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Coxeter systems with 2–dimensional Davis complexes, growth rates and Perron numbers

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We study growth rates of Coxeter systems with Davis complexes of dimension at most 2. We show that if the Euler characteristic χ of the nerve of a Coxeter system is vanishing (resp. positive), then its growth rate is a Salem (resp. Pisot) number. In this way, we extend results due to Floyd (1992) and Parry (1993). In the case where χ is negative, we provide infinitely many nonhyperbolic Coxeter systems whose growth rates are Perron numbers.

20F55, 20F65

1 Introduction

Let Γ be a finitely generated group with generating set S. For an element $x \in \Gamma$, we write $|x|_S$ for the word length with respect to S. The growth rate of (Γ, S) is defined by

$$\tau(\Gamma, S) = \limsup_{\ell\to\infty} \sqrt[\ell]{a_\ell},$$

where a_ℓ is the number of elements of Γ of word length ℓ. Gromov's polynomial growth theorem [1981] states that Γ has a nilpotent subgroup of finite index if and only if there exist positive constants $C > 0$ and $d > 0$ such that $a_\ell \le C\ell^d$ for $\ell \ge 0$. If (Γ, S) satisfies the latter property, then we say that (Γ, S) has *polynomial growth*. In this case, one has $\tau(\Gamma, S) = 1$. The pair (Γ, S) is said to have *exponential growth* when $\tau(\Gamma, S) > 1$. Note that there exist pairs of groups and finite generating sets which have neither polynomial growth nor exponential growth (see [Grigorchuk 1984] for example).

Suppose that (Γ, S) is an abstract Coxeter system; that is, Γ is generated by S and has the presentation

$$\Gamma = \langle s_1, \dots, s_N \mid (s_is_j)^{k_{ij}} \text{ for } 1 \le i, j \le N\rangle,$$

where $k_{ii} = 1$ and $k_{ij} \ge 2$ (see Section 2.1). There are three types of Coxeter systems: spherical, affine, and otherwise. If (Γ, S) is spherical or affine, then it has polynomial growth. Therefore, our interest lies in the growth rates of nonspherical, nonaffine Coxeter systems. For instance, cofinite hyperbolic Coxeter systems are such Coxeter systems (see Section 2.2).

In the study of the growth rates of hyperbolic Coxeter systems, three kinds of real algebraic integers appear: Salem numbers, Pisot numbers, and Perron numbers (see Section 2.3). By results of Parry [1993], the growth rates of 2– and 3–dimensional cocompact hyperbolic Coxeter systems are Salem numbers. Floyd [1992] showed that the growth rates of 2–dimensional cofinite hyperbolic Coxeter systems are Pisot numbers. Moreover, their growth rates are limits of growth rates of 2–dimensional cocompact hyperbolic Coxeter systems. Yukita [2017; 2018] proved that the growth rates of 3–dimensional cofinite hyperbolic Coxeter systems are Perron numbers. Kolpakov [2012] proved that the growth rates of particular 3–dimensional cofinite hyperbolic Coxeter systems are Pisot numbers. With all the above considerations, we are interested in the relation between the geometric properties of Coxeter systems and the arithmetic nature of their growth rates as follows.

Let (Γ, S) be an abstract Coxeter system. Its *nerve* $L(\Gamma, S)$ is the abstract simplicial complex defined as follows (see Section 2.2). The vertex set is S. For a nonempty subset $T = \{s_{i_1}, \dots, s_{i_n}\} \subset S$, the vertices $s_{i_1}, \dots, s_{i_n}$ span an $(n-1)$–simplex if and only if T generates a finite subgroup of Γ. By abuse of notation, we write $L(\Gamma, S)$ for its geometric realization (see [Munkres 1984, Chapter 1, Section 3] for details). The *dimension of* (Γ, S) is defined as the maximal rank of a spherical parabolic subgroup of Γ, that is a subgroup generated by a subset of S. It coincides with the dimension of the Davis complex of (Γ, S); see [Davis 2008; Felikson and Tumarkin 2010].

In this paper, we study the arithmetic nature of the growth rates of nonspherical, nonaffine Coxeter systems (Γ, S) of dimension at most 2. We will prove the following main theorems.

Theorem A *If $\chi(L(\Gamma, S)) = 0$, then the growth rate $\tau(\Gamma, S)$ is a Salem number.*

Theorem B *If $\chi(L(\Gamma, S)) \geq 1$, then the growth rate $\tau(\Gamma, S)$ is a Pisot number. Moreover, there exists a sequence of Coxeter systems (Γ_n, S_n) with vanishing Euler characteristic such that the growth rate $\tau(\Gamma_n, S_n)$ converges to $\tau(\Gamma, S)$ from below.*

This paper is organized as follows. In Section 2, we provide the necessary background about Coxeter systems, their nerves, and their growth rates. Theorem A is discussed in Section 3 where we consider Coxeter systems with vanishing Euler characteristic. This extends the result by Parry [1993]. Section 4 is devoted to the study of Coxeter systems with positive Euler characteristic where we prove Theorem B generalizing Floyd's result [1992]. In Section 5, we provide some examples of infinite sequences of Coxeter systems with negative Euler characteristic whose growth rates are Perron numbers; see Proposition 5.1.

2 Preliminaries

2.1 Coxeter systems

For a group Γ with generating set $S = \{s_1, \dots, s_N\}$, the pair (Γ, S) is called a *Coxeter system* if Γ has the presentation

$$\Gamma = \langle s_1, \dots, s_N \mid (s_i s_j)^{k_{ij}} \text{ for } 1 \leq i, j \leq N \rangle,$$

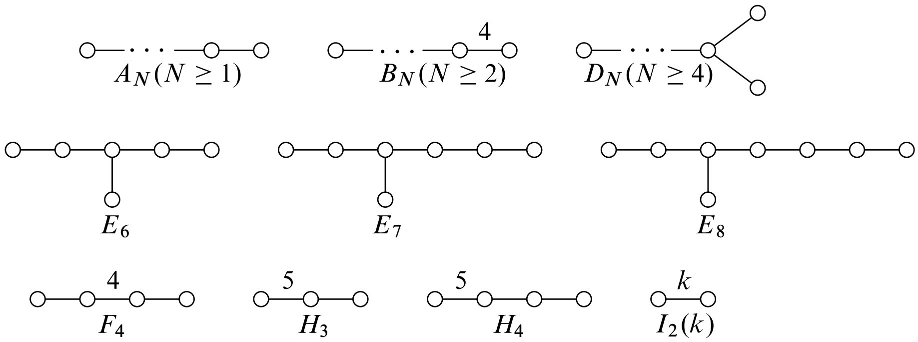

Figure 1: Irreducible spherical Coxeter systems of rank N.

where $k_{ii} = 1$ and $k_{ij} \geq 2$. In the case where $s_i s_j$ has infinite order, we put $k_{ij} = \infty$. The *rank* of a Coxeter system (Γ, S) is defined as the cardinality $\#S$ of S. For a subset $T \subset S$, the subgroup Γ_T of Γ generated by T is called a *parabolic subgroup* of Γ, with $\Gamma_\varnothing = \{1\}$ by convention.

Given a Coxeter system (Γ, S) of rank N, define the *cosine matrix* associated to (Γ, S) as the symmetric matrix $C_{(\Gamma,S)} = (c_{ij}) \in M_N(\mathbb{R})$ with entries

$$c_{ij} = \begin{cases} -\cos(\pi/k_{ij}) & \text{if } k_{ij} < \infty, \\ -1 & \text{if } k_{ij} = \infty. \end{cases}$$

The Coxeter system (Γ, S) is said to be *spherical* (resp. *affine*), if $C_{(\Gamma,S)}$ is positive definite (resp. positive semidefinite).

In this paper, a graph X is said to be *simple* if X has no loops or multiple edges. We associate to a Coxeter system (Γ, S) two kinds of edge-labeled simple graphs: the *Coxeter diagram* $\mathrm{Cox}(\Gamma, S)$ and the *presentation diagram* $X(\Gamma, S)$.

The *Coxeter diagram* $\mathrm{Cox}(\Gamma, S)$ is defined as follows. The vertex set is S. Two vertices s_i and s_j are connected by an edge if and only if $k_{ij} \geq 3$. The edge between s_i and s_j is labeled by k_{ij} if $k_{ij} \in \{4, 5, \dots\} \cup \{\infty\}$. A Coxeter system (Γ, S) is said to be *irreducible* if the underlying graph of $\mathrm{Cox}(\Gamma, S)$ is connected. It is known that a spherical (resp. affine) Coxeter system decomposes into a

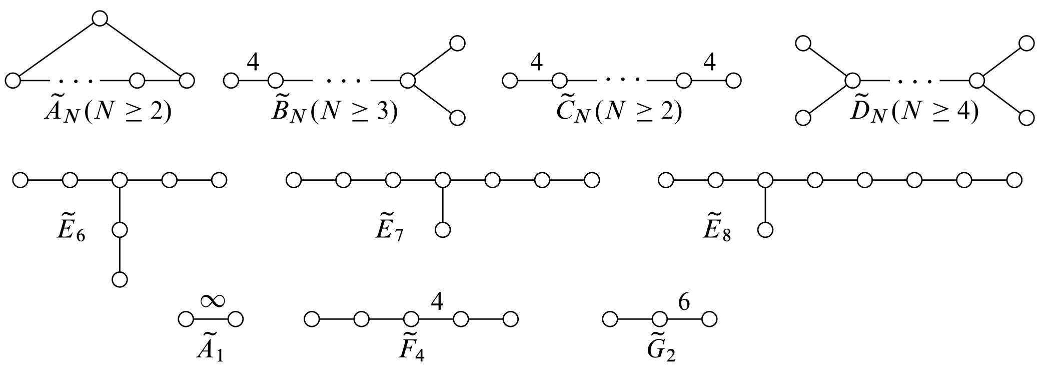

Figure 2: Irreducible affine Coxeter systems of rank $N + 1$.

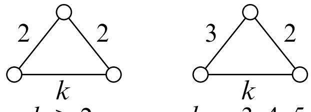

Figure 3: The presentation diagrams of the spherical Coxeter systems of rank 3.

direct product of irreducible spherical (resp. spherical and affine) Coxeter systems. The Coxeter diagrams of irreducible spherical and affine Coxeter systems are depicted in Figures 1 and 2, respectively (see [Humphreys 1990, pages 32 and 34]).

The *presentation diagram* $X(\Gamma, S)$ is defined as follows. The vertex set is S. Two vertices s_i and s_j are connected by an edge labeled by k_{ij} when $k_{ij} < \infty$. It follows that the underlying graphs of the presentation diagrams of spherical Coxeter systems of rank N are complete graphs with N vertices. For example, Figure 3 shows the presentation diagrams of the spherical Coxeter systems of rank 3.

Remark 2.1 If the Coxeter diagram $\mathrm{Cox}(\Gamma, S)$ (resp. the presentation diagram $X(\Gamma, S)$) of a Coxeter system (Γ, S) is disconnected, then (Γ, S) is a direct product (resp. a free product) of the Coxeter systems corresponding to the connected components.

2.2 Geometric Coxeter groups and nerves

For more details about geometric Coxeter groups and nerves of Coxeter systems we refer to [Davis 2008; Ratcliffe 1994].

Let us denote by $\mathbb{X}^n$ the n–dimensional spherical space $\mathbb{S}^n$, Euclidean space $\mathbb{E}^n$, or hyperbolic space $\mathbb{H}^n$. An *n–dimensional Coxeter polytope* $P \subset \mathbb{X}^n$ is the intersection of finitely many half-spaces whose interior is nonempty and dihedral angles are of the form π/k for $k \geq 2$ or equal to zero. Given an n–dimensional Coxeter polytope $P \subset \mathbb{X}^n$, the set S_P of the reflections in the bounding hyperplanes of P generates a discrete subgroup Γ_P of $\mathrm{Isom}\,(\mathbb{X}^n)$. The pair (Γ_P, S_P) is a Coxeter system, and is called an *n–dimensional geometric Coxeter system associated with* P. The group Γ_P is called the *n–dimensional geometric Coxeter group associated with* P. It is known that P is a fundamental polytope for Γ_P and the orbit $\{gP \mid g \in \Gamma_P\}$ of P gives rise to an exact tessellation of $\mathbb{X}^n$. Furthermore, Γ_P is said to be *cocompact* (resp. *cofinite*) when P is compact (resp. not compact but of finite volume). For a hyperbolic Coxeter polytope P, we say that Γ_P is *ideal* when every vertex of P lies on the boundary at infinity $\partial\mathbb{H}^n$. For each irreducible spherical (resp. affine) Coxeter system (Γ, S), there exists a spherical (resp. compact Euclidean) Coxeter polytope P such that $(\Gamma, S) = (\Gamma_P, S_P)$. Therefore, if (Γ, S) is a spherical (resp. affine) Coxeter system, then Γ is finite (resp. virtually nilpotent). In contrast to this, if (Γ, S) is nonspherical and nonaffine, then Γ contains a free group of rank at least 2; see [de la Harpe 1987].

Let (Γ, S) be an abstract Coxeter system. The *nerve* $L(\Gamma, S)$ is an abstract simplicial complex defined as follows. The vertex set is S, and for a nonempty subset $T = \{s_{i_1}, \dots, s_{i_n}\} \subset S$, the vertices $s_{i_1}, \dots, s_{i_n}$

span an $(n-1)$–simplex if and only if the parabolic subgroup Γ_T is finite. For simplicity of notation, we continue to write $L(\Gamma, S)$ for its geometric realization (see [Munkres 1984, Chapter 1, Section 3] for details). The *dimension of* (Γ, S), denoted by $\dim(\Gamma, S)$, is defined as the maximal rank of a spherical parabolic subgroup of Γ, that is a subgroup generated by a subset of S. It coincides with the dimension of the Davis complex of (Γ, S); see [Davis 2008; Felikson and Tumarkin 2010].

In this paper, we consider Coxeter systems of dimension at most 2. In particular, such a class of Coxeter systems contains hyperbolic Coxeter groups of dimension 2 and ideal hyperbolic Coxeter groups of dimension 3. Indeed, for such groups, maximal spherical subgroups are of rank at most 2. For a Coxeter system (Γ, S) of dimension at most 2, it is easy to see that the underlying graph of $X(\Gamma, S)$ is the geometric realization of the nerve $L(\Gamma, S)$. Therefore the Euler characteristic $\chi(L(\Gamma, S))$ equals the one of the underlying graph of $X(\Gamma, S)$. It is known that the Euler characteristic of a graph is the number of vertices minus the number of edges.

2.3 Growth rates of Coxeter systems

Let (Γ, S) be a Coxeter system. For $x \in \Gamma$, we define its *word length with respect to* S by

$$|x|_S = \min\{n \in \mathbb{N} \mid x = s_1 \cdots s_n \ (s_1, \dots, s_n \in S)\}.$$

By convention, $|1|_S = 0$. The *growth series* $f_{(\Gamma,S)}(z)$ of (Γ, S) is defined by

$$f_{(\Gamma,S)}(z) = \sum_{\ell \geq 0} a_\ell z^\ell,$$

where a_ℓ is the number of the elements of Γ of word length ℓ. If (Γ, S) is spherical, then $f_{(\Gamma,S)}(z)$ is a polynomial and called the *growth polynomial* of (Γ, S).

By a result of Solomon [1966], the growth polynomials of spherical Coxeter systems can be computed in terms of its exponents. For the list of exponents, see [Humphreys 1990]. For example, the exponents of A_N are given by $1, 2, \dots, N$, and those of $I_2(k)$ are $1, k-1$. For positive integers $m, m_1, \dots, m_r$, we put

$$[m] = 1 + z + \cdots + z^{m-1} \quad \text{and} \quad [m_1, \dots, m_r] = [m_1] \cdots [m_r].$$

Solomon's formula states that for a spherical Coxeter system (Γ, S) with the exponents $m_1, \dots, m_r$, one has $f_{(\Gamma,S)}(z) = [m_1 + 1, \dots, m_r + 1]$.

If (Γ, S) is nonspherical, then the inverse of the radius of convergence of $f_{(\Gamma,S)}(z)$ is called the *growth rate* of (Γ, S), denoted by $\tau(\Gamma, S)$. The Cauchy–Hadamard formula gives

$$\tau(\Gamma, S) = \limsup_{\ell \to \infty} \sqrt[\ell]{a_\ell}.$$

Since free abelian groups of finite rank have polynomial growth [Wolf 1968], and any affine Coxeter system contains a free abelian subgroup of finite rank and finite index, the growth rate of an affine Coxeter system is 1.

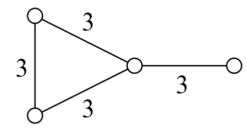

Figure 4: The presentation diagram of $(\Gamma_\star, S_\star)$.

Remark 2.2 If a Coxeter system (Γ, S) decomposes as $(\Gamma_1, S_1) \times (\Gamma_2, S_2) \times \cdots \times (\Gamma_l, S_l)$, then its growth series satisfies $f_{(\Gamma,S)} = \Pi_i f_{(\Gamma_i,S_i)}$. It follows for the growth rate that $\tau(\Gamma, S) = \max_i \tau(\Gamma_i, S_i)$. This does not hold when (Γ, S) decomposes as a free product.

The following formula, established by Steinberg, is an important tool to compute the growth series of Coxeter systems.

Theorem 2.3 (Steinberg's formula [1968]) *Let (Γ, S) be a Coxeter system. Then the identity*

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = \sum_{\substack{T\subset S \\ \#\Gamma_T<\infty}} \frac{(-1)^{\#T}}{f_{(\Gamma_T,T)}(z)} \tag{2-1}$$

holds for the growth series $f_{(\Gamma,S)}(z)$.

Steinberg's formula implies that the growth series is a rational function and satisfies that

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = \frac{P(z)}{Q(z)},$$

where $P(z)$ and $Q(z)$ are monic polynomials with integer coefficients. It follows that the growth rate $\tau(\Gamma, S)$ is the real root of $P(z)$ whose modulus is maximal among the roots of $P(z)$, and hence $\tau(\Gamma, S) \geq 1$ is a real algebraic integer.

Example 2.4 Consider the abstract Coxeter system $(\Gamma_\star, S_\star)$ whose presentation diagram is depicted in Figure 4. The spherical subgroups are A_1 and A_2, both with multiplicity four. By Steinberg's formula (2-1), we compute its growth series

$$\frac{1}{f_{(\Gamma_\star,S_\star)}(z^{-1})} = 1 - \frac{4}{[2]} + \frac{4}{[2,3]} = \frac{[2,3]-4[3]+4}{[2,3]}.$$

We write $P(z)$ for the numerator of $1/f_{(\Gamma_\star,S_\star)}(z^{-1})$; that is,

$$P(z) = 1 - 2z - 2z^2 + z^3.$$

One easily sees that $P(-1) = 0$ and that the greatest positive root of $P(z)$ is given by

$$\tau(\Gamma_\star, S_\star) = \frac{3+\sqrt{5}}{2} = \frac{1}{(\varphi-1)^2},$$

where φ is the golden ratio.

Example 2.5 If (Γ, S) is a Coxeter system of rank N whose presentation diagram $X(\Gamma, S)$ has no edges, then $\tau(\Gamma, S) = N - 1$. Indeed, we compute by Steinberg's formula (2-1)

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = 1 - \frac{N}{[2]} = \frac{z-(N-1)}{[2]}.$$

Example 2.6 If (Γ, S) is a Coxeter system of rank N whose presentation diagram $X(\Gamma, S)$ is a tree with edges labeled by 2 only, then

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = 1 - \frac{N}{[2]} + \frac{N-1}{[2,2]} = \frac{[2,2]-N[2]+N-1}{[2,2]} = \frac{z(z-(N-2))}{(1+z)^2}.$$

Observe that the growth series does not depend on the isomorphism type of the tree, only on the number of its vertices. Therefore, the growth rate is given by $\tau(\Gamma, S) = N - 2$.

From now on, we focus on the growth rates of nonspherical, nonaffine Coxeter systems. Three kinds of real algebraic integers appear in the study of the growth rates of hyperbolic Coxeter systems: Salem numbers, Pisot numbers, and Perron numbers (see [Bertin et al. 1992, page 84]).

An algebraic integer $\tau > 1$ is called a *Salem number* if it is a quadratic unit or is such that the inverse τ^{-1} is a Galois conjugate of τ and the other Galois conjugates lie on the unit circle. The minimal polynomial of a Salem number is called a *Salem polynomial*. Parry showed that the growth rates of 2– and 3–dimensional cocompact hyperbolic Coxeter systems are Salem numbers [Parry 1993].

An algebraic integer $\tau > 1$ is called a *Pisot number* if τ is an integer or if all of its other Galois conjugates are contained in the unit open disk. The minimal polynomial of a Pisot number is called a *Pisot polynomial*. Floyd showed that the growth rates of 2–dimensional cofinite hyperbolic Coxeter systems are Pisot numbers [Parry 1993]. Moreover, for a 2–dimensional cofinite hyperbolic Coxeter systems (Γ, S), there exists a sequence of 2–dimensional cocompact hyperbolic Coxeter systems (Γ_n, S_n) whose growth rates $\tau(\Gamma_n, S_n)$ converges to $\tau(\Gamma, S)$ from below.

An algebraic integer $\tau > 1$ is called a *Perron number* if τ is an integer or if all of its other Galois conjugates are strictly less than τ in absolute value. Note that Salem numbers and Pisot numbers are Perron numbers. Yukita [2017; 2018]] showed that the growth rates of 3–dimensional cofinite hyperbolic Coxeter systems are Perron numbers. Note that Komori and Yukita [2015] and Nonaka and Kellerhals [2017] showed that the growth rates of cofinite 3–dimensional hyperbolic ideal Coxeter systems are Perron numbers. For a 4–dimensional cocompact Coxeter system (Γ_P, S_P), Kellerhals and Perren [2011] proved that the growth rates are Perron numbers for $\#S_P = 5$ and 6. In particular, they conjectured that the growth rates of hyperbolic Coxeter systems are Perron numbers.

This is a motivation to relate geometric properties of Coxeter systems to the arithmetic nature of their growth rates. The aim of this paper is to extend the results of Floyd and Parry to *nonspherical*, *nonaffine*, *and nonhyperbolic* Coxeter systems of dimension at most 2. Note that Charney and Davis [1991] studied the relationship between the geometry of nerves and reciprocity of the growth series.

We use the partial order on the set of Coxeter systems defined by McMullen [2002]. Let (Γ, S) and (Γ', S') be Coxeter systems. Write $(\Gamma, S) \preceq (\Gamma', S')$ when there exists an injection $\iota : S \to S'$ such that $k(s,t) \le k'(\iota(s), \iota(t))$, where $k(s,t)$ and $k'(\iota(s), \iota(t))$ are the orders of st and $\iota(s)\iota(t)$, respectively.

Theorem 2.7 [Terragni 2016, Corollary 3.2] *If $(\Gamma, S) \preceq (\Gamma', S')$, then $\tau(\Gamma, S) \le \tau(\Gamma', S')$.*

For a finitely generated group Γ with ordered finite generating set S with $\#S = N$, we call the pair (Γ, S) an *N–marked group*. Given two N–marked groups (Γ, S) and (Γ', S') we say that they are isomorphic as marked groups when the map $\iota : S \to S'$ sending s_i to s_i' extends to a group isomorphism between Γ and Γ'. The *space of N–marked groups* is the set of isomorphism classes of N–marked groups equipped with a metric topology, given by the Chabauty–Grigorchuk topology; see [Grigorchuk 1984]. Let us denote by $\mathscr{C}_N$ the set of marked Coxeter systems of rank N. Yukita [2024] studied the space $\mathscr{C}_N$ and showed that $\mathscr{C}_N$ is compact.

Theorem 2.8 [Yukita 2024, Theorems 3.2 and 3.5] *Let $\{(\Gamma_n, S_n)\}$ and (Γ, S) be marked Coxeter systems of rank N. We write $k_{ij}(n)$ (resp. k_{ij}) for the order of $s_i(n)s_j(n)$ in Γ_n (resp. $s_i s_j$ in Γ).*

(1) *The sequence $\{(\Gamma_n, S_n)\}$ converges to (Γ, S) if and only if $\lim_{n\to\infty} k_{ij}(n) = k_{ij}$ for $1 \le i, j \le N$.*

(2) *If $\lim_{n\to\infty}(\Gamma_n, S_n) = (\Gamma, S)$, then $\lim_{n\to\infty} \tau(\Gamma_n, S_n) = \tau(\Gamma, S)$.*

3 Growth rates of Coxeter systems with vanishing Euler characteristic

Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2 such that $\chi(L(\Gamma, S)) = 0$, where $L(\Gamma, S)$ denotes the geometric realization of its nerve. In this section, we prove that the growth rate $\tau(\Gamma, S)$ is a Salem number.

We write N (resp. E) for the number of vertices (resp. edges) of the presentation diagram $X(\Gamma, S)$. Recall that the Euler characteristic of a graph is the number of vertices minus the number of edges. Since the dimension of (Γ, S) is at most 2, the underlying graph of $X(\Gamma, S)$ coincides with $L(\Gamma, S)$, and hence $N = E$. Suppose that the set of labels of the edges of $X(\Gamma, S)$ is $\{k_1, \dots, k_r\}$. Let us denote by E_i the number of edges of $X(\Gamma, S)$ labeled by k_i.

We obtain the equality

$$\begin{aligned}\frac{1}{f_{(\Gamma,S)}(z^{-1})} = 1 - \frac{N}{[2]} + \sum_{i=1}^{r} \frac{E_i}{[2,k_i]} &= 1 - \frac{E_1 + \cdots + E_r}{[2]} + \sum_{i=1}^{r} \frac{E_i}{[2,k_i]} \\ &= 1 + \sum_{i=1}^{r} \frac{E_i}{[2]}\left(\frac{1}{[k_i]} - 1\right) \\ &= 1 + \sum_{i=1}^{r} \frac{E_i}{[2]}\left(\frac{z-1}{z^{k_i}-1} - 1\right) = 1 + \sum_{i=1}^{r} E_i \frac{z - z^{k_i}}{(z+1)(z^{k_i}-1)}\end{aligned}$$

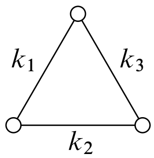

Figure 5: The presentation diagram in the case $N = 3$.

by Steinberg's formula (2-1); see also [Parry 1993, page 413]. Hence,

$$\frac{z+1}{(z-1)f_{(\Gamma,S)}(z^{-1})} = \frac{z+1}{z-1} + \sum_{i=1}^{r} E_i \frac{z - z^{k_i}}{(z-1)(z^{k_i}-1)}. \tag{3-1}$$

The following lemma is fundamental for the proof.

Lemma 3.1 [Parry 1993, Corollary 1.8] *Given integers $k_1, \dots, k_r \geq 2$ and $E_1, \dots, E_r \geq 1$, suppose that*

$$\sum_{i=1}^{r} \Big(1 - \frac{1}{k_i}\Big) E_i > 2. \tag{3-2}$$

Let $R(z)$ be the rational function defined by

$$R(z) = \frac{z+1}{z-1} + \sum_{i=1}^{r} E_i \frac{z - z^{k_i}}{(z-1)(z^{k_i}-1)}.$$

Then $R(z) = P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are relatively prime monic polynomials with integer coefficients and equal degrees, and $P(z)$ is a product of distinct irreducible cyclotomic polynomials and exactly one Salem polynomial.

Theorem 3.2 *Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2. If $\chi(L(\Gamma, S)) = 0$, then the growth rate $\tau(\Gamma, S)$ is a Salem number.*

Proof We apply Lemma 3.1 to (3-1). The proof is divided into three cases: the cases $N = 3$, $N = 4$, and $N \geq 5$.

(i) Assume $N = 3$. By assumption, $N = E = 3$, and hence the presentation diagram of $X(\Gamma, S)$ is as in Figure 5.

Since (Γ, S) is nonspherical and nonaffine,

$$\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} < 1.$$

Therefore,

$$\Big(1 - \frac{1}{k_1}\Big) + \Big(1 - \frac{1}{k_2}\Big) + \Big(1 - \frac{1}{k_3}\Big) > 2.$$

(ii) Assume $N = 4$. The presentation diagram $X(\Gamma, S)$ is one of the diagrams in Figure 6. We show that one of the labels of $X(\Gamma, S)$ is at least 3.

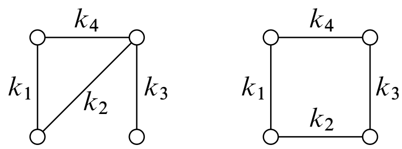

Figure 6: The presentation diagrams in the case $N = 4$.

Suppose that $X(\Gamma, S)$ is the diagram in Figure 6, left. If $k_1 = k_2 = k_4 = 2$, then the vertices of the triangle generates a spherical parabolic subgroup of Γ of rank 3. This contradicts the fact that the dimension of (Γ, S) is at most 2. Therefore, one of the labels is at least 3. Suppose that $X(\Gamma, S)$ is the diagram in Figure 6, right. If $k_1 = k_2 = k_3 = k_4 = 2$, then the Coxeter diagram $\mathrm{Cox}(\Gamma, S)$ is made of two connected components $\widetilde{A}_1$ (see Figure 2 for $\widetilde{A}_1$). This is a contradiction to the fact that (Γ, S) is nonspherical and nonaffine. Therefore, one of the labels is at least 3. Hence,

$$\left(1-\frac{1}{k_1}\right)+\left(1-\frac{1}{k_2}\right)+\left(1-\frac{1}{k_3}\right)+\left(1-\frac{1}{k_4}\right) \geq 3\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{3}\right) > 2.$$

(iii) Assume $N \geq 5$. It follows that

$$\sum_{i=1}^{r}\left(1-\frac{1}{k_i}\right)E_i = \sum_{i=1}^{r} E_i - \sum_{i=1}^{r}\frac{E_i}{k_i} = N - \sum_{i=1}^{r}\frac{E_i}{k_i} \geq N - \sum_{i=1}^{r}\frac{E_i}{2} = \frac{N}{2} \geq \frac{5}{2} > 2.$$

Therefore, (3-2) holds, and the assertion follows from Lemma 3.1. □

For later use, we show the following.

Lemma 3.3 *Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2. Suppose that the growth series $f_{(\Gamma,S)}(z)$ satisfies the equality*

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = \frac{P(z)}{[2, k_1, \dots, k_r]},$$

where $P(z)$ is a monic polynomial with integer coefficients. If $\chi(L(\Gamma, S)) = 0$, then $P(z)$ is a product of cyclotomic polynomials and exactly one Salem polynomial.

Proof As in the proof of Theorem 3.2, we apply Lemma 3.1 to (3-1):

$$\frac{z+1}{(z-1)f_{(\Gamma,S)}(z^{-1})} = \frac{P_0(z)}{Q_0(z)},$$

where $P_0(z)$ and $Q_0(z)$ are the relatively prime polynomials with integer coefficients. P_0 is a product of distinct irreducible cyclotomic polynomials and exactly one Salem polynomial. By assumption, we have

$$\frac{P(z)}{[2, k_1, \dots, k_r]} = \frac{(z-1)P_0(z)}{(z+1)Q_0(z)}. \tag{3-3}$$

Since every factor of the polynomial $[2, k_1, \dots, k_r]$ is a cyclotomic polynomial, the equality (3-3) implies that $P(z)$ is a product of cyclotomic polynomials and exactly one Salem polynomial. □

4 Growth rates of Coxeter systems with positive Euler characteristic

Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2 such that $\chi(L(\Gamma, S)) \geq 1$, where $L(\Gamma, S)$ denotes the geometric realization of its nerve. Recall that $\chi(L(\Gamma, S))$ equals the Euler characteristic of the underlying graph of $X(\Gamma, S)$. In this section, we prove that the growth rate $\tau(\Gamma, S)$ is a Pisot number.

Lemma 4.1 *Let (Γ, S) be a nonspherical, nonaffine marked Coxeter system of dimension at most 2 and rank N. Suppose that either the presentation diagram $X(\Gamma, S)$ is disconnected, or has an edge labeled by $k \geq 3$. If $\chi(L(\Gamma, S)) \geq 1$, then there exists a sequence of marked Coxeter systems $\{(\Gamma_n, S_n)\}_{n \geq 7}$ of rank N such that for $n \geq 7$,*

(1) $(\Gamma_n, S_n) \preceq (\Gamma_{n+1}, S_{n+1}) \preceq (\Gamma, S)$;

(2) $\dim(\Gamma_n, S_n) \leq 2$;

(3) $\chi(L(\Gamma_n, S_n)) = \chi(L(\Gamma, S)) - 1$;

(4) *the sequence $\{(\Gamma_n, S_n)\}_{n \geq 7}$ converges to (Γ, S) in the space $\mathscr{C}_N$ of marked Coxeter systems of rank N.*

Proof Set $S = \{s_1, \ldots, s_N\}$. We denote by E and k_{ij} the number of edges of $X(\Gamma, S)$ and the order of the product $s_i s_j$, respectively.

Suppose first that the underlying graph of the presentation diagram $X(\Gamma, S)$ is disconnected. Let s_p and s_q be two vertices of different connected components of the underlying graph of $X(\Gamma, S)$. It follows that $k_{pq} = \infty$. For $n \geq 7$, we define a marked Coxeter system (Γ_n, S_n) of rank N by the presentation

$$\Gamma_n = \langle s_1(n), \ldots, s_N(n) \mid (s_i(n) s_j(n))^{k_{ij}(n)} = 1 \text{ for } 1 \leq i, j \leq N \rangle,$$

where

$$k_{ij}(n) = \begin{cases} n & \text{if } \{i, j\} = \{p, q\}, \\ k_{ij} & \text{otherwise.} \end{cases}$$

We will show that (Γ_n, S_n) satisfies the desired properties. For $1 \leq i, j \leq N$ and $n \geq 7$, we have $k_{ij}(n) \leq k_{ij}(n+1) \leq k_{ij}$, so

$$(\Gamma_n, S_n) \preceq (\Gamma_{n+1}, S_{n+1}) \preceq (\Gamma, S).$$

In order to show that $\dim(\Gamma_n, S_n) \leq 2$, it is sufficient to see that the presentation diagram $X(\Gamma_n, S_n)$ does not contain any of the diagrams depicted in Figure 3. Since $\dim(\Gamma, S) \leq 2$, no such diagram is contained in $X(\Gamma, S)$. The presentation diagram $X(\Gamma_n, S_n)$ is obtained from $X(\Gamma, S)$ by adding an edge between s_p and s_q labeled by n (see Figure 7). In Figure 7, we do not put labels of the edges other than the added edge for simplicity. Since the vertices s_p and s_q lie in different connected components of the underlying graph of $X(\Gamma, S)$, every cycle of the underlying graph of $X(\Gamma_n, S_n)$ comes from one of $X(\Gamma, S)$. Hence we see that $X(\Gamma_n, S_n)$ does not contain any of the diagrams depicted in Figure 3. The Euler characteristics of the underlying graphs of $X(\Gamma_n, S_n)$ and $X(\Gamma, S)$ are equal to $\chi(L(\Gamma_n, S_n))$

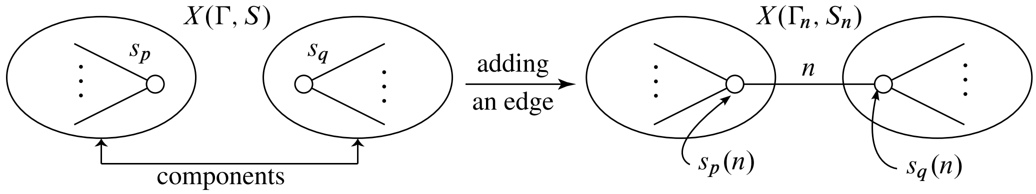

Figure 7: Adding an edge between s_p and s_q.

and $\chi(L(\Gamma, S))$, respectively. This observation implies that $\{(\Gamma_n, S_n)\}_{n\geq 7}$ satisfies the property (3). By definition of (Γ_n, S_n), we have $\lim_{n\to\infty} k_{ij}(n) = k_{ij}$ for $1 \leq i, j \leq N$. Property (1) of Theorem 2.8 implies that $\{(\Gamma_n, S_n)\}_{n\geq 7}$ converges to (Γ, S) in $\mathscr{C}_N$.

Suppose next that the underlying graph of $X(\Gamma, S)$ is connected, and let us show that the underlying graph is a tree. Since every connected graph with the Euler characteristic 1 is a tree, it is sufficient to show that $\chi(L(\Gamma, S)) = 1$. By the connectivity of the underlying graph of $X(\Gamma, S)$, there exists a spanning tree T of the graph. We denote by N_T and E_T the number of vertices and of edges of T, respectively. It follows that $N = N_T$, $E_T \leq E$, and $N_T - E_T = 1$. Since $\chi(L(\Gamma, S)) = N - E \geq 1$,

$$1 \leq N - E \leq N - E_T = N_T - E_T = 1,$$

and hence $\chi(L(\Gamma, S)) = 1$.

Since Coxeter systems of rank at most 2 are spherical or affine, our assumption implies that $N \geq 3$. Also by assumption, there exists an edge e between vertices s_p and s_q of $X(\Gamma, S)$, labeled by $k_{pq} \geq 3$. Since the underlying graph of $X(\Gamma, S)$ is a tree with at least 3 vertices, we can find an edge e' incident with e. Without loss of generality we can assume that e and e' share the vertex s_q. We write s_r for the endpoint of e' other than s_q. Since the underlying graph of $X(\Gamma, S)$ is a tree, the vertices s_p and s_r are not joined by an edge. It follows that $k_{pr} = \infty$. For $n \geq 7$, we define a marked Coxeter system (Γ_n, S_n) of rank N by the presentation

$$\Gamma_n = \langle s_1(n), \dots, s_N(n) \mid (s_i(n)s_j(n))^{k_{ij}(n)} = 1 \text{ for } 1 \leq i, j \leq N \rangle,$$

where

$$k_{ij}(n) = \begin{cases} n & \text{if } \{i, j\} = \{p, r\}, \\ k_{ij} & \text{otherwise.} \end{cases}$$

We will show that (Γ_n, S_n) satisfies the desired properties. For $1 \leq i, j \leq N$ and $n \geq 7$, we have $k_{ij}(n) \leq k_{ij}(n+1) \leq k_{ij}$, so $(\Gamma_n, S_n) \preceq (\Gamma_{n+1}, S_{n+1}) \preceq (\Gamma, S)$.

The presentation diagram $X(\Gamma_n, S_n)$ is obtained from $X(\Gamma, S)$ by adding an edge between s_p and s_r labeled by n (see Figure 8). In Figure 8, we do not put labels of the edges other than three edges joining two of s_p, s_q, and s_r for simplicity.

Since the underlying graph of $X(\Gamma, S)$ is a tree, the graph of $X(\Gamma_n, S_n)$ has only one cycle and the cycle consists of three edges joining two of s_p, s_q, and s_r. Therefore, the presentation diagram $X(\Gamma_n, S_n)$ does

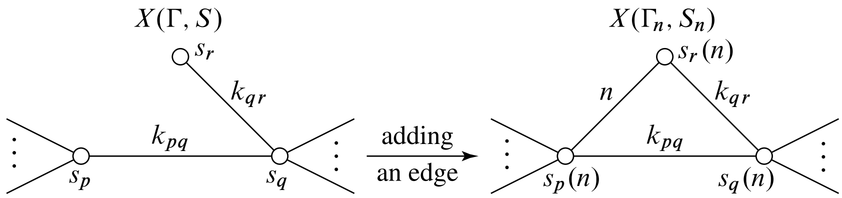

Figure 8: Adding an edge between s_p and s_r.

not contain any of the diagrams in Figure 3, which is due to the fact that $k_{pq} \geq 3$ and $n \geq 7$. It follows that $\dim(\Gamma_n, S_n) \leq 2$. The same reasoning as before allows one to conclude that

$$\chi(L(\Gamma_n, S_n)) = \chi(L(\Gamma, S)) - 1.$$

By definition of (Γ_n, S_n), we have $\lim_{n\to\infty} k_{ij}(n) = k_{ij}$ for $1 \leq i, j \leq n$, and Property (1) of Theorem 2.8 implies that $\{(\Gamma_n, S_n)\}_{n\geq 7}$ converges to (Γ, S) in $\mathscr{C}_N$. □

Remark 4.2 Suppose that (Γ, S) is a nonspherical, nonaffine Coxeter system of at most dimension 2 such that $\chi(L(\Gamma, S)) \geq 1$. If (Γ, S) does not satisfy the hypothesis in Lemma 4.1, the presentation diagram $X(\Gamma, S)$ is connected and its edges are labeled by 2 only. As shown in the proof, in this case, the positivity of the Euler characteristic forces $X(\Gamma, S)$ to be a tree.

Corollary 4.3 *Let (Γ, S) be a nonspherical, nonaffine marked Coxeter system of dimension at most 2 and rank N such that $\chi(L(\Gamma_n, S_n)) \geq 1$. Suppose that either the presentation diagram $X(\Gamma, S)$ is disconnected, or has an edge labeled by $k \geq 3$. Then there exists a sequence of marked Coxeter systems $\{(\Gamma_n, S_n)\}_{n\geq 7}$ of rank N such that for $n \geq 7$,*

(1) $(\Gamma_n, S_n) \preceq (\Gamma_{n+1}, S_{n+1}) \preceq (\Gamma, S)$;

(2) $\dim(\Gamma_n, S_n) \leq 2$;

(3) $\chi(L(\Gamma_n, S_n)) = 0$;

(4) *the sequence $\{(\Gamma_n, S_n)\}$ converges to (Γ, S) in the space $\mathscr{C}_N$ of marked Coxeter systems of rank N.*

Proof We take a sequence of marked Coxeter systems $\{(\Gamma_{n_1}, S_{n_1})\}_{n_1\geq 7}$ of rank N as in Lemma 4.1. If $\chi(L(\Gamma, S)) = 1$, then for $n_1 \geq 7$,

$$\chi(L(\Gamma_{n_1}, S_{n_1})) = \chi(L(\Gamma, S)) - 1 = 0.$$

Hence the sequence $\{(\Gamma_{n_1}, S_{n_1})\}_{n_1\geq 7}$ satisfies the properties in Corollary 4.3.

Suppose that $\chi(L(\Gamma, S)) \geq 2$. The presentation diagram $X(\Gamma_{n_1}, S_{n_1})$ has an edge labeled by $n_1 \geq 7$ and $\chi(L(\Gamma_{n_1}, S_{n_1})) = \chi(L(\Gamma, S)) - 1 \geq 1$. For each $n_1 \geq 7$, by applying Lemma 4.1 to (Γ_{n_1}, S_{n_1}), there exists a sequence of marked Coxeter systems $\{(\Gamma_{n_1,n_2}, S_{n_1,n_2})\}_{n_2\geq 7}$ of rank N satisfying the properties in

Lemma 4.1. Moreover, we may assume that $(\Gamma_{n_1,n_2}, S_{n_1,n_2}) \preceq (\Gamma_{n'_1,n'_2}, S_{n'_1,n'_2})$ for $n_1 \le n'_1$ and $n_2 \le n'_2$. If $\chi(L(\Gamma,S)) = 2$, then for $n_1, n_2 \ge 7$,

$$\chi(L(\Gamma_{n_1,n_2}, S_{n_1,n_2})) = \chi(L(\Gamma_{n_1}, S_{n_1})) - 1 = \chi(L(\Gamma,S)) - 2 = 0.$$

Therefore, the diagonal subsequence $\{(\Gamma_{n,n}, S_{n,n})\}_{n\ge 7}$ satisfies the properties in Corollary 4.3. Repeating this procedure until the Euler characteristic vanishes completes the proof. □

Let (Γ, S) be a nonspherical, nonaffine marked Coxeter system of dimension at most 2 with $\chi(L(\Gamma,S)) \ge 1$. For simplicity, we write χ instead of $\chi(L(\Gamma,S))$.

We denote by N (resp. E) the number of vertices (resp. edges) of the presentation diagram $X(\Gamma,S)$. It follows that $N - E = \chi \ge 1$. Suppose that the set of labels of the edges of $X(\Gamma,S)$ is $\{k_1,\dots,k_r\}$. Let us write E_i for the number of edges of $X(\Gamma,S)$ labeled by k_i, so $E = E_1 + \cdots + E_r$.

We obtain the equality

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = 1 - \frac{N}{[2]} + \sum_{i=1}^{r} \frac{E_i}{[2,k_i]} = 1 - \frac{E+\chi}{[2]} + \sum_{i=1}^{r} \frac{E_i}{[2,k_i]}$$

by Steinberg's formula (2-1); see also [Floyd 1992, page 479]. Therefore,

$$\begin{aligned}\frac{1}{f_{(\Gamma,S)}(z^{-1})} &= \frac{[2,k_1,\dots,k_r] - (E+\chi)[k_1,\dots,k_r] + \sum_{i=1}^r E_i[k_1,\dots,\hat{k}_i,\dots,k_r]}{[2,k_1,\dots,k_r]}\\ &= \frac{[2,k_1,\dots,k_r] + \sum_{i=1}^r E_i(1-[k_i])[k_1,\dots,\hat{k}_i,\dots,k_r] - \chi[k_1,\dots,k_r]}{[2,k_1,\dots,k_r]}\\ &= \frac{[2,k_1,\dots,k_r] - \sum_{i=1}^r E_i z[k_i-1][k_1,\dots,\hat{k}_i,\dots,k_r] - \chi[k_1,\dots,k_r]}{[2,k_1,\dots,k_r]}\\ &= \frac{[2,k_1,\dots,k_r] - \sum_{i=1}^r E_i z[k_1,\dots,k_i-1,\dots,k_r] - \chi[k_1,\dots,k_r]}{[2,k_1,\dots,k_r]}.\end{aligned}$$

If $\chi = 1$, then

$$\begin{aligned}\frac{1}{f_{(\Gamma,S)}(z^{-1})} &= \frac{([2]-1)[k_1,\dots,k_r] - \sum_{i=1}^r E_i z[k_1,\dots,k_i-1,\dots,k_r]}{[2,k_1,\dots,k_r]}\\ &= \frac{z\big([k_1,\dots,k_r] - \sum_{i=1}^r E_i[k_1,\dots,k_i-1,\dots,k_r]\big)}{[2,k_1,\dots,k_r]}.\end{aligned}$$

We define the polynomial $P(z)$ as

$$P(z) = \begin{cases}[k_1,\dots,k_r] - \sum_{i=1}^r E_i[k_1,\dots,k_i-1,\dots,k_r] & \text{if } \chi = 1,\\ [2,k_1,\dots,k_r] - \sum_{i=1}^r E_i z[k_1,\dots,k_i-1,\dots,k_r] - \chi[k_1,\dots,k_r] & \text{if } \chi \ge 2.\end{cases}$$

It follows that

$$\frac{1}{f_{(\Gamma,S)}(z^{-1})} = \begin{cases} zP(z)/[2,k_1,\dots,k_r] & \text{if } \chi = 1,\\ P(z)/[2,k_1,\dots,k_r] & \text{if } \chi \ge 2.\end{cases}$$

In order to show that $P(z)$ is a product of cyclotomic polynomials and exactly one Pisot polynomial, we use the following; see [Floyd 1992].

Lemma 4.4 [Floyd 1992, Lemma 1] *Let $P(z)$ be a monic polynomial with integer coefficients. We denote the reciprocal polynomial of $P(z)$ by $\widetilde{P}(z)$; that is, $\widetilde{P}(z) = z^{\deg P} P(z^{-1})$. Suppose that $P(z)$ satisfies*

(i) $P(0) \neq 0$ *and* $P(1) < 0$;

(ii) $P(z) \neq \widetilde{P}(z)$;

(iii) *for sufficiently large integer m, $(z^m P(z) - \widetilde{P}(z))/(z-1)$ is a product of cyclotomic polynomials and exactly one Salem polynomial.*

Then the polynomial $P(z)$ is a product of cyclotomic polynomials and exactly one Pisot polynomial.

Theorem 4.5 *Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2 with $\chi(L(\Gamma, S)) \geq 1$. Then the growth rate $\tau(\Gamma, S)$ is a Pisot number.*

Proof Assume that (Γ, S) of rank N satisfies the hypothesis of the theorem. Since (Γ, S) is a nonspherical, nonaffine Coxeter system, we have that $N \geq 3$. If the presentation diagram $X(\Gamma, S)$ has no edges, the growth rate $\tau(\Gamma, S) = N - 1 \geq 2$ is a Pisot number; see Example 2.5. From now on, we assume that the presentation diagram $X(\Gamma, S)$ has at least one edge. Denote by $E \geq 1$ the number of edges of $X(\Gamma, S)$.

Considering Remark 4.2, we divide the proof into two cases: the presentation diagram $X(\Gamma, S)$ is a tree all of whose edges are labeled by 2, and otherwise.

In the first case, we have $E = N - 1$. Without loss of generality, we can assume that $N \geq 4$ since (Γ, S) is nonaffine. Therefore, by Example 2.6, the growth rate $\tau(\Gamma, S) = N - 2 \geq 2$ is a Pisot number.

In the other case, either the presentation diagram $X(\Gamma, S)$ is disconnected or it has an edge labeled by $k \geq 3$. We fix an ordering of the generating set S. Let us take a sequence of marked Coxeter systems $\{(\Gamma_n, S_n)\}_{n \geq 7}$ of rank N as in Corollary 4.3. It follows from property (3) that the number of edges of $X(\Gamma_n, S_n)$ equals $E + \chi(L(\Gamma, S))$. In particular, for every $n \geq 7$ different from $k_1, \dots, k_r$, the number of edges of $X(\Gamma_n, S_n)$ labeled by n is equal to $\chi(L(\Gamma, S))$. For simplicity, we write χ instead of $\chi(L(\Gamma, S))$. By Steinberg's formula (2-1),

$$\frac{1}{f_{(\Gamma_n, S_n)}(z^{-1})} = 1 - \frac{N}{[2]} + \sum_{i=1}^{r} \frac{E_i}{[2, k_i]} + \frac{\chi}{[2, n]} = \frac{P_n(z)}{[2, k_1, \dots, k_r, n]},$$

where

$$P_n(z) = [2, k_1, \dots, k_r, n] - N[k_1, \dots, k_r, n] + \sum_{i=1}^{r} E_i[k_1, \dots, \hat{k}_i, \dots, k_r, n] + \chi[k_1, \dots, k_r].$$

From the equality $N = E_1 + \dots + E_r + \chi$, we obtain that

$$P_n(z) = [2, k_1, \dots, k_r, n] - \sum_{i=1}^{r} E_i z[k_1, \dots, k_i - 1, \dots, k_r, n] - \chi z[k_1, \dots, k_r, n-1].$$

Define the polynomials $P(z)$ as

$$P(z) = \begin{cases} [k_1, \dots, k_r] - \sum_{i=1}^{r} E_i [k_1, \dots, k_i - 1, \dots, k_r] & \text{if } \chi = 1, \\ [2, k_1, \dots, k_r] - \sum_{i=1}^{r} E_i z [k_1, \dots, k_i - 1, \dots, k_r] - \chi [k_1, \dots, k_r] & \text{if } \chi \geq 2, \end{cases}$$

and $\widetilde{P}(z) = z^{\deg P} P(z^{-1})$. Then

$$(z-1) P_n(z) = \begin{cases} z^{n+1} P(z) - \widetilde{P}(z) & \text{if } \chi = 1, \\ z^n P(z) - \widetilde{P}(z) & \text{if } \chi \geq 2. \end{cases}$$

Since $\chi(L(\Gamma_n, S_n)) = 0$, by Lemma 3.3, the polynomial $P_n(z)$ is a product of cyclotomic polynomials and exactly one Salem polynomial. In order to apply Lemma 4.4 to $P(z)$, we need to show that $P(0) \neq 0$, $P(1) < 0$, and that $P(z)$ is not reciprocal. First,

$$P(0) = \begin{cases} 1 - E & \text{if } \chi = 1, \\ 1 - \chi & \text{if } \chi \geq 2. \end{cases}$$

It follows that $P(0) \neq 0$. Since $P(z)$ is monic, we also conclude that $P(z)$ is not reciprocal. Finally, we see that $P(1) < 0$ as follows.

In the case $\chi = 1$,

$$P(1) = \prod_{i=1}^{r} k_i - \sum_{i=1}^{r} \left(E_i \cdot \prod_{j=1}^{r} k_j \cdot \frac{k_i - 1}{k_i} \right) = \prod_{i=1}^{r} k_i \cdot \left\{ 1 - \sum_{i=1}^{r} E_i \left(1 - \frac{1}{k_i} \right) \right\}.$$

If $N \geq 4$, then

$$\sum_{i=1}^{r} E_i \left(1 - \frac{1}{k_i} \right) \geq \sum_{i=1}^{r} E_i \left(1 - \frac{1}{2} \right) = \frac{E}{2} = \frac{N-1}{2} \geq \frac{3}{2} > 1.$$

It follows that $P(1) < 0$ from

$$1 - \sum_{i=1}^{r} E_i \left(1 - \frac{1}{k_i} \right) < 0.$$

For $N = 3$, the presentation diagram is made of two edges with labels k_1 and k_2. We necessarily have $k_1 \geq 3$ or $k_2 \geq 3$, so

$$1 - \left(1 - \frac{1}{k_1} \right) - \left(1 - \frac{1}{k_2} \right) \leq 1 - \frac{1}{2} - \frac{2}{3} = -\frac{1}{6} < 0.$$

Hence $P(1) < 0$.

In the case $\chi \geq 2$,

$$P(1) = 2 \prod_{i=1}^{r} k_i - \sum_{i=1}^{r} \left(E_i \cdot \prod_{j=1}^{r} k_j \cdot \frac{k_i - 1}{k_i} \right) - \chi \prod_{i=1}^{r} k_i = \prod_{i=1}^{r} k_i \cdot \left\{ 2 - \chi - \sum_{i=1}^{r} E_i \left(1 - \frac{1}{k_i} \right) \right\}.$$

Since $X(\Gamma, S)$ has at least one edge,

$$P(1) < \prod_{i=1}^{r} k_i \cdot (2 - \chi) \leq 0.$$

By Lemma 4.4, the polynomial $P(z)$ is a product of cyclotomic polynomials and exactly one Pisot polynomial, and hence the growth rate $\tau(\Gamma, S)$ is a Pisot number. □

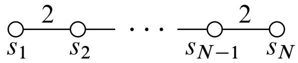

Figure 9: The presentation diagram $X(\widehat{\Gamma}, \widehat{S})$.

Theorem 4.6 *Let (Γ, S) be a nonspherical, nonaffine Coxeter system of dimension at most 2 with $\chi(L(\Gamma, S)) \geq 1$. Then, there exists a sequence of Coxeter systems (Γ_n, S_n) of dimension at most 2 with vanishing Euler characteristic such that the growth rate $\tau(\Gamma_n, S_n)$ converges to $\tau(\Gamma, S)$ from below.*

Proof We denote by N the rank of (Γ, S). As in the proof of Theorem 4.5, we divide the proof into two cases: either the presentation diagram $X(\Gamma, S)$ is disconnected or has an edge labeled by $k \geq 3$, and otherwise.

In the first case, we fix an ordering of S and we take a sequence of marked Coxeter systems $\{(\Gamma_n, S_n)\}_{n \geq 7}$ of rank N as in Corollary 4.3. By combining Theorems 2.7, 2.8, and 3.2, we conclude that the growth rate $\tau(\Gamma_n, S_n)$ is a Salem number and the sequence $\{\tau(\Gamma_n, S_n)\}_{n \geq 7}$ converges to $\tau(\Gamma, S)$ from below.

In the other case, by Remark 4.2, the presentation diagram $X(\Gamma, S)$ is a tree with all edges labeled by 2. Since (Γ, S) is nonspherical and nonaffine, it forces $N \geq 4$. It was shown in Example 2.6 that the growth rate of (Γ, S) does not depend on the isomorphism type of the tree, only on the number of its vertices, and that $\tau(\Gamma, S) = N - 2 \geq 2$.

Consider the marked Coxeter system $(\widehat{\Gamma}, \widehat{S})$ of rank N whose presentation diagram $X(\widehat{\Gamma}, \widehat{S})$ is depicted in Figure 9.

Let (Γ_n, S_n) be the marked Coxeter system of rank N whose presentation diagram $X(\Gamma_n, S_n)$ is obtained by adding an edge labeled by $n \geq 3$ between s_1 and s_N. As a direct consequence, (Γ_n, S_n) converges to $(\widehat{\Gamma}, \widehat{S})$ in the space of marked Coxeter systems $\mathscr{C}_N$ of rank N. Since $\tau(\widehat{\Gamma}, \widehat{S}) = \tau(\Gamma, S)$, by combining Theorems 2.7, 2.8, and 3.2, the assertion follows. □

Remark 4.7 We mention that for hyperbolic groups, Fujiwara and Sela [2023] have studied the convergence properties of growth rates with respect to all their finite generating sets; see also [Yukita 2024]. However, they did not characterize the arithmetic nature of growth rates.

5 Examples for the growth rates of Coxeter systems with negative Euler characteristic

In this section, we consider Coxeter systems of dimension at most 2 with negative Euler characteristic. We provide some infinite sequences of such Coxeter systems, and prove by a classical approach that their growth rates are Perron numbers; see also Remark 5.2.

Let $(\Gamma_\star, S_\star)$ be the Coxeter system with presentation diagram depicted in Figure 10. As discussed in Example 2.4, the radius of convergence of its growth series is given by $r_\star = 1/\tau(\Gamma_\star, S_\star) = (\varphi - 1)^2$, where φ is the golden ratio.

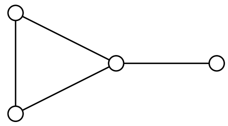

Figure 10: The presentation diagram of $(\Gamma_\star, S_\star)$.

For all the Coxeter systems (Γ, S) discussed below, we assume that $(\Gamma_\star, S_\star) \preceq (\Gamma, S)$; see Section 2.3.

We provide examples in terms of the underlying graphs of their presentation diagrams; see Figure 11. For terminology, we refer to [Gallian 1998]. Such Coxeter systems all satisfy $\chi \le -1$. For instance, the family of *wheel graphs* W_N, for all $N \ge 6$, formed by a cycle of length $N-1$ and a universal vertex, that is, a central vertex linked to each other vertex. In that case the number of edges of the graph is given by $E = 2(N-1)$. The same goes for the *windmill graphs* of type $\mathcal{W}(4, l)$, with $l \ge 2$, made of l copies of complete graphs K_4 joined at common central vertex. The family of *friendship graphs* $F_l = \mathcal{W}(3, l)$ for $l \ge 3$ satisfies $E = \frac{3}{2}(N-1)$. Several variations of those graphs can be constructed. For example, we defined the *triangulated bouquet* $\mathcal{T}(c, l)$ as the graph formed by l copies of c–cycles glued in a common vertex v, such that any other vertex is linked to v. In this case, v is universal and one has

$$E = \frac{2c-1}{c-1}(N-1).$$

Proposition 5.1 *Let $(\Gamma_{k,N}, S)$ be a nonspherical, nonaffine Coxeter system of dimension at most 2 and rank N, such that all edges of the presentation diagram $X(\Gamma_{k,N}, S)$ are labeled by the same $k \ge 3$. Denote by E the number of edges of $X(\Gamma_{k,N}, S)$.*

If $(\Gamma_{k,N}, S)$ satisfies that

(i) $(\Gamma_\star, S_\star) \preceq (\Gamma_{k,N}, S)$,

(ii) $E = a(N-1)$ *for a rational number* $1 < a \le \frac{1}{3}(1+\varphi)^2$,

then the growth rate $\tau(\Gamma_{k,N}, S)$ is a Perron number.

Proof We give an outline of the proof, which is classical, and omit details. Assume that $(\Gamma_{k,N}, S)$ satisfies the hypothesis of Proposition 5.1. In what follows, we denote by $f_{k,N}(z) = Q_{k,N}(z)/P_{k,N}(z)$ the growth series of $(\Gamma_{k,N}, S)$, by $r_{k,N}$ its radius of convergence, and by $\tau_{k,N}$ the growth rate of $(\Gamma_{k,N}, S)$. Recall that $r_{k,N}$ is the smallest positive real root of $P_{k,N}(z)$.

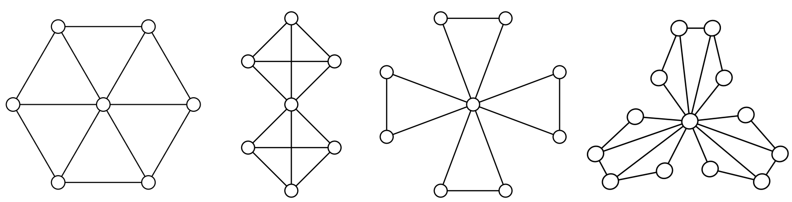

Figure 11: The graphs W_7, $\mathcal{W}(4,2)$, $F_4 = \mathcal{W}(3,4)$, and $\mathcal{T}(5,3)$.

By Steinberg's formula (2-1),

$$\frac{1}{f_{k,N}(z^{-1})} = 1 - \frac{N}{[2]} + \frac{a(N-1)}{[2,k]} = \frac{[2,k] - N[k] + a(N-1)}{[2,k]}. \tag{5-1}$$

Therefore, the denominator of $f_{k,N}(z)$ is given by

$$P_{k,N}(z) = 1 + (2-N)(z + z^2 + \cdots + z^{k-1}) + (a-1)(N-1)z^k = h_N(z) + R_{k,N}(z),$$

where $h_N(z)$ is the quadratic polynomial $h_N(z) = 1 + (2-N)(z+z^2)$ and $R_{k,N}(z)$ is the remaining part.

By hypothesis, $(\Gamma_\star, S_\star) \preceq (\Gamma_{k,N}, S)$; therefore by Theorem 2.8, we conclude that $\tau(\Gamma_\star, S_\star) \leq \tau_{k,N}$. It follows that the associated radii of convergence all satisfy $r_{k,N} \leq r_\star$. In order to prove that $r_{k,N}$ is the unique root with smallest modulus of $P_{k,N}(z)$, we use Rouché's theorem on the open disk $D(0, r_\star)$. We first observe that $h_N(z)$ has a unique root in $D(0, r_\star)$, and we prove $|h_N(z)| - |R_{k,N}| > 0$ on $|z| = r_\star$.

We assume that $N \geq 9$; the case where $N \leq 8$ can be done by applying similar reasoning. An easy analysis of the roots shows that for any N, $h_N(z)$ admits a unique root in the open disk $D(0, r_\star)$. Moreover, on the circle $|z| = r_\star$, one has

$$|h_N(z)| \geq |1 + (2-N)(r_\star^2 - r_\star)|. \tag{5-2}$$

Let z be such that $|z| = r_\star$, and put $\Delta_{k,N}(z) = |h_{k,N}(z)| - |R_{k,N}(z)|$. Since $a > 1$, by the triangle inequality,

$$|R_{k,N}(z)| \leq (N-2)(r_\star^3 + \cdots + r_\star^{k-1}) + (a-1)(N-1)r_\star^k.$$

Also, by (5-2), one has $|h_N(z)| \geq |1 + (2-N)(r_\star^2 - r_\star)| \geq 1 + (2-N)(r_\star^2 - r_\star)$. It follows that

$$\Delta_{k,N}(z) \geq N\left(1 + 2r - \frac{1-r_\star^k}{1-r_\star} - (a-1)r_\star^k\right) - 3 - 4r_\star + 2\frac{1-r_\star^k}{1-r_\star} + (a-1)r_\star^k.$$

By analysis of each term, one can prove that $\Lambda_{k,N}$ increases with respect to N for all $k \geq 3$, and that $\Lambda_{k,N}$ decreases with respect to k for all $N \geq 9$. Therefore,

$$\Delta_{k,N}(z) \geq \lim_{k\to\infty} \Lambda_{k,N} = N\left(1 + 2r_\star - \frac{1}{1-r_\star}\right) - 3 - 4r_\star + \frac{2}{1-r_\star}.$$

We obtain that $\Delta_{k,N}(z) > 0$ when

$$N > \frac{3 + 4r_\star - 2\frac{1}{1-r_\star}}{1 + 2r_\star - \frac{1}{1-r_\star}} = \frac{11 + \sqrt{45}}{2}.$$

This is true for any $N \geq 9$, which finishes the proof. □

Remark 5.2 A Coxeter system is said to be ∞–spanned if there exists a spanning tree of its Coxeter diagram with edges labeled ∞ only. Kolpakov and Talambutsa [2022] proved that the growth rate of ∞–spanned Coxeter systems are Perron numbers. By the existence of a universal vertex in the presentation diagram of the Coxeter systems discussed above, such a spanning tree cannot be found in the corresponding

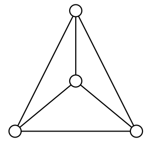

Figure 12: The presentation diagram of (Γ_0, S_0).

Coxeter diagrams. However, the growth series of such a Coxeter system (Γ, S) of dimension 2 at most coincides with the growth series of a ∞–spanned Coxeter system obtained as follows.[1] Construct a complete graph with N vertices, with E of its edges labeled by $k \geq 3$ and the remaining ones by ∞. If a vertex is chosen so that all its emanating edges are labeled by ∞, the resulting graph encodes a ∞–spanned Coxeter system whose growth series equals the original growth series.

In Theorems 3.2 and 4.5, we proved that growth rates of Coxeter systems of dimension at most 2 with positive and vanishing Euler characteristic are Salem and Pisot numbers respectively. By Proposition 5.1 and Remark 5.2, the growth rates of infinitely many Coxeter systems with negative Euler characteristic are Perron numbers.

Note that, there exist Coxeter systems of dimension at most 2 such that $\chi \leq -1$ whose growth rates are Perron numbers but *are neither* Pisot numbers nor Salem numbers. For instance, the 3–dimensional hyperbolic *ideal* Coxeter system (Γ_0, S_0) whose presentation diagram admits labels 3 only and is depicted in Figure 12.

Inspired by these observations, we make the following claim.

Conjecture *The growth rate of any Coxeter system of dimension at most* 2 *is a Perron number.*

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