Algebraic & Geometric Topology

Volume 24 (2024)

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Algebraic & Geometric Topology **24**:4 (2024) 1863–1901
DOI: 10.2140/agt.2024.24.1863
Published: 16 July 2024

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Generalizing the ideas of $\mathbb{Z}_k$–manifolds from Sullivan and stratifolds from Kreck, we define $\mathbb{Z}_k$–stratifolds. We show that the bordism theory of $\mathbb{Z}_k$–stratifolds is sufficient to represent all homology classes of a CW–complex with coefficients in $\mathbb{Z}_k$. We present a geometric interpretation of the Bockstein long exact sequences and the Atiyah–Hirzebruch spectral sequence for $\mathbb{Z}_k$–bordism for k an odd number. Finally, for p an odd prime, we give geometric representatives of all classes in $H_*(B\mathbb{Z}_p;\mathbb{Z}_p)$ using $\mathbb{Z}_p$–stratifolds.

57R90, 58A35, 58A40; 55N20

1 Introduction

Various geometric models of homology classes use the notion of bordism. For instance, Baas [3] constructs a generalized homology theory using the bordism of manifolds with singularities. Buoncristiano, Rourke and Sanderson [5] give a geometric treatment of generalized homology. Certain singular spaces called $\mathbb{Z}_k$–manifolds were introduced initially by Sullivan [18; 19; 20], although Morgan and Sullivan [15] gave the first formal study of this subject. The theory of $\mathbb{Z}_k$–manifolds gives a geometric model for $\mathbb{Z}_k$–homology classes, but Sullivan pointed out that $\mathbb{Z}_k$–manifolds are not general enough to represent $\mathbb{Z}_k$–homology. For example, the generator of $H_8(K(\mathbb{Z},3);\mathbb{Z}_3)$ is not represented by a $\mathbb{Z}_3$–manifold; see Sullivan [21]. Moreover, Brumfiel [4] shows that the nonzero classes in $H_{2p}(K(\mathbb{Z}_p,1);\mathbb{Z}_p)$ cannot be represented by $\mathbb{Z}_p$–manifolds whenever p is prime. In this work, we show that for an odd prime number p, there exists a class $\alpha_{2i}\in H_{2i}(B\mathbb{Z}_p;\mathbb{Z}_p)$, with $i\geq p$, that cannot be represented by $\mathbb{Z}_p$–manifolds. Thus a geometric model is needed to represent every homology class with $\mathbb{Z}_k$–coefficients. For this purpose, we focus on the theory of stratifolds developed by Kreck [12], where the homology groups with $\mathbb{Z}$–coefficients and $\mathbb{Z}_2$–coefficients are represented by the bordism theories of stratifold homology $SH_*(X)$ and stratifold homology with $\mathbb{Z}_2$–coefficients (this only works for $\mathbb{Z}_2$–coefficients).

We consider the generalized homology theory of bordism of $\mathbb{Z}_k$–manifolds with continuous maps to X, denoted by $\Omega_*(X;\mathbb{Z}_k)$. There is a long exact sequence satisfying the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_n(X) \arrow[r, "\times k"] \arrow[d, "h"] & \Omega_n(X) \arrow[r, "r"] \arrow[d, "h"] & \Omega_n(X;\mathbb{Z}_k) \arrow[r, "\delta"] \arrow[d, "h_{\mathbb{Z}_k}"] & \Omega_{n-1}(X) \arrow[r] \arrow[d, "h"] & \cdots \\
\cdots \arrow[r] & H_n(X) \arrow[r, "\times k"] & H_n(X) \arrow[r, "r"] & H_n(X;\mathbb{Z}_k) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd} \tag{1}$$

where $\delta\colon \Omega_*(X;\mathbb{Z}_k) \to \Omega_{n-1}(X)$ is the Bockstein homomorphism, $r\colon \Omega_n(X) \to \Omega_n(X;\mathbb{Z}_k)$ is the reduction homomorphism obtained by considering a closed manifold as a $\mathbb{Z}_k$–manifold with empty Bockstein, and $h_{\mathbb{Z}_k}\colon \Omega_*(X;\mathbb{Z}_k) \to H_*(X;\mathbb{Z}_k)$ is the Hurewicz homomorphism provided by the existence of fundamental $\mathbb{Z}_k$–homology classes.

Generalizing the ideas of Sullivan and Kreck, we define the bordism theory of $\mathbb{Z}_k$–stratifolds, and we can consider the generalized homology theory of bordism of $\mathbb{Z}_k$–stratifolds with continuous maps to X, denoted by $SH_*(X;\mathbb{Z}_k)$. We call this theory *$\mathbb{Z}_k$–stratifold homology*. Again, we have a long exact sequence satisfying the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & SH_n(X) \arrow[r, "\times k"] \arrow[d, "h"] & SH_n(X) \arrow[r, "r"] \arrow[d, "h"] & SH_n(X;\mathbb{Z}_k) \arrow[r, "\delta"] \arrow[d, "h_{\mathbb{Z}_k}"] & SH_{n-1}(X) \arrow[r] \arrow[d, "h"] & \cdots \\
\cdots \arrow[r] & H_n(X) \arrow[r, "\times k"] & H_n(X) \arrow[r, "r"] & H_n(X;\mathbb{Z}_k) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd} \tag{2}$$

In this case, the Hurewicz homomorphism $h_{\mathbb{Z}_k}\colon SH_*(X;\mathbb{Z}_k) \to H_*(X;\mathbb{Z}_k)$ is constructed in the same vein as in the theory of $\mathbb{Z}_k$–manifolds. We show that $\mathbb{Z}_k$–stratifold homology satisfies the Eilenberg–Steenrod axioms on CW–complexes, in particular, we show that the Mayer–Vietoris sequence axiom holds by using a regularity argument for $\mathbb{Z}_k$–stratifolds; see Kreck [12]. The main result of this paper is the following.

Theorem 1.1 *An isomorphism exists between $\mathbb{Z}_k$–stratifold homology theory and singular homology with $\mathbb{Z}_k$–coefficients. This isomorphism is valid for all CW–complexes and is compatible with the Bockstein homomorphisms.*

Führing [9] develops a smooth version of the Baas–Sullivan theory of manifolds with singularities that is applied to the positive scalar curvature problem. In a way, stratifolds and $\mathbb{Z}_k$–stratifolds are another kind of smooth version of the Baas–Sullivan theory of manifolds with singularities. One of the advantages of stratifolds and $\mathbb{Z}_k$–stratifolds is a very concrete description of the filtration of the Atiyah–Hirzebruch spectral sequence (AHSS) for oriented bordism and $\mathbb{Z}_k$–bordism. This geometric description of the AHSS for $\mathbb{Z}$–coefficients was given by Tene [23], and for $\mathbb{Z}_k$–coefficients has the following form.

Theorem 1.2 *For k an odd number, the filtration for the AHSS of $\mathbb{Z}_k$–bordism*

$$E^{\infty}_{n,0} \subseteq \cdots \subseteq E^{r+2}_{n,0} \subseteq \cdots \subseteq E^2_{n,0} \cong H_n(X;\mathbb{Z}_k) = SH_n(X;\mathbb{Z}_k) \tag{3}$$

coincides with the set of classes generated by singular $\mathbb{Z}_k$–stratifolds in X, where the singular part is of dimension at most $n-r-2$.

A fascinating application is the existence of homology classes $\alpha_{2i} \in H_{2p}(B\mathbb{Z}_p;\mathbb{Z}_p)$, for an odd prime number p and $i \geq p$, that cannot be represented by a $\mathbb{Z}_p$–manifold. This is similar to the counterexample of Thom for the Steenrod problem [24, Chapter III], which we explain geometrically in [2].

We organize the article as follows: Section 2 outlines some basic facts about $\mathbb{Z}_k$–manifolds studied by Morgan and Sullivan [15]. In Section 3, we briefly introduce the language of stratifolds from Kreck [12; 13]. Section 4 introduces the main theorems of this work, where we combine the theory of $\mathbb{Z}_k$–manifolds from Sullivan and the theory of stratifolds from Kreck. Then we define $\mathbb{Z}_k$–stratifolds and develop the basic theory of these objects. We show that the usual properties of stratifolds still remain valid. We show that $\mathbb{Z}_k$–stratifold homology satisfies the Eilenberg–Steenrod axioms on CW–complexes. Section 6 develops the existence of the fundamental class, and we postpone the proof of the existence of the Mayer–Vietoris sequence until the appendix. In Section 7, we apply the results of Tene [23] to give a geometric description of the Atiyah–Hirzebruch spectral sequence for $\mathbb{Z}_k$–bordism, for k an odd number. In Section 8, we use this description to find homology classes with $\mathbb{Z}_k$–coefficients that cannot be represented by $\mathbb{Z}_k$–manifolds. Finally, in Section 9, the two possible ways to represent homology with $\mathbb{Z}_2$–coefficients using stratifolds are related, providing an explicit isomorphism between the two theories.

Acknowledgements We thank the Math Institute UNAM-Oaxaca and Universidad de los Andes for the hospitality and financial support that made this collaboration possible. Ángel acknowledges and thanks the hospitality and financial support provided by the Max Planck Institute for Mathematics in Bonn. This work was partially supported by the grant #INV-2019-84-1860 from the Fondo de Investigaciones de la Facultad de Ciencias de la Universidad de los Andes. Segovia is supported by cátedras CONACYT and Proyecto CONACYT Ciencias básicas 2016, #284621. Torres' PhD thesis [25] contains part of these results under the supervision of Ángel. The maturity of the present paper is due to the guidance of Segovia during two visits by Torres to the Math Institute UNAM-Oaxaca; without this invaluable contribution, this work would not have been possible. Torres would like to thank the Universidad Pontificia Javeriana for the help provided after his PhD, and especially the Universidad Externado de Colombia, where he has been a professor in the mathematics department since 2020. Finally, we thank the reviewer for the careful reading of our manuscript. We sincerely appreciate all the valuable comments and suggestions which helped us improve its quality.

2 $\mathbb{Z}_k$–manifolds

Suppose that $k \geq 2$ is a positive integer. In what follows, we outline some basic facts about $\mathbb{Z}_k$–manifolds introduced by Morgan and Sullivan [15].

Note 2.1 Unless otherwise indicated, let us set the convention that the manifolds are oriented and compact. Also, all the diffeomorphisms and embeddings are orientation-preserving.

Definition 2.2 A closed n–dimensional $\mathbb{Z}_k$*–manifold* is given by the triple $\mathcal{M} = (M, \delta M, \theta_i)$, where

(1) M is a compact n–manifold, with boundary ∂M,

(2) δM is a compact $(n-1)$–manifold without boundary, called the Bockstein, and

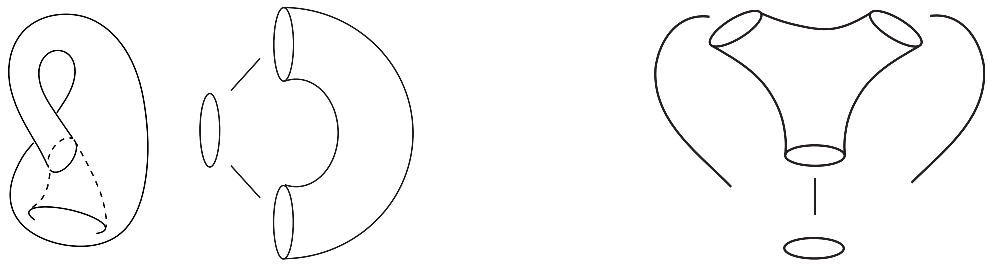

Figure 1: Left: a representation of the Klein bottle as the quotient space of a $\mathbb{Z}_2$–manifold. Right: a closed $\mathbb{Z}_3$–manifold.

(3) $\theta_i\colon \delta M \hookrightarrow \partial M$, with $i \in \mathbb{Z}_k$, are k disjoint embeddings such that we have a diffeomorphism $\partial M = \bigsqcup_{i\in\mathbb{Z}_k} \theta_i(\delta M)$.

Definition 2.3 There is an associated *quotient* space $\widetilde{M}$ given by the identification on M of the k copies of δM together using the embeddings θ_i.

Example 2.4 A closed oriented manifold is a $\mathbb{Z}_0$–manifold (or equivalently a $\mathbb{Z}$–manifold) where the Bockstein δM is empty.

Example 2.5 The typical example of a $\mathbb{Z}_2$–manifold is the cylinder $M = S^1\times[0,1]$, $\delta M = S^1$ and embeddings $\theta_1,\theta_2\colon S^1 \hookrightarrow S^1\times\{0\}\sqcup S^1\times\{1\}$, with $\theta_1(S^1) = S^1\times\{0\}$ and $\theta_2(S^1) = S^1\times\{1\}$ (with the reverse orientation on $S^1\times\{1\}$). The quotient space $K := \widetilde{M}$ is the well-known Klein bottle; see Figure 1, left.

Here we observe that even though the second integral homology group is zero for the Klein bottle, we can obtain a fundamental class after we change to $\mathbb{Z}_2$ coefficients, ie $H_2(K;\mathbb{Z}_2)\cong\mathbb{Z}_2$. In Section 6, we show this fundamental class always exists for a $\mathbb{Z}_k$–stratifold.

Example 2.6 Consider the pair of pants P with boundary $\partial P = S^1\sqcup S^1\sqcup S^1$ and Bockstein $\delta P = S^1$; see Figure 1, right.

Definition 2.7 An $(n+1)$–dimensional $\mathbb{Z}_k$*–manifold with boundary* is given by the triple $\mathcal{B} = (B,\delta B,\psi_i)$, where

(1) B is a compact $(n+1)$–dimensional manifold, with boundary ∂B,

(2) δB is a compact n–dimensional manifold, called the Bockstein, with boundary $\partial\delta B$, and

(3) $\psi_i\colon \delta B \hookrightarrow \partial B$, with $i\in\mathbb{Z}_k$, are k disjoint embeddings such that the triple

$$\Bigl(\partial B - \operatorname{int}\Bigl(\bigsqcup_{i\in\mathbb{Z}_k}\psi_i(\delta B)\Bigr), \partial\delta B, \psi_i|_{\partial\delta B}\Bigr)$$

defines a closed n–dimensional $\mathbb{Z}_k$–manifold $(M,\delta M,\theta_i)$.

This closed n–dimensional $\mathbb{Z}_k$–manifold is called the $\mathbb{Z}_k$*–boundary* of the $\mathbb{Z}_k$–manifold with boundary $\mathcal{B}$ and is denoted by $\partial\mathcal{B} = (M,\delta M,\theta_i)$.

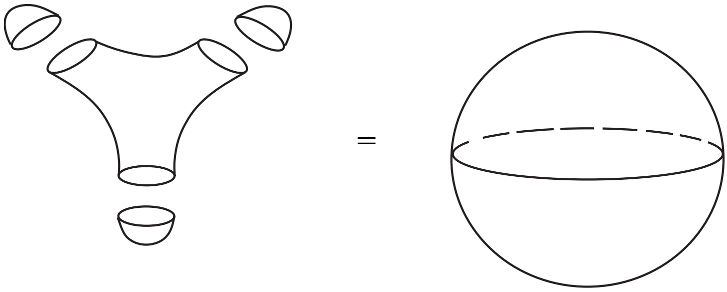

Figure 2: A $\mathbb{Z}_3$–manifold with boundary.

Definition 2.8 As before, there is the *quotient* space $\widetilde{B}$ which results from the identification on B of the k embedded copies of δB together using the embeddings ψ_i.

Example 2.9 Consider the three-dimensional $\mathbb{Z}_3$–manifold with boundary $\mathcal{B} = (B, \delta B, \psi_i)$, where $B = D^3$ is the three-dimensional closed ball (hence $\partial B = S^2$), $\delta B = D^2$ is the two-dimensional closed disc and the $\psi_i \colon D^2 \to S^2$ for $i \in \mathbb{Z}_3$ are given by three disjoint embedded discs inside the sphere. The $\mathbb{Z}_3$–boundary $\partial\mathcal{B} = (M, \delta M, \theta_i)$ is the two-dimensional $\mathbb{Z}_3$–manifold from Example 2.6, where M is the pair of pants and δM is the circle. See Figure 2 for an illustration.

Example 2.10 Consider the two-dimensional $\mathbb{Z}_3$–manifold with boundary $\mathcal{B} = (B, \delta B, \psi_i)$, where B is a connected surface of genus one with only one boundary circle, the Bockstein δB is the interval $[0, 1]$, and the $\psi_i \colon [0, 1] \to \partial B = S^1$ for $i \in \mathbb{Z}_3$ are given by three disjoint embedded intervals inside the circle. The $\mathbb{Z}_3$–boundary of the $\mathbb{Z}_3$–manifold $\mathcal{B}$ is a one-dimensional $\mathbb{Z}_3$–manifold $\partial\mathcal{B} = (M, \delta M, \theta_i)$, where M is the disjoint union of three copies of the interval, δM is the disjoint union of two points and the embeddings θ_i are given by the restrictions $\psi_i|_{\delta M}$. In Figure 3, we illustrate the $\mathbb{Z}_3$–stratifold $(B, \delta B, \psi_i)$, where on the right-hand side we depict the boundary ∂B after the quotient.

Definition 2.11 Let X be a topological space and n a natural number. An n–dimensional *singular $\mathbb{Z}_k$–manifold* in X is a closed n–dimensional $\mathbb{Z}_k$–manifold $\mathcal{M} = (M, \delta M, \theta_i)$ together with a continuous map $f \colon M \to X$ such that $f \circ \theta_i = f \circ \theta_j$ for $i, j \in \mathbb{Z}_k$. A *singular $\mathbb{Z}_k$–bordism* between two n–dimensional

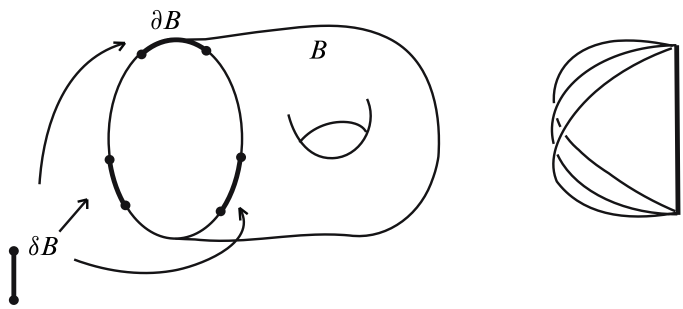

Figure 3: A $\mathbb{Z}_3$–manifold with boundary, left, and the boundary ∂B after quotient, right.

singular $\mathbb{Z}_k$–manifolds $(\mathcal{M}, f)$ and $(\mathcal{M}', f')$ is a $\mathbb{Z}_k$–manifold with boundary $\mathcal{B} = (B, \delta B, \psi_i)$, with $\mathbb{Z}_k$–boundary $\partial\mathcal{B} = (M + M', \delta M + \delta M', f + f')$ together with a continuous map $F\colon B \to X$ such that $F \circ \psi_i = F \circ \psi_j$ for $i, j \in \mathbb{Z}_k$, extending f and f'. Recall that the $\mathbb{Z}_k$–manifolds are oriented. In this definition, the sum of $\mathbb{Z}_k$–manifolds is given by

$$(M + M', \delta M + \delta M', f + f') = (M \sqcup -M', \delta M \sqcup -\delta M', f \sqcup f').$$

The $\mathbb{Z}_k$*–bordism group* group $\Omega_n(X; \mathbb{Z}_k)$ is given by the equivalence classes of n–dimensional singular $\mathbb{Z}_k$–manifolds $(\mathcal{M}, f)$ under this $\mathbb{Z}_k$–bordism relation. The elements of this group are denoted by $[\mathcal{M}, f]$.

The $\mathbb{Z}_k$–bordism groups $\Omega_n(X; \mathbb{Z}_k)$ are a generalized homology theory (this follows by Section 4 or see [5, Chapter III]). The existence of the fundamental class $[\mathcal{M}]_{\mathbb{Z}_k} \in H_n(\widetilde{M}; \mathbb{Z}_k)$, see Section 6, induces the Hurewicz homomorphism $h_{\mathbb{Z}_k}\colon \Omega_n(X; \mathbb{Z}_k) \to H_n(X; \mathbb{Z}_k)$. In addition, we have the reduction map $r\colon \Omega_n(X) \to \Omega_n(X; \mathbb{Z}_k)$. This map considers an n–dimensional closed manifold as a $\mathbb{Z}_k$–manifold with $\delta M = \varnothing$. Moreover, we have the Bockstein sequence, which fits into the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_n(X) \arrow[r, "\times k"] \arrow[d, "h"] & \Omega_n(X) \arrow[r, "r"] \arrow[d, "h"] & \Omega_n(X; \mathbb{Z}_k) \arrow[r, "\delta"] \arrow[d, "h_{\mathbb{Z}_k}"] & \Omega_{n-1}(X) \arrow[r] \arrow[d, "h"] & \cdots \\
\cdots \arrow[r] & H_n(X) \arrow[r, "\times k"] & H_n(X) \arrow[r, "r"] & H_n(X; \mathbb{Z}_k) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd} \tag{4}$$

for $n \geq 1$.

3 Stratifolds

We briefly introduce the language of stratifolds from Kreck [12; 13]. For this purpose, we need the notion of differential space [17; 12; 13].

Definition 3.1 A *differential space* is a pair $(X, \mathscr{C})$ where X is a topological Hausdorff space with a countable basis and $\mathscr{C} \subset C^0(X)$ is a sheaf of real-valued continuous functions such that for $f_1, \dots, f_k$ in $\mathscr{C}$ and f a smooth function on $\mathbb{R}^k$, the composition $f(f_1, \dots, f_k)$ is in $\mathscr{C}$.

For a differential space, each point $x \in X$ has associated a tangent space, denoted by $T_x X$, which is the space of all derivations of the germ $\Gamma_x(\mathscr{C})$ of smooth functions at x. A smooth manifold is a natural example of a differential space, which is locally diffeomorphic to $\mathbb{R}^n$ equipped with the sheaf of all smooth functions.

Definition 3.2 [13, Definition 1] An n–dimensional *stratifold* is a differential space $(S, \mathscr{C})$ where the sheaf $\mathscr{C}$ induces a suitable stratification $S^k := \{x \in S : \dim T_x S = k\}$. The union of all strata of dimension $\leq k$ is called the k–skeleton S_k. In addition, we assume:

(i) For each k, the stratum S^k, together with the restriction sheaf $\mathscr{C}|_{S^k}$, is a smooth k–dimensional manifold as a differential space.

(ii) All skeleta are closed subsets of S.

(iii) All strata of dimension $> n$ are empty.

(iv) For each $x \in S$ and open neighborhood U with $x \in U$, there is a so-called bump function $\rho\colon S \to \mathbb{R}_{\geq 0}$ in $\mathscr{C}$ such that $\operatorname{supp}\rho \subset U$ and $\rho(x) > 0$.

(v) For each $x \in S^k$, the restriction gives an isomorphism $\Gamma_x(\mathscr{C}) \to \Gamma_x(\mathscr{C}|_{S^k})$.

Definition 3.3 A continuous map $f\colon (S, \mathscr{C}) \to (S', \mathscr{C}')$ is *smooth* if the precomposition by f sends every element of $\mathscr{C}'$ to an element of $\mathscr{C}$. If f and the inverse f^{-1} are smooth, then f is called a *diffeomorphism of stratifolds*. Similarly, we can define the notion of a (smooth) *embedding of stratifolds* by requiring that the restriction to the image is a diffeomorphism of stratifolds.

Example 3.4 [12, Example 1, page 19] The open cone of an n–dimensional manifold,

$$\mathring{CM} := M \times [0,1)/_{M\times\{0\}},$$

is an example of an $(n+1)$–dimensional stratifold, where $\mathscr{C}$ consists of all continuous functions on $\mathring{CM}$ which are constant on some open neighborhood of the point produced by collapsing $M \times \{0\}$, and whose restriction to $M \times (0,1)$ is smooth.

Definition 3.5 Let W be a smooth manifold. A *collar* is a homeomorphism $c\colon \partial W \times [0,\epsilon) \to U$ with $\epsilon > 0$, where U is an open neighborhood of ∂W in W such that $c|_{\partial W\times\{0\}} = \mathrm{id}_{\partial W}$ and $c|_{\partial W\times(0,\epsilon)}$ is a diffeomorphism onto $U - \partial W$.

Definition 3.6 Let $(T, \partial T)$ be a pair of topological spaces. Assume $\mathring{T} = T - \partial T$ and ∂T are stratifolds of dimensions n and $n-1$, with $\partial T \subset T$ a closed subspace. A *collar* of ∂T into T is a homeomorphism $c\colon \partial T \times [0,\epsilon) \to U$ with $\epsilon > 0$, where U is an open neighborhood of ∂T in T such that $c|_{\partial T\times\{0\}} = \mathrm{id}_{\partial T}$ and $c|_{\partial T\times(0,\epsilon)}$ is a diffeomorphism of stratifolds onto $U - \partial T$.

Definition 3.7 An $(n+1)$–dimensional *stratifold with boundary* is a pair of topological spaces $(T, \partial T)$, together with a collar c of ∂T into T, where $T - \partial T$ is an $(n+1)$–dimensional stratifold and ∂T is an n–dimensional stratifold, which is a closed subspace of T. We call ∂T the *boundary* of T.

The following example is crucial in the theory of stratifolds.

Example 3.8 [12, page 36] The closed cone $C(S)$ of a stratifold S has underlying topological space $T = S \times [0,1]/_{S\times\{0\}}$, whose interior is $S \times [0,1)/_{S\times\{0\}}$ and whose boundary is $S \times \{1\}$. The collar is given by the map $S \times \left[0, \frac{1}{2}\right) \to C(S)$ mapping (x,t) to $(x, 1-t)$.

Now, we define some important classes of stratifolds [12].

Definition 3.9 [12, page 79] An n–dimensional stratifold S is *oriented* if the top stratum S^n is an oriented manifold and the stratum S^{n-1} is empty.

Definition 3.10 [12, page 43] An n–dimensional stratifold S is *regular* if for each $x \in S^i$, where $0 \le i \le n$, there is an open neighborhood U of x in S, a stratifold F with F^0 a single point, an open subset V of S^i, and a diffeomorphism of stratifolds $\phi\colon V \times F \to U$, whose restriction to $V \times F^0$ is the identity.

Remark 3.11 [12, page 24] In this paper, we restrict to a special class of stratifolds called *p–stratifolds*. The construction of a p–stratifold is as follows: we start with a zero-dimensional p–stratifold, which is a zero-dimensional manifold. Assume we construct by induction a $(k-1)$–dimensional p–stratifold $(S, \mathscr{C})$ and let W be a k–dimensional manifold with a smooth and proper map $f\colon \partial W \to S$. Then we define the k–dimensional p–stratifold $(W \sqcup_f S, \mathscr{C}')$, where $\mathscr{C}'$ is constructed using a collar $c\colon \partial W \times [0,\epsilon) \to U$. More precisely, the function g belongs to $\mathscr{C}'$ if and only if $g|_S$ and $g|_{W-\partial W}$ are smooth and for some $\delta < \epsilon$ we have $gc(x,t) = gf(x)$ for all $x \in \partial W$ and $t < \delta$.

Note 3.12 A stratifold with boundary T is an oriented/regular stratifold if both $T - \partial T$ and ∂T are oriented/regular stratifolds (the collar preserves the product orientation for oriented stratifolds). Similarly, T is a p–stratifold if both $T - \partial T$ and ∂T are p–stratifolds.

From Section 4, until the end of this paper, all statements about stratifolds are meant as statements about p–stratifolds; see Note 4.1.

As Kreck mentions in [13, page 303]: "The following observation is central for our construction of the zoo of bordism groups." For two stratifolds T and T' with the same boundary $\partial T = \partial T'$, there is a stratifold structure for the gluing of stratifolds $T \cup_{\partial T} T'$, where the two collars are combined to produce a *bicollar*; see the details in [12, pages 36–37].

Definition 3.13 Let X be a topological space and n a natural number. An n–dimensional *singular stratifold* in X is a closed (compact without boundary) n–dimensional stratifold S together with a continuous map $f\colon S \to X$. A *singular bordism* between two n–dimensional singular stratifolds (S, f) and (S', f') is a compact stratifold with boundary T, with boundary $(S + S', f + f')$ together with a continuous map $F\colon T \to X$ extending f and f'. The sum of oriented stratifolds is given by

$$(S + S', f + f') = (S \sqcup -S', f \sqcup f).$$

Since one can glue n–dimensional singular stratifolds over a common boundary component, singular bordism is an equivalence relation. The *oriented stratifold homology* group $SH_n(X)$ consists of the equivalence classes of n–dimensional oriented singular stratifolds (S, f) under this bordism relation. The elements of these groups are denoted by $[S, f]$.

The significance of the previous bordism groups lies in the positive solution for the Steenrod problem [7] of showing that a geometric object represents integral homology classes. The precise statement is:

Theorem 3.14 (Kreck [12, Theorem 20.1, page 186]) *The functor SH_* defines a homology theory. Moreover, there exists a natural transformation h from $SH_*(\cdot)$ to singular homology $H_*(\cdot\,;\mathbb{Z})$ such that h is an isomorphism for all CW–complexes.*

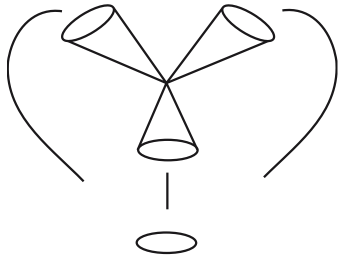

Figure 4: A closed $\mathbb{Z}_3$–stratifold.

4 $\mathbb{Z}_k$–stratifolds

Now we combine the theory of $\mathbb{Z}_k$–manifolds from Sullivan and the theory of stratifolds from Kreck.

Note 4.1 Unless otherwise indicated, let us set the convention that the stratifolds are oriented, regular p–stratifolds. Also, all the diffeomorphisms and embeddings of stratifolds are orientation-preserving.

Definition 4.2 A closed n–dimensional $\mathbb{Z}_k$*–stratifold* is given by the triple $\mathcal{S} = (S, \delta S, \theta_i)$, where

(1) S is a compact, n–dimensional stratifold, with boundary ∂S,

(2) δS is a compact $(n-1)$–dimensional stratifold without boundary, called the Bockstein, and

(3) the $\theta_i : \delta S \to \partial S$ for $i \in \mathbb{Z}_k$ are k disjoint embeddings of stratifolds such that we have a diffeomorphism of stratifolds $\partial S = \bigsqcup_{i \in \mathbb{Z}_k} \theta_i(\delta S)$.

Definition 4.3 There is an associated *quotient* space $\widetilde{S}$ given by the identification on S of the k copies of δS together using the embeddings θ_i.

Example 4.4 The class of closed stratifolds and the class of $\mathbb{Z}_k$–manifolds are the first examples of $\mathbb{Z}_k$–stratifolds.

Example 4.5 Consider the two-dimensional $\mathbb{Z}_3$–stratifold given by the closed cone of the disjoint union of three circles $S = C(S^1 \sqcup S^1 \sqcup S^1)$, where the boundary is $\partial S = S^1 \sqcup S^1 \sqcup S^1$, and the Bockstein is $\delta S = S^1$; see Figure 4.

Definition 4.6 An $(n+1)$–dimensional $\mathbb{Z}_k$*–stratifold with boundary* is given by the triple $\mathcal{T} = (T, \delta T, \psi_i)$, where

(1) T is a compact $(n+1)$–dimensional stratifold, with boundary ∂T,

(2) δT is a compact n–dimensional stratifold with boundary, called the Bockstein, with boundary $\partial \delta T$, and

(3) the $\psi_i : \delta T \hookrightarrow \partial T$ for $i \in \mathbb{Z}_k$ are k disjoint embeddings of stratifolds such that the triple

$$\Bigl(\partial T - \operatorname{int}\Bigl(\bigsqcup_{i\in\mathbb{Z}_k} \psi_i(\delta T)\Bigr), \partial\delta T, \psi_i|_{\partial\delta T}\Bigr)$$

defines a closed n–dimensional $\mathbb{Z}_k$–stratifold $(S, \delta S, \theta_i)$.

This closed n–dimensional $\mathbb{Z}_k$–stratifold is called the $\mathbb{Z}_k$*–boundary* of the $\mathbb{Z}_k$–stratifold $\mathcal{T}$ and is denoted by $\partial\mathcal{T} = (S, \delta S, \theta_i)$.

Definition 4.7 There is a *quotient* space $\widetilde{T}$ resulting from the identification on T of the k copies of δT together using the embeddings ψ_i.

Example 4.8 A $\mathbb{Z}_k$–manifold with boundary is an example of a $\mathbb{Z}_k$–stratifold with boundary.

Example 4.9 Consider the three-dimensional $\mathbb{Z}_3$–stratifold with boundary $\mathcal{T} = (T, \delta T, \psi_i)$, where T is the wedge of three closed balls $D^3 \vee D^3 \vee D^3$ by the north pole of the boundary spheres, hence the boundary is $\partial T = S^2 \vee S^2 \vee S^2$. The stratifold structure over the wedge point is given by the open cone of the disjoint union of three discs. The Bockstein is the two-dimensional closed disc $\delta T = D^2$, and the $\psi_i : D^2 \to S^2 \vee S^2 \vee S^2$ for $i \in \mathbb{Z}_3$ are given by the embeddings of D^2 on each of the three southern hemispheres. The $\mathbb{Z}_3$–boundary $\partial\mathcal{T} = (S, \delta S, \theta_i)$ is the two-dimensional $\mathbb{Z}_3$–stratifold from Example 4.5, where $S = C(S^1 \sqcup S^1 \sqcup S^1)$ and the Bockstein is $\delta S = S^1$. See Figure 5 for an illustration.

Definition 4.10 The *cone* of a $\mathbb{Z}_k$–stratifold $(S, \delta S, \theta_i)$ is defined as follows: take the closed cone $C(\delta S)$ (see [12, page 36] or Example 3.8) and use k copies $kC(\delta S) := \bigsqcup_{i\in\mathbb{Z}_k}(C(\delta S) \times \{i\})$ to get the closed stratifold $S' := kC(\delta S) \sqcup_{\partial S} S$. Now take the cone $C(S')$, which is an $(n+1)$–dimensional stratifold. The cone of the $\mathbb{Z}_k$–stratifold $(S, \delta S, \theta_i)$ is given by the $(n+1)$–dimensional $\mathbb{Z}_k$–stratifold with boundary $\mathcal{T} := (C(S'), C(\delta S), \psi_i)$, where ψ_i is the canonical inclusion in the i–component. The $\mathbb{Z}_k$–boundary of $\mathcal{T}$ is the original $\mathbb{Z}_k$–stratifold $(S, \delta S, \theta_i)$.

Note 4.11 For an n–dimensional $\mathbb{Z}_k$–stratifold $(S, \delta S, \theta_i)$, we need $n \geq 2$ in order to for $C(S')$ and $C(\delta S)$ to be oriented stratifolds.

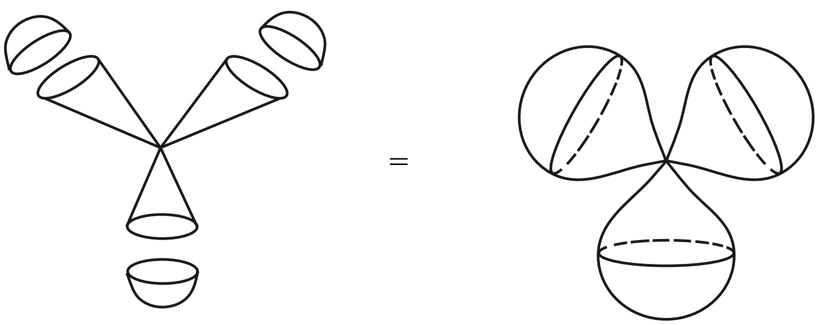

Figure 5: A $\mathbb{Z}_3$–stratifold with boundary.

The technique to show that the cartesian product of two differentiable manifolds has a differentiable structure is called *straightening the angle*. We follow the exposition given by Conner and Floyd in [6, Section I.3]. Let $\mathbb{R}_+ \subset \mathbb{R}$ consist of all nonnegative real numbers. We have the homeomorphism $\tau\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \times \mathbb{R}_+$, defined using polar coordinates by $\tau(\rho, \theta) = (\rho, 2\theta)$ with $0 \le \theta \le \pi/2$, such that the restriction τ is a diffeomorphism of $\mathbb{R}_+ \times \mathbb{R}_+ \setminus (0,0)$ onto $\mathbb{R} \times \mathbb{R}_+ \setminus (0,0)$. Consider the product of two differentiable manifolds B_1 and B_2 with collars U_1 and U_2 of the boundaries ∂B_1 and ∂B_2, respectively. There are diffeomorphisms $\Phi_1\colon U_1 \to \partial B_1 \times \mathbb{R}_+$ and $\Phi_2\colon U_2 \to \partial B_2 \times \mathbb{R}_+$. Let $U = U_1 \times U_2$. Then $\Phi = \Phi_1 \times \Phi_2$ is a homeomorphism of U onto $\partial B_1 \times \partial B_2 \times \mathbb{R}_+ \times \mathbb{R}_+$ and the composition with $\tau' = \mathrm{id} \times \tau$ produces a homeomorphism $\tau' \circ \Phi\colon U \to \partial B_1 \times \partial B_2 \times \mathbb{R} \times \mathbb{R}_+$. The differentiable structure of $\partial B_1 \times \partial B_2 \times \mathbb{R} \times \mathbb{R}_+$ induces a differentiable structure on U such that $\tau' \circ \Phi$ is a diffeomorphism. Then U and $B_1 \times B_2 \setminus \partial B_1 \times \partial B_2$ have differentiable structures, and they induce the same differentiable structure on their intersection. This structure is referred to as obtained by straightening the angle.

Proposition 4.12 *If $\mathcal{S} = (S, \delta S, \theta_i)$ is a closed n–dimensional $\mathbb{Z}_k$–stratifold, then after straightening the angle we obtain an $(n+1)$–dimensional $\mathbb{Z}_k$–stratifold with boundary $\mathcal{S} \times [0,1] := (S \times [0,1], \delta S \times [0,1], \psi_i)$, where the $\mathbb{Z}_k$–boundary $(S', \delta S', \theta_i')$ is given by*

- $S' = S \times \{0\} \sqcup -S \times \{1\}$,
- $\delta S' = \delta S \times \{0\} \sqcup -\delta S \times \{1\}$,
- $\theta_i' = \theta_i \times \{0\} \sqcup \theta_i \times \{1\}$.

Proof The technique of straightening the angle works similarly for the product of two stratifolds with boundary. In fact, from Kreck [12, Sections A.1–A.2], we can use local retractions to show that the product of stratifolds has a stratifold structure.

Consequently, the product space $S \times [0,1]$ has the structure of compact $(n+1)$–dimensional stratifold with boundary, where $\partial\,(S \times [0,1]) = (\partial S \times [0,1]) \cup (S \times \{0,1\})$ is also a stratifold with a collar into $S \times [0,1]$. Similarly, the product $\delta S \times [0,1]$ is a compact n–dimensional stratifold with boundary, and we have embeddings $\theta_i \times \mathrm{id}_{[0,1]}\colon \delta S \times [0,1] \hookrightarrow \partial S \times [0,1]$ for $i \in \mathbb{Z}_k$. Denote by ψ_i the embedding obtained as the composition of $\theta_i \times \mathrm{id}_{[0,1]}$ with the inclusion $\partial S \times [0,1] \hookrightarrow \partial\,(S \times [0,1])$. We associate the $\mathbb{Z}_k$–stratifold with boundary $(T, \delta T, \psi_i)$, where $T := S \times [0,1]$ and the Bockstein $\delta T := \delta S \times [0,1]$.

From Definition 4.6, it remains to show that the triple $(S', \delta S', \theta_i') := (\partial T - \mathrm{int}(\partial S \times [0,1]), \partial\delta T, \psi_i|_{\partial\delta T})$ is a closed n–dimensional $\mathbb{Z}_k$–stratifold. We have $S' = S \times \{0,1\}$, $\delta S' = \delta S \times \{0,1\}$ and the embeddings are $\theta_i' = \psi_i|_{\delta S'} = \theta_i \times \{0,1\}$. The orientation of $S \times [0,1]$ induces opposite orientations for the two copies of S associated to $\{0,1\}$, and similarly for δS. The embedding $\theta_i \times \{0\}$ preserves the orientation, while the embedding $\theta_i \times \{1\}$ reverses the orientation. This shows that $(S', \delta S', \theta_i')$ is a $\mathbb{Z}_k$–stratifold which is the $\mathbb{Z}_k$–boundary of $\mathcal{S} \times [0,1]$. □

Now we state a gluing lemma for $\mathbb{Z}_k$–stratifolds. This result is a direct application of Proposition A.1 in Kreck's book [12, page 194].

Lemma 4.13 *Let $\mathcal{T} := (T, \delta T, \psi_i)$ and $\mathcal{T}' := (T', \delta T', \psi_i')$ be $\mathbb{Z}_k$–stratifolds with $\mathbb{Z}_k$–boundaries $\partial\mathcal{T} = \mathcal{S} \sqcup \mathcal{S}'$ and $\partial\mathcal{T}' = \mathcal{S} \sqcup \mathcal{S}''$, where $\mathcal{S} = (S, \delta S, \theta_i)$, $\mathcal{S}' = (S', \delta S', \theta_i')$ and $\mathcal{S}'' = (S'', \delta S'', \theta_i'')$ are closed $\mathbb{Z}_k$–stratifolds. Then there is a $\mathbb{Z}_k$–stratifold with boundary*

$$\mathcal{T} \sqcup_{\mathcal{S}} \mathcal{T}' := (T \sqcup_S T', \delta T \sqcup_{\delta S} \delta T', \psi_i \sqcup_{\delta S} \psi_i'),$$

where the $\mathbb{Z}_k$–boundary is $\mathcal{S}' \sqcup \mathcal{S}''$.

Proof We consider the stratifolds $Y_1 := S' \sqcup_{\partial S'} \bigsqcup_{i \in \mathbb{Z}_k} \psi_i(\delta T)$ and $Y_2 := S'' \sqcup_{\partial S''} \bigsqcup_{i \in \mathbb{Z}_k} \psi_i'(\delta T')$. Thus the boundary of the stratifold T and T' are $\partial T = S \sqcup_{\partial S} Y_1$ and $\partial T' = S \sqcup_{\partial S} Y_2$, respectively. The work of Kreck [12, Proposotion A.1, page 194] implies that the gluing $T \sqcup_S T'$ is a stratifold with boundary, where $\partial(T \sqcup_S T') = Y_1 \sqcup_{\partial S} Y_2$. Similarly, the gluing $\delta T \sqcup_{\delta S} \delta T'$ is a stratifold with boundary, which is the Bockstein. Thus the $\mathbb{Z}_k$–boundary is precisely $(S' \sqcup S'', \delta S' \sqcup \delta S'', \theta_i' \sqcup \theta_i'')$, and the lemma follows. □

Definition 4.14 Let X be a topological space and n a natural number. An n–dimensional *singular $\mathbb{Z}_k$–stratifold* in X is a closed n–dimensional $\mathbb{Z}_k$–stratifold $\mathcal{S} = (S, \delta S, \theta_i)$ together with a continuous map $f : S \to X$ such that $f \circ \theta_i = f \circ \theta_j$ for $i, j \in \mathbb{Z}_k$. A *singular $\mathbb{Z}_k$–bordism* between two n–dimensional singular $\mathbb{Z}_k$–stratifolds $(\mathcal{S}, f)$ and $(\mathcal{S}', f')$ is a $\mathbb{Z}_k$–stratifold with boundary $\mathcal{T} = (T, \delta T, \psi_i)$, with $\mathbb{Z}_k$–boundary $\partial\mathcal{T} = (S + S', \delta S + \delta S', f + f')$ together with a continuous map $F : T \to X$ such that $F \circ \psi_i = F \circ \psi_j$ for $i, j \in \mathbb{Z}_k$, extending f and f'. Recall that the $\mathbb{Z}_k$–stratifolds consist of oriented, regular p–stratifolds. In this definition, the sum of $\mathbb{Z}_k$–stratifolds is given by

$$(S + S', \delta S + \delta S', f + f') = (S \sqcup -S', \delta S \sqcup -\delta S', f \sqcup f').$$

Again, one can glue n–dimensional singular $\mathbb{Z}_k$–stratifolds over a common boundary component. We state in Proposition 4.15 that singular $\mathbb{Z}_k$–bordism is an equivalence relation. The *$\mathbb{Z}_k$–stratifold homology* group $SH_n(X; \mathbb{Z}_k)$ is given by the equivalence classes of n–dimensional singular $\mathbb{Z}_k$–stratifolds $(\mathcal{S}, f)$ under the $\mathbb{Z}_k$–stratifold bordism relation. We denote by $[\mathcal{S}, f]$ the elements of this group.

As a consequence of Proposition 4.12 and the gluing result of Lemma 4.13, we obtain the following.

Proposition 4.15 *The $\mathbb{Z}_k$–stratifold bordism relation is an equivalence relation.*

To any closed n–dimensional stratifold S, there is an associated closed n–dimensional stratifold given by the disjoint union $kS := \bigsqcup_{i \in \mathbb{Z}_k} S \times \{i\}$. This assignment produces the homomorphism

$$\times k : SH_n(X) \to SH_n(X). \tag{5}$$

To any closed n–dimensional $\mathbb{Z}_k$–stratifold $\mathcal{S} = (S, \delta S, \theta_i)$, there is an associated closed n–dimensional $\mathbb{Z}_k$–stratifold given by the disjoint union $kS := \bigsqcup_{i \in \mathbb{Z}_k} S \times \{i\}$, where the Bockstein is the whole boundary ∂S and the embeddings $\psi_i \colon \partial S \to \bigsqcup_{i\in\mathbb{Z}_k} \partial S \times \{i\}$ are the canonical inclusions. Moreover, the boundary $\partial S = \bigsqcup_{i\in\mathbb{Z}_k} \theta_i(\delta S)$ can be considered as a k–disjoint union and we can denote $(kS, k\delta S, \psi_i) := (kS, \partial S, \psi_i)$. This assignment produces the homomorphism

$$\times k^k \colon SH_n(X;\mathbb{Z}_k) \to SH_n(X;\mathbb{Z}_k), \tag{6}$$

which we show below is trivial.

Proposition 4.16 *For every integer $n \geq 0$, the homomorphism $\times k^k \colon SH_n(X;\mathbb{Z}_k) \to SH_n(X;\mathbb{Z}_k)$ is zero.*

Proof Take $(\mathcal{S}, f) = ((S, \delta S), f)$ a closed singular $\mathbb{Z}_k$–stratifold. Consider the stratifold with boundary given by the cylinder $T := kS \times [0,1]$ and the Bockstein $\delta T := (\partial S \times [0,1]) \sqcup_{\partial S\times\{1\}} (-S \times \{1\})$ with embeddings

$$\psi_i \colon \delta T \hookrightarrow \partial T = \bigl[(S \times \{0\}) \sqcup_{\partial S\times\{0\}} (\partial S \times [0,1]) \sqcup_{\partial S\times\{1\}} (-S \times \{1\})\bigr] \times \{i\},$$

which are the canonical inclusions. The $\mathbb{Z}_k$–boundary of the $\mathbb{Z}_k$–stratifold $(T, \delta T, \psi_i)$ is the k–disjoint union of $(S, \delta S)$. $\square$

Similar to the work of Morgan and Sullivan [15], we have the Bockstein sequence, which fits into the commutative diagram

$$\begin{tikzcd}
\arrow[r] & SH_n(X) \arrow[r, "\times k"] \arrow[d, "h"] & SH_n(X) \arrow[r, "r"] \arrow[d, "h"] & SH_n(X;\mathbb{Z}_k) \arrow[r, "\delta"] \arrow[d, "h_{\mathbb{Z}_k}"] & SH_{n-1}(X) \arrow[r] \arrow[d, "h"] & \cdots SH_0(X;\mathbb{Z}_k) \arrow[d] \\
\arrow[r] & H_n(X) \arrow[r, "\times k"] & H_n(X) \arrow[r, "r"] & H_n(X;\mathbb{Z}_k) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots H_0(X;\mathbb{Z}_k)
\end{tikzcd} \tag{7}$$

The description of the maps is as follows:

- The reduction $r \colon SH_n(X) \to SH_n(X;\mathbb{Z}_k)$ is obtained by considering an n–dimensional closed stratifold as a $\mathbb{Z}_k$–stratifold, ie $(S, \delta S, \theta_i)$ with $\delta S = \varnothing$.
- Multiplication $\times k \colon SH_n(X) \to SH_n(X)$ takes a singular stratifold (S, f) in X and assigns the class of the k–disjoint union of S, denoted by $[kS, kf]$.
- The Bockstein $\delta \colon SH_n(X;\mathbb{Z}_k) \to SH_{n-1}(X)$ assigns to a singular $\mathbb{Z}_k$–stratifold $(\mathcal{S}, f)$, where $\mathcal{S} = (S, \delta S, \theta_i)$, the class $[\delta S, f|_{\delta S}]$.
- The Hurewicz homomorphism for stratifolds, $h \colon SH_n(X) \to H_n(X)$ for $n \geq 0$, was constructed by Kreck [12, pages 186–187].
- The Hurewicz homomorphism for $\mathbb{Z}_k$–stratifolds, $h_{\mathbb{Z}_k} \colon SH_n(X;\mathbb{Z}_k) \to H_n(X;\mathbb{Z}_k)$ for $n \geq 0$, is constructed in Section 6, where we show the existence of the fundamental class for $\mathbb{Z}_k$–stratifolds.

We leave the proof of the exactness of (7) for Section 5, where the commutativity follows after we construct the fundamental class in Section 6.

Finally, we spend the rest of the section discussing the properties of $SH_*(\,\cdot\,;\mathbb{Z}_k)$ as a functor. Kreck [12] proves the Eilenberg–Steenrod axioms for the bordism groups $SH_*(\,\cdot\,)$ in the category of CW–complexes. We have a functor, ie $\mathrm{id}_* = \mathrm{id}$ and $(g\circ f)_* = g_*\circ f_*$, which is homotopy invariant, has the Mayer–Vietoris sequence, $SH_n(*) = 0$ for $n \neq 0$ and $SH_0(*) = \mathbb{Z}$. Similarly, the $\mathbb{Z}_k$–stratifold homology satisfies the Eilenberg–Steenrod axioms, that we show in detail below. The proof of the Mayer–Vietoris sequence is in Section A.2.

Definition 4.17 A continuous map $g\colon X\to Y$ defines a morphism between the $\mathbb{Z}_k$–stratifold bordism groups by

$$g_*\colon SH_n(X;\mathbb{Z}_k)\to SH_n(Y;\mathbb{Z}_k),\quad [\mathcal{S},f]\mapsto[\mathcal{S},g\circ f],$$

for $\mathcal{S} = (S,\delta S,\theta_i)$ a closed n–dimensional $\mathbb{Z}_k$–stratifold.

This defines a functor which is homotopy invariant, as in the following proposition.

Proposition 4.18 *If g and g' are homotopic maps from X to Y, then*

$$g_* = {g'}_*\colon SH_n(X;\mathbb{Z}_k)\to SH_n(Y;\mathbb{Z}_k).$$

Proof There is a homotopy $G\colon X\times[0,1]\to Y$ between g and g'. Take $[\mathcal{S},f]\in SH_n(X;\mathbb{Z}_k)$, and hence $[\mathcal{S}\times[0,1], G\circ(f\times\mathrm{id})]$ is a singular $\mathbb{Z}_k$–stratifold bordism (see Proposition 4.12) between $g_*([\mathcal{S},f])$ and ${g'}_*([\mathcal{S},f])$. □

Proposition 4.19 *For the $\mathbb{Z}_k$–stratifold bordism group, we have*

$$SH_n(*;\mathbb{Z}_k) = \begin{cases}\mathbb{Z}_k & \text{for } n=0,\\ 0 & \text{for } n\neq 0.\end{cases}$$

Proof An important assumption here is that every n–dimensional $\mathbb{Z}_k$–stratifold $(S,\delta S)$ is formed by oriented stratifolds S and δS. For $n\geq 2$, we use the first horizontal long exact sequence of (7), with $SH_n(*) = 0$ and $SH_{n-1}(*) = 0$, and we conclude $SH_n(*;\mathbb{Z}_k) = 0$. For $n = 1$, the sequence (7) becomes

$$0\to SH_1(*;\mathbb{Z}_k)\to\mathbb{Z}\xrightarrow{\times k}\mathbb{Z}\xrightarrow{r} SH_0(*;\mathbb{Z}_k)\to 0,$$

then $SH_1(*;\mathbb{Z}_k) = 0$ and $SH_0(*;\mathbb{Z}_k) = \mathbb{Z}_k$. □

A geometric approach for the previous proposition is as follows: for any closed n–dimensional $\mathbb{Z}_k$–stratifold $\mathcal{S} = (S,\delta S,\theta_i)$, with $n>1$, we take the cone as in Definition 4.10. Thus we consider the usual cone $C(\delta S)$ and use k copies $kC(\delta S)$ to get the closed stratifold $S' := kC(\delta S)\sqcup_{\partial S} S$. Then we form the $(n{+}1)$–dimensional $\mathbb{Z}_k$–stratifold with boundary $\mathcal{T} := (C(S'), C(\delta S),\psi_i)$ where ψ_i is the canonical inclusion on the i^{th} component. The $\mathbb{Z}_k$–boundary of $\mathcal{T}$ is the original $\mathbb{Z}_k$–stratifold $(S,\delta S,\theta_i)$. For $n = 1$, we have a disjoint union of circles and intervals with orientation. Since each interval has the boundary $\{+,-\}$, then the number of intervals must be divided by k. Thus, after capping the circles with discs by Proposition 4.16, this element is trivial in $SH_1(*;\mathbb{Z}_k)$. Finally, for $n = 0$, the generator of $SH_0(*;\mathbb{Z}_k)$ is the closed zero-dimensional $\mathbb{Z}_k$–stratifold $(*,\varnothing,\mathrm{id}_\varnothing)$, where we use Proposition 4.16.

5 The Bockstein sequence

Previously, we have defined the k–disjoint union homomorphisms for stratifolds and $\mathbb{Z}_k$–stratifolds. These homomorphisms are as follows $\times k\colon SH_n(X) \to SH_n(X)$ and $\times k^k\colon SH_n(X;\mathbb{Z}_k) \to SH_n(X;\mathbb{Z}_k)$, defined in (5) and (6), respectively. The second is the trivial homomorphism by Proposition 4.16. There is a third k–disjoint union homomorphism of the form

$$\times k^{k^2}\colon SH_n(X;\mathbb{Z}_k) \to SH_n(X;\mathbb{Z}_{k^2}), \tag{8}$$

which assigns to an n–dimensional $\mathbb{Z}_k$–stratifold $(S,\delta S)$ the n–dimensional $\mathbb{Z}_{k^2}$–stratifold $(kS,\delta S)$. There is a projection homomorphism

$$p\colon SH_n(X;\mathbb{Z}_{k^2}) \to SH_n(X;\mathbb{Z}_k)$$

which assigns to an n–dimensional $\mathbb{Z}_{k^2}$–stratifold $(S,\delta S)$ the n–dimensional $\mathbb{Z}_k$–stratifold $(S,k\delta S)$.

We skip the embeddings and singular maps in defining these homomorphisms to simplify the notation.

These homomorphisms satisfy a compatibility condition with the reduction and the Bockstein homomorphisms from the last section.

Proposition 5.1 *Let $r\colon SH_n(X) \to SH_n(X;\mathbb{Z}_k)$ and $r\colon SH_n(X) \to SH_n(X;\mathbb{Z}_{k^2})$ be the reduction homomorphisms and let $\delta\colon SH_n(X;\mathbb{Z}_{k^2}) \to SH_{n-1}(X)$ be the Bockstein homomorphism for $\mathbb{Z}_{k^2}$–stratifolds. We have the following commutative diagrams*:

$$\begin{tikzcd} SH_n(X;\mathbb{Z}_{k^2}) \arrow[r, "p"] \arrow[d, "\delta"] & SH_n(X;\mathbb{Z}_k) \arrow[d, "\delta"] \\ SH_{n-1}(X) \arrow[r, "\times k"] & SH_{n-1}(X) \end{tikzcd} \qquad \begin{tikzcd} SH_n(X) \arrow[r, "\times k"] \arrow[d, "r"] & SH_n(X) \arrow[d, "r"] \\ SH_n(X;\mathbb{Z}_k) \arrow[r, "\times k^k"] & SH_n(X;\mathbb{Z}_k) \end{tikzcd}$$

$$\begin{tikzcd} SH_n(X) \arrow[r, "\times k"] \arrow[d, "r"] & SH_n(X) \arrow[d, "r"] \\ SH_n(X;\mathbb{Z}_k) \arrow[r, "\times k^{k^2}"] & SH_n(X;\mathbb{Z}_{k^2}) \end{tikzcd}$$

$$\begin{tikzcd} SH_n(X) \arrow[d, "r"] \arrow[dr, "r"] & \\ SH_n(X;\mathbb{Z}_{k^2}) \arrow[r, "p"] & SH_n(X;\mathbb{Z}_k) \end{tikzcd} \qquad \begin{tikzcd} SH_n(X;\mathbb{Z}_{k^2}) \arrow[dr, "p"] & \\ SH_n(X;\mathbb{Z}_k) \arrow[u, "\times k^{k^2}"] \arrow[r, "\times k^k"] & SH_n(X;\mathbb{Z}_k) \end{tikzcd}$$

Proof We show the commutativity of the first three squares. Take $(S,\delta S)$ an n–dimensional $\mathbb{Z}_{k^2}$–stratifold. We have $k\delta S := \times k(\delta S) = \times k \circ \delta(S,\delta S)$ and $k\delta S = \delta(S,k\delta S) = \delta \circ p(S,\delta S)$. Now, for S a closed n–dimensional stratifold, we obtain $r \circ \times k(S) = (kS,\varnothing)$ and $\times k^k \circ r(S) = \times k^k(S,\varnothing) = (kS,\varnothing)$ in $SH_n(X;\mathbb{Z}_k)$. Similarly, we can show the commutativity of the third diagram with $(kS,\varnothing)$

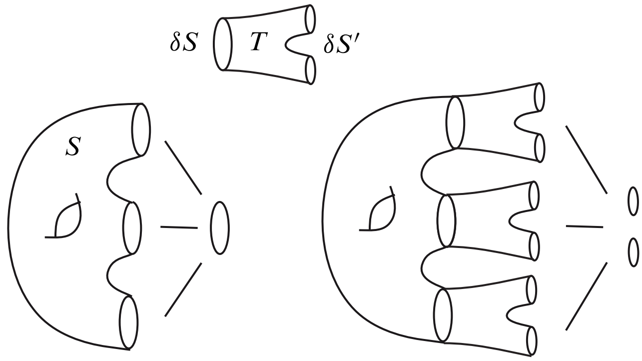

Figure 6: The bordism T from δS and $\delta S'$ and the two $\mathbb{Z}_k$–bordant $\mathbb{Z}_k$–stratifolds.

in $SH_n(X;\mathbb{Z}_{k^2})$. Finally, we show the commutativity of the last two diagrams. We have $r(S)=(S,\varnothing)=p(S,\varnothing)=p(r(S))$ and $p\circ\times k^{k^2}(S,\delta S)=p(kS,\delta S)=(kS,k\delta S)=\times k^k(S,\delta S)$. The commutativity of the second and fifth diagrams means that the composition is trivial by Proposition 4.16. □

The following result shows how a stratifold bordism gives rise to a $\mathbb{Z}_k$–stratifold bordism.

Proposition 5.2 *Assume that δS and $\delta S'$ are two n–dimensional closed stratifolds such that there is a bordism of stratifolds T with boundary $\partial T=\delta S\sqcup -\delta S'$. In addition, suppose the pair $(S,\delta S)$ is an n–dimensional $\mathbb{Z}_k$–stratifold. Then $(S,\delta S)$ is $\mathbb{Z}_k$–bordant to $(S\sqcup_{\partial S}-kT,\delta S')$.*

Proof This is similar to Proposition 4.12. Consider the product space $T':=(S\sqcup_{\partial S}-kT)\times[0,1]$ and the Bockstein $\delta T':=(\delta S'\times[0,1])\sqcup_{\delta S'\times\{1\}}-T$ with embeddings $\psi_i\colon\delta T'\hookrightarrow\partial T'$, where

$$\partial T'=\big((S\sqcup_{\partial S}-kT)\times\{0\}\big)\sqcup_{k\delta S'\times\{0\}}k(\delta S'\times[0,1])\sqcup_{k\delta S'\times\{1\}}\big((S\sqcup_{\partial S}-kT)\times\{1\}\big).$$

The $\mathbb{Z}_k$–stratifold $(T',\delta T',\psi_i)$ is a $\mathbb{Z}_k$–bordism between $(S,\delta S)$ and $(S\sqcup_{\partial S}-kT,\delta S')$. □

Remark 5.3 Because of the relevance of the previous result for our work, in Figure 6 we illustrate two $\mathbb{Z}_k$–stratifolds that are $\mathbb{Z}_k$–bordant by the previous proposition. Notice that, whenever it is possible to connect δS to the empty set by a bordism T, then the $\mathbb{Z}_k$–stratifold $(S,\delta S)$ is $\mathbb{Z}_k$–bordant to $(S\sqcup_{\partial S}-kT,\varnothing)$.

Similar to the work of Morgan and Sullivan [15], the $\mathbb{Z}_k$–stratifolds bordisms groups have a Bockstein exact sequence associated with $0\to\mathbb{Z}\xrightarrow{\times k}\mathbb{Z}\to\mathbb{Z}_k\to 0$. There is also the other Bockstein exact sequence associated with $0\to\mathbb{Z}_k\xrightarrow{\times k}\mathbb{Z}_{k^2}\to\mathbb{Z}_k\to 0$. These two sequences are part of the commutative diagram

$$\tag{9}\begin{array}{ccccccccc}
\longrightarrow & SH_n(X) & \xrightarrow{\times k} & SH_n(X) & \xrightarrow{r} & SH_n(X;\mathbb{Z}_k) & \xrightarrow{\delta} & SH_{n-1}(X) & \longrightarrow\\
& \downarrow r & & \downarrow r & & \downarrow = & & \downarrow r & \\
\longrightarrow & SH_n(X;\mathbb{Z}_k) & \xrightarrow{\times k} & SH_n(X;\mathbb{Z}_{k^2}) & \xrightarrow{p} & SH_n(X;\mathbb{Z}_k) & \xrightarrow{\tilde\delta} & SH_{n-1}(X;\mathbb{Z}_k) & \longrightarrow
\end{array}$$

The primary purpose of the present section is to show the exactness of the two Bockstein exact sequences.

Proposition 5.4 *The sequence*

$$\cdots \to SH_n(X) \xrightarrow{\times k} SH_n(X) \xrightarrow{r} SH_n(X;\mathbb{Z}_k) \xrightarrow{\delta} SH_{n-1}(X) \xrightarrow{\times k} \cdots$$

is exact.

Proof We have $r\circ(\times k) = (\times k^k)\circ r = 0$ by Proposition 5.1. In addition, we obtain $\delta\circ r = 0$ since the Bockstein of a (closed) stratifold is empty. Moreover, $\times k\circ\delta = 0$ since we start with a $\mathbb{Z}_k$–stratifold $(S,\delta S,\theta_i)$, where the boundary ∂S is diffeomorphic to $\bigsqcup_{i\in\mathbb{Z}_k}\theta_i(\delta S)$.

Now, we show exactness.

- **$\ker r \subset \operatorname{im}(\times k)$** Consider an n–dimensional singular stratifold (S,f) with $r([S,f]) = 0$. Then there is an $(n+1)$–dimensional $\mathbb{Z}_k$–bordism $(\mathcal{T},F) = ((T,\delta T),F)$ such that the $\mathbb{Z}_k$–boundary $\partial(T,\delta T) = (S,\varnothing)$ and F extends f. Consequently, we obtain $\partial\delta T = \delta S = \varnothing$ and hence $\partial T = S\sqcup k\delta T$, and we can take the singular stratifolds given by $(\delta T, F|_{\partial T})$ with the reverse orientation. We have $k[-\delta T, -F|_{\partial T}] = [S,f]$.
- **$\ker\delta \subset \operatorname{im} r$** Consider an n–dimensional singular $\mathbb{Z}_k$–stratifold $(\mathcal{S},f) = ((S,\delta S),f)$ such that $\delta([\mathcal{S},f]) = 0$. Then $(\delta S, f|_{\delta S})$ is the boundary of an n–dimensional singular stratifold (T,F), ie $\partial T = \delta S$ and F extends $f|_{\partial S}$. Proposition 5.2 and Remark 5.3 imply that the $\mathbb{Z}_k$–stratifold $(S\sqcup_{\partial S} -kT,\varnothing)$ is $\mathbb{Z}_k$–bordant to $\mathbb{Z}_k$–stratifold $(S,\delta S)$. There is a map $f'\colon S\sqcup_{\partial S} -kT \to X$ which extends the singular map f. Therefore, the singular $\mathbb{Z}_k$–stratifold $((S\sqcup_{\partial S} -kT,\varnothing), f')$ is $\mathbb{Z}_k$–bordant to the original singular $\mathbb{Z}_k$–stratifold $((S,\delta S),f)$.
- **$\ker(\times k) \subset \operatorname{im}\delta$** Consider an $(n-1)$–dimensional singular stratifold (S,f) with $\times k([S,f]) = 0$. Then there exists an n–dimensional singular stratifold (T,F) with $\partial T = kS$ and F extends kf. Thus we can take the n–dimensional singular $\mathbb{Z}_k$–stratifold $((T,S),F)$ and we obtain $\delta([(T,S),F]) = [S,f]$. □

Denote by $\widetilde{\delta}$ the composition $SH_n(X;\mathbb{Z}_k) \xrightarrow{\delta} SH_{n-1}(X) \xrightarrow{r} SH_{n-1}(X;\mathbb{Z}_k)$.

Proposition 5.5 *The sequence*

$$\cdots \to SH_n(X;\mathbb{Z}_k) \xrightarrow{\times k^{k^2}} SH_n(X;\mathbb{Z}_{k^2}) \xrightarrow{p} SH_n(X;\mathbb{Z}_k) \xrightarrow{\widetilde{\delta}} SH_{n-1}(X;\mathbb{Z}_k) \xrightarrow{\times k^{k^2}} \cdots$$

is exact.

Proof We have $p\circ(\times k^{k^2}) = \times k^k = 0$ by Proposition 5.1. Again we use Proposition 5.1, and we get

$$\widetilde{\delta}\circ p = r\circ\delta\circ p = (r\circ(\times k))\circ\delta = 0.$$

Similarly, we obtain

$$(\times k^{k^2})\circ\widetilde{\delta} = (\times k^{k^2})\circ r\circ\delta = (r\circ(\times k))\circ\delta = 0.$$

Now we show exactness.

- **$\ker p \subset \operatorname{im}(\times k^{k^2})$** Consider an n–dimensional singular $\mathbb{Z}_{k^2}$–stratifold $(\mathcal{S},f) = ((S,\delta S),f)$ with $p([\mathcal{S},f]) = 0$. Then there exists an $(n+1)$–dimensional singular $\mathbb{Z}_k$–stratifold with boundary $(\mathcal{T},F) = ((T,\delta T),F)$ such that the $\mathbb{Z}_k$–boundary is $\partial\mathcal{T} = (S,k\delta S)$. Thus we can consider k copies of δT with the reverse orientation, which are glued with S to form a closed stratifold $S \sqcup_{\partial S} -k\delta T$, which is the boundary of T. There are k disjoint embeddings $c_i\colon \delta S \times [0,\epsilon) \hookrightarrow \delta T$ induced by the collar of ∂S into the k copies of δT. Write $\overline{\delta T} := \delta T - \bigsqcup_{i\in\mathbb{Z}_k} c_i(\delta S \times [0,\epsilon/2])$. We consider the $\mathbb{Z}_{k^2}$–stratifold with boundary $(T, \delta S \times [0,\epsilon/2], \psi_i)$, where $\psi_i = c_i|_{\delta S\times[0,\epsilon/2]}$. This is a $\mathbb{Z}_{k^2}$–bordism between $(S,\delta S)$ and $(k\overline{\delta T}, \delta S)$. This means that $(\times k^{k^2})([\overline{\delta T}, \delta S]) = [k\overline{\delta T}, \delta S] = [S, \delta S]$.

- **$\ker \widetilde{\delta} \subset \operatorname{im} p$** Consider an n–dimensional singular $\mathbb{Z}_k$–stratifold $(\mathcal{S}, f) = ((S,\delta S), f)$ such that $\widetilde{\delta}([\mathcal{S}, f]) = 0$. Since $\widetilde{\delta} = r \circ \delta$, this means that there exists an n–dimensional singular $\mathbb{Z}_k$–bordism $(\mathcal{T}, F) = ((T,\delta T), F)$ such that the $\mathbb{Z}_k$–boundary is $((\delta S, \varnothing), f|_{\delta S})$. Therefore, $\partial T = \delta S \sqcup k\delta T$, F extends $f|_{\delta S}$ and $\partial\delta T = \varnothing$. Consequently, we consider k copies of T with the reverse orientation, glued with S to form the n–dimensional stratifold with boundary $S' = -kT \sqcup_{\partial S} S$. There is a map $f'\colon S' \to X$ also constructed by the gluing. Thus we have an n–dimensional singular $\mathbb{Z}_{k^2}$–stratifold $((S', \delta T), f')$. We have $p([(S',\delta T), f']) = [(S', k\delta T), f']$, which is equal to $(\mathcal{S}, f)$ by Proposition 5.2.
- **$\ker(\times k^{k^2}) \subset \operatorname{im}(\widetilde{\delta})$** Consider an $(n-1)$–dimensional singular $\mathbb{Z}_k$–stratifold $(\mathcal{S}, f) = ((S,\delta S,\theta_i), f)$ with $\times k^{k^2}([\mathcal{S}, f]) = 0$. Then there is an n–dimensional singular $\mathbb{Z}_{k^2}$–stratifold $(\mathcal{T}, F) = ((T,\delta T,\psi_i), F)$ with $\mathbb{Z}_{k^2}$–boundary $((kS,\delta S), kf)$. Therefore, $\partial T = kS \sqcup_{\partial kS} -k^2\delta T$ is a closed n–dimensional stratifold. By the definition of the $\mathbb{Z}_{k^2}$–boundary of a $\mathbb{Z}_{k^2}$–stratifold with boundary (Definition 4.6), hence $\delta S = \partial\delta T$ and the embeddings are $\theta_i = \psi_i|_{\partial\delta T}$. Therefore, the gluing $S \sqcup_{\partial S} k\delta T$ is a closed n–dimensional stratifold and, in addition, we obtain ∂T is the disjoint union of k copies of $S \sqcup_{\partial S} k\delta T$. Consequently, we take the $(n+1)$–dimensional singular $\mathbb{Z}_k$–stratifold $((T, S\sqcup_{\partial S} k\delta T), F)$ and $\widetilde{\delta}([(T, S\sqcup_{\partial S} k\delta T), F]) = [(S\sqcup_{\partial S} k\delta T, \varnothing), F|_{S\sqcup_{\partial S} k\delta T}]$, which is $\mathbb{Z}_k$–bordant to $((S,\delta S,\theta_i), f)$ by Proposition 5.2. □

6 Fundamental classes

Recall from Section 2 that a closed $\mathbb{Z}_k$–manifold $(M, \delta M, \theta_i)$ has an associated quotient space $\widetilde{M}$. Similarly, we write $\widetilde{\partial M}$ to mean the quotient space given by the identification on ∂M of the k copies of δM. Notice that in this case, we have $\widetilde{\partial M} \cong \delta M$. Similarly, for a $\mathbb{Z}_k$–manifold with boundary $(B, \delta B, \psi_i)$, we denote by $\widetilde{B}$ and $\widetilde{\partial B}$ the quotient spaces obtained by the identification of the k copies of δB on B and ∂B, respectively.

In this section, we will construct a natural transformation from $\mathbb{Z}_k$–bordism stratifold homology to homology with $\mathbb{Z}_k$–coefficients

$$\Phi\colon SH_*(X;\mathbb{Z}_k) \to H_*(X;\mathbb{Z}_k). \tag{10}$$

We can define this natural transformation for $\mathbb{Z}_k$–manifolds [15]. There is no formal proof of this fact in the literature, so we provide a detailed argument below. The case of $\mathbb{Z}_k$–stratifolds uses some results of Tene [22]. We give the details of these statements at the end of this section.

Assume that $\mathcal{M} = (M, \partial M, \theta_i)$ is a closed n–dimensional $\mathbb{Z}_k$–manifold and that there is a continuous map $f\colon M \to X$ to the topological space X. There exists the *fundamental class* $[\mathcal{M}]_{\mathbb{Z}_k} \in H_n(\widetilde{M};\mathbb{Z}_k)$, and for an element $[\mathcal{M}, f] \in \Omega_n(X;\mathbb{Z}_k)$, there is a natural transformation defined by

$$\Phi([\mathcal{M}, f]) = \tilde{f}_*([\mathcal{M}]_{\mathbb{Z}_k}), \tag{11}$$

where $\tilde{f}\colon \widetilde{M} \to X$ is the induced map from the quotient space $\widetilde{M}$.

We can find the fundamental class $[\mathcal{M}]_{\mathbb{Z}_k}$ using the commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & H_n(\partial M;\mathbb{Z}_k) \arrow[r] \arrow[d, "q_*"] & H_n(M;\mathbb{Z}_k) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_n(M,\partial M;\mathbb{Z}_k) \arrow[r, "\partial"] \arrow[d, "q_*"] & H_{n-1}(\partial M;\mathbb{Z}_k) \arrow[r] \arrow[d, "q_*"] & {} \\
{} \arrow[r] & H_n(\widetilde{\partial M};\mathbb{Z}_k) \arrow[r] & H_n(\widetilde{M};\mathbb{Z}_k) \arrow[r, "i_*"] & H_n(\widetilde{M},\widetilde{\partial M};\mathbb{Z}_k) \arrow[r, "\partial"] & H_{n-1}(\widetilde{\partial M};\mathbb{Z}_k) \arrow[r] & {}
\end{tikzcd} \tag{12}$$

In the previous diagram, the rows are the long exact sequences associated with the pairs $(M, \partial M)$ and $(\widetilde{M}, \widetilde{\partial M})$. The quotient map induces the vertical morphisms. We start with the well-known fundamental class $[M, \partial M] \in H_n(M, \partial M;\mathbb{Z}_k)$ which satisfies $\partial([M, \partial M]) = [\partial M]$ and

$$H_{n-1}(\widetilde{\partial M};\mathbb{Z}_k) \xrightarrow{\cong} H_{n-1}(\delta M;\mathbb{Z}_k), \quad q_*([\partial M]) \mapsto k[\delta M]. \tag{13}$$

Thus $q_*([\partial M]) = 0$ by the coefficients. We have the isomorphism $q_*\colon H_n(M, \partial M;\mathbb{Z}_k) \to H_n(\widetilde{M}, \widetilde{\partial M};\mathbb{Z}_k)$ and $H_n(\widetilde{\partial M};\mathbb{Z}_k) \cong H_n\,(\delta M;\mathbb{Z}_k) = 0$. Therefore, there exists a unique class $[\mathcal{M}]_{\mathbb{Z}_k} \in H_n(\widetilde{M};\mathbb{Z}_k)$ with the property

$$i_*([\mathcal{M}]_{\mathbb{Z}_k}) = q_*([M, \partial M]). \tag{14}$$

The following lemma is needed to show the existence of relative fundamental classes for $\mathbb{Z}_k$–manifolds.

Lemma 6.1 *Let M be a closed compact oriented manifold of dimension n. Assume M is the gluing of two compact oriented manifolds with boundary of dimension n, ie*

$$M = M_1 \sqcup_{\partial M_1 = \partial M_2} M_2. \tag{15}$$

Then the composition

$$H_n(M) \xrightarrow{i_*} H_n(M, M_1) \xrightarrow{\cong} H_n(M_2, \partial M_2)$$

sends the fundamental class $[M] \in H_n(M)$ to the relative fundamental class $[M_2, \partial M_2] \in H_n(M_2, \partial M_2)$, where the isomorphism $H_n(M, M_1) \xrightarrow{\cong} H_n(M_2, \partial M_2)$ is provided by excision.

Proof We have the commutative diagram

$$\begin{tikzcd}
& H_n(M_2, \partial M_2) \arrow[r] \arrow[d, "\mathrm{exc}"] & H_n(M_2, M_2 - \{x\}) \arrow[d, "\cong"] \\
H_n(M) \arrow[r] & H_n(M, M_1) \arrow[r] & H_n(M, M - \{x\})
\end{tikzcd} \tag{16}$$

where $x \in \mathring{M}_2 = M_2 - \partial M_2$. By classic algebraic topology [11, Lemma 3.27], the two rows send the fundamental classes to the generators associated with the point x, which shows the lemma. □

Now we show the existence of a *relative fundamental class* of an $(n+1)$–dimensional $\mathbb{Z}_k$–manifold with boundary $\mathcal{B}=(B,\partial B,\psi_i)$, where the $\mathbb{Z}_k$–boundary is $\partial\mathcal{B}=(M,\delta M,\theta_i)$. We find the fundamental class $[\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k}$ using the commutative diagram

$$\begin{tikzcd}
H_{n+1}(\partial B,M;\mathbb{Z}_k) \arrow[r] \arrow[d, "q_*"] & H_{n+1}(B,M;\mathbb{Z}_k) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_{n+1}(B,\partial B;\mathbb{Z}_k) \arrow[r, "\partial"] \arrow[d, "q_*"] & H_n(\partial B,M;\mathbb{Z}_k) \arrow[d, "q_*"] \\
H_{n+1}(\widetilde{\partial B},\widetilde{M};\mathbb{Z}_k) \arrow[r] & H_{n+1}(\widetilde{B},\widetilde{M};\mathbb{Z}_k) \arrow[r, "i_*"] & H_{n+1}(\widetilde{B},\widetilde{\partial B};\mathbb{Z}_k) \arrow[r, "\partial"] & H_n(\widetilde{\partial B},\widetilde{M};\mathbb{Z}_k)
\end{tikzcd} \tag{17}$$

In the previous diagram, the rows are the long exact sequences associated with the triples $(B,\partial B,M)$ and $(\widetilde{B},\widetilde{\partial B},\widetilde{M})$, respectively, and the quotient map induces the vertical morphisms. We start with the relative fundamental class $[B,\partial B]\in H_n(B,\partial B;\mathbb{Z}_k)$ and using Lemma 6.1 we have $\partial[B,\partial B]=[k\delta B,\partial M]$, where $k\delta B:=\bigsqcup_{i\in\mathbb{Z}_k}\psi_i(\delta B)$, and

$$H_n(\widetilde{\partial B},\widetilde{M};\mathbb{Z}_k)\xrightarrow{\cong}H_n(\delta B,\delta M;\mathbb{Z}_k),\qquad q_*[k\delta B,\partial M]\mapsto k[\delta B,\delta M]. \tag{18}$$

Thus $q_*[k\delta B,\partial M]=0$ by the coefficients. We have isomorphisms of the form

$$q_*\colon H_{n+1}(B,\partial B;\mathbb{Z}_k)\xrightarrow{\cong}H_{n+1}(\widetilde{B},\widetilde{\partial B};\mathbb{Z}_k)\quad\text{and}\quad H_{n+1}(\widetilde{\partial B},\widetilde{M};\mathbb{Z}_k)\cong H_{n+1}(\delta B,\delta M;\mathbb{Z}_k)=0.$$

Therefore, there exists a unique class $[\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k}\in H_{n+1}(\widetilde{B},\widetilde{M};\mathbb{Z}_k)$ with the property

$$i_*([\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k})=q_*([B,\partial B]). \tag{19}$$

Proposition 6.2 *Let $\mathcal{B}=(B,\partial B,\psi_i)$ be an $(n+1)$–dimensional $\mathbb{Z}_k$–manifold with boundary, where the $\mathbb{Z}_k$–boundary is $\partial\mathcal{B}=(M,\delta M,\theta_i)$. Then the class $[\partial\mathcal{B}]_{\mathbb{Z}_k}$ is the image of $[\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k}$ under the map $\partial\colon H_{n+1}(\widetilde{B},\widetilde{M};\mathbb{Z}_k)\to H_n(\widetilde{M};\mathbb{Z}_k)$.*

Proof We apply the differential maps to the middle square in (17), and we obtain the commutative cube

$$\begin{tikzcd}
H_{n+1}(B,M;\mathbb{Z}_k) \arrow[rr, "i_*"] \arrow[dr, "\partial"] \arrow[dd, "q_*"] & & H_{n+1}(B,\partial B;\mathbb{Z}_k) \arrow[dr, "\partial"] \arrow[dd, "q_*"] & \\
& H_n(M;\mathbb{Z}_k) \arrow[rr, "i_*"] \arrow[dd, "q_*"] & & H_n(\partial B;\mathbb{Z}_k) \arrow[dd, "q_*"] \\
H_{n+1}(\widetilde{B},\widetilde{M};\mathbb{Z}_k) \arrow[rr, "i_*"] \arrow[dr, "\partial"] & & H_{n+1}(\widetilde{B},\widetilde{\partial B};\mathbb{Z}_k) \arrow[dr, "\partial"] & \\
& H_n(\widetilde{M};\mathbb{Z}_k) \arrow[rr, "i_*"] & & H_n(\widetilde{\partial B};\mathbb{Z}_k)
\end{tikzcd} \tag{20}$$

We continue with the long exact sequence of the pairs $(\partial B,k\delta B)$ and $(\widetilde{\partial B},k\widetilde{\partial B})$ for the front square of (20), and we obtain the middle square in the commutative diagram

$$\begin{tikzcd}
H_n(M;\mathbb{Z}_k) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_n(\partial B;\mathbb{Z}_k) \arrow[r, "j_*"] \arrow[d, "q_*"] & H_n(\partial B,k\delta B;\mathbb{Z}_k) \arrow[r, "\cong", "\text{exc}"'] \arrow[d, "q_*"] & H_n(M,\partial M;\mathbb{Z}_k) \arrow[d, "q*", "\cong"'] \\
H_n(\widetilde{M};\mathbb{Z}_k) \arrow[r, "i_*"] & H_n(\widetilde{\partial B};\mathbb{Z}_k) \arrow[r, "j*"] & H_n(\widetilde{\partial B},k\tilde{\delta}B;\mathbb{Z}_k) \arrow[r, "\cong", "\text{exc}"'] & H_n(\widetilde{M},\widetilde{\partial M};\mathbb{Z}_k)
\end{tikzcd} \tag{21}$$

In the previous commutative diagram, we use excision for the third square on the right. Notice that the composition of the horizontal maps in (21) are the maps $i_*\colon H_n(M;\mathbb{Z}_k) \to H_n(M,\partial M;\mathbb{Z}_k)$ and $i_*\colon H_n(\widetilde{M};\mathbb{Z}_k) \to H_n(\widetilde{M},\partial\widetilde{M};\mathbb{Z}_k)$.

We chase the class $[\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k} \in H_{n+1}(\widetilde{B},\widetilde{M};\mathbb{Z}_k)$ in the diagrams (20) and (21), where we obtain, as consequences,

$$i_*\partial([\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k}) = \partial i_*([\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k}) = \partial q_*([B,\partial B]) = q_*\partial([B,\partial B]) = q_*([\partial B]).$$

By Lemma 6.1, we have the equation $j_*([\partial B]) = [M,\partial M]$. Thus, we obtain the property (14) and the result follows. □

Proposition 6.3 *The natural transformation $\Phi\colon \Omega_*(X;\mathbb{Z}_k) \to H_*(X;\mathbb{Z}_k)$ is well defined.*

Proof For an n–dimensional singular $\mathbb{Z}_k$–manifold $(\mathcal{M}, f)$ which is null $\mathbb{Z}_k$–bordant, there exists an $(n+1)$–dimensional $\mathbb{Z}_k$–bordism $(\mathcal{B}, F)$ with $\partial\mathcal{B} = \mathcal{M}$, where F extends f. We have the commutative diagram

$$\begin{tikzcd}
{[\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k} \in H_n(\widetilde{B},\widetilde{M};\mathbb{Z}_k)} \arrow[r] \arrow[d, "\partial"] & H_n(X,X;\mathbb{Z}_k) = 0 \arrow[d, "\partial"] \\
{[\mathcal{M}]_{\mathbb{Z}_k} \in H_n(\widetilde{M};\mathbb{Z}_k)} \arrow[r] & H_n(X;\mathbb{Z}_k)
\end{tikzcd} \tag{22}$$

This ends the proposition. □

In the case of stratifolds, the fundamental classes are defined by Tene [22]. More precisely, let S be a compact oriented regular p–stratifold of dimension n and denote by $(M,\partial M)$ the smooth manifold we attach as top stratum. We have isomorphisms

$$H_n(M,\partial M) \xrightarrow[\text{exc}]{\cong} H_n(S,S_{n-2}) \xleftarrow{\cong} H_n(S), \tag{23}$$

where S_{n-2} is the $(n-2)$–skeleton of S. The *fundamental class* $[S] \in H_n(S)$ is defined as the image of $[M,\partial M] \in H_n(M,\partial M)$.

Let $(T,\partial T)$ be a compact oriented regular p–stratifold of dimension $n+1$ with boundary and denote by $(B,\partial B)$ the smooth manifold with boundary and collar attached as the top stratum. Then

$$H_{n+1}(B,\partial B) \xrightarrow[\text{exc}]{\cong} H_{n+1}(T,T_{n-1}\cup\partial T) \xleftarrow{\cong} H_{n+1}(T,\partial T), \tag{24}$$

where T_{n-1} is the $(n-1)$–skeleton of T. The *relative fundamental class* $[T,\partial T] \in H_{n+1}(T,\partial T)$ is defined as the image of $[B,\partial B] \in H_{n+1}(B,\partial B)$.

Proposition 6.4 [22, Lemma 3.9] *Let T be a compact oriented regular stratifold of dimension $n+1$, where the boundary is ∂T. Then the image of $[T,\partial T]$ under the map $\partial\colon H_{n+1}(T,\partial T) \to H_n(\partial T)$ is the class $[\partial T]$.*

Assume $\mathcal{S} = (S, \delta S, \theta_i)$ is a closed n–dimensional $\mathbb{Z}_k$–stratifold, where both S and δS are compact oriented regular p–stratifolds. Similarly as in diagram (12), we can find *the fundamental class* $[\mathcal{S}]_{\mathbb{Z}_k}$ in $H_n(\widetilde{S}; \mathbb{Z}_k)$ using the commutative diagram

$$\begin{tikzcd}
H_n(\partial S; \mathbb{Z}_k) \arrow[r] \arrow[d, "q_*"] & H_n(S; \mathbb{Z}_k) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_n(S, \partial S; \mathbb{Z}_k) \arrow[r, "\partial"] \arrow[d, "q_*"] & H_{n-1}(\partial S; \mathbb{Z}_k) \arrow[d, "q_*"] \\
H_n(\widetilde{\partial S}; \mathbb{Z}_k) \arrow[r] & H_n(\widetilde{S}; \mathbb{Z}_k) \arrow[r, "i_*"] & H_n(\widetilde{S}, \widetilde{\partial S}; \mathbb{Z}_k) \arrow[r, "\partial"] & H_{n-1}(\widetilde{\partial S}; \mathbb{Z}_k)
\end{tikzcd} \tag{25}$$

In the previous diagram, the rows are the long exact sequences associated with the pairs $(S, \partial S)$ and $(\widetilde{S}, \widetilde{\partial S})$. The quotient map induces the vertical morphisms. Again, we have the isomorphism $q_*\colon H_n(S, \partial S; \mathbb{Z}_k) \to H_n(\widetilde{S}, \widetilde{\partial S}; \mathbb{Z}_k)$ and $H_n(\widetilde{\partial S}; \mathbb{Z}_k) \cong H_n(\delta S; \mathbb{Z}_k) = 0$. The same arguments as those for $\mathbb{Z}_k$–manifolds, show that there exists a unique fundamental class $[\mathcal{S}]_{\mathbb{Z}_k} \in H_n(\widetilde{S}; \mathbb{Z}_k)$ with the property

$$i_*([\mathcal{S}]_{\mathbb{Z}_k}) = q_*([S, \partial S]). \tag{26}$$

The local orientations at each point define the fundamental class of a manifold. This property also follows for stratifolds considering points inside the interior of the top stratum. Therefore, we use this fact to generalize Lemma 6.1 for stratifolds. More precisely, let S be a compact oriented regular p–stratifold of dimension n, which is the gluing $S = S' \sqcup_{\partial S' = \partial S''} S''$, then in the next diagram, we have that the fundamental classes are mapped to the generators associated with the point x:

$$\begin{tikzcd}
{[S'', \partial S''] \in H_n(S'', \partial S'')} \arrow[r, "\cong"] & H_n(S'', (S'')_{n-2} \cup \partial S'') \arrow[r] & H_n(S'', S'' - \{x\}) \arrow[dd, "\cong"] \\
H_n(S, S') \arrow[u, "\cong"', "\mathrm{exc}"] & & \\
{[S] \in H_n(S)} \arrow[u] \arrow[r, "\cong"] & H_n(S, S_{n-2}) \arrow[r] & H_n(S, S - \{x\})
\end{tikzcd} \tag{27}$$

Here $(S'')_{n-2}$ and S_{n-2} are the $(n-2)$–skeletons of S'' and S.

Similarly, we show the existence of a *relative fundamental class* of an $(n+1)$–dimensional $\mathbb{Z}_k$–stratifold with boundary $\mathcal{T} = (T, \partial T, \psi_i)$. The $\mathbb{Z}_k$–boundary is $\partial\mathcal{T} = (S, \delta S, \theta_i)$ and all stratifolds are compact oriented regular p–stratifolds. We can find the fundamental class $[\mathcal{T}, \partial\mathcal{T}]_{\mathbb{Z}_k}$ using the commutative diagram

$$\begin{tikzcd}
H_{n+1}(\partial T, S; \mathbb{Z}_k) \arrow[r] \arrow[d, "q_*"] & H_{n+1}(T, S; \mathbb{Z}_k) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_{n+1}(T, \partial T; \mathbb{Z}_k) \arrow[r, "\partial"] \arrow[d, "q_*"] & H_n(\partial T, S; \mathbb{Z}_k) \arrow[d, "q_*"] \\
H_{n+1}(\widetilde{\partial T}, \widetilde{S}; \mathbb{Z}_k) \arrow[r] & H_{n+1}(\widetilde{T}, \widetilde{S}; \mathbb{Z}_k) \arrow[r, "i_*"] & H_{n+1}(\widetilde{T}, \widetilde{\partial T}; \mathbb{Z}_k) \arrow[r, "\partial"] & H_n(\widetilde{\partial T}, \widetilde{S}; \mathbb{Z}_k)
\end{tikzcd} \tag{28}$$

where the rows are the long exact sequences associated with the triples $(T, \partial T, S)$ and $(\widetilde{T}, \widetilde{\partial T}, \widetilde{S})$, respectively, and the vertical morphisms are induced by considering the quotient spaces. The same arguments show the existence of the fundamental class $[\mathcal{T},\partial\mathcal{T}]_{\mathbb{Z}_k} \in H_{n+1}(\widetilde{T},\widetilde{S};\mathbb{Z}_k)$ with the property

$$i_*([\mathcal{T},\partial\mathcal{T}]_{\mathbb{Z}_k}) = q_*([T,\partial T]). \tag{29}$$

The same arguments as those for $\mathbb{Z}_k$–manifolds, show that the image of $[\mathcal{T},\partial\mathcal{T}]_{\mathbb{Z}_k}$ under the map $\partial\colon H_{n+1}(\widetilde{T},\widetilde{S};\mathbb{Z}_k) \to H_n(\widetilde{S};\mathbb{Z}_k)$ is the class $[\partial\mathcal{T}]_{\mathbb{Z}_k}$.

As a consequence, the following result is straightforward.

Proposition 6.5 *There is a well-defined natural transformation $\Phi'\colon SH_*(X;\mathbb{Z}_k) \to H_*(X;\mathbb{Z}_k)$, which fits into the commutative diagram*

$$\begin{tikzcd} \Omega_*(X;\mathbb{Z}_k) \arrow[r, "\Phi"] \arrow[d] & H_*(X;\mathbb{Z}_k) \\ SH_*(X;\mathbb{Z}_k) \arrow[ur, "\Phi'"'] & \end{tikzcd} \tag{30}$$

In addition, Φ' is an isomorphism for all CW–complexes.

7 A geometric description of the Atiyah–Hirzebruch spectral sequence for $\mathbb{Z}_k$–coefficients

We assume all spaces are CW–complexes, and for a CW–complex X we denote by X^k its k^{th} skeleton. For a generalized homology theory h, a Postnikov tower is a sequence of homology theories $h^{(r)}$ and natural transformations

$$\begin{tikzcd} & h \arrow[d] \arrow[drrr] \arrow[drrrr] \arrow[drrrrr] & & & & & \\ \cdots \arrow[r] & h^{(r)} \arrow[r] & \cdots \arrow[r] & h^{(2)} \arrow[r] & h^{(1)} \arrow[r] & h^{(0)} \end{tikzcd} \tag{31}$$

such that

- $h_n(*) \to h_n^{(r)}(*)$ is an isomorphism for $n \le r$, and
- $h_n^{(r)}(*)$ is trivial for $n > r$.

These properties determine $h^{(r)}$ completely, see [16, Chapter II, 4.13-4.18].

Every generalized homology theory h, has an associated Atiyah–Hirzebruch spectral sequence $(E^r_{s,t}, d^r_{s,t})$. For $r \ge 2$, Tene [23] constructs a natural isomorphism of spectral sequences $E^r_{s,t} \to \widehat{E}^r_{s,t}$, where

$$E^r_{s,t} = \frac{\operatorname{Im}\big(h_{s+t}(X^s, X^{s-r}) \to h_{s+t}(X^s, X^{s-1})\big)}{\operatorname{Im}\big(h_{s+t+1}(X^{s+r-1}, X^s) \to h_{s+t}(X^s, X^{s-1})\big)}, \qquad \widehat{E}^r_{s,t} = \operatorname{Im}\big(h^{(t+r-2)}_{s+t}(X^s) \to h^{(t)}_{s+t}(X^{s+r-1})\big).$$

The argument of Tene [23, Section 4] that gives the isomorphisms

$$E^r_{s,t} = \frac{\operatorname{Im}(f')}{\operatorname{Im}(f)} \cong \operatorname{Im}(f_1) \cong \operatorname{Im}(f_2) \cong \operatorname{Im}(f_3) = \widehat{E}^r_{s,t}$$

we now explain with diagram (32):

$$
\begin{tikzcd}
& & h^{(t+r-2)}_{s+t}(X^s) \arrow[r, "f_3"] \arrow[d, two heads] & h^{(t)}_{s+t}(X^{s+r-1}) \arrow[dd, "\cong"] \\
& & h^{(t+r-2)}_{s+t}(X^s, X^{s-r}) \arrow[dr] & \\
& h_{s+t}(X^s, X^{s-r}) \arrow[r] \arrow[ur, two heads] \arrow[rr, bend left, "f_2"] \arrow[d, "f'"] \arrow[dr, "f_1"] & h_{s+t}(X^{s+r-1}, X^{s-r}) \arrow[r] \arrow[d] & h^{(t)}_{s+t}(X^{s+r-1}, X^{s-r}) \arrow[d, tail] \\
h_{s+t+1}(X^{s+r-1}, X^s) \arrow[r, "f"] \arrow[ur] & h_{s+t}(X^s, X^{s-1}) \arrow[r] & h_{s+t}(X^{s+r-1}, X^{s-1}) \arrow[r, "\cong"] & h^{(t)}_{s+t}(X^{s+r-1}, X^{s-1})
\end{tikzcd}
\tag{32}
$$

The differential $\hat{d}^r_{s,t} : \hat{E}^r_{s,t} \to \hat{E}^r_{s-r,t+r-1}$ is the homomorphism induced by the diagram

$$
\begin{tikzcd}
& h^{(t+r-2)}_{s+t}(X^s) \arrow[r] \arrow[d, "\Phi"] & h^{(t)}_{s+t}(X^{s+r-1}) \arrow[d, "\Phi"] \\
& h_{s+t-1}(X^{s-r+1}) \arrow[r] \arrow[d, "\Psi"] & h_{s+t-1}(X^{s-1}) \arrow[d, "\Psi"] \\
h^{(t+2r-3)}_{s+t-1}(X^{s-r}) \arrow[r] & h^{(t+2r-3)}_{s+t-1}(X^{s-r+1}) \arrow[r] & h^{(t+r-1)}_{s+t-1}(X^{s-1})
\end{tikzcd}
\tag{33}
$$

where the natural transformation Φ is defined by the composition

$$h^{(r)}_n(X) \to h^{(r)}_n(X, X^{n-r-1}) \xrightarrow{\cong} h_n(X, X^{n-r-1}) \to h_{n-1}(X^{n-r-1}),$$

and Ψ is the natural transformation given by the composition of the natural transformations in the Postnikov tower.

For oriented bordism Ω_*, Tene [23] has a geometric description of the Atiyah–Hirzebruch spectral sequence, coming from a geometric description of Postnikov tower $SH^{(r)}$. This description of the spectral sequence is similar in spirit to the Conner–Floyd spectral sequence appearing in equivariant bordism [6] and the spectral sequence for orbifold cobordism of [1]. The bordism theory $SH^{(r)}$ is defined using oriented p–stratifolds, with all strata of codimension $0 < k < r+2$ empty. Thus, a singular stratifold S in X, of the form $f : S \to X$, gives an element of $SH^{(r)}_n(X)$ if S is an n–dimensional stratifold with singular part of dimension at most $n-r-2$. We put a similar restriction to the stratifold bordisms, which are $(n+1)$–dimensional stratifolds with boundary, and the singular part is of dimension at most $n-r-1$.

Therefore, we have natural transformations $\Omega_n \to SH^{(r)}_n$ such that $\Omega_n(*) \to SH^{(r)}_n(*)$ are isomorphisms for $n \le r$, and $SH^{(r)}_n(*)$ is trivial for $n > r$. Among other properties, we obtain that $SH^{(r)}_n(X^k)$ is trivial for $k + r < n$.

For $r \geq 2$, write

$$\hat{E}^r_{s,t} = \operatorname{Im}(SH^{(t+r-2)}_{s+t}(X^s) \to SH^{(t)}_{s+t}(X^{s+r-1})), \tag{34}$$

and the differential $\hat{d}^r_{s,t} \colon \hat{E}^r_{s,t} \to \hat{E}^r_{s-r,t+r-1}$ is the homomorphism induced by the diagram

$$\begin{tikzcd}
 & SH^{(t+r-2)}_{s+t}(X^s) \arrow[r] \arrow[d, "\Phi"'] & SH^{(t)}_{s+t}(X^{s+r-1}) \arrow[d, "\Phi"'] \\
 & \Omega_{s+t-1}(X^{s-r+1}) \arrow[r] \arrow[d, "\Psi"'] & \Omega_{s+t-1}(X^{s-1}) \arrow[d, "\Psi"'] \\
SH^{(t+2r-3)}_{s+t-1}(X^{s-r}) \arrow[r] & SH^{(t+2r-3)}_{s+t-1}(X^{s-r+1}) \arrow[r] & SH^{(t+r-1)}_{s+t-1}(X^{s-1})
\end{tikzcd} \tag{35}$$

where Φ is a natural transformation defined by

$$SH^{(r)}_n(X) \to SH^{(r)}_n(X, X^{n-r-1}) \xrightarrow{\cong} \Omega_n(X, X^{n-r-1}) \to \Omega_{n-1}(X^{n-r-1}). \tag{36}$$

The isomorphism $SH^{(r)}_n(X, X^{n-r-1}) \xrightarrow{\cong} \Omega_n(X, X^{n-r-1})$ is the restriction to the top stratum and the map $\Omega_n(X, X^{n-r-1}) \to \Omega_{n-1}(X^{n-r-1})$ is the boundary homomorphism. The natural transformation Ψ is the composition of the natural transformations in the Postnikov tower. Therefore, for a stratifold S of dimension $s+t$, with a map $f \colon S \to X^s$, the image of the differential $d^r_{s,t}$ is induced by

$$[f \colon S \to X^s] \mapsto [f|_{\operatorname{sing}(S)} \circ g \colon \partial W \to X^{s-1}], \tag{37}$$

where W is the top stratum of S and $g \colon \partial W \to \operatorname{sing}(S)$ is the attaching map used to glue W to the singular part $\operatorname{sing}(S)$.

The $\mathbb{Z}_k$–bordism groups $\Omega_n(X;\mathbb{Z}_k)$ form a generalized homology theory (this follows by Section 6 or see [5, Chapter III]). The authors define bordism theory for resolutions with abelian groups in that book. The standard resolution for $\mathbb{Z}_k$ and the theory of this section coincide with that given by the definition of $\mathbb{Z}_k$–manifolds. We construct a Postnikov tower $SH^{(r)}(\,\cdot\,;\mathbb{Z}_k)$ defined with oriented $\mathbb{Z}_k$–stratifolds, with all strata of codimension $0 < k < r+2$ empty. Thus a singular $\mathbb{Z}_k$–stratifold in X, of the form $f \colon (S, \delta S) \to X$, represents an element of $SH^{(r)}_n(X;\mathbb{Z}_k)$ if

- S is an n–dimensional $\mathbb{Z}_k$–stratifold with singular part of dimension at most $n-r-2$, and
- δS is an $(n-1)$–dimensional $\mathbb{Z}_k$–stratifold with singular part of dimension at most $n-r-3$.

Similarly, the stratifold bordism $(T, \delta T)$ should be such that

- T is an $(n+1)$–dimensional $\mathbb{Z}_k$–stratifold with boundary, the singular part is of dimension at most $n-r-1$, and
- δT is an n–dimensional $\mathbb{Z}_k$–stratifold with boundary, and the singular part is of dimension at most $n-r-2$.

Notice that we obtain $SH^{(0)}(\,\cdot\,;\mathbb{Z}_k) = SH(\,\cdot\,;\mathbb{Z}_k)$. In what follows, we use the important property that $\Omega_*(*)$ has no odd torsion and just 2–torsion; see [14].

Theorem 7.1 *For k an odd number, the homology theories $SH_*^{(r)}(\cdot;\mathbb{Z}_k)$ give the Postnikov tower of the generalized homology theory $\Omega_*(\cdot;\mathbb{Z}_k)$.*

Proof We have natural transformations

$$\begin{tikzcd}
& \Omega_*(\cdot;\mathbb{Z}_k) \arrow[d] \arrow[rrd] \arrow[rrrd] \arrow[rrrrd] & & & & \\
\arrow[r] & SH_*^{(r)}(\cdot;\mathbb{Z}_k) \arrow[r] & \cdots \arrow[r] & SH_*^{(2)}(\cdot;\mathbb{Z}_k) \arrow[r] & SH_*^{(1)}(\cdot;\mathbb{Z}_k) \arrow[r] & SH_*^{(0)}(\cdot;\mathbb{Z}_k)
\end{tikzcd} \tag{38}$$

The conditions of the Postnikov tower are proven as follows:

- Assume $n \le r$, hence $n-r-2 \le -2$ and $n-r-1 \le -1$. Thus the $\mathbb{Z}_k$–stratifolds are $\mathbb{Z}_k$–manifolds and the $\mathbb{Z}_k$–stratifolds bordism are $\mathbb{Z}_k$–manifolds with boundary. Therefore, the maps $\Omega_n(*,\mathbb{Z}_k) \to SH_n^{(r)}(*,\mathbb{Z}_k)$ are isomorphisms for $n \le r$.
- Assume $n > r+1$, hence $n-r-1 \ge 1$ and $n-r-2 \ge 0$. Thus for an n–dimensional $\mathbb{Z}_k$–stratifold $(S,\delta S)$ in $SH_n^{(r)}(*;\mathbb{Z}_k)$, we construct the cone as in Definition 4.10. As a consequence, $SH_n^{(r)}(*;\mathbb{Z}_k) = 0$ for $n > r+1$.
- Assume $n = r+1$, hence $n-r-2 = -1$ and $n-r-3 = -2$. Thus an n–dimensional $\mathbb{Z}_k$–stratifold in $SH_n^{(r)}(*;\mathbb{Z}_k)$ is a $\mathbb{Z}_k$–manifold $(M,\delta M)$. Because $n-r-1 = 0$ and $n-r-2 = -1$, we allow $\mathbb{Z}_k$–stratifold bordisms with singular points of dimension at most 0 and the Bockstein has to be an n–dimensional manifold with boundary. In $\Omega_{n-1}(*)$ we have $k[\delta M] = 0$, but since Ω_* *has no odd torsion*, then there exists an n–dimensional manifold with boundary N where $\partial N = \delta M$. Consider the $\mathbb{Z}_k$–stratifold bordism $(C(kN \sqcup_{\partial M} M), N)$ where $C(kN \sqcup_{\partial M} M)$ is the closed cone. The $\mathbb{Z}_k$–boundary is precisely the $\mathbb{Z}_k$–manifold $(M,\delta M)$ which shows that $SH_n^{(r)}(*;\mathbb{Z}_k) = 0$ for $n = r+1$.

For $k = 2$, this argument fails, and we cannot work around it using the cone of δM because we obtain singular points of dimension ≥ 1. □

The same arguments of Tene [23] give a geometric description of the Atiyah–Hirzebruch spectral sequence for $\mathbb{Z}_k$–bordism. For $r \ge 2$ and X a CW–complex, define

$$\widehat{E}^r_{s,t} = \operatorname{Im}(SH_{s+t}^{(t+r-2)}(X^s;\mathbb{Z}_k) \to SH_{s+t}^{(t)}(X^{s+r-1};\mathbb{Z}_k)), \tag{39}$$

and the differential $\widehat{d}^r_{s,t}\colon \widehat{E}^r_{s,t} \to \widehat{E}^r_{s-r,t+r-1}$ is the homomorphism induced by the diagram

$$\begin{tikzcd}
& SH_{s+t}^{(t+r-2)}(X^s;\mathbb{Z}_k) \arrow[r] \arrow[d, "\Phi"'] & SH_{s+t}^{(t)}(X^{s+r-1};\mathbb{Z}_k) \arrow[d, "\Phi"'] \\
& \Omega_{s+t-1}(X^{s-r+1};\mathbb{Z}_k) \arrow[r] \arrow[d, "\Psi"'] & \Omega_{s+t-1}(X^{s-1};\mathbb{Z}_k) \arrow[d, "\Psi"'] \\
SH_{s+t-1}^{(t+2r-3)}(X^{s-r};\mathbb{Z}_k) \arrow[r] & SH_{s+t-1}^{(t+2r-3)}(X^{s-r+1};\mathbb{Z}_k) \arrow[r] & SH_{s+t-1}^{(t+r-1)}(X^{s-1};\mathbb{Z}_k)
\end{tikzcd} \tag{40}$$

Therefore, for a singular $\mathbb{Z}_k$–stratifold $((S,\delta S), f\colon S\to X^s)$, we consider the top stratum, which is a $\mathbb{Z}_k$–manifold with boundary $(W,\delta W)$. Denote the $\mathbb{Z}_k$–boundary by $(M,\delta M) := \partial(W,\delta W)$ and $g\colon M\to \operatorname{sing}(S)$ the attaching map used to glue W to the singular part which is of dimension at most $s-r$. The image of the differential $d^r_{s,t}$ is induced by

$$[(S,\delta S), f\colon S\to X^s]\longmapsto [(M,\delta M), f|_{\operatorname{sing}(S)}\circ g\colon M\to X^{s-r}]. \tag{41}$$

We have finally proved:

Theorem 7.2 *For k an odd number, the filtration of the Atiyah–Hirzebruch spectral sequence of $\mathbb{Z}_k$–bordism*

$$E^\infty_{n,0}\subseteq\cdots\subseteq E^{r+2}_{n,0}\subseteq\cdots\subseteq E^2_{n,0}\cong H_n(X;\mathbb{Z}_k), \tag{42}$$

coincides with

$$E^r_{n,0}=\operatorname{Im}\big(SH_n^{(r-2)}(X;\mathbb{Z}_k)\to SH_n^{(0)}(X;\mathbb{Z}_k)\cong H_n(X;\mathbb{Z}_k)\big), \tag{43}$$

ie the set of classes generated by singular $\mathbb{Z}_k$–stratifolds in X with singular part of dimension at most $n-r-2$.

Notice that the Atiyah–Hirzebruch spectral sequence is trivial for $k=2$; hence, the last theorem does not apply.

8 Geometric representatives of nonrepresentable classes

The present section is motivated by the authors' counterexamples of the Steenrod problem in [2].

The Steenrod problem [7] states the following: if $z\in H_n(X)$ is an integral homology class, does there exist an oriented manifold M and a map $f\colon M\to X$ such that z is the image of the generator of $H_n(M)$?

Conner and Floyd [6] rephrased the Steenrod realization problem in terms of the Atiyah–Hirzebruch spectral sequence $(E^r_{s,t}, d^r_{s,t})$. More precisely, the homomorphism from oriented bordism to integral homology $\Omega_*(X)\to H_*(X)$ is an epimorphism if and only if the differentials $d^r_{s,t}\colon E^r_{s,t}\to E^r_{s-r,t+r-1}$ are trivial for all $r\ge 2$.

Using the previous section, the Steenrod realization problem for $\mathbb{Z}_k$–coefficients has the following form.

Theorem 8.1 *If X is a CW–complex and k an odd number, then for the Atiyah–Hirzebruch spectral sequence $(E^r_{s,t}, d^r_{s,t})$, the differentials $d^r_{s,t}\colon E_{s,t}\to E_{s-r,t+r-1}$ are trivial for all $r\ge 2$ if and only if the map $\mu\colon \Omega_n(X;\mathbb{Z}_k)\to H_n(X;\mathbb{Z}_k)$ is an epimorphism for all $n\ge 0$.*

For the rest of this section, we assume that k is an odd prime number p. Following Conner and Floyd [6], we identify stratifolds with maps to $B\mathbb{Z}_p$ with stratifolds with free actions of $\mathbb{Z}_p$.

The Bockstein exact sequence of $B\mathbb{Z}_p$ implies the isomorphisms

$$H_{2n-1}(B\mathbb{Z}_p)\overset{\text{mod } p}{\cong} H_{2n-1}(B\mathbb{Z}_p;\mathbb{Z}_p)\quad\text{and}\quad H_{2n}(B\mathbb{Z}_p;\mathbb{Z}_p)\overset{\beta}{\cong} H_{2n-1}(B\mathbb{Z}_p) \tag{44}$$

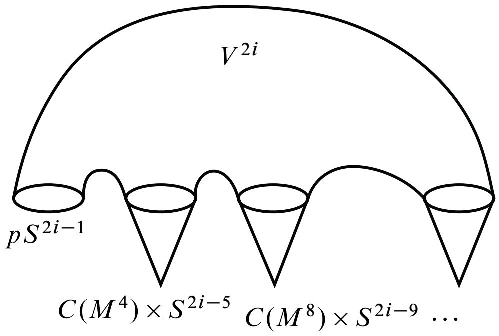

Figure 7: The class α_{2i}.

for $n>0$ (the map β was formerly denoted by δ). Take generators $\alpha_i \in H_i(B\mathbb{Z}_p;\mathbb{Z}_p)$ such that $\beta(\alpha_i)=\alpha_{i-1}$ for i even, and $\beta(\alpha_i)=0$ for i odd. The odd generators are represented by spheres. The generator α_{2i} is determined by the identity $\beta(\alpha_{2i})=\alpha_{2i-1}$. From Conner and Floyd [6, page 144], we know that the following equation holds in bordism of $B\mathbb{Z}_p$:

(45) $$p\alpha_{2i-1}+[M^4]\alpha_{2i-5}+[M^8]\alpha_{2i-9}+\cdots=0 \quad \text{for } i\geq 1.$$

The manifolds M^{4k} for $k=1,2,\dots$ are constructed inductively in [6]. Therefore, there is a compact oriented manifold V^{2i}, with a free action of $\mathbb{Z}_p$, such that

(46) $$\partial V^{2i}=pS^{2i-1}\cup(M^4\times S^{2i-5})\cup(M^8\times S^{2i-9})\cup\cdots.$$

There are two representations of the generator α_{2i} by $\mathbb{Z}_p$–stratifolds, which we will show are $\mathbb{Z}_p$–bordant:

(i) Denote by $C(M^{4l})$ the cone of M^{4l} for $l=1,2,\dots$, and take the gluing of V^{2i} with
$$(C(M^4)\times S^{2i-5})\cup(C(M^8)\times S^{2i-9})\cup\cdots.$$
The boundary of this construction is pS^{2i-1} and therefore the Bockstein is α_{2i-1}. We obtain a $2i$–dimensional $\mathbb{Z}_p$–stratifold $(S,\delta S)$, where $S=V\cup(C(M^4)\times S^{2i-5})\cup(C(M^8)\times S^{2i-9})\cup\cdots$ is a $2i$–dimensional $\mathbb{Z}_p$–stratifold with singular part $S^{2i-5}\cup S^{2i-9}\cup\cdots$, and the Bockstein $\delta S=S^{2i-1}$ is a $(2i-1)$–dimensional $\mathbb{Z}_p$–stratifold with empty singular part. We illustrate this construction in Figure 7.

(ii) The manifolds M^{4l}, with $4l<2p-2$, belong to $p\Omega_*$; see the paper by Floyd [8, page 336]. Therefore, there exist manifolds $M_l\in\Omega_{4l}$ such that $M^{4l}=pM_l$. For $p=2k+1$, consider the cone $C(M^{4m})$ for $m=k,k+1,\dots$, and take the gluing of V^{2i} with
$$(C(M^{4k})\times S^{2i-4k-1})\cup(C(M^{4(k+1)})\times S^{2i-4(k+1)-1})\cup\cdots.$$
We obtain a $2i$–dimensional $\mathbb{Z}_p$–stratifold $(S,\delta S)$, where
$$S=V\cup(C(M^{4k})\times S^{2i-4k-1})\cup(C(M^{4(k+1)})\times S^{2i-4(k+1)-1})\cup\cdots$$

is a $2i$–dimensional $\mathbb{Z}_p$–stratifold with singular part $S^{2i-4k-1} \cup S^{2i-4(k+1)-1} \cup \cdots$, and the Bockstein $\delta S = S^{2i-1} \cup (M_1 \times S^{2i-5}) \cup \cdots \cup (M_{k-1} \times S^{2i-4(k-1)-1})$ is a $(2i-1)$–dimensional $\mathbb{Z}_p$–stratifold with empty singular part.

Notice that for a generic $\mathbb{Z}_p$–stratifold $(S, \delta S) \in H_{2i}(B\mathbb{Z}_p \times B\mathbb{Z}_p; \mathbb{Z}_p)$ the singular parts of S and δS are allowed up to dimensions $2i-2$ and $2i-3$, respectively. The two previous $\mathbb{Z}_p$–stratifolds satisfy these dimension conditions for the singular parts. The bordism of $\mathbb{Z}_p$–stratifolds is of the form $(T, \delta T)$, where the singular parts of T and δT are allowed up to dimensions $2i-1$ and $2i-2$, respectively. If we show the two $\mathbb{Z}_p$–stratifolds in (i) and (ii) are $\mathbb{Z}_p$–bordant, we will have two representations of the generator α_{2i}. Apply Proposition 5.2 using the bordism in stratifolds between $M_1 \times S^{2i-5} \cup \cdots \cup M_{k-1} \times S^{2i-4(k-1)-1}$ and the empty stratifold produced by the cone $C(M_1) \times S^{2i-5} \cup \cdots \cup C(M_{k-1}) \times S^{2i-4(k-1)-1}$, which has singular part $S^{2i-5} \cup \cdots S^{2i-4(k-1)-1}$. The proof of Proposition 5.2 uses a product with the interval producing a $\mathbb{Z}_p$–stratifold $(T', \delta T')$ with the singular parts of T' and $\delta T'$ of dimensions $2i-4$ and $2i-5$. This shows that the two $\mathbb{Z}_p$–stratifolds in (i) and (ii) are $\mathbb{Z}_p$–bordant, consequently both $\mathbb{Z}_p$–stratifolds represent the generator α_{2i}.

Theorem 8.2 *For $X = B\mathbb{Z}_p$, the differentials $d^r_{s,t}$ are trivial for $r \leq 2p-2$. In particular, the differential $d^{2p-1}_{2i,0}$ is of the form*

$$d^{2p-1}_{2i,0} \colon H_{2i}(B\mathbb{Z}_p; \Omega_0(*; \mathbb{Z}_p)) \to H_{2p-2i+1}(B\mathbb{Z}_p; \Omega_{2p-2}(*; \mathbb{Z}_p)),$$

and the image of the class $\alpha_{2i} \in H_{2i}(B\mathbb{Z}_p; \mathbb{Z}_p)$ with $i \geq p$ under the differential d^{2p-1} is nontrivial.

Proof We can restrict to the differentials $d^r_{2i,0}$ since those starting on coordinates $(2i+1, 0)$ are always trivial since the classes α_{2i+1} are represented by spheres. From Section 7, the differential $d^r_{2i,0} \colon E^r_{2i,0} \to E^r_{2i-r,r-1}$ has the form

$$\begin{array}{c}
\mathrm{Im}\big(SH^{(r-2)}_{2i}(X^{2i}; \mathbb{Z}_p) \to SH^{(0)}_{2i}(X^{2i+r-1}; \mathbb{Z}_p)\big) \\
{\scriptstyle d^r}\big\downarrow \\
\mathrm{Im}\big(SH^{(2r-3)}_{2i-1}(X^{2i-r}; \mathbb{Z}_p) \to SH^{(r-1)}_{2i-1}(X^{2i-1}; \mathbb{Z}_p)\big)
\end{array}$$

For $p = 2k+1$, recall the representation of the generator α_{2i} by a $2i$–dimensional $\mathbb{Z}_p$–stratifold $(S, \delta S)$, where

- $S = V \cup (C(M^{4k}) \times S^{2i-4k-1}) \cup (C(M^{4(k+1)}) \times S^{2i-4(k+1)-1}) \cup \cdots$ is a $2i$–dimensional $\mathbb{Z}_p$–stratifold with singular part $S^{2i-4k-1} \cup S^{2i-4(k+1)-1} \cup \cdots$, and
- $\delta S = S^{2i-1} \cup (M_1 \times S^{2i-5}) \cup \cdots \cup (M_{k-1} \times S^{2i-4(k-1)-1})$ is a $(2i-1)$–dimensional $\mathbb{Z}_p$–stratifold with empty singular part.

Since $r \leq 2p-2 = 4k$, we obtain $2i-4k-1 \leq 2i-(r-2)-2$; hence α_{2i} belongs to $SH^{(r-2)}_{2i}(X^{2i}; \mathbb{Z}_p)$. From Section 7, the representation of the differential $d^r(\alpha_{2i})$ is calculated with the top stratum, which is the

$\mathbb{Z}_p$–manifold $(V \cup M^{4k} \times [0,1] \times S^{2i-4k-1} \cup \cdots, S^{2i-1} \cup (M_1 \times S^{2i-5}) \cup \cdots (M_{k-1} \times S^{2i-4(k-1)-1}))$, which has $\mathbb{Z}_p$–boundary $((M^{4k} \times S^{2i-4k-1}) \cup (M^{4(k+1)} \times S^{2i-4(k+1)-1}) \cup \cdots, \varnothing)$. Therefore, $d^r(\alpha_{2i}) = (M^{4k} \times S^{2i-4k-1}) \cup (M^{4(k+1)} \times S^{2i-4(k+1)-1}) \cup \cdots$ and we can cone all the M^{4m} since the singular parts of the bordisms in $SH^{(r-1)}_{2i-1}(X^{2i-1}; \mathbb{Z}_k)$ are allowed up to dimension $2i-r-1$ and $2i-4k-1 \leq 2i-r-1$ precisely when $r \leq 4k$. Therefore, the differential $d^r(\alpha_{2i})$ is zero for $r \leq 4k = 2p-2$. In fact, we have $E^2 \cong \cdots \cong E^{2p-1}$ because we have a commutative diagram

$$\begin{tikzcd} E^r_{s,0} \otimes \Omega_t(*; \mathbb{Z}_p) \arrow[r] \arrow[d, "d^r \otimes \mathrm{id}"'] & E^r_{s,t} \arrow[d, "d^r"'] \\ E^r_{s-r,r-1} \otimes \Omega_t(*; \mathbb{Z}_p) \arrow[r] & E^r_{s-r,t+r-1} \end{tikzcd} \tag{47}$$

as in Conner and Floyd [6, pages 17 and 41], and we have by induction that the rows are isomorphisms for $r \leq 2p-2$. Finally, for $r = 2p-1$, the element $d^{2p-1}_{2i,0}(\alpha_{2i}) = M^{2p-2} \times S^{2i-2p+1}$ is not zero in $H_{2i-2p+1}(B\mathbb{Z}_p; \Omega_{2p-2}(*; \mathbb{Z}_p))$, since M^{2p-2} is a Milnor generator of $\Omega/p\Omega$. For $p=3$, M^4 can be taken to be $\mathbb{CP}^2$ and we find the obstruction to realizability with d^5. □

9 $\mathbb{Z}_2$–stratifold homology is stratifold homology with $\mathbb{Z}_2$–coefficients

Kreck [12, Chapter 4] introduces the theory of $\mathbb{Z}_2$–oriented stratifolds in order to represent homology with $\mathbb{Z}_2$–coefficients. He calls this theory *stratifold homology with $\mathbb{Z}_2$–coefficients*, denoted by $S\mathcal{H}_*(X; \mathbb{Z}_2)$. The elements are bordism classes of singular stratifolds where the stratum of codimension 1 is empty, but there is no requirement of an orientation of the top stratum. There is a natural isomorphism

$$S\mathcal{H}_*(X; \mathbb{Z}_2) \to H_*(X; \mathbb{Z}_2) \tag{48}$$

that, for a singular stratifold $(S, f : S \to X)$, takes the pushforward of the fundamental class $[S]$ in $H_*(S; \mathbb{Z}_2)$.

This article introduces the theory of $\mathbb{Z}_2$–stratifolds, which also represent homology with $\mathbb{Z}_2$–coefficients. This is called *$\mathbb{Z}_2$–stratifold homology*, denoted by $SH_*(X; \mathbb{Z}_2)$. The elements are $\mathbb{Z}_2$–bordism classes of singular $\mathbb{Z}_2$–stratifolds where the stratum of codimension 1 is empty, but we require an orientation of the top stratum. There is a natural isomorphism

$$SH_*(X; \mathbb{Z}_2) \to H_*(X; \mathbb{Z}_2) \tag{49}$$

that, for a singular $\mathbb{Z}_2$–stratifold $((S, \delta S), f : S \to X)$, takes the pushforward of the fundamental class $[S]_{\mathbb{Z}_2} \in H_n(\widetilde{S}; \mathbb{Z}_2)$.

Therefore, we have the commutative diagram

$$\begin{tikzcd} SH_*(X; \mathbb{Z}_2) \arrow[rr, "q"] \arrow[dr, "\cong"'] & & S\mathcal{H}_*(X; \mathbb{Z}_2). \arrow[dl, "\cong"] \\ & H_*(X; \mathbb{Z}_2) & \end{tikzcd} \tag{50}$$

To define the map q, note that for an n–dimensional $\mathbb{Z}_2$–stratifold $(S, \delta S, \theta_i)$, the quotient space $\widetilde{S}$ is an n–dimensional $\mathbb{Z}_2$–oriented stratifold. This is true because the two disjoint collars associated with the two embedded copies of the Bockstein δS are combined to produce a bicollar on the quotient space $\widetilde{S}$. For $(\mathcal{S}, f)$ an n–dimensional singular $\mathbb{Z}_2$–stratifold with $\mathcal{S} = (S, \delta S, \theta_i)$, we have the map $q\colon SH_n(X;\mathbb{Z}_2) \to S\mathcal{H}_n(X;\mathbb{Z}_2)$ defined by $q([\mathcal{S}, f]) = [\widetilde{S}, \widetilde{f}]$, where $\widetilde{f}$ is the quotient map.

The description of the inverse for the isomorphism $q\colon SH_*(X;\mathbb{Z}_2) \to S\mathcal{H}_*(X;\mathbb{Z}_2)$ is an open question. Wall [26] shows a description for an n–dimensional manifold whose first Stiefel–Whitney class ω_1 in $H^1(M;\mathbb{Z}_2)$ is the restriction mod 2 of a class with integer coefficients. Thus there is a map $f\colon M \to K(\mathbb{Z},1) = S^1$, which can be approximated by a smooth map. Take a regular value t and consider the cutting $f^{-1}(t)$. The manifold with boundary $M - f^{-1}(t)$ is orientable, and in that case $f^{-1}(t)$ is also orientable; this describes q^{-1} for this particular case.

Appendix

A.1 Regular values for $\mathbb{Z}_k$–stratifolds

In [12, page 27], Kreck defines a *regular value* for a smooth map $f\colon S \to N$ from a closed stratifold S to a boundaryless manifold N as a point $x \in N$ such that for all $y \in f^{-1}(x)$ the differential df_y is surjective, or, equivalently, x is a regular value of $f|_{S^i}$ for all i. Kreck [12, Propositions 2.6 and 2.7, pages 27–29] shows that the set of regular values of f is dense in N, and $f^{-1}(x)$ is a stratifold of dimension $\dim S - \dim N$.

In [12, page 35], Kreck defines a *smooth map* $f\colon T \to N$ from a stratifold with boundary T to a boundaryless manifold N as a continuous function whose restriction to $\mathring{T} = T - \partial T$ and to ∂T is smooth and which commutes with the collar $c\colon \partial T \times [0,\epsilon) \to U$, ie there is a $\delta > 0$ with $\delta \leq \epsilon$ such that $fc(x,t) = f(x)$ for all $x \in \partial T$ and $t < \delta$. Kreck [12, page 38] says $x \in N$ is a *regular value* if x is a regular value for $f|_{T-\partial T}$ and $f|_{\partial T}$. In this case, the preimage $f^{-1}(x)$ is a stratifold with boundary of dimension $\dim T - \dim N$. This fact is a generalization of a result of [12, Proposition 2.7] using local retractions for $T - \partial T$ and ∂T, together with Theorem A.1. Also, by Theorem A.1, the set of regular values is dense in N.

Theorem A.1 [10, pages 60–62] *Let $f\colon M \to N$ be a smooth map of a manifold M with boundary onto a boundaryless manifold N and let $x \in N$ a regular value of both f and ∂f. Then the preimage $f^{-1}(x)$ is a submanifold of M with boundary $f^{-1}(x) \cap \partial M$ of dimension* $\dim M - \dim N$. *Moreover, the set of critical values of both f and ∂f has measure zero.*

In what follows, we obtain the version for stratifolds with boundary of Propositions 4.2 and 4.3 of Kreck [12].

Proposition A.2 *Let T be an oriented, regular stratifold with boundary, $f\colon T\to\mathbb{R}$ a smooth function and t a regular value. Then $f^{-1}(t)$ is an oriented, regular stratifold with boundary.*

Proof We use the work of Kreck [12, Proposition 4.2, page 44] in order to show that $f|_{T-\partial T}{}^{-1}(t)$ and $f|_{\partial T}{}^{-1}(t)$ are regular stratifolds. We induce the collar by restriction. We notice $f^{-1}(t)$ is an oriented stratifold, since $T^{n-1}=\varnothing$ and the intersection with the top stratum is an oriented manifold. □

Remark A.3 In the case T is a p–stratifold with boundary, see Remark 3.11; hence the preimage $f^{-1}(t)$ is also a p–stratifold with boundary, for t a regular value. The construction of this p–stratifold is as follows: for t a regular value, on each stratum T_i the preimage $f|_{T_i}^{-1}(t)$ is a submanifold of T_i with boundary $f|_{T_i}^{-1}\cap\partial T_i$ by Theorem A.1. Similarly, the preimage $\partial f|_{\partial T_i}^{-1}(t)$ is a submanifold of ∂T_i. Moreover, these submanifolds come with collars and attaching maps that construct this p–stratifold with boundary inductively.

Proposition A.4 *Let T be a regular stratifold with boundary. Then the set of regular points of a smooth map $f\colon T\to\mathbb{R}$ is an open subset of T. If, in addition, T is compact, the regular values form an open set.*

Proof We know the regular points of $f|_{T-\partial T}$ and $f|_{\partial T}$ are open in $T-\partial T$ and ∂T, respectively. By definition $fc(x,t)=f(x)$ for some collar c in T. So, the regular points of $f|_{\partial T}$ extend to the collar by an open set. Thus, we obtain the first statement. Now, in the case T is compact, the singular points that are the complement of the regular points, form a closed set which is compact. Thus, the image under f is closed, implying that the regular values are an open set. □

A crucial fact for the Mayer–Vietoris sequence for stratifolds is the following:

Proposition A.5 [12, Proposition 2.8] *Let S be a closed n–dimensional, connected stratifold and A and B disjoint closed nonempty subsets of S. Then there is a nonempty $(n-1)$–dimensional stratifold P with $P\subset S-(A\cup B)$. That is, P separates A and B.*

Remark A.6 More precisely, Kreck [12, Proposition 2.4, page 26] constructs a smooth function $f\colon S\to\mathbb{R}$ which maps A to 1 and B to -1. The stratifold P is the preimage $f^{-1}(t)$ of a regular value $t\in(-1,1)$ such that $f^{-1}(t)\subset S-(A\cup B)$ and $A\subset f^{-1}(t,\infty)$ and $B\subset f^{-1}(-\infty,t)$. After composition with an appropriate translation, we can assume $t=0$.

We extend Proposition A.5 to the theory of $\mathbb{Z}_k$–stratifolds. However, it is not enough to consider stratifolds with boundary. The reason is that the smooth function must be $\mathbb{Z}_k$–invariant on the boundary. One needs a smooth function that factors as

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}"'] & & \mathbb{R} \\ & \widetilde{S} \arrow[ur, "\tilde f"'] & \end{tikzcd}$$

We need a $\mathbb{Z}_k$–stratifold version of the following result.

Proposition A.7 [12, Proposition 2.4] *Let $A \subset S$ be a closed subset of a stratifold S, let U be an open neighborhood of A, and $f : U \to \mathbb{R}$ a smooth function. Then there is a smooth function $g : S \to \mathbb{R}$ such that $g|_A = f|_A$.*

Proposition A.8 *Let $\mathcal{S} = (S, \delta S, \theta_i)$ be an n–dimensional compact closed $\mathbb{Z}_k$–stratifold, $A \subseteq \widetilde{S}$ a closed subset of the quotient space, U an open neighborhood of A and $f : U \to \mathbb{R}$ a smooth function. Then there exists a smooth function $G : S \to \mathbb{R}$ that factors through the quotient space $\widetilde{S}$ such that $G|_A = f|_A$ in the quotient space.*

Proof We construct a smooth function on S, which is the gluing of the following two functions:

- For the first function, consider δS inside the quotient space $\widetilde{S}$. By normality of S, there exists a closed subset $A_1 \subset \delta S$ such that $A \cap \delta S \subset \operatorname{int} A_1$ and $A_1 \subset \delta S \cap U$. By compactness and using the collar, $\mathrm{pr} : \delta S \times [0, \epsilon) \to \widetilde{S}$, we find $0 < t < \epsilon$ such that
$$\mathrm{pr}^{-1}(A) \cap (\partial S \times [0, 2t)) \subset \mathrm{pr}^{-1}(A_1) \times [0, 2t) \subset \mathrm{pr}^{-1}(U).$$
Proposition A.7 implies that it is possible to construct a smooth function $f_1 : \delta S \to \mathbb{R}$ such that A_1 maps to 1 and $f_1(x) = 0$ for $x \in \delta S - U \cap \delta S$. Lift f_1 to a smooth function on the whole boundary ∂S and take the smooth function $g_1 : \partial S \times [0, 2t) \to \mathbb{R}$ by writing $g_1(x, s) = f_1(x)$.
- For the second function, take the stratifold $S_1 := S - (\partial S \times [0, t])$ and again by Proposition A.7 we can construct a smooth function $g_2 : S_1 \to \mathbb{R}$ such that $A \cap S_1$ maps to 1 and $g_2(x) = 0$ for $x \in S_1 - U \cap S_1$.

A partition of unity glues these two functions together into a smooth function $G : S \to \mathbb{R}$, which is $\mathbb{Z}_k$–invariant. Thus it descends to the quotient and sends A to 1 and $\widetilde{S} - U$ to 0. Using Proposition 2.4 of Kreck [12] (Proposition A.7), we apply the previous process to construct the function $G : S \to \mathbb{R}$, which is $\mathbb{Z}_k$–invariant and is such that $G|_A = f|_A$ in the quotient space. □

In conclusion, we obtain the $\mathbb{Z}_k$–stratifold version of Kreck [12, Proposition 2.8] (Proposition A.5).

Proposition A.9 *Let $(S, \delta S)$ be an n–dimensional, compact, connected $\mathbb{Z}_k$–stratifold and A and B disjoint closed nonempty subsets of the quotient space $\widetilde{S}$. Then there is a nonempty $(n-1)$–dimensional $\mathbb{Z}_k$–stratifold $(P, \delta P)$ with $\widetilde{P} \subset \widetilde{S} - (A \cup B)$ and $\delta P \subset \delta S - ((A \cup B) \cap \delta S)$.*

We construct a smooth function $G : S \to \mathbb{R}$ that factors through the quotient space $\widetilde{S}$, and maps A to 1 and B to -1. The $\mathbb{Z}_k$–stratifold $(P, \delta P)$ is provided by a regular value $t \in (-1, 1)$ of both S and ∂S, and we have $P = G^{-1}(t)$ and $\delta P = G|_{\delta S}^{-1}(t)$. The pair $(P, \delta P)$ is a $\mathbb{Z}_k$–stratifold because we choose a regular value by Proposition A.4 and the preimage $P = G^{-1}(t)$ is a stratifold with boundary, where $\partial P = G^{-1}(t) \cap \partial S = \bigsqcup_{i \in \mathbb{Z}_k} \theta_i(G^{-1}(t) \cap \delta S) = \bigsqcup_{i \in \mathbb{Z}_k} \theta_i(G|_{\delta S}^{-1}(t))$ and the Bockstein is $\delta P = G|_{\delta S}^{-1}(t)$.

A.2 Mayer–Vietoris sequence

Let U and V be open subsets of a space X. In this section we show that the long exact sequence

$$\cdots \xrightarrow{d} SH_n(U\cap V;\mathbb{Z}_k) \xrightarrow{i_*} SH_n(U;\mathbb{Z}_k)\oplus SH_n(V;\mathbb{Z}_k) \xrightarrow{j_*} SH_n(U\cup V;\mathbb{Z}_k) \xrightarrow{d} SH_{n-1}(U\cap V;\mathbb{Z}_k) \xrightarrow{i_*} \cdots \tag{51}$$

is exact for $n \geq 1$. Denote by i_U and i_V the inclusions $U\cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$. Denote by j_U and j_V the inclusions $U \hookrightarrow U\cup V$ and $V \hookrightarrow U \cup V$. We describe the homomorphisms as follows:

- $i_*\colon SH_n(U\cap V;\mathbb{Z}_k) \to SH_n(U;\mathbb{Z}_k)\oplus SH_n(V;\mathbb{Z}_k)$ is given by (i_{U*}, i_{V*}).
- $j_*\colon SH_n(U;\mathbb{Z}_k)\oplus SH_n(V;\mathbb{Z}_k) \to SH_n(U\cup V;\mathbb{Z}_k)$ is given by $j_{U*} - j_{V*}$.
- The connecting homomorphism $d\colon SH_n(U\cup V;\mathbb{Z}_k) \to SH_{n-1}(U\cap V;\mathbb{Z}_k)$ considers an element $[(S,\delta S), g] \in SH_n(U\cup V;\mathbb{Z}_k)$. For the projection $\mathrm{pr}\colon S \to \widetilde{S}$, we obtain disjoint closed subsets of $\widetilde{S}$ given by $A := \mathrm{pr}(g^{-1}(X-V))$ and $B := \mathrm{pr}(g^{-1}(X-U))$. By Proposition A.9, we obtain an $(n-1)$–dimensional $\mathbb{Z}_k$–stratifold $(P,\delta P)$ such that $\widetilde{P} \subset \widetilde{S}-(A\cup B)$ and $\delta P \subset \delta S - ((A\cup B)\cap \delta S)$. We define

$$d([(S,\delta S), g]) = [(P,\delta P), g|_P]. \tag{52}$$

  In the case that A or B is empty, the $\mathbb{Z}_k$–stratifold $(P,\delta P)$ is empty, and the differential is zero.

Proof that d is well defined It was pointed out by Kreck [13, page 304] that in the case of bordism of smooth manifolds, the connecting homomorphism for the Mayer–Vietoris sequence is well defined because of the existence of a *bicollar* for $P := G^{-1}(0)$, ie an isomorphism with $P\times(-\epsilon,\epsilon)$, where 0 is a regular value by the composition of a translation. For a stratifold S, this is only possible up to bordism where we naively change S by $S - P \cup (P\times(-\epsilon,\epsilon))$. The formal statement is [12, Lemma B.1, page 197], and the proof is as follows. Kreck's Proposition 4.3 in [12] (our Proposition A.4) allows us to choose $\delta > 0$ such that $(-\delta,\delta)$ consists only of regular values of G. Consider a monotone smooth map $\mu\colon \mathbb{R}\to\mathbb{R}$ which is the identity for $|t| > \delta/2$ and 0 for $|t| < \delta/4$. Take $\eta\colon S\times\mathbb{R}\to\mathbb{R}$ mapping $(x,t)\mapsto G(x)-\mu(t)$, which has 0 as regular value. Kreck's Proposition 4.2 in [12] implies that $S' = \eta^{-1}(0)$ is a regular stratifold containing $P\times(-\delta/4,\delta/4)$, which is the required bicollar. It remains to construct a bordism between S and S'. Now take the function $\gamma\colon S\times\mathbb{R}\times[0,1]\to\mathbb{R}$ defined by

$$(x,t,s)\mapsto G(x) - (\zeta(s)\mu(t) + (1-\zeta(s))t),$$

where $\zeta\colon [0,1]\to\mathbb{R}$ is 0 near 0, and 1 near 1. This map has 0 as a regular value, and the preimage $Q := \gamma^{-1}(0)$ is the bordism between S and S'.

For the case of $\mathbb{Z}_k$–stratifolds, we start with a closed $\mathbb{Z}_k$–stratifold $(S,\delta S,\theta_i)$ and we need to separate this $\mathbb{Z}_k$–stratifold by a bicollar over the regular $\mathbb{Z}_k$–stratifold

$$(P,\delta P,\theta_i|_{\delta P}) = (G^{-1}(0), G|_{\delta S}^{-1}(0), \theta_i|_{G|_{\delta S}^{-1}(0)}).$$

In such a case, the bicollar consists of a pair of embedded cylinders $(G^{-1}(0)\times(-\epsilon,\epsilon), G|_{\delta S}^{-1}(0)\times(-\epsilon,\epsilon))$ which are consistent with respect to the embeddings θ_i. In order to reproduce Kreck's [12, Lemma B.1] in the context of $\mathbb{Z}_k$–stratifolds, we observe that the map $\eta\colon S\times\mathbb{R}\to\mathbb{R}$ is $\mathbb{Z}_k$–invariant for our case and we take $(S',\delta S') := (\eta^{-1}(0), \eta|_{\delta S\times\mathbb{R}}^{-1}(0))$, which is a regular $\mathbb{Z}_k$–stratifold by Proposition A.2. We construct the bicollar taking $\epsilon=\delta/4$ with δ as in the previous paragraph. The $\mathbb{Z}_k$–bordism between $((S,\delta S),\mathrm{id})$ and $((S',\delta S'),\pi_1)$, where π_1 is the projection on the first variable, is constructed similarly, as in the case of stratifolds.

The remaining steps to show d is well defined are analogous to the case of stratifolds [12, pages 199–200]. The idea is to assume that $[(S,\delta S),g]$ is trivial, then $[(S',\delta S'),g\circ\pi_1]$ is also trivial. For the modified $\mathbb{Z}_k$–stratifold $(S',\delta S')$, we can take the separating function given by the projection on the second variable. This means that there exists a $\mathbb{Z}_k$–bordism $(T,\delta T)$ that has as $\mathbb{Z}_k$–boundary $(S',\delta S')$. Moreover, the separating function extends to T. This function has a regular value t very close to 0, then $(P\times\{t\},\delta P\times\{t\})$ is null $\mathbb{Z}_k$–bordant taking the preimage of t. However, this last $\mathbb{Z}_k$–stratifold is $\mathbb{Z}_k$–bordant to $(P,\delta P)$. □

The following results are required to show that (51) is exact.

Proposition A.10 *Suppose M is a manifold with boundary of dimension n and $g\colon M\to\mathbb{R}$ a smooth map with regular value 0. Then the preimage $g^{-1}(-\infty,0]$ is a manifold with boundary, and the boundary has the form*

$$g^{-1}(0)\sqcup_{(g^{-1}(0)\cap\partial M)}(g^{-1}(-\infty,0]\cap\partial M).$$

In addition, if M is oriented, then $g^{-1}(-\infty,0]$ is oriented.

Proof Here we will dismiss the orientation of the manifolds, which is understood depending on the case. From [10, page 62], we have that for a manifold N without boundary and $f\colon N\to\mathbb{R}$ a smooth map, the preimage $f^{-1}(-\infty,0]$ is a manifold with boundary given by $f^{-1}(0)$. Thus the restriction to the boundary $g|_{\partial M}$ is such that $g|_{\partial M}^{-1}(-\infty,0] = g^{-1}(-\infty,0]\cap\partial M$ is a manifold whose boundary is $g|_{\partial M}^{-1}(0) = g^{-1}(0)\cap\partial M$. Furthermore, we use Theorem A.1 (or [10, pages 60–62]) which shows that $g^{-1}(0)$ is also a manifold with boundary $g^{-1}(0)\cap\partial M$. Then we glue these two manifolds obtaining a boundaryless smooth manifold of dimension $n-1$. In Figure 8 we illustrate the boundary of $g^{-1}(-\infty,0]$.

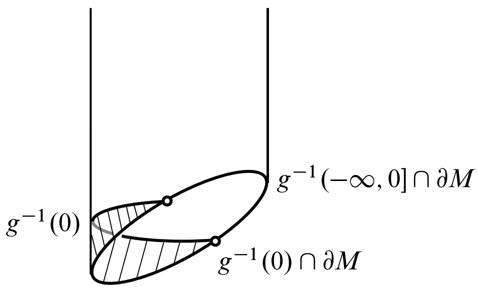

Figure 8: The boundary of $g^{-1}(-\infty,0]$.

Now we consider the restriction $g|_{M-\partial M}$ and we obtain a smooth structure for

$$g|_{M-\partial M}^{-1}(-\infty,0] = g^{-1}(-\infty,0] - (g^{-1}(-\infty,0] \cap \partial M)$$

with boundary $g^{-1}(0) - (g^{-1}(0) \cap \partial M)$. We can establish a collar around $g^{-1}(0)$. As g commutes with the collar of ∂M, there is a collar around $(g^{-1}(-\infty,0] \cap \partial M)$. Finally, similar to the proof of Proposition 4.15, we combine the two collars of $g^{-1}(0)$ and $(g^{-1}(-\infty,0] \cap \partial M)$, where we smooth the corners by straightening the angle [6, pages 9–10] (or see Section 4). □

Proposition A.10 follows for stratifolds with boundary (all p–stratifolds). Notice that

$$g^{-1}(-\infty,0] \cap \partial T = (g^{-1}(-\infty,0] \cap S) \cup (g^{-1}(-\infty,0] \cap k\delta T)$$

and hence $g^{-1}(-\infty,0]$ is a stratifold with boundary where

$$\partial g^{-1}(-\infty,0] = g^{-1}(0) \cup (g^{-1}(-\infty,0] \cap S) \cup (g^{-1}(-\infty,0] \cap k\delta T).$$

Thus we obtain the following application for $\mathbb{Z}_k$–stratifolds.

Corollary A.11 *Suppose $(T,\delta T)$ is a $\mathbb{Z}_k$–stratifold with boundary of dimension n, where the $\mathbb{Z}_k$–boundary is denoted by $(S,\delta S)$. Let $g\colon T \to \mathbb{R}$ be a smooth map which factors to the quotient space $\widetilde{T}$ with 0 as a regular value for g. Then the preimage*

$$\big(g^{-1}(-\infty,0], g^{-1}(-\infty,0] \cap \delta T\big)$$

is a $\mathbb{Z}_k$–stratifold whose $\mathbb{Z}_k$–boundary is the $\mathbb{Z}_k$–stratifold

$$\big(g^{-1}(0) \cup (g^{-1}(-\infty,0] \cap S), (g^{-1}(0) \cap \delta T) \cup (g^{-1}(-\infty,0]) \cap \delta S)\big).$$

Now we use these tools to show the exactness of the Mayer–Vietoris sequence.

Proof of exactness of (51) We follow the arguments used for the case of stratifolds [12, pages 200–208], where we will specify the additional details used for the case of $\mathbb{Z}_k$–stratifolds.

To show that we have a complex, we notice that both $j_U \circ i_U$ and $j_V \circ i_V$ are the canonical inclusion $U \cap V \hookrightarrow U \cup V$, therefore $j_* \circ i_* = 0$. We show the other cases $i_* \circ d = 0$ and $d \circ j_* = 0$ in what follows: for the first identity, we choose a representative for the homology class (with $\mathbb{Z}_k$–coefficients) in $U \cap V$ such that we can cut along the separating $\mathbb{Z}_k$–stratifold defining the boundary operator. The two pieces separated by this $\mathbb{Z}_k$–stratifold induce the null $\mathbb{Z}_k$–bordisms on the homology groups (with $\mathbb{Z}_k$–coefficients) associated with U and V. For the second identity, if $[(S,\delta S), g] \in SH(U;\mathbb{Z}_k)$, we can choose a smooth function and the regular value such that the separating regular $\mathbb{Z}_k$–stratifold is empty, therefore, $d(j_{U*}) = 0$. By the same argument $d(j_{V*}) = 0$.

Now we show exactness.

- **ker $j_* \subset$ im i_*** Consider $[(S,\delta S), f] \in SH_n(U;\mathbb{Z}_k)$ and $[(S',\delta S'), f'] \in SH_n(V;\mathbb{Z}_k)$ which are such that $j_{U*}([(S,\delta S), f]) = j_{V*}([(S',\delta S'), f'])$. There exists a $\mathbb{Z}_k$–bordism $((T,\delta T), F)$ between

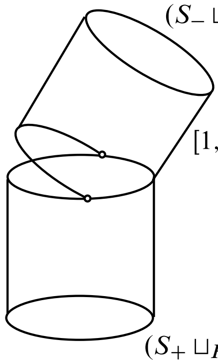

Figure 9: The $\mathbb{Z}_k$–bordism T.

$[(S, \delta S), j_U f]$ and $[(S', \delta S'), j_V f']$, where $F = \widetilde{F} \circ \mathrm{pr}$ for the quotient $\widetilde{F}\colon \widetilde{T} \to U \cup V$. For the closed disjoint subsets $A_T = \widetilde{S} \cup \widetilde{F}^{-1}(X - V)$ and $B_T = \widetilde{S}' \cup \widetilde{F}^{-1}(X - U)$, we construct a separating function $G\colon T \to \mathbb{R}$ which is $\mathbb{Z}_k$–invariant with $G(A_T) = 1$ and $G(B_T) = -1$ and with a regular value $-1 < s < 1$ (we can assume that $s = 0$) such that $(G^{-1}(0), G^{-1}(0) \cap \delta T)$ is a separating $\mathbb{Z}_k$–stratifold. We can find a bicollar around $G^{-1}(0)$ similarly to when we show that d is well defined. Therefore, Corollary A.11 implies that $((S, \delta S), f)$ and $((G^{-1}(0), G^{-1}(0) \cap \delta T), F|_{G^{-1}(0)})$ are $\mathbb{Z}_k$–bordant in U by the $\mathbb{Z}_k$–bordism $((G^{-1}[0,\infty), G^{-1}[0,\infty) \cap \delta T), F|_{G^{-1}[0,\infty)})$, and $((G^{-1}(0), G^{-1}(0) \cap \delta T), F|_{G^{-1}(0)})$ and $((S', \delta S'), f')$ are $\mathbb{Z}_k$–bordant in V by the $\mathbb{Z}_k$–bordism $((G^{-1}(-\infty,0], G^{-1}(-\infty,0] \cap \delta T), F|_{G^{-1}(-\infty,0]})$. Thus,

$$i_{U*}\big([(G^{-1}(0), G^{-1}(0) \cap \delta T), F|_{G^{-1}(0)}]\big) = [(S, \delta S), f],$$
$$i_{V*}\big([(G^{-1}(0), G^{-1}(0) \cap \delta T), F|_{G^{-1}(0)}]\big) = [(S', \delta S'), f'].$$

- **ker $\boldsymbol{i_*} \subset$ im $\boldsymbol{d}$** Suppose we have $[(P, \delta P), r] \in SH_{n-1(U\cap V;\mathbb{Z}_k)}$ which satisfies $i_{U*}([(P, \delta P), r]) = 0$ and $i_{V*}([(P, \delta P), r]) = 0$. Then there exist null $\mathbb{Z}_k$–bordisms $((T_1, \delta T_1), R_1)$ and $((T_2, \delta T_2), R_2)$ of $i_{U*}([(P, \delta P), r])$ and $i_{V*}([(P, \delta P), r])$, respectively. We construct $((T_1 \sqcup_P T_2, \delta T_1 \sqcup_{\delta P} \delta T_2), R_1 \sqcup_r R_2)$ with image under d equal to $[(P, \delta P), r]$.

- **ker $\boldsymbol{d} \subset$ im $\boldsymbol{j_*}$** Consider $[(S, \delta S), f] \in SH_n(U \cup V; \mathbb{Z}_k)$ with $d([(S, \delta S), f]) = 0$. For a separating function G with regular value s as in the definition of d, write $(P, \delta P) = (G^{-1}(s), G|_{\delta S}^{-1}(s))$, which has a bicollar. We put

$$(S_+, \delta S_+) = (G^{-1}[s,\infty), G|_{\delta S}^{-1}[s,\infty)) \quad \text{and} \quad (S_-, \delta S_-) = (G^{-1}(-\infty,0], G|_{\delta S}^{-1}(-\infty,0]).$$

Then $S = S_+ \sqcup_P S_-$ and $\delta S = \delta S_+ \sqcup_{\delta P} \delta S_-$. By the assumptions, there is $((Z, \delta Z), r)$ with $r\colon Z \to U \cap V$, which has the $\mathbb{Z}_k$–boundary $(P, \delta P)$ and $f|_P = r|_P$. Consider the continuous maps $f_+\colon S_+ \sqcup_P Z \to U$ and $f_-\colon S_- \sqcup_P Z \to V$. The gluing $T := ((S_+ \sqcup_P Z) \times [0,1]) \sqcup_Z ((S_- \sqcup_P Z) \times [1,2])$ (similarly for the Bockstein δT) gives a $\mathbb{Z}_k$–bordism between

$$j_{U*}\big(((S_+ \sqcup_P Z, \delta S_+ \sqcup_{\delta P} \delta Z), f_+)\big) - j_{V*}\big(((S_- \sqcup_P Z, \delta S_- \sqcup_{\delta P} \delta Z), f_-)\big)$$

and $((S, \delta S), f)$. We show an illustrative picture of the $\mathbb{Z}_k$–bordism $(T, \partial T)$ in Figure 9. □

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Received: 13 November 2020 Revised: 10 March 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 4 (pages 1809–2387) 2024