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Dehn twists and the Nielsen realization problem for spin 4–manifolds

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We prove that for a closed oriented smooth spin 4–manifold X with nonzero signature, the Dehn twist about a $(+2)$– or (-2)–sphere in X is not homotopic to any finite-order diffeomorphism. In particular, we negatively answer the Nielsen realization problem for each group generated by the mapping class of a Dehn twist. We also show that there is a discrepancy between the Nielsen realization problems in the topological category and smooth category for connected sums of copies of $K3$ and $S^2 \times S^2$. The main ingredients of the proofs are Y Kato's $10/8$–type inequality for involutions and a refinement of it.

57S17

1 Main results

Given a smooth manifold X, let $\mathrm{Diff}(X)$ denote the group of diffeomorphisms. The *Nielsen realization problem* asks whether a given finite subgroup G of the mapping class group $\pi_0(\mathrm{Diff}(X))$ can be realized as a subgroup of $\mathrm{Diff}(X)$, ie whether there exists a (group-theoretic) section $s\colon G \to \mathrm{Diff}(X)$ of the natural map $\mathrm{Diff}(X) \to \pi_0(\mathrm{Diff}(X))$ over G. If there is a section, we say that G is *realizable in* $\mathrm{Diff}(X)$. When X is of $\dim = 2$ and oriented closed, which is the classical case of the Nielsen realization problem, Kerckhoff [18] proved that every G is realizable.

In contrast, Raymond and Scott [30] showed that, in every dimension ≥ 3, there are nilmanifolds for which the Nielsen realization fails essentially using their nontrivial fundamental groups. Focusing on dimension 4 and simply connected manifolds, it was recently proven by Baraglia and the author [4] and Farb and Looijenga [9] that the Nielsen realization fails for $K3$, the underlying smooth 4–manifold of a $K3$ surface. However, to the best of the author's knowledge, the nilmanifolds in [30] and $K3$ are the only examples of 4–manifolds X that are shown to admit finite subgroups of $\pi_0(\mathrm{Diff}(X))$ that are not realizable in $\mathrm{Diff}(X)$. The purpose of this paper is to expand the list of such 4–manifolds considerably. In particular, we give infinitely many examples of simply connected 4–manifolds with distinct intersection forms for which the Nielsen realization fails.

For a general 4–manifold, it is not obvious to find a potential example of nonrealizable finite subgroups of the mapping class group. Following Farb and Looijenga [9], we consider *Dehn twists*, which are sources of interesting examples. Given a $(+2)$– or (-2)–sphere S embedded in a 4–manifold X, one has a diffeomorphism $T_S\colon X \to X$ called the Dehn twist, whose mapping class $[T_S]$ generates an order-2 subgroup of $\pi_0(\mathrm{Diff}(X))$ (see Section 5.1). Our first main result is:

Theorem 1.1 *Let X be a closed oriented smooth spin 4–manifold with nonzero signature and S be a smoothly embedded 2–sphere in X with $[S]^2 = 2$ or $[S]^2 = -2$. Then the Dehn twist $T_S\colon X \to X$ about S is not homotopic to any finite-order diffeomorphism of X. In particular, the order-2 subgroup of $\pi_0(\mathrm{Diff}(X))$ generated by the mapping class $[T_S]$ is not realizable in $\mathrm{Diff}(X)$.*

Theorem 1.1 generalizes the case when $X = K3$ due to Farb and Looijenga [9, Corollary 1.10] (see Remark 5.2 for the comparison), and Theorem 1.1 immediately implies that the Nielsen realization fails for quite many 4–manifolds, such as $\#_m K3 \#_n S^2 \times S^2$ with $m > 0$ and also infinitely many examples of irreducible 4–manifolds. See Example 5.3 for details.

Theorem 1.1 makes a striking contrast to a recent result by Lee [21, Corollary 1.5, Remark 1.7], which implies that the Dehn twist about every (± 2)–sphere in $\mathbb{CP}^2 \# n(-\mathbb{CP}^2)$ with $n \leq 8$ is topologically isotopic (hence homotopic) to a smooth involution. This means that an analogous statement to Theorem 1.1 does not hold for *nonspin* 4–manifolds.

Another result of this paper concerns a comparison between the Nielsen realization problems in the topological category and the smooth category. Let $\mathrm{Homeo}(X)$ denote the group of homeomorphisms of a manifold X. As well as the smooth Nielsen realization, we say that a subgroup G of $\pi_0(\mathrm{Homeo}(X))$ is *realizable in* $\mathrm{Homeo}(X)$ if there is a section $s\colon G \to \mathrm{Homeo}(X)$ of the natural map

$$\mathrm{Homeo}(X) \to \pi_0(\mathrm{Homeo}(X))$$

over G. In [4, Theorem 1.2], Baraglia and the author showed that some order-2 subgroup of $\pi_0(\mathrm{Diff}(K3))$ is not realizable in $\mathrm{Diff}(K3)$, even when the corresponding subgroup in $\pi_0(\mathrm{Homeo}(K3))$ is realizable in $\mathrm{Homeo}(K3)$. We generalize this result to connected sums of copies of $K3$ and $S^2 \times S^2$:

Theorem 1.2 *For $m > 0$ and $n \geq 0$, set $X = mK3 \# nS^2 \times S^2$. Then there exists an order-2 subgroup G of $\pi_0(\mathrm{Diff}(X))$ with the following properties:*

- *The group G is not realizable in $\mathrm{Diff}(X)$. Moreover, a representative of the generator of G is not homotopic to any finite-order diffeomorphism of X.*
- *The subgroup $G' \subset \pi_0(\mathrm{Homeo}(X))$ defined as the image o G under the natural map*

$$\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))$$

is a nontrivial group, and G' is realizable in $\mathrm{Homeo}(X)$.

In other words, a representative $g \in \mathrm{Diff}(X)$ of the generator of G in Theorem 1.2 is not homotopic to any finite-order diffeomorphism, although g^2 is smoothly isotopic to the identity and g is topologically isotopic to some topological involution with nontrivial mapping class. Theorem 1.2 gives also an alternative proof of a result by Baraglia [2, Proposition 1.2] about the realization problem along $\mathrm{Diff}(X) \to \mathrm{Aut}(H_2(X;\mathbb{Z}))$ (see Section 7).

Theorems 1.1 and 1.2 shall be derived from the following constraint on the induced actions of finite-order diffeomorphisms on homology. Let $\sigma(X)$ denote the signature of an oriented closed 4–manifold

X and $b_+(X)$ denote the maximal-dimension of positive-definite subspaces of $H_2(X;\mathbb{R})$. For an involution φ on the intersection lattice, we denote by $b_+^{\varphi}(X)$ (resp. $b_-^{\varphi}(X)$) the maximal-dimension of positive-definite (resp. negative-definite) subspaces of the φ–invariant part $H_2(X;\mathbb{R})^{\varphi}$, and we set $\sigma^{\varphi}(X)=b_+^{\varphi}(X)-b_-^{\varphi}(X)$.

Theorem 1.3 *Let X be a closed oriented smooth 4–manifold with $\sigma(X)<0$, and let $\mathfrak{s}$ be a spin structure on X. Let $g\colon X\to X$ be a finite-order diffeomorphism that preserves orientation of X and $\mathfrak{s}$, and let $\varphi\colon H_2(X;\mathbb{Z})/\mathrm{Tor}\to H_2(X;\mathbb{Z})/\mathrm{Tor}$ denote the action on homology induced from g. Suppose that φ is of order 2 and that $\sigma^{\varphi}(X)\neq\sigma(X)/2$. Then*

$$-\tfrac{1}{16}\sigma(X)\le b_+(X)-b_+^{\varphi}(X). \tag{1}$$

Moreover, if $b_+(X)-b_+^{\varphi}(X)>0$, then

$$-\tfrac{1}{16}\sigma(X)+1\le b_+(X)-b_+^{\varphi}(X).$$

The main ingredients of the proof of Theorem 1.3 are Y Kato's $10/8$–type inequality for involutions [17] (Theorem 2.2) coming from Seiberg–Witten theory and a refinement of it (Theorem 3.1). This refinement is necessary to show the "moreover" part of Theorem 1.3, which shall be used to obtain the results on Dehn twists (Theorem 1.1) for both $(+2)$– and (-2)–spheres.

Here is an outline of the contents of this paper. In Section 2, we recall Kato's $10/8$–type inequality for a smooth involution on a spin 4–manifold. In Section 3, we give a refinement of Kato's inequality by proving a new Borsuk–Ulam-type theorem using equivariant K–theory. In Section 4, we prove Theorem 1.3 based on Kato's inequality and the refinement of it in Section 3. Sections 5 and 6 are devoted to prove Theorems 1.1 and 1.2 respectively. We conclude by giving remarks on another kind of Dehn twist and other variants of the Nielsen realization problem in Section 7.

2 Kato's $10/8$–type inequality for involutions

Henceforth, for an oriented closed 4–manifold X, we identify $H_2(X)$ with $H^2(X)$ via Poincaré duality. For an involution ι on X, we set $b_+^{\iota}(X)=b_+^{\iota^*}(X)$, and similarly define $b_-^{\iota}(X)$ and $\sigma^{\iota}(X)$. Note that, if X has nonvanishing signature, all diffeomorphisms of X are orientation-preserving, namely, we have $\mathrm{Diff}(X)=\mathrm{Diff}^+(X)$, the group of orientation-preserving diffeomorphisms.

First, we recall the notion of even and odd involutions following [1; 6]. Let X be an oriented closed smooth 4–manifold and $\mathfrak{s}$ be a spin structure on X. Let $\iota\colon X\to X$ be an orientation-preserving diffeomorphism of order 2, and suppose that ι preserves (the isomorphism class of) $\mathfrak{s}$. Then there are exactly two lifts of ι to $\mathfrak{s}$ as automorphisms of the spin structure. We have either both lifts are of order 2 or both are of order 4. We say that the involution ι is *of even type* if the lifts are of order 2, and say that ι is *of odd type* if the lifts are of order 4. When the fixed-point set X^{ι} is nonempty, the codimension of all components of X^{ι} are the same, which is either 4 or 2, and the parity of ι determines which of them arises: X^ι is of codimension-4 if ι is of even type, and X^ι is of codimension-2 if ι is of odd type [1, Proposition 8.46]; see also [31].

Lemma 2.1 *Let X be an oriented closed smooth 4–manifold and $\mathfrak{s}$ be a spin structure on X. Let $\iota\colon X \to X$ be an orientation-preserving diffeomorphism of order 2, and suppose that ι preserves (the isomorphism class of) $\mathfrak{s}$ and is of even type. Then $\sigma^\iota(X) = \sigma(X)/2$.*

Proof By Hirzebruch's signature theorem (see for example [16, equation (12), page 177]), $\sigma^\iota(X)$ can be obtained by adding $\sigma(X)/2$ to contributions from fixed surfaces of ι. (Note that, for a general involution, the contribution from isolated fixed points is zero.) However, X^ι does not contain surfaces since ι is even. □

An important ingredient of this paper is the following 10/8–type constraint on odd smooth involutions, proven by Y Kato [17] using Seiberg–Witten theory and $\mathbb{Z}/4$–equivariant K–theory:

Theorem 2.2 (Kato [17, Theorem 2.3]) *Let $(X, \mathfrak{s})$ be a smooth closed oriented spin 4–manifold. Let $\iota\colon X \to X$ be a smooth orientation-preserving involution, and suppose that ι preserves $\mathfrak{s}$ and is of odd type. Then*

$$-\tfrac{1}{16}\sigma(X) \le b_+(X) - b_+^\iota(X). \tag{2}$$

Remark 2.3 In [17], the result corresponding to Theorem 2.2 is stated using a quantity $b_+^I(X)$, where I acts on $H^2(X;\mathbb{R})$ as $I = -\iota^*$. By Poincaré duality, it immediately follows that $b_+^I(X) = b_+(X) - b_+^\iota(X)$.

3 A refinement of Kato's inequality

To deal with Dehn twists about both $(+2)$– and (-2)–spheres in Theorem 1.1, we shall need the following refinement of Kato's inequality (Theorem 2.2), which we call the *refined Kato's inequality*:

Theorem 3.1 *Let $(X, \mathfrak{s})$ be a smooth closed oriented spin 4–manifold. Let $\iota\colon X \to X$ be a smooth orientation-preserving involution, and suppose that ι preserves $\mathfrak{s}$ and is of odd type. Suppose that $b_+(X) - b_+^\iota(X) > 0$. Then*

$$-\tfrac{1}{16}\sigma(X) + 1 \le b_+(X) - b_+^\iota(X).$$

This shall be proven in Section 3.2 using a Borsuk–Ulam-type theorem (Theorem 3.3), which we first give in Section 3.1.

3.1 $\mathbb{Z}/4$–equivariant K–theory

To show Theorem 3.1, we prove a new Borsuk–Ulam-type theorem using $\mathbb{Z}/4$–equivariant K–theory. As in Kato's argument [17], the following approach is modeled on Bryan's argument [6] for Pin(2)–equivariant K–theory. A difference from Kato's argument is that we shall use the structure of the whole representation ring $R(\mathbb{Z}/4)$.

Set $G = \mathbb{Z}/4$ and let j denote a generator; $G = \{1, j, -1, -j\}$. (The symbol j stands for a unit quaternion $j \in \mathrm{Pin}(2) \subset \mathbb{H}$, which is a symmetry that the Seiberg–Witten equations admit.) Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_-$ be complex 1–dimensional representations of G determined by

$$\mathrm{tr}_j\, \mathbb{C} = 1, \quad \mathrm{tr}_j\, \mathbb{C}_+ = i, \quad \mathrm{tr}_j\, \mathbb{C}_- = -i,$$

where tr_j denotes the trace of the action of j and $i = \sqrt{-1}$. Namely, $\mathbb{C}$ is the trivial 1–dimensional representation, and $\mathbb{C}_\pm$ are representations given as $\pm i$–multiplication of the fixed generator of G. Let $\widetilde{\mathbb{R}}$ denote a real 1–dimensional representation of G defined through the surjective homomorphism $G \to \mathbb{Z}/2$ and multiplication of $\mathbb{Z}/2 = \{\pm 1\}$. Set $\widetilde{\mathbb{C}} = \widetilde{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. Recall that the complex representation ring $R(G)$ is given by

$$R(G) = \mathbb{Z}[t]/(t^4 - 1), \tag{3}$$

where $t = \mathbb{C}_+$.

Here we recall a general fact, which holds for any compact Lie group G, called tom Dieck's formula by Bryan [6]. Let V and W be finite-dimensional unitary representations of G. Let V^+ denote the one-point compactification of V, naturally acted by G. We regard the point at infinity as the base point of V^+. Let $f : V^+ \to W^+$ be a pointed G–continuous map. By the equivariant K–theoretic Thom isomorphism, we have that $\widetilde{K}_G(V^+)$ and $\widetilde{K}_G(W^+)$ are free $\widetilde{K}_G(S^0) = R(G)$–modules generated by the equivariant K–theoretic Thom classes $\tau_G^K(V)$ and $\tau_G^K(W)$ respectively, and thus one may define the equivariant K–theoretic mapping degree $\alpha_f \in R(G)$ of f characterized by

$$f^* \tau_G^K(W) = \alpha_f \tau_G^K(V).$$

For an element $g \in G$, let V^g and W^g denote the fixed-point set for g, and let $(V^g)^\perp$ and $(W^g)^\perp$ denote the orthogonal complement of V^g and W^g in V and W respectively. Let $d(f^g) \in \mathbb{Z}$ denote the mapping degree, defined using just the ordinary cohomology, of the fixed-point set map $f^g : (V^g)^+ \to (W^g)^+$. For $\beta \in R(G)$, define $\lambda_{-1}\beta \in R(G)$ to be $\sum_{i \ge 0} (-1)^i \Lambda^i \beta$. Then tom Dieck's formula is:

Proposition 3.2 ([7, Proposition 9.7.2], see also [6, Theorem 3.3]) *In the above setup, we have*

$$\mathrm{tr}_g(\alpha_f) = d(f^g)\, \mathrm{tr}_g(\lambda_{-1}((W^g)^\perp - (V^g)^\perp)).$$

Now we are ready to prove the Borsuk–Ulam-type theorem we need:

Theorem 3.3 *Let $G = \mathbb{Z}/4$. For natural numbers $m_0, m_1, n_0, n_1 \ge 0$ with $m_0 < m_1$, suppose that there exists a G–equivariant pointed continuous map*

$$f : (\widetilde{\mathbb{C}}^{m_0} \oplus (\mathbb{C}_+ \oplus \mathbb{C}_-)^{n_0})^+ \to (\widetilde{\mathbb{C}}^{m_1} \oplus (\mathbb{C}_+ \oplus \mathbb{C}_-)^{n_1})^+ \tag{4}$$

with $f(0) = 0$. Then

$$n_0 - n_1 + 1 \le m_1 - m_0. \tag{5}$$

Remark 3.4 This Borsuk–Ulam-type theorem, Theorem 3.3, may be of independent interest. Especially, it is worth noting that Theorem 3.3 allows us to give a proof of Furuta's celebrated $10/8$–inequality [11] using only the $\mathbb{Z}/4$–symmetry of the Seiberg–Witten equations, while the original proof used a bigger symmetry, $\mathrm{Pin}(2)$. See Remark 3.5 for further comments on this.

Also, Theorem 3.3 generalizes a result by Pfister and Stolz [28, Theorem, page 286], where they proved Theorem 3.3 for the case that $m_0 = 0$ and $n_1 = 0$. The argument of Pfister and Stolz is also based on K–theory, but in a slightly different way than ours.

Proof of Theorem 3.3 Let $\alpha = \alpha_f \in R(G)$ denote the equivariant K–theoretic mapping degree of f. We shall obtain constraints on α from the actions of -1 and j. First, note that the (-1)–fixed point set map for f is given by $f^{-1}\colon (\tilde{\mathbb{C}}^{m_0})^+ \to (\tilde{\mathbb{C}}^{m_1})^+$, and thus the assumption $m_0 < m_1$ implies $d(f^{-1}) = 0$. Hence it follows from Proposition 3.2 that $\mathrm{tr}_{-1}(\alpha) = 0$. Thanks to the ring structure (3) of $R(G)$, α can be expressed in the form

$$\alpha = \sum_{k=0}^{3} a_k t^k, \tag{6}$$

where $a_k \in \mathbb{Z}$. Since $\mathrm{tr}_{-1}(t) = -1$, it follows that $\mathrm{tr}_{-1}(\alpha) = a_0 - a_1 + a_2 - a_3$. Thus,

$$a_0 - a_1 + a_2 - a_3 = 0. \tag{7}$$

Next, let us consider the j–action on α. First, note that f^j is just the identity map on $S^0 = \{0\} \cup \{\infty\}$, and hence $d(f^j) = 1$. In general, for complex rank 1 (virtual) representations $L_1, \dots, L_N \in R(G)$, one has $\lambda_{-1}\bigl(\sum_{i=1}^N L_i\bigr) = \prod_{i=1}^N \lambda_{-1} L_i$. Thus, again using Proposition 3.2,

$$\begin{aligned}
\mathrm{tr}_j(\alpha) &= \mathrm{tr}_j\bigl(\lambda_{-1}(\tilde{\mathbb{C}}^{m_1-m_0} \oplus (\mathbb{C}_+ \oplus \mathbb{C}_-)^{n_1-n_0})\bigr) \\
&= \mathrm{tr}_j\bigl(\lambda_{-1}((m_1-m_0)t^2 + (n_1-n_0)t + (n_1-n_0)t^3)\bigr) \\
&= \mathrm{tr}_j\bigl((1-t^2)^{m_1-m_0}(1-t)^{n_1-n_0}(1-t^3)^{n_1-n_0}\bigr) \\
&= (1+1)^{m_1-m_0}(1-i)^{n_1-n_0}(1+i)^{n_1-n_0} \\
&= 2^{m_1-m_0+n_1-n_0}.
\end{aligned} \tag{8}$$

On the other hand, from the expression (6) of α, we have $\mathrm{tr}_j(\alpha) = a_0 - a_2 + (a_1 - a_3)i$. Since $\mathrm{tr}_j(\alpha) \in \mathbb{R}$ by (8), we have $a_1 - a_3 = 0$, and this combined with (7) implies that

$$\mathrm{tr}_j(\alpha) = a_0 - a_2 = 2(a_1 - a_2). \tag{9}$$

Since $a_1 - a_2 \in \mathbb{Z}$, the desired inequality (5) follows from (8) and (9). □

Note that the divisibility by 2 over $\mathbb{Z}$ of the right-hand side of (9) contributes to the "$+1$" term in the inequality (5), which is the source of the refined Kato's inequality.

3.2 Proof of Theorem 3.1

Now we are ready to prove the refined Kato's inequality:

Proof of Theorem 3.1 Set $G = \mathbb{Z}/4$. Kato proved in [17] that the odd involution ι gives rise to an involutive symmetry I on the Seiberg–Witten equations on $(X, \mathfrak{s})$, and the complexification of a finite-dimensional approximation of the I–invariant part of the Seiberg–Witten equations is a G–equivariant pointed continuous map f of the form (4) with $f(0) = 0$, where the natural numbers m_0, m_1, n_0 and n_1 in (4) satisfy

$$m_1 - m_0 = b_+(X) - b_+^\iota(X), \quad n_0 - n_1 = -\tfrac{1}{16}\sigma(X).$$

By the assumption $b_+(X) - b_+^\iota(X) > 0$, we may apply Theorem 3.3 to this f. □

Remark 3.5 Furuta's $10/8$–inequality [11] was proved using the $\mathrm{Pin}(2)$–symmetry of the Seiberg–Witten equations for a closed spin 4–manifold X. Using our Borsuk–Ulam-type theorem, Theorem 3.3, we may recover Furuta's $10/8$–inequality using only the $\mathbb{Z}/4$–symmetry of the Seiberg–Witten equations as follows. Note that $G = \mathbb{Z}/4 = \langle j \rangle$ is a subgroup of $\mathrm{Pin}(2) = S^1 \cup jS^1 \subset \mathbb{H}$. Restricting the $\mathrm{Pin}(2)$–symmetry to the $\mathbb{Z}/4$–symmetry in Furuta's construction [11], we have that the complexification of a finite-dimensional approximation of the Seiberg–Witten equations is a G–equivariant pointed continuous map f of the form (4) with $f(0) = 0$ for natural numbers m_0, m_1, n_0 and n_1 with

$$m_1 - m_0 = b_+(X), \quad n_0 - n_1 = -\tfrac{1}{8}\sigma(X).$$

Applying Theorem 3.3 to f, we obtain

$$-\tfrac{1}{8}\sigma(X) + 1 \le b_+(X)$$

provided that $b_+(X) > 0$. This inequality is equivalent to the $10/8$–inequality [11, Theorem 1].

4 Proof of Theorem 1.3

Proof of Theorem 1.3 First, we reduce the problem to involutions following [9, Proof of Corollary 1.10]. Since the subgroup of $\mathrm{Diff}(X)$ generated by g has a surjective homomorphism onto $\langle \varphi \rangle \cong \mathbb{Z}/2$, the order of g is even. Let $2m$ be the order of g; then g^m is a smooth involution. Set $\iota = g^m$. Since $g_* = \varphi$ is of order 2, either $\iota_* = \varphi$ or $\iota_* = \mathrm{id}$. By the condition that $g^*\mathfrak{s} \cong \mathfrak{s}$, ι also preserves $\mathfrak{s}$.

If $\iota_* = \varphi$, we have $\sigma^\iota(X) \ne \sigma(X)/2$ from the assumption that $\sigma^\varphi(X) \ne \sigma(X)/2$. If $\iota_* = \mathrm{id}$, we have $\sigma^\iota(X) \ne \sigma(X)/2$ since we supposed $\sigma(X) \ne 0$. Thus, in any of these cases, $\sigma^\iota(X) \ne \sigma(X)/2$, and hence it follows from Lemma 2.1 that ι is of odd type. It then follows from Kato's inequality, Theorem 2.2, that

$$-\tfrac{1}{16}\sigma(X) \le b_+(X) - b_+^\iota(X) \le b_+(X) - b_+^\varphi(X). \tag{10}$$

To see the "moreover" part of the theorem, suppose that $b_+(X) - b_+^\iota(X) > 0$. Then we can replace the left-hand side of (10) with $-\sigma(X)/16 + 1$ by the refined Kato's inequality, Theorem 3.1. □

5 Proof of Theorem 1.1

5.1 Dehn twists about (±2)–spheres

First, we recall 4–dimensional Dehn twists associated with (±2)–spheres. We refer readers to a lecture note by Seidel [32, Section 2] for details. While the construction of the Dehn twist in [32] is described for a Lagrangian sphere in a symplectic 4–manifold, which is always a (−2)–sphere, the construction works for any (−2)–sphere in a general 4–manifold without any change, and it is easy to obtain an analogous diffeomorphism for a (+2)–sphere, described below.

Given a (−2)–sphere S in an oriented 4–manifold X, namely a smoothly embedded 2–dimensional sphere S with $[S]^2=-2$, one may construct a diffeomorphism $T_S\colon X\to X$ called the *Dehn twist* about S, which is supported in a tubular neighborhood of S in X, as follows. First, note that a tubular neighborhood of S is diffeomorphic to T^*S^2 since S is a (−2)–sphere, and fix an embedding $T^*S^2\hookrightarrow X$. The Dehn twist T_S is the extension by the identity of some compactly supported diffeomorphism τ of T^*S^2 called the *model Dehn twist*, which is given as the monodromy around the nodal singular fiber of the family $\mathbb{C}^3\to\mathbb{C}$, $(z_1,z_2,z_3)\mapsto z_1^2+z_2^2+z_3^2$ over the origin of $\mathbb{C}$. The model Dehn twist τ acts on the zero-section S^2 as the antipodal map and τ^2 is smoothly isotopic to the identity through compactly supported diffeomorphisms of T^*S^2 [32, Proposition 2.1]. Hence the induced action of T_S on homology is nontrivial, more precisely, $(T_S)_*\colon H_2(X;\mathbb{Z})\to H_2(X;\mathbb{Z})$ is given as

$$(T_S)_*(x)=x+(x\cdot[S])[S],$$

and T_S^2 is smoothly isotopic to the identity. Thus the mapping class $[T_S]$ is nontrivial and it generates an order-2 subgroup of $\pi_0(\mathrm{Diff}(X))$.

Next, consider the situation that a (+2)–sphere S in an oriented 4–manifold X is given. Then a tubular neighborhood of S is diffeomorphic to TS^2. Via an isomorphism between TS^2 and T^*S^2 obtained by fixing a metric on S^2, we may implant the model Dehn twist into X as well as the (−2)–sphere case above. We denote by $T_S\colon X\to X$ also this diffeomorphism, and call T_S the Dehn twist as well. This Dehn twist also generates an order-2 subgroup of $\pi_0(\mathrm{Diff}(X))$, since the corresponding statement for a (−2)–sphere follows just from a property of the model Dehn twist, and the action on $H_2(X)$ is given by

$$(T_S)_*(x)=x-(x\cdot[S])[S].$$

We note that every Dehn twist preserves every spin structure:

Lemma 5.1 *Let X be a closed oriented smooth 4–manifold, and suppose that X admits a spin structure $\mathfrak{s}$. Let S be a (+2)– or (−2)–sphere in X. Then the Dehn twist T_S preserves $\mathfrak{s}$.*

Proof Recall that T_S is just the identity map on the complement of a tubular neighborhood of S in X, which is diffeomorphic to the disk cotangent bundle $D(T^*S^2)$. Thus it suffices to show that, given a spin structure $\mathfrak{t}$ on $\partial D(T^*S^2)=S(T^*S^2)$, an extension of $\mathfrak{t}$ to $D(T^*S^2)$ is unique. By the relative obstruction theory for a natural fibration $B(\mathbb{Z}/2) \to B\,\mathrm{Spin}(4) \to B\mathrm{SO}(4)$, it follows that the extensions of $\mathfrak{t}$ are classified by $H^1(D(T^*S^2), S(T^*S^2); \mathbb{Z}/2)$, which is the trivial group by the mod 2 Thom isomorphism for $T^*S^2 \to S^2$. □

5.2 Proof of Theorem 1.1

Now we are ready to prove our main result on Dehn twists:

Proof of Theorem 1.1 By reversing the orientation, we may suppose that $\sigma(X) < 0$. Note that a (± 2)–sphere turns into a (∓ 2)–sphere if we reverse the orientation of X. First we consider the case that a (-2)–sphere is given in X with $\sigma(X) < 0$. Let S be a (-2)–sphere, and let φ denote the induced automorphism of $H_2(X;\mathbb{Z})$ from the Dehn twist T_S. Let us calculate b_+^φ, b_-^φ and σ^φ. As described above, φ is given by $\varphi(x) = x + (x \cdot [S])[S]$, namely, φ acts on $H_2(X)$ as the reflection with respect to the orthogonal complement of the subspace generated by $[S]$. Here the orthogonal complement is with respect to the intersection form, and hence the complement contains a maximal-dimensional positive-definite subspace. Thus,

$$b_+^\varphi(X) = b_+(X), \quad b_-^\varphi(X) = b_-(X) - 1, \quad \sigma^\varphi(X) = \sigma(X) + 1.$$

From this we have that $\sigma^\varphi(X) \neq \sigma(X)/2$, since we supposed that $\sigma(X) < 0$ and hence $|\sigma(X)| \geq 8$ since $H_2(X;\mathbb{Z})$ is an even lattice. Moreover, we also have $-\sigma(X)/16 > b_+(X) - b_+^\varphi(X)$, again by $\sigma(X) < 0$. Now the claim of Theorem 1.1 for (-2)–spheres in X with $\sigma(X) < 0$ follows from Theorem 1.3 combined with Lemma 5.1.

Next, we consider the case that a $(+2)$–sphere S in X with $\sigma(X) < 0$ is given. Note that, as in the (-2)–sphere case above, $\varphi = (T_S)_*$ is the reflection with respect to the orthogonal complement of the subspace generated by $[S]$, but now $[S]$ has positive self-intersection. Thus,

$$b_+^\varphi(X) = b_+(X) - 1, \quad b_-^\varphi(X) = b_-(X), \quad \sigma^\varphi(X) = \sigma(X) - 1.$$

Again because $|\sigma(X)| \geq 8$, it follows that $\sigma^\varphi(X) \neq \sigma(X)/2$. Moreover,

$$b_+(X) - b_+^\varphi(X) = 1 < -\tfrac{1}{16}\sigma(X) + 1.$$

Now the desired claim follows from the "moreover" part of Theorem 1.3 combined with Lemma 5.1. □

Note that the "moreover" part of Theorem 1.3, which was derived from the refined Kato's inequality (Theorem 3.1), was effectively used to deal with $(+2)$–spheres in X with $\sigma(X) < 0$ in the above proof of Theorem 1.1.

Remark 5.2 For $X = K3$, the above proof of Theorem 1.1 gives an alternative proof of [9, Corollary 1.10] by Farb and Looijenga. They gave two different proofs of [9, Corollary 1.10], and one of them is based on Seiberg–Witten theory. We also used Seiberg–Witten theory, but in a slightly different manner: our proof uses Kato's result [17], rather than a result due to Bryan [6] used by Farb and Looijenga.

Kato's inequality (2) is useful to obtain a result for general spin 4–manifolds as in Theorem 1.1, not only $K3$. This is essentially because b_+ is replaced with $b_+ - b_+^\iota$ in Kato's inequality (2).

Example 5.3 Theorem 1.1 tells us that quite many spin 4–manifolds X have (many) nonrealizable order-2 subgroups of $\pi_0(\mathrm{Diff}(X))$. Indeed, there are many spin 4–manifolds that admit (± 2)–spheres. For example, $S^2 \times S^2$ admits both $(+2)$– and (-2)–spheres. A $K3$ surface, more generally, a spin complete intersection surface M admits a (-2)–sphere. Except for $M = S^2 \times S^2$ we have $\sigma(M) < 0$ for such M, and thus we may apply Theorem 1.1 to M and obtain a nonrealizable subgroup, and, of course, we may apply Theorem 1.1 also to the connected sum of M with any spin 4–manifold with $\sigma \leq 0$. (For the fact that M contains a (-2)–sphere, see the proof of Theorem 1.5 in [32, page 255]. In fact, one may find a Lagrangian sphere in M, whose self-intersection is always -2. See also [15, pages 23–24] for the topology of M, including when a complete intersection is spin.)

6 Proof of Theorem 1.2

Given an oriented closed simply connected smooth 4–manifold X, let $\mathrm{Aut}(H_2(X;\mathbb{Z}))$ denote the automorphism group of $H_2(X;\mathbb{Z})$ equipped with the intersection form. Since the space of maximal-dimensional positive-definite subspaces of $H^2(X;\mathbb{R})$ is known to be contractible, it makes sense whether a given $\varphi \in \mathrm{Aut}(H_2(X;\mathbb{Z}))$ preserves a given orientation of the positive part of $H^2(X;\mathbb{R})$. Let us recall the following classical fact:

Theorem 6.1 [5; 8; 23] *Let $\Gamma(K3) \subset \mathrm{Aut}(H_2(K3;\mathbb{Z}))$ denote the image of the natural map*

$$\pi_0(\mathrm{Diff}(K3)) \to \mathrm{Aut}(H_2(K3;\mathbb{Z})).$$

Then $\Gamma(K3)$ is the index-2 subgroup of $\mathrm{Aut}(H_2(K3;\mathbb{Z}))$ which consists of automorphisms that preserve a given orientation of $H^+(K3)$.

We shall also use:

Theorem 6.2 [4, Theorem 1.1] *There exists a (group-theoretic) section $s\colon \Gamma(K3) \to \pi_0(\mathrm{Diff}(K3))$ of the natural map $\pi_0(\mathrm{Diff}(K3)) \to \mathrm{Aut}(H_2(K3;\mathbb{Z}))$.*

Proof of Theorem 1.2 First, we recall a construction of a topological involution f_K on $K3$ (ie order-2 element of $\mathrm{Homeo}(K3)$) in [4, Section 3]. Let $-E_8$ denote the negative-definite E_8–manifold, namely, simply connected closed oriented topological 4–manifold whose intersection form is the negative-definite E_8–lattice. Let $f_S\colon S^2 \times S^2 \to S^2 \times S^2$ be the diffeomorphism defined by $(x,y) \mapsto (y,x)$. Since f_S has nonempty fixed-point set, which is of codimension-2, we can form an equivariant connected sum of three copies of $(S^2 \times S^2, f_S)$. Take a point x_0 of $3S^2 \times S^2$ outside the fixed-point set of $\#_3 f_S$, and attach two copies of $-E_8$ with $3S^2 \times S^2$ at x_0 and $(\#_3 f_S)(x_0)$. Now we have got a topological involution $\tilde{f}\colon 3S^2 \times S^2 \# 2(-E_8) \to 3S^2 \times S^2 \# 2(-E_8)$. Let $h\colon K3 \to 3S^2 \times S^2 \# 2(-E_8)$ be a homeomorphism obtained from Freedman theory [10], and define $f_K\colon K3 \to K3$ by $f_K = h^{-1}\circ \tilde{f}\circ h$, which is a topological involution on $K3$.

Define a topological involution $f\colon X \to X$ by an equivariant connected sum $f = \#_m f_K \#_n f_S$ on $X = mK3 \# nS^2\times S^2$ along fixed points, which acts on homology as follows. Recall that $H^+(S^2\times S^2)$ is generated by $[S^2\times \mathrm{pt}]+[\mathrm{pt}\times S^2]$ and $H^-(S^2\times S^2)$ is generated by $[S^2\times \mathrm{pt}]-[\mathrm{pt}\times S^2]$. Hence f_0 acts trivially on $H^+(S^2\times S^2)$, and acts on $H^-(S^2\times S^2)$ by (-1)–multiplication. Thus, $b_+^{f_S}(S^2\times S^2)=1$ and $b_-^{f_S}(S^2\times S^2)=0$, and hence

$$b_+^{f_K}(K3)=3,\qquad b_-^{f_K}(K3)=8,\qquad \sigma^{f_K}(K3)=-5, \tag{11}$$

$$b_+^{f}(X)=3m+n,\qquad b_-^{f}(X)=8m,\qquad \sigma^{f}(X)=-5m+n. \tag{12}$$

It follows from (11) that $(f_K)_*$ preserves an orientation of $H^+(K3)$, and hence $(f_K)_*$ lies in $\Gamma(K3)$ by Theorem 6.1. Using the section $s\colon \Gamma(K3)\to \pi_0(\mathrm{Diff}(K3))$ given in Theorem 6.2, set $\Phi = s((f_K)_*)$. Then Φ is a nontrivial element of $\pi_0(\mathrm{Diff}(K3))$ of order 2, and hence a representative $g_K\colon K3\to K3$ of Φ is a diffeomorphism whose square g_K^2 is smoothly isotopic to the identity. By smooth isotopy, we may take g_K such that g_K pointwise fixes a 4–disk in $K3$. Similarly, we may obtain a diffeomorphism g_S of $S^2\times S^2$ which is smoothly isotopic to f_S and which fixes a 4–disk pointwise. Fixing disjoint disks $D^4_1,\dots,D^4_{m+n}$ in S^4, form a diffeomorphism

$$g = \#_m g_K \#_n g_S\colon X\to X$$

by attaching g_K's and g_S's with (S^4,id_{S^4}) along the fixed disks of the g_K's and g_S's and $D^4_1,\dots,D^4_{m+n}$. It is clear that g is supported outside $S^4_0 := S^4\setminus \bigsqcup_{i=1}^{m+n} D^4_i$.

We claim that g^2 is smoothly isotopic to the identity. First, for a simply connected closed oriented 4–manifold M, let $\mathrm{Diff}(M,D^4)$ denote the group of diffeomorphisms fixing pointwise an embedded 4–disk D^4 in M. It follows from [12, Proposition 3.1] that we have an exact sequence

$$1\to \ker p\to \pi_0(\mathrm{Diff}(M,D^4)) \xrightarrow{p} \pi_0(\mathrm{Diff}(M))\to 1,$$

where p is an obvious homomorphism and $\ker p$ is isomorphic to either $\mathbb{Z}/2$ or 0, which is generated by the mapping class of the Dehn twist τ_M along the 3–sphere parallel to the boundary. Set $\tau_K=\tau_{K3}$ and $\tau_S=\tau_{S^2\times S^2}$. Note that the relative mapping class $[\tau_K]_\partial$ is nontrivial in $\pi_0(\mathrm{Diff}(K3,D^4))$ by [19, Proposition 1.2], while $[\tau_S]_\partial$ is trivial since τ_S can be absorbed into the S^1–action on $S^2\times S^2$ given by the rotation of one S^2–component. Thus we obtain from $[g_K]^2=1$ and $[g_S]^2=1$ that $[g_K]^2_\partial=[\tau_K]_\partial\neq 1$ and $[g_S]^2_\partial=1$. Hence $[g]^2$ is the product of the Dehn twists along necks between m–copies of $K3$ and S^4_0.

On the other hand, let $\tau_{S^4_0}\colon S^4_0\to S^4_0$ be the diffeomorphism defined as the simultaneous Dehn twists near all ∂D^4_i. It follows from Lemma 6.3 below that $\tau_{S^4_0}$ is smoothly isotopic to the identity relative to ∂S^4_0. Thus, $[g]^2=[(\tau_{S^4_0}\#\mathrm{id}_{X\setminus S^4_0})\circ g^2]$. Note that $\tau_{S^4_0}$ restricted to the neck between each $K3$ and S^4_0 cancels the Dehn twist τ_K, but $\tau_{S^4_0}$ yields the Dehn twist on each of the necks between the $S^2\times S^2$'s and S^4_0. As a result, $[g]^2$ is the product of the Dehn twists along the necks between all of the $S^2 \times S^2$ and S^4_0. But each of these Dehn twists can be absorbed into the rotation of $S^2 \times S^2$ as above. Thus we get $[g]^2 = 1$.

Let G be the subgroup of $\pi_0(\mathrm{Diff}(X))$ generated by the mapping class $[g]$. We claim that this group G is the desired one. First, by construction, $g_* = f_*$ on $H_2(X;\mathbb{Z})$. By a theorem of Quinn [29] and Perron [27], this implies that g and f are topologically isotopic to each other. Thus the image G' of G under the map $\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X))$ lifts to the order-2 subgroup of $\mathrm{Homeo}(X)$ generated by f. Since G' is a nontrivial group as g acts homology nontrivially, this proves the statement on G' in the theorem.

What remains to prove is that g is not homotopic to any finite-order diffeomorphism of X. However, using $g_* = f_*$, (12), and $m > 0$, it is straightforward to see that $\varphi = g_*$ violates the inequality (1) and that $\sigma^{\varphi}(X) \neq \sigma(X)/2$. Thus the desired assertion follows from Theorem 1.3. □

The following lemma and how to use it in the proof of Theorem 1.2 were suggested to the author by David Baraglia:

Lemma 6.3 *Let $N > 0$ and S^4_0 be an N–punctured 4–sphere, $S^4_0 = S^4 \setminus \bigsqcup_{i=1}^N D^4_i$. Let $\tau_{S^4_0}\colon S^4_0 \to S^4_0$ be the diffeomorphism defined as the simultaneous Dehn twists near all ∂D^4_i. Then $\tau_{S^4_0}$ is smoothly isotopic to the identity relative to ∂S^4_0.*

Proof Regard S^4 as the unit sphere of $\mathbb{R}^5 = \mathbb{R}^2 \oplus \mathbb{R}^3$, and let S^1 act on S^4 by the standard rotation of the $\mathbb{R}^2$–component. The fixed-point set Σ of the S^1–action is given by $S(0 \oplus \mathbb{R}^3) \cong S^2$. We may assume that D^4_i are embedded disks in S^4 whose centers p_i are on Σ. Then the normal tangent space N_{p_i} of Σ at p_i in S^4 is acted on by S^1 as the standard rotation.

Pick a disk $\hat{D}^4_i$ in S^4 that contains D^4_i such that $\hat{D}^4_i \setminus D^4_i$ is diffeomorphic to the annulus $S^3 \times [0,1]$. Set $\hat{S}^4_0 = S^4 \setminus \bigsqcup_{i=1}^N \hat{D}^4_i$. The S^1–action on S^4 gives rise to an isotopy $\{\varphi_t\}_{t \in [0,1]} \subset \mathrm{Diff}(\hat{S}^4_0)$ from $\mathrm{id}_{\hat{S}^4_0}$ to itself such that $\{\varphi_t|_{\partial \hat{D}^4_i}\}_t$ gives the homotopically nontrivial loop in $\mathrm{SO}(4) \subset \mathrm{Diff}(S^3) \cong \mathrm{Diff}(\partial \hat{D}^4_i)$.

On the other hand, recall that the Dehn twist τ on $S^3 \times [0,1]$ is defined by $\tau(y,t) = (g(t)\cdot y, t)$, where $g\colon [0,1] \to \mathrm{SO}(4)$ is the homotopically nontrivial loop in $\mathrm{SO}(4)$. By definition, τ is isotopic to $\mathrm{id}_{S^3 \times [0,1]}$ by an isotopy

$$\psi_t \in \mathrm{Diff}(S^3 \times [0,1], S^3 \times \{1\}),$$

through the diffeomorphism group fixing $S^3 \times \{1\}$ pointwise, such that $\{\psi_t|_{S^3 \times \{0\}}\}_t$ gives the homotopically nontrivial loop in $\mathrm{Diff}^+(S^3)$.

Let ψ^i_t be copies of ψ_t, regarded as isotopies on $\hat{D}^4_i \setminus D^4_i$. By gluing φ_t with ψ^i_t along $\bigsqcup_{i=1}^N \partial \hat{D}^4_i$, we obtain an isotopy from $\tau_{S^4_0}$ to $\mathrm{id}_{S^4_0}$ relative to ∂S^4_0. □

Remark 6.4 For $X = K3$, the above proof of Theorem 1.2 gives a slight alternative proof of [4, Theorem 1.2], which used the adjunction inequality rather than Kato's result [17].

7 Additional remarks

7.1 Another kind of Dehn twist

A kind of Dehn twist different from that in Theorem 1.1 is the Dehn twist along an embedded annulus $S^3 \times [0,1]$ in a 4–manifold, defined using the generator of $\pi_1(\mathrm{SO}(4)) \cong \mathbb{Z}/2$, as described in the previous section. The square of the Dehn twist of this kind is smoothly isotopic to the identity. Recently, Kronheimer and Mrowka [19] proved that the Dehn twist τ along the neck of $K3 \# K3$ is not smoothly isotopic to the identity, and J Lin [22] showed that the extension of τ to $K3 \# K3 \# S^2 \times S^2$ by the identity of $S^2 \times S^2$ is also not smoothly isotopic to the identity. Hence it turns out that these Dehn twists generate order-2 subgroups of the mapping class groups. We remark that these subgroups also give counterexamples to the Nielsen realization problem:

Proposition 7.1 (i) *Let τ be the Dehn twist along the neck of $K3 \# K3$. Then the order-2 subgroup of $\pi_0(\mathrm{Diff}(K3 \# K3))$ generated by the mapping class of τ is not realized in* $\mathrm{Diff}(K3 \# K3)$.

(ii) *Let τ' be the extension of τ by the identity to $K3 \# K3 \# S^2 \times S^2$. Then the order-2 subgroup of $\pi_0(\mathrm{Diff}(K3 \# K3 \# S^2 \times S^2))$ generated by the mapping class of τ' is not realized in* $\mathrm{Diff}(K3 \# K3 \# S^2 \times S^2)$.

Proof By a result of Matumoto [24] and Ruberman [31], a simply connected closed spin 4–manifold with nonzero signature does not admit a homologically trivial locally linear involution. Since the Dehn twist τ is homologically trivial, the claim of the proposition immediately follows. □

7.2 Other variants of the realization problem

Given a manifold X of any dimension, one may also consider the realization problem for *infinite* subgroups of $\pi_0(\mathrm{Diff}(X))$ along $\mathrm{Diff}(X) \to \pi_0(\mathrm{Diff}(X))$ (or along $\mathrm{Diff}^+(X) \to \pi_0(\mathrm{Diff}^+(X))$ when $\mathrm{Diff}(X) \neq \mathrm{Diff}^+(X)$). To answer this problem negatively, several authors developed cohomological obstructions, which can be thought of as descendants of an argument started by Morita [25] for surfaces. In dimension 4, concrete results on the nonrealization were obtained in [14; 33] in this direction (see also [13]). Concretely, Giansiracusa, Kupers and Tshishiku [14] studied $X = K3$, and Tshishiku [33] considered manifolds of any dimension, but especially the result [33, Theorem 9.1] treated 4–manifolds whose fundamental groups are isomorphic to nontrivial lattices, which does not have overlap with 4–manifolds that we considered in this paper.

Another variant of the realization problem is about the realization along the natural map

$$\mathrm{Diff}^+(X) \to \pi_0(\mathrm{Homeo}^+(X))$$

for a subgroup of the image of this map. If X is a simply connected 4–manifold, the natural map $\pi_0(\mathrm{Homeo}^+(X)) \to \mathrm{Aut}(H_2(X;\mathbb{Z}))$ is isomorphic [27; 29], and hence this version of realization problem is equivalent to the realization along the map $\mathrm{Diff}^+(X) \to \mathrm{Aut}(H_2(X;\mathbb{Z}))$, which has been extensively studied by Nakamura [26], Baraglia [2; 3], and Lee [20; 21]. As noted in Section 1, Theorem 1.2 gives an alternative proof of [2, Proposition 1.2] about the realization of an involution of $H_2(X;\mathbb{Z})$.

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