't have an unstable algebra over the mod 3 Steenrod algebra.*

Proof Let A^* be the polynomial ring with the degrees of generators are equal to $\{4, 16\} \cup \{2, \dots, 2\}$, and let x be the generator with degree 16 in A^*. We assume that $A^* \otimes \mathbb{Z}/3$ has an unstable algebra structure over the mod 3 Steenrod algebra. By the Adem relation, there is an equation $\mathcal{P}^8 = -\mathcal{P}^1\mathcal{P}^7$. Since $\mathcal{P}^8(x) = x^3$, it follows that x^3 is in $\mathrm{Im}(\mathcal{P}^1)$. On the other hand since there is no generator y with $|y| \equiv 12 \bmod 16$, the term x^i is not included in the image $\mathcal{P}^1$. This is a contradiction. □

Proposition 5.6 *Let A^* be a nontrivial graded polynomial algebra over $\mathbb{Z}$ such that*

- *there is at most one generator with degree* 4, *and*
- *for all prime numbers p, $A \otimes \mathbb{Z}/p$ has an unstable algebra structure over the mod p Steenrod algebra.*

Then the degree of the generators of A^* is equal to the one of the following list as a multiset for some n:

- $\{2, 2, \dots, 2\}$
- $\{4, 6, \dots, 2n\} \cup \{2, 2, \dots, 2\}$
- $\{4, 8, \dots, 4n\} \cup \{2, 2, \dots, 2\}$
- $\{4, 8, \dots, 2^{n+1} - 4, 2^n\} \cup \{2, 2, \dots, 2\}$

Proof By Proposition 5.2 and the first condition, the degree of the generator of A^* is equal to the union of the one of the table in Proposition 5.2 and the copies of $\{2\}$. By Lemmas 5.4 and 5.5, the cases except for the cases $\{4, 6, \dots, 2n\}$, $\{4, 8, \dots, 4n\}$ or $\{4, 8, \dots, 2^{n+1} - 4, 2^n\}$ don't satisfies the second condition. Thus we obtain the proposition. □

Example 5.7 Let $\mathbb{Z}/2[t_1, \dots, t_{2^n}] \cong H^*(BT^{2^n}; \mathbb{Z}/2)$ for $n \geq 2$. Take a subring of $H^*(BT^{2^n}; \mathbb{Z}/2)$ as

$$\mathbb{Z}/2[t_1, t_2, \dots, t_{2^n - 1}]^{W(\mathrm{Sp}(2^n - 1))} \otimes \mathbb{Z}/2[{t_{2^n}}^{2^{n-1}}],$$

where $W(\mathrm{Sp}(2^n - 1))$ is the Weyl group of $\mathrm{Sp}(2^n - 1)$ and $\mathbb{Z}/2[t_1, t_2, \dots, t_{2^n}]^{W(\mathrm{Sp}(2^n - 1))}$ is the invariant ring of the canonical $W(\mathrm{Sp}(2^n - 1))$–action. Since $\mathbb{Z}/2[t_1, t_2, \dots, t_{2^n}]^{W(\mathrm{Sp}(2^n - 1))}$ is isomorphic to $H^*(B\mathrm{Sp}(2^n - 1); \mathbb{Z}/2)$, this subring preserve the action of mod 2 Steenrod operations, and the degree of generators of this subring is $\{4, 8, \dots, 2^{n+1} - 4, 2^n\}$. This subring has the unstable algebra structure over the mod 2 Steenrod algebra induced by $H^*(BT^{2^n}; \mathbb{Z}/2)$.

When p is an odd prime number, the cohomology ring $H^*(B\mathrm{Spin}(2^n); \mathbb{Z}/p)$ is isomorphic to the polynomial ring with generator's degree $\{4, 8, \dots, 2^{n+1} - 4, 2^n\}$ (cf [12, Chapter III, Theorem 3.19]), and has the unstable algebra structure over the mod p Steenrod algebra. And the ring in this example has the unstable algebra structure over the mod 2 Steenrod algebra. Therefore by only using the method in this section, we cannot remove the case $\{4, 8, \dots, 2^{n+1} - 4, 2^n\}$ in Proposition 5.6.

6 Stanley–Reisner ring and Steenrod algebra

Let K be a simplicial set with the vertex set V, and $\phi\colon V \to 2\mathbb{Z}_{>0}$. For a polynomial $f \in \mathrm{SR}(K, \phi)$ and a monomial g, we write $g < f$ when $g \neq 0$ in $\mathrm{SR}(K, \phi)$ and the coefficient of g in f is not equal to 0. This notation is well-defined because the ideal I is generated by monomials.

Lemma 6.1 *Let X be a space such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$ for some graded Stanley–Reisner ring, and $\sigma \in K$ be a maximal simplex. Then for any prime number p the ideal $(V \setminus \sigma)$ in $H^*(X;\mathbb{Z}/p) \cong \mathrm{SR}(K,\phi) \otimes \mathbb{Z}/p$ preserves the action of the mod p Steenrod algebra.*

Proof We assume that the ideal $(V \setminus \sigma)$ doesn't preserve the action of mod p Steenrod algebra. Then by the Cartan formula, there is $x \in V \setminus \sigma$ and a monomial $f \in \mathrm{SR}(K,\phi)$ such that $f \notin (V \setminus \sigma)$ and $f < \mathcal{P}^i(x)$ for some i. Now we can take f with i being minimal, ie $\mathcal{P}^j(x) \in (V \setminus \sigma)$ for $j < i$. We write $g = \prod_{y \in \sigma} y$. Since f is a monomial generated by σ, we get $fg \neq 0$ in $\mathrm{SR}(K,\phi)$. Then

$$fg < \mathcal{P}^i(x)g,$$

and since i is minimal,

$$fg \not< \sum_{j>0} \mathcal{P}^{i-j}(x)\mathcal{P}^j(g).$$

Therefore, by the Cartan formula,

$$fg < \mathcal{P}^i(x)g + \sum_{j>0} \mathcal{P}^{i-j}(x)\mathcal{P}^j(g) = \mathcal{P}^i(xg),$$

and we obtain

$$\mathcal{P}^i(xg) \neq 0.$$

Since $xg = 0$ in $\mathrm{SR}(K,\phi)$, this is a contradiction, so the assumption is false. This completes the proof. □

Proposition 6.2 *Let X be a space such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$ for some graded Stanley–Reisner ring. Let $\sigma_1, \dots, \sigma_m \in K$ be maximal simplexes. Then for any prime number p, the ring*

$$\mathrm{SR}(K,\phi) \otimes \mathbb{Z}/p\mathbb{Z}/(V \setminus \sigma_1 \cap \dots \cap \sigma_m)$$

has an unstable algebra structure over the mod p Steenrod algebra induced by the quotient map

$$H^*(X;\mathbb{Z}/p) \cong \mathrm{SR}(K,\phi) \otimes \mathbb{Z}/p\mathbb{Z} \to \mathrm{SR}(K,\phi) \otimes \mathbb{Z}/p\mathbb{Z}/(V \setminus \sigma_1 \cap \dots \cap \sigma_m).$$

Proof By Lemma 6.1, for all $x \in V \setminus \sigma_k$ and i, we obtain

$$\mathcal{P}^i(x) \in (V \setminus \sigma_k) \subset (V \setminus \sigma_1 \cap \dots \cap \sigma_n).$$

Therefore the ideal $(V \setminus \sigma_1 \cap \dots \cap \sigma_m)$ preserves the action of the mod p Steenrod algebra. □

Theorem 6.3 *For a graded Stanley–Reisner ring $\mathrm{SR}(K,\phi)$, let X be a space such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$. We assume that there is no pair of generators $x, y \in V$ such that $\phi(x) = \phi(y) = 4$ and $xy \neq 0$ in $\mathrm{SR}(K,\phi)$. Then for $\sigma \in P_{\max}(K)$, the set $\{\phi(x) \mid x \in \sigma\}$ is equal to*

- $\{2, \dots, 2\}$,
- $\{4, 6, \dots, 2n+2\} \cup \{2, \dots, 2\}$,
- $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$, *or*
- $\{4, 8, \dots, 2^{n+2}-8, 2^{n+2}-4, 2^{n+1}\} \cup \{2, \dots, 2\}$

as a multiset for some $n \geq 1$.

Proof Since $\sigma \in K$, there is no relation between the generators in σ. Therefore there is an isomorphism

$$\mathrm{SR}(K,\phi)/(V \setminus \sigma) \cong \mathbb{Z}[\sigma].$$

By the definition of $P_{\max}(K)$ there are maximal simplexes $\sigma_1, \dots, \sigma_m \in K$ such that $\sigma = \bigcap_i \sigma_i$. Thus $\mathbb{Z}[\sigma]$ satisfies the condition of Proposition 6.2. By the assumption in the statement, for any $\sigma \in P_{\max}(K)$ there is at most one generator with degree 4 in σ. By Proposition 6.2 and this condition, the polynomial ring $\mathbb{Z}[\sigma]$ satisfies the condition in Proposition 5.6. Therefore the set $\{\phi(x) \mid x \in \bigcap_i \sigma_i\}$ is equal, as a multiset, to

- $\{2, \dots, 2\}$,
- $\{4, 6, \dots, 2n+2\} \cup \{2, \dots, 2\}$,
- $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$, or
- $\{4, 8, \dots, 2^{n+2}-8, 2^{n+2}-4, 2^{n+1}\} \cup \{2, \dots, 2\}$. □

Example 6.4 Let $\mathrm{SR}(K,\phi) \cong \mathbb{Z}[x_4, x_6]/(x_4x_6)$, where $|x_i| = i$. Then $P_{\max}(K) = \{\{4\}, \{6\}\}$. By Theorem 6.3, there is no space X with $H^*(X;\mathbb{Z}) \cong \mathbb{Z}[x_4, x_6]/(x_4x_6)$.

7 Proof of the main theorem

By combining Corollary 3.4 and Theorem 6.3, we can prove Theorem 1.1.

Proof of Theorem 1.1 In Corollary 3.4, we prove that if $\mathrm{SR}(K,\phi)$ satisfies these conditions then there is a space X such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$.

On the other hand, we assume that there is a space X such that $H^*(X;\mathbb{Z}) \cong \mathrm{SR}(K,\phi)$. By assumption, in the statement for $i = 2$, $\mathrm{SR}(K,\phi)$ satisfies the condition of Theorem 6.3. By Theorem 6.3, for any $\sigma \in P_{\max}(K)$ the set $\{\phi(x) \mid x \in \sigma\}$ is equal to $\{2, \dots, 2\}$, $\{4, 6, \dots, 2n+2\} \cup \{2, \dots, 2\}$, $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$ or $\{4, 8, \dots, 2^{n+2}-8, 2^{n+2}-4, 2^{n+1}\} \cup \{2, \dots, 2\}$ as a multiset. By the assumption in the statement, there is no pair of generators x, y such that $|x| = |y| = 2^n$ for some $n \geq 3$ and $xy \neq 0$ in $\mathrm{SR}(K,\phi)$. Since the case $\{4, 8, \dots, 2^{n+2}-8, 2^{n+1}-4, 2^n\} \cup \{2, \dots, 2\}$ for $n \geq 3$ includes such a pair of generators, this case doesn't appear. Therefore for any $\sigma \in P_{\max}(K)$ the set $\{\phi(x) \mid x \in \sigma\}$ is equal to $\{2, \dots, 2\}$, $\{4, 6, \dots, 2n+2\} \cup \{2, \dots, 2\}$ or $\{4, 8, \dots, 4n\} \cup \{2, \dots, 2\}$ as a multiset.

By combining these, the proof is complete. □

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Received: 24 July 2022 Revised: 9 February 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 3 (pages 1225–1808) 2024