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Parametrized motion planning algorithms have a high degree of universality and flexibility; they generate the motion of a robotic system under a variety of external conditions. The latter are viewed as parameters and constitute part of the input of the algorithm. The concept of sequential parametrized topological complexity $\mathsf{TC}_r[p\colon E \to B]$ is a measure of the complexity of such algorithms. It was studied by Cohen, Farber and Weinberger (2021, 2022) for $r = 2$ and by Farber and Paul (2022) for $r \geq 2$. We analyze the dependence of the complexity $\mathsf{TC}_r[p\colon E \to B]$ on an initial bundle with structure group G and on its fibre X viewed as a G–space. Our main results estimate $\mathsf{TC}_r[p\colon E \to B]$ in terms of certain invariants of the bundle and the action on the fibre. Moreover, we also obtain estimates depending on the base and the fibre. Finally, we develop a calculus of sectional categories featuring a new invariant $\mathsf{secat}_f[p\colon E \to B]$ which plays an important role in the study of sectional category of towers of fibrations.

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1 Introduction

Motion planning algorithms of robotics control autonomous robots in engineering; see [LaValle 2006]. A motion planning algorithm takes as input the initial and the desired states of the system and produces as output a motion of the system starting at the initial states and ending at the desired states. A robot is "told" where it needs to go and the execution of this task, including selection of a specific route of motion, is made by the robot itself, ie by the robot's motion planning algorithm. Typically it is understood that the external conditions (such as the positions of the obstacles and the geometry of the enclosing domain) are known and are constant during the motion.

In [Cohen et al. 2021; 2022], motion planning algorithms of a new type were analyzed. These are *parametrized motion planning algorithms*, which, besides the initial and desired states, take as input the parameters characterizing the external conditions. The output of a parametrized motion planning algorithm is a continuous motion of the system from the initial to the desired state, respecting the given external conditions. The papers [Cohen et al. 2021; 2022] laid out the new formalism and analyzed in full detail the problem of moving an arbitrary number n of robots in the domain with m a priori unknown obstacles.

The recent paper [Farber and Paul 2022] developed a generalization where the robot must perform *a sequence* of tasks. The topological complexity of such an algorithm is called *sequential parametrized topological complexity* $\mathsf{TC}_r[p\colon E \to B]$, where $r = 2, 3, \ldots$, and the case $r = 2$ corresponds to the situations analyzed in [Cohen et al. 2021; 2022]. Formally, $\mathsf{TC}_r[p\colon E \to B]$ is an integer associated with a fibration $p\colon E \to B$ where the points of the base $b \in B$ parametrize the external conditions (for example, positions of the obstacles) and for each $b \in B$ the fibre $X_b = p^{-1}(b) \subset E$ is the space of all admissible configurations of the system under the external conditions b. To make the present work independent, we include the definition of the concept $\mathsf{TC}_r[p\colon E \to B]$ and its major properties in Section 2.

In this paper we further analyze the invariant $\mathsf{TC}_r[p\colon E \to B]$ trying to understand its dependence on classical invariants of the initial bundle $p\colon E \to B$; in particular, on its base B and on its fibre X. As with all such invariants, exact calculation is generally hard and the development of lower and upper bounds is an essential part of the subject. This is the focus of our work.

We first show that, if the bundle $p\colon E \to B$ has structure group G with fibre X a G–space, then the equivariant sequential topological complexity of X — developed in [Bayeh and Sarkar 2020; Colman and Grant 2012] — serves as an upper bound for $\mathsf{TC}_r[p\colon E \to B]$; see (9). The case when G acts freely on X is especially interesting and leads to several somewhat surprising estimates. But using equivariant topological complexity as an upper bound is fraught with danger since it can be infinite in what appear to be innocuous situations.

As an alternative we develop the notion of *weak sequential equivariant complexity*, denoted by $\mathsf{TC}^w_{r,G}(X)$, and its variant $\mathsf{TC}^w_{r,G}(X; P)$ which we are tempted (but loathe) to call *weak sequential equivariant complexity with coefficients* P (see Section 7). We will give several examples showing that these invariants are finite even when equivariant topological complexity is infinite, so they offer the opportunity for effective estimation in many situations. Indeed, our main result Theorem 8.1 gives lower and upper bounds for $\mathsf{TC}_r[p\colon E \to B]$ in terms of these invariants. To state it one needs to recall the invariant

$$G\text{–}\mathsf{cat}[p\colon E \to B]$$

introduced by I M James [1978, page 342]. It is defined as the smallest integer $k \geq 0$ such that the base B admits an open cover $B = U_0 \cup U_1 \cup \cdots \cup U_k$ with the property that over each set U_i the bundle E is trivial *as a G–bundle*. Clearly, G–$\mathsf{cat}[p\colon E \to B]$ equals the sectional category $\mathsf{secat}[\tau\colon P \to B]$ of the associated principal bundle that constructs $p\colon E \to B$. In general,

$$G\text{–}\mathsf{cat}[p\colon E \to B] \leq \mathsf{cat}(B) \leq \dim B \tag{1}$$

and if the group G is 2–connected (as is the case for simply connected compact Lie groups for instance) then we can say

$$G\text{–}\mathsf{cat}[p\colon E \to B] \leq \left\lceil \tfrac{1}{4}(\dim B - 3) \right\rceil \tag{2}$$

as follows by applying [Schwartz 1962, Theorem 5] to $\mathsf{secat}[\tau\colon P \to B]$. If the structure group G is discrete then instead of (1) one has a stronger inequality

$$G\text{–}\mathsf{cat}[p\colon E \to B] \leq \mathsf{cat}_1(B), \tag{3}$$

where $\mathsf{cat}_1(B)$ is the sectional category of the universal cover $\widetilde{B} \to B$. Our main result (Theorem 8.1) then is the following.

Theorem *For a locally trivial bundle $p\colon E = X \times_G P \to B = P/G$ one has the inequalities*

$$\mathsf{TC}^w_{r,G}(X; P) \leq \mathsf{TC}_r[p\colon E \to B] \leq G\text{–}\mathsf{cat}[p\colon E \to B] + \mathsf{TC}^w_{r,G}(X). \tag{4}$$

Note that the first summand in the right-hand side of (4) is independent of r and is bounded above by the Lusternik–Schnirelmann category of the base B; the second term is the weak equivariant sequential topological complexity of the fibre X. In our view, this estimate gets at the heart of the matter. After all, what *is* a bundle? It is just a principal bundle together with an action of the structure group on the fibre and our upper bound is expressed exactly in numerical quantities derived from these objects. In Example 8.4 we shall see that the right-hand side can be an equality, so at least in some cases the upper bound can be sharp. Such a posteriori knowledge then warrants a deeper study of the invariants $\mathsf{TC}^w_{r,G}(X)$ and $\mathsf{TC}^w_{r,G}(X; P)$ and we hope the present work elicits this.

Beyond defining and applying the new invariants $\mathsf{TC}^w_{r,G}(X)$ and $\mathsf{TC}^w_{r,G}(X; P)$, in Sections 4, 5 and 6 we develop a calculus of sectional categories, including a new notion denoted by $\mathsf{secat}_f[p\colon E \to B]$ where $f\colon B \to C$ is a continuous map. It is this notion that allows us to estimate the sectional category of towers of fibrations which serves as the crucial technical tool in the proof of our main results. We believe that $\mathsf{secat}_f[p\colon E \to B]$ holds independent interest and should find application in many situations orbiting the twin galaxies of Lusternik–Schnirelmann category and topological complexity.

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2 The concept of sequential parametrized topological complexity

In this section we recall the notion of sequential parametrized topological complexity introduced in [Farber and Paul 2022]. It is a generalization of the concept of topological complexity [Farber 2003] and its parametrized version [Cohen et al. 2021].

Let $p\colon E \to B$ be a Hurewicz fibration with fibre X. Fix an integer $r \geq 2$ and set

$$E^r_B = \{(e_1, \dots, e_r) \in E^r \mid p(e_1) = \cdots = p(e_r)\}.$$

The symbol $I = [0,1]$ denotes the unit interval. Let $E^I_B \subset E^I$ be the space of all paths $\gamma\colon I \to E$ such that the path $p \circ \gamma\colon I \to B$ is constant. Fix r points

$$0 \leq t_1 < t_2 < \cdots < t_r \leq 1$$

and consider the evaluation map

$$\Pi_r\colon E^I_B \to E^r_B, \quad \Pi_r(\gamma) = (\gamma(t_1), \gamma(t_2), \dots, \gamma(t_r)). \tag{5}$$

Π_r is a Hurewicz fibration; see [Cohen et al. 2022, Appendix]. The fibre of Π_r is $(\Omega X)^{r-1}$, the Cartesian $(r-1)^{\text{st}}$ power of the based loop space ΩX. A section $s\colon E^r_B \to E^I_B$ of the fibration Π_r can be interpreted as *a parametrized motion planning algorithm*, ie a function which assigns to every sequence of points $(e_1, e_2, \dots, e_r) \in E^r_B$ a continuous path $\gamma\colon I \to E$ (representing motion of the system) satisfying $\gamma(t_i) = e_i$ for every $i = 1, 2, \dots, r$ and such that the path $p \circ \gamma\colon I \to B$ is constant. The latter condition means that the system moves under *constant external conditions* (such as positions of the obstacles etc).

Typically, the fibration Π_r does not admit continuous sections; see [Farber and Weinberger 2023a, Corollary 1 and Lemma 1], which deal with the case $r = 2$; when $r > 2$ the arguments are similar. Therefore the motion planning algorithms are necessarily discontinuous in most situations.

The following definition [Farber and Paul 2022] gives a measure of complexity of sequential parametrized motion planning algorithms.

Definition 2.1 The r^{th} *sequential parametrized topological complexity* of the fibration $p\colon E \to B$, denoted by $\mathsf{TC}_r[p\colon E \to B]$, is defined as the sectional category of the fibration Π_r, ie

$$\mathsf{TC}_r[p\colon E \to B] := \mathsf{secat}(\Pi_r). \tag{6}$$

In more detail, $\mathsf{TC}_r[p\colon E \to B]$ is the minimal integer k such that there is an open cover $\{U_0, U_1, \dots, U_k\}$ of E^r_B with the property that each open set U_i admits a continuous section $s_i\colon U_i \to E^I_B$ of Π_r.

Under some mild assumptions, instead of open covers one can consider totally general partitions:

Proposition 2.2 [Farber and Paul 2022, Proposition 3.6] *Let E and B be metrizable separable ANRs and let $p\colon E \to B$ be a locally trivial fibration. Then the sequential parametrized topological complexity $\mathsf{TC}_r[p\colon E \to B]$ equals the smallest integer n such that E^r_B admits a partition*

$$E^r_B = F_0 \sqcup F_1 \sqcup \cdots \sqcup F_n, \qquad F_i \cap F_j = \varnothing \quad \text{for } i \neq j,$$

with the property that on each set F_i there exists a continuous section $s_i\colon F_i \to E^I_B$ of Π_r.

If two fibrations $p\colon E \to B$ and $p'\colon E' \to B$ are fibrewise homotopy equivalent then

$$\mathsf{TC}_r[p\colon E \to B] = \mathsf{TC}_r[p'\colon E' \to B];$$

see [Farber and Paul 2022, Corollary 4.2].

The following upper bound is a reformulation of [Farber and Paul 2022, Proposition 6.1]:

Proposition 2.3 *Let $p\colon E \to B$ be a locally trivial fibration with fibre X, where E, B and X are CW–complexes. Assume that the fibre X is k–connected, where $k \geq 0$. Then*

$$\mathsf{TC}_r[p\colon E \to B] \leq \left\lceil \frac{r \dim X + \dim B - k}{1+k} \right\rceil. \tag{7}$$

We refer the reader to [Farber and Paul 2022] for proofs and further detail.

3 Relation with the equivariant sequential topological complexity

In this section we show that $\mathsf{TC}_r[p\colon E \to B]$ admits as an upper bound the sequential equivariant topological complexity [Bayeh and Sarkar 2020; Colman and Grant 2012] of the fibre X. This leads to simple estimates in terms of the dimension of the fibre in the case when the structure group G of the fibration acts freely on X; see Lemma 3.5 and Corollary 3.6.

3.1 Equivariant topological complexity

We shall recall a sequential analogue of the notion of equivariant topological complexity introduced by M Bayeh and S Sarkar [2020]; it generalizes the concept of equivariant topological complexity originally introduced and studied by H Colman and M Grant [2012].

Let G be a topological group acting on a topological space X from the left. The papers [Bayeh and Sarkar 2020; Colman and Grant 2012] require G to be compact but we do not impose this assumption at this stage.

The symbol X^I denotes the space of all continuous paths $\gamma\colon I \to X$ where $I = [0,1]$ is equipped with the compact–open topology. The group G acts naturally on X^I, where $(g\gamma)(t) = g\gamma(t)$ for $t \in I$.

Fix an integer $r \geq 2$ and consider the Cartesian power $X^r = X \times X \times \cdots \times X$ (r times). We shall consider the diagonal action of G on X^r.

Fix r points $0 = t_1 < t_2 < \cdots < t_r = 1$ in the unit interval $I = [0,1]$ and consider the evaluation map

$$\rho_r : X^I \to X^r, \tag{8}$$

where $\rho_r(\gamma) = (\gamma(t_1), \dots, \gamma(t_r))$. Clearly, ρ_r a G–equivariant map.

Definition 3.1 For a path-connected G–space X, we denote by $\mathsf{TC}_{r,G}(X)$ the smallest integer $k \geq 0$ such that the Cartesian power $X^r = X \times X \times \cdots \times X$ (r times) admits an open cover $X^r = U_0 \cup U_1 \cup \cdots \cup U_k$ with the following properties: each set U_i is G–invariant and admits a continuous G–equivariant section $s_i : U_i \to X^I$ of the fibration ρ_r. If no such cover exists we set $\mathsf{TC}_{r,G}(X) = \infty$.

The invariant $\mathsf{TC}_{2,G}(X)$ coincides with the equivariant topological complexity $\mathsf{TC}_G(X)$ of Colman and Grant [2012].

It is obvious from Definition 3.1 that

$$\mathsf{TC}_r(X) \leq \mathsf{TC}_{r,G}(X),$$

where $\mathsf{TC}_r(X)$ is the sequential topological complexity of X introduced by Rudyak [2010].

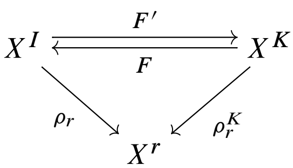
An alternative definition of $\mathsf{TC}_{r,G}(X)$ is obtained as follows (compare [Farber and Paul 2022, Lemma 3.5]). Let K be a path-connected locally compact metrizable space and let $k_1, k_2, \dots, k_r \in K$ be a set of r pairwise distinct points. Consider the set X^K of continuous maps $\alpha : K \to X$ equipped with compact–open topology. The evaluation map

$$\rho_r^K : X^K \to X^r,$$

where $\Pi_r^K(\alpha) = (\alpha(k_1), \dots, \alpha(k_r)) \in X^r$, is continuous and G–equivariant, where we view X^r with the diagonal action of G.

Lemma 3.2 *For any path-connected locally compact metrizable space K, the number $\mathsf{TC}_{r,G}(X)$ equals the smallest integer $k \geq 0$ such that the Cartesian power X^r admits an open cover $X^r = U_0 \cup U_1 \cup \cdots \cup U_k$ with the following properties: each set U_i is G–invariant and admits a continuous G–equivariant section $s_i : U_i \to X^K$ of ρ_r^K.*

Proof Consider the commutative diagram



where the maps $F : X^K \to X^I$ and $F' : X^I \to X^K$ are defined as follows. Fix a path $\gamma : I \to K$ satisfying $\gamma(t_i) = k_i$ for all $i = 1, \dots, r$. Then $F(\alpha) = \alpha \circ \gamma : I \to X$, where $\alpha \in X^K$.

To define the map $F'\colon X^I \to X^K$ we first construct a continuous function $f : K \to I$ satisfying $f(k_i) = t_i$ for all $i = 1, \dots, r$. Applying the Tietze extension theorem we find continuous functions $\psi_j : K \to [0, 1]$ with $\psi_j(t_i) = \delta_{ij}$ where $j = 1, \dots, r$. Then the function $f = \min\{1, \sum_{i=1}^{r} t_i \psi_r\}$, $f : K \to I$, has the required properties. The map $F'\colon X^I \to X^K$ is defined by $F'(\alpha) = \alpha \circ f$ where $\alpha \in X^I$.

Clearly the maps F and F' are G–equivariant. For an open G–invariant subset $U \subset X^r$ any G–equivariant section $s\colon U \to X^I$ of ρ_r defines the G–equivariant section $s' = F' \circ s\colon U \to X^K$ of ρ_r^K. And vice versa, any G–equivariant section $s'\colon U \to X^K$ defines $s = F \circ s'\colon U \to X^I$, an equivariant section of ρ_r. □

Yet another equivalent characterization of $\mathsf{TC}_{r,G}(X)$ is given by the following (see [Bayeh and Sarkar 2020]):

Lemma 3.3 *For a G–space X and $r \geq 2$ the integer $\mathsf{TC}_{r,G}(X)$ equals the smallest $k \geq 0$ such that X^r admits an open cover $X^r = U_0 \cup U_1 \cup \dots \cup U_k$ by G–invariant open sets U_i with the property that each inclusion $U_j \to X^r$ is G–homotopic to a map with values in the diagonal $X \subset X^r$.*

Now we can state our result relating the sequential parametrized topological complexity of a fibration with the equivariant sequential topological complexity of the fibre:

Theorem 3.4 *Consider a locally trivial bundle $p\colon E \to B$ with path-connected fibre X and structure group G. Let $\tau\colon P \to B$ be a G–principal bundle such that $p\colon E \to B$ coincides with the associated bundle $p\colon E = X \times_G P = (X \times P)/G \to P/G = B$. Then the sequential parametrized topological complexity $\mathsf{TC}_r[p\colon E \to B]$ is bounded above by $\mathsf{TC}_{r,G}(X)$, ie*

$$\mathsf{TC}_r[p\colon E \to B] \leq \mathsf{TC}_{r,G}(X). \tag{9}$$

Note that the right-hand side of inequality (9) depends only on the fibre X viewed as a G–space, where G is the structure group of the bundle.

Proof First we note that there exists the commutative diagram

$$\begin{array}{ccc} X^I \times_G P & \xrightarrow{\ \alpha\ } & E_B^I \\ {\scriptstyle \rho_r \times_G 1}\downarrow & & \downarrow{\scriptstyle \Pi_r} \\ X^r \times_G P & \xrightarrow[\ \beta\ ]{} & E_B^r \end{array}$$

where α and β are homeomorphisms. Therefore,

$$\mathsf{TC}_r[p\colon E \to B] = \mathsf{secat}[\Pi_r : E_B^I \to E_B^r] = \mathsf{secat}[\rho_r \times_G 1\colon X^I \times_G P \to X^r \times_G P].$$

For $k = \mathsf{TC}_{r,G}(X)$ let $X^r = U_0 \cup U_1 \cup \dots \cup U_k$ be an open cover as in Definition 3.1. Consider the sets

$$W_i = (U_i \times P)/G \subset (X^r \times P)/G.$$

They are open and cover $(X^r \times P)/G$. Any G–equivariant section $s_i\colon U_i \to X^I$ of the fibration ρ_r obviously defines the section $\sigma_i : W_i \to (X^I \times P)/G$ of the orbit spaces; here σ_i is the map induced by $s_i \times 1_P$ on the spaces of orbits. This shows that

$$\mathsf{TC}_r[p\colon E \to B] = \mathsf{secat}[\rho_r \times_G 1\colon X^I \times_G P \to X^r \times_G P] \le k. \qquad \square$$

As mentioned in [Bayeh and Sarkar 2020; Colman and Grant 2012], in some cases the number $\mathsf{TC}_{r,G}(X)$ is infinite. In particular, one has $\mathsf{TC}_{r,G}(X) = \infty$ if for a subgroup $H \subset G$ the fixed-point set X^H is not path-connected. In such situations the upper bound (9) becomes meaningless. We discuss below situations when the number $\mathsf{TC}_{r,G}(X)$ is finite and admits useful upper bounds.

The following lemma uses the notion of G–equivariant homotopy lifting property (G–HLP) applied to a map $q\colon X \to X/G$. This property means that the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\ f\ } & X \\ {\scriptstyle \text{inc}}\downarrow & & \downarrow{\scriptstyle q} \\ Y \times I & \xrightarrow[F]{} & X/G \end{array}$$

where X and Y are separable metric spaces and the map $f : Y \to X$ is G–equivariant, can be completed by a G–equivariant map $H : Y \times I \to X$ extending f and such that $q \circ H = F$. A theorem of R Palais (see [Bredon 1972, Theorem II.7.3]) states that this property is automatically satisfied for free actions of compact Lie groups.

Lemma 3.5 *Consider a locally trivial bundle $p\colon E \to B$ with fibre X (a path-connected separable metric space) and structure group G. Assume that the group G acts freely on X and, moreover, that the quotient map $q_r : X^r \to X^r/G$ possesses the G–HLP. Then*

$$\mathsf{TC}_r[p\colon E \to B] \le \mathsf{TC}_{r,G}(X) \le \mathsf{cat}(X^r/G) \le \dim(X^r/G).$$

Proof In view of Theorem 3.4 we only need to prove the inequality $\mathsf{TC}_{r,G}(X) \le \mathsf{cat}(X^r/G)$. Consider an open covering $X^r/G = V_0 \cup V_1 \cup \cdots \cup V_k$, where $k = \mathsf{cat}(X^r/G)$ and each inclusion $U_i \subset X^r/G$ is homotopic to the constant map into a point $x_0 \in X/G \subset X^r/G$; here $X \subset X^r$ is the diagonal. By our assumption, the projection $q_r : X^r \to X^r/G$ has the G–homotopy lifting property. The sets $U_i = q_r^{-1}(V_i) \subset X^r$ are G–invariant, where $i = 0, 1, \ldots, k$, and applying the G–homotopy lifting property to the homotopy of V_i to x_0 we find a homotopy $h_t^i\colon U_i \to X^r$ (where $t \in [0, 1]$ and $i \in \{0, 1, \ldots, r\}$) such that h_0^i is the inclusion $U_i \to X^r$, each map h_t^i is G–equivariant and $h_1^i(U_i) \subset X \subset X^r$. Applying Lemma 3.3 we obtain $\mathsf{TC}_{r,G}(X) \le k$. $\square$

Corollary 3.6 *Consider a locally trivial bundle $p\colon E \to B$ with fibre X (which is a path-connected separable metric space) and a compact Lie group G acting freely on X, as the structure group. Then for any $r \ge 2$,*

(10) $$\mathsf{TC}_r[p\colon E \to B] \le \mathsf{cat}(X^r/G) \le r \dim X - \dim G.$$

Proof First we note that due to the theorem of Palais [Bredon 1972, Theorem II.7.3] the assumptions of Lemma 3.5 are satisfied. We are only left to note that $\dim X^r/G = \dim X^r - \dim G \le r \dim X - \dim G$; see [Palais 1960, Corollary 1.7.32]. □

One can use Lemma 3.5 to give an alternative proof of [Farber and Paul 2022, Proposition 3.3] — see also [Cohen et al. 2021, Proposition 4.3] — with some minor additional assumptions:

Corollary 3.7 *Let $G \to P \xrightarrow{\tau} B$ be a principal bundle, where G is a path-connected topological group which has the topology of a separable metric space. Then*

$$\mathsf{TC}_r[\tau\colon P \to B] = \mathsf{cat}(G^{r-1}) \quad \text{for any } r \ge 2.$$

Proof By [Farber and Paul 2022, Section 3] we know that $\mathsf{TC}_r[\tau\colon P \to B] \ge \mathsf{TC}_r(G) = \mathsf{cat}(G^{r-1})$. We view the fibre G as acting on itself by left translations and acting diagonally on G^r. The quotient map $q_r\colon G^r \to G^r/G$ admits a section $s\colon G^r/G \to G^r$, given by

$$s(g_1, g_2, \dots, g_r) = (e, g_1^{-1}g_2, g_1^{-1}g_3, \dots, g_1^{-1}g_r).$$

Therefore, we explicitly obtain a G–homeomorphism $G^r \cong G^r/G \times G$, so q_r is a trivial bundle $G^r/G \times G \to G^r/G$ and, as such, has the G–HLP. Lemma 3.5 then applies and gives the upper bound $\mathsf{TC}_r[\tau\colon P \to B] \le \mathsf{cat}(G^r/G) = \mathsf{cat}(G^{r-1})$. Comparing, we see that both bounds are in fact equalities. □

4 Calculus of sectional categories

In this section we introduce a new invariant $\mathsf{secat}_f[p\colon E \to B]$ which generalizes the concept of sectional category of a fibration. This invariant plays a role in estimating sectional category of towers of fibrations, see Theorem 5.1.

Let $p\colon E \to B$ be a fibration and let $f\colon B \to C$ be a continuous map.

Definition 4.1 We define the invariant

$$\mathsf{secat}_f[p\colon E \to B]$$

to be the smallest integer $k \ge 0$ such that C admits a family of open subsets $U_0, U_1, \dots, U_k$ with the properties

(a) $U_0 \cup U_1 \cup \dots \cup U_k \supset f(B)$ or, equivalently, $B = \bigcup_{i=1}^k f^{-1}(U_i)$;

(b) the fibration $p\colon E \to B$ admits a continuous section over each open set $f^{-1}(U_i)$ for $i = 0, 1, \dots, k$.

We set $\mathsf{secat}_f[p\colon E \to B] = \infty$ if no such family exists.

Open sets of the form $f^{-1}(U) \subset B$, where $U \subset C$, can be called *f–saturated*. Definition 4.1 can be rephrased as dealing with covers of the base B by f–saturated open sets admitting continuous sections of the fibration $p\colon E \to B$.

4.1 Finiteness

The following lemma summarizes information about finiteness of the invariant $\mathsf{secat}_f[p\colon E \to B]$.

Lemma 4.2 *Let $p\colon E \to B$ be a fibration and let $f : B \to C$ be a continuous map.*

(A) *If $\mathsf{secat}[p\colon p^{-1} f^{-1}(x) \to f^{-1}(x)] > 0$ for some $x \in f(B) \subset C$ then $\mathsf{secat}_f[p\colon E \to B] = \infty$.*

(B) *If B is compact and every point $x \in f(B) \subset C$ has an open neighbourhood $U \subset C$ such that $\mathsf{secat}[p\colon p^{-1} f^{-1}(U) \to f^{-1}(U)] = 0$ then $\mathsf{secat}_f[p\colon E \to B] < \infty$.*

Proof Under assumption (A) there is no open set $U \subset C$ containing x with $f^{-1}(U)$ having a continuous section. Statement (B) is obvious. □

In our applications we shall typically have the map $f : B \to C$ be surjective, and more specifically, it will often be the quotient map with respect to a group action. However, it is convenient to make no additional assumptions at this stage.

4.2 Dependence on f

In the special case when the map $f : B \to C = B$ is the identity map, the number $\mathsf{secat}_f[p\colon E \to B]$ turns into the usual sectional category $\mathsf{secat}[p\colon E \to B]$. In general, obviously,

$$\mathsf{secat}[p\colon E \to B] \le \mathsf{secat}_f[p\colon E \to B] \tag{11}$$

and

$$\mathsf{secat}[p\colon E \to B] = \mathsf{secat}_f[p\colon E \to B] \tag{12}$$

assuming that $\mathsf{secat}[p\colon E \to B] = 0$.

Moreover, for $B \xrightarrow{f} C \xrightarrow{g} C'$ one clearly has

$$\mathsf{secat}_f[p\colon E \to B] \le \mathsf{secat}_{gf}[p\colon E \to B]. \tag{13}$$

Lemma 4.3 *Let $p\colon E \to B$ be a fibration and let $f : B \to C$ and $f' : B \to C'$ be two continuous maps.*

(a) *If there is a continuous map $h\colon C \to C'$ such that $f' = h \circ f$, then*

$$\mathsf{secat}_f[p\colon E \to B] \le \mathsf{secat}_{f'}[p\colon E \to B].$$

(b) *Moreover, if the restriction of $h\colon C \to C'$ induces a homeomorphism $f(B) \to f'(B)$, then*

$$\mathsf{secat}_f[p\colon E \to B] = \mathsf{secat}_{f'}[p\colon E \to B].$$

Proof Statement (a) follows from inequality (13). To prove (b) assume that $U \subset C$ is an open subset with the property that $f^{-1}(U)$ admits a section of p. Then

$$h(U \cap f(B)) = U' \subset f'(B)$$

is an open subset of $f'(B)$ and hence there exists an open subset $V \subset C'$ with $V \cap f'(B) = U'$. Then $f'^{-1}(V) = f^{-1}(U)$ admits a section of p. Thus any family of open sets $U_0 \cup U_1 \cup \cdots \cup U_k \supset f(B)$

such that $f^{-1}(U_i)$ admits a section of p determines a family of open subsets of the same cardinality, $V_0 \cup V_1 \cup \cdots \cup V_k \supset f'(B)$ with the preimages $f'^{-1}(V_j)$ admitting sections of p. This shows the inverse inequality $\mathsf{secat}_f[p\colon E \to B] \geq \mathsf{secat}_{f'}[p\colon E \to B]$. □

4.3 Induced fibrations

Lemma 4.4 *Assume that a fibration $p\colon E' \to B'$ is induced from the fibration $p\colon E \to B$ via the map $\alpha\colon B' \to B$ as shown on the diagram*

$$\begin{array}{ccccc} E' & \xrightarrow{\beta} & E & & \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} & & \\ B' & \xrightarrow[\alpha]{} & B & \xrightarrow[f]{} & C \end{array}$$

For $f: B \to C$ set $f' = f \circ \alpha$. Then

$$\mathsf{secat}_{f'}[p'\colon E' \to B'] \leq \mathsf{secat}_f[p\colon E \to B].$$

Proof Assuming that there is a continuous section $s\colon f^{-1}(U) \to E$ of $p\colon E \to B$, for $U \subset C$ open, define $\phi\colon f'^{-1}(U) \to E$ by $\phi = s \circ \alpha$. Then we have $p \circ \phi = \alpha$ and by the pullback property there is a continuous map $s'\colon f'^{-1}(U) \to E'$ with $p' \circ s' =$ inclusion, ie s' is a section of p'. Since $f'(B) \subset f(B) \subset C$, we see that the statement of the lemma follows. □

Lemma 4.5 (maps of fibrations) *If for two fibrations $p\colon E \to B$ and $p'\colon E' \to B$ over the same base B there exists a map $\phi\colon E \to E'$ such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & B & \end{array}$$

commutes up to homotopy, then $\mathsf{secat}_f[p'\colon E' \to B] \leq \mathsf{secat}_f[p\colon E \to B]$.

Proof If $U \subset C$ is such that p admits a continuous section s over $f^{-1}(U) \subset B$ then p' admits a homotopy section $\phi \circ s$ over the same subset. Since p' satisfies the homotopy lifting property, the homotopy section can be made a genuine section. The statement now follows from the definition. □

Lemma 4.6 *Suppose that for two fibrations $p\colon E \to B$ and $p'\colon E' \to B'$ there exist continuous maps G, α, β and $\hat{\alpha}$ shown on the diagram*

$$\begin{array}{ccccc} & & E' & \xrightarrow{G} & E \\ & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[\alpha]{} & B' & \xrightarrow[\beta]{} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \\ C & \xrightarrow[\hat{\alpha}]{} & C' & & \end{array}$$

such that the bottom left square is commutative, the upper right square is homotopy commutative and $\beta \circ \alpha\colon B \to B$ is homotopic to the identity $\mathrm{Id}_B\colon B \to B$. Then $\mathsf{secat}_f[p\colon E \to B] \leq \mathsf{secat}_{f'}[p'\colon E' \to B']$.

Proof Consider the fibration $q\colon \bar{E} \to B$ induced by the map $\alpha\colon B \to B'$ from $p'\colon E' \to B'$. It appears in the commutative diagram

$$\begin{CD} \bar{E} @>{\psi}>> E' \\ @V{q}VV @VV{p'}V \\ B @>>{\alpha}> B' \end{CD}$$

Using Lemmas 4.3 and 4.4 one obtains

(14) $$\mathsf{secat}_f[q\colon \bar{E} \to B] \le \mathsf{secat}_{\hat{\alpha}\circ f}[q\colon \bar{E} \to B] \le \mathsf{secat}_{f'}[p'\colon E' \to B'].$$

Next we note that the diagram

$$\begin{array}{ccc} \bar{E} & \xrightarrow{\;G\circ\psi\;} & E \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

homotopy commutes:

$$p \circ G \circ \psi \simeq \beta \circ p' \circ \psi = \beta \circ \alpha \circ q \simeq q.$$

Applying Lemma 4.5 we obtain the inequality $\mathsf{secat}_f[p\colon E \to B] \le \mathsf{secat}_f[q\colon \bar{E} \to B]$ which together with (14) implies $\mathsf{secat}_f[p\colon E \to B] \le \mathsf{secat}_{f'}[p'\colon E' \to B']$, as claimed. □

Corollary 4.7 *Assume that in the diagram*

$$\begin{CD} E @>{F}>> E' @>{G}>> E \\ @V{p}VV @VV{p'}V @VV{p}V \\ B @>>{\alpha}> B' @>>{\beta}> B \\ @V{f}VV @VV{f'}V @VV{f}V \\ C @>>{\hat{\alpha}}> C' @>>{\hat{\beta}}> C \end{CD}$$

the maps p and p' are fibrations, the lower squares are commutative, the upper squares are homotopy commutative and the maps α and β are mutually inverse homotopy equivalences. Then

$$\mathsf{secat}_f[p\colon E \to B] = \mathsf{secat}_{f'}[p'\colon E' \to B'].$$

Proof This follows from applying Lemma 4.6 twice: to the diagram

$$\begin{CD} @. E @>{F}>> E' \\ @. @VV{p}V @VV{p'}V \\ B' @>>{\beta}> B @>>{\alpha}> B' \\ @V{f'}VV @VV{f}V @. \\ C' @>>{\hat{\beta}}> C @. \end{CD}$$

and to the diagram of Lemma 4.6. □

Corollary 4.8 *Suppose that in the commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\alpha} & B' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ C & \xrightarrow{\hat\alpha} & C' \end{array}$$

the maps p', f and f' are fibrations and p is the induced fibration. If α and $\hat\alpha$ are homotopy equivalences then

$$\mathsf{secat}_f[p\colon E\to B] = \mathsf{secat}_{f'}[p'\colon E'\to B'].$$

Proof By Lemmas 4.3 and 4.4 we have $\mathsf{secat}_{f'}[p'\colon E'\to B'] \geq \mathsf{secat}_{\hat\alpha f}[p\colon E\to B] \geq \mathsf{secat}_f[p\colon E\to B]$ so we must only show the inverse inequality. Since f and f' are fibrations and α and $\hat\alpha$ are homotopy equivalences, applying the proposition on page 53 of [May 1999] we see that there exist homotopy inverses β and $\hat\beta$ for α and $\hat\alpha$, respectively, such that the diagram

$$\begin{array}{ccc} B' & \xrightarrow{\beta} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C' & \xrightarrow{\hat\beta} & C \end{array}$$

commutes. We obtain the commutative diagram

$$\begin{array}{ccccc} & & E & \xrightarrow{F} & E' \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B' & \xrightarrow[\beta]{} & B & \xrightarrow[\alpha]{} & B' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \\ C' & \xrightarrow[\hat\beta]{} & C & & \end{array}$$

with the composition $\alpha\circ\beta\colon B'\to B'$ homotopic to the identity map. Lemma 4.6 now gives

$$\mathsf{secat}_{f'}[p'\colon E'\to B'] \leq \mathsf{secat}_f[p\colon E\to B]. \qquad \square$$

4.4 Homotopical dimension

For a topological space A having the homotopy type of a finite-dimensional CW–complex we shall denote by $\mathsf{hdim}(A)$ *the homotopical dimension of* A; it is defined as the minimal dimension of a CW–complex homotopy equivalent to A.

The following lemma will be used later.

Lemma 4.9 *Consider a locally trivial bundle $p\colon E \to B$ where E and B are separable metric spaces and the base B and the fibre F have the homotopy type of finite-dimensional CW–complexes. Assume also that the fibre F of $p\colon E \to B$ has finite covering dimension $\dim F$. Then the total space E has the homotopy type of a finite-dimensional CW–complex and, moreover,*

$$\mathsf{hdim}(E) \le \mathsf{hdim}(B) + \dim F. \tag{15}$$

Proof Let $g\colon B' \to B$ be a homotopy equivalence where B' is a CW–complex satisfying $\dim B' = \mathsf{hdim} B$. Consider the diagram

$$\begin{array}{ccc} E' & \xrightarrow{G} & E \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ B' & \xrightarrow[g]{} & B \end{array}$$

where $p'\colon E' \to B'$ is the fibration induced by g. Clearly G is a homotopy equivalence and

$$\dim E' \le \dim B' + \dim F.$$

By [Fritsch and Piccinini 1990, Theorem 5.4.2] the space E' has homotopy type of a CW–complex. Hence,

$$\mathsf{hdim}(E) = \mathsf{hdim}(E') \le \dim(E') \le \dim B' + \dim F = \mathsf{hdim}(B) + \dim F. \qquad \square$$

4.5 An upper bound

The following statement gives a useful upper bound for the invariant $\mathsf{secat}_f[p\colon E \to B]$.

Proposition 4.10 *Assume that E, B and C are separable metric spaces. Let $p\colon E \to B$ be a fibration and let $f : B \to C$ be a locally trivial bundle such that*

(a) *the space C and the fibre F_0 of $f : B \to C$ have the homotopy type of CW–complexes;*

(b) *the fibre F_1 of the fibration $p\colon E \to B$ is $(k-1)$–connected, where $k \ge 0$;*

(c) *the fibre F_0 of the fibration $f : B \to C$ is d–dimensional, where $0 \le d \le k$.*

Then one has

$$\mathsf{secat}_f[p\colon E \to B] \le \left\lceil \frac{\dim B - k}{1 + k - d} \right\rceil. \tag{16}$$

Proof First we shall prove the statement under an additional assumption that C is a simplicial complex. We shall remove this assumption afterwards.

Consider the skeleta $C^{(i)} \subset C$ of C, where $i = 0, 1, \dots$. We know that for any two integers $0 \le i < j$ the complement $C^{(i)} - C^{(j)}$ is homotopy equivalent to a simplicial complex of dimension at most $i - j - 1$; see for example [Farber et al. 2019, Corollary 5.3].

We may find a chain of open subsets $U_0 \subset U_1 \subset U_2 \subset \cdots$ of C such that each set U_i contains $C^{(i)}$ as a strong deformation retract.

Setting $r = k - d$, consider the skeleta

$$C^{(r)} \subset C^{(2r+1)} \subset C^{(3r+2)} \subset \cdots \subset C^{((c+1)r+c)},$$

where c is the smallest integer satisfying $\dim C \le (c+1)r + c$, ie

$$c = \left\lceil \frac{\dim C - r}{1+r} \right\rceil = \left\lceil \frac{\dim B - k}{1+k-d} \right\rceil.$$

Each complement,

$$X_i = C^{((i+1)r+i)} - C^{(ir+i-1)}, \quad i = 0, 1, \dots, c,$$

has the homotopy type of a simplicial complex of dimension $\le r$. The open set

$$Y_i = U_{(i+1)r+i} - C^{(ir+i-1)} \subset C$$

deformation retracts onto X_i and therefore $\mathsf{hdim}(Y_i) \le r$. Applying Lemma 4.9 we obtain

$$\mathsf{hdim}(V_i) \le r + d = k,$$

where

$$V_i = f^{-1}(Y_i) \subset B, \quad i = 0, 1, \dots, c.$$

The fibre F_1 of $p\colon E \to B$ is $(k-1)$–connected, and thus we may apply the well-known result of the obstruction theory stating that the fibration $p\colon E \to B$ admits a continuous section over each open set V_i, where $i = 0, 1, \dots, c$. Since $B = V_0 \cup V_1 \cup \cdots \cup V_c$, it shows that $\mathsf{secat}_f[p\colon E \to B] \le c$. This completes the proof in the case when C is a simplicial complex.

Consider now the general case, ie we shall only assume that C has the homotopy type of a CW–complex. We can find a simplicial complex C' and a homotopy equivalence $\hat{\alpha}\colon C' \to C$; see [Fritsch and Piccinini 1990, Theorem 5.2.1]. Consider the fibration $f'\colon B' \to C'$ induced by $\hat{\alpha}$ from $f\colon B \to C$. The map α shown on the diagram

$$\begin{array}{ccc} E' & \xrightarrow{F} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow[\alpha]{} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C' & \xrightarrow[\hat{\alpha}]{} & C \end{array}$$

is a homotopy equivalence. The map α induces the fibration $p'\colon E' \to B'$. Applying Corollary 4.8 we obtain that

$$\mathsf{secat}_{f'}[p'\colon E' \to B'] = \mathsf{secat}_f[p\colon E \to B].$$

Hence the upper bound (16) which we proved above for $\mathsf{secat}_{f'}[p'\colon E' \to B']$ applies to $\mathsf{secat}_f[p\colon E \to B]$ as well. □

Remark 4.11 In [Farber et al. 2019] an upper bound for topological complexity was derived that made use of an invariant which was called

$$\widetilde{\mathsf{TC}}(X) = \widetilde{\mathsf{secat}}(E \xrightarrow{p} \overline{X} \xrightarrow{q} X)$$

there, but which we recognize in fact to be $\mathsf{secat}_q[p\colon E \to \overline{X}]$ here. In [Farber et al. 2019] it was further shown that $\widetilde{\mathsf{TC}}(X)$ could be identified with the notion of strongly invariant topological complexity $\mathsf{TC}^*_\pi(\widetilde{X})$ introduced by A Dranishnikov [2015] earlier. A K Paul and D Sen [2020] extended both the invariant $\widetilde{\mathsf{TC}}(X)$ and the strongly invariant topological complexity to the realm of sequential topological complexity and proved the analogous identification. This identification, in some sense, was the genesis of our calculus of sectional categories and together with Theorem 3.4 begs the question of exactly how parametrized topological complexity and various forms of equivariant topological complexity are intertwined, especially in the case of locally trivial fibre bundles.

5 Sectional category of towers of fibrations

Consider a tower of fibrations

$$E_r \xrightarrow{p_r} E_{r-1} \xrightarrow{p_{r-1}} E_{r-2} \to \cdots \xrightarrow{p_1} E_0$$

and the total fibration

$$p = p_1 p_2 \cdots p_r \colon E_r \to E_0.$$

We shall assume that all spaces E_i are normal.

Theorem 5.1 *The sectional category* $\mathsf{secat}[p\colon E_r \to E_0]$ *of the total fibration admits the lower and upper bounds*

(17)
$$\begin{aligned}\mathsf{secat}[p_1\colon E_1 \to E_0] &\le \mathsf{secat}[p\colon E_r \to E_0]\\ &\le \mathsf{secat}[p_1\colon E_1 \to E_0] + \sum_{i=1}^{r-1} \mathsf{secat}_{(p_1 p_2 \cdots p_i)}[p_{i+1}\colon E_{i+1} \to E_i].\end{aligned}$$

Here $p_1 p_2 \cdots p_i\colon E_i \to E_0$ *denotes the composition.*

Lemma 5.2 below will be used in the proof of Theorem 5.1.

Lemma 5.2 *Let* C *be a normal space. Consider properties* $A_1, A_2, \dots, A_r$ *of open subsets of* C*, such that each property* A_i *is inherited by open subsets and disjoint unions. Assume that for each* $i = 1, 2, \dots, r$ C *admits an open cover consisting of* $n_i + 1$ *open sets satisfying the property* A_i*. Then* C *admits an open cover consisting of* $N + 1$ *open sets, where* $N = \sum_{i=1}^r n_i$*, satisfying all the properties* $A_1, \dots, A_r$*.*

Proof For $r = 2$ this statement was proven in [Oprea and Strom 2011, Lemma 4.3]. The case $r > 2$ follows from this by induction. □

Proof of Theorem 5.1 Since the left inequality in (17) is obvious we shall concentrate on the right one and use Lemma 5.2 to prove it. Consider the following properties $A_1, A_2, \dots, A_r$ of open subsets of E_0. We shall say that an open subset $U \subset E_0$ satisfies A_1 if U has a continuous section of the fibration p_1. For $2 \le i \le r$, we shall say that an open subset $U \subset E_0$ satisfies the property A_i if the open set $p_{i-1}^{-1} \cdots p_2^{-1} p_1^{-1}(U) \subset E_{i-1}$ admits a continuous section of p_i. By definition, for any $i = 1, 2, \dots, r$, the set E_0 admits an open cover of cardinality $\mathrm{secat}_{(p_1 p_2 \cdots p_{i-1})}[p_i \colon E_i \to E_{i-1}] + 1$ with each set satisfying A_i. Applying Lemma 5.2, we obtain that E_0 admits an open cover $\{U_j\}$ of cardinality $\sum_{i=1}^r n_i + 1$ such that each set U_j satisfies all the properties $A_1, \dots, A_r$. This means that there exists a continuous section $s_0 \colon U_j \to E_1$ of p_1 and for any $i = 1, 2, \dots, r-1$, there exists a continuous section

$$s_i \colon p_i^{-1} \cdots p_2^{-1} p_1^{-1}(U_j) \to E_{i+1}$$

of the fibration p_i. Hence, the composition

$$s = s_{r-1} s_{r-2} \cdots s_1 s_0 \colon U_j \to E_r$$

is a well-defined continuous section of the composition $p = p_1 p_2 \cdots p_r \colon E_r \to E_0$. This gives the inequality (17). ◻

For convenience of references, we state below the special case $r = 2$ of Theorem 5.1 which we combine with the dimension-connectivity upper bound of Proposition 4.10:

Corollary 5.3 *Consider a tower of fibrations $E_2 \xrightarrow{p_2} E_1 \xrightarrow{p_1} E_0$ of separable metric spaces. Assume that $p_1 \colon E_1 \to E_0$ is locally trivial. Then the sectional category $\mathrm{secat}[p \colon E_2 \to E_0]$ of the total bundle*

$$p = p_2 \circ p_1 \colon E_2 \to E_0$$

lies between $\mathrm{secat}[p_1 \colon E_1 \to E_0]$ and

(18) $$\mathrm{secat}[p_1 \colon E_1 \to E_0] + \mathrm{secat}_{p_1}[p_2 \colon E_2 \to E_1].$$

Moreover, under the additional assumptions that

(a) *the fibre of $p_2 \colon E_2 \to E_1$ is $(k-1)$–connected,*

(b) *the space E_0 and the fibre of $p_1 \colon E_1 \to E_0$ have the homotopy type of CW–complexes,*

(c) *the fibre of $p_1 \colon E_1 \to E_0$ has dimension $\le d$ where $0 \le d \le k$,*

one has

(19) $$\mathrm{secat}_{p_1}[p_2 \colon E_2 \to E_1] \le \left\lceil \frac{\dim E_1 - k}{1 + k - d} \right\rceil.$$

6 Product inequalities

Lemma 5.2 distills the main results of [Dranishnikov 2009; 2010; Oprea and Strom 2011; Ostrand 1965], but for the product inequalities which we describe below we need more specific information about open covers.

An open cover $\mathcal{W} = \{W_0, \dots, W_{m+k}\}$ of a space C is an $(m+1)$–*cover* if every subcollection

$$\{W_{j_0}, W_{j_1}, \dots, W_{j_m}\}$$

of $m+1$ sets from $\mathcal{W}$ also covers C. The following simple observation (see [Farber et al. 2019] for instance) is the basis for many arguments in this approach.

Lemma 6.1 *A cover $\mathcal{W} = \{W_0, W_1, \dots, W_{k+m}\}$ is an $(m+1)$–cover of C if and only if each $x \in C$ is contained in at least $k+1$ sets of $\mathcal{W}$.*

An open cover can be lengthened to a $(k+1)$–cover, while retaining certain essential properties of the sets in the cover.

Theorem 6.2 [Dranishnikov 2009; Ostrand 1965] *Let $\mathcal{U} = \{U_0, \dots, U_k\}$ be an open cover of a normal space C. Then, for any $m = k, k+1, \dots, \infty$, there is an open $(k+1)$–cover of C, $\{U_0, \dots, U_m\}$, extending $\mathcal{U}$ such that for $n > k$, U_n is a disjoint union of open sets that are subsets of the U_j, $0 \le j \le k$.*

We use these facts to obtain inequalities for product fibrations.

Lemma 6.3 (product inequality, I) *Let $p\colon E \to B$ and $p'\colon E' \to B'$ be fibrations and let $f : B \to C$ and $f'\colon B' \to C'$ be continuous maps. Assume that the spaces $f(B)$ and $f'(B')$ with topology induced from C and C', respectively, are normal. Then the sectional category of the product fibration*

$$\mathsf{secat}_{f \times f'}[p \times p'\colon E \times E' \to B \times B']$$

is bounded above by the sum

$$\mathsf{secat}_f[p\colon E \to B] + \mathsf{secat}_{f'}[p'\colon E' \to B']$$

and it is bounded below by

$$\max\{\mathsf{secat}_f[p\colon E \to B], \mathsf{secat}_{f'}[p'\colon E' \to B']\}.$$

Proof First we deal with the lower bounds. Fix a point $b'_0 \in B'$ and embed B into $B \times B'$ via $b \mapsto (b, b'_0)$; also, embed C into $C \times C'$ via $x \mapsto (x, x'_0)$ where $x'_0 = f'(b'_0)$. For an open subset $U \subset C \times C'$, a section of $p \times p'$ over $(f \times f')^{-1}(U) \subset B \times B'$ determines obviously a section of p over $f^{-1}(U \cap (C \times x'_0))$. This implies the inequality $\mathsf{secat}_{f \times f'}[p \times p'\colon E \times E' \to B \times B'] \ge \mathsf{secat}_f[p\colon E \to B]$. Similarly, one obtains $\mathsf{secat}_{f \times f'}[p \times p'\colon E \times E' \to B \times B'] \ge \mathsf{secat}_{f'}[p'\colon E' \to B']$.

Now we prove the upper bound. Let $\mathsf{secat}_f[p\colon E \to B] = k$ be realized by open sets $U_0, \dots, U_k \subset C$ covering $f(B) \subset C$, with continuous sections $s_j : f^{-1}(U_j) \to E$ of p, and let $\mathsf{secat}_{f'}[p'\colon E' \to B'] = m$ be realized by open sets $V_0, \dots, V_m \subset C'$ covering $f'(B')$, with sections $s'_j : f'^{-1}(V_j) \to E'$ of p'. By Theorem 6.2 we can extend the family $U_0, \dots, U_k$ to a family of open subsets $U_0, \dots, U_{k+m}$ of C such that any $k+1$ members of this family cover $f(B)$. Similarly, we can find a family $V_0, \dots, V_{k+m}$ of open subsets of C' extending the initial family $V_0, \dots, V_m$ such that any $m+1$ members of this extended family cover $f'(B')$. Theorem 6.2 guarantees that every set of the form $f^{-1}(U_j)$ or $f'^{-1}(V_j)$ admits a continuous section of p or p' respectively, where $j = 0, 1, \dots, k+m$.

Letting $W_j = U_j \times V_j$, where $j = 0, \dots, k+m$, we see that each set $(f \times f')^{-1}(W_j) = f^{-1}(U_j) \times f'^{-1}(V_j)$ admits a continuous section of $p \times p'$. We show below that the sets W_j cover $f(B) \times f'(B')$, which implies that $\mathsf{secat}_{f \times f'}[p \times p' \colon E \times E' \to B \times B'] \le k+m$.

Suppose that a point $(x, y) \in f(B) \times f'(B')$ is not in any of the sets W_j, where $j = 0, \dots, k+m$. Since any $k+1$ sets U_j cover $f(B)$, we know that x belongs to at least $m+1$ of the U_j, by Lemma 6.1. Without loss of generality, we may assume that $x \in U_0 \cap U_1 \cap \dots \cap U_m$. Then $y \notin V_0 \cup V_1 \cup \dots \cup V_m$, in view of our assumption. Therefore, y can only lie in the sets $V_{m+1}, \dots, V_{k+m}$ which is a contradiction since y belongs to at least $k+1$ of the sets V_j, by Lemma 6.1. □

Next we state another product inequality dealing with fibrations over the same base.

Lemma 6.4 (product inequality, II) *Let $p \colon E \to B$ and $p' \colon E' \to B$ be two fibrations, and let $f : B \to C$. We shall assume that $f(B)$ is normal in the topology induced from C. Then the sectional category*

$$\mathsf{secat}_f[p \times_B p' \colon E \times_B E' \to B]$$

of the fibrewise product is bounded below by

$$\max\{\mathsf{secat}_f[p \colon E \to B], \mathsf{secat}_f[p' \colon E' \to B]\}$$

and is bounded above by the sum

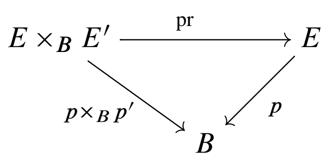
$$\mathsf{secat}_f[p \colon E \to B] + \mathsf{secat}_f[p' \colon E' \to B].$$

Moreover,

$$\mathsf{secat}_f[p \times_B p' \colon E \times_B E' \to B] = \mathsf{secat}_f[p \colon E \to B]$$

if $\mathsf{secat}[p' \colon E' \to B] = 0$, ie if p' admits a section.

Proof The projection $\mathrm{pr} \colon E \times_B E' \to E$ appears in the commutative diagram



and Lemma 4.5 gives $\mathsf{secat}_f[p \times_B p' \colon E \times_B E' \to B] \ge \mathsf{secat}_f[p \colon E \to B]$. Similarly one gets the lower bound using $\mathsf{secat}_f[p' \colon E' \to B]$, which proves the statement concerning the lower bound. Next we note that

(20) $$\mathsf{secat}_f[p \times_B p' \colon E \times_B E' \to B] \le \mathsf{secat}_{f \times f}[p \times p' \colon E \times E' \to B \times B].$$

Indeed, the fibration $p \times_B p' \colon E \times_B E' \to B$ is induced from the product fibration $p \times p' \colon E \times E' \to B \times B$ by the diagonal map $\Delta \colon C \to C \times C$. Lemma 4.4 gives the inequality

$$\mathsf{secat}_{(f \times f) \circ \Delta}[p \times_B p' \colon E \times_B E' \to B] \leq \mathsf{secat}_{f \times f}[p \times p' \colon E \times E' \to B \times B].$$

Finally, we can apply Lemma 4.3 and replace $(f \times f) \circ \Delta$ by f. Combining (20) with Lemma 6.3 we obtain the upper bound.

The last statement obviously follows by combining the lower and upper bounds. □

7 Weak equivariant topological complexity $\mathsf{TC}^w_{r,G}(X)$

Let $p \colon E \to B$ be a bundle with fibre X and structure group G which is associated to a principal bundle $\tau \colon P \to B$. In other words, $E = X \times_G P$.

As in Section 2, we fix $r \geq 2$ points $0 = t_1 < t_2 < \cdots < t_r = 1$ and consider the evaluation map

$$\rho_r \colon X^I \to X^r, \qquad \rho_r(\gamma) = (\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_r)), \quad \text{where } \gamma \in X^I.$$

Consider also the quotient map

$$q_r \colon X^r \to X^r/G,$$

where we view G acting diagonally on X^r.

The following invariant plays an important role in our main Theorem 8.1:

$$\mathsf{TC}^w_{r,G}(X) = \mathsf{secat}_{q_r}[\rho_r \colon X^I \to X^r]. \tag{21}$$

Explicitly, we have:

Definition 7.1 The invariant $\mathsf{TC}^w_{r,G}(X)$ equals the smallest integer $k \geq 0$ such that X^r admits an open cover $X^r = U_0 \cup U_1 \cup \cdots \cup U_k$ by G–invariant open sets such that for each $i = 0, 1, \ldots, k$ there is a continuous section $s_i \colon U_i \to X^I$ of π_r.

Note that the section s_i in Definition 7.1 is not required to be G–equivariant, unlike in the case of $\mathsf{TC}_{r,G}(X)$. This explain the adjective "weak" and the symbol "w" in the notation. We obviously have

$$\mathsf{TC}_r(X) \leq \mathsf{TC}^w_{r,G}(X) \leq \mathsf{TC}_{r,G}(X), \tag{22}$$

where the left inequality is a special case of (11). All these inequalities become equalities when the action of G is trivial.

Lemma 7.2 *For any G–space P,*

$$\mathsf{TC}^w_{r,G}(X) = \mathsf{secat}_{q_r \times \epsilon}[\rho_r \times 1 \colon X^I \times P \to X^r \times P],$$

where $\epsilon \colon P \to *$ is the map onto a singleton.

Proof This follows from Lemma 6.3 since clearly $\mathsf{secat}_\epsilon[1 \colon P \to P] = 0$. □

Next we state the dimension-connectivity upper bound:

Lemma 7.3 *Assume that X is a k–connected simplicial complex and G is a topological group homeomorphic to a CW–complex acting freely on X and such that the map $q_r\colon X^r \to X^r/G$ is a locally trivial bundle. If $\dim G \le k$ then*

$$\mathsf{TC}^w_{r,G}(X) \le \left\lceil \frac{r\dim X - k}{1+k-\dim G} \right\rceil. \tag{23}$$

Proof We apply Proposition 4.10 having in mind that the fibre $(\Omega X)^{r-1}$ of fibration ρ_r is $(k-1)$–connected. □

As a special case of Lemma 7.3 we mention:

Corollary 7.4 *If X is k–connected, where $k \ge 0$, and the group G is discrete and the quotient map $q_r\colon X^r \to X^r/G$ is a covering map then*

$$\mathsf{TC}^w_{r,G}(X) \le \left\lceil \frac{r\dim X - k}{1+k} \right\rceil. \tag{24}$$

We shall be discussing yet another invariant $\mathsf{TC}^w_{r,G}(X;P)$ given by

$$\mathsf{TC}^w_{r,G}(X;P) = \mathsf{secat}_Q[\rho_r \times 1\colon X^I \times P \to X^r \times P] \tag{25}$$

with $Q\colon X^r \times P \to X^r \times_G P$ being the natural projection; here X and P are G–spaces and ρ_r is the fibration (8). Comparing with Lemma 7.2 we see that it is similar to $\mathsf{TC}^w_{r,G}(X)$ with the only distinction that the map $q_r \times \epsilon$ is replaced by Q.

Lemma 7.5 *One has*

$$\mathsf{TC}_r(X) \le \mathsf{TC}^w_{r,G}(X;P) \le \mathsf{TC}^w_{r,G}(X). \tag{26}$$

Proof Consider the commutative diagram

$$\begin{array}{ccccc}
X^I \times P & \xrightarrow{\rho_r \times 1} & X^r \times P & \xrightarrow{Q} & X^r \times_G P \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} \\
X^I & \xrightarrow[\rho_r]{} & X^r & \xrightarrow[q_r]{} & X^r/G
\end{array}$$

where the maps p_1, p_2 and p_3 are projections on the first factor. Since the fibration $\rho_r \times 1$ is induced from ρ_r via p_2, we may apply Lemma 4.4 to conclude

$$\begin{aligned}
\mathsf{TC}^w_{r,G}(X) &= \mathsf{secat}_{q_r}[\rho_r\colon X^I \to X^r] \\
&\ge \mathsf{secat}_{q_r \circ p_2}[\rho_r \times 1\colon X^I \times P \to X^r \times P] \\
&= \mathsf{secat}_{p_3 \circ Q}[\rho_r \times 1\colon X^I \times P \to X^r \times P] \\
&\ge \mathsf{secat}_Q[\rho_r \times 1\colon X^I \times P \to X^r \times P] \\
&= \mathsf{TC}^w_{r,G}(X;P).
\end{aligned}$$

On the third line we used Lemma 4.3(a). This proves the right inequality in (26). The left inequality follows from

$$\begin{aligned}\mathsf{TC}^w_{r,G}(X;P) &= \mathsf{secat}_Q[\rho_r \times 1\colon X^I \times P \to X^r \times P]\\ &\geq \mathsf{secat}[\rho_r \times 1\colon X^I \times P \to X^r \times P]\\ &= \mathsf{secat}[\rho_r\colon X^I \to X^r]\\ &= \mathsf{TC}_r(X),\end{aligned}$$

where on the second line we used inequality (11) and on the third line Lemma 6.3. □

The next result gives a dimension-connectivity upper bound for $\mathsf{TC}^w_{r,G}(X;P)$ which holds for weaker assumptions on X compared to Lemma 7.3.

Lemma 7.6 *Assume that X is a k–connected simplicial complex and G is a topological group homeomorphic to a CW–complex. Suppose that $P \to P/G$ is a locally trivial bundle. If* $\dim G \leq k$ *then*

$$\mathsf{TC}^w_{r,G}(X;P) \leq \left\lceil \frac{r\dim X + \dim P - k}{1 + k - \dim G} \right\rceil. \tag{27}$$

Proof This follows by applying Proposition 4.10 to the definition (25). □

Example 7.7 Consider the unit circle $S^1 \subset \mathbb{C}$ with the action of the cyclic group of order two $G = \mathbb{Z}_2$ acting as the complex conjugation, $z \mapsto \bar{z}$. We know from [Colman and Grant 2012] that in this case $\mathsf{TC}_{2,G}(S^1)$ is infinite due to the fact that the set of fixed points is disconnected.

On the other hand one can consider the open cover $S^1 \times S^1 = U_0 \cup U_1$ where $U_0 = \{(z_1, z_2) \mid z_1 \neq -z_2\}$ and $U_1 = \{(z_1, z_2) \mid z_1 \neq z_2\}$. These sets are G–invariant and over each of these sets one has the well-known continuous sections. Thus, $\mathsf{TC}^w_{2,G}(S^1) = 1$.

Example 7.8 Consider the more general case of a sphere S^n, where $n \geq 1$, with an action of a discrete group G. First we apply the upper bound (24) with $k = n - 1$ to obtain

$$\mathsf{TC}^w_{r,G}(S^n) \leq r \quad \text{for any } r \geq 2.$$

Second, using (26) and the result of Y Rudyak [2010] (stating that $\mathsf{TC}_r(S^n)$ equals r for n even and $r-1$ for n odd), we obtain that for any even n

$$\mathsf{TC}^w_{r,G}(S^n) = r. \tag{28}$$

For n odd our inequalities imply that $\mathsf{TC}^w_{r,G}(S^n)$ equals either $r-1$ or r.

Example 7.9 Let S^1 act on S^2 by rotations about the z–axis. The fixed-point set of the action is the disconnected set $\{N, S\}$, where N and S are the north and south poles, respectively, so the equivariant topological complexity is infinite: $\mathsf{TC}_{r,S^1}(S^2) = \infty$ for all $r \geq 2$.

Let us now examine the weak equivariant topological complexity $\mathsf{TC}^w_{2,S^1}(S^2)$. Fix an orbit $O \subset S^2$ given by the equator and fix an orientation of O. Consider the open cover $S^2 \times S^2 = U_0 \cup U_1 \cup U_2$ where

$$U_0 = \{(x,y) \mid x \neq -y\},$$
$$U_1 = \{(x,y) \mid x \neq y\} - \{(N,S),(S,N)\},$$
$$U_2 = \{(x,y) \mid x \notin O \text{ and } y \notin O\}.$$

Clearly, the sets U_0, U_1 and U_2 are S^1–invariant. We may define the motion planning rules over each of the sets U_i as follows. For $(x,y) \in U_0$, go from x to y along the shortest geodesic arc. For $(x,y) \in U_1$ the point x moves along the shortest geodesic arc first to the closest point of O, then along O in the positive direction to the point closest to y, and finally to y. For $(x,y) \in U_2$ the point x moves along the shortest geodesic arc to the closest pole (N or S), then to the closest pole to y along a fixed path and then to y; the first and the third portions are along the shortest geodesic arc on the sphere S^2. Hence $\mathsf{TC}^w_{2,S^1}(S^2) \leq 2$. Since $2 = \mathsf{TC}(S^2) \leq \mathsf{TC}^w_{2,S^1}(S^2)$, we see that $\mathsf{TC}^w_{2,S^1}(S^2) = 2$.

8 Bounds for the sequential parametrized topological complexity

Finally we are in position to state and prove the main result of this paper:

Theorem 8.1 *Let $p\colon E \to B$ be a locally trivial fibre bundle with structure group G, the fibre X and the associated principal bundle $\tau\colon P \to B$. Then the sequential parametrized topological complexity $\mathsf{TC}_r[p\colon E \to B]$ admits the upper and lower bounds*

(29) $$\mathsf{TC}^w_{r,G}(X;P) \leq \mathsf{TC}_r[p\colon E \to B] \leq G\text{–}\mathsf{cat}[p\colon E \to B] + \mathsf{TC}^w_{r,G}(X;P).$$

Proof Since $E = X \times_G P$,

$$E^r_B = X^r \times_G P \quad \text{and} \quad E^I_B = X^I \times_G P \quad \text{for any } r \geq 2.$$

The map $\Pi_r\colon E^I_B \to E^r_B$ becomes $\rho_r \times 1\colon X^I \times_G P \to X^r \times_G P$, where $\rho_r(\gamma) = (\gamma(t_0),\dots,\gamma(t_r))$. Consider the commutative diagram

(30) $$\begin{array}{ccc} X^I \times P & \xrightarrow{Q'} & X^I \times_G P \\ {\scriptstyle \rho_r \times 1}\downarrow & & \downarrow{\scriptstyle \rho_r \times_G 1} \\ X^r \times P & \xrightarrow[Q]{} & X^r \times_G P \end{array}$$

where $Q\colon X^r \times P \to X^r \times_G P$ and $Q'\colon X^I \times P \to X^I \times_G P$ are the natural projections. Using Lemma 4.5 and Theorem 5.1,

$$\begin{aligned} \mathsf{TC}_r[p\colon E \to B] &= \mathsf{secat}[\rho_r \times_G 1\colon X^I \times_G P \to X^r \times_G P] \\ &\leq \mathsf{secat}[(\rho_r \times_G 1) \circ Q'\colon X^I \times P \to X^r \times_G P] \\ &= \mathsf{secat}[(Q \circ (\rho_r \times 1)\colon X^I \times P \to X^r \times_G P] \\ &\leq \mathsf{secat}[Q\colon X^r \times P \to X^r \times_G P] + \mathsf{secat}_Q[\rho_r \times 1\colon X^I \times P \to X^r \times P]. \end{aligned}$$

Next we observe that

$$\mathsf{secat}[X^r \times P \to X^r \times_G P] \le \mathsf{secat}[\tau\colon P \to B] = G\text{–}\mathsf{cat}[p\colon E \to B]$$

and

$$\mathsf{secat}_Q[X^I \times P \to X^r \times P] = \mathsf{TC}^w_{r,G}(X;P).$$

Thus, we obtain the right inequality in (29).

For the left inequality in (29) we consider again diagram (30) and observe that the fibration

$$\pi_r \times 1\colon X^I \times P \to X^r \times P$$

is induced from $\pi_r \times_G 1\colon X^I \times_G P \to X^r \times_G P$ via Q. Therefore, using Lemma 4.4 we obtain

$$\begin{aligned}\mathsf{TC}_r[p\colon E \to B] &= \mathsf{secat}[\rho_r \times 1\colon X^I \times_G P \to X^r \times_G P]\\ &\ge \mathsf{secat}_Q[\rho_r \times 1\colon X^I \times P \to X^r \times P]\\ &= \mathsf{TC}^w_{r,G}(X;P).\end{aligned}$$

□

Remark 8.2 Due to the right inequality in (26), the upper bound in (29) gives

$$\mathsf{TC}_r[p\colon E \to B] \le G\text{–}\mathsf{cat}[p\colon E \to B] + \mathsf{TC}^w_{r,G}(X). \tag{31}$$

The right-hand side of this inequality has two terms, one depending only on the initial bundle $p\colon E \to B$ and the other depending only on the fibre, X viewed as a G–space.

Theorem 8.1 implies that for the trivial bundle $p\colon E \to B$ with fibre X one has $\mathsf{TC}_r[p\colon E \to B] = \mathsf{TC}_r(X)$; see [Farber and Paul 2022, Example 3.2]. Indeed, in this case

$$G\text{–}\mathsf{cat}[p\colon E \to B] = 0 \quad \text{and} \quad \mathsf{TC}^w_{r,G}(X,P) = \mathsf{TC}_r(X);$$

hence the statement follows from (29).

Example 8.3 The Klein bottle K is the total space of the bundle $p\colon K = S^1 \times_{\mathbb{Z}/2} S^1 \to S^1$ with the associated principal bundle the 2–fold covering $\tau\colon S^1 \to S^1$ and the action of $G = \mathbb{Z}/2$ on the fibre S^1 being given by reflection in the last coordinate. The inequality (31) with $r = 2$ and the result of Example 7.7 give

$$\mathsf{TC}[p\colon K \to S^1] = \mathsf{TC}_2[p\colon K \to S^1] \le 1 + 1 = 2. \tag{32}$$

Mark Grant observed that (32) is in fact an equality. The inequality $\mathsf{TC}[p\colon K \to S^1] \ge 2$ can be obtained by applying [Farber and Weinberger 2023a, Theorem 2]. The bundle $p\colon K \to S^1$ is the unit sphere bundle of a rank 2 vector bundle ξ over the circle S^1. One has $\mathrm{w}_2(\xi) = 0$ (for dimensional reasons) and $\mathrm{w}_1(\xi) \ne 0$ (since ξ is not orientable) and therefore the relative height $\mathfrak{h}(\mathrm{w}_1(\xi) \mid \mathrm{w}_2(\xi))$ equals one. Theorem 2 from [Farber and Weinberger 2023a] now applies and gives an equality $\mathsf{TC}[p\colon K \to S^1] = 2$.

Example 8.4 Consider the principal G–bundle $\tau\colon P \to B$ where $G = S^1$, $P = S^{2n+1}$ and $B = \mathbb{CP}^n$ (the Hopf bundle). Here the sphere S^{2n+1} is viewed as the unit sphere in $\mathbb{C}^{n+1}$ and the circle S^1 acts on it by complex multiplication. Let $X = S^2$ with S^1–action given by rotations about the z–axis, as in Example 7.9. Consider the fibre bundle $p\colon E \to B$ with fibre $X = S^2$ where $E = X \times_G P$. Applying (31) with $r = 2$ we obtain

$$\mathsf{TC}[p\colon E \to B] = \mathsf{TC}_2[p\colon E \to B] \le \mathsf{secat}[\tau\colon P \to B] + \mathsf{TC}^w_{2,G}(X) \tag{33}$$

and from Example 7.9 we know that $\mathsf{TC}^w_{2,G}(X) = 2$. On the other hand, since $\mathsf{cat}(\mathbb{CP}^n) = n$, we have $\mathsf{secat}[\tau\colon P \to B] \le \mathsf{cat}(B) = n$ (in fact, this is an equality by a cup-length argument). Thus, (33) gives $\mathsf{TC}[p\colon E \to B] \le n + 2$.

In [Farber and Weinberger 2023b] the authors studied parametrized topological complexity of sphere bundles. The sphere bundle $p\colon E \to B$ which was discussed in the previous paragraph is the unit sphere bundle associated with the rank 3 vector bundle over $B = \mathbb{CP}^n$ which is the Whitney sum $\eta \oplus \epsilon$ where η is the canonical complex line bundle over $\mathbb{CP}^n$ and ϵ is a trivial real line bundle. The result of [Farber and Weinberger 2023b, Example 20] states that $\mathsf{TC}[p\colon E \to B] \le n+2$ and moreover $\mathsf{TC}[p\colon E \to B] = n+2$ for any even n.

Here the point is that, in the example above, the upper bound (31) is in fact sharp; that is, we have an equality

$$\mathsf{TC}[p\colon E \to B] = G\text{–}\mathsf{cat}[p\colon E \to B] + \mathsf{TC}^w_{2,G}(S^2).$$

In fact, since in general $\mathsf{TC}^w_{r,G}(X; P) \le \mathsf{TC}^w_{r,G}(X)$, we see that (29) in this case is an equality as well. This emphasizes the fact that these upper bounds can sometimes detect parametrized topological complexity precisely.

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