

AG
T

*Algebraic & Geometric
Topology*

Volume 24 (2024)

$\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^n \times S^n$

JIM FOWLER

COURTNEY THATCHER



$\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^n \times S^n$

JIM FOWLER

COURTNEY THATCHER

We determine the homotopy type of quotients of $S^n \times S^n$ by free actions of $\mathbb{Z}/p \times \mathbb{Z}/p$ where $2p > n + 3$. Much like free \mathbb{Z}/p actions, they can be classified via the first p -localized k -invariant, but there are restrictions on the possibilities, and these restrictions are sufficient to determine every possibility in the $n = 3$ case. We use this to complete the classification of free $\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^3 \times S^3$ for $p > 3$ by reducing the problem to the simultaneous classification of pairs of binary quadratic forms. Although the restrictions are not sufficient to determine which k -invariants are realizable in general, they can sometimes be used to rule out free actions by groups that contain $\mathbb{Z}/p \times \mathbb{Z}/p$ as a normal abelian subgroup.

57N65, 57S25

1 Introduction

The topological spherical space form problem asks: what groups can act freely on the sphere and how can these group actions be classified? Conditions for which groups can act were determined during the middle of the last century; see e.g. Smith [28], Milnor [22] and Madsen, Thomas and Wall [18]. The question of *how* free cyclic groups can act on spheres was addressed in the study of lens spaces, with the classification of all free cyclic group actions being completed recently; see Macko and Wegner [16; 17].

This question can easily be extended to actions on products of spheres. What groups can act has been addressed in a number of papers (see e.g. Conner [7], Heller [12], Oliver [26], Adem and Smith [1], Benson and Carlson [3], Hambleton and Ünlü [11] and Okay and Yalçın [25]), while the classification of *how* the simplest of groups do act on products of spheres and what invariants distinguish them has largely been skipped. Here we focus specifically on the *how* question.

To begin addressing how groups act, one might consider the simplest group actions. Free \mathbb{Z}/p actions on $S^n \times S^n$ for $p > \frac{1}{2}(n + 3)$ were addressed by Thatcher [30] — the homotopy type is determined completely by the homotopy groups and the first k -invariant. We consider quotients of free actions of $\mathbb{Z}/p \times \mathbb{Z}/p$ on $S^n \times S^n$ with $n > 1$ odd and $p > \frac{1}{2}(n + 3)$. It turns out that the homotopy classification is similar to the \mathbb{Z}/p case — the classes are determined by the first k -invariants, but the k -invariants are more complicated. A significant insight is the usefulness of localizing at a large prime — while the homotopy groups of spheres are replete with torsion, $\pi_i S^n$ has no p -torsion for $i \leq 2n$ when p is reasonably large.

From this we see that only a couple of nontrivial stages in the localized Postnikov tower carry all the relevant data for our study.

We begin with a review of the cohomology of $\mathbb{Z}/p \times \mathbb{Z}/p$ in Section 2 and then proceed with the classification. In Section 3 we determine the homotopy type in terms of a single k -invariant, or equivalently, in terms of the transgression in a certain spectral sequence, which the reader might also prefer to think of as an Euler class. The homotopy classification of $\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^n \times S^n$ then amounts to a choice of parameters in \mathbb{Z}/p .

In Section 4 we find that there are strong restrictions on the possible k -invariants. In Section 5 we provide constructions of the possible homotopy classes based on these restrictions, and in Section 6 we show that this is the full homotopy classification of $\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^3 \times S^3$ by reducing the classification to that of pairs of binary quadratic forms. One of our main results is the following:

Theorem 6.6 *Let $p > 3$ be prime. If $p \equiv 1 \pmod{4}$, then there are four homotopy classes of quotients of $S^3 \times S^3$ by free $\mathbb{Z}/p \times \mathbb{Z}/p$ actions. If $p \equiv 3 \pmod{4}$, then there are two classes.*

Finally, in Section 8 we show that these restrictions can be used to rule out free actions by groups containing $\mathbb{Z}/p \times \mathbb{Z}/p$ as a normal abelian subgroup. This is consistent with the results about $\text{Qd}(p)$ in a recent paper by Okay and Yalçın [25].

We note that a subsequent paper will provide the homeomorphism classification of these quotients in the case of linear actions.

Acknowledgements

The authors thank Ian Hambleton for helpful conversations and the reviewer for many useful suggestions, including repairing a mistake in an earlier version of this manuscript.

2 The cohomology of $\mathbb{Z}/p \times \mathbb{Z}/p$

To begin, we will need the integral cohomology of $X = (S^n \times S^n)/(\mathbb{Z}/p \times \mathbb{Z}/p)$. To determine this, we first need to consider the ring structure of the integral cohomology of $\mathbb{Z}/p \times \mathbb{Z}/p$. It is known that $H^*(\mathbb{Z}/p; \mathbb{Z}/p) = \mathbb{F}_p[a] \otimes \wedge(u)$, where $|u| = 1$, $|a| = 2$, and $\beta(u) = a$ with β the Bockstein homomorphism, and that $H^*(\mathbb{Z}/p; \mathbb{Z}) = \mathbb{Z}[a]/(pa)$, where $|a| = 2$. It follows from the Künneth theorem that $H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}/p) \cong \mathbb{F}_p[a, b] \otimes \wedge(u, v)$, where $|u| = |v| = 1$ and $|a| = |b| = 2$, but $H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z})$ requires a bit more work.

The homology and cohomology groups themselves can be determined using the Künneth theorem and universal coefficients.

Proposition 2.1 *The integral homology groups of $\mathbb{Z}/p \times \mathbb{Z}/p$ are*

$$H_k(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ (\mathbb{Z}/p)^{(k+3)/2} & \text{for } k > 0 \text{ odd,} \\ (\mathbb{Z}/p)^{k/2} & \text{for } k > 0 \text{ even.} \end{cases}$$

The integral cohomology groups are

$$H^k(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ 0 & \text{for } k = 1, \\ (\mathbb{Z}/p)^{(k-1)/2} & \text{for } k > 1 \text{ odd,} \\ (\mathbb{Z}/p)^{(k+2)/2} & \text{for } k > 1 \text{ even.} \end{cases}$$

The ring structure can then be determined by piecing together the exact sequences in cohomology associated to the short exact sequences $0 \rightarrow \mathbb{Z}/p \rightarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p \rightarrow 0$ and $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/p \rightarrow 0$. We take $G = \mathbb{Z}/p \times \mathbb{Z}/p$ for notational ease. Then the triangle in the diagram

$$\begin{array}{ccccccc} H^n(G; \mathbb{Z}) & \xrightarrow{\rho} & H^n(G; \mathbb{Z}/p) & \xrightarrow{\tilde{\beta}} & H^{n+1}(G; \mathbb{Z}) & \xrightarrow{p} & H^{n+1}(G; \mathbb{Z}) \\ & & & \searrow \beta & \downarrow \rho & & \\ & & & & H^{n+1}(G; \mathbb{Z}/p) & & \end{array}$$

commutes, where β is the Bockstein associated to the first short exact sequence above, $\tilde{\beta}$ is the Bockstein associated to the second one, ρ is the homomorphism induced by the map $\mathbb{Z} \rightarrow \mathbb{Z}/p$, and p is the map induced by multiplication by p . This along with the ring structure of $H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}/p)$ allows one to find the ring structure of $H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z})$. This ring structure is given, among other places, in [6; 27].

Theorem 2.2 *The integral cohomology ring of $\mathbb{Z}/p \times \mathbb{Z}/p$ is*

$$H^*(\mathbb{Z}/p \times \mathbb{Z}/p; \mathbb{Z}) \cong \mathbb{Z}[a, b, c]/(pa, pb, pc, c^2),$$

where $|a| = |b| = 2$ and $|c| = 3$.

3 Homotopy equivalence and the k -invariants

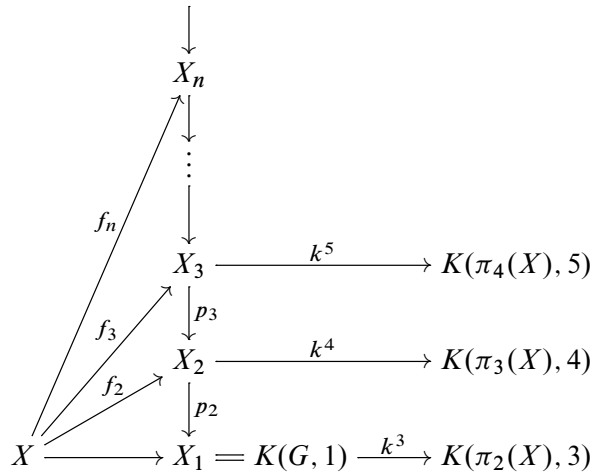
Let $G = \mathbb{Z}/p \times \mathbb{Z}/p$ act freely on $S^n \times S^n$, and let X be the resulting quotient manifold, which may only be a TOP manifold. A simple example of such an action is given by the first \mathbb{Z}/p acting freely on the first S^n and the second \mathbb{Z}/p acting freely on the second S^n in such a way that the resulting quotient manifold is the product of two lens spaces. We wish to determine when two arbitrary free actions of $\mathbb{Z}/p \times \mathbb{Z}/p$ result in homotopy equivalent quotients.

For $p > 3$, the fundamental group $\pi_1(X) = G$ acts trivially on the homology of the universal cover of X because $GL_2(\mathbb{Z})$ has no p -torsion. So by [13, Remark 2.19], it follows that X is nilpotent, and hence X has a Postnikov tower that admits principal refinements and X can be p -localized.

Definition 3.1 A connected space X n -simple if $\pi_1(X)$ is abelian and acts trivially on $\pi_i(X)$ for $1 < i \leq n$.

An n -simple space has a Postnikov tower that consists of principal fibrations through the n^{th} stage. We briefly describe the construction, but more specific details can be found in [19]. The first stage is taken to be $X_1 = K(\pi_1(X), 1)$, with $f_1: X \rightarrow X_1$ inducing an isomorphism on π_1 . The map $p_i: X_i \rightarrow X_{i-1}$ is constructed iteratively as the fibration induced from the path space fibration over $K(\pi_i X, i + 1)$ by the map $k^{i+1}: X_{i-1} \rightarrow K(\pi_i X, i + 1)$. The k^{i+1} are called k -invariants, and are thought of as cohomology classes. There are maps $f_i: X \rightarrow X_i$ for $1 \leq i \leq n$ such that $p_i \circ f_i = f_{i-1}$, and each f_i induces an isomorphism on π_k for all $k \leq i$. Additionally, $\pi_k(X_i) = 0$ for $k > i$.

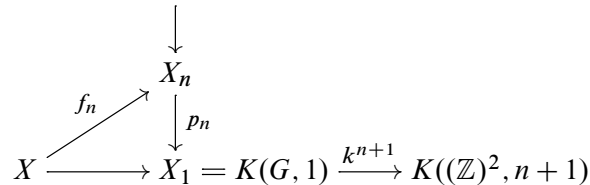
The bottom of the Postnikov tower for an n -simple space generically looks like:



Lemma 3.2 Let $n \geq 3$. For $p > 3$, $X = (S^n \times S^n)/(\mathbb{Z}/p \times \mathbb{Z}/p)$ is n -simple.

Proof Since $\pi_i(X) \cong \pi_i(S^n \times S^n) \cong \pi_i(S^n) \times \pi_i(S^n)$ we see that $\pi_2(X) \cong \pi_3(X) \cong \dots \cong \pi_{n-1}(X) = 0$, and hence there is one nontrivial homotopy group $\pi_i X$ for $1 < i < n + 1$: $\pi_n(X) \cong \pi_n(S^n \times S^n) = \mathbb{Z}^2$. Since $\text{Aut}(\mathbb{Z}^2)$ only has 2- and 3-torsion and $p > 3$, π_1 acts trivially on $\pi_i(X)$ for $1 < i \leq n$. \square

Since $\pi_i(X)$ is trivial for $1 < i < n$, $X_1 \simeq X_2 \simeq \dots \simeq X_{n-1}$, and the bottom of the Postnikov tower becomes



As X is nilpotent, the Postnikov tower above the n^{th} step admits principal refinements. Specifically, using the notation in [20], there is a central $\pi_1(X)$ -series $1 = G_{j,r_j} \subset \dots \subset G_{j,0} = \pi_j(X)$ for each $j > n$ such that $A_{j,l} = G_{j,l}/G_{j,l+1}$ for $0 \leq l < r_j$ is abelian and $\pi_1(X)$ acts trivially on $A_{j,l}$. The $(n+1)^{\text{st}}$ stage is then a finite collection of spaces $X_{n+1,l}$ constructed from maps $k^{n+2,l}: X_{n+1,l} \rightarrow K(A_{n+1,l}, n+2)$

and with $X_{n+1,0} = X_n$. Similarly, the $(n+i)^{\text{th}}$ stage is a finite collection of spaces $X_{n+i+1,l}$ constructed from maps $k^{n+i+1,l}: X_{n+i,l} \rightarrow K(A_{n+i,l}, n+i+1)$ and with $X_{n+i,0} = X_{n+i-1,r_{n+i-1}}$.

Additionally, since X is nilpotent, X can be p -localized. This is done by inductively p -localizing the Postnikov tower, i.e. the $(X_j)_{(p)}$ are inductively constructed using fibrations with the $K(\pi, j)$, where each π is a $\mathbb{Z}_{(p)}$ -module; see for example [20, Theorem 5.3.2] or Sullivan’s notes [29]. Specifically we localize the first stage $(X_1)_{(p)} = (K(\pi_1(X), 1))_{(p)} = K((\pi_1(X))_{(p)}, 1) = K((\mathbb{Z}/p)^2, 1) = X_1$ and localize the n^{th} homotopy group $(\pi_n X)_{(p)} = \pi_n X \otimes \mathbb{Z}_{(p)} = (\mathbb{Z}_{(p)})^2$, and then consider the following diagram:

$$\begin{array}{ccccccc}
 K(\pi_n X, n) & \longrightarrow & X_n & \longrightarrow & X_1 & \xrightarrow{k^{n+1}} & K(\pi_n X, n+1) \\
 \downarrow & & \downarrow \phi_{n+1} & & \downarrow \phi_n & & \downarrow \\
 K((\pi_n X)_{(p)}, n) & \longrightarrow & (X_n)_{(p)} & \longrightarrow & (X_1)_{(p)} & \xrightarrow{(k^{n+1})_{(p)}} & K((\pi_n X)_{(p)}, n+1)
 \end{array}$$

Here $(k^{n+1})_{(p)}$ is the p -localization of k^{n+1} . The right square commutes up to homotopy and there exists a map ϕ_{n+1} , that is, localization of X_n at p , such that the middle and left squares commute up to homotopy. Similar arguments can be made for the stages above n , and then we take $X_{(p)} = \lim(X_i)_{(p)}$ and $\phi = \lim \phi_i: X \rightarrow X_{(p)}$.

We note that the unique map (up to homotopy) ϕ localizes the homotopy and homology groups of X . In particular, $\phi_*(\pi_i X) = (\pi_i X)_{(p)}$, and further, $\phi_*: [X_{(p)}, Z] \rightarrow [X, Z]$ is an isomorphism for any p -local space Z [13; 20].

By [4] the unstable homotopy group $\pi_i(S^n)$ has no p -torsion for $i < 2p + n - 3$. We restrict to $p > \frac{1}{2}(n + 3)$, so that $\pi_i(X)$ has no p -torsion for $i \leq 2n$. It follows that $A_{j,l}$ has no p -torsion for $n < j \leq 2n$, and since $A_{j,l}$ is finite for all $j > n$, $(A_{j,l})_{(p)} = A_{j,l} \otimes \mathbb{Z}_{(p)} = 0$, $K((A_{j,l})_{(p)}, j+1)$ is a point, and $(k^{m+1})_{(p)} = 0$, where $(k^{m+1})_{(p)}$ is the p -localized k -invariant associated with the m^{th} stage ($X_m = X_{j,l}$). Since the construction of the tower becomes formal after the dimension of X (after $2n$), the only nontrivial k -invariant in the localized Postnikov tower before it becomes formal is $(k^{n+1})_{(p)} \in H^{n+1}(X_1; (\mathbb{Z}_{(p)})^2) \cong (\mathbb{Z}/p)^{n+3}$. Given an identification of π_1 and π_n , this p -localized first k -invariant then determines the homotopy type of the localization. In fact, the first nontrivial k -invariant characterizes X up to homotopy as well.

To state Theorem 3.3, we define G_n following [2]. For $n = 1, 3, 7$ define $G_n = \text{GL}_2(\mathbb{Z})$, and for other positive odd n define G_n to be the subgroup of $\text{GL}_2(\mathbb{Z})$ generated by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Theorem 3.3 *Let X and Y be quotients of free $\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^n \times S^n$ with odd $n \geq 3$, where $p > 3$ satisfies $2p + n - 3 > 2n$, and let k_X^{n+1} and k_Y^{n+1} denote the first nontrivial k -invariant. The*

spaces X and Y are homotopy equivalent if and only if there are isomorphisms $g_1: \pi_1 X \rightarrow \pi_1 Y$ and $g_n: \pi_n X \rightarrow \pi_n Y$ with $g_n \in G_n$ and such that

$$\begin{CD} K(\pi_1(X), 1) @>k_X^{n+1}>> K(\pi_n(X), n+1) \\ @Vg_{1\star}VV @VVg_{n\star}V \\ K(\pi_1(Y), 1) @>k_Y^{n+1}>> K(\pi_n(Y), n+1) \end{CD}$$

commutes up to homotopy, i.e. $k_X^{n+1} \in H^{n+1}(\pi_1 X; \pi_n X)$ and $k_Y^{n+1} \in H^{n+1}(\pi_1 Y; \pi_n Y)$ are identified through the maps induced by g_1 and g_n .

Lemmas 3.4 and 3.5 are used to prove this, and are related to [30, Lemmas 1 and 2], respectively.

Lemma 3.4 *Let X and Y be n -simple spaces with identifications $\pi_1(X) \cong \pi_1(Y) \cong G$ and $\pi_n(X) \cong \pi_n(Y) \cong H$. Further suppose $\pi_i(X) = \pi_i(Y) = 0$ for $1 < i < n$. If, as in Theorem 3.3, the identifications on π_1 and π_n provide an identification of the first nontrivial k -invariants of X and Y in $H^{n+1}(G; H)$, then the n^{th} stages of the Postnikov towers for X and Y are homotopy equivalent, i.e. $X_n \simeq Y_n$.*

Proof We have isomorphisms $g_1: \pi_1 X \rightarrow \pi_1 Y$ and $g_n: \pi_n X \rightarrow \pi_n Y$, and k_X^{n+1} and k_Y^{n+1} are the first nontrivial k -invariants of X and Y , respectively. The k -invariant is regarded as a map

$$k_X^{n+1}: K(\pi_1(X), 1) \rightarrow K(\pi_n(X), n+1).$$

The isomorphism g_1 induces a homotopy equivalence $g_{1\star}: K(\pi_1(X), 1) \rightarrow K(\pi_1(Y), 1)$. Similarly, the isomorphism g_n induces a homotopy equivalence $g_{n\star}: K(\pi_n(X), n+1) \rightarrow K(\pi_n(Y), n+1)$. The identification of the first nontrivial k -invariant means that $g_{n\star} \circ k_X^{n+1}$ is homotopic to $k_Y^{n+1} \circ g_{1\star}$.

The n^{th} stage X_n of the Postnikov tower is constructed as the pullback of the path-space fibration over $K(\pi_n(X), n+1)$ and k_X^{n+1} :

$$\begin{CD} X_n @>>> (K(\pi_n(X), n+1))^I \\ @VVV @VVV \\ K(\pi_1(X), 1) @>k_X^{n+1}>> K(\pi_n(X), n+1) \end{CD}$$

A similar construction is performed for Y_n . We have the following map of fibrations, and we want to define a map f on the fibers:

$$\begin{CD} X_n @>>> K(\pi_1(X), 1) @>k_X^{n+1}>> K(\pi_n(X), n+1) \\ @VfVV @VVg_{1\star}V @VVg_{n\star}V \\ Y_n @>>> K(\pi_1(Y), 1) @>k_Y^{n+1}>> K(\pi_n(Y), n+1) \end{CD}$$

The identification of the first nontrivial k -invariants means the square on the right commutes up to homotopy. Let $h: K(\pi_1(X), 1) \times I \rightarrow K(\pi_n(Y), n+1)$ be a homotopy from $g_{n\star} \circ k_X^{n+1}$ to $k_Y^{n+1} \circ g_{1\star}$. With X_n defined as a pullback, a point in X_n consists of a pair (x, q) with $x \in K(\pi_1(X), 1)$ and

$q: I \rightarrow K(\pi_n(X), n + 1)$ satisfying $q(1) = k_X^{n+1}(x)$. Define $f: X_n \rightarrow Y_n$ by $f(x, q) = (y, r)$ with $y = g_{1\star}(x)$ and $r: I \rightarrow K(\pi_n(Y), n + 1)$ given by

$$r(t) = \begin{cases} g_{n\star}q(t) & \text{if } t \leq \frac{1}{2}, \\ h(x, 2t - 1) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

This provides a construction for $f: X_n \rightarrow Y_n$, and by a theorem of Milnor, the fibers X_n and Y_n have the homotopy types of CW complexes. Therefore we have a commuting diagram of homotopy groups

$$\begin{array}{ccccccccc} \pi_{j+1}K(\pi_1(X), 1) & \longrightarrow & \pi_{j+1}K(\pi_n(X), n + 1) & \longrightarrow & \pi_j X_n & \longrightarrow & \pi_j K(\pi_1(X), 1) & \longrightarrow & \pi_j K(\pi_n(X), n + 1) \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ \pi_{j+1}K(\pi_1(Y), 1) & \longrightarrow & \pi_{j+1}K(\pi_n(Y), n + 1) & \longrightarrow & \pi_j Y_n & \longrightarrow & \pi_j K(\pi_1(Y), 1) & \longrightarrow & \pi_j K(\pi_n(Y), n + 1) \end{array}$$

The five lemma gives us that $\pi_j X_n \cong \pi_j Y_n$ for all j . Thus we have a weak equivalence between spaces having the homotopy type of CW complexes, so we have a homotopy equivalence. \square

Lemma 3.5 *Let M and N be nilpotent spaces such that $H^n(M; \mathbb{Z}) = 0$ and $H^n(N; \mathbb{Z}) = 0$ for $n > m$, for some $m > 0$. If the m^{th} stage of the Postnikov tower for M is homotopy equivalent to the m^{th} stage of the Postnikov tower for N , then M is homotopy equivalent to N , i.e. if $M_m \simeq N_m$ then $M \simeq N$.*

We note that this lemma is essentially [30, Lemma 2] — the difference being the change of “ m -dimensional” to the cohomology requirement above — and the obstruction argument proof works exactly as written.

Proof of Theorem 3.3 As has been our convention, let $G = \mathbb{Z}/p \times \mathbb{Z}/p$.

In one direction, we assume there is a homotopy equivalence from X to Y . On π_1 , the homotopy equivalence provides an isomorphism which then yields a homotopy equivalence between the first stage of a Postnikov tower of X and the same of Y . The next nontrivial stage is stage n , and we have a commutative square

$$\begin{array}{ccc} X_n & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & Y_1 \end{array}$$

The commutativity of this square follows by using a functorial model for the Postnikov tower; see [10, Chapter VI.2]. The vertical maps are fibrations, and taking the cofibers of these vertical maps yields the commutative square displayed in the statement of Theorem 3.3. The condition that g_n , regarded as an element of $GL_2(\mathbb{Z})$, lies in G_n is a consequence of [2, Theorem 6.3].

To prove the other direction, we assume $\pi_1 X$ and $\pi_1 Y$ are identified with G and that this gives an isomorphism $g_1: \pi_1 X \rightarrow \pi_1 Y$, and $\pi_n X$ and $\pi_n Y$ are identified with $(\mathbb{Z})^2$ and that this gives an isomorphism $g_n: \pi_n X \rightarrow \pi_n Y$. These maps induce identifications $\pi_1 X_{(p)} \cong \pi_1 Y_{(p)} \cong G \otimes \mathbb{Z}_{(p)} \cong G$ and $\pi_n X_{(p)} \cong \pi_n Y_{(p)} \cong (\mathbb{Z})^2 \otimes \mathbb{Z}_{(p)} \cong (\mathbb{Z}_{(p)})^2$ after localizing the Postnikov systems of both X and Y at p . We have that $(X_1)_{(p)} = K(\pi_1 X_{(p)}, 1) \simeq (Y_1)_{(p)} = K(\pi_1 Y_{(p)}, 1)$. Let k_X^{n+1} and k_Y^{n+1} be

the first nontrivial k -invariants of X and Y , respectively, and take $(k_X^{n+1})_{(p)}$ and $(k_Y^{n+1})_{(p)}$ to be the p -localized first k -invariants, respectively. Since k_X^{n+1} and k_Y^{n+1} are in the same homotopy class of maps in $[K(G, 1) : K((\mathbb{Z})^2, n + 1)]$, $(k_X^{n+1})_{(p)}$ and $(k_Y^{n+1})_{(p)}$ will be in the same homotopy class of maps in $[K(G, 1) : K((\mathbb{Z}_{(p)})^2, n + 1)]$ by construction. Since $p > 3$, X and Y are n -simple by Lemma 3.2. Given that the localization of both spaces and their homotopy groups preserves this property, $X_{(p)}$ and $Y_{(p)}$ are both n -simple as well, and we can apply Lemma 3.4. It follows that $(X_n)_{(p)} \simeq (Y_n)_{(p)}$.

Since we are assuming $2p + n - 3 > 2n$, we have that $(X_{2n+1,0})_{(p)} \simeq (X_n)_{(p)} \simeq (Y_n)_{(p)} \simeq (Y_{2n+1,0})_{(p)}$. It follows from Lemma 3.5 that $X_{(p)} \simeq Y_{(p)}$. The maps $l_1 : X_{(p)} \rightarrow X_{(0)}$ and $l_2 : Y_{(p)} \rightarrow Y_{(0)}$, given by inverting p , give via the naturality of localization a homotopy equivalence $\iota : X_{(0)} \xrightarrow{\simeq} Y_{(0)}$ and identifications of $\pi_n X_{(0)}$ and $\pi_n Y_{(0)}$ with $(\mathbb{Q})^2$. The following commutes up to homotopy:

$$\begin{array}{ccc} X_{(p)} & \xrightarrow{\simeq} & Y_{(p)} \\ \downarrow l_1 & & \downarrow l_2 \\ X_{(0)} & \xrightarrow{\iota} & Y_{(0)} \end{array}$$

On the other hand, we can consider localization away from p . For X we have the commutative diagram

$$\begin{array}{ccc} S^n \times S^n & \longrightarrow & (S^n \times S^n)[1/p] \\ q \downarrow & & \downarrow q[1/p] \\ X & \longrightarrow & X[1/p] \end{array}$$

Since $\pi_1(X[1/p]) = G \otimes \mathbb{Z}[1/p] = 0$, we see that $\pi_j(X[1/p]) \cong \pi_j((S^n \times S^n)[1/p])$ for all j . Thus $q[1/p]$ induces an isomorphism on every homotopy group, and is a homotopy equivalence since $(S^n \times S^n)[1/p]$ and $X[1/p]$ both have the homotopy types of CW complexes. Similarly we have a homotopy equivalence $(S^n \times S^n)[1/p] \simeq Y[1/p]$. Invoking [2, Theorem 6.3], we can realize any element of $GL_2(\mathbb{Z})$ via a homotopy equivalence $S^n \times S^n \simeq S^n \times S^n$. Composing these equivalences yields

$$X[1/p] \simeq (S^n \times S^n)[1/p] \simeq (S^n \times S^n)[1/p] \simeq Y[1/p],$$

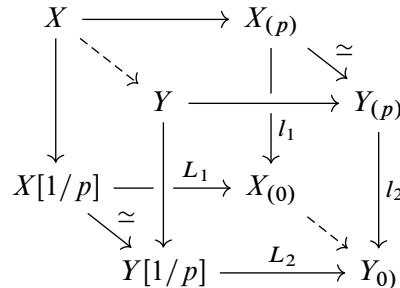
so the map $X[1/p] \rightarrow Y[1/p]$ on π_n is $g_n \otimes \mathbb{Z}[1/p]$.

Since we have maps $X[1/p] \rightarrow X_{(0)}$ and $Y[1/p] \rightarrow Y_{(0)}$ given by inverting everything else, the naturality of localization gives us a map $\iota' : X_{(0)} \xrightarrow{\simeq} Y_{(0)}$. It is a homotopy equivalence because it induces an isomorphism on all of the homotopy groups. We have a diagram that commutes up to homotopy:

$$\begin{array}{ccc} X[1/p] & \xrightarrow{\simeq} & Y[1/p] \\ \downarrow L_1 & & \downarrow L_2 \\ X_{(0)} & \xrightarrow{\iota'} & Y_{(0)} \end{array}$$

Since $X_{(0)}$ and $Y_{(0)}$ are $K((\mathbb{Q})^2, n)$, homotopy classes of maps from $X_{(0)}$ to $Y_{(0)}$ are identified with elements of $\text{Hom}(\pi_n X \otimes \mathbb{Q}, \pi_n Y \otimes \mathbb{Q})$, but by construction, ι and ι' are identified by their action on π_n .

The space X is the homotopy pullback of $X_{(p)}$ and $X[1/p]$ along l_1 and L_1 . For $x \in X$, write x_1 and x_2 for the images of x in $X_{(p)}$ and $X[1/p]$, respectively, so $l_1(x_1)$ and $L_1(x_2)$ are connected by a path in $X_{(0)}$. To map into Y , a homotopy pullback, it is enough to provide maps $X \rightarrow Y_{(p)}$ and $X \rightarrow Y[1/p]$ which agree up to a path in $Y_{(0)}$. Combining the localization squares for X and Y and all of the maps we have constructed between the squares, we have the following cube that commutes up to homotopy, thereby providing maps $X \rightarrow Y_{(p)}$ and $X \rightarrow Y[1/p]$ which agree up to homotopy:



From this we obtain maps of short exact sequences on homotopy for all j :

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \pi_j X & \longrightarrow & \pi_j X_{(p)} \oplus \pi_j X[1/p] & \longrightarrow & \pi_j X_{(0)} & \longrightarrow & 0 \\
 \parallel & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \parallel \\
 0 & \longrightarrow & \pi_j Y & \longrightarrow & \pi_j Y_{(p)} \oplus \pi_j Y[1/p] & \longrightarrow & \pi_j Y_{(0)} & \longrightarrow & 0
 \end{array}$$

The five lemma gives isomorphisms on the homotopy groups of X and Y . This then gives a homotopy equivalence from X to Y as they are both CW complexes. □

4 Restrictions on the first k -invariant

Throughout this section we will continue to let $G = \mathbb{Z}/p \times \mathbb{Z}/p$ and $X := (S^n \times S^n)/G$, where $p > 3$ is an odd prime and $n \geq 3$ is odd. The first stage of the Postnikov system provides a fibration: $K(\pi_n(X), n) \xrightarrow{j} X_n \rightarrow X_1 = K(\pi_1 X, 1)$. The space X_n is induced from the path-space fibration over $K(\pi_n(X), n + 1)$, so the fundamental group $\pi_1(X_1) = G$ acts trivially on the homology of $K(\pi_n(X), n)$. This results in an exact sequence

$$\dots \rightarrow H^n(X_n; \pi_n(X)) \xrightarrow{j^*} H^n(K(\pi_n(X), n); \pi_n(X)) \xrightarrow{\tau} H^{n+1}(X_1; \pi_n(X)),$$

where τ is the transgression. By [21, Section 6.2], τ is also the differential $\tau = d_{n+1} : E_{n+1}^{0,n} \rightarrow E_{n+1}^{n+1,0}$ in the Serre spectral sequence of the fibration. As described in [8, Section 3.7], the fundamental classes of the fiber $K(\pi_n(X), n)$ and the base X_1 correspond under the transgression. If $\iota \in H^n(K(\pi_n(X), n); \pi_n(X))$ is the fundamental class of the fiber, the k -invariant $k^{n+1} \in H^{n+1}(X_1; \pi_n(X))$ is the pullback of the fundamental class of the base space, and $\tau(\iota) = k^{n+1}$.

On the other hand, consider the Borel fibration

$$S^n \times S^n \xrightarrow{i} (S^n \times S^n)_{hG} \rightarrow BG,$$

where $(S^n \times S^n)_{hG} := (EG \times S^n \times S^n)/G \simeq (S^n \times S^n)/G = X$.

There is a map of fibrations

$$\begin{array}{ccccc} S^n \times S^n & \xrightarrow{i} & X & \xrightarrow{f_1} & BG \\ \downarrow \phi_n & & \downarrow f_n & & \downarrow = \\ K(\pi_n(X), n) & \xrightarrow{j} & X_n & \xrightarrow{p_n} & BG \end{array}$$

where the map $\phi_n: S^n \times S^n \rightarrow K(\pi_n(X), n)$ classifies the fundamental class in $H^n(S^n \times S^n; \mathbb{Z}^2)$, and $f_n: X \rightarrow X_n$ is the n -equivalence in the Postnikov tower. Since $\pi_1(BG) = G$ is a finite group generated by odd-order elements, it acts trivially on the cohomology of the fiber (see [25]), and we obtain maps between the induced exact sequences in cohomology:

$$\begin{array}{ccccc} \cdots \rightarrow H^n(X; \pi_n(X)) & \xrightarrow{i^*} & H^n(S^n \times S^n; \pi_n(X)) & \xrightarrow{\bar{\tau}} & H^{n+1}(X_1; \pi_n(X)) \\ & \uparrow f_n^* & & & \uparrow = \\ \cdots \rightarrow H^n(X_n; \pi_n(X)) & \xrightarrow{j^*} & H^n(K(\pi_n(X), n); \pi_n(X)) & \xrightarrow{\tau} & H^{n+1}(X_1; \pi_n(X)) \end{array}$$

It follows that $\bar{\tau}(\phi_n^*(\iota)) = \tau(\iota) = k^{n+1}$ for the fundamental class $\iota \in H^n(K(\pi_n(X), n), \pi_n(X))$, which corresponds to the identity map under the equivalence $H^n(K(\pi_n(X), n), \pi_n(X)) \cong \text{Hom}(\pi_n(X), \pi_n(X))$ by the universal coefficient theorem. Further, since $H_{n-1}(S^n \times S^n) = 0$, the universal coefficient theorem also gives $H^n(S^n \times S^n; \mathbb{Z}^2) \cong H^n(S^n \times S^n; \mathbb{Z}) \oplus H^n(S^n \times S^n; \mathbb{Z})$.

We write $(0, 1)$ for the element of $\text{Hom}(\mathbb{Z}^2, \mathbb{Z})$ sending (x, y) to y , write $(1, 0)$ for the element of $\text{Hom}(\mathbb{Z}^2, \mathbb{Z})$ sending (x, y) to x , and set $\iota = (1, 0) \oplus (0, 1) \in H^n(K(\pi_n(X), n); \pi_n(X)) \cong \text{Hom}(\mathbb{Z}^2, \mathbb{Z}^2) \cong \text{Hom}(\mathbb{Z}^2, \mathbb{Z}) \oplus \text{Hom}(\mathbb{Z}^2, \mathbb{Z})$. Then $\phi_n^*(\iota) = \phi_n^*((1, 0) \oplus (0, 1)) = (\alpha, 0) \oplus (0, \gamma) \in H^n(S^n \times S^n; \mathbb{Z}^2) \cong H^n(S^n \times S^n; \mathbb{Z}) \oplus H^n(S^n \times S^n; \mathbb{Z})$. Here α and γ are preferred generators for $H^n(S^n \times S^n; \mathbb{Z}) \cong \mathbb{Z}^2$. It can now be seen that $k^{n+1} = \bar{\tau}((\alpha, 0) \oplus (0, \gamma))$.

It suffices to examine the transgression from the Serre spectral sequence with integral coefficients for the Borel fibration in order to find out information about the first nontrivial k -invariant, k^{n+1} . In particular, for $S^n \times S^n \rightarrow X \rightarrow BG$,

$$E_2^{p,q} = H^p(BG; H^q(S^n \times S^n; \mathbb{Z})) \Rightarrow H^{p+q}(X; \mathbb{Z}).$$

The first nontrivial differential is d_{n+1} , and the transgression

$$d_{n+1}: H^0(BG; H^n(S^n \times S^n; \mathbb{Z})) \rightarrow H^{n+1}(BG; H^0(S^n \times S^n; \mathbb{Z}))$$

here satisfies $d_{n+1}(\alpha) = \bar{\tau}(\alpha, 0)$ and $d_{n+1}(\gamma) = \bar{\tau}(0, \gamma)$. It follows that $k^{n+1} = d_{n+1}(\alpha) \oplus d_{n+1}(\gamma)$. The cohomology ring of $H^*(BG; \mathbb{Z})$ is given in Theorem 2.2 and we use the same notation by taking the generators to be a, b and c , with $|a| = |b| = 2, |c| = 3$ and $pa = pb = pc = c^2 = 0$. Additionally,

we take α and γ to be the generators in degree n of $H^*(S^n \times S^n; \mathbb{Z})$ with $\alpha^2 = \gamma^2 = 0$, as described above. We see that the $E_2 \cong E_{n+1}$ page reads

$2n$	$\alpha\gamma$	0	$\alpha\gamma b$ $\alpha\gamma a$	$\alpha\gamma c$	3 gens	$\alpha\gamma ac$ $\alpha\gamma bc$	$\frac{1}{2}(n+3)$ gens
n	α, γ	0	$\alpha a, \gamma a$ $\alpha b, \gamma b$	αc γc	6 gens	$\alpha ac, \alpha bc$ $\gamma ac, \gamma bc$	$n+3$ gens
0	1	0	a, b	c	$a^2, ab,$ b^2	ac bc	$a^{(n+1)/2}, \dots$ $\dots, b^{(n+1)/2}$
	0	1	2	3	4	5	\dots \dots $n+1$

where $\alpha\gamma b$ is $\alpha\gamma \otimes b$, etc, by abuse of notation. Note that the blank entries are not necessarily 0.

By virtue of its codomain being generated by suitable powers of a and b , the transgression

$$d_{n+1}: E_{n+1}^{0,n} = H^0(BG; H^n(S^n \times S^n; \mathbb{Z})) \rightarrow E_{n+1}^{n+1,0} = H^{n+1}(BG; H^0(S^n \times S^n; \mathbb{Z}))$$

satisfies

$$d_{n+1}(\alpha) = \sum_{i=0}^{(n+1)/2} q_{\alpha,i} a^{(n+1)/2-i} b^i \quad \text{and} \quad d_{n+1}(\gamma) = \sum_{j=0}^{(n+1)/2} q_{\gamma,j} a^{(n+1)/2-j} b^j,$$

where the $q_{\alpha,i}$ and $q_{\gamma,j}$ are elements of \mathbb{Z}/p .

This spectral sequence converges to the integral cohomology of X , and since X is a finite manifold of dimension $2n$, there are restrictions on what the coefficients $q_{\alpha,i}$ and $q_{\gamma,j}$ can be.

Proposition 4.1 *The coefficients $q_{\alpha,0}$ and $q_{\gamma,0}$ (which are coefficients for $a^{(n+1)/2}$) cannot both be zero. Similarly, the coefficients $q_{\alpha,(n+1)/2}$ and $q_{\gamma,(n+1)/2}$ (which are coefficients for $b^{(n+1)/2}$) cannot both be zero.*

Proof Since G acts freely and $H^{2n}((S^n \times S^n)/G; \mathbb{Z}) \cong \mathbb{Z}$, only quotients of the groups generated by the $E_2^{p,q} \cong E_{n+1}^{p,q}$ terms with $p+q < 2n$ or $p=0$ and $q=2n$ can survive. Assume the transgression $d_{n+1}: E_{n+1}^{0,n} \rightarrow E_{n+1}^{n+1,0}$ satisfies $d_{n+1}(\alpha) = q_{\alpha,1} a^{(n-1)/2} b + \dots + q_{\alpha,(n+1)/2} b^{(n+1)/2}$ and $d_{n+1}(\gamma) = q_{\gamma,1} a^{(n-1)/2} b + \dots + q_{\gamma,(n+1)/2} b^{(n+1)/2}$ for some $q_{\alpha,i}, q_{\gamma,j} \in \mathbb{Z}/p$ with $1 \leq i, j \leq \frac{1}{2}(n+1)$. In other words, both $q_{\alpha,0}$ and $q_{\gamma,0}$ vanish.

The $(n+1)^{\text{st}}$ differential takes the generators in $E_{n+1}^{n-1,n}$ to combinations of the generators in $E_{n+1}^{2n,0}$. By Leibniz, d_{n+1} sends $\alpha \otimes a^{(n-1)/2}$ to $q_{\alpha,1} a^{n-1} b + \dots + q_{\alpha,(n+1)/2} a^{(n-1)/2} b^{(n+1)/2}$, and similarly for the other generators. It is not hard to see that the only other nontrivial differential, d_{n+1} , does not hit the subgroup generated by a^{n+1} , and there are no other differentials that map to this group. Therefore

the generated \mathbb{Z}/p is present in $H^{2n}((S^n \times S^n)/G; \mathbb{Z})$ and other cohomology groups in higher degrees. Since $H^{2n}((S^n \times S^n)/G; \mathbb{Z})$ is torsion free and the highest nontrivial degree, we get a contradiction.

The argument for $q_{\alpha,(n+1)/2}$ and $q_{\gamma,(n+1)/2}$ both being nontrivial is similar. □

Observe that Proposition 4.1 also implies that neither $d_{n+1}(\alpha)$ nor $d_{n+1}(\gamma)$ can map to 0. We also see that it holds after replacing the specified generators with their images under an automorphism of G .

Corollary 4.2 For nonzero $\lambda \in H^2(G; \mathbb{Z})$, either $d_{n+1}(\alpha)$ or $d_{n+1}(\gamma)$ is nonzero in

$$H^{n+1}(G; \mathbb{Z})/\lambda^{(n+1)/2}.$$

Proof Suppose φ is an automorphism of G chosen so that $\varphi_*\lambda = a \in H^2(G; \mathbb{Z})$. After twisting by φ the action of G on $S^n \times S^n$, the resulting quotient is homeomorphic (albeit not equivariantly homeomorphic) to the original quotient space. In particular, in that quotient the coefficients $q_{\alpha,0}$ and $q_{\gamma,0}$, namely the coefficients for $a^{(n+1)/2}$, cannot both be zero, which corresponds in the original space to the condition in the corollary. □

5 Constructions

Now we construct examples which are more complicated than lens spaces cross lens spaces. In this section, we take the dimension of the spheres we are acting on to be $n = 2m - 1$ to avoid fractions appearing in subscripts. Let $R = (r_1, \dots, r_m, r'_1, \dots, r'_m)$ and $Q = (q_1, \dots, q_m, q'_1, \dots, q'_m)$ be elements of $(\mathbb{Z}/p)^{2m}$ so that R and Q together generate a copy of $(\mathbb{Z}/p)^2$ inside $(\mathbb{Z}/p)^{2m}$. We refer to these $4m$ parameters as “rotation numbers” in analogy with the case of a lens space.

Let S^{2m-1} be the unit sphere in \mathbb{C}^m , so $S^{2m-1} \times S^{2m-1}$ is a submanifold of $\mathbb{C}^m \times \mathbb{C}^m$. Then R acts on $S^{2m-1} \times S^{2m-1}$ by

$$\begin{aligned} R \cdot (z, z') &= (r, r') \cdot (z, z') = (r, r') \cdot (z_1, \dots, z_m, z'_1, \dots, z'_m) \\ &= (e^{2\pi i r_1/p} z_1, \dots, e^{2\pi i r_m/p} z_m, e^{2\pi i r'_1/p} z'_1, \dots, e^{2\pi i r'_m/p} z'_m), \end{aligned}$$

and similarly Q acts on $S^{2m-1} \times S^{2m-1}$. This provides an action of the group $(\mathbb{Z}/p)^2 \cong \langle R, Q \rangle$ on $S^{2m-1} \times S^{2m-1}$. In analogy with the lens space case, we call such actions “linear” and we write the quotient as $L(p, p; R, Q)$. In the case of lens spaces, the k -invariant is the product of rotation numbers. We now compute the first nontrivial k -invariant in the case of $L(p, p; R, Q)$. We will denote this first nontrivial k -invariant by k in what follows.

Lemma 5.1 Let $L = L(p, p; R, Q)$ and suppose $p > m$. Then $k(L) \in H^{2m}((\mathbb{Z}/p)^2; \mathbb{Z}^2)$ is

$$\left(\prod_{i=1}^m (r_i a + q_i b), \prod_{i=1}^m (r'_i a + q'_i b) \right),$$

where a and b are generators of $H^2((\mathbb{Z}/p)^2; \mathbb{Z})$ as described in Section 2.

In keeping with the analogy to the lens space, Lemma 5.1 states that the k -invariant is the product of rotation classes in $H^2((\mathbb{Z}/p)^2; \mathbb{Z})$.

Proof The k -invariant $k(L) \in H^{2m}(K(G, 1); \mathbb{Z}^2)$ is a homotopy class of maps

$$K(\pi_1 L, 1) \rightarrow K(\pi_{2m-1} L, 2m).$$

The proof makes use of the naturality of the k -invariant. Suppose a \mathbb{Z}/p subgroup of $(\mathbb{Z}/p)^2$ is generated by (α, β) . Then we have a cover

$$\bar{L} = (S^{2m-1} \times S^{2m-1})/\mathbb{Z}/p \rightarrow L(p, p; R, Q).$$

By [30, page 396], the k -invariant $k(\bar{L}) \in H^{2m}(\mathbb{Z}/p; \mathbb{Z}^2)$ associated to the quotient of $S^{2m-1} \times S^{2m-1}$ by the subgroup $\langle (\alpha, \beta) \rangle \cong \mathbb{Z}/p$ is

$$k(\bar{L}) = \left(\prod_{i=1}^m (r_i \alpha + q_i \beta) \omega, \prod_{i=1}^m (r'_i \alpha + q'_i \beta) \omega \right),$$

where ω is the generator in $H^2(\mathbb{Z}/p; \mathbb{Z})$, which is identified with the generator of \mathbb{Z}/p via $H^2(\mathbb{Z}/p; \mathbb{Z}) \cong \text{Ext}(H_1(\mathbb{Z}/p; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/p$.

By universal coefficients and the fact that the cohomology (except in degree zero) of \mathbb{Z}/p and $(\mathbb{Z}/p)^2$ is torsion, we have $\text{Ext}(H_1((\mathbb{Z}/p)^2; \mathbb{Z}), \mathbb{Z}) \cong H^2((\mathbb{Z}/p)^2; \mathbb{Z}) \cong (\mathbb{Z}/p)^2$ and $\text{Ext}(H_1(\mathbb{Z}/p; \mathbb{Z}), \mathbb{Z}) \cong H^2(\mathbb{Z}/p; \mathbb{Z}) \cong \mathbb{Z}/p$, and $H^2((\mathbb{Z}/p)^2; \mathbb{Z}) \rightarrow H^2(\mathbb{Z}/p; \mathbb{Z})$ is dual to the inclusion map $\mathbb{Z}/p \hookrightarrow (\mathbb{Z}/p)^2$; the inclusion map sends the generator of \mathbb{Z}/p to (α, β) , so the dual map sends $xa + yb \in H^2((\mathbb{Z}/p)^2; \mathbb{Z})$ to $(\alpha a + \beta b)\omega$.

By naturality of the k -invariant, the map $H^{2m}((\mathbb{Z}/p)^2; \mathbb{Z}^2) \rightarrow H^{2m}(\mathbb{Z}/p; \mathbb{Z}^2)$ sends $k(L)$ to $k(\bar{L})$. We consider only the left-hand factor of $k(L)$; this is some homogeneous polynomial of degree n in the classes $a, b \in H^2((\mathbb{Z}/p)^2; \mathbb{Z})$. Write this polynomial as $f(a, b)$.

Then the map $H^{2m}((\mathbb{Z}/p)^2; \mathbb{Z}) \rightarrow H^{2m}(\mathbb{Z}/p; \mathbb{Z})$ sends $f(a, b)$ to

$$f(\alpha, \beta)\omega \in H^{2m}(\mathbb{Z}/p; \mathbb{Z}),$$

and therefore, for $\alpha, \beta \in \mathbb{Z}/p$,

$$f(\alpha, \beta) = \prod_{i=1}^m (r_i \alpha + q_i \beta).$$

Now assuming $m < p$, this equality of polynomials as functions gives rise to the desired equality

$$f(a, b) = \prod_{i=1}^m (r_i a + q_i b).$$

The right-hand factor of $k(L)$ is computed the same way. □

6 The $S^3 \times S^3$ classification

Suppose $p > 3$. We now classify $\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^3 \times S^3$ up to homotopy. By Theorem 3.3, this boils down to the k -invariants encoded by the transgression

$$d_4(\alpha) = q_{\alpha,0}a^2 + q_{\alpha,1}ab + q_{\alpha,2}b^2 \quad \text{and} \quad d_4(\gamma) = q_{\gamma,0}a^2 + q_{\gamma,1}ab + q_{\gamma,2}b^2.$$

We therefore package $(d_4(\alpha), d_4(\gamma))$ as a pair (Q_1, Q_2) of binary quadratic forms over \mathbb{Z}/p . The homotopy classification of $(S^3 \times S^3)/(\mathbb{Z}/p \times \mathbb{Z}/p)$ amounts, algebraically, to classifying pairs of binary quadratic forms over \mathbb{Z}/p , up to the action of automorphisms of \mathbb{Z}^2 on the pair (Q_1, Q_2) . For example, the pair (Q_1, Q_2) determines the same equivariant oriented homotopy type as $(Q_1 + Q_2, Q_1)$. Note that $\text{Aut}(\mathbb{Z}^2)$ amounts to the action of

$$\text{SL}_2^\pm(\mathbb{Z}/p) := \{M \in \text{GL}_2(\mathbb{Z}/p) \mid \det M = \pm 1\}$$

on pairs (Q_1, Q_2) . In what follows regard this as a *left* action of $\text{SL}_2^\pm(\mathbb{Z}/p)$ so that $M = (m_{ij}) \in \text{SL}_2^\pm(\mathbb{Z}/p)$ acts via

$$(*) \quad M \cdot (Q_1, Q_2) = (m_{11}Q_1 + m_{12}Q_2, m_{21}Q_1 + m_{22}Q_2).$$

Now we determine the classification disregarding the identification of $\mathbb{Z}/p \times \mathbb{Z}/p$ with π_1 . On the levels of quadratic forms, we may replace the pair (Q_1, Q_2) by (Q'_1, Q'_2) where Q_1 and Q'_1 (as well as Q_2 and Q'_2) are related by a common change of coordinates, i.e. an automorphism of $\mathbb{Z}/p \times \mathbb{Z}/p$, which amounts to $\text{GL}_2(\mathbb{Z}/p)$. In what follows, regard this as a *right* action of $\text{GL}_2(\mathbb{Z}/p)$ on pairs (Q_1, Q_2) .

Lemma 6.1 *Let z be a quadratic nonresidue in \mathbb{Z}/p . A pair of binary quadratic forms (Q_1, Q_2) satisfying the condition in Proposition 4.1 is equivalent to (xa^2, yb^2) or equivalent to $(a^2 + xb^2, 2ab)$ for $x, y \in \mathbb{Z}/p$.*

Proof There are five [24, Theorem IV.10] equivalence classes of binary quadratic forms modulo p , namely the trivial form $Q(a, b) = 0$, two degenerate forms a^2 and za^2 , and two nondegenerate quadratic forms $a^2 + b^2$ and $a^2 + zb^2$.

Suppose Q_1 is degenerate, so $Q_1(a, b) = a^2$ or $Q_1(a, b) = za^2$. Through an automorphism of \mathbb{Z}^2 subtracting a multiple of Q_1 , the form Q_2 becomes $xab + yb^2$ for some $x, y \in \mathbb{Z}/p$. By Proposition 4.1, it cannot be that $y = 0$. Since $y \neq 0$, the automorphism of $\mathbb{Z}/p \times \mathbb{Z}/p$ sending a to a and b to $-ax/(2y) + b$ preserves Q_1 but transforms Q_2 into $yb^2 - (x^2/(4y))a^2$. Subtracting off a multiple of Q_1 via an automorphism of \mathbb{Z}^2 finally transforms Q_2 into yb^2 . Therefore $(Q_1, Q_2) \simeq (xa^2, yb^2)$ for some $x, y \in \mathbb{Z}/p$.

On the other hand, suppose Q_1 is nondegenerate, meaning $Q_1(a, b) = a^2 + b^2$ or $Q_1(a, b) = a^2 + zb^2$. As before, by subtracting off a multiple of Q_1 , the form Q_2 becomes $xab + yb^2$ for some $x, y \in \mathbb{Z}/p$. Either $y \neq 0$ or $y = 0$. If $y \neq 0$, then as before Q_2 is equivalent to $yb^2 - (x^2/(4y))a^2$, which via an automorphism of \mathbb{Z}^2 is transformed into a multiple of b^2 , and this case is then handled by the above case in which Q_1 is degenerate. If $y = 0$, then we are in the situation $(a^2 + zb^2, xab)$ for nonzero z

and x . We apply the automorphism given by $a \mapsto a$ and $b \mapsto 2b/x$ to reduce to a situation of the form $(a^2 + wb^2, 2ab)$ for some nonzero w . □

Proposition 6.2 *A pair of the form (xa^2, yb^2) for nonzero $x, y \in \mathbb{Z}/p$ is equivalent to $(a^2 + wb^2, 2ab)$ for $w \in \mathbb{Z}/p$.*

Proof By independently scaling a and b , and depending on whether or not x and y are quadratic residues, the pair (xa^2, yb^2) is equivalent to

$$(a^2, b^2), \quad (a^2, zb^2), \quad (za^2, b^2) \quad \text{or} \quad (za^2, zb^2)$$

for a quadratic nonresidue $z \in \mathbb{Z}/p^\star$. By exchanging the roles of a and b and swapping the components of the tuple, the pair (a^2, zb^2) is equivalent to (za^2, b^2) . It is also the case that $(za^2, zb^2) \simeq (a^2, b^2)$ because

$$(za^2, zb^2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1/z \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \cdot (a^2, b^2).$$

To conclude the proof, we show $(a^2, wb^2) \simeq (a^2 + 4w^2b^2, 2ab)$ for $w \in \mathbb{Z}/p$. To see this, applying the equivalence given by $a \mapsto a/(2w) - b$ and $b \mapsto a + 2wb$ shows

$$(a^2, wb^2) \simeq \left(\frac{1}{4w^2}a^2 - \frac{1}{w}ab + b^2, wa^2 + 4w^2ab + 4w^3b^2 \right),$$

and then applying the automorphism of \mathbb{Z}^2 corresponding to

$$\begin{pmatrix} 1/(4w^2) & -1/(2w) \\ w & 2w^2 \end{pmatrix} \in \text{SL}_2(\mathbb{Z}/p)$$

implies that

$$(a^2 + 4w^2b^2, 2ab) \simeq \left(\frac{1}{4w^2}a^2 - \frac{1}{w}ab + b^2, wa^2 + 4w^2ab + 4w^3b^2 \right),$$

so $(a^2, wb^2) \simeq (a^2 + 4w^2b^2, 2ab)$. □

It remains to check that $(a^2 + b^2, 2ab)$ is *not* equivalent to $(a^2 + zb^2, 2ab)$.

Lemma 6.3 *If $(a^2 + \delta b^2, 2ab)$ is equivalent to $(a^2 + \delta' b^2, 2ab)$ for nonzero δ and δ' , then $\delta'/\delta \in \mathbb{Z}/p^{\star 4}$.*

Proof We follow the argument in [9]. Suppose $(a^2 + \delta b^2, 2ab)$ is equivalent to $(a^2 + \delta' b^2, 2ab)$ for nonzero δ and δ' . Then there is an $R \in \text{GL}_2(\mathbb{Z}/p)$ and $S \in \text{SL}_2^\pm(\mathbb{Z}/p)$ such that

$$(1) \quad (a^2 + \delta b^2, 2ab) \cdot R = S \cdot (a^2 + \delta' b^2, 2ab).$$

Equality of the first component in each tuple yields

$$(2) \quad (r_{11}^2 + \delta r_{21}^2)a^2 + 2(r_{11}r_{12} + \delta r_{21}r_{22})ab + (r_{12}^2 + \delta r_{22}^2)b^2 = s_{11}a^2 + 2s_{12}ab + \delta' s_{11}b^2.$$

Equality of the coefficients of a^2 and b^2 in (2) yields

$$s_{11} = \delta r_{21}^2 + r_{11}^2 \quad \text{and} \quad \delta' s_{11} = \delta r_{22}^2 + r_{12}^2,$$

respectively, and therefore

$$(3) \quad \delta r_{22}^2 + r_{12}^2 = \delta' \delta r_{21}^2 + \delta' r_{11}^2.$$

Equality of the second component in (1) yields

$$(4) \quad 2r_{11}r_{21}a^2 + (2r_{12}r_{21} + 2r_{11}r_{22})ab + 2r_{12}r_{22}b^2 = s_{21}a^2 + 2s_{22}ab + \delta' s_{21}b^2.$$

Equality of the coefficients of a^2 and b^2 in (4) yields

$$s_{21} = 2r_{11}r_{21} \quad \text{and} \quad \delta' s_{21} = 2r_{12}r_{22},$$

respectively. We conclude

$$(5) \quad r_{12}r_{22} = \delta' r_{11}r_{21}.$$

Squaring both sides of (3) and subtracting 4δ times (5) squared yields

$$(\delta r_{22}^2 - r_{12}^2)^2 = (\delta' \delta r_{21}^2 - \delta' r_{11}^2)^2,$$

and so

$$(6) \quad \delta r_{22}^2 - r_{12}^2 = \pm(\delta' \delta r_{21}^2 - \delta' r_{11}^2).$$

The sign in (6) cannot be positive; if it were, then adding (6) to (3) yields

$$2\delta r_{22}^2 = 2\delta' \delta r_{21}^2,$$

so $r_{22}^2 = \delta' r_{21}^2$. But multiply both sides of (5) by r_{21}^2 and we deduce

$$r_{12}r_{22}r_{21}^2 = \delta' r_{11}r_{21}^2 = r_{22}^2 r_{11}.$$

So either $r_{22} = 0$, in which case $r_{21} = 0$ and the second row of R is zero, or $r_{12}r_{21} = r_{22}r_{11}$ and so $\det R = 0$. In either case we contradict the assumption $R \in \text{GL}_2(\mathbb{Z}/p)$, and so the sign in (6) must be negative, meaning

$$(7) \quad \delta r_{22}^2 - r_{12}^2 = -\delta' \delta r_{21}^2 + \delta' r_{11}^2.$$

The difference of (3) and (7) yields

$$2r_{12}^2 = 2\delta\delta' r_{21}^2,$$

so $\delta\delta'$ is a square in \mathbb{Z}/p . And if our only requirement is that $R \in \text{GL}_2(\mathbb{Z}/p)$, then the necessary condition that $\delta'\delta \in \mathbb{Z}/p^{\star 2}$ would suffice, but we also required $S \in \text{SL}_2^{\pm}(\mathbb{Z}/p)$, or equivalently that $(\det S)^2 = 1$. From (2) and (4),

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} \delta r_{21}^2 + r_{11}^2 & \delta r_{21}r_{22} + r_{11}r_{12} \\ 2r_{11}r_{21} & r_{12}r_{21} + r_{11}r_{22} \end{pmatrix},$$

which means

$$\det S = (\delta r_{21}^2 - r_{11}^2)(r_{12}r_{21} - r_{11}r_{22}).$$

Squaring $\det R$ results in

$$\begin{aligned} (\det R)^2 &= (r_{11}r_{22} - r_{12}r_{21})^2 = r_{11}^2 r_{22}^2 - 2r_{11}r_{12}r_{21}r_{22} + r_{12}^2 r_{21}^2 = r_{11}^4 \frac{\delta'}{\delta} - 2\delta' r_{11}^2 r_{21}^2 + \delta\delta' r_{21}^4 \\ &= \frac{\delta'}{\delta} (\delta r_{21}^2 - r_{11}^2)^2 \end{aligned}$$

by invoking (5) and applying the identities $r_{12}^2 = \delta\delta'r_{21}^2$ and $\delta r_{22}^2 = \delta'r_{11}^2$, which follow from taking the sum and difference of (3) and (7). Consequently,

$$(\det S)^2 = (\delta r_{21}^2 - r_{11}^2)^2 (\det R)^2 = \frac{\delta'}{\delta} (\delta r_{21}^2 - r_{11}^2)^4,$$

so δ'/δ must be a fourth power. □

In particular, $(a^2 + b^2, 2ab)$ is not equivalent to $(a^2 + zb^2, 2ab)$ because z was chosen specifically to be a quadratic nonresidue.

Lemma 6.4 For nonzero $\delta, w \in \mathbb{Z}/p$, the pair $(a^2 + \delta b^2, 2ab)$ is equivalent to $(a^2 + \delta w^4 b^2, 2ab)$.

Proof Choose $r_1, r_2 \in \mathbb{Z}/p$ so that

$$(8) \quad \delta r_1^2 - r_2^2 \equiv 1/w^3 \pmod{p}.$$

This is possible; in fact, there are $p - (\delta/p)$ solutions to (8). Then set

$$R := \begin{pmatrix} w^2 r_2 & \delta w^2 r_1 \\ r_1 & r_2 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} \delta w^4 r_1^2 + w^4 r_2^2 & 2\delta w^4 r_1 r_2 \\ 2w^2 r_1 r_2 & \delta w^2 r_1^2 + w^2 r_2^2 \end{pmatrix}.$$

Because of (8), we have

$$\det R = -w^2(\delta r_1^2 - r_2^2) = -1/w \neq 0 \quad \text{and} \quad \det S = w^6(\delta r_1^2 - r_2^2)^2 = 1,$$

so $R \in \text{GL}_2(\mathbb{Z}/p)$ and $S \in \text{SL}_2(\mathbb{Z}/p)$.

We finish the proof by verifying

$$(9) \quad (a^2 + \delta w^4 b^2, 2ab) \cdot R = S \cdot (a^2 + \delta b^2, 2ab).$$

Comparing the first coordinates each side of (9) shows

$$(r_2 w^2 a + \delta r_1 w^2 b)^2 + \delta w^4 (r_1 a + r_2 b)^2 = (\delta w^4 r_1^2 + w^4 r_2^2) \cdot (a^2 + \delta b^2) + 2\delta w^4 r_1 r_2 \cdot 2ab.$$

Similarly, the second coordinates are equal because

$$2(w^2 r_2 a + \delta w^2 r_1 b)(r_1 a + r_2 b) = 2w^2 r_1 r_2 (a^2 + b^2 \delta) + (\delta w^2 r_1^2 + w^2 r_2^2) 2ab. \quad \square$$

It is easier to see that $(a^2 + \delta b^2, 2ab)$ is equivalent to $(a^2 + \delta w^8 b^2, 2ab)$. Simply replace a by aw and b by b/w^3 to show $(a^2 + \delta w^8 b^2, 2ab) \simeq (w^2 a^2 + \delta w^2 b^2, (2/w^2)ab)$, and then scale the first by $1/w^2$ and the second by w^2 to see that this is equivalent to $(a^2 + \delta b^2, 2ab)$. The challenge of Lemma 6.4 lies in replacing w^8 with w^4 .

Combining Lemmas 6.3 and 6.4 yields the following:

Proposition 6.5 Equivalence classes of pairs of the form $(a^2 + wb^2, 2ab)$ are in one-to-one correspondence with elements of $\mathbb{Z}/p^\times / (\mathbb{Z}/p^\times)^4$, where \mathbb{Z}/p^\times denotes units modulo p .

Observe that the size of $\mathbb{Z}/p^\times / (\mathbb{Z}/p^\times)^4$ depends on $p \pmod{4}$. Specifically, for $p \equiv 1 \pmod{4}$, there are four equivalence classes. These are given by $(a^2 + zb^2, 2ab)$ for z representatives of classes $\mathbb{Z}/p^\times / \mathbb{Z}/p^{\star 4}$.

For $p \equiv 3 \pmod{4}$, there are *two* equivalence classes. For nonzero $x, x', y, y' \in \mathbb{Z}/p$, the pair (xa^2, yb^2) is equivalent to $(x'a^2, y'b^2)$, and every pair is equivalent to either $(a^2 + b^2, 2ab)$ or $(a^2 + zb^2, 2ab)$ for a quadratic nonresidue z . So the only possibilities are $(a^2 + b^2, 2ab) \simeq (a^2, b^2)$ and $(a^2 + zb^2, 2ab)$.

All of this algebra encodes the homotopy type of the quotients, as summarized in the following:

Theorem 6.6 *Let $p > 3$ be prime. If $p \equiv 1 \pmod{4}$, then there are four homotopy classes of quotients of $S^3 \times S^3$ by free $\mathbb{Z}/p \times \mathbb{Z}/p$ actions. If $p \equiv 3 \pmod{4}$, then there are two classes.*

Proof We must construct quotients of $S^3 \times S^3$ by free $\mathbb{Z}/p \times \mathbb{Z}/p$ actions which exhibit these possible k -invariants. For this, we rely on Lemma 5.1. We note that $(a^2 + wb^2, 2ab)$ is equivalent to

$$(a^2 + wb^2 + (1+w)ab, 2ab) = ((a+b)(a+wb), 2ab),$$

so let $R = (1, 1, 2, 0)$ and $Q = (1, w, 0, 1)$, and then $L(p, p; R, Q)$ has k -invariant equivalent to $(a^2 + wb^2, 2ab)$. We must impose the additional condition $w \neq 0$ in order to ensure that this is a *free* action. With this construction in hand, the classification of quotients then follows from Proposition 6.5. \square

Remark 6.7 There are precedents for considering the simultaneous equivalence of forms. The case of simultaneous equivalence of forms over \mathbb{Z} is discussed in [23], but our situation over \mathbb{Z}/p is easier. To make the situation even more concrete, instead of forms, consider matrices; equivalence of forms amounts to congruence of matrices. That setup fits into the work of Corbas and Williams [9] which considers the action of $\mathrm{GL}_2(\mathbb{Z}/p) \times \mathrm{GL}_2(\mathbb{Z}/p)$ on pairs (A, B) of matrices, where GL_2 acts on the right by congruence and on the left as in $(*)$.

7 Lens cross lens

Section 6 completed the classification of $\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^3 \times S^3$, but now we narrow in on a special case. Consider $L_3(p; 1, x) \times L_3(p; 1, y)$, i.e. the product of two lens spaces with rotation numbers x and y , respectively. Viewed as a quotient of $S^3 \times S^3$ by $\mathbb{Z}/p \times \mathbb{Z}/p$, this product has k -invariant (xa^2, yb^2) .

We can classify $L_3(p; 1, x) \times L_3(p; 1, y)$ up to (simple) homotopy equivalence. When $p \equiv 3 \pmod{4}$, any product of 3-dimensional lens spaces is (simple) homotopy equivalent to any other such product.

Proposition 7.1 *Suppose $p \equiv 3 \pmod{4}$. Then for nonzero $x, x', y, y' \in \mathbb{Z}/p$, the pair (xa^2, yb^2) is equivalent to $(x'a^2, y'b^2)$.*

Proof As in the proof of Proposition 6.2, the pair (xa^2, yb^2) is equivalent to

$$(a^2, b^2) \simeq (za^2, zb^2) \quad \text{or} \quad (a^2, zb^2) \simeq (za^2, b^2)$$

for a quadratic nonresidue $z \in \mathbb{Z}/p^\star$. But when $p \equiv 3 \pmod{4}$, the quantity $-z$ is a square, and so

$$(a^2, zb^2) \simeq (a^2, -zb^2) \simeq (a^2, b^2),$$

meaning *all* pairs of the form (xa^2, yb^2) are equivalent. \square

When $p \equiv 1 \pmod{4}$, since

$$(xa^2, yb^2) \simeq (a^2, (y/x)b^2) \simeq (a^2 + 4(y/x)^2b^2, 2ab),$$

the classification boils down to whether or not $2(y/x)$ is a square modulo p .

This is related to work of Kwasik and Schultz; they proved squares of lens spaces are diffeomorphic.

Theorem 7.2 [14] *For p odd and rotation numbers r and q , there is a diffeomorphism*

$$L_3(p; 1, r) \times L_3(p; 1, r) \cong L_3(p; 1, q) \times L_3(p; 1, q).$$

A future paper completes the homeomorphism classification of spaces resulting from “linear” actions such as these products of lens spaces.

8 Some comments on groups containing $\mathbb{Z}/p \times \mathbb{Z}/p$

While we know that \mathbb{Z}/p and $\mathbb{Z}/p \times \mathbb{Z}/p$ can act freely on $S^n \times S^n$, the exact conditions for a group to be able to act freely on $S^n \times S^n$ remains open. Conner [7] and Heller [12] showed that for a group to act freely on $S^n \times S^n$ the group must have rank at most 2, but Oliver [26] showed that A_4 cannot act on $S^n \times S^n$, and so every rank-2 simple group is also ruled out [1]. Explicit examples of free actions by subgroups of a nonabelian extension of S^1 by $\mathbb{Z}/p \times \mathbb{Z}/p$ have been constructed [11], but Okay and Yalçın [25] have shown that $\text{Qd}(p) = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes SL_2(\mathbb{F}_p)$ cannot act freely on $S^n \times S^n$. In this section we show how the restrictions on the k -invariant as described in Section 4 can be useful in determining whether or not a group G containing $\mathbb{Z}/p \times \mathbb{Z}/p$ as a normal abelian subgroup can act freely on $X = S^n \times S^n$. We continue to take $p > 3$ to be an odd prime and $n \geq 3$ to be odd. We align some of our notation with that in [25] to better show the parallel calculations.

Similar to the approach in Section 4, we can consider the Borel fibration

$$X \xrightarrow{i} X_{hG} \rightarrow BG,$$

and the associated Serre spectral sequence

$$E_2^{p,q} = H^p(BG; H^q(X; \mathbb{Z})) \Rightarrow H^{p+q}(X_{hG}; \mathbb{Z})$$

with the first nontrivial differential d_{n+1} . If α and γ are the generators in degree n of $H^*(X; \mathbb{Z})$ with $\alpha^2 = \gamma^2 = 0$, then $d_{n+1}(\alpha) = \bar{\tau}(\alpha, 0)$, $d_{n+1}(\gamma) = \bar{\tau}(0, \gamma)$ and $k^{n+1} = d_{n+1}(\alpha) \oplus d_{n+1}(\gamma)$.

Set K to be the normal abelian subgroup of $\mathbb{Z}/p \times \mathbb{Z}/p$ in G , and consider the restriction of the spectral sequence associated to the Borel fibration to the K action. Then Proposition 4.1 and Corollary 4.2 can sometimes be used to determine if G can act freely on X .

The transgression for the first nontrivial differential of the restriction of the spectral sequence associated to the Borel fibration to K is

$$(d_{n+1})_K: H^0(BK; H^n(X; \mathbb{Z})) \rightarrow H^{n+1}(BK; H^0(X; \mathbb{Z})).$$

Let $\text{Res}_K^G : H^*(G) \rightarrow H^*(K)$ be induced by the inclusion of K into G . Since the Borel construction is natural, it follows that the k -invariant in the restricted case is $k_K^{n+1} = \text{Res}_K^G(d_{n+1}(\alpha)) \oplus \text{Res}_K^G(d_{n+1}(\gamma))$.

Suppose G acts freely on X , so $H^*(X_{hG}; \mathbb{Z}) \cong H^*(X/G; \mathbb{Z})$ is finite-dimensional in each degree and vanishes above $2n$. It follows that the restriction to K gives that $H^*(X_{hK}; \mathbb{Z}) \cong H^*(X/K; \mathbb{Z})$ is also finite-dimensional in each degree and vanishes above $2n$ as K acts freely. If both $(d_{n+1})_K(\alpha)$ and $(d_{n+1})_K(\gamma)$ are zero in $H^{n+1}(K; \mathbb{Z})/\lambda^{(n+1)/2}$, for some nonzero $\lambda \in H^2(K; \mathbb{Z})$, then X/K will fail to be finite-dimensional by Corollary 4.2, and we get a contradiction. Hence G cannot act freely.

As an example, consider $G = \text{Qd}(p) = (\mathbb{Z}/p)^2 \rtimes SL_2(\mathbb{Z}/p)$. We show that one can use the restrictions on the k -invariants and some of the arguments in [25] to see that $\text{Qd}(p)$ cannot act freely on $S^n \times S^n$ for p an odd prime and n odd. This result is consistent with [25, Theorem 5.1].

Since cohomology is taken with \mathbb{Z}/p coefficients in [25], we first set up a relationship between generators from the different coefficient groups. Suppose the first nontrivial differential takes α and γ , also the generators of $H^n(S^n \times S^n; \mathbb{Z}/p)$ by slight abuse of notation, to μ_1 and μ_2 in $H^{n+1}(G; \mathbb{Z}/p)$. Taking K to be the normal elementary abelian subgroup $\mathbb{Z}/p \times \mathbb{Z}/p$ in $G = \text{Qd}(p)$ and restricting the action to K , we have that $\theta_1, \theta_2 \in H^{n+1}(K; \mathbb{Z}/p)$ are such that $\theta_1 = \text{Res}_K^G(\mu_1)$ and $\theta_2 = \text{Res}_K^G(\mu_2)$.

Recall the commuting triangle, from Section 2,

$$\begin{array}{ccccc}
 H^n(K; \mathbb{Z}) & \xrightarrow{\rho} & H^n(K; \mathbb{Z}/p) & \xrightarrow{\tilde{\beta}} & H^{n+1}(K; \mathbb{Z}) & \xrightarrow{p} & H^{n+1}(K; \mathbb{Z}) \\
 & & & \searrow \beta & \downarrow \rho & & \\
 & & & & H^{n+1}(K; \mathbb{Z}/p) & &
 \end{array}$$

Since p is the 0 map, the vertical ρ is injective and $\tilde{\beta}$ is surjective. We can write $H^*(K; \mathbb{Z}/p) = \mathbb{F}_p[x, y] \otimes \wedge(u, v)$, where $|x| = |y| = 2, |u| = |v| = 1, \beta(u) = x, \beta(v) = y$ and $H^*(K; \mathbb{Z}) = \mathbb{F}_p[a, b] \otimes \wedge(c)$, with $|a| = |b| = 2$ and $|c| = 3$. It is not hard to see that $\tilde{\beta}(x) = a, \tilde{\beta}(y) = b$ and $\tilde{\beta}(uv) = c$.

Now the Bockstein generally satisfies $\beta(\delta\varepsilon) = \beta(\delta)\varepsilon + (-1)^{|\delta|}\delta\beta(\varepsilon) = \delta\beta(\varepsilon)$ for δ being $x^i y^j$ and ε being u, v or uv . We see that

$$\beta(H^n(K; \mathbb{Z}/p)) \subseteq \langle x^{(n+1)/2}, x^{(n-1)/2}y, \dots, y^{(n+1)/2} \rangle \subseteq \mathbb{F}_p[x, y],$$

since $n + 1$ is even. Similarly, $\tilde{\beta}$ satisfies $\tilde{\beta}(\delta\varepsilon) = \delta\tilde{\beta}(\varepsilon)$ for δ being $x^i y^j$ and ε being u, v or uv . Again we see that $\tilde{\beta}(H^n(K; \mathbb{Z}/p)) \subseteq \langle a^{(n+1)/2}, a^{(n-1)/2}b, \dots, b^{(n+1)/2} \rangle \subseteq \mathbb{F}_p[a, b]$. As $\tilde{\beta}$ is surjective, ρ is injective, and $\beta = \rho(\tilde{\beta})$. It follows that the k -invariant $\theta_1 \oplus \theta_2$ comes from elements in $H^{n+1}(K; \mathbb{Z})$ for some K action on $S^n \times S^n$: $k^{n+1} = \rho^{-1}(\theta_1) \oplus \rho^{-1}(\theta_2)$.

In [25] it is shown that the ideal generated by θ_1 and θ_2 is in fact generated by $\zeta^{(n+1)/2(p+1)}$, where $\zeta = xy^p - yx^p$ (which is in part based on calculations in [15]). Since no power of ζ will contain $x^{(n+1)/2}$ or $y^{(n+1)/2}$, we see that $d_{n+1}(\alpha)$ and $d_{n+1}(\gamma)$, where α and γ generate $H^n(S^n \times S^n; \mathbb{Z})$, have both $q_{\alpha,0}$ and $q_{\gamma,0}$ zero (where $q_{\alpha,0}$ and $q_{\gamma,0}$ are the coefficients in Proposition 4.1). We derive a contradiction.

It is worth noting that, in [25], the calculations show that the free actions of $\text{Qd}(p)$ must have p smaller than n , and $n + 1$ divisible by $2(p + 1)$. The argument also finds a contradiction to finiteness, but relies on [5]. We also note that while we take p to be large in our homotopy type calculations, the only restrictions that were required in Section 4 (and hence in this section) were that $p > 3$ be an odd prime and $n \geq 3$ be odd. Further, there may be a way to show a contradiction to finiteness using Proposition 4.1 more directly (without needing to make arguments with \mathbb{Z}/p coefficients).

A similar argument could hold for any group containing $(\mathbb{Z}/p)^2$ that has a restriction that forces the transgression to behave in such a way.

References

- [1] **A Adem, JH Smith**, *On spaces with periodic cohomology*, Electron. Res. Announc. Amer. Math. Soc. 6 (2000) 1–6 MR Zbl
- [2] **HJ Baues**, *On the group of homotopy equivalences of a manifold*, Trans. Amer. Math. Soc. 348 (1996) 4737–4773 MR Zbl
- [3] **DJ Benson, JF Carlson**, *Complexity and multiple complexes*, Math. Z. 195 (1987) 221–238 MR Zbl
- [4] **C-F Bödighheimer, H-W Henn**, *A remark on the size of $\pi_q(S^n)$* , Manuscripta Math. 42 (1983) 79–83 MR Zbl
- [5] **G Carlsson**, *On the rank of abelian groups acting freely on $(S^n)^k$* , Invent. Math. 69 (1982) 393–400 MR Zbl
- [6] **GR Chapman**, *The cohomology ring of a finite abelian group*, Proc. Lond. Math. Soc. 45 (1982) 564–576 MR Zbl
- [7] **PE Conner**, *On the action of a finite group on $S^n \times S^n$* , Ann. of Math. 66 (1957) 586–588 MR Zbl
- [8] **GE Cooke**, *The Hauptvermutung according to Casson and Sullivan*, from “The Hauptvermutung book”, K–Monogr. Math. 1, Kluwer, Dordrecht (1996) 165–187 MR Zbl
- [9] **B Corbas, GD Williams**, *Congruence of two-dimensional subspaces in $M_2(K)$ (characteristic $\neq 2$)*, Pacific J. Math. 188 (1999) 225–235 MR Zbl
- [10] **PG Goerss, JF Jardine**, *Simplicial homotopy theory*, Progr. Math. 174, Birkhäuser, Basel (1999) MR Zbl
- [11] **I Hambleton, Ö Ünlü**, *Free actions of finite groups on $S^n \times S^n$* , Trans. Amer. Math. Soc. 362 (2010) 3289–3317 MR Zbl
- [12] **A Heller**, *A note on spaces with operators*, Illinois J. Math. 3 (1959) 98–100 MR Zbl
- [13] **P Hilton, G Mislin, J Roitberg**, *Localization of nilpotent groups and spaces*, North-Holland Math. Stud. 15, North-Holland, Amsterdam (1975) MR Zbl
- [14] **S Kwasik, R Schultz**, *All \mathbb{Z}_q lens spaces have diffeomorphic squares*, Topology 41 (2002) 321–340 MR Zbl
- [15] **JH Long**, *The cohomology rings of the special affine group of \mathbb{F}_p^2 and of $\text{PSL}(3, p)$* , PhD thesis, University of Maryland, College Park (2008) Available at <https://www.proquest.com/docview/304560975>
- [16] **T Macko, C Wegner**, *On fake lens spaces with fundamental group of order a power of 2*, Algebr. Geom. Topol. 9 (2009) 1837–1883 MR Zbl

- [17] **T Macko, C Wegner**, *On the classification of fake lens spaces*, Forum Math. 23 (2011) 1053–1091 MR Zbl
- [18] **I Madsen, C B Thomas, C T C Wall**, *The topological spherical space form problem, II: Existence of free actions*, Topology 15 (1976) 375–382 MR Zbl
- [19] **J P May**, *A concise course in algebraic topology*, Univ. of Chicago Press (1999) MR Zbl
- [20] **J P May, K Ponto**, *More concise algebraic topology: localization, completion, and model categories*, Univ. Chicago Press (2012) MR Zbl
- [21] **J McCleary**, *A user's guide to spectral sequences*, 2nd edition, Cambridge Stud. Adv. Math. 58, Cambridge Univ. Press (2001) MR Zbl
- [22] **J Milnor**, *Groups which act on S^n without fixed points*, Amer. J. Math. 79 (1957) 623–630 MR Zbl
- [23] **J Morales**, *The classification of pairs of binary quadratic forms*, Acta Arith. 59 (1991) 105–121 MR Zbl
- [24] **M Newman**, *Integral matrices*, Pure Appl. Math. 45, Academic, New York (1972) MR Zbl
- [25] **C Okay, E Yalçın**, *Dimension functions for spherical fibrations*, Algebr. Geom. Topol. 18 (2018) 3907–3941 MR Zbl
- [26] **R Oliver**, *Free compact group actions on products of spheres*, from “Algebraic topology” (JL Dupont, IH Madsen, editors), Lecture Notes in Math. 763, Springer (1979) 539–548 MR Zbl
- [27] **C D Roberts**, *The cohomology ring of a finite abelian group*, PhD thesis, University of Waterloo (2013) Available at <http://hdl.handle.net/10012/7276>
- [28] **P A Smith**, *Permutable periodic transformations*, Proc. Nat. Acad. Sci. U.S.A. 30 (1944) 105–108 MR Zbl
- [29] **D P Sullivan**, *Geometric topology: localization, periodicity and Galois symmetry*, K–Monogr. Math. 8, Springer (2005) MR Zbl
- [30] **C M Thatcher**, *On free \mathbb{Z}_p actions on products of spheres*, Geom. Dedicata 148 (2010) 391–415 MR Zbl

*Department of Mathematics, The Ohio State University
Columbus, OH, United States*

*Department of Mathematics and Computer Science, University of Puget Sound
Tacoma, WA, United States*

fowler@math.osu.edu, courtneythatcher@gmail.com

<https://math.osu.edu/~fowler.291/>

Received: 28 August 2020 Revised: 16 December 2022

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Kathryn Hess
kathryn.hess@epfl.ch
École Polytechnique Fédérale de Lausanne

BOARD OF EDITORS

Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Robert Lipshitz	University of Oregon lipshitz@uoregon.edu
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu	Norihiko Minami	Yamato University minami.norihiko@yamato-u.ac.jp
Tara E Brendle	University of Glasgow tara.brendle@glasgow.ac.uk	Andrés Navas	Universidad de Santiago de Chile andres.navas@usach.cl
Indira Chatterji	CNRS & Univ. Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr	Thomas Nikolaus	University of Münster nikolaus@uni-muenster.de
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Robert Oliver	Université Paris 13 bobol@math.univ-paris13.fr
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Jessica S Purcell	Monash University jessica.purcell@monash.edu
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Birgit Richter	Universität Hamburg birgit.richter@uni-hamburg.de
David Futер	Temple University dfuter@temple.edu	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk	Vesna Stojanoska	Univ. of Illinois at Urbana-Champaign vesna@illinois.edu
Ian Hambleton	McMaster University ian@math.mcmaster.ca	Zoltán Szabó	Princeton University szabo@math.princeton.edu
Matthew Hedden	Michigan State University mhedden@math.msu.edu	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Nathalie Wahl	University of Copenhagen wahl@math.ku.dk
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu	Chris Wendl	Humboldt-Universität zu Berlin wendl@math.hu-berlin.de
Thomas Koberda	University of Virginia thomas.koberda@virginia.edu	Daniel T Wise	McGill University, Canada daniel.wise@mcgill.ca
Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr		


See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US \$705/year for the electronic version, and \$1040/year (+\$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<https://msp.org/>

© 2024 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 4 (pages 1809–2387) 2024

Möbius structures, quasimetrics and completeness	1809
MERLIN INCERTI-MEDICI	
$\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^n \times S^n$	1841
JIM FOWLER and COURTNEY THATCHER	
\mathbb{Z}_k -stratifolds	1863
ANDRÉS ÁNGEL, CARLOS SEGOVIA and ARLEY FERNANDO TORRES	
Relative systoles in hyperelliptic translation surfaces	1903
CORENTIN BOISSY and SLAVYANA GENINSKA	
Smooth singular complexes and diffeological principal bundles	1913
HIROSHI KIHARA	
Natural symmetries of secondary Hochschild homology	1953
DAVID AYALA, JOHN FRANCIS and ADAM HOWARD	
The shape of the filling-systole subspace in surface moduli space and critical points of the systole function	2011
YUE GAO	
Moduli spaces of geometric graphs	2039
MARA BELOTTI, ANTONIO LERARIO and ANDREW NEWMAN	
Classical shadows of stated skein representations at roots of unity	2091
JULIEN KORINMAN and ALEXANDRE QUESNEY	
Commensurators of thin normal subgroups and abelian quotients	2149
THOMAS KOBERDA and MAHAN MJ	
Pushouts of Dwyer maps are $(\infty, 1)$ -categorical	2171
PHILIP HACKNEY, VIKTORIYA OZORNOVA, EMILY RIEHL and MARTINA ROVELLI	
A variant of a Dwyer–Kan theorem for model categories	2185
BORIS CHORNY and DAVID WHITE	
Integral generalized equivariant cohomologies of weighted Grassmann orbifolds	2209
KOUSHIK BRAHMA and SOUMEN SARKAR	
Projective modules and the homotopy classification of (G, n) -complexes	2245
JOHN NICHOLSON	
Realization of Lie algebras of derivations and moduli spaces of some rational homotopy types	2285
YVES FÉLIX, MARIO FUENTES and ANICETO MURILLO	
On the positivity of twisted L^2 -torsion for 3-manifolds	2307
JIANRU DUAN	
An algebraic C_2 -equivariant Bézout theorem	2331
STEVEN R COSTENOBLE, THOMAS HUDSON and SEAN TILSON	
Topologically isotopic and smoothly inequivalent 2-spheres in simply connected 4-manifolds whose complement has a prescribed fundamental group	2351
RAFAEL TORRES	
Remarks on symplectic circle actions, torsion and loops	2367
MARCELO S ATALLAH	
Correction to the article Hopf ring structure on the mod p cohomology of symmetric groups	2385
LORENZO GUERRA	