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**The multiplicative structures on motivic homotopy groups**

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# The multiplicative structures on motivic homotopy groups

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We reconcile the multiplications on the homotopy rings of motivic ring spectra used by Voevodsky and Dugger. While the connection is elementary and similar phenomena have been observed in situations like supersymmetry, neither we nor other researchers we consulted were aware of the conflicting definitions and the potential consequences. Hence this short note.

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The homotopy groups of a motivic spectrum  $E$  form a  $\mathbb{Z} \times \mathbb{Z}$ -graded abelian group  $\pi_{*,*}E$ . If  $E$  is a motivic ring spectrum, then the multiplication induces a ring structure on  $\pi_{*,*}E$ , which, if  $E$  is commutative, should be graded commutative, as explained in [Dugger 2014]. Voevodsky [2003] displays the dual Steenrod algebra  $\mathcal{A}_{*,*}$  as a ring with graded commutativity  $x \cdot y = y \cdot x \cdot (-1)^{ac}$  for  $x \in \mathcal{A}_{a,b}$  and  $y \in \mathcal{A}_{c,d}$  — the same convention is used in [Hoyois et al. 2017; Spitzweck 2018] — while [Dugger 2014] yields  $x \cdot y = y \cdot x \cdot (-1)^{(a-b)(c-d)} \cdot (-1)^{bd}$ . These are different formulas: for instance, Voevodsky claims  $\tau_0 \tau = \tau \tau_0$  and, according to [Dugger 2014], we must have that  $\tau_0 \tau = -\tau \tau_0$ .

The authors were distressed to discover this, and, worryingly enough, none of those we consulted had discovered the discrepancy (although [Dugger 2014] claims that the Betti realization is not a ring map). Was there a subtle mistake buried in the literature somewhere? Something was surely wrong. But what?

## Don't panic

Fortunately, the results are not irreconcilable, and in fact the solution is already to be found in [Dugger 2014, Proposition 7.2]:

“The” homotopy ring of a motivic ring spectrum  $A$  is not canonical.

Let us recall the outline of this story:

- (1) Taking as given the usual bigraded family of spheres  $S^{p,q}$ , one obtains a bigraded abelian group  $\pi_{*,*}A = \bigoplus_{p,q} \pi_{p,q}A$ . But equipping this with a product requires fixing a choice of isomorphisms  $\phi_{a,b}: S^{a_1,a_2} \wedge S^{b_1,b_2} \cong S^{a_1+b_1,a_2+b_2}$  in the stable homotopy category. For the product to be associative, a set of familiar pentagonal diagrams has to commute; when this happens, let us say that the collection of  $\phi$ -isomorphisms is *coherent*.

- (2) Let  $\mathbb{S}$  denote the motivic sphere spectrum. The set of coherent collections of  $\phi$ -isomorphisms is a torsor for the group  $Z^2(\mathbb{Z} \times \mathbb{Z}, (\pi_{0,0}\mathbb{S})^*)$  of reduced 2-cocycles on the group  $\mathbb{Z} \times \mathbb{Z}$  with values in the group of units in the ring  $\pi_{0,0}\mathbb{S}$ . In other words, if we fix one collection of coherent  $\phi$ -isomorphisms, then any other such collection differs from it by such a reduced 2-cocycle. Recall here that a function  $\alpha: \mathbb{Z} \times \mathbb{Z} \rightarrow (\pi_{0,0}\mathbb{S})^*$  is a 2-cocycle when  $\alpha(u + v, w) \cdot \alpha(u, v) = \alpha(v, w) \cdot \alpha(u, v + w)$  for  $u, v, w \in \mathbb{Z}^2$ , and is reduced when  $\alpha(0, 0) = 1$ .
- (3) Two different choices of coherent  $\phi$ -isomorphisms typically lead to two different ring structures on  $\pi_{*,*}A$ . The difference 2-cocycle is a coboundary precisely when there is a bigraded isomorphism between these rings that multiplies elements of each bidegree  $a = (a_1, a_2)$  by a fixed unit  $e_a \in \pi_{0,0}(\mathbb{S})^*$ . Such isomorphisms are called *standard isomorphisms* in [Dugger 2014].

See [Dugger 2014, Section 7] for details on the above.

It turns out that the  $\phi$ -isomorphisms chosen in [Dugger 2014] lead to a different ring structure on  $\pi_{*,*}A$  than the one used by Voevodsky, even up to standard isomorphism. Of course, we can still translate between the two rings, and it is not exactly that one choice is right and one is wrong — if a person keeps their wits about them as far as remembering the different conventions, there are no contradictions. But below we will analyze a collection of different choices and make some suggestions about which ones seem ideal. We stress that the underlying symmetric monoidal structure of motivic spectra and the definition of homotopy groups are the same in [Dugger 2014; Voevodsky 2003]; it is only the choice of coherent  $\phi$ -isomorphisms (not explicitly spelled out in [Voevodsky 2003], but in some sense there implicitly) that differs.

That multigraded objects have flexibility in sign conventions has been observed in situations other than motivic homotopy theory, for instance in supersymmetry [Deligne and Morgan 1999]. We comment on this, as well as on the connection to equivariant theory, in Remarks 2 and 3 below.

## The signs they are a-changin’

Regardless of the base scheme,  $\pi_{0,0}\mathbb{S}$  always contains the following four (not necessarily distinct) square roots of 1: 1,  $-1$ ,  $\epsilon$  and  $-\epsilon$ , where  $-1$  and  $\epsilon$  are given by  $g \mapsto g^{-1}$  on the topological and Tate circles,  $S^1$  and  $\mathbb{G}_m$ , respectively. When choosing the coherent isomorphisms

$$S^{a_1, a_2} \wedge S^{b_1, b_2} \cong S^{a_1 + b_1, a_2 + b_2},$$

where  $S^{a_1, a_2} = (S^1)^{\wedge(a_1 - a_2)} \wedge \mathbb{G}_m^{\wedge a_2}$ , the convention in [Dugger 2014] was as follows: every time two  $S^1$ ’s are moved past each other, the sign  $-1$  appears, and every time two  $\mathbb{G}_m$ ’s are moved past each other, we get an  $\epsilon$ . But swapping  $S^1$ ’s and  $\mathbb{G}_m$ ’s is *not* assigned any punishment in [Dugger 2014]. This convention makes sense if  $S^1$  and  $\mathbb{G}_m$  are regarded as generic objects without any special relation between them, which was the case in the more general settings treated in [Dugger 2014]. However, this particular choice raises a problem: when the ground field is the complex numbers, Betti realization sends  $\mathbb{G}_m$  to  $\mathbb{G}_m(\mathbb{C}) \simeq S^1$ , so moving a  $\mathbb{G}_m$  past an  $S^1$  is detected in topology. Consequently, with these

conventions, the Betti realization map  $\pi_{*,\star} X \rightarrow \pi_* X(\mathbb{C})$  is not a ring homomorphism — there is an annoying sign that comes up (see [Dugger 2014, Proposition 1.19]).

A better approach is to recognize that the isomorphism  $S^{a_1, a_2} \wedge S^{b_1, b_2} \cong S^{a_1+b_1, a_2+b_2}$  should involve  $a_2(b_1 - b_2)$  swaps of  $\mathbb{G}_m$ 's past  $S^1$ 's and we can choose to include a “generalized sign” factor to track this. To this end, choose once and for all a unit  $u \in \pi_{0,0}\mathbb{S}$ . In our applications we will have  $u^2 = 1$  and  $u$  will play the role of a “generalized sign”, but the basic setup only needs  $u$  to be invertible. If  $A$  is any motivic ring spectrum with unit map  $\eta: \mathbb{S} \rightarrow A$ , we may consider the  $\mathbb{Z} \times \mathbb{Z}$ -graded ring  $(\pi_{*,\star} A, \cdot)$  provided by [Dugger 2014] and we may consider the alternative  $(\pi_{*,\star} A, \cdot_u)$  with

$$x \cdot_u y = x \cdot y \cdot \eta u^{a_2(b_1 - b_2)}$$

when  $x \in \pi_{a_1, a_2} A$  and  $y \in \pi_{b_1, b_2} A$  (“punishing” each swap of  $\mathbb{G}_m$ 's past  $S^1$ 's by multiplying with  $u$ ). Here  $\alpha_u((a_1, a_2), (b_1, b_2)) = \eta u^{a_2(b_1 - b_2)}$  is the 2-cocycle from our story. The cocycle condition gives associativity of  $\cdot_u$ , and the other axioms for a ring follow readily. If  $A$  is commutative then the same proof as for [Dugger 2014, Proposition 1.18] shows that  $x \cdot y = y \cdot x \cdot (-1)^{(a_1 - a_2)(b_1 - b_2)} \epsilon^{a_2 b_2}$ . So

$$\begin{aligned} x \cdot_u y &= y \cdot_u x \cdot (-1)^{(a_1 - a_2)(b_1 - b_2)} \eta(\epsilon^{a_2 b_2} u^{a_2(b_1 - b_2)} u^{-b_2(a_1 - a_2)}) \\ &= y \cdot_u x \cdot (-1)^{(a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2)} \eta(\epsilon^{a_2 b_2} u^{a_2 b_1 - a_1 b_2}) \\ &= y \cdot_u x \cdot (-1)^{a_1 b_1} \eta(-u)^{a_2 b_1 - a_1 b_2} \eta(-\epsilon)^{a_2 b_2}. \end{aligned}$$

In particular, if  $\eta(\epsilon) = \eta(u) = -1$  then  $\eta(-\epsilon) = \eta(-u) = 1$  and thus

$$x \cdot_u y = y \cdot_u x \cdot (-1)^{a_1 b_1}.$$

This is exactly Voevodsky’s convention for commutativity in the dual Steenrod algebra: graded commutativity with respect to the total grading (see [Voevodsky 2003, Theorem 2.2]).

**Remark 1** We used a special 2-cocycle in the above computations, but this wasn’t necessary. For any reduced 2-cocycle  $\alpha$ , we can define  $x \cdot_\alpha y = x \cdot y \cdot \alpha((a_1, a_2), (b_1, b_2))$ , and then there is an associated commutativity formula of the form

$$x \cdot_\alpha y = y \cdot_\alpha x \cdot w((a_1, a_2), (b_1, b_2)),$$

where  $w$  is a 2-cocycle that is skew-symmetric in the sense of  $w(a, b) = w(b, a)^{-1}$ . In fact,

$$w(a, b) = (-1)^{(a_1 - a_2)(b_1 - b_2)} \epsilon^{a_2 b_2} \alpha(a, b)^{-1} \alpha(b, a).$$

**Proposition** An invertible element  $u$  in  $\pi_{0,0}\mathbb{S}$  gives a functor  $A \mapsto (\pi_{*,\star} A, \cdot_u)$  from motivic ring spectra to  $\mathbb{Z} \times \mathbb{Z}$ -graded rings.

Choosing  $u = -1$  or  $u = \epsilon$  gives graded rings conforming with Voevodsky’s commutativity formulas for ring spectra  $A$  having the property that  $\eta_A(\epsilon) = -1$ . Choosing  $u = 1$  gives the multiplication in [Dugger 2014].

Also, over the complex numbers, when choosing  $u = -1$  or  $u = \epsilon$ , Betti realization gives a map of (commutative) graded rings by forgetting weight.

For a given choice of  $u$ , we can ask whether the rings  $(\pi_{*,\star} A, \cdot)$  and  $(\pi_{*,\star} A, \cdot_u)$  happen to be isomorphic via a standard isomorphism. Deciding this is equivalent to checking whether  $\alpha_u$  is a coboundary. But, if  $\beta$  is a 1-cochain, then  $(\delta\beta)(a, b) = \beta(a) - \beta(a + b) + \beta(b)$  and is therefore symmetric in  $a$  and  $b$ . As  $\alpha_u$  is not symmetric, it is not a coboundary.

If  $u^2 = 1$  then the subgroup  $\langle u \rangle$  of  $(\pi_{0,0}\mathbb{S})^*$  is just  $\mathbb{Z}/2$ , and since  $B(\mathbb{Z} \times \mathbb{Z})$  is the 2-torus we have  $H^2(\mathbb{Z} \times \mathbb{Z}; \mathbb{Z}/2) = \mathbb{Z}/2$ . So, as far as twisting by  $u$  goes (once  $u$  is fixed), there are only two different standard isomorphism classes of homotopy rings that can arise: these are represented by the products  $\cdot$  and  $\cdot_u$  that we saw above. Allowing arbitrary twists from the subgroup  $\{1, -1, \epsilon, -\epsilon\}$  increases the number of possibilities to four.

**Remark 2** Of course, these considerations hold in situations other than motivic homotopy theory. An interesting example is that of  $C_2$ -equivariant spectra. When over the real numbers, evaluating at complex points gives a symmetric monoidal functor from motivic spectra to  $C_2$ -equivariant spectra, where  $S^1(\mathbb{C})$  corresponds to the trivial representation and  $\mathbb{G}_m(\mathbb{C})$  to the sign representation  $\sigma$ . Thus, choosing your  $u$ 's in the same way in the motivic and in the  $C_2$ -equivariant setting gives that Betti realization induces a map of (commutative) bigraded rings.

In this  $C_2$ -equivariant context, in addition to the forgetful map to nonequivariant spectra there is also the fixed-point functor  $A \mapsto \phi A$ . This induces maps of groups  $\pi_{p,q}(A) \rightarrow \pi_{p-q}(\phi A)$ , and so we can ask whether  $\pi_{*,\star}(A) \rightarrow \pi_*(\phi A)$  is a ring homomorphism. For  $u = -1$  it is not, but for  $u = \epsilon$  it is. For this reason we suggest that  $u = \epsilon$  is the best choice for both motivic and  $C_2$ -equivariant homotopy. With this convention, the graded-commutativity formula for the homotopy ring of a ring spectrum is

$$xy = yx \cdot (-1)^{a_1 b_1} (-\epsilon)^{a_2 b_1 + a_1 b_2 + a_2 b_2} = yx \cdot (-\epsilon)^{(a_1 - a_2)(b_1 - b_2)} \cdot \epsilon^{a_1 b_1}$$

for  $x \in \pi_{a_1, a_2} A$  and  $y \in \pi_{b_1, b_2} A$ .

**Remark 3** We mention a connection to supersymmetry. Choosing  $u = 1$  corresponds to the ‘‘Deligne convention’’ (see [Deligne and Morgan 1999, 1.2.8]), where commuting something in degree  $a + b\sigma$  with something in degree  $c + d\sigma$  would introduce the penalty  $(-1)^{ac + bd\sigma}$  (where  $(-1)^\sigma$  is the twist on the sign representation) while choosing  $u = \epsilon$  would result in the ‘‘Bernstein convention’’ with sign  $(-1)^{(a+b\sigma)(c+d\sigma)}$ .

**Remark 4** Another approach to these issues is to grade the stable homotopy ring by invertible objects rather than by isomorphism classes of invertible objects. This is sometimes referred to as a Pic-grading, though that terminology can be confusing since Pic is often used for isomorphism classes of invertible objects. For example, in the context of  $G$ -equivariant stable homotopy theory, one has to remember that  $\pi_{V-W} X$  depends on the pair of representations  $(V, W)$  and not just on the class  $V - W$  in  $RO(G)$ . In the Pic-grading,  $\pi_{V \oplus W} X$  and  $\pi_{W \oplus V} X$  are different groups, albeit isomorphic ones.

The Pic-grading eliminates all questions of sign choices: everything works out canonically. However, the cost is that one does not have a ring graded by a manageable collection of objects, so this approach is not

conducive for computation. The sign issues considered in this paper arise when one tries to reduce the Pic-grading to something practical for computation.

## The effect on the motivic stable homotopy ring

These considerations led us to wonder whether any well-known relations in the motivic stable homotopy groups change under the different sign conventions. For example, do any of the relations in [Dugger and Isaksen 2013] depend on the sign convention? See [Isaksen and Østvær 2020] for a recent survey article on motivic stable homotopy groups.

First of all, the relation  $(1 - \epsilon)\eta^2 = 0$  witnesses that  $\epsilon$  *must* play a role in graded commutativity. When we commute the element  $\eta$  in  $\pi_{1,1}$  past itself, a factor of  $\epsilon$  appears. Note that  $2\eta^2$  is not zero in general; this is detected in the  $\mathbb{R}$ -motivic homotopy groups.

Consider the list

$$\rho, \quad \eta, \quad \nu, \quad \sigma, \quad \eta_{\text{top}}, \quad \nu_{\text{top}}, \quad \sigma_{\text{top}}$$

of elements of degrees

$$(-1, -1), \quad (1, 1), \quad (3, 2), \quad (7, 4), \quad (1, 0), \quad (3, 0), \quad (7, 0),$$

respectively. These seven elements are defined in the motivic stable homotopy ring over any base. As far as we are aware, the only way to produce additional “universal” examples is to assemble these elements with Toda brackets.

By inspection, it turns out the commutativity relations amongst these elements are the same when  $u = 1$  or  $u = \epsilon$ . The “error” factor  $\epsilon^{a_2b_1+a_1b_2}$  is not equal to one in some cases. However, in all such cases, we are saved by the relations  $(1 - \epsilon)\rho = 0$  and  $(1 - \epsilon)\eta = 0$ .

This observation led us to search further for an explicit example where the cases  $u = 1$  and  $u = \epsilon$  give different commutativity relations in the motivic stable homotopy ring. We inspected the 2-complete  $\mathbb{R}$ -motivic stable homotopy ring in a large range [Belmont and Isaksen 2022], and we found no possible differences. Similarly, a brief, speculative investigation of 3-complete homotopy yielded no examples.

On the other hand, assume that  $\tau$  detects a stable homotopy element of degree  $(0, -1)$ . This assumption holds, for example, in the  $p$ -complete context over the field  $\mathbb{C}$ . Then the cases  $u = 1$  and  $u = \epsilon$  give different commutativity relations. For example, if  $u = 1$ , then  $\tau\nu = -\nu\tau$  in  $\pi_{3,1}$ ; but if  $u = \epsilon$ , then  $\tau\nu = -\epsilon\nu\tau$ .

This investigation led us to notice a pattern in the 2-complete  $\mathbb{R}$ -motivic stable homotopy groups that had not been previously observed.

**Conjecture** *Let  $\alpha$  have degree  $(s, w)$  in the 2-complete  $\mathbb{R}$ -motivic stable homotopy ring. If  $(1 - \epsilon)\alpha$  is nonzero, then  $w$  is even.*

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