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We study cross ratios from an axiomatic viewpoint, also known as the study of Möbius spaces. We characterise cross ratios induced by quasimetrics in terms of topological properties of their image. Furthermore, we generalise the notions of Cauchy sequences and completeness to Möbius spaces and prove the existence of a unique completion under an extra assumption that, again, can be expressed in terms of the image of the cross ratio.

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1 Introduction

Let Z be a set, ρ a metric on Z, possibly with a point at infinity; see Section 2 for definitions. We can define the cross ratio induced by ρ with the formula

$$\operatorname{cr}(z_1, z_2, z_3, z_4) := \frac{\rho(z_1, z_2)\rho(z_3, z_4)}{\rho(z_1, z_3)\rho(z_2, z_4)}, \tag{1-1}$$

where the quotient of any two infinite distances equals 1, ie infinite distances cancel each other. Provided that no three points in the quadruple $(z_1, \dots, z_4)$ coincide, this yields a well-defined number in $[0, \infty]$.

Cross ratios arise naturally in the study of negatively curved spaces: If X is a CAT(-1) space, we can define its boundary at infinity, which can be endowed with a family of metrics $\{\rho_x\}_{x \in X}$, called visual metrics. It is a classical result by Bourdon that, for a CAT(-1) space, all visual metrics induce the same cross ratio on the boundary. Therefore, the cross ratio provides us with an intrinsic geometric structure on the boundary at infinity. This allows us to think about the pair $(\partial X, \mathrm{cr})$ as a topological space with a geometric structure of its own, which leads to the study of cross ratios from an axiomatic viewpoint; see for example [Hamenstädt 1997; Buyalo 2016]. In this context, cross ratios are also referred to as Möbius structures and a set equipped with a Möbius structure will be called a Möbius space.

Buyalo [2016] showed how Möbius structures give rise to a topology, called Möbius topology. Furthermore, he showed that every Möbius structure is induced by a semimetric, ie every Möbius structure arises from formula (1-1) if ρ is a semimetric, that is, it satisfies the same properties as a metric, except for the triangle inequality. Between semimetrics and metrics there is the notion of a K–quasimetric, which satisfies a weak triangle inequality; see Section 2 for precise definitions. Quasimetrics are of particular interest in the study of cross ratios because of involutions. Given a metric ρ, its involution at a point $o \in Z$ is defined by

$$\rho_o(z, z') = \frac{\rho(z, z')}{\rho(z, o)\rho(o, z')}.$$

A direct computation shows that ρ_o induces the same cross ratio as ρ. However, if ρ is a metric, the map ρ_o may no longer be a metric which leads to technical complications when studying cross ratios purely from a metric point of view. Quasimetrics have the advantage that, given a quasimetric ρ, the involution ρ_o is again a quasimetric; cf Proposition 5.3.6 in [Buyalo and Schroeder 2007]. Quasimetrics are weaker than metrics in many ways. For example, they do not enjoy the same continuity properties as metrics, as we will see in Example 4.5. However, Möbius structures induced by quasimetrics have several nice topological features, which, together with the observation on involutions above, motivates their study.

When studying Möbius structures that appear on boundaries at infinity, there are many results that only require for one to 'roughly' know the Möbius structure. More specifically, a map $f: Z \to Z'$ between metric spaces — which induce cross ratios cr and cr′ — is called a *quasi-Möbius map* if there exists a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ such that for all quadruples Q of distinct points in Z, we have $\mathrm{cr}'(f(Q)) \le \eta(\mathrm{cr}(Q))$. It is called a *quasi-Möbius equivalence* if it is invertible and the inverse is quasi-Möbius as well. There are instances where it is much easier to define a Möbius structure only up to quasi-Möbius equivalence (eg on boundaries of δ–hyperbolic spaces) and, in fact, sometimes we only know how to define the cross ratio up to quasi-Möbius equivalences (at the time of writing, this is the case for Morse boundaries [Charney et al. 2019]). Studying the quasi-Möbius class of a Möbius structure is of interest as the quasi-Möbius class of the Möbius structure on a boundary often characterises the interior space up to quasi-isometry [Paulin 1996; Charney et al. 2019]. If one wishes to determine a (sufficiently) negatively curved space from its boundary more precisely, one needs to utilise a finer structure on the boundary than the quasi-Möbius class. When the Möbius structure can be defined (eg on boundaries of CAT(-1) spaces [Bourdon 1995], Roller boundaries of CAT(0) cube complexes [Beyrer et al. 2021], or boundaries of rank-one Hadamard manifolds [Incerti-Medici 2020]), one can obtain stronger rigidity results, where the Möbius structure on the boundary determines the interior space up to a $(1, C)$–quasi-isometry or even up to isometry; see [Biswas 2015; Beyrer et al. 2021; Incerti-Medici 2020]. For this reason both the Möbius structure and its quasi-Möbius class have become separate objects of interest and study. Some of their properties are shared or analogous, but there are also some notable differences. For example, we will see in Example 4.12 that Theorem A does not hold for the quasi-Möbius class.

In this paper, we put our attention to Möbius structures. We provide a characterisation of those Möbius structures that are induced by quasimetrics in terms of the image of the cross ratio. We then study the Möbius topology introduced by Buyalo and show that, if the cross ratio is induced by a metric, the metric topology and the Möbius topology coincide. Finally, if a Möbius structure is induced by a quasimetric that satisfies an additional symmetry condition, we can define the notion of Cauchy sequences for such a Möbius structure. The main results of this paper are the following:

Theorem A *Let (Z,ρ) be a metric space, M the Möbius structure induced by ρ. Denote the metric topology induced by ρ by $\mathcal{T}_\rho$ and the Möbius topology induced by M by $\mathcal{T}_M$. Then $\mathcal{T}_\rho = \mathcal{T}_M$.*

Theorem B *Let (Z,ρ) be a (possibly extended) metric space and denote the induced Möbius structure by M. The following are equivalent:*

1. *(Z,M) is complete as a Möbius space.*
2. *(Z,ρ) is complete as a metric space and is either bounded or has a point at infinity.*

Theorem C *Let (Z,M) be a Möbius space that satisfies the symmetry condition. Then there exists a complete Möbius space $(\bar{Z},\bar{M})$ with a Möbius embedding $i_Z\colon Z \hookrightarrow \bar{Z}$ such that $i_Z(Z)$ is dense in $\bar{Z}$.*

Furthermore, if (Z',M') is a complete Möbius space with a Möbius embedding $i\colon Z \hookrightarrow Z'$ such that $i(Z)$ is dense in Z', then there exists a unique Möbius equivalence $f\colon \bar{Z} \to Z'$ such that $i = f \circ i_Z$.

The rest of the paper is organised as follows. In Section 2, we give precise definitions for the terminology we will require. In Section 3, we show the characterisation of Möbius structures induced by quasimetrics. In Section 4, we review Buyalo's definition of the Möbius topology and prove Theorem A. In Section 5, we introduce Cauchy sequences and prove Theorem B. In Section 6, we construct the completion and prove Theorem C.

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2 Preliminaries

Let Z be a set, $\rho\colon Z \times Z \to \mathbb{R}$ a map. We say that ρ is a *semimetric* if it is symmetric, nonnegative and $\rho(z,z') = 0$ if and only if $z = z'$. We say that ρ is a *K–quasimetric*, where $K \geq 1$, if it is a semimetric and for all $x,y,z \in Z$, we have $\rho(x,z) \leq K \max(\rho(x,y), \rho(y,z))$. Finally, we say ρ is a *metric* if it is a semimetric and for all $x,y,z \in Z$, we have $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. Generalising the definition of a metric, we say that $\rho\colon Z \times Z \to [0,\infty]$ is an *extended metric* if there exists exactly one point $\omega \in Z$, such that for all $x \in Z \setminus \{\omega\}$, $\rho(x,\omega) = \infty$, $\rho(\omega,\omega) = 0$ and the restriction of ρ to $(Z \setminus \{\omega\}) \times (Z \setminus \{\omega\})$ is a metric. We call ω the *point at infinity* with respect to ρ. A motivating example for this notion is the Riemannian sphere, seen as the union $\mathbb{C} \cup \{\infty\}$. We define the notions of extended semimetrics and extended K–quasimetrics analogously.

We call an n–tuple $(z_1,\dots,z_n) \in Z^n$ *nondegenerate* if and only if for all $i \neq j$, we have $z_i \neq z_j$.

Given a semimetric ρ, we can define a cross ratio. The cross ratio will be defined on admissible quadruples.

Definition 2.1 A quadruple $(z_1,z_2,z_3,z_4) \in Z^4$ is admissible if there exists no triple $i \neq j \neq k \neq i$ such that $z_i = z_j = z_k$. We denote the set of admissible quadruples by $\mathscr{A}$.

We define the cross ratio induced by ρ as follows: for $(z_1,z_2,z_3,z_4) \in \mathscr{A}$,

$$\operatorname{cr}(z_1,z_2,z_3,z_4) := \frac{\rho(z_1,z_2)\rho(z_3,z_4)}{\rho(z_1,z_3)\rho(z_2,z_4)} \in [0,\infty].$$

Admissible quadruples are exactly those quadruples, for which the expression above does not yields division of zero by zero for any permutation of the points z_i.

We also define the cross ratio triple. Write

$$\Delta := \{(a:b:c) \in \mathbb{R}P^2 \mid a,b,c > 0\}, \quad \overline{\Delta} := \Delta \cup \{(0:1:1),(1:0:1),(1:1:0)\}.$$

The cross ratio triple induced by ρ is a map $\operatorname{crt}\colon \mathscr{A} \to \overline{\Delta}$ defined by

$$\operatorname{crt}(z_1,z_2,z_3,z_4) := (\rho(z_1,z_2)\rho(z_3,z_4) : \rho(z_1,z_3)\rho(z_2,z_4) : \rho(z_1,z_4)\rho(z_2,z_3)).$$

Admissible quadruples are exactly those quadruples, for which at most one entry of the cross ratio triple is zero.

We can generalise these definitions to extended semimetrics by using the following convention. Let $\omega \in Z$ be the point at infinity with respect to ρ. Fractions of the form $\rho(\omega,z)/\rho(\omega,z')$ for $z,z' \in Z \setminus \{\omega\}$ can be replaced by 1, based on the principle that "infinite distances cancel each other". In other words, if $z_1,z_2,z_3 \in Z \setminus \{\omega\}$, then

$$\operatorname{cr}(z_1,z_2,z_3,\omega) = \frac{\rho(z_1,z_2)}{\rho(z_1,z_3)}, \quad \operatorname{cr}(z_1,z_2,\omega,\omega) = 0, \quad \operatorname{cr}(z_1,\omega,\omega,z_2) = 1,$$
$$\operatorname{crt}(z_1,z_2,z_3,\omega) = (\rho(z_1,z_2) : \rho(z_1,z_3) : \rho(z_2,z_3)), \quad \operatorname{crt}(z_1,z_2,\omega,\omega) = (0:1:1).$$

It turns out that the maps cr and crt determine each other. If $\operatorname{crt}(z_1,z_2,z_3,z_4) = (a:b:c)$, then $\operatorname{cr}(z_1,z_2,z_3,z_4) = a/b$. On the other hand, if we write

$$\operatorname{cr}(z_1,z_3,z_4,z_2) := \alpha, \quad \operatorname{cr}(z_1,z_4,z_2,z_3) := \beta, \quad \operatorname{cr}(z_1,z_2,z_3,z_4) := \gamma,$$

then

$$\operatorname{crt}(z_1,z_2,z_3,z_4) = (\gamma^{1/3}\beta^{-1/3} : \alpha^{1/3}\gamma^{-1/3} : \beta^{1/3}\alpha^{-1/3}).$$

In order to study the properties of the cross ratio, it is sometimes useful to reformulate the cross ratio in an additive manner. Write

$$\overline{L}_4 := \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\} \cup \{(0,\infty,-\infty),(-\infty,0,\infty),(\infty,-\infty,0)\}.$$

We define the cross difference $M : \mathcal{A} \to \overline{L}_4$ induced by ρ to be

$$M(z_1,z_2,z_3,z_4) := \big(\ln(\mathrm{cr}(z_1,z_3,z_4,z_2)), \ln(\mathrm{cr}(z_1,z_4,z_2,z_3)), \ln(\mathrm{cr}(z_1,z_2,z_3,z_4))\big).$$

The maps M and cr determine each other.

We end this section with a construction that allows us to construct different semimetrics that induce the same cross ratio. Let ρ be an extended semimetric and let $o \in Z$ be a point such that for all $z \neq o$, $\rho(z,o) > 0$. We define the *involution of* ρ *at* o by

$$\rho_o(x,y) := \frac{\rho(x,y)}{\rho(x,o)\rho(o,y)}.$$

Note that o lies at infinity with respect to ρ_o and, if ω is a point at infinity with respect to ρ, then

$$\rho_o(x,\omega) = \frac{1}{\rho(x,o)}.$$

Note that, if ρ was an extended semimetric, then ρ_o is again an extended semimetric. Buyalo and Schroeder [2007, Proposition 5.3.6] prove that for any extended K–quasimetric ρ, its involution ρ_o is a K'^2–quasimetric for some $K' \geq K$. A direct computation shows that ρ and ρ_o induce the same cross ratio.

3 Möbius structures and quasimetrics

Consider the ordered triple $\big((12)(34),(13)(42),(14)(23)\big)$. The symmetric group of four elements $\mathcal{S}_4$ acts on this triple by permuting the numbers 1–4. Whenever $\sigma \in \mathcal{S}_4$ acts on the numbers, it induces a permutation on the triple. Define $\varphi(\sigma) \in \mathcal{S}_3$ to be the permutation on the triple induced by the action of σ. It is easy to check that $\varphi : \mathcal{S}_4 \to \mathcal{S}_3$ is a group homomorphism. One can interpret the expression $(12)(34)$ as denoting two opposite edges of a tetrahedron whose corners are labelled by the numbers 1–4. In this interpretation, φ is the group homomorphism that sends a permutation of the corners to the induced permutation of pairs of opposite edges.

Let Z be a set with at least three points. For any semimetric, denote its set of admissible quadruples by $\mathcal{A}$ (recall that all semimetrics have the same admissible quadruples). We can now define a cross ratio axiomatically.

Definition 3.1 Let Z be a set with at least three points. A map $M : \mathcal{A} \to \overline{L}_4$ is called a *Möbius structure* if and only if it satisfies the following conditions:

(1) For all $P \in \mathcal{A}$ and all $\pi \in \mathcal{S}_4$, we have

$$M(\pi P) = \mathrm{sgn}(\pi)\varphi(\pi)M(P).$$

(2) For $P \in \mathscr{A}$, $M(P) \in L_4$ if and only if P is nondegenerate.

(3) For $P = (x, x, y, z)$, we have $M(P) = (0, \infty, -\infty)$.

(4) Let $(x, y, \omega, \alpha, \beta)$ be an admissible 5–tuple $(x, y, \omega, \alpha, \beta)$ such that (ω, α, β) is a nondegenerate triple, $\alpha \neq x \neq \beta$ and $\alpha \neq y \neq \beta$. Then there exists some $\lambda = \lambda(x, y, \omega, \alpha, \beta) \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$M(\alpha x \omega \beta) + M(\alpha \omega y \beta) - M(\alpha x y \beta) = (\lambda, -\lambda, 0).$$

Moreover, when (ω, α, β) is nondegenerate, $x \neq \beta$ and $y \neq \alpha$, the first component of the left-hand side expression is well defined. Analogously, the second component of the left-hand side expression is well defined when (ω, α, β) is nondegenerate, $x \neq \alpha$ and $y \neq \beta$.

The pair (Z, M) is called a *Möbius space*.

Given M, we obtain a map $\mathrm{cr}\colon \mathscr{A} \to [0, \infty]$ and a map $\mathrm{crt}\colon \mathscr{A} \to \overline{\Delta}$ using the formulas from Section 2. Abusing notation, we will also refer to (Z, cr) and (Z, crt) as Möbius spaces.

It is a straightforward computation to show that for any semimetric ρ, the induced cross difference M is a Möbius structure. Buyalo [2016] proved that the converse is true as well: Every Möbius structure is the cross difference of a semimetric. We also have a characterisation of Möbius structures that are induced by quasimetrics.

Definition 3.2 Let Z be a set with at least three points. A map $M\colon \mathscr{A} \to \overline{L}_4$ is called a *strong Möbius structure* if it is a Möbius structure and the induced map crt satisfies the following condition:

Corner There exist open neighbourhoods of $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$, such that the image of crt doesn't intersect these neighbourhoods.

The remainder of this section is devoted to proving the following result.

Proposition 3.3 *Let (Z, M) be a Möbius structure. There exists an extended quasimetric ρ inducing M if and only if M is a strong Möbius structure.*

Furthermore, whenever there exists an extended K–quasimetric inducing M, there exists a bounded K^2–quasimetric inducing M.

We begin by proving that quasimetrics induce strong Möbius structures.

Lemma 3.4 *Let Z be a set, ρ a quasimetric on Z and* crt *the cross ratio induced by ρ. Then* crt *satisfies the corner condition and, therefore, the induced cross difference M is a strong Möbius structure.*

Proof Let ρ be a K–quasimetric on Z, M the induced Möbius structure and crt the induced cross ratio triple. Let (w, x, y, z) be an admissible quadruple. We want to show that $\mathrm{crt}(w, x, y, z)$ cannot be close to any of the three corner points. We will show this for the corner point $(0:0:1)$. The others work analogously.

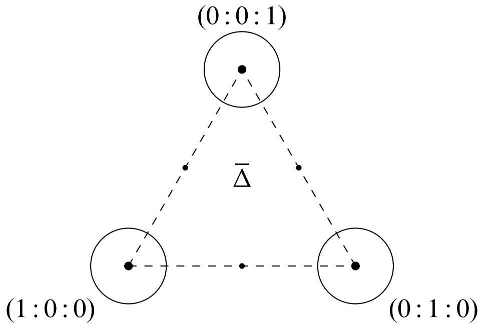

Figure 1: A Möbius structure crt satisfies the corner condition if and only if we can find open neighbourhoods, as depicted above, such that the image of crt in $\overline{\Delta}$ doesn't intersect these neighbourhoods.

In order for the point $\mathrm{crt}(w,x,y,z)$ to be close to $(0:0:1)$, the ratio between the first and third component has to be small, as does the ratio between the second and the third component. We will show that this cannot happen. To prove this, we need to make several case distinctions. We leave it to the reader to check that all cases can be handled analogously by simply permuting the roles and properties of w,x,y,z.

Let $\epsilon > 0$. Consider $\mathrm{crt}(w,x,y,z) = (\rho(w,x)\rho(y,z) : \rho(w,y)\rho(x,z) : \rho(w,z)\rho(x,y))$ and suppose

$$\max(\rho(w,x)\rho(y,z), \rho(w,y)\rho(x,z)) = \epsilon.$$

We want to bound $\rho(w,z)\rho(x,y)$ in terms of ϵ, proving that the ratios

$$\frac{\rho(w,x)\rho(y,z)}{\rho(w,z)\rho(x,y)} \quad \text{and} \quad \frac{\rho(w,y)\rho(x,z)}{\rho(w,z)\rho(x,y)}$$

cannot become too small. Assume without loss of generality that

$$\rho(w,x) \le \rho(y,z) \quad \text{and} \quad \rho(w,y) \le \rho(x,z),$$

and thus

$$\rho(w,x) \le \sqrt{\epsilon} \quad \text{and} \quad \rho(w,y) \le \sqrt{\epsilon}.$$

Since ρ is a K–quasimetric, we have

$$\rho(x,y) \le K \max(\rho(w,x), \rho(w,y)).$$

Swapping x and y if necessary (which does not change any of the inequalities obtained above), we may assume without loss of generality that $\rho(w,x) \ge \rho(w,y)$, and hence

$$\rho(x,y) \le K\rho(w,x).$$

Further, we have

$$\rho(z,w) \le K \max(\rho(z,y), \rho(y,w)).$$

We now combine the inequalities above, distinguishing between two cases. If $\rho(z,y) \ge \rho(y,w)$, then

$$\rho(x,y)\rho(z,w) \le K^2\rho(w,x)\rho(z,y) \le K^2\epsilon,$$

as $\epsilon = \max(\rho(w,x)\rho(y,z), \rho(w,y)\rho(x,z))$. If $\rho(z,y) < \rho(y,w)$, then we use the previously obtained inequalities $\rho(w,x), \rho(w,y) \leq \sqrt{\epsilon}$ to estimate

$$\rho(x,y)\rho(z,w) \leq K^2\rho(w,x)\rho(y,w) \leq K^2\epsilon.$$

We see that, in either case, $\rho(x,y)\rho(z,w) \leq K^2\epsilon$. We use this to show that crt stays away from the corner points. Consider the triple

$$(a,b,c) := (\rho(w,x)\rho(y,z), \rho(w,y)\rho(x,z), \rho(w,z)\rho(x,y)) \in \mathbb{R}^3.$$

The argument above shows that

$$c \leq K^2 \max(a,b).$$

Projecting (a,b,c) to projective space, this implies that

$$(a:b:c) \notin \left\{ (a':b':1) \in \mathbb{R}P^2 \;\middle|\; a' < \frac{1}{K^2}, b' < \frac{1}{K^2} \right\},$$

which is an open neighbourhood of $(0:0:1)$ in $\mathbb{R}P^2$. Since $(a:b:c) = \operatorname{crt}(w,x,y,z)$, we found an open neighbourhood of $(0:0:1)$ that doesn't intersect with Im(crt). Using analogous arguments, we find neighbourhoods of $(1:0:0)$ and $(0:1:0)$ that don't intersect with Im(crt). This completes the proof. □

The other direction of the characterisation is based on the following result.

Lemma 3.5 *Let ρ be a semimetric on the set Z such that ρ has a point at infinity. Then ρ is a quasimetric if and only if its induced Möbius structure is a strong Möbius structure.*

Proof Lemma 3.4 immediately implies one direction of the proof. Suppose now crt satisfies the corner condition. We want to show that ρ is a quasimetric.

Denote the point at infinity with respect to ρ by ω. Let $x,y,z \in Z$. If two of the points are the same, or if one of the three points equals ω, then the inequality for quasimetrics is immediately satisfied. So assume x,y,z are mutually different and different from ω. Then (x,y,z,ω) is a nondegenerate quadruple and we can look at the cross ratio triple

$$\operatorname{crt}(x,y,z,\omega) = (\rho(x,y) : \rho(x,z) : \rho(y,z)).$$

Since crt satisfies the corner condition, we know that there is an open neighbourhood of $(1:0:0)$, independent of x, y and z, such that $\operatorname{crt}(x,y,z,\omega)$ doesn't lie within that neighbourhood. We find $\epsilon > 0$ such that $\operatorname{crt}(x,y,z,\omega) \notin N_\epsilon$, where

$$N_\epsilon := \{(1:b:c) \mid b,c \in (-\epsilon,\epsilon)\}.$$

This implies that

$$\max\left(\frac{\rho(x,z)}{\rho(x,y)}, \frac{\rho(y,z)}{\rho(x,y)}\right) \geq \epsilon,$$

or, equivalently,

$$\frac{1}{\epsilon}\max(\rho(x,z), \rho(z,y)) \geq \rho(x,y).$$

Thus, ρ is a $(1/\epsilon)$–quasimetric. □

Lemmas 3.4 and 3.5 together with Buyalo's result that every Möbius structure is induced by a semimetric prove the first part of Proposition 3.3. We are left to prove the second part.

Proof of Proposition 3.3 Let ρ be a K–quasimetric on Z with a point at infinity. Denote the point at infinity by ω. Choose a base point $o \in Z$. Now define, for all $x, y, z \in Z$,

$$\lambda(z) := \max(1, \rho(z, o)) \quad \text{and} \quad \widetilde{\rho}(x, y) := \frac{\rho(x, y)}{\lambda(x)\lambda(y)}.$$

By Proposition 5.3.6 from [Buyalo and Schroeder 2007], $\widetilde{\rho}$ is a K'^2–quasimetric for some $K' \geq K$. Furthermore,

$$\widetilde{\rho}(x, y) = \frac{\rho(x, y)}{\lambda(x)\lambda(y)} \leq K \frac{\max(\rho(x, o), \rho(o, y))}{\lambda(x)\lambda(y)} \leq K,$$

and thus, $\widetilde{\rho}$ is a bounded quasimetric on Z. A straightforward computation shows that ρ and $\widetilde{\rho}$ induce the same cross ratio and therefore, the same M. □

4 The Möbius topology

Let (Z, M) be a Möbius space. In order to construct a topology on Z, we will recall Buyalo's construction of a family of extended semimetrics, each of which induces M. We will then use those semimetrics to define a topology.

Since $M(w, x, y, z) \in \overline{L}_4$ is a triple, we write $M = (a, b, c)$, where $a, b, c : \mathscr{A} \to [-\infty, \infty]$ are the components of M. Condition (4) in the definition of Möbius structures now implies that for all nondegenerate triples (ω, α, β) and $x, y \in Z \setminus \{\omega\}$, we have

$$a(\alpha, x, \omega, \beta) + a(\alpha, \omega, y, \beta) - a(\alpha, x, y, \beta) = b(\alpha, x, y, \beta) - b(\alpha, x, \omega, \beta) - b(\alpha, \omega, y, \beta).$$

Therefore, writing $A := (\omega, \alpha, \beta)$, we can define

$$\rho_A(x, y) := \begin{cases} 0 & \text{if } x = y, \\ e^{a(\alpha,x,\omega,\beta)+a(\alpha,\omega,y,\beta)-a(\alpha,x,y,\beta)} & \text{if } x \neq \beta \text{ and } y \neq \alpha, \\ e^{b(\alpha,x,y,\beta)-b(\alpha,x,\omega,\beta)-b(\alpha,\omega,y,\beta)} & \text{if } x \neq \alpha \text{ and } y \neq \beta. \end{cases}$$

Buyalo [2016] proved the following properties of ρ_A.

Theorem 4.1 [Buyalo 2016] *Let (Z, M) be a Möbius space, and ρ_A the map induced by A for any nondegenerate triple A in Z. Let M_A be the cross difference induced by ρ_A. Then the following hold:*

1. *Every ρ_A is an extended semimetric on Z, ie ρ_A is symmetric, nonnegative and nondegenerate.*
2. *For all $x \neq \omega$, $\rho_{(\omega,\alpha,\beta)}(x, \omega) = \infty$. Moreover, $\rho_{(\omega,\alpha,\beta)}(\alpha, \beta) = 1$.*
3. *Let $A = (\omega, \alpha, \beta)$, $A' = (\omega, \beta, \alpha)$, $A'' = (\beta, \alpha, \omega)$. Then*

$$\rho_A = \rho_{A'} \quad \text{and} \quad \rho_{A''}(x, y) = \frac{\rho_A(x, y)}{\rho_A(x, \beta)\rho_A(\beta, y)}.$$

(4) *Let* $(\omega, \alpha, \beta, b)$ *be a nondegenerate quadruple in* Z*. Then* $\rho_{(\omega,\alpha,\beta)} = \lambda \rho_{(\omega,\alpha,b)}$ *for some constant* $\lambda > 0$.

(5) *For each nondegenerate triple* A, $M_A = M$.

The following result is a straightforward computation.

Lemma 4.2 *If* M *is induced by a semimetric* ρ*, then for every nondegenerate triple* A *and for all* $x \neq y$,

$$\rho_A(x, y) = \frac{\rho(x, y)}{\rho(x, \omega)\rho(\omega, y)} \frac{\rho(\alpha, \omega)\rho(\omega, \beta)}{\rho(\alpha, \beta)}.$$

Proof Let A be a nondegenerate triple and let $x, y \in Z$. Suppose, $x \neq \beta$ and $y \neq \alpha$. Then

$$\begin{aligned} \rho_A(x, y) &= e^{a(\alpha,x,\omega,\beta)+a(\alpha,\omega,y,\beta)-a(\alpha,x,y,\beta)} \\ &= \operatorname{cr}(\alpha, \omega, \beta, x) \cdot \operatorname{cr}(\alpha, y, \beta, \omega) \cdot \operatorname{cr}(\alpha, y, \beta, x)^{-1} \\ &= \frac{\rho(\alpha, \omega)\rho(\beta, x)\rho(\alpha, y)\rho(\beta, \omega)\rho(\alpha, \beta)\rho(x, y)}{\rho(\alpha, \beta)\rho(\omega, x)\rho(\alpha, \beta)\rho(y, \omega)\rho(\alpha, y)\rho(\beta, x)} \\ &= \frac{\rho(x, y)}{\rho(x, \omega)\rho(\omega, y)} \frac{\rho(\alpha, \omega)\rho(\omega, \beta)}{\rho(\alpha, \beta)}. \end{aligned}$$

The case when $x \neq \alpha$ and $y \neq \beta$ is analogous. □

We see that $\{\rho_A\}_A$ is a family of extended semimetrics that can be constructed from a Möbius structure M. In [Buyalo 2016], these semimetrics are used to define the following topology.

Let $A = (\omega, \alpha, \beta)$ be a nondegenerate triple. For $y \in Z \setminus \{\omega\}$ and $r > 0$, define

$$B_{A,r}(y) := \{x \in Z \mid \rho_A(x, y) < r\}$$

to be the *open ball* around y of radius r with respect to ρ_A. We take the family of all open balls for all nondegenerate triples A, all positive radii r and all points $y \in Z \setminus \{\omega\}$ as a subbasis to define a topology $\mathcal{T}_M$ on Z. This is the *topology on* Z *induced by* M. From now on, whenever we speak of a Möbius space (Z, M), we assume it to be endowed with the topology induced by M, unless stated otherwise.

Lemma 4.3 *Consider* $[0, \infty]$ *with the topology where open neighbourhoods of* ∞ *are complements of compact sets in* $[0, \infty)$ *and open neighbourhoods of other points are just the standard euclidean open neighbourhoods. Let* (Z, M) *be a Möbius space,* A *a nondegenerate triple in* Z *and* $y \in Z$*. Then the maps* $\rho_A(\cdot, y), \rho_A(y, \cdot) \colon Z \to [0, \infty]$ *are continuous with respect to* $\mathcal{T}_M$.

Proof First note that if $y = \omega$, then $\rho_A(\cdot, y) \equiv \infty$ is constant and hence continuous. If $y \neq \omega$, we start by defining the set

$$C_{A,r}(y) := \{x \in Z \mid \rho_A(x, y) > r\},$$

which can be thought of as the complement of a "closed" ball (again $y \neq \omega$). Let (a, b) be an open interval in $\mathbb{R}$ (possibly unbounded) and consider the map $f := \rho_A(\cdot\,, y)$ for some fixed $y \neq \omega$. Then $f^{-1}((a,b)) = B_{A,b}(y) \cap C_{A,a}(y)$ and continuity of f follows, if $C_{A,a}(y)$ is open for all $a \geq 0$.

By Theorem 4.1, we know that for any nondegenerate triple (ω, α, y) and every $x \in Z \setminus \{y, \omega\}$,

$$\rho_{(\omega,\alpha,y)}(x, y)\rho_{(y,\alpha,\omega)}(x, \omega) = 1 \quad \text{and} \quad \rho_{(\omega,\alpha,\beta)}(x, y) = \lambda\rho_{(\omega,\alpha,o)}(x, y).$$

Therefore, we see that

$$\rho_{(y,\alpha,o)}(x, \omega) = \lambda\rho_{(y,\alpha,\omega)}(x, \omega) = \frac{\lambda}{\rho_{(\omega,\alpha,y)}(x, y)} = \frac{\lambda}{\mu\rho_{(\omega,\alpha,\beta)}(x, y)}$$

for $y, \omega, \alpha, \beta, o$ mutually different and $\lambda, \mu > 0$ depending only on α, ω, y, o and α, β, ω, y, respectively. This immediately implies that $B_{(\omega,\alpha,\beta),r}(y) = C_{(y,\alpha,o),\lambda/(\mu r)}(\omega)$ for some $\lambda, \mu > 0$ (notice that the points ω and y behave nicely). Since this is true for all ω, α, β, y, o and r as above, we see that $C_{A,r}(y)$ is open for all nondegenerate triples A, all $r > 0$ and all $y \in Z$. This implies the lemma. □

Remark 4.4 The proof of the continuity of ρ_A relies on the fact that we take the open balls of *all* semimetrics ρ_A. It is not sufficient to take just one — or some — of the nondegenerate triples. Only collectively can they define a topology such that $\rho_A(\cdot\,, y)$ is continuous. In particular, the involution plays a critical role. The following example illustrates how the topology induced by a single quasimetric does not have this.

Example 4.5 Let $X = [0, 1]$ and define

$$\rho(x, y) := \begin{cases} |x - y| & \text{if } |x - y| < 1, \\ 2|x - y| & \text{if } |x - y| \geq 1. \end{cases}$$

Since for all $x, y, z \in X$ we have

$$\rho(x, y) \leq 2|x - y| \leq 2(|x - z| + |z - y|) \leq 4\max(\rho(x, z), \rho(z, y)),$$

we see that ρ is a 4–quasimetric. Consider the sequence $x_n := 1 - 1/n$ and the topology generated by the "open balls" $B_r(x) := \{y \in Z \mid \rho(x, y) < r\}$. The sequence x_n converges to 1 in the topology induced by ρ. However,

$$\rho(0, x_n) = 1 - \frac{1}{n} \xrightarrow{n\to\infty} 1 \neq 2 = \rho(0, 1),$$

and therefore, ρ is not continuous with respect to the topology it induces. This is in significant contrast to metric spaces or the maps ρ_A with the Möbius topology.

Lemma 4.6 *The topological space $(Z, \mathcal{T}_M)$ is Hausdorff.*

Proof Let $x, y \in Z$ be two different points. Choose a point $\alpha \in Z \setminus \{x, y\}$. We know that for every $z \in Z$,

$$\rho_{(y,\alpha,x)}(x, z) = \frac{1}{\rho_{(x,\alpha,y)}(y, z)}.$$

Therefore, the intersection of the two open balls $B_{(y,\alpha,x),1}(x)$, $B_{(x,\alpha,y),1}(y)$ is empty. □

Consider two Möbius spaces (Z, M) and (Z', M'). We want to have a notion of maps that are compatible with the Möbius structures.

Definition 4.7 Let (Z, M) and (Z', M') be Möbius spaces. A map $f : Z \to Z'$ is called a *Möbius map* if and only if for every admissible quadruple $(w, x, y, z) \in \mathcal{A}$, we have

$$M(w, x, y, z) = M'(f(w), f(x), f(y), f(z)).$$

If a Möbius map f is bijective, we call it a *Möbius equivalence*.

Lemma 4.8 *Let (Z, M) and (Z', M') be two Möbius spaces and $f : Z \to Z'$ a Möbius equivalence. Then f is a homeomorphism when we equip Z and Z' with their respective Möbius topologies.*

Proof Let $A = (\omega, \alpha, \beta)$ be a nondegenerate triple in Z. Since f is a bijection, it sends A to a nondegenerate triple, denoted by $f(A)$, in Z'. Looking at the definition of the semimetric ρ_A, we immediately see that, since f preserves the Möbius structure, we have for all $x, y \in Z$ that

$$\rho_A(x, y) = \rho_{f(A)}(f(x), f(y)).$$

Thus, the map f sends an open ball $B_{A,r}(x)$ in Z to the open ball $B_{f(A),r}(f(x))$ in Z' and a subbasis of $\mathcal{T}_M$ to a subbasis of $\mathcal{T}_{M'}$. The same is true for f^{-1}, which proves the lemma. □

Classically, Möbius structures arise in the study of metric spaces. When a Möbius structure arises from a metric, the topology constructed above coincides with the topology induced by the metric.

Theorem 4.9 *Let (Z, ρ) be a metric space. Let $\mathcal{T}_\rho$ denote the topology on Z induced by ρ, and denote the induced Möbius structure by M. Let $\mathcal{T}_M$ be the topology induced by M and let $\{\rho_A\}_A$ be the family of semimetrics induced by M. Then $\mathcal{T}_\rho = \mathcal{T}_M$.*

Proof Since Z is a metric space, Lemma 4.2 tells us that for all nondegenerate triples A and for all $x \neq y$, we have

$$\rho_A(x, y) = \frac{\rho(x, y)}{\rho(x, \omega)\rho(\omega, y)} \frac{\rho(\alpha, \omega)\rho(\omega, \beta)}{\rho(\alpha, \beta)}.$$

In particular, $\rho_A(x, y)$ is continuous in x with respect to $\mathcal{T}_\rho$ as long as $x \in Z \setminus \{\omega\}$.

We need to show that the open balls in ρ are open with respect to $\mathcal{T}_M$, and that the open balls with respect to the ρ_A are open with respect to $\mathcal{T}_\rho$. We denote by

$$B_s(y) := \{x \in Z \mid \rho(x, y) < s\}$$

the open ball of radius s with respect to ρ, and by

$$B_{A,s}(y) := \{x \in Z \mid \rho_A(x, y) < s\}$$

the open ball of radius s with respect to ρ_A. These sets generate $\mathcal{T}_\rho$ and $\mathcal{T}_M$, respectively.

We first show that $B_{A,r}(y)$ is open with respect to $\mathcal{T}_\rho$ for all nondegenerate triples A, and all $r > 0$ and $y \in Z \setminus \{\omega\}$. Let $x \in B_{A,r}(y)$, ie $\rho_A(x, y) < r$. Since ρ_A is continuous with respect to $\mathcal{T}_\rho$, there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subset B_{A,r}(y)$. We conclude that $B_{A,r}(y)$ is open in $\mathcal{T}_\rho$ and that $\mathcal{T}_\rho$ is finer than $\mathcal{T}_M$.

In order to show that $\mathcal{T}_M$ is finer than $\mathcal{T}_\rho$, we consider the open ball $B_r(y)$ for $r > 0$ and $y \in Z$. Let $x \in B_r(y)$. Since ρ is a metric, there exists a smaller ball around x contained in $B_r(y)$, ie there exists $r' < r$ such that $B_{r'}(x) \subset B_r(y)$. Replacing r' by ϵ, it is now enough to show that for every $\epsilon > 0$, we can find $\delta > 0$ and a nondegenerate triple A such that $B_{A,\delta}(x) \subset B_\epsilon(x)$.

Choose $\omega, \alpha \in Z$ such that $A := (\omega, \alpha, x)$ is nondegenerate. We claim that there exist $\delta > 0$ and $C > 0$ such that $\rho(z, \omega) < C$ for all $z \in B_{A,\delta}(x)$. Suppose not. Then we find a sequence z_n such that $\rho_A(z_n, x) \to 0$ and $\rho(z_n, \omega) \to \infty$. However,

$$0 \leftarrow \rho_A(z_n, x) = \frac{\rho(z_n, x)\rho(\alpha, \omega)}{\rho(z_n, \omega)\rho(\alpha, x)} \geq \frac{\rho(z_n, \omega) - \rho(\omega, x)}{\rho(z_n, \omega)} \frac{\rho(\alpha, \omega)}{\rho(\alpha, x)} \to \frac{\rho(\alpha, \omega)}{\rho(\alpha, x)} \neq 0.$$

Let

$$0 < \delta' < \min\left(\epsilon \frac{\rho(\alpha, \omega)}{C\rho(\alpha, x)}, \delta\right) \quad \text{and} \quad z \in B_{A,\delta'}(x).$$

Then

$$\rho(z, x) = \frac{\rho(z, x)\rho(\alpha, \omega)}{\rho(z, \omega)\rho(\alpha, x)} \frac{\rho(z, \omega)\rho(\alpha, x)}{\rho(\alpha, \omega)} \leq \rho_A(z, x) \frac{C\rho(\alpha, x)}{\rho(\alpha, \omega)} \leq \epsilon.$$

In other words, $B_{A,\delta'}(x) \subseteq B_\epsilon(x)$. This implies that balls with respect to ρ are open with respect to $\mathcal{T}_M$. Hence $\mathcal{T}_M$ is finer than $\mathcal{T}_\rho$, which completes the proof of Theorem 4.9. □

Remark 4.10 This proof easily extends to extended metric spaces which have a point at infinity: Let ∞ denote the point at infinity in the metric space (Z, ρ). Then, for any nondegenerate triple $A = (\infty, \alpha, \beta)$, we have $\rho_A = \lambda\rho$ for some positive number λ. This immediately implies that $\mathcal{T}_\rho \subseteq \mathcal{T}_M$. To prove equality, one modifies the proof provided above.

Applying Lemma 4.8 in the context of Theorem 4.9 immediately yields the following corollary.

Corollary 4.11 *Let (Z, ρ) and (Z', ρ') be — possibly extended — metric spaces, let M_ρ and $M_{\rho'}$ be the induced Möbius structures, and $f : (Z, M_\rho) \to Z'$ a Möbius equivalence. Then f is a homeomorphism with respect to the metric topologies $\mathcal{T}_\rho$ and $\mathcal{T}_{\rho'}$.*

Proof We know from Lemma 4.8 that f is a homeomorphism with respect to the topologies $\mathcal{T}_M, \mathcal{T}_{M'}$. By Theorem 4.9, the Möbius topologies and the metric topologies coincide, ie $\mathcal{T}_M = \mathcal{T}_\rho$ and $\mathcal{T}_{M'} = \mathcal{T}_{\rho'}$. The statement follows. □

It is worth noting that the Möbius topology is not preserved under quasi-Möbius equivalences; see Section 1 for the definition. This is illustrated by the following example.

Example 4.12 Let $X = [0,1]$ and define

$$\rho(x,y) := \begin{cases} 3|x-y| & \text{if } |x-y| < 1, \\ |x-y| & \text{if } |x-y| = 1. \end{cases}$$

One easily checks that ρ is a quasimetric and bi-Lipschitz equivalent to the standard metric on X, which we shall denote by d. Since d and ρ are bi-Lipschitz equivalent, their induced Möbius structures are quasi-Möbius equivalent. Let $\mathcal{T}_M$ denote the Möbius topology coming from the Möbius structure induced by ρ and $\mathcal{T}_{\text{std}}$ denote the standard topology, which is the one induced by d. By Theorem 4.9, $\mathcal{T}_{\text{std}}$ is the Möbius topology of the Möbius structure induced by d. We will now show that $\mathcal{T}_{\text{std}} \neq \mathcal{T}_M$, providing an example where two quasi-Möbius equivalent Möbius structures do not induce the same topology.

We will show our claim by proving that $1 \in X$ is an isolated point with respect to $\mathcal{T}_M$. Let $A = \left(\frac{1}{2}, 0, 1\right)$ and compute

$$\rho_A(x,y) := \begin{cases} \dfrac{3|x-y|}{9\left|x-\frac{1}{2}\right|\left|y-\frac{1}{2}\right|} C_A & \text{if } |x-y| < 1, \\ \dfrac{1}{9\frac{1}{4}} \cdot C_A & \text{if } |x-y| = 1, \end{cases}$$

where $C_A = \rho\left(0, \frac{1}{2}\right)\rho\left(1, \frac{1}{2}\right)/\rho(0,1)$ depends on A but not on x, y. Theorem 4.1(2) implies that $C_A = \frac{9}{4}$. If we fix $x = 0$, we obtain

$$\rho_A(0,y) := \begin{cases} \dfrac{3}{2}\dfrac{y}{\left|y-\frac{1}{2}\right|} & \text{if } |x-y| < 1, \\ 1 & \text{if } |x-y| = 1. \end{cases}$$

Since $\frac{3}{2}y/\left|y-\frac{1}{2}\right| \geq \frac{3}{2}$ for all $y > \frac{1}{2}$, we see that $B_{1+\epsilon,A}(0) = [0,t) \cup \{1\}$ for some $\epsilon > 0$ sufficiently small and $t < \frac{1}{2}$ depending on ϵ. On the other hand, we have

$$\rho_A(1,y) := \begin{cases} \dfrac{3}{2}\dfrac{|1-y|}{\left|y-\frac{1}{2}\right|} & \text{if } |x-y| < 1, \\ 1 & \text{if } |x-y| = 1, \end{cases}$$

which only approaches zero for $y \to 1$. We see that for ϵ sufficiently small, $B_{1+\epsilon,A}(0) \cap B_{\epsilon,A}(1) = \{1\}$, implying that $\{1\}$ is an open set in the Möbius topology of the Möbius structure induced by ρ.

We conclude that $\mathcal{T}_M \neq \mathcal{T}_{\text{std}}$, showing that the Möbius topology is not preserved under quasi-Möbius maps in general, even if the inducing quasimetrics ρ, d are bi-Lipschitz equivalent.

5 Cauchy sequences and completeness

The next two sections are devoted to the notion of Cauchy sequences. We show how to define Cauchy sequences on strong Möbius spaces in a way that is compatible with the situation when the strong Möbius structure is induced by a metric space. In the next section, we show how to construct a completion under an additional symmetry assumption.

Let (Z,ρ) be a metric space, M its induced strong Möbius structure. We recall that a Cauchy sequence — in its usual sense on a metric space — is a sequence $(x_n)_n$ in Z such that for all $\epsilon > 0$ there exists a natural number N_ϵ such that for all $m, n \geq N_\epsilon$, we have $\rho(x_m, x_n) < \epsilon$. Our goal is to generalise this notion to strong Möbius spaces. It may be tempting to simply generalise the statement above to quasi- and semimetrics and use that as a definition, but since a Möbius structure can be induced by many different semimetrics, a definition relying only on the Möbius structure itself is more desirable.

Before we formulate the key insight, we need some notation. Let ρ be a (possibly extended) semimetric that induces M. If ρ has a point at infinity, we denote that point by ω. We write $(y \mid z) := -\ln(\rho(y,z))$ for all $y, z \in Z$. Further, consider a sequence $(x_{n,m})_{n,m\in\mathbb{N}}$ in Z. We say that $\lim_{n,m\to\infty} x_{n,m} = y$, if and only if for all $\epsilon > 0$ there exists an N_ϵ such that for all $n, m \geq N_\epsilon$, we have $\rho(x_{n,m}, y) < \epsilon$.

In what follows, we will often consider a sequence $(x_n)_n$ and a pair of points $y, z \in Z \setminus \{\omega\}$ such that $y \neq z$ and neither $\rho(x_n, y)$ nor $\rho(x_n, z)$ converges to zero. Given a sequence $(x_n)_n$, we will refer such a pair y, z as a *good pair*.

Recall that we write $M = (a, b, c)$, where a, b, c denote the components of M. We can now characterise Cauchy sequences in terms of the Möbius structure.

Lemma 5.1 *Let (Z,ρ) be a metric space, and $(x_n)_{n\in\mathbb{N}}$ a sequence in Z. The following are equivalent*:

(1) *The sequence $(x_n)_n$ is either a Cauchy sequence, or $\rho(x_n, y) \xrightarrow{n\to\infty} \infty$ for all $y \in Z$.*

(2) *There exists a good pair $y, z \in Z$ such that $\lim_{n,m\to\infty} \mathrm{crt}(x_n, x_m, y, z) = (0:1:1)$.*

(3) *There exists a good pair $y, z \in Z$ such that $\lim_{n,m\to\infty} c(x_n, x_m, y, z) = -\infty$.*

Further, if (1) *holds, then* (2) *and* (3) *hold for all good pairs $y, z \in Z$. In addition,* (2) *holds for a good pair y, z if and only if* (3) *holds for the same good pair y, z.*

The equivalence of (1) and (2) is stated in Lemma 2.2 of [Beyrer and Schroeder 2017]. Furthermore, it is easy to see from the proof that (1) implies (2) for every good pair. We are left to prove (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1). For this, we require an auxiliary result. Since it is our goal to generalise Cauchy sequences beyond the realm of metric spaces, we will formulate this result in a more general context.

Lemma 5.2 *Let (Z, M) be a strong Möbius structure and ρ a quasimetric that induces M. Let $(x_n)_n$ be a sequence in Z and suppose there exists a good pair $y, z \in Z$ such that $c(x_n, x_m, y, z) \xrightarrow{n,m\to\infty} -\infty$. Then one of the following two statements holds*:

(a) *For every $x \in Z \setminus \{\omega\}$, there exists some $B_x > 0$ such that $\rho(x_n, x) < B_x$ for all n. Furthermore, $\rho(x_n, x_m) \xrightarrow{n,m\to\infty} 0$. We say that x_n is bounded.*

(b) *For every $x \in Z \setminus \{\omega\}$, we have $\rho(x_n, x) \xrightarrow{n\to\infty} \infty$. We say that x_n diverges to infinity and write $x_n \to \infty$.*

Lemma 5.2 is a generalisation of the statement (3) $\Longrightarrow$ (1) in Lemma 5.1.

Remark 5.3 Lemmas 5.1 and 5.2 also hold for extended metric spaces. One can prove (1) $\Longrightarrow$ (2) for the case $y=\omega$ separately (and, by symmetry, the same proof works for $z=\omega$). The proof of (2) $\Longrightarrow$ (3) that we see below immediately generalises to extended metric spaces. For (3) $\Longrightarrow$ (1), we can use the fact that by Lemma 5.2, this statement also holds for quasimetrics. If $y=\omega$ for a given quasimetric, we can perform involution of ρ at any point $x \in Z \setminus \{y,z\}$. This provides us with a quasimetric that induces the same strong Möbius structure, but neither y nor z lies at infinity.

Proof of Lemma 5.2 Let $(x_n)_n$ be a sequence in the strong Möbius space (Z,M), let ρ be a quasimetric that induces M and let y,z be a good pair such that $c(x_n,x_m,y,z) \xrightarrow{n,m\to\infty} -\infty$. By definition of the Möbius structure induced by ρ, we can write

$$c(x_n,x_m,y,z) = (x_n|y)+(x_m|z)-(x_n|x_m)-(y|z) = \ln\left(\frac{\rho(x_n,x_m)\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)}\right).$$

Using this equality, the statement $c(x_n,x_m,y,z) \xrightarrow{n,m\to\infty} -\infty$ becomes equivalent to

$$\frac{\rho(x_n,x_m)\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)} \xrightarrow{n,m\to\infty} 0. \tag{5-1}$$

We will distinguish between two cases, which will turn out to be exactly the distinction between case (a) and case (b). Suppose there exists some $x \in Z \setminus \{\omega\}$ and some constant $B>0$ such that $\rho(x_n,x)<B$ for all n. We want to show that we are in case (a).

Since ρ is a quasimetric, we have that for all $x' \in Z \setminus \{\omega\}$,

$$\rho(x_n,x') \le K\max(\rho(x_n,x),\rho(x,x')) \le K\max(B,\rho(x,x')).$$

Therefore, we see that $\rho(x_n,x')$ is bounded for all $x' \in Z \setminus \{\omega\}$. In particular, $\rho(x_n,y)$ and $\rho(x_n,z)$ are both bounded by some constant $B>0$. We obtain

$$\frac{\rho(x_n,x_m)\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)} \ge \rho(x_n,x_m)\frac{\rho(y,z)}{B^2}.$$

Since the left-hand side of this equation goes to zero by assumption, the right-hand side has to go to zero as well. Hence we see that $\rho(x_n,x_m) \xrightarrow{n,m\to\infty} 0$.

We are left to show that we end up in case (b) whenever there is no $x \in Z \setminus \{\omega\}$ such that $\rho(x_n,x)$ is bounded. Suppose $\rho(x_n,x)$ is unbounded for all $x \in Z \setminus \{\omega\}$. Then there exists a subsequence $(x_{n_i})_i$ of $(x_n)_n$ such that $\rho(x_{n_i},x)\to\infty$ for one (and hence all, since ρ is a quasimetric) $x \in Z \setminus \{\omega\}$.

Suppose by contradiction that $\rho(x_n,x)$ does not converge to infinity for one and hence all $x \in Z \setminus \{\omega\}$. Then we find another subsequence $(x_{m_j})_j$ of $(x_n)_n$, which is bounded. In particular, we find a constant $B>0$ such that

$$\rho(x_{m_j},y) \le B \quad \text{and} \quad \rho(x_{m_j},z) \le B$$

for all j. From our treatment of case (a), we know that for this subsequence, $\rho(x_{m_j}, x_{m_{j'}}) \xrightarrow{j,j'\to\infty} 0$. In particular, we find a number J such that for all $j, j' \geq J$, we have

$$\rho(x_{m_j}, x_{m_{j'}}) < 1.$$

Now we estimate the distance between the two subsequences $(x_{m_j})_j$ and $(x_{n_i})_i$. For this, we need to take x_{m_J} as an auxiliary point. Since x_{n_i} diverges to infinity, there is a number I such that

$$\rho(x_{m_J}, x_{n_i}) > \max(K, K\cdot B) \quad \text{for all } i \geq I.$$

Now we use the fact that ρ is a quasimetric to get that for all $i \geq I$ and $j \geq J$ we have

$$\max(K, K\cdot B) \leq \rho(x_{m_J}, x_{n_i}) \leq K\max(\rho(x_{m_J}, x_{m_j}), \rho(x_{m_j}, x_{n_i})) = K\rho(x_{m_j}, x_{n_i}),$$

where the last equality follows from the fact that $\rho(x_{m_J}, x_{m_j}) < 1$ for all $j \geq J$. Now consider, for $i \geq I$ and $j \geq J$,

$$\begin{aligned}\frac{\rho(x_{m_j}, x_{n_i})\rho(y,z)}{\rho(x_{m_j}, y)\rho(x_{n_i}, z)} \geq \frac{\rho(x_{m_j}, x_{n_i})\rho(y,z)}{B\rho(x_{n_i}, z)} &\geq \frac{\rho(x_{m_j}, x_{n_i})\rho(y,z)}{BK\max(\rho(x_{n_i}, x_{m_j}), \rho(x_{m_j}, z))}\\ &= \frac{\rho(x_{m_j}, x_{n_i})\rho(y,z)}{BK\rho(x_{n_i}, x_{m_j})}\\ &= \frac{\rho(y,z)}{BK},\end{aligned}$$

where in the second-to-last step we use the fact that $\rho(x_{n_i}, x_{m_j}) \geq \max(1, B) \geq B \geq \rho(x_{m_j}, z)$ for all $i \geq I$ and $j \geq J$. This inequality shows that $\rho(x_{m_j}, x_{n_i})\rho(y,z)/(\rho(x_{m_j}, y)\rho(x_{n_i}, z))$ is bounded from below by a positive constant. But by assumption, $\rho(x_{m_j}, x_{n_i})\rho(y,z)/(\rho(x_{m_j}, y)\rho(x_{n_i}, z))$ converges to zero, a contradiction. We see that, if a subsequence $(x_{n_i})_i$ diverges to infinity, the sequence $(x_n)_n$ has to diverge to infinity as well. Thus, we are in case (b), which completes the proof. □

Proof of Lemma 5.1 Let (Z, ρ) be a nonextended metric space, $(x_n)_n$ a sequence in Z and $y, z \in Z$ such that $\lim_{n\to\infty} x_n \neq y, z$.

(1) $\Longrightarrow$ (2) Instead of proving just (1) $\Longrightarrow$ (2), which follows directly from [Beyrer and Schroeder 2017], we will also prove the second part of the lemma, ie that $\lim_{n,m\to\infty} \operatorname{crt}(x_n, x_m, y, z) = (0:1:1)$ for all good pairs y, z.

Step 1 We start by proving that for every Cauchy sequence, we have

$$\lim_{n,m\to\infty} \operatorname{crt}(x_n, x_m, y, z) = (0:1:1).$$

Suppose (x_n) is a Cauchy sequence. Note that this implies that $\rho(x_n, x)$ converges for all $x \in Z$. Let $\epsilon > 0$. We find some $N_\epsilon \in \mathbb{N}$ such that for all $n, m \geq N_\epsilon$, we have $\rho(x_n, x_m) < \epsilon$. Since y, z is a good pair, we can choose ϵ sufficiently small such that there is an N_ϵ such that, additionally, $\rho(x_n, y), \rho(x_n, z) > \epsilon^{1/4}$ for all $n \geq N_\epsilon$. Therefore, we get

$$\frac{\rho(x_n, x_m)\rho(y,z)}{\rho(x_n, y)\rho(x_m, z)} < \frac{\epsilon\rho(y,z)}{\rho(x_n, y)\rho(x_m, z)} < \frac{\epsilon}{\sqrt{\epsilon}}\rho(y,z) = \sqrt{\epsilon}\rho(y,z).$$

Thus we see that

$$\frac{\rho(x_n,x_m)\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)} \xrightarrow{n,m\to\infty} 0.$$

For symmetry reasons, we immediately see that also

$$\frac{\rho(x_n,x_m)\rho(y,z)}{\rho(x_n,z)\rho(x_m,y)} \xrightarrow{n,m\to\infty} 0.$$

We are left to show that

$$\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} \xrightarrow{n,m\to\infty} 1$$

in order to prove that $\mathrm{crt}(x_n,x_m,y,z) \xrightarrow{n,m\to\infty} (0:1:1)$. Since y,z is a good pair, we have

$$\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} \le \frac{\rho(x_n,y)(\rho(x_n,z)+\rho(x_n,x_m))}{\rho(x_n,z)(\rho(x_n,y)-\rho(x_n,x_m))} = \frac{1+\dfrac{\rho(x_n,x_m)}{\rho(x_n,z)}}{1-\dfrac{\rho(x_n,x_m)}{\rho(x_n,y)}} \xrightarrow{n,m\to\infty} 1,$$

$$\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} \ge \frac{\rho(x_n,y)(\rho(x_n,z)-\rho(x_n,x_m))}{\rho(x_n,z)(\rho(x_n,y)+\rho(x_n,x_m))} = \frac{1-\dfrac{\rho(x_n,x_m)}{\rho(x_n,z)}}{1+\dfrac{\rho(x_n,x_m)}{\rho(x_n,y)}} \xrightarrow{n,m\to\infty} 1.$$

It follows that

$$\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} \xrightarrow{n,m\to\infty} 1$$

and hence $\mathrm{crt}(x_n,x_m,y,z) \xrightarrow{n,m\to\infty} (0:1:1)$. Note that we relied on the triangle inequality for this part of the proof.

Step 2 We show that if (x_n) diverges to infinity, we get

$$\lim_{n,m\to\infty} \mathrm{crt}(x_n,x_m,y,z) = (0:1:1).$$

Suppose that $\rho(x_n,x)\to\infty$ for all $x\in Z$ as n goes to infinity (except for the point $x\in Z$ that may lie at infinity). Then, for any $y,z\in Z$ that do not lie at infinity, we have

$$\begin{aligned}\frac{\rho(x_n,x_m)\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)} &\le \frac{(\rho(x_n,y)+\rho(y,x_m))\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)}\\ &= \frac{\rho(y,z)}{\rho(x_m,z)} + \frac{\rho(x_m,y)\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)}\\ &\le \frac{\rho(y,z)}{\rho(x_m,z)} + \frac{(\rho(x_m,z)+\rho(z,y))\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)}\\ &= \frac{\rho(y,z)}{\rho(x_m,z)} + \frac{\rho(y,z)}{\rho(x_n,y)} + \frac{\rho(y,z)^2}{\rho(x_n,y)\rho(x_m,z)} \xrightarrow{n,m\to\infty} 0.\end{aligned}$$

We are left to show that $\rho(x_n,y)\rho(x_m,z)/(\rho(x_n,z)\rho(x_m,y)) \xrightarrow{n,m\to\infty} 1$. For this, we do the estimate

$$\begin{aligned}\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} &\le \frac{(\rho(x_n,z)+\rho(y,z))(\rho(x_m,y)+\rho(y,z))}{\rho(x_n,z)\rho(x_m,y)}\\ &= 1+\frac{\rho(y,z)}{\rho(x_n,z)}+\frac{\rho(y,z)}{\rho(x_m,y)}+\frac{\rho(y,z)^2}{\rho(x_n,z)\rho(x_m,y)} \xrightarrow{n,m\to\infty} 1.\end{aligned}$$

In the same way, we have

$$\begin{aligned}\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} &\ge \frac{(\rho(x_n,z)-\rho(y,z))(\rho(x_m,y)-\rho(y,z))}{\rho(x_n,z)\rho(x_m,y)}\\ &= 1-\frac{\rho(y,z)}{\rho(x_n,z)}-\frac{\rho(y,z)}{\rho(x_m,y)}+\frac{\rho(y,z)^2}{\rho(x_n,z)\rho(x_m,y)} \xrightarrow{n,m\to\infty} 1.\end{aligned}$$

From these two estimates, we conclude that $\rho(x_n,y)\rho(x_m,z)/(\rho(x_m,y)\rho(x_n,z)) \xrightarrow{n,m\to\infty} 1$. This concludes the proof of Step 2 and the proof that (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3) Recall that, by definition,

$$c(w,x,y,z) = \ln\left(\frac{\rho(w,x)\rho(y,z)}{\rho(w,y)\rho(x,z)}\right),$$

which is a continuous map with respect to the metric topology. In particular, if $\mathrm{crt}(w,x,y,z)\to(0:1:1)$, then

$$\ln\left(\frac{\rho(w,x)\rho(y,z)}{\rho(w,y)\rho(x,z)}\right) \to -\infty.$$

We see that (2) $\Longrightarrow$ (3). In particular, if (2) holds for a given pair y,z then (3) holds for the same pair y,z.

(3) $\Longrightarrow$ (1) This is a special case of Lemma 5.2. Since we have seen that (1) $\Longrightarrow$ (2) for all good pairs y,z, we also see that, if (3) holds for a good pair y,z, then (2) holds for the same good pair y,z. This concludes the proof of Lemma 5.1 □

Among other things, Lemma 5.1 tells us that for metric spaces, we only need to find one good pair y,z that satisfies condition (2) or (3) to get the same condition for all good pairs y,z that aren't the limit of $(x_n)_n$. It would be good to have the same condition in any strong Möbius space that isn't necessarily induced by a metric. Then we could define a sequence in a strong Möbius space to be a Cauchy sequence if for one good pair y,z and hence all good pairs, we have $\mathrm{crt}(x_n,x_m,y,z)\to(0:1:1)$, which would be much easier to check in practice than if we had to check all good pairs. The next lemma tells us that this is actually true in the case of condition (3).

Lemma 5.4 *Let (Z,M) be a strong Möbius space. Let $(x_n)_n$ be a sequence in Z. Suppose there is a good pair y,z such that*

$$c(x_n,x_m,y,z) \xrightarrow{n,m\to\infty} -\infty.$$

Then the same holds for all good pairs $y',z'\in Z$.

Proof Let ρ be a quasimetric that induces M. By Lemma 5.2, we know that (x_n) is either bounded or diverges to infinity. Let y', z' be a good pair. As we have seen in the proofs of Lemma 5.1 and 5.2, we get the right convergence of $c(x_n, x_m, y', z')$ if $\rho(x_n, x_m)\rho(y', z')/(\rho(x_n, y')\rho(x_m, z'))$ converges to zero.

Case 1 Suppose $(x_n)_n$ is bounded. Since y', z' is a good pair, we find some $\epsilon > 0$ and a subsequence $(x_{n_i})_i$ such that $\rho(x_{n_i}, y') \geq \epsilon$ for all i. From Lemma 5.2, we know that $\rho(x_n, x_m) \xrightarrow{n,m\to\infty} 0$ and we find a number N such that for all $n, m \geq N$, $\rho(x_n, x_m) < \epsilon/(2K)$. Thus, we have for all $n \geq N$,

$$\epsilon \leq \rho(x_{n_i}, y') \leq K \max(\rho(x_{n_i}, x_n), \rho(x_n, y')).$$

Since $K\rho(x_{n_i}, x_n) \leq \frac{1}{2}\epsilon < \epsilon$, we see that

$$\frac{\epsilon}{K} \leq \frac{1}{K}\rho(x_{n_i}, y') \leq \rho(x_n, y')$$

for $n \geq N$. This implies the sequence $(x_n)_n$ stays away from y' for large n; specifically, $\rho(x_n, y') \geq \epsilon/K$ for $n \geq N$. The same is true for $(x_n)_n$ and z' and some other $\tilde{\epsilon} > 0$. Hence, we have

$$\frac{\rho(x_n, x_m)\rho(y', z')}{\rho(x_n, y')\rho(x_m, z')} \leq K^2 \frac{\rho(x_n, x_m)\rho(y', z')}{\epsilon\tilde{\epsilon}} \xrightarrow{n,m\to\infty} 0.$$

We see that $\rho(x_n, x_m)\rho(y', z')/(\rho(x_n, y')\rho(x_m, z'))$ converges to zero; hence $c(x_n, x_m, y', z') \to -\infty$.

Case 2 Suppose x_n diverges to infinity. We can find a number N such that $\rho(x_n, y') \geq \rho(y', z')$ and $\rho(x_n, z') \geq \rho(y', z')$ for all $n \geq N$. Then we have

$$\begin{aligned}\frac{\rho(x_n, x_m)\rho(y', z')}{\rho(x_n, y')\rho(x_m, z')} &\leq \frac{K \max(\rho(x_n, y'), \rho(y', x_m))\rho(y', z')}{\rho(x_n, y')\rho(x_m, z')} \\ &\leq \frac{K^2 \max(\rho(x_n, y'), \rho(y', z'), \rho(z', x_m))\rho(y', z')}{\rho(x_n, y')\rho(x_m, z')} \\ &= K^2 \frac{\rho(y', z')}{\min(\rho(x_n, y'), \rho(x_m, z'))} \to 0.\end{aligned}$$

Hence, we see that also in this case, $\rho(x_n, x_m)\rho(y', z')/(\rho(x_n, y')\rho(x_m, z'))$ converges to zero and, therefore, $c(x_n, x_m, y', z') \to -\infty$. This completes the proof. □

One might hope that an analogous statement for condition (2) holds. However, the following example illustrates that Lemmas 5.2 and 5.4 are the best that we can hope for.

Example 5.5 Consider the circle, represented as $S^1 = \mathbb{R}/4\mathbb{Z}$. We will mostly use representatives in $[-2, 4]$ to represent points on the circle. Consider the space $Z := S^1 \setminus \{[0]\}$ and define a map $\rho\colon Z \times Z \to [0, \infty)$ by

$$\rho([x], [y]) := \begin{cases} |x-y| & \text{if } (x, y) \in (0, 2]^2 \cup [1, 3]^2 \cup [2, 4)^2 \cup ([-1, 1] \setminus \{0\})^2 \\ 2|x-y| & \text{if } (x, y) \in ((0, 1) \times (2, 3)) \cup ((2, 3) \times (0, 1)) \cup ((1, 2) \times (3, 4)) \cup ((3, 4) \times (1, 2)). \end{cases}$$

Notice the use of different representatives depending on the case. Geometrically, (Z, ρ) can be thought of as follows. Think of Z as a subset of the circle of circumference 4 with the shortest path metric. This circle can be embedded into $\mathbb{R}^2$ such that it is centred at the origin, ie it is the boundary of a disk centred at the origin.

We can now consider the intersection of the circle with each quarter of $\mathbb{R}^2$. We call them the upper-right, upper-left, lower-left and lower-right segments of S^1, based on their position in the standard coordinate system of $\mathbb{R}^2$.

The distance $\rho(x, y)$ between two points x and y is now defined to be the same as on S^1 if x and y lie on the same segment of S^1 or if they lie on segments that are neighbours of each other. If x and y lie on segments of S^1 that lie opposite to each other, then $\rho(x, y)$ is exactly twice the length of the path from x to y that passes through the point $(0, -1)$.

A straightforward computation with several case-distinctions shows that ρ is a 12–quasimetric. Thus, we get a strong Möbius space (Z, M_ρ). Consider now the following sequence in Z:

$$x_n = \left[\frac{1}{n}(-1)^n\right].$$

One can show that there is a good pair for $(x_n)_n$ that satisfies condition (3), but not condition (2). Furthermore, one can even find another good pair for $(x_n)_n$ that satisfies both conditions (2) and (3). Specifically, choose $y = 1.5$, $z = -1.5$ for the first case, and $y = 1.5$, $z = 1.6$ for the second case.

The issue at hand is that even if we understand the convergence behaviour of

$$\frac{\rho(x_n, x_m)\rho(y, z)}{\rho(x_n, y)\rho(x_m, z)},$$

we cannot control the convergence behaviour of

$$\frac{\rho(x_n, y)\rho(x_m, z)}{\rho(x_n, z)\rho(x_m, y)}$$

if ρ is not a metric. So we have found a quasimetric — and thus a strong Möbius structure M_ρ — for which the statement "(3) $\Longrightarrow$ (2)", that we have proven for metrics in Lemma 5.1, does not hold.

This example illustrates the relationship between the different possible conditions one could use to define Cauchy sequences in a strong Möbius space. If condition (2) holds for one good pair y, z, this does not imply that condition (2) holds for all good pairs, unless we work with a metric space. In the same way, if condition (3) holds for all good pairs, this doesn't imply the same for condition (2). However, from Lemma 5.4 we know that, if condition (3) holds for one good pair, it holds for all of them.

Example 5.5 leads us to the following definition of Cauchy sequences in a strong Möbius space.

Definition 5.6 Let (Z, M) be a strong Möbius space. A sequence $(x_n)_n$ in Z is called a *Cauchy sequence* if and only if for one (and hence all) good pairs y, z in Z, we have

$$c(x_n, x_m, y, z) \xrightarrow{n,m\to\infty} -\infty.$$

Definition 5.7 A strong Möbius space (Z, M) is called *complete* if and only if all Cauchy sequences in (Z, M) converge.

Using the previous lemma, the following results are easy to see.

Proposition 5.8 *Let (Z, M) and (Z', M') be two strong Möbius spaces, and let $f : Z \to Z'$ be a Möbius equivalence between them.*

(1) *Let $(x_n)_n$ be a sequence in Z. Then $(x_n)_n$ is a Cauchy sequence in (Z, M) if and only if $(f(x_n))_n$ is a Cauchy sequence in (Z', M').*

(2) *The strong Möbius space (Z, M) is complete if and only if (Z', M') is.*

Proof (1) The sequence $(x_n)_n$ is a Cauchy sequence if and only if for some good pair y, z in Z,

$$c(x_n, x_m, y, z) \to -\infty.$$

Since f is a Möbius equivalence, this implies

$$c'(f(x_n), f(x_m), f(y), f(z)) = c(x_n, x_m, y, z) \to -\infty.$$

Since f is a homeomorphism by Lemma 4.8 and y, z is a good pair, so is $f(y), f(z)$ for $(f(x_n))_n$. Thus, $(f(x_n))_n$ is a Cauchy sequence in (Z', M').

(2) Suppose (Z, M) is complete and let $(x'_n)_n$ be a Cauchy sequence in (Z', M'). By part (1), $(f^{-1}(x'_n))_n$ is a Cauchy sequence in (Z, M) which converges to some $x \in Z$ by completeness. Since f is a homeomorphism, $(x'_n)_n$ has to converge to $f(x)$. This implies completeness. □

The notion of completeness defined above compares to the notion of completeness defined in metric spaces as follows:

Theorem 5.9 *Let (Z, ρ) be a (possibly extended) metric space, and denote the induced Möbius structure by M. The following are equivalent:*

(1) *(Z, M) is complete as a strong Möbius space.*

(2) *(Z, ρ) is complete as a metric space and is either bounded or has a point at infinity.*

Proof (1) $\Longrightarrow$ (2) Suppose (Z, M) is complete as a strong Möbius space and let $(x_n)_n$ be a Cauchy sequence in the metric sense. By Lemma 5.1, $(x_n)_n$ is also a Cauchy sequence in the Möbius sense. Hence, (x_n) has to converge in the Möbius topology. Since the Möbius topology is the same as the metric topology on a metric space by Theorem 4.9, $(x_n)_n$ converges in the metric topology and (Z, ρ) is complete in the metric sense.

(2) $\Longrightarrow$ (1) Suppose (Z, ρ) is complete as a metric space and let $(x_n)_n$ be a Cauchy sequence in the Möbius sense. By Lemma 5.1, $(x_n)_n$ is either a Cauchy sequence in the metric sense, or it diverges to infinity. If it is a Cauchy sequence in the metric sense, it converges in the metric topology (and thus in the Möbius topology) by metric completeness. If x_n diverges to infinity, the metric space cannot be bounded. Hence, it has a point at infinity by assumption, and x_n converges to the point at infinity in the metric and Möbius topologies. □

6 Constructing the completion

Now that we have a notion of Cauchy sequences and a notion of completeness for strong Möbius spaces, an obvious question is whether every strong Möbius space has a naturally unique completion, as metric spaces do.

Certainly, if we take a metric space (Z,ρ) and consider the induced Möbius structure M, the metric completion $(\bar{Z},\bar{\rho})$ is either complete with respect to the induced Möbius structure $\overline{M}$, which is just an extension of M, or one has to add one point at infinity to make it complete in the Möbius sense. Adding a point at infinity doesn't change that Z is dense in its completion and it is easy to see that uniqueness up to isometry for the metric case implies uniqueness up to Möbius equivalence (even up to isometry) in the Möbius sense.

We want to see whether we can create a completion beyond the metric case. It turns out that this requires an extra condition. We start by doing the same construction that is used to obtain the metric completion. We will point out where the construction fails, and distil the extra condition needed. Let (Z,crt) be a strong Möbius space. Define the set

$$\bar{Z} := \{(x_n)_n \mid (x_n) \text{ a Cauchy sequence in } (Z,\mathrm{crt})\}/{\sim},$$

where $(x_n)\sim(x_n')$ if and only if, for every pair $y\neq z$ in Z that is a good pair for both (x_n) and (x_n'), we have

$$c(x_n,x_n',y,z)\to-\infty.$$

There is a canonical embedding of Z into $\bar{Z}$ defined by sending x to the constant sequence $x_n=x$. This is clearly a Cauchy sequence and the map $x\mapsto[(x)_n]$ is injective, since two different constant sequences are not equivalent in the sense defined above.

The next step is to extend the Möbius structure crt to $\bar{Z}$. We would like to define

$$\overline{\mathrm{crt}}\big([(w_n)],[(x_n)],[(y_n)],[(z_n)]\big) := \lim_{n\to\infty}\mathrm{crt}(w_n,x_n,y_n,z_n).$$

There are two questions that arise immediately when stating this definition. Does the limit on the right-hand side exist and is it independent of the choice of representative of a point $[(w_n)]\in\bar{Z}$? In general, the answer to these two questions is no. The reason for that has already appeared in Example 5.5, namely that, if $\rho(x_n,x_m)\to0$, we cannot make sure that $\rho(x_n,y)$ converges for all $y\in Z$. Specifically, the sequence x_n discussed in Example 5.5 satisfies $\mathrm{crt}(x_{2n},x_{2n+1},y,z)\to(0:1:4)$ and $\mathrm{crt}(x_{2n},x_{2n+2},y,z)\to(0:1:1)$. Therefore $\lim_{n,m\to\infty}\mathrm{crt}(x_n,x_m,y,z)$ does not exist. This example is a special case that will appear in the definition of $\overline{\mathrm{crt}}$ given above and makes this construction not well defined in general.

As mentioned in Example 5.5, the problem at hand is that we cannot control the behaviour of the ratio $\rho(x_n,y)\rho(x_m,z)/(\rho(x_m,y)\rho(x_n,z))$ for a Cauchy sequence (x_n). If we knew that $\mathrm{crt}(w_n,x_n,y_n,z_n)$ could only converge to points in $\mathbb{R}P^2$ that are allowed to be obtained by a Möbius structure, then we could resolve this problem (as we will see below). The following property makes sure that these issues cannot arise.

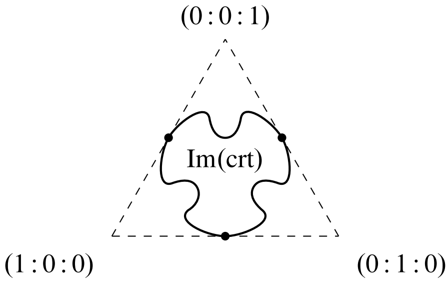

Figure 2: A Möbius structure crt satisfies the symmetry condition if and only if no point in the boundary of $\overline{\Delta}$ can be approximated by a sequence of points in $\mathrm{Im}(\Delta)$ except for $\left(\frac{1}{2} : \frac{1}{2} : 0\right)$, $\left(\frac{1}{2} : 0 : \frac{1}{2}\right)$ and $\left(0 : \frac{1}{2} : \frac{1}{2}\right)$. In other words, the image doesn't touch the boundary at any other than those three points.

Definition 6.1 A Möbius structure crt or a Möbius space (Z, crt) satisfies the *symmetry* condition if and only if

$$\overline{\mathrm{Im}(\mathrm{crt})} \subseteq \overline{\Delta} = \{(a : b : c) \mid a, b, c > 0\} \cup \{(0 : 1 : 1), (1 : 0 : 1), (1 : 1 : 0)\},$$

where $\overline{\mathrm{Im}(\mathrm{crt})}$ denotes the closure of the image of crt in $\mathbb{R}P^2$.

To interpret this definition, it is useful to think of $\Delta \subset \mathbb{R}P^2$ as a triangle. Specifically, consider the triangle $\{(x, y, z) \in \mathbb{R} \mid x + y + z = 1,\ x, y, z \geq 0\}$. The projection of this triangle onto $\mathbb{R}P^2$ is exactly the topological closure of Δ. The symmetry condition tells us that any sequence of cross ratio triples $\mathrm{crt}(w_n, x_n, y_n, z_n)$ can only accumulate at points in the interior of this triangle or at one of the three distinct points on the boundary of the triangle that are assumed by degenerate quadruples. It turns out that this is the property needed to construct a completion.

Theorem 6.2 *Let (Z, M) be a Möbius space that satisfies the symmetry condition. Then there exists a complete strong Möbius space $(\overline{Z}, \overline{\mathrm{crt}})$ with a Möbius embedding $i_Z \colon Z \hookrightarrow \overline{Z}$ — that is, satisfying $\overline{\mathrm{crt}}(i_Z(w), i_Z(x), i_Z(y), i_Z(z)) = \mathrm{crt}(w, x, y, z)$ for all admissible quadruples (w, x, y, z) — such that $i_Z(Z)$ is dense in $\overline{Z}$.*

Furthermore, if (Z', crt') is a complete strong Möbius space such that there exists a Möbius embedding $i \colon Z \hookrightarrow Z'$ such that $i(Z)$ is dense in Z', then there exists a unique Möbius equivalence $f \colon \overline{Z} \to Z'$ such that $i = f \circ i_Z$.

The space $(\overline{Z}, \overline{\mathrm{crt}})$ is going to be the one constructed above. Suppose (Z, crt) satisfies the symmetry condition. Let ρ be a quasimetric inducing crt, (x_n) a Cauchy sequence in the Möbius sense and y, z a good pair for (x_n). By symmetry of x_n, x_m we see that $\rho(x_n, x_m)\rho(y, z)/(\rho(x_n, y)\rho(x_m, z))$ and $\rho(x_n, x_m)\rho(y, z)/(\rho(x_m, y)\rho(x_n, z))$ both converge to zero as n and m tend to infinity. Therefore, the sequence $\mathrm{crt}(x_n, x_m, y, z)$ can be written in the form $(a_n : b_n : c_n)$ with all three entries being nonnegative, where we scale a_n, b_n, c_n so that $a_n + b_n + c_n = 2$. By the convergence statements above, a_n has to converge to zero. Since crt satisfies the symmetry condition, the only point $(0:b:c)$ that can be approximated arbitrarily well in Im(crt) is $(0:1:1)$. Therefore, $\mathrm{crt}(x_n, x_m, y, z) \xrightarrow{n,m\to\infty} (0:1:1)$.

Remark 6.3 Theorem 6.2 has an analogue for the quasi-Möbius class. Given a strong Möbius space (Z, M), one can choose a bounded quasimetric ρ that induces the given Möbius structure by Proposition 3.3. Using the fact that 2–quasimetrics can be deformed into metrics (see for example [Heinonen 2005]), we find that there exists some $\epsilon > 0$ and a metric d such that d is bi-Lipschitz-equivalent to ρ^ϵ and the Möbius structures induced by ρ and d respectively are quasi-Möbius equivalent. Since d is a metric, it has a completion, which is still bounded, and by Theorem 5.9 the Möbius space induced by the completion of (Z, d) is complete as a Möbius space. In other words, every strong Möbius space is quasi-Möbius equivalent to a Möbius space that is induced by a metric and admits a completion. This is in contrast to the situation where we stay within the same Möbius class, where not every strong Möbius structure admits a completion, as Example 5.5 shows.

The symmetry condition allows us to prove a result about convergence that will be useful in proving Theorem 6.2.

Proposition 6.4 *Let (Z, crt) be a strong Möbius structure satisfying the symmetry condition. Let (x_n) and (y_n) be Cauchy sequences in Z, let $y \in Z$ and let ρ be a quasimetric that induces* crt *and has a point at infinity (eg $\rho = \rho_A$). Then $\rho(x_n, y)$ and $\rho(x_n, y_n)$ converge, possibly to infinity.*

Recall that every sequence $(x_{n,m})$ in $\mathbb{R}$ parametrised by $\mathbb{N}^2$ with the property that $\lim_{n\to\infty} x_{n,m}$ exists for every m, $\lim_{m\to\infty} x_{n,m}$ exists for every n and $\lim_{n,m\to\infty} x_{n,m}$ exists, satisfies

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m} = \lim_{n,m\to\infty} x_{n,m}.$$

Proof Denote the point at infinity with respect to ρ by ∞. By Lemma 5.2, x_n is either bounded or diverges to infinity. If x_n diverges to infinity with respect to ρ, then $\rho(x_n, y) \to \infty$. Now assume the Cauchy sequence x_n is bounded with respect to ρ. By Lemma 5.2, we know that $\rho(x_n, x_m) \xrightarrow{n,m\to\infty} 0$. In particular, since ρ is a quasimetric, either $\rho(x_n, y) \xrightarrow{n\to\infty} 0$, or there exists $\epsilon > 0$, such that $\rho(x_n, y) \geq \epsilon$ for all n sufficiently large. Suppose $\rho(x_n, y)$ does not converge to zero. Then y, ∞ are a good pair for (x_n), $c(x_n, x_m, y, \infty) \xrightarrow{n,m\to\infty} -\infty$ and, by the symmetry condition,

$$\mathrm{crt}(x_n, x_m, y, \infty) \xrightarrow{n,m\to\infty} (0:1:1).$$

This implies that

$$\frac{\rho(x_n, y)}{\rho(x_m, y)} \xrightarrow{n,m\to\infty} 1. \tag{6-1}$$

We can now use this to prove that $\rho(x_n, y)$ converges for every Cauchy sequence (x_n) and any $y \in Z$. If (x_n) converges to y, then $\rho(x_n, y) \to 0$ by definition. If (x_n) diverges to infinity with respect to ρ, then

$\rho(x_n, y) \to \infty$. If (x_n) is bounded with respect to ρ, then $0 \le \rho(x_n, y) \le B$ and hence — by compactness — has a convergent subsequence $\rho(x_{n_i}, y)$. Applying equation (6-1) in the case $m = n_i$ yields

$$\frac{\rho(x_n, y)}{\rho(x_{n_i}, y)} \xrightarrow{n,i\to\infty} 1.$$

Since $\rho(x_{n_i}, y)$ converges, this implies that the limit of $\rho(x_n, y)$ exists and

$$\lim_{n\to\infty} \rho(x_n, y) = \lim_{i\to\infty} \rho(x_{n_i}, y).$$

Now consider the two Cauchy sequences (x_n) and (y_n). If one of the sequences is bounded and the other diverges to infinity, then $\rho(x_n, y_n) \to \infty$. If both sequences diverge to infinity, replace ρ with an involution ρ_o at any point $o \in Z$. Both (x_n) and (y_n) are bounded with respect to ρ_o. Convergence of $\rho_o(x_n, y_n)$ and the fact that ρ is the involution of ρ_o at the point $\infty \in Z$ will imply convergence of $\rho(x_n, y_n)$.

We are left to prove convergence of $\rho(x_n, y_n)$ when both sequences are bounded. In this situation, we know that $\rho(x_n, x_m), \rho(y_n, y_m) \xrightarrow{n,m\to\infty} 0$. Suppose $\rho(x_n, y_n)$ does not converge to zero. Then the limits above and the fact that ρ is a quasimetric imply that there exists some $\epsilon > 0$ such that for all n sufficiently large, $\rho(x_n, y_n) > \epsilon$. We conclude that

$$\operatorname{cr}(x_n, x_m, y_n, \infty) = \frac{\rho(x_n, x_m)}{\rho(x_n, y_n)} \xrightarrow{n,m\to\infty} 0.$$

Since crt satisfies the symmetry condition, this implies that

$$\frac{\rho(y_n, x_n)}{\rho(y_n, x_m)} = \operatorname{cr}(y_n, x_n, x_m, \infty) \xrightarrow{n,m\to\infty} 1. \tag{6-2}$$

Furthermore, replacing either n or m by a subsequence does not change this convergence behaviour. The same argument with the roles of (x_n) and (y_n) swapped implies

$$\frac{\rho(x_n, y_n)}{\rho(x_n, y_m)} \xrightarrow{n,m\to\infty} 1.$$

Since (x_n) and (y_n) are bounded, there exist subsequences (x_{n_i}) and (y_{n_i}) such that $\rho(x_{n_i}, y_{n_i})$ converges. Equation (6-2) now implies that

$$\lim_{i,m\to\infty} \frac{\rho(y_{n_i}, x_{n_i})}{\rho(y_{n_i}, x_m)} = 1$$

and, therefore,

$$\lim_{i\to\infty} \rho(y_{n_i}, x_{n_i}) = \lim_{i,n\to\infty} \rho(y_{n_i}, x_n).$$

Using equation (6-2) with the roles of (x_n), (y_n) swapped, we obtain

$$\lim_{i,n\to\infty} \frac{\rho(x_n, y_n)}{\rho(x_n, y_{n_i})} = 1$$

and, therefore,

$$\lim_{n\to\infty} \rho(x_n, y_n) = \lim_{i,n\to\infty} \rho(x_n, y_{n_i}).$$

This implies that $\rho(x_n, y_n)$ converges whenever both sequences are Cauchy sequences (provided that ρ has a point at infinity). □

Proof of Theorem 6.2 Let $\bar{Z}$ and $\overline{\mathrm{crt}}$ be as defined before. We start by proving that $\overline{\mathrm{crt}}$ is well defined. Let (w_n), (x_n), (y_n) and (z_n) be Cauchy sequences in Z. By Proposition 6.4, $\rho(\cdot_n, \cdot_n)$ converges for any two of the sequences. Therefore, $\mathrm{crt}(w_n, x_n, y_n, z_n)$ converges as well and, by the symmetry condition, it converges to a point in $\overline{\mathrm{Im(crt)}} \subseteq \bar{\Delta}$.

We are left to show that $\lim_{n\to\infty} \mathrm{crt}(w_n, x_n, y_n, z_n) = \lim_{n\to\infty} \mathrm{crt}(w'_n, x'_n, y'_n, z'_n)$ for $(w_n) \sim (w'_n)$, $(x_n) \sim (x'_n)$, $(y_n) \sim (y'_n)$ and $(z_n) \sim (z'_n)$. Again, we will prove the statement for $\rho(x_n, y)$ and a quasimetric ρ that induces crt and has a point at infinity. Repeating this argument then implies, as above, that the statement for $\mathrm{crt}(w_n, x_n, y_n, z_n)$ holds.

So let ρ be a quasimetric that induces crt and has a point at infinity, denoted by ∞. Let $(x_n) \sim (x'_n)$. Since $c(x_n, x'_n, y, z) \to -\infty$ for all good pairs, it is easy to see that either $\rho(x_n, x'_n) \to 0$ or x_n and x'_n both diverge to infinity.

If (x_n) diverges to ∞, then x'_n has to diverge to infinity too; hence $\lim_{n\to\infty} \rho(x_n, y) = \lim_{n\to\infty} \rho(x'_n, y)$ for all $y \in Z$.

Now suppose (x_n) does not diverge to ∞. It has to be bounded by Lemma 5.2, and $\rho(x_n, x'_n) \xrightarrow{n\to\infty} 0$. By Proposition 6.4, $\rho(x_n, y)$ and $\rho(x'_n, y)$ both converge. Suppose $\rho(x_n, y) \xrightarrow{n\to\infty} 0$. Then

$$\rho(x'_n, y) \le K \max(\rho(x'_n, x_n), \rho(x_n, y)) \xrightarrow{n\to\infty} 0.$$

Thus, $\lim_{n\to\infty} \rho(x'_n, y) = 0 = \lim_{n\to\infty} \rho(x_n, y)$.

Finally, suppose $\rho(x_n, y) \to r$ for some positive real number. Then, by swapping x_n and x'_n in the argument above, $\rho(x'_n, y)$ doesn't converge to zero. Therefore and because (x_n) and (x'_n) are both bounded, y, ∞ is a good pair for both sequences. Since the two sequences are equivalent by assumption,

$$c(x_n, x'_n, y, \infty) \to -\infty.$$

The symmetry condition implies

$$\mathrm{crt}(x_n, x'_n, y, \infty) \to (0:1:1).$$

In other words,

$$\mathrm{crt}(x_n, x'_n, y, \infty) = \frac{\rho(x_n, y)}{\rho(x'_n, y)} \to 1$$

and, therefore,

$$\lim_{n\to\infty} \rho(x_n, y) = \lim_{n\to\infty} \rho(x'_n, y).$$

Analogously to the second half of the proof of Proposition 6.4, we show that $\lim_{n\to\infty} \rho(x_n, y_n) = \lim_{n\to\infty} \rho(x'_n, y_n)$ for all Cauchy sequences $(x_n) \sim (x'_n), (y_n)$. Thus, $\lim_{n\to\infty} \mathrm{crt}(w_n, x_n, y_n, z_n) = \lim_{n\to\infty} \mathrm{crt}(w'_n, x'_n, y'_n, z'_n)$ and therefore, $\overline{\mathrm{crt}}$ is well defined.

Given a Möbius space (Z, crt) that satisfies the symmetry condition, we have constructed a new strong Möbius space $(\bar{Z}, \overline{\mathrm{crt}})$. We also have a canonical map of Z into $\bar{Z}$ that preserves the Möbius structure (hence it is also a topological embedding).

We are left to show that $\overline{Z}$ is complete and that $\overline{Z}$ is unique. We prove completeness first. Suppose that $\xi^m = [(x_n^{(m)})_n] \in \overline{Z}$ is such that $(\xi^m)_m$ is a Cauchy sequence in $\overline{Z}$. We will often identify ξ^m with the representative $(x_n^{(m)})$. Choose a quasimetric ρ on Z that induces crt and let $\overline{\rho}$ be the extension to $\overline{Z}$. Clearly, $\overline{\rho}$ induces $\overline{\mathrm{crt}}$. By Lemma 5.2, $(\xi^m)_m$ either diverges to infinity, or it is bounded with respect to $\overline{\rho}$.

We analyse the point at infinity in $\overline{Z}$ with respect to $\overline{\rho}$. Let it be represented by a Cauchy sequence (z_n) in Z. Then $\overline{\rho}((z_n),(y_n)) = \infty$ for all Cauchy sequences (y_n) in Z that are not equivalent to (z_n). This means that

$$\infty = \overline{\rho}((z_n),(y_n)) = \lim_{n\to\infty} \rho(z_n, y_n),$$

which is the same as saying that (z_n) diverges to infinity. So the point at infinity with respect to $\overline{\rho}$ is the equivalence class of all sequences in Z that diverge to infinity with respect to ρ.

Before we study the convergence of our sequence $(\xi^m)_m$, we need to take a look at convergence in the Möbius topology. Given a strong Möbius space (Z', M'), a sequence x_n in Z' converges to x if and only if, for all nondegenerate triples $A = (\omega,\alpha,\beta)$ in Z' and all $y \in Z'$ such that y does not lie at infinity with respect to ρ_A, we have $\rho_A(x_n, y) \to \rho_A(x, y)$. By Lemmas 3.5 and 4.2, if a Möbius structure crt is induced by a quasimetric ρ, then the induced semimetrics ρ_A are quasimetrics and have the form

$$\rho_A(x,y) = \frac{\rho(x,y)}{\rho(x,\omega)\rho(\omega,y)} \frac{\rho(\alpha,\omega)\rho(\omega,\beta)}{\rho(\alpha,\beta)}.$$

We see that, as long as x_n does not diverge to infinity with respect to ρ, it is sufficient to prove that $\rho(x_n, y) \to \rho(x, y)$ for all y. In particular, since every strong Möbius structure is induced by a bounded quasimetric ρ by Proposition 3.3, we can simply use such a quasimetric to study convergence.

Returning to the space $(\overline{Z}, \overline{\mathrm{crt}})$ constructed above, if we pick a bounded quasimetric ρ that induces crt, then $\overline{\rho}$ will be a bounded quasimetric as well. The discussion above implies that a sequence $(\xi^m)_m$ converges to a point ξ if and only if $\overline{\rho}(\xi^m, \eta) \to \overline{\rho}(\xi, \eta)$ for all $\eta \in \overline{Z}$.

Back to the sequence $(\xi^m)_m$. Since we assume ρ to be bounded, any Cauchy sequence in (Z, M) is bounded with respect to ρ. We need to find a Cauchy sequence $(x_l)_l$ in Z such that $(\xi^m)_m$ converges to that sequence in the Möbius topology as m tends to infinity. Since ρ is bounded, $\xi^m = [(x_n^{(m)})_n]$ can be represented by a bounded Cauchy sequence for every m. By Lemma 5.2,

$$\rho(x_n^{(m)}, x_{n'}^{(m)}) \xrightarrow{n,n'\to\infty} 0.$$

Thus, for every fixed m and every $\epsilon > 0$, we find a natural number N_m such that for all $n, n' \geq N_m$, we have

$$\rho(x_n^{(m)}, x_{n'}^{(m)}) < \epsilon.$$

Let (y_n) be a Cauchy sequence in Z. Since $\overline{\rho}$ is bounded, the sequence $(\xi^m)_m$ is bounded and we find some constant $B > 0$ such that $\overline{\rho}(\xi^m, (y_n)) < B$ for all $m \in \mathbb{N}$. Therefore, for every m we find some natural number $\overline{N}_m$ such that for all $n \geq \overline{N}_m$, we have

$$\rho(x_n^{(m)}, y_n) \leq 2B.$$

Since $(\xi^m)_m$ is a bounded Cauchy sequence by assumption, we also find for every $\epsilon > 0$ a natural number M such that for all $m, m' \geq M$,

$$\overline{\rho}(\xi^m, \xi^{m'}) < \epsilon.$$

We now use the following technical lemma.

Lemma 6.5 *There exists a sequence $(x_l)_l$ in Z satisfying the following properties*:

(1) $x_l = x_{n_l}^{(m_l)}$.

(2) *The sequences m_l and n_l are increasing.*

(3) *For every $l \in \mathbb{N}$ and all $n \geq n_l$, we have $\rho(x_{n_l}^{(m_l)}, x_n^{(m_l)}) < 1/(lK)$.*

(4) *For every $l \in \mathbb{N}$ and all $m, m' \geq m_l$, we have $\overline{\rho}(\xi^m, \xi^{m'}) \leq 1/(2lK)$.*

(5) *For all $l \leq l' \in \mathbb{N}$ and all $n \geq n_{l'}$, we have $\rho(x_n^{(m_l)}, x_n^{(m_{l'})}) < 1/(lK)$.*

We first show how the lemma completes the proof of Theorem 6.2. Given such a sequence $(x_l)_l$, one immediately sees that for all l and all $l' \geq l$, we have

$$\rho(x_l, x_{l'}) = \rho(x_{n_l}^{(m_l)}, x_{n_{l'}}^{(m_{l'})}) \leq K \max\bigl(\rho(x_{n_l}^{(m_l)}, x_{n_{l'}}^{(m_l)}), \rho(x_{n_{l'}}^{(m_l)}, x_{n_{l'}}^{(m_{l'})})\bigr) \leq K \frac{1}{lK} = \frac{1}{l}.$$

This implies that x_l is bounded and a Cauchy sequence. Furthermore, for any $l_0 \in \mathbb{N}^+$ and $m \geq m_{l_0}$,

$$\begin{aligned}\overline{\rho}(\xi^m, (x_l)_l) &= \lim_{l\to\infty} \rho(x_l^{(m)}, x_{n_l}^{(m_l)}) \\ &\leq \lim_{l\to\infty} K^3 \max\bigl(\rho(x_l^{(m)}, x_l^{(m_{l_0})}), \rho(x_l^{(m_{l_0})}, x_{n_{l_0}}^{(m_{l_0})}), \rho(x_{n_{l_0}}^{(m_{l_0})}, x_{n_l}^{(m_{l_0})}), \rho(x_{n_l}^{(m_{l_0})}, x_{n_l}^{(m_l)})\bigr).\end{aligned}$$

For sufficiently large l, we can estimate each of the four expressions in the maximum. By property (4) above, the limit of the first expression is at most $1/(2l_0K)$. The second and third expression are both bounded by $1/(l_0K)$ due to property (3) for $l \geq \max(n_{l_0}, l_0)$. The fourth expression is bounded by $1/(l_0K)$ due to property (5) for $l \geq l_0$. We conclude

$$\begin{aligned}\overline{\rho}(\xi^m, (x_l)_l) &\leq \lim_{l\to\infty} K^3 \max\bigl(\rho(x_l^{(m)}, x_l^{(m_{l_0})}), \rho(x_l^{(m_{l_0})}, x_{n_{l_0}}^{(m_{l_0})}), \rho(x_{n_{l_0}}^{(m_{l_0})}, x_{n_l}^{(m_{l_0})}), \rho(x_{n_l}^{(m_{l_0})}, x_{n_l}^{(m_l)})\bigr) \\ &\leq \lim_{l\to\infty} \frac{K^2}{l_0} = \frac{K^2}{l_0}.\end{aligned}$$

Thus $\overline{\rho}(\xi^m, (x_l)_l) \xrightarrow{m\to\infty} 0$ and for any other point $(y_l)_l \in \overline{Z}$, we find ϵ_y such that $\overline{\rho}(\xi^m, (y_l)_l) > \epsilon_y$ for m sufficiently large. This implies that

$$\overline{\mathrm{crt}}(\xi^m, (x_l)_l, (y_l)_l, (z_l)_l) \xrightarrow{m\to\infty} (0:1:1) \quad \text{for all } (y_l)_l, (z_l)_l \in \overline{Z} \setminus \{(x_l)_l\}.$$

Since we assume that $(\xi^m)_m$ does not diverge to infinity, we have that $(x_l)_l \neq \infty$ and we can choose $(z_l)_l = \infty$ (by having chosen the original ρ to have a point at infinity). Then, writing $y := (y_l)_l$ and $\infty = (\infty)_l$, this limit takes the form

$$\overline{\mathrm{crt}}(\xi^m, (x_l)_l, y, \infty) \xrightarrow{m\to\infty} (0:1:1).$$

By the definition of $\overline{\mathrm{crt}}$ this implies
$$\frac{\bar{\rho}(\xi^m, y)}{\bar{\rho}((x_l)_l, y)} \xrightarrow{m\to\infty} 1.$$
In other words, $\lim_{m\to\infty} \bar{\rho}(\xi^m, y) = \bar{\rho}((x_l)_l, y)$. This implies that ξ^m converges to (x_l).

We are left to prove the technical lemma and to show that the completion $(\bar{Z}, \overline{\mathrm{crt}})$ is unique up to unique Möbius equivalence. Let (Z', crt') be a complete strong Möbius space and $i : Z \hookrightarrow Z'$ a Möbius embedding, ie an injective map that is a Möbius equivalence onto its image. Further, assume $i(Z)$ is dense in Z' with its Möbius topology. Denote the canonical inclusion of Z into $\bar{Z}$ by i_Z. Since i and i_Z are both injective, we get a bijection $f : i(Z) \to i_Z(Z)$ which sends $i(x)$ to $i_Z(x)$. Since i and i_Z are Möbius equivalences onto their images, they are also homeomorphisms onto their images. Therefore, the map f is a homeomorphism with respect to the subspace topology on $i(Z)$ and $i_Z(Z)$. Since f preserves the Möbius structure and therefore Cauchy sequences and equivalent Cauchy sequences, it extends to a bijection $F : Z' \to \bar{Z}$.

We claim that F is a Möbius equivalence. Let (w, x, y, z) be a nondegenerate quadruple in Z' (clearly, F preserves the Möbius structure on degenerate, admissible quadruples). Then we can approximate these four points by sequences w_n, x_n, y_n, z_n in $i(Z)$. By definition of F,
$$F(w) = \lim_{n\to\infty} F(w_n), \quad F(x) = \lim_{n\to\infty} F(x_n), \quad F(y) = \lim_{n\to\infty} F(y_n), \quad F(z) = \lim_{n\to\infty} F(z_n),$$
and hence
$$\begin{aligned}\overline{\mathrm{crt}}(F(w)F(x)F(y)F(z)) &= \lim_{n\to\infty} \mathrm{crt}(F(w_n)F(x_n)F(y_n)F(z_n)) = \lim_{n\to\infty} \mathrm{crt}(f(w_n)f(x_n)f(y_n)f(z_n)) \\ &= \lim_{n\to\infty} \mathrm{crt}'(w_n, x_n, y_n, z_n) = \mathrm{crt}'(w, x, y, z).\end{aligned}$$
This shows that F preserves the Möbius structure on nondegenerate quadruples. Hence, F is a Möbius equivalence. Since all Möbius equivalences are homeomorphisms, uniqueness follows from the fact that $F|_{i(Z)} = f$ is given and the fact that $i(Z)$ is dense in Z'. This completes the proof of Theorem 6.2 up to the proof of Lemma 6.5. □

Proof of Lemma 6.5 We are left to construct the sequence x_l. We construct x_l inductively. The induction starts as follows: Since $(\xi^m)_m$ is a bounded Cauchy sequence, we find natural numbers $M_1 < M_2$ such that
$$\bar{\rho}(\xi^m, \xi^{m'}) < \begin{cases} \dfrac{1}{2K} & \text{for all } m, m' \geq M_1, \\ \dfrac{1}{4K} & \text{for all } m, m' \geq M_2. \end{cases}$$
Now we fix $m = M_1, m' = M_2$. We find a natural number N_1 such that
$$\rho(x_n^{(M_1)}, x_n^{(M_2)}) < \frac{1}{K} \quad \text{for all } n \geq N_1.$$
Since $(x_n^{(M_1)})_n$ is a bounded Cauchy sequence in Z, we can choose N_1 such that, additionally,
$$\rho(x_n^{(M_1)}, x_{n'}^{(M_1)}) < \frac{1}{K} \quad \text{for all } n, n' \geq N_1.$$

Set

$$x_1 := x_{N_1}^{(M_1)}.$$

We see that x_1 satisfies conditions (3) and (4) from above. Now we do the inductive construction.

Suppose we are given points $x_1, \dots, x_l$ in Z satisfying properties (1)–(5). Since $(\xi^m)_m$ is a Cauchy sequence in $\bar{Z}$, we find some $M_{l+1} > m_l$ such that

$$\bar{\rho}(\xi^m, \xi^{m'}) < \frac{1}{2(l+1)K} \quad \text{for all } m, m' \geq M_{l+1}.$$

Put $m_{l+1} := M_{l+1}$. Since we have chosen $M_{l+1} > m_l$, condition (2) stays satisfied for $(m_l)_l$. Furthermore, m_{l+1} satisfies condition (4). Since $\xi^{m_{l+1}}$ is a Cauchy sequence, we find some natural number N_0 such that

$$\rho(x_n^{(m_{l+1})}, x_{n'}^{(m_{l+1})}) < \frac{1}{(l+1)K} \quad \text{for all } n, n' \geq N_0.$$

Thus condition (3) is satisfied if we choose $n_{l+1} \geq N_0$. By condition (4), we know that

$$\bar{\rho}(\xi^{m_i}, \xi^{m_{l+1}}) < \frac{1}{2iK} \quad \text{for all } i < l+1.$$

Therefore, we find some natural numbers N_i such that

$$\rho(x_n^{(m_i)}, x_n^{(m_{l+1})}) < \frac{1}{iK} \quad \text{for all } n \geq N_i.$$

We put $N := \max(N_0, N_1, \dots, N_l, n_l)$ and get

$$\rho(x_n^{(m_i)}, x_n^{(m_{l+1})}) < \frac{1}{iK} \quad \text{for all } n \geq N \text{ and } i < l+1.$$

Put $n_{l+1} := N$ and put

$$x_{l+1} := x_{n_{l+1}}^{(m_{l+1})}.$$

By the definition of N, the sequence $(n_l)_l$ satisfies condition (2). Condition (3) is satisfied since $n_{l+1} \geq N_0$. Condition (4) is satisfied by choice of m_{l+1}. Finally, condition (5) is satisfied because $n_{l+1} \geq \max(N_1, \dots, N_l)$. Condition (1) is trivially satisfied and hence we have constructed a sequence with properties (1)–(5). We have seen before that such a sequence is a Cauchy sequence in (Z, crt) and $(\xi^m)_m$ converges to $(x_l)_l$ in $(\bar{Z}, \overline{\mathrm{crt}})$. Hence the Cauchy sequence $(\xi^m)_m$ converges. This implies that $(\bar{Z}, \overline{\mathrm{crt}})$ is complete. □

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