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$\mathbb{Z}/_p \times \mathbb{Z}/_p$ actions on $S^n \times S^n$

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We determine the homotopy type of quotients of $S^n \times S^n$ by free actions of $\mathbb{Z}/_p \times \mathbb{Z}/_p$ where $2p > n+3$. Much like free $\mathbb{Z}/_p$ actions, they can be classified via the first p–localized k–invariant, but there are restrictions on the possibilities, and these restrictions are sufficient to determine every possibility in the $n = 3$ case. We use this to complete the classification of free $\mathbb{Z}/_p \times \mathbb{Z}/_p$ actions on $S^3 \times S^3$ for $p > 3$ by reducing the problem to the simultaneous classification of pairs of binary quadratic forms. Although the restrictions are not sufficient to determine which k–invariants are realizable in general, they can sometimes be used to rule out free actions by groups that contain $\mathbb{Z}/_p \times \mathbb{Z}/_p$ as a normal abelian subgroup.

57N65, 57S25

1 Introduction

The topological spherical space form problem asks: what groups can act freely on the sphere and how can these group actions be classified? Conditions for which groups can act were determined during the middle of the last century; see e.g. Smith [28], Milnor [22] and Madsen, Thomas and Wall [18]. The question of *how* free cyclic groups can act on spheres was addressed in the study of lens spaces, with the classification of all free cyclic group actions being completed recently; see Macko and Wegner [16; 17].

This question can easily be extended to actions on products of spheres. What groups can act has been addressed in a number of papers (see e.g. Conner [7], Heller [12], Oliver [26], Adem and Smith [1], Benson and Carlson [3], Hambleton and Ünlü [11] and Okay and Yalçin [25]), while the classification of *how* the simplest of groups do act on products of spheres and what invariants distinguish them has largely been skipped. Here we focus specifically on the *how* question.

To begin addressing how groups act, one might consider the simplest group actions. Free $\mathbb{Z}/p$ actions on $S^n \times S^n$ for $p > \frac{1}{2}(n+3)$ were addressed by Thatcher [30] — the homotopy type is determined completely by the homotopy groups and the first k–invariant. We consider quotients of free actions of $\mathbb{Z}/_p \times \mathbb{Z}/_p$ on $S^n \times S^n$ with $n > 1$ odd and $p > \frac{1}{2}(n+3)$. It turns out that the homotopy classification is similar to the $\mathbb{Z}/_p$ case — the classes are determined by the first k–invariants, but the k–invariants are more complicated. A significant insight is the usefulness of localizing at a large prime — while the homotopy groups of spheres are replete with torsion, $\pi_i S^n$ has no p–torsion for $i \le 2n$ when p is reasonably large.

From this we see that only a couple of nontrivial stages in the localized Postnikov tower carry all the relevant data for our study.

We begin with a review of the cohomology of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ in Section 2 and then proceed with the classification. In Section 3 we determine the homotopy type in terms of a single k–invariant, or equivalently, in terms of the transgression in a certain spectral sequence, which the reader might also prefer to think of as an Euler class. The homotopy classification of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions on $S^n \times S^n$ then amounts to a choice of parameters in $\mathbb{Z}_{/p}$.

In Section 4 we find that there are strong restrictions on the possible k–invariants. In Section 5 we provide constructions of the possible homotopy classes based on these restrictions, and in Section 6 we show that this is the full homotopy classification of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions on $S^3 \times S^3$ by reducing the classification to that of pairs of binary quadratic forms. One of our main results is the following:

Theorem 6.6 *Let $p > 3$ be prime. If $p \equiv 1 \bmod 4$, then there are four homotopy classes of quotients of $S^3 \times S^3$ by free $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions. If $p \equiv 3 \bmod 4$, then there are two classes.*

Finally, in Section 8 we show that these restrictions can be used to rule out free actions by groups containing $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ as a normal abelian subgroup. This is consistent with the results about $\mathrm{Qd}(p)$ in a recent paper by Okay and Yalçin [25].

We note that a subsequent paper will provide the homeomorphism classification of these quotients in the case of linear actions.

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2 The cohomology of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$

To begin, we will need the integral cohomology of $X = (S^n \times S^n)/(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p})$. To determine this, we first need to consider the ring structure of the integral cohomology of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$. It is known that $H^*(\mathbb{Z}_{/p}; \mathbb{Z}_{/p}) = \mathbb{F}_p[a] \otimes \wedge(u)$, where $|u| = 1$, $|a| = 2$, and $\beta(u) = a$ with β the Bockstein homomorphism, and that $H^*(\mathbb{Z}_{/p}; \mathbb{Z}) = \mathbb{Z}[a]/(pa)$, where $|a| = 2$. It follows from the Künneth theorem that $H^*(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z}_{/p}) \cong \mathbb{F}_p[a, b] \otimes \wedge(u, v)$, where $|u| = |v| = 1$ and $|a| = |b| = 2$, but $H^*(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z})$ requires a bit more work.

The homology and cohomology groups themselves can be determined using the Künneth theorem and universal coefficients.

Proposition 2.1 *The integral homology groups of* $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ *are*

$$H_k(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ (\mathbb{Z}_{/p})^{(k+3)/2} & \text{for } k > 0 \text{ odd}, \\ (\mathbb{Z}_{/p})^{k/2} & \text{for } k > 0 \text{ even}. \end{cases}$$

The integral cohomology groups are

$$H^k(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ 0 & \text{for } k = 1, \\ (\mathbb{Z}_{/p})^{(k-1)/2} & \text{for } k > 1 \text{ odd}, \\ (\mathbb{Z}_{/p})^{(k+2)/2} & \text{for } k > 1 \text{ even}. \end{cases}$$

The ring structure can then be determined by piecing together the exact sequences in cohomology associated to the short exact sequences $0 \to \mathbb{Z}_{/p} \to \mathbb{Z}_{/p}^2 \to \mathbb{Z}_{/p} \to 0$ and $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_{/p} \to 0$. We take $G = \mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ for notational ease. Then the triangle in the diagram

$$\begin{array}{ccccccc} H^n(G;\mathbb{Z}) & \xrightarrow{\rho} & H^n(G;\mathbb{Z}_{/p}) & \xrightarrow{\tilde{\beta}} & H^{n+1}(G;\mathbb{Z}) & \xrightarrow{p} & H^{n+1}(G;\mathbb{Z}) \\ & & & \searrow^{\beta} & \downarrow{\scriptstyle \rho} & & \\ & & & & H^{n+1}(G;\mathbb{Z}_{/p}) & & \end{array}$$

commutes, where β is the Bockstein associated to the first short exact sequence above, $\tilde{\beta}$ is the Bockstein associated to the second one, ρ is the homomorphism induced by the map $\mathbb{Z} \to \mathbb{Z}_{/p}$, and p is the map induced by multiplication by p. This along with the ring structure of $H^*(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z}_{/p})$ allows one to find the ring structure of $H^*(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z})$. This ring structure is given, among other places, in [6; 27].

Theorem 2.2 *The integral cohomology ring of* $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ *is*

$$H^*(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z}) \cong \mathbb{Z}[a, b, c]/(pa, pb, pc, c^2),$$

where $|a| = |b| = 2$ *and* $|c| = 3$.

3 Homotopy equivalence and the k–invariants

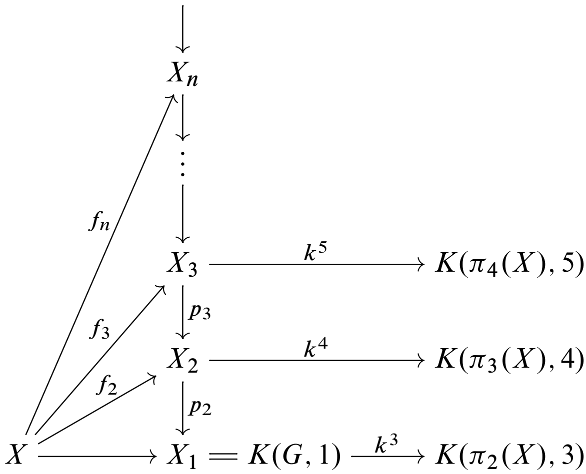
Let $G = \mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ act freely on $S^n \times S^n$, and let X be the resulting quotient manifold, which may only be a TOP manifold. A simple example of such an action is given by the first $\mathbb{Z}_{/p}$ acting freely on the first S^n and the second $\mathbb{Z}_{/p}$ acting freely on the second S^n in such a way that the resulting quotient manifold is the product of two lens spaces. We wish to determine when two arbitrary free actions of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ result in homotopy equivalent quotients.

For $p > 3$, the fundamental group $\pi_1(X) = G$ acts trivially on the homology of the universal cover of X because $\mathrm{GL}_2(\mathbb{Z})$ has no p–torsion. So by [13, Remark 2.19], it follows that X is nilpotent, and hence X has a Postnikov tower that admits principal refinements and X can be p–localized.

Definition 3.1 A connected space X *n–simple* if $\pi_1(X)$ is abelian and acts trivially on $\pi_i(X)$ for $1 < i \leq n$.

An n–simple space has a Postnikov tower that consists of principal fibrations through the n^{th} stage. We briefly describe the construction, but more specific details can be found in [19]. The first stage is taken to be $X_1 = K(\pi_1(X), 1)$, with $f_1 \colon X \to X_1$ inducing an isomorphism on π_1. The map $p_i \colon X_i \to X_{i-1}$ is constructed iteratively as the fibration induced from the path space fibration over $K(\pi_i X, i+1)$ by the map $k^{i+1} \colon X_{i-1} \to K(\pi_i X, i+1)$. The k^{i+1} are called k–invariants, and are thought of as cohomology classes. There are maps $f_i \colon X \to X_i$ for $1 \le i \le n$ such that $p_i \circ f_i = f_{i-1}$, and each f_i induces an isomorphism on π_k for all $k \le i$. Additionally, $\pi_k(X_i) = 0$ for $k > i$.

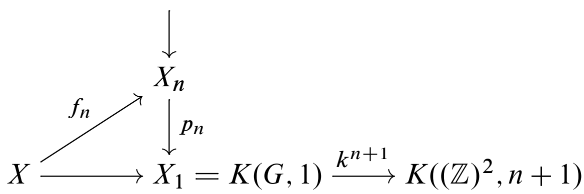
The bottom of the Postnikov tower for an n–simple space generically looks like:



Lemma 3.2 *Let $n \ge 3$. For $p > 3$, $X = (S^n \times S^n)/(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p})$ is n–simple.*

Proof Since $\pi_i(X) \cong \pi_i(S^n \times S^n) \cong \pi_i(S^n) \times \pi_i(S^n)$ we see that $\pi_2(X) \cong \pi_3(X) \cong \cdots \cong \pi_{n-1}(X) = 0$, and hence there is one nontrivial homotopy group $\pi_i X$ for $1 < i < n+1$: $\pi_n(X) \cong \pi_n(S^n \times S^n) = \mathbb{Z}^2$. Since $\mathrm{Aut}(\mathbb{Z}^2)$ only has 2– and 3–torsion and $p > 3$, π_1 acts trivially on $\pi_i(X)$ for $1 < i \le n$. □

Since $\pi_i(X)$ is trivial for $1 < i < n$, $X_1 \simeq X_2 \simeq \cdots \simeq X_{n-1}$, and the bottom of the Postnikov tower becomes



As X is nilpotent, the Postnikov tower above the n^{th} step admits principal refinements. Specifically, using the notation in [20], there is a central $\pi_1(X)$–series $1 = G_{j,r_j} \subset \cdots \subset G_{j,0} = \pi_j(X)$ for each $j > n$ such that $A_{j,l} = G_{j,l}/G_{j,l+1}$ for $0 \le l < r_j$ is abelian and $\pi_1(X)$ acts trivially on $A_{j,l}$. The $(n+1)^{\text{st}}$ stage is then a finite collection of spaces $X_{n+1,l}$ constructed from maps $k^{n+2,l} \colon X_{n+1,l} \to K(A_{n+1,l}, n+2)$

and with $X_{n+1,0} = X_n$. Similarly, the $(n+i)^{\text{th}}$ stage is a finite collection of spaces $X_{n+i+1,l}$ constructed from maps $k^{n+i+1,l} : X_{n+i,l} \to K(A_{n+i,l}, n+i+1)$ and with $X_{n+i,0} = X_{n+i-1,r_{n+i-1}}$.

Additionally, since X is nilpotent, X can be p–localized. This is done by inductively p–localizing the Postnikov tower, i.e. the $(X_j)_{(p)}$ are inductively constructed using fibrations with the $K(\pi, j)$, where each π is a $\mathbb{Z}_{(p)}$–module; see for example [20, Theorem 5.3.2] or Sullivan's notes [29]. Specifically we localize the first stage $(X_1)_{(p)} = (K(\pi_1(X),1))_{(p)} = K((\pi_1(X))_{(p)},1) = K((\mathbb{Z}/_p)^2,1) = X_1$ and localize the n^{th} homotopy group $(\pi_n X)_{(p)} = \pi_n X \otimes \mathbb{Z}_{(p)} = (\mathbb{Z}_{(p)})^2$, and then consider the following diagram:

$$\begin{array}{ccccccc}
K(\pi_n X, n) & \longrightarrow & X_n & \longrightarrow & X_1 & \xrightarrow{k^{n+1}} & K(\pi_n X, n+1)) \\
\downarrow & & \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} & & \downarrow \\
K((\pi_n X)_{(p)}, n) & \longrightarrow & (X_n)_{(p)} & \longrightarrow & (X_1)_{(p)} & \xrightarrow{(k^{n+1})_{(p)}} & K((\pi_n X)_{(p)}, n+1))
\end{array}$$

Here $(k^{n+1})_{(p)}$ is the p–localization of k^{n+1}. The right square commutes up to homotopy and there exists a map ϕ_{n+1}, that is, localization of X_n at p, such that the middle and left squares commute up to homotopy. Similar arguments can be made for the stages above n, and then we take $X_{(p)} = \lim (X_i)_{(p)}$ and $\phi = \lim \phi_i : X \to X_{(p)}$.

We note that the unique map (up to homotopy) ϕ localizes the homotopy and homology groups of X. In particular, $\phi_*(\pi_i X) = (\pi_i X)_{(p)}$, and further, $\phi_* : [X_{(p)}, Z] \to [X, Z]$ is an isomorphism for any p–local space Z [13; 20].

By [4] the unstable homotopy group $\pi_i(S^n)$ has no p–torsion for $i < 2p+n-3$. We restrict to $p > \frac{1}{2}(n+3)$, so that $\pi_i(X)$ has no p–torsion for $i \le 2n$. It follows that $A_{j,l}$ has no p–torsion for $n < j \le 2n$, and since $A_{j,l}$ is finite for all $j > n$, $(A_{j,l})_{(p)} = A_{j,l} \otimes Z_{(p)} = 0$, $K((A_{j,l})_{(p)}, j+1)$ is a point, and $(k^{m+1})_{(p)} = 0$, where $(k^{m+1})_{(p)}$ is the p–localized k–invariant associated with the m^{th} stage $(X_m = X_{j,l})$. Since the construction of the tower becomes formal after the dimension of X (after $2n$), the only nontrivial k–invariant in the localized Postnikov tower before it becomes formal is $(k^{n+1})_{(p)} \in H^{n+1}(X_1; (\mathbb{Z}_{(p)})^2) \cong (\mathbb{Z}/_p)^{n+3}$. Given an identification of π_1 and π_n, this p–localized first k–invariant then determines the homotopy type of the localization. In fact, the first nontrivial k–invariant characterizes X up to homotopy as well.

To state Theorem 3.3, we define G_n following [2]. For $n = 1,3,7$ define $G_n = \mathrm{GL}_2(\mathbb{Z})$, and for other positive odd n define G_n to be the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ generated by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Theorem 3.3 *Let X and Y be quotients of free $\mathbb{Z}/_p \times \mathbb{Z}/_p$ actions on $S^n \times S^n$ with odd $n \ge 3$, where $p > 3$ satisfies $2p+n-3 > 2n$, and let k_X^{n+1} and k_Y^{n+1} denote the first nontrivial k–invariant. The* *spaces X and Y are homotopy equivalent if and only if there are isomorphisms $g_1\colon \pi_1 X \to \pi_1 Y$ and $g_n\colon \pi_n X \to \pi_n Y$ with $g_n \in G_n$ and such that*

$$\begin{CD} K(\pi_1(X),1) @>{k_X^{n+1}}>> K(\pi_n(X),n+1) \\ @V{g_{1\star}}VV @VV{g_{n\star}}V \\ K(\pi_1(Y),1) @>{k_Y^{n+1}}>> K(\pi_n(Y),n+1) \end{CD}$$

commutes up to homotopy, i.e. $k_X^{n+1} \in H^{n+1}(\pi_1 X; \pi_n X)$ and $k_Y^{n+1} \in H^{n+1}(\pi_1 Y; \pi_n Y)$ are identified through the maps induced by g_1 and g_n.

Lemmas 3.4 and 3.5 are used to prove this, and are related to [30, Lemmas 1 and 2], respectively.

Lemma 3.4 *Let X and Y be n–simple spaces with identifications $\pi_1(X) \cong \pi_1(Y) \cong G$ and $\pi_n(X) \cong \pi_n(Y) \cong H$. Further suppose $\pi_i(X) = \pi_i(Y) = 0$ for $1 < i < n$. If, as in Theorem 3.3, the identifications on π_1 and π_n provide an identification of the first nontrivial k–invariants of X and Y in $H^{n+1}(G; H)$, then the n^{th} stages of the Postnikov towers for X and Y are homotopy equivalent, i.e. $X_n \simeq Y_n$.*

Proof We have isomorphisms $g_1\colon \pi_1 X \to \pi_1 Y$ and $g_n\colon \pi_n X \to \pi_n Y$, and k_X^{n+1} and k_Y^{n+1} are the first nontrivial k–invariants of X and Y, respectively. The k–invariant is regarded as a map

$$k_X^{n+1}\colon K(\pi_1(X),1) \to K(\pi_n(X),n+1).$$

The isomorphism g_1 induces a homotopy equivalence $g_{1\star}\colon K(\pi_1(X),1) \to K(\pi_1(Y),1)$. Similarly, the isomorphism g_n induces a homotopy equivalence $g_{n\star}\colon K(\pi_n(X),n+1) \to K(\pi_n(Y),n+1)$. The identification of the first nontrivial k–invariant means that $g_{n\star} \circ k_X^{n+1}$ is homotopic to $k_Y^{n+1} \circ g_{1\star}$.

The n^{th} stage X_n of the Postnikov tower is constructed as the pullback of the path-space fibration over $K(\pi_n(X),n+1)$ and k_X^{n+1}:

$$\begin{CD} X_n @>>> (K(\pi_n(X),n+1))^I \\ @VVV @VVV \\ K(\pi_1(X),1) @>{k_X^{n+1}}>> K(\pi_1(X),n+1) \end{CD}$$

A similar construction is performed for Y_n. We have the following map of fibrations, and we want to define a map f on the fibers:

$$\begin{CD} X_n @>>> K(\pi_1(X),1) @>{k_X^{n+1}}>> K(\pi_n(X),n+1) \\ @V{f}VV @V{g_{1\star}}VV @VV{g_{n\star}}V \\ Y_n @>>> K(\pi_1(Y),1) @>{k_Y^{n+1}}>> K(\pi_n(Y),n+1) \end{CD}$$

The identification of the first nontrivial k–invariants means the square on the right commutes up to homotopy. Let $h\colon K(\pi_1(X),1) \times I \to K(\pi_n(Y),n+1)$ be a homotopy from $g_{n\star} \circ k_X^{n+1}$ to $k_Y^{n+1} \circ g_{1\star}$. With X_n defined as a pullback, a point in X_n consists of a pair (x,q) with $x \in K(\pi_1(X),1)$ and

$q\colon I \to K(\pi_n(X), n+1)$ satisfying $q(1) = k_X^{n+1}(x)$. Define $f\colon X_n \to Y_n$ by $f(x,q) = (y,r)$ with $y = g_{1\star}(x)$ and $r\colon I \to K(\pi_n(Y), n+1)$ given by

$$r(t) = \begin{cases} g_{n\star}q(t) & \text{if } t \le \frac{1}{2}, \\ h(x, 2t-1) & \text{if } t \ge \frac{1}{2}. \end{cases}$$

This provides a construction for $f\colon X_n \to Y_n$, and by a theorem of Milnor, the fibers X_n and Y_n have the homotopy types of CW complexes. Therefore we have a commuting diagram of homotopy groups

$$\begin{array}{ccccccccc} \pi_{j+1}K(\pi_1(X),1) & \longrightarrow & \pi_{j+1}K(\pi_n(X),n+1) & \longrightarrow & \pi_j X_n & \longrightarrow & \pi_j K(\pi_1(X),1) & \longrightarrow & \pi_j K(\pi_n(X),n+1) \\ \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\ \pi_{j+1}K(\pi_1(Y),1) & \longrightarrow & \pi_{j+1}K(\pi_n(Y),n+1) & \longrightarrow & \pi_j Y_n & \longrightarrow & \pi_j K(\pi_1(Y),1) & \longrightarrow & \pi_j K(\pi_n(Y),n+1) \end{array}$$

The five lemma gives us that $\pi_j X_n \cong \pi_j Y_n$ for all j. Thus we have a weak equivalence between spaces having the homotopy type of CW complexes, so we have a homotopy equivalence. □

Lemma 3.5 *Let M and N be nilpotent spaces such that $H^n(M;\mathbb{Z}) = 0$ and $H^n(N;\mathbb{Z}) = 0$ for $n > m$, for some $m > 0$. If the m^{th} stage of the Postnikov tower for M is homotopy equivalent to the m^{th} stage of the Postnikov tower for N, then M is homotopy equivalent to N, i.e. if $M_m \simeq N_m$ then $M \simeq N$.*

We note that this lemma is essentially [30, Lemma 2] — the difference being the change of "m–dimensional" to the cohomology requirement above — and the obstruction argument proof works exactly as written.

Proof of Theorem 3.3 As has been our convention, let $G = \mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$.

In one direction, we assume there is a homotopy equivalence from X to Y. On π_1, the homotopy equivalence provides an isomorphism which then yields a homotopy equivalence between the first stage of a Postnikov tower of X and the same of Y. The next nontrivial stage is stage n, and we have a commutative square

$$\begin{array}{ccc} X_n & \longrightarrow & Y_n \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & Y_1 \end{array}$$

The commutativity of this square follows by using a functorial model for the Postnikov tower; see [10, Chapter VI.2]. The vertical maps are fibrations, and taking the cofibers of these vertical maps yields the commutative square displayed in the statement of Theorem 3.3. The condition that g_n, regarded as an element of $\mathrm{GL}_2(\mathbb{Z})$, lies in G_n is a consequence of [2, Theorem 6.3].

To prove the other direction, we assume $\pi_1 X$ and $\pi_1 Y$ are identified with G and that this gives an isomorphism $g_1\colon \pi_1 X \to \pi_1 Y$, and $\pi_n X$ and $\pi_n Y$ are identified with $(\mathbb{Z})^2$ and that this gives an isomorphism $g_n\colon \pi_n X \to \pi_n Y$. These maps induce identifications $\pi_1 X_{(p)} \cong \pi_1 Y_{(p)} \cong G \otimes \mathbb{Z}_{(p)} \cong G$ and $\pi_n X_{(p)} \cong \pi_n Y_{(p)} \cong (\mathbb{Z})^2 \otimes \mathbb{Z}_{(p)} \cong (\mathbb{Z}_{(p)})^2$ after localizing the Postnikov systems of both X and Y at p. We have that $(X_1)_{(p)} = K(\pi_1 X_{(p)}, 1) \simeq (Y_1)_{(p)} = K(\pi_1 Y_{(p)}, 1)$. Let k_X^{n+1} and k_Y^{n+1} be the first nontrivial k–invariants of X and Y, respectively, and take $(k_X^{n+1})_{(p)}$ and $(k_Y^{n+1})_{(p)}$ to be the p–localized first k–invariants, respectively. Since k_X^{n+1} and k_Y^{n+1} are in the same homotopy class of maps in $[K(G,1):K((\mathbb{Z})^2,n+1)]$, $(k_X^{n+1})_{(p)}$ and $(k_Y^{n+1})_{(p)}$ will be in the same homotopy class of maps in $[K(G,1):K((\mathbb{Z}_{(p)})^2,n+1)]$ by construction. Since $p>3$, X and Y are n–simple by Lemma 3.2. Given that the localization of both spaces and their homotopy groups preserves this property, $X_{(p)}$ and $Y_{(p)}$ are both n–simple as well, and we can apply Lemma 3.4. It follows that $(X_n)_{(p)} \simeq (Y_n)_{(p)}$.

Since we are assuming $2p+n-3>2n$, we have that $(X_{2n+1,0})_{(p)} \simeq (X_n)_{(p)} \simeq (Y_n)_{(p)} \simeq (Y_{2n+1,0})_{(p)}$. It follows from Lemma 3.5 that $X_{(p)} \simeq Y_{(p)}$. The maps $l_1\colon X_{(p)} \to X_{(0)}$ and $l_2\colon Y_{(p)} \to Y_{(0)}$, given by inverting p, give via the naturality of localization a homotopy equivalence $\iota\colon X_{(0)} \xrightarrow{\simeq} Y_{(0)}$ and identifications of $\pi_n X_{(0)}$ and $\pi_n Y_{(0)}$ with $(\mathbb{Q})^2$. The following commutes up to homotopy:

$$\begin{array}{ccc} X_{(p)} & \xrightarrow{\simeq} & Y_{(p)} \\ \downarrow{\scriptstyle l_1} & & \downarrow{\scriptstyle l_2} \\ X_{(0)} & \xrightarrow{\iota} & Y_{(0)} \end{array}$$

On the other hand, we can consider localization away from p. For X we have the commutative diagram

$$\begin{array}{ccc} S^n \times S^n & \longrightarrow & (S^n \times S^n)[1/p] \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q[1/p]} \\ X & \longrightarrow & X[1/p] \end{array}$$

Since $\pi_1(X[1/p]) = G \otimes \mathbb{Z}[1/p] = 0$, we see that $\pi_j(X[1/p]) \cong \pi_j((S^n \times S^n)[1/p])$ for all j. Thus $q[1/p]$ induces an isomorphism on every homotopy group, and is a homotopy equivalence since $(S^n \times S^n)[1/p]$ and $X[1/p]$ both have the homotopy types of CW complexes. Similarly we have a homotopy equivalence $(S^n \times S^n)[1/p] \simeq Y[1/p]$. Invoking [2, Theorem 6.3], we can realize any element of $\mathrm{GL}_2(\mathbb{Z})$ via a homotopy equivalence $S^n \times S^n \simeq S^n \times S^n$. Composing these equivalences yields

$$X[1/p] \simeq (S^n \times S^n)[1/p] \simeq (S^n \times S^n)[1/p] \simeq Y[1/p],$$

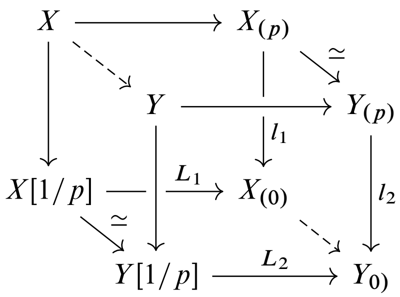
so the map $X[1/p] \to Y[1/p]$ on π_n is $g_n \otimes \mathbb{Z}[1/p]$.

Since we have maps $X[1/p] \to X_{(0)}$ and $Y[1/p] \to Y_{(0)}$ given by inverting everything else, the naturality of localization gives us a map $\iota'\colon X_{(0)} \xrightarrow{\simeq} Y_{(0)}$. It is a homotopy equivalence because it induces an isomorphism on all of the homotopy groups. We have a diagram that commutes up to homotopy:

$$\begin{array}{ccc} X[1/p] & \xrightarrow{\simeq} & Y[1/p] \\ \downarrow{\scriptstyle L_1} & & \downarrow{\scriptstyle L_2} \\ X_{(0)} & \xrightarrow{\iota'} & Y_{(0)} \end{array}$$

Since $X_{(0)}$ and $Y_{(0)}$ are $K((\mathbb{Q})^2,n)$, homotopy classes of maps from $X_{(0)}$ to $Y_{(0)}$ are identified with elements of $\mathrm{Hom}(\pi_n X \otimes \mathbb{Q}, \pi_n Y \otimes \mathbb{Q})$, but by construction, ι and ι' are identified by their action on π_n.

The space X is the homotopy pullback of $X_{(p)}$ and $X[1/p]$ along l_1 and L_1. For $x \in X$, write x_1 and x_2 for the images of x in $X_{(p)}$ and $X[1/p]$, respectively, so $l_1(x_1)$ and $L_1(x_2)$ are connected by a path in $X_{(0)}$. To map into Y, a homotopy pullback, it is enough to provide maps $X \to Y_{(p)}$ and $X \to Y[1/p]$ which agree up to a path in $Y_{(0)}$. Combining the localization squares for X and Y and all of the maps we have constructed between the squares, we have the following cube that commutes up to homotopy, thereby providing maps $X \to Y_{(p)}$ and $X \to Y[1/p]$ which agree up to homotopy:



From this we obtain maps of short exact sequences on homotopy for all j:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_j X & \longrightarrow & \pi_j X_{(p)} \oplus \pi_j X[1/p] & \longrightarrow & \pi_j X_{(0)} & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \| \\ 0 & \longrightarrow & \pi_j Y & \longrightarrow & \pi_j Y_{(p)} \oplus \pi_j Y[1/p] & \longrightarrow & \pi_j Y_{(0)} & \longrightarrow & 0 \end{array}$$

The five lemma gives isomorphisms on the homotopy groups of X and Y. This then gives a homotopy equivalence from X to Y as they are both CW complexes. □

4 Restrictions on the first k–invariant

Throughout this section we will continue to let $G = \mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ and $X := (S^n \times S^n)/G$, where $p > 3$ is an odd prime and $n \geq 3$ is odd. The first stage of the Postnikov system provides a fibration: $K(\pi_n(X), n) \xrightarrow{j} X_n \to X_1 = K(\pi_1 X, 1)$. The space X_n is induced from the path-space fibration over $K(\pi_n(X), n+1)$, so the fundamental group $\pi_1(X_1) = G$ acts trivially on the homology of $K(\pi_n(X), n)$. This results in an exact sequence

$$\cdots \to H^n(X_n; \pi_n(X)) \xrightarrow{j^*} H^n(K(\pi_n(X), n); \pi_n(X)) \xrightarrow{\tau} H^{n+1}(X_1; \pi_n(X)),$$

where τ is the transgression. By [21, Section 6.2], τ is also the differential $\tau = d_{n+1} \colon E^{0,n}_{n+1} \to E^{n+1,0}_{n+1}$ in the Serre spectral sequence of the fibration. As described in [8, Section 3.7], the fundamental classes of the fiber $K(\pi_n(X), n)$ and the base X_1 correspond under the transgression. If $\iota \in H^n(K(\pi_n(X), n); \pi_n(X))$ is the fundamental class of the fiber, the k–invariant $k^{n+1} \in H^{n+1}(X_1; \pi_n(X))$ is the pullback of the fundamental class of the base space, and $\tau(\iota) = k^{n+1}$.

On the other hand, consider the Borel fibration

$$S^n \times S^n \xrightarrow{i} (S^n \times S^n)_{hG} \to BG,$$

where $(S^n \times S^n)_{hG} := (EG \times S^n \times S^n)/G \simeq (S^n \times S^n)/G = X$.

There is a map of fibrations

$$\begin{array}{ccccc} S^n \times S^n & \xrightarrow{i} & X & \xrightarrow{f_1} & BG \\ \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle =} \\ K(\pi_n(X), n) & \xrightarrow{j} & X_n & \xrightarrow{p_n} & BG \end{array}$$

where the map $\phi_n \colon S^n \times S^n \to K(\pi_n(X), n)$ classifies the fundamental class in $H^n(S^n \times S^n; \mathbb{Z}^2)$, and $f_n \colon X \to X_n$ is the n–equivalence in the Postnikov tower. Since $\pi_1(BG) = G$ is a finite group generated by odd-order elements, it acts trivially on the cohomology of the fiber (see [25]), and we obtain maps between the induced exact sequences in cohomology:

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^n(X; \pi_n(X)) & \xrightarrow{i^*} & H^n(S^n \times S^n; \pi_n(X)) & \xrightarrow{\bar\tau} & H^{n+1}(X_1; \pi_n(X)) \\ & {\scriptstyle f_n^*}\uparrow & & {\scriptstyle \phi_n^*}\uparrow & & {\scriptstyle =}\uparrow \\ \cdots \longrightarrow & H^n(X_n; \pi_n(X)) & \xrightarrow{j^*} & H^n(K(\pi_n(X), n); \pi_n(X)) & \xrightarrow{\tau} & H^{n+1}(X_1; \pi_n(X)) \end{array}$$

It follows that $\bar\tau(\phi_n^*(\iota)) = \tau(\iota) = k^{n+1}$ for the fundamental class $\iota \in H^n(K(\pi_n(X), n), \pi_n(X))$, which corresponds to the identity map under the equivalence $H^n(K(\pi_n(X), n), \pi_n(X)) \cong \mathrm{Hom}(\pi_n(X), \pi_n(X))$ by the universal coefficient theorem. Further, since $H_{n-1}(S^n \times S^n) = 0$, the universal coefficient theorem also gives $H^n(S^n \times S^n; \mathbb{Z}^2) \cong H^n(S^n \times S^n; \mathbb{Z}) \oplus H^n(S^n \times S^n; \mathbb{Z})$.

We write $(0,1)$ for the element of $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$ sending (x, y) to y, write $(1,0)$ for the element of $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$ sending (x, y) to x, and set $\iota = (1,0) \oplus (0,1) \in H^n(K(\pi_n(X), n); \pi_n(X)) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^2) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$. Then $\phi_n^*(\iota) = \phi_n^*((1,0) \oplus (0,1)) = (\alpha, 0) \oplus (0, \gamma) \in H^n(S^n \times S^n; \mathbb{Z}^2) \cong H^n(S^n \times S^n; \mathbb{Z}) \oplus H^n(S^n \times S^n; \mathbb{Z})$. Here α and γ are preferred generators for $H^n(S^n \times S^n; \mathbb{Z}) \cong \mathbb{Z}^2$. It can now be seen that $k^{n+1} = \bar\tau((\alpha, 0) \oplus (0, \gamma))$.

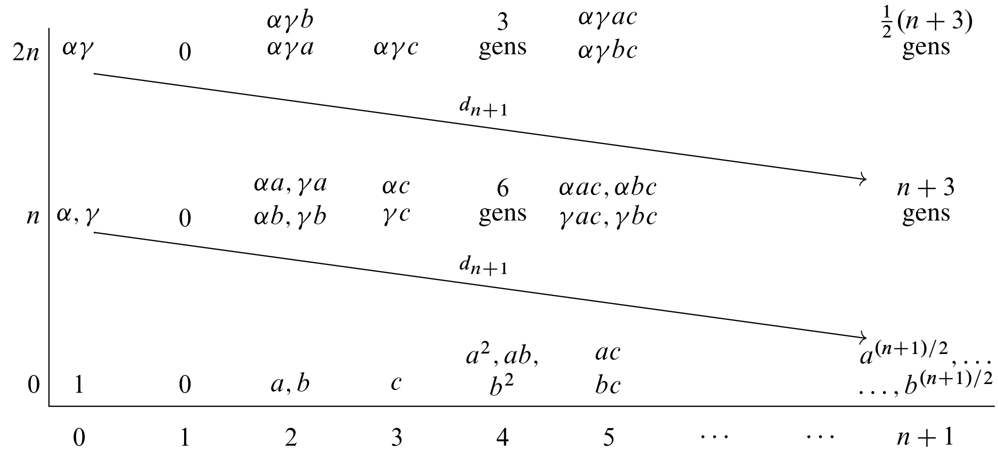
It suffices to examine the transgression from the Serre spectral sequence with integral coefficients for the Borel fibration in order to find out information about the first nontrivial k–invariant, k^{n+1}. In particular, for $S^n \times S^n \to X \to BG$,

$$E_2^{p,q} = H^p(BG; H^q(S^n \times S^n; \mathbb{Z})) \Rightarrow H^{p+q}(X; \mathbb{Z}).$$

The first nontrivial differential is d_{n+1}, and the transgression

$$d_{n+1} \colon H^0(BG; H^n(S^n \times S^n; \mathbb{Z})) \to H^{n+1}(BG; H^0(S^n \times S^n; \mathbb{Z}))$$

here satisfies $d_{n+1}(\alpha) = \bar\tau(\alpha, 0)$ and $d_{n+1}(\gamma) = \bar\tau(0, \gamma)$. It follows that $k^{n+1} = d_{n+1}(\alpha) \oplus d_{n+1}(\gamma)$. The cohomology ring of $H^*(BG; \mathbb{Z})$ is given in Theorem 2.2 and we use the same notation by taking the generators to be a, b and c, with $|a| = |b| = 2$, $|c| = 3$ and $pa = pb = pc = c^2 = 0$. Additionally, we take α and γ to be the generators in degree n of $H^*(S^n \times S^n; \mathbb{Z})$ with $\alpha^2 = \gamma^2 = 0$, as described above. We see that the $E_2 \cong E_{n+1}$ page reads



where $\alpha\gamma b$ is $\alpha\gamma \otimes b$, etc, by abuse of notation. Note that the blank entries are not necessarily 0.

By virtue of its codomain being generated by suitable powers of a and b, the transgression

$$d_{n+1} \colon E_{n+1}^{0,n} = H^0(BG; H^n(S^n \times S^n; \mathbb{Z})) \to E_{n+1}^{n+1,0} = H^{n+1}(BG; H^0(S^n \times S^n; \mathbb{Z}))$$

satisfies

$$d_{n+1}(\alpha) = \sum_{i=0}^{(n+1)/2} q_{\alpha,i} a^{(n+1)/2-i} b^i \quad \text{and} \quad d_{n+1}(\gamma) = \sum_{j=0}^{(n+1)/2} q_{\gamma,j} a^{(n+1)/2-j} b^j,$$

where the $q_{\alpha,i}$ and $q_{\gamma,j}$ are elements of $\mathbb{Z}_{/p}$.

This spectral sequence converges to the integral cohomology of X, and since X is a finite manifold of dimension $2n$, there are restrictions on what the coefficients $q_{\alpha,i}$ and $q_{\gamma,j}$ can be.

Proposition 4.1 *The coefficients* $q_{\alpha,0}$ *and* $q_{\gamma,0}$ (*which are coefficients for* $a^{(n+1)/2}$) *cannot both be zero. Similarly, the coefficients* $q_{\alpha,(n+1)/2}$ *and* $q_{\gamma,(n+1)/2}$ (*which are coefficients for* $b^{(n+1)/2}$) *cannot both be zero.*

Proof Since G acts freely and $H^{2n}((S^n \times S^n)/G; \mathbb{Z}) \cong \mathbb{Z}$, only quotients of the groups generated by the $E_2^{p,q} \cong E_{n+1}^{p,q}$ terms with $p + q < 2n$ or $p = 0$ and $q = 2n$ can survive. Assume the transgression $d_{n+1} \colon E_{n+1}^{0,n} \to E_{n+1}^{n+1,0}$ satisfies $d_{n+1}(\alpha) = q_{\alpha,1} a^{(n-1)/2} b + \cdots + q_{\alpha,(n+1)/2} b^{(n+1)/2}$ and $d_{n+1}(\gamma) = q_{\gamma,1} a^{(n-1)/2} b + \cdots + q_{\gamma,(n+1)/2} b^{(n+1)/2}$ for some $q_{\alpha,i}, q_{\gamma,j} \in \mathbb{Z}_{/p}$ with $1 \le i, j \le \frac{1}{2}(n+1)$. In other words, both $q_{\alpha,0}$ and $q_{\gamma,0}$ vanish.

The $(n+1)^{\text{st}}$ differential takes the generators in $E_{n+1}^{n-1,n}$ to combinations of the generators in $E_{n+1}^{2n,0}$. By Leibniz, d_{n+1} sends $\alpha \otimes a^{(n-1)/2}$ to $q_{\alpha,1} a^{n-1} b + \cdots + q_{\alpha,(n+1)/2} a^{(n-1)/2} b^{(n+1)/2}$, and similarly for the other generators. It is not hard to see that the only other nontrivial differential, d_{n+1}, does not hit the subgroup generated by a^{n+1}, and there are no other differentials that map to this group. Therefore the generated $\mathbb{Z}_{/p}$ is present in $H^{2n}((S^n \times S^n)/G; \mathbb{Z})$ and other cohomology groups in higher degrees. Since $H^{2n}((S^n \times S^n)/G; \mathbb{Z})$ is torsion free and the highest nontrivial degree, we get a contradiction.

The argument for $q_{\alpha,(n+1)/2}$ and $q_{\gamma,(n+1)/2}$ both being nontrivial is similar. □

Observe that Proposition 4.1 also implies that neither $d_{n+1}(\alpha)$ nor $d_{n+1}(\gamma)$ can map to 0. We also see that it holds after replacing the specified generators with their images under an automorphism of G.

Corollary 4.2 *For nonzero $\lambda \in H^2(G; \mathbb{Z})$, either $d_{n+1}(\alpha)$ or $d_{n+1}(\gamma)$ is nonzero in*

$$H^{n+1}(G; \mathbb{Z})/\lambda^{(n+1)/2}.$$

Proof Suppose φ is an automorphism of G chosen so that $\varphi_\star\lambda = a \in H^2(G; \mathbb{Z})$. After twisting by φ the action of G on $S^n \times S^n$, the resulting quotient is homeomorphic (albeit not equivariantly homeomorphic) to the original quotient space. In particular, in that quotient the coefficients $q_{\alpha,0}$ and $q_{\gamma,0}$, namely the coefficients for $a^{(n+1)/2}$, cannot both be zero, which corresponds in the original space to the condition in the corollary. □

5 Constructions

Now we construct examples which are more complicated than lens spaces cross lens spaces. In this section, we take the dimension of the spheres we are acting on to be $n = 2m - 1$ to avoid fractions appearing in subscripts. Let $R = (r_1, \dots, r_m, r'_1, \dots, r'_m)$ and $Q = (q_1, \dots, q_m, q'_1, \dots, q'_m)$ be elements of $(\mathbb{Z}_{/p})^{2m}$ so that R and Q together generate a copy of $(\mathbb{Z}_{/p})^2$ inside $(\mathbb{Z}_{/p})^{2m}$. We refer to these $4m$ parameters as "rotation numbers" in analogy with the case of a lens space.

Let S^{2m-1} be the unit sphere in $\mathbb{C}^m$, so $S^{2m-1} \times S^{2m-1}$ is a submanifold of $\mathbb{C}^m \times \mathbb{C}^m$. Then R acts on $S^{2m-1} \times S^{2m-1}$ by

$$\begin{aligned} R \cdot (z, z') &= (r, r') \cdot (z, z') = (r, r') \cdot (z_1, \dots, z_m, z'_1, \dots, z'_m) \\ &= (e^{2\pi i r_1/p} z_1, \dots, e^{2\pi i r_1/p} z_m, e^{2\pi i r'_1/p} z'_1, \dots, e^{2\pi i r'_m/p} z'_m), \end{aligned}$$

and similarly Q acts on $S^{2m-1} \times S^{2m-1}$. This provides an action of the group $(\mathbb{Z}_{/p})^2 \cong \langle R, Q\rangle$ on $S^{2m-1} \times S^{2m-1}$. In analogy with the lens space case, we call such actions "linear" and we write the quotient as $L(p, p; R, Q)$. In the case of lens spaces, the k–invariant is the product of rotation numbers. We now compute the first nontrivial k–invariant in the case of $L(p, p; R, Q)$. We will denote this first nontrivial k–invariant by k in what follows.

Lemma 5.1 *Let $L = L(p, p; R, Q)$ and suppose $p > m$. Then $k(L) \in H^{2m}((\mathbb{Z}_{/p})^2; \mathbb{Z}^2)$ is*

$$\left(\prod_{i=1}^{m}(r_i a + q_i b), \prod_{i=1}^{m}(r'_i a + q'_i b)\right),$$

where a and b are generators of $H^2((\mathbb{Z}_{/p})^2; \mathbb{Z})$ as described in Section 2.

In keeping with the analogy to the lens space, Lemma 5.1 states that the k–invariant is the product of rotation *classes* in $H^2((\mathbb{Z}_{/p})^2; \mathbb{Z})$.

Proof The k–invariant $k(L) \in H^{2m}(K(G,1); \mathbb{Z}^2)$ is a homotopy class of maps

$$K(\pi_1 L, 1) \to K(\pi_{2m-1} L, 2m).$$

The proof makes use of the naturality of the k–invariant. Suppose a $\mathbb{Z}_{/p}$ subgroup of $(\mathbb{Z}_{/p})^2$ is generated by (α, β). Then we have a cover

$$\bar{L} = (S^{2m-1} \times S^{2m-1})/\mathbb{Z}_{/p} \to L(p, p; R, Q).$$

By [30, page 396], the k–invariant $k(\bar{L}) \in H^{2m}(\mathbb{Z}_{/p}; \mathbb{Z}^2)$ associated to the quotient of $S^{2m-1} \times S^{2m-1}$ by the subgroup $\langle(\alpha, \beta)\rangle \cong \mathbb{Z}_{/p}$ is

$$k(\bar{L}) = \left(\prod_{i=1}^{m} (r_i \alpha + q_i \beta)\omega, \prod_{i=1}^{m} (r_i' \alpha + q_i' \beta)\omega \right),$$

where ω is the generator in $H^2(\mathbb{Z}_{/p}; \mathbb{Z})$, which is identified with the generator of $\mathbb{Z}_{/p}$ via $H^2(\mathbb{Z}_{/p}; \mathbb{Z}) \cong \operatorname{Ext}(H_1(\mathbb{Z}_{/p}; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}_{/p}$.

By universal coefficients and the fact that the cohomology (except in degree zero) of $\mathbb{Z}_{/p}$ and $(\mathbb{Z}_{/p})^2$ is torsion, we have $\operatorname{Ext}(H_1((\mathbb{Z}_{/p})^2; \mathbb{Z}), \mathbb{Z}) \cong H^2((\mathbb{Z}_{/p})^2; \mathbb{Z}) \cong (\mathbb{Z}_{/p})^2$ and $\operatorname{Ext}(H_1(\mathbb{Z}_{/p}; \mathbb{Z}), \mathbb{Z}) \cong H^2(\mathbb{Z}_{/p}; \mathbb{Z}) \cong \mathbb{Z}_{/p}$, and $H^2((\mathbb{Z}_{/p})^2; \mathbb{Z}) \to H^2(\mathbb{Z}_{/p}; \mathbb{Z})$ is dual to the inclusion map $\mathbb{Z}_{/p} \hookrightarrow (\mathbb{Z}_{/p})^2$; the inclusion map sends the generator of $\mathbb{Z}_{/p}$ to (α, β), so the dual map sends $xa + yb \in H^2((\mathbb{Z}_{/p})^2; \mathbb{Z})$ to $(\alpha a + \beta b)\omega$.

By naturality of the k–invariant, the map $H^{2m}((\mathbb{Z}_{/p})^2; \mathbb{Z}^2) \to H^{2m}(\mathbb{Z}_{/p}; \mathbb{Z}^2)$ sends $k(L)$ to $k(\bar{L})$. We consider only the left-hand factor of $k(L)$; this is some homogeneous polynomial of degree n in the classes $a, b \in H^2((\mathbb{Z}_{/p})^2; \mathbb{Z})$. Write this polynomial as $f(a, b)$.

Then the map $H^{2m}((\mathbb{Z}_{/p})^2; \mathbb{Z}) \to H^{2m}(\mathbb{Z}_{/p}; \mathbb{Z})$ sends $f(a, b)$ to

$$f(\alpha, \beta)\omega \in H^{2m}(\mathbb{Z}_{/p}; \mathbb{Z}),$$

and therefore, for $\alpha, \beta \in \mathbb{Z}_{/p}$,

$$f(\alpha, \beta) = \prod_{i=1}^{m} (r_i \alpha + q_i \beta).$$

Now assuming $m < p$, this equality of polynomials as functions gives rise to the desired equality

$$f(a, b) = \prod_{i=1}^{m} (r_i a + q_i b).$$

The right-hand factor of $k(L)$ is computed the same way. □

6 The $S^3 \times S^3$ classification

Suppose $p > 3$. We now classify $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions on $S^3 \times S^3$ up to homotopy. By Theorem 3.3, this boils down to the k–invariants encoded by the transgression

$$d_4(\alpha) = q_{\alpha,0}a^2 + q_{\alpha,1}ab + q_{\alpha,2}b^2 \quad \text{and} \quad d_4(\gamma) = q_{\gamma,0}a^2 + q_{\gamma,1}ab + q_{\gamma,2}b^2.$$

We therefore package $(d_4(\alpha), d_4(\gamma))$ as a pair (Q_1, Q_2) of binary quadratic forms over $\mathbb{Z}_{/p}$. The homotopy classification of $(S^3 \times S^3)/(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p})$ amounts, algebraically, to classifying pairs of binary quadratic forms over $\mathbb{Z}_{/p}$, up to the action of automorphisms of $\mathbb{Z}^2$ on the pair (Q_1, Q_2). For example, the pair (Q_1, Q_2) determines the same equivariant oriented homotopy type as $(Q_1 + Q_2, Q_1)$. Note that $\operatorname{Aut}(\mathbb{Z}^2)$ amounts to the action of

$$\mathrm{SL}_2^{\pm}(\mathbb{Z}_{/p}) := \{M \in \mathrm{GL}_2(\mathbb{Z}_{/p}) \mid \det M = \pm 1\}$$

on pairs (Q_1, Q_2). In what follows regard this as a *left* action of $\mathrm{SL}_2^{\pm}(\mathbb{Z}_{/p})$ so that $M = (m_{ij}) \in \mathrm{SL}_2^{\pm}(\mathbb{Z}_{/p})$ acts via

$$(*) \qquad M \cdot (Q_1, Q_2) = (m_{11}Q_1 + m_{12}Q_2, m_{21}Q_1 + m_{22}Q_2).$$

Now we determine the classification disregarding the identification of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ with π_1. On the levels of quadratic forms, we may replace the pair (Q_1, Q_2) by (Q_1', Q_2') where Q_1 and Q_1' (as well as Q_2 and Q_2') are related by a common change of coordinates, i.e. an automorphism of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$, which amounts to $\mathrm{GL}_2(\mathbb{Z}_{/p})$. In what follows, regard this as a *right* action of $\mathrm{GL}_2(\mathbb{Z}_{/p})$ on pairs (Q_1, Q_2).

Lemma 6.1 *Let z be a quadratic nonresidue in $\mathbb{Z}_{/p}$. A pair of binary quadratic forms (Q_1, Q_2) satisfying the condition in Proposition 4.1 is equivalent to (xa^2, yb^2) or equivalent to $(a^2 + xb^2, 2ab)$ for $x, y \in \mathbb{Z}_{/p}$.*

Proof There are five [24, Theorem IV.10] equivalence classes of binary quadratic forms modulo p, namely the trivial form $Q(a,b) = 0$, two degenerate forms a^2 and za^2, and two nondegenerate quadratic forms $a^2 + b^2$ and $a^2 + zb^2$.

Suppose Q_1 is degenerate, so $Q_1(a,b) = a^2$ or $Q_1(a,b) = za^2$. Through an automorphism of $\mathbb{Z}^2$ subtracting a multiple of Q_1, the form Q_2 becomes $xab + yb^2$ for some $x, y \in \mathbb{Z}_{/p}$. By Proposition 4.1, it cannot be that $y = 0$. Since $y \neq 0$, the automorphism of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ sending a to a and b to $-ax/(2y) + b$ preserves Q_1 but transforms Q_2 into $yb^2 - (x^2/(4y))a^2$. Subtracting off a multiple of Q_1 via an automorphism of $\mathbb{Z}^2$ finally transforms Q_2 into yb^2. Therefore $(Q_1, Q_2) \simeq (xa^2, yb^2)$ for some $x, y \in \mathbb{Z}_{/p}$.

On the other hand, suppose Q_1 is nondegenerate, meaning $Q_1(a,b) = a^2 + b^2$ or $Q_1(a,b) = a^2 + zb^2$. As before, by subtracting off a multiple of Q_1, the form Q_2 becomes $xab + yb^2$ for some $x, y \in \mathbb{Z}_{/p}$. Either $y \neq 0$ or $y = 0$. If $y \neq 0$, then as before Q_2 is equivalent to $yb^2 - (x^2/(4y))a^2$, which via an automorphism of $\mathbb{Z}^2$ is transformed into a multiple of b^2, and this case is then handled by the above case in which Q_1 is degenerate. If $y = 0$, then we are in the situation $(a^2 + zb^2, xab)$ for nonzero z

and x. We apply the automorphism given by $a \mapsto a$ and $b \mapsto 2b/x$ to reduce to a situation of the form $(a^2 + wb^2, 2ab)$ for some nonzero w. □

Proposition 6.2 *A pair of the form* (xa^2, yb^2) *for nonzero* $x, y \in \mathbb{Z}_{/p}$ *is equivalent to* $(a^2 + wb^2, 2ab)$ *for* $w \in \mathbb{Z}_{/p}$.

Proof By independently scaling a and b, and depending on whether or not x and y are quadratic residues, the pair (xa^2, yb^2) is equivalent to

$$(a^2, b^2), \quad (a^2, zb^2), \quad (za^2, b^2) \quad \text{or} \quad (za^2, zb^2)$$

for a quadratic nonresidue $z \in \mathbb{Z}^\star_{/p}$. By exchanging the roles of a and b and swapping the components of the tuple, the pair (a^2, zb^2) is equivalent to (za^2, b^2). It is also the case that $(za^2, zb^2) \simeq (a^2, b^2)$ because

$$(za^2, zb^2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1/z \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \cdot (a^2, b^2).$$

To conclude the proof, we show $(a^2, wb^2) \simeq (a^2 + 4w^2b^2, 2ab)$ for $w \in \mathbb{Z}_{/p}$. To see this, applying the equivalence given by $a \mapsto a/(2w) - b$ and $b \mapsto a + 2wb$ shows

$$(a^2, wb^2) \simeq \Big(\frac{1}{4w^2}a^2 - \frac{1}{w}ab + b^2, wa^2 + 4w^2ab + 4w^3b^2\Big),$$

and then applying the automorphism of $\mathbb{Z}^2$ corresponding to

$$\begin{pmatrix} 1/(4w^2) & -1/(2w) \\ w & 2w^2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_{/p})$$

implies that

$$(a^2 + 4w^2b^2, 2ab) \simeq \Big(\frac{1}{4w^2}a^2 - \frac{1}{w}ab + b^2, wa^2 + 4w^2ab + 4w^3b^2\Big),$$

so $(a^2, wb^2) \simeq (a^2 + 4w^2b^2, 2ab)$. □

It remains to check that $(a^2 + b^2, 2ab)$ is *not* equivalent to $(a^2 + zb^2, 2ab)$.

Lemma 6.3 *If* $(a^2 + \delta b^2, 2ab)$ *is equivalent to* $(a^2 + \delta' b^2, 2ab)$ *for nonzero* δ *and* δ', *then* $\delta'/\delta \in \mathbb{Z}^{\star 4}_{/p}$.

Proof We follow the argument in [9]. Suppose $(a^2 + \delta b^2, 2ab)$ is equivalent to $(a^2 + \delta' b^2, 2ab)$ for nonzero δ and δ'. Then there is an $R \in \mathrm{GL}_2(\mathbb{Z}_{/p})$ and $S \in \mathrm{SL}_2^{\pm}(\mathbb{Z}_{/p})$ such that

(1) $$(a^2 + \delta b^2, 2ab) \cdot R = S \cdot (a^2 + \delta' b^2, 2ab).$$

Equality of the first component in each tuple yields

(2) $$(r_{11}^2 + \delta r_{21}^2)a^2 + 2(r_{11}r_{12} + \delta r_{21}r_{22})ab + (r_{12}^2 + \delta r_{22}^2)b^2 = s_{11}a^2 + 2s_{12}ab + \delta' s_{11}b^2.$$

Equality of the coefficients of a^2 and b^2 in (2) yields

$$s_{11} = \delta r_{21}^2 + r_{11}^2 \quad \text{and} \quad \delta' s_{11} = \delta r_{22}^2 + r_{12}^2,$$

respectively, and therefore

$$\delta r_{22}^2 + r_{12}^2 = \delta'\delta r_{21}^2 + \delta' r_{11}^2. \tag{3}$$

Equality of the second component in (1) yields

$$2r_{11}r_{21}a^2 + (2r_{12}r_{21} + 2r_{11}r_{22})ab + 2r_{12}r_{22}b^2 = s_{21}a^2 + 2s_{22}ab + \delta' s_{21}b^2. \tag{4}$$

Equality of the coefficients of a^2 and b^2 in (4) yields

$$s_{21} = 2r_{11}r_{21} \quad \text{and} \quad \delta' s_{21} = 2r_{12}r_{22},$$

respectively. We conclude

$$r_{12}r_{22} = \delta' r_{11}r_{21}. \tag{5}$$

Squaring both sides of (3) and subtracting 4δ times (5) squared yields

$$(\delta r_{22}^2 - r_{12}^2)^2 = (\delta'\delta r_{21}^2 - \delta' r_{11}^2)^2,$$

and so

$$\delta r_{22}^2 - r_{12}^2 = \pm(\delta'\delta r_{21}^2 - \delta' r_{11}^2). \tag{6}$$

The sign in (6) cannot be positive; if it were, then adding (6) to (3) yields

$$2\delta r_{22}^2 = 2\delta'\delta r_{21}^2,$$

so $r_{22}^2 = \delta' r_{21}^2$. But multiply both sides of (5) by r_{21}^2 and we deduce

$$r_{12}r_{22}r_{21}^2 = \delta' r_{11}r_{21}^2 = r_{22}^2 r_{11}.$$

So either $r_{22} = 0$, in which case $r_{21} = 0$ and the second row of R is zero, or $r_{12}r_{21} = r_{22}r_{11}$ and so $\det R = 0$. In either case we contradict the assumption $R \in \mathrm{GL}_2(\mathbb{Z}_{/p})$, and so the sign in (6) must be negative, meaning

$$\delta r_{22}^2 - r_{12}^2 = -\delta'\delta r_{21}^2 + \delta' r_{11}^2. \tag{7}$$

The difference of (3) and (7) yields

$$2r_{12}^2 = 2\delta\delta' r_{21}^2,$$

so $\delta\delta'$ is a square in $\mathbb{Z}_{/p}$. And if our only requirement is that $R \in \mathrm{GL}_2(\mathbb{Z}_{/p})$, then the necessary condition that $\delta'\delta \in \mathbb{Z}_{/p}^{\star 2}$ would suffice, but we also required $S \in \mathrm{SL}_2^{\pm}(\mathbb{Z}_{/p})$, or equivalently that $(\det S)^2 = 1$. From (2) and (4),

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} \delta r_{21}^2 + r_{11}^2 & \delta r_{21}r_{22} + r_{11}r_{12} \\ 2r_{11}r_{21} & r_{12}r_{21} + r_{11}r_{22} \end{pmatrix},$$

which means

$$\det S = (\delta r_{21}^2 - r_{11}^2)(r_{12}r_{21} - r_{11}r_{22}).$$

Squaring $\det R$ results in

$$\begin{aligned}(\det R)^2 = (r_{11}r_{22} - r_{12}r_{21})^2 = r_{11}^2 r_{22}^2 - 2r_{11}r_{12}r_{21}r_{22} + r_{12}^2 r_{21}^2 = r_{11}^4 \frac{\delta'}{\delta} - 2\delta' r_{11}^2 r_{21}^2 + \delta\delta' r_{21}^4 \\ = \frac{\delta'}{\delta}(\delta r_{21}^2 - r_{11}^2)^2\end{aligned}$$

by invoking (5) and applying the identities $r_{12}^2 = \delta\delta' r_{21}^2$ and $\delta r_{22}^2 = \delta' r_{11}^2$, which follow from taking the sum and difference of (3) and (7). Consequently,

$$(\det S)^2 = (\delta r_{21}^2 - r_{11}^2)^2 (\det R)^2 = \frac{\delta'}{\delta}(\delta r_{21}^2 - r_{11}^2)^4,$$

so δ'/δ must be a fourth power. □

In particular, $(a^2 + b^2, 2ab)$ is not equivalent to $(a^2 + zb^2, 2ab)$ because z was chosen specifically to be a quadratic nonresidue.

Lemma 6.4 *For nonzero* $\delta, w \in \mathbb{Z}_{/p}$, *the pair* $(a^2 + \delta b^2, 2ab)$ *is equivalent to* $(a^2 + \delta w^4 b^2, 2ab)$.

Proof Choose $r_1, r_2 \in \mathbb{Z}_{/p}$ so that

(8) $$\delta {r_1}^2 - r_2^2 \equiv 1/w^3 \pmod{p}.$$

This is possible; in fact, there are $p - (\delta/p)$ solutions to (8). Then set

$$R := \begin{pmatrix} w^2 r_2 & \delta w^2 r_1 \\ r_1 & r_2 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} \delta w^4 {r_1}^2 + w^4 r_2^2 & 2\delta w^4 r_1 r_2 \\ 2w^2 r_1 r_2 & \delta w^2 r_1^2 + w^2 r_2^2 \end{pmatrix}.$$

Because of (8), we have

$$\det R = -w^2(\delta {r_1}^2 - {r_2}^2) = -1/w \neq 0 \quad \text{and} \quad \det S = w^6(\delta {r_1}^2 - {r_2}^2)^2 = 1,$$

so $R \in \mathrm{GL}_2(\mathbb{Z}_{/p})$ and $S \in \mathrm{SL}_2(\mathbb{Z}_{/p})$.

We finish the proof by verifying

(9) $$(a^2 + \delta w^4 b^2, 2ab) \cdot R = S \cdot (a^2 + \delta b^2, 2ab).$$

Comparing the first coordinates each side of (9) shows

$$(r_2 w^2 a + \delta r_1 w^2 b)^2 + \delta w^4 (r_1 a + r_2 b)^2 = (\delta w^4 {r_1}^2 + w^4 {r_2}^2) \cdot (a^2 + \delta b^2) + 2\delta w^4 r_1 r_2 \cdot 2ab.$$

Similarly, the second coordinates are equal because

$$2(w^2 r_2 a + \delta w^2 r_1 b)(r_1 a + r_2 b) = 2w^2 r_1 r_2 (a^2 + b^2 \delta) + (\delta w^2 {r_1}^2 + w^2 {r_2}^2) 2ab.$$ □

It is easier to see that $(a^2 + \delta b^2, 2ab)$ is equivalent to $(a^2 + \delta w^8 b^2, 2ab)$. Simply replace a by aw and b by b/w^3 to show $(a^2 + \delta w^8 b^2, 2ab) \simeq (w^2 a^2 + \delta w^2 b^2, (2/w^2)ab)$, and then scale the first by $1/w^2$ and the second by w^2 to see that this is equivalent to $(a^2 + \delta b^2, 2ab)$. The challenge of Lemma 6.4 lies in replacing w^8 with w^4.

Combining Lemmas 6.3 and 6.4 yields the following:

Proposition 6.5 *Equivalence classes of pairs of the form* $(a^2 + wb^2, 2ab)$ *are in one-to-one correspondence with elements of* $\mathbb{Z}_{/p}^\times / (\mathbb{Z}_{/p}^\times)^4$, *where* $\mathbb{Z}_{/p}^\times$ *denotes units modulo* p.

Observe that the size of $\mathbb{Z}_{/p}^\star / (\mathbb{Z}_{/p}^\times)^4$ depends on $p \bmod 4$. Specifically, for $p \equiv 1 \bmod 4$, there are *four* equivalence classes. These are given by $(a^2 + zb^2, 2ab)$ for z representatives of classes $\mathbb{Z}_{/p}^\star / \mathbb{Z}_{/p}^{\star 4}$.

For $p \equiv 3 \bmod 4$, there are *two* equivalence classes. For nonzero $x, x', y, y' \in \mathbb{Z}_{/p}$, the pair (xa^2, yb^2) is equivalent to $(x'a^2, y'b^2)$, and every pair is equivalent to either $(a^2+b^2, 2ab)$ or $(a^2+zb^2, 2ab)$ for a quadratic nonresidue z. So the only possibilities are $(a^2+b^2, 2ab) \simeq (a^2, b^2)$ and $(a^2+zb^2, 2ab)$.

All of this algebra encodes the homotopy type of the quotients, as summarized in the following:

Theorem 6.6 *Let $p > 3$ be prime. If $p \equiv 1 \bmod 4$, then there are four homotopy classes of quotients of $S^3 \times S^3$ by free $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions. If $p \equiv 3 \bmod 4$, then there are two classes.*

Proof We must construct quotients of $S^3 \times S^3$ by free $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions which exhibit these possible k–invariants. For this, we rely on Lemma 5.1. We note that $(a^2+wb^2, 2ab)$ is equivalent to

$$(a^2+wb^2+(1+w)ab, 2ab) = ((a+b)(a+wb), 2ab),$$

so let $R = (1,1,2,0)$ and $Q = (1,w,0,1)$, and then $L(p,p;R,Q)$ has k–invariant equivalent to $(a^2+wb^2, 2ab)$. We must impose the additional condition $w \neq 0$ in order to ensure that this is a *free* action. With this construction in hand, the classification of quotients then follows from Proposition 6.5. □

Remark 6.7 There are precedents for considering the simultaneous equivalence of forms. The case of simultaneous equivalence of forms over $\mathbb{Z}$ is discussed in [23], but our situation over $\mathbb{Z}_{/p}$ is easier. To make the situation even more concrete, instead of forms, consider matrices; equivalence of forms amounts to congruence of matrices. That setup fits into the work of Corbas and Williams [9] which considers the action of $\mathrm{GL}_2(\mathbb{Z}_{/p}) \times \mathrm{GL}_2(\mathbb{Z}_{/p})$ on pairs (A, B) of matrices, where GL_2 acts on the right by congruence and on the left as in $(*)$.

7 Lens cross lens

Section 6 completed the classification of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions on $S^3 \times S^3$, but now we narrow in on a special case. Consider $L_3(p;1,x) \times L_3(p;1,y)$, i.e. the product of two lens spaces with rotation numbers x and y, respectively. Viewed as a quotient of $S^3 \times S^3$ by $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$, this product has k–invariant (xa^2, yb^2).

We can classify $L_3(p;1,x) \times L_3(p;1,y)$ up to (simple) homotopy equivalence. When $p \equiv 3 \bmod 4$, any product of 3–dimensional lens spaces is (simple) homotopy equivalent to any other such product.

Proposition 7.1 *Suppose $p \equiv 3 \bmod 4$. Then for nonzero $x, x', y, y' \in \mathbb{Z}_{/p}$, the pair (xa^2, yb^2) is equivalent to $(x'a^2, y'b^2)$.*

Proof As in the proof of Proposition 6.2, the pair (xa^2, yb^2) is equivalent to

$$(a^2, b^2) \simeq (za^2, zb^2) \quad \text{or} \quad (a^2, zb^2) \simeq (za^2, b^2)$$

for a quadratic nonresidue $z \in \mathbb{Z}^\star_{/p}$. But when $p \equiv 3 \bmod 4$, the quantity $-z$ is a square, and so

$$(a^2, zb^2) \simeq (a^2, -zb^2) \simeq (a^2, b^2),$$

meaning *all* pairs of the form (xa^2, yb^2) are equivalent. □

When $p \equiv 1 \mod 4$, since

$$(xa^2, yb^2) \simeq (a^2, (y/x)b^2) \simeq (a^2 + 4(y/x)^2 b^2, 2ab),$$

the classification boils down to whether or not $2(y/x)$ is a square modulo p.

This is related to work of Kwasik and Schultz; they proved squares of lens spaces are diffeomorphic.

Theorem 7.2 [14] *For p odd and rotation numbers r and q, there is a diffeomorphism*

$$L_3(p;1,r) \times L_3(p;1,r) \cong L_3(p;1,q) \times L_3(p;1,q).$$

A future paper completes the homeomorphism classification of spaces resulting from "linear" actions such as these products of lens spaces.

8 Some comments on groups containing $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$

While we know that $\mathbb{Z}_{/p}$ and $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ can act freely on $S^n \times S^n$, the exact conditions for a group to be able to act freely on $S^n \times S^n$ remains open. Conner [7] and Heller [12] showed that for a group to act freely on $S^n \times S^n$ the group must have rank at most 2, but Oliver [26] showed that A_4 cannot act on $S^n \times S^n$, and so every rank-2 simple group is also ruled out [1]. Explicit examples of free actions by subgroups of a nonabelian extension of S^1 by $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ have been constructed [11], but Okay and Yalçin [25] have shown that $\mathrm{Qd}(p) = (\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}) \rtimes SL_2(\mathbb{F}_p)$ cannot act freely on $S^n \times S^n$. In this section we show how the restrictions on the k–invariant as described in Section 4 can be useful in determining whether or not a group G containing $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ as a normal abelian subgroup can act freely on $X = S^n \times S^n$. We continue to take $p > 3$ to be an odd prime and $n \geq 3$ to be odd. We align some of our notation with that in [25] to better show the parallel calculations.

Similar to the approach in Section 4, we can consider the Borel fibration

$$X \xrightarrow{i} X_{hG} \to BG,$$

and the associated Serre spectral sequence

$$E_2^{p,q} = H^p(BG; H^q(X;\mathbb{Z})) \Rightarrow H^{p+q}(X_{hG};\mathbb{Z})$$

with the first nontrivial differential d_{n+1}. If α and γ are the generators in degree n of $H^*(X;\mathbb{Z})$ with $\alpha^2 = \gamma^2 = 0$, then $d_{n+1}(\alpha) = \bar{\tau}(\alpha, 0)$, $d_{n+1}(\gamma) = \bar{\tau}(0, \gamma)$ and $k^{n+1} = d_{n+1}(\alpha) \oplus d_{n+1}(\gamma)$.

Set K to be the normal abelian subgroup of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ in G, and consider the restriction of the spectral sequence associated to the Borel fibration to the K action. Then Proposition 4.1 and Corollary 4.2 can sometimes be used to determine if G can act freely on X.

The transgression for the first nontrivial differential of the restriction of the spectral sequence associated to the Borel fibration to K is

$$(d_{n+1})_K \colon H^0(BK; H^n(X;\mathbb{Z})) \to H^{n+1}(BK; H^0(X;\mathbb{Z})).$$

Let $\mathrm{Res}^G_K \colon H^*(G) \to H^*(K)$ be induced by the inclusion of K into G. Since the Borel construction is natural, it follows that the k–invariant in the restricted case is $k^{n+1}_K = \mathrm{Res}^G_K(d_{n+1}(\alpha)) \oplus \mathrm{Res}^G_K(d_{n+1}(\gamma))$.

Suppose G acts freely on X, so $H^*(X_{hG};\mathbb{Z}) \cong H^*(X/G;\mathbb{Z})$ is finite-dimensional in each degree and vanishes above $2n$. It follows that the restriction to K gives that $H^*(X_{hK};\mathbb{Z}) \cong H^*(X/K;\mathbb{Z})$ is also finite-dimensional in each degree and vanishes above $2n$ as K acts freely. If both $(d_{n+1})_K(\alpha)$ and $(d_{n+1})_K(\gamma)$ are zero in $H^{n+1}(K;\mathbb{Z})/\lambda^{(n+1)/2}$, for some nonzero $\lambda \in H^2(K;\mathbb{Z})$, then X/K will fail to be finite-dimensional by Corollary 4.2, and we get a contradiction. Hence G cannot act freely.

As an example, consider $G = \mathrm{Qd}(p) = (\mathbb{Z}_{/p})^2 \rtimes SL_2(\mathbb{Z}_{/p})$. We show that one can use the restrictions on the k–invariants and some of the arguments in [25] to see that $\mathrm{Qd}(p)$ cannot act freely on $S^n \times S^n$ for p an odd prime and n odd. This result is consistent with [25, Theorem 5.1].

Since cohomology is taken with $\mathbb{Z}_{/p}$ coefficients in [25], we first set up a relationship between generators from the different coefficient groups. Suppose the first nontrivial differential takes α and γ, also the generators of $H^n(S^n \times S^n;\mathbb{Z}_{/p})$ by slight abuse of notation, to μ_1 and μ_2 in $H^{n+1}(G;\mathbb{Z}_{/p})$. Taking K to be the normal elementary abelian subgroup $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ in $G = \mathrm{Qd}(p)$ and restricting the action to K, we have that $\theta_1, \theta_2 \in H^{n+1}(K;\mathbb{Z}_{/p})$ are such that $\theta_1 = \mathrm{Res}^G_K(\mu_1)$ and $\theta_2 = \mathrm{Res}^G_K(\mu_2)$.

Recall the commuting triangle, from Section 2,

$$\begin{array}{ccccccc} H^n(K;\mathbb{Z}) & \xrightarrow{\rho} & H^n(K;\mathbb{Z}_{/p}) & \xrightarrow{\tilde\beta} & H^{n+1}(K;\mathbb{Z}) & \xrightarrow{p} & H^{n+1}(K;\mathbb{Z}) \\ & & & \searrow^{\beta} & \downarrow{\scriptstyle\rho} & & \\ & & & & H^{n+1}(K;\mathbb{Z}_{/p}) & & \end{array}$$

Since p is the 0 map, the vertical ρ is injective and $\tilde\beta$ is surjective. We can write $H^*(K;\mathbb{Z}/p) = \mathbb{F}_p[x,y] \otimes \wedge(u,v)$, where $|x| = |y| = 2$, $|u| = |v| = 1$, $\beta(u) = x$, $\beta(v) = y$ and $H^*(K;\mathbb{Z}) = \mathbb{F}_p[a,b] \otimes \wedge(c)$, with $|a| = |b| = 2$ and $|c| = 3$. It is not hard to see that $\tilde\beta(x) = a$, $\tilde\beta(y) = b$ and $\tilde\beta(uv) = c$.

Now the Bockstein generally satisfies $\beta(\delta\varepsilon) = \beta(\delta)\varepsilon + (-1)^{|\delta|}\delta\beta(\varepsilon) = \delta\beta(\varepsilon)$ for δ being $x^i y^j$ and ε being u, v or uv. We see that

$$\beta(H^n(K;\mathbb{Z}_{/p})) \subseteq \langle x^{(n+1)/2}, x^{(n-1)/2} y, \dots, y^{(n+1)/2} \rangle \subseteq \mathbb{F}_p[x,y],$$

since $n+1$ is even. Similarly, $\tilde\beta$ satisfies $\tilde\beta(\delta\varepsilon) = \delta\tilde\beta(\varepsilon)$ for δ being $x^i y^j$ and ε being u, v or uv. Again we see that $\tilde\beta(H^n(K;\mathbb{Z}_{/p})) \subseteq \langle a^{(n+1)/2}, a^{(n-1)/2} b, \dots, b^{(n+1)/2} \rangle \subseteq \mathbb{F}_p[a,b]$. As $\tilde\beta$ is surjective, ρ is injective, and $\beta = \rho(\tilde\beta)$. It follows that the k–invariant $\theta_1 \oplus \theta_2$ comes from elements in $H^{n+1}(K;\mathbb{Z})$ for some K action on $S^n \times S^n$: $k^{n+1} = \rho^{-1}(\theta_1) \oplus \rho^{-1}(\theta_2)$.

In [25] it is shown that the ideal generated by θ_1 and θ_2 is in fact generated by $\zeta^{(n+1)/2(p+1)}$, where $\zeta = xy^p - yx^p$ (which is in part based on calculations in [15]). Since no power of ζ will contain $x^{(n+1)/2}$ or $y^{(n+1)/2}$, we see that $d_{n+1}(\alpha)$ and $d_{n+1}(\gamma)$, where α and γ generate $H^n(S^n \times S^n;\mathbb{Z})$, have both $q_{\alpha,0}$ and $q_{\gamma,0}$ zero (where $q_{\alpha,0}$ and $q_{\gamma,0}$ are the coefficients in Proposition 4.1). We derive a contradiction.

It is worth noting that, in [25], the calculations show that the free actions of $\mathrm{Qd}(p)$ must have p smaller than n, and $n+1$ divisible by $2(p+1)$. The argument also finds a contradiction to finiteness, but relies on [5]. We also note that while we take p to be large in our homotopy type calculations, the only restrictions that were required in Section 4 (and hence in this section) were that $p > 3$ be an odd prime and $n \geq 3$ be odd. Further, there may be a way to show a contradiction to finiteness using Proposition 4.1 more directly (without needing to make arguments with $\mathbb{Z}_{/p}$ coefficients).

A similar argument could hold for any group containing $(\mathbb{Z}_{/p})^2$ that has a restriction that forces the transgression to behave in such a way.

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