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Relative systoles in hyperelliptic translation surfaces

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We prove that the systole function on a connected component of area-1 translation surfaces admits a local maximum that is not a global maximum if and only if the connected component is not hyperelliptic.

32G15; 30F30

1 Introduction

We deal with flat metrics defined by abelian differentials on compact Riemann surfaces (*translation surfaces*). Such flat metrics have conical singularities of angle $(k+1)2\pi$, where k is the order of the zero of the corresponding abelian differential. A stratum of the moduli space of abelian differentials corresponds to translation surfaces that share the same combinatorics of zeroes.

Connected components of the strata have been classified by Kontsevich and Zorich in [6]. In each genus $g \geq 2$, there are exactly two components that consist of hyperelliptic translation surfaces, the so-called *hyperelliptic connected components*.

A saddle connection on a translation surface S is a geodesic joining two singularities (possibly the same) and with no singularity in its interior. We define the *relative systole* $\mathrm{Sys}(S)$ to be the length of the shortest saddle connection of S. A sequence of area-1 translation surfaces $(S_n)_{n\in\mathbb{N}}$ in a stratum of the moduli space of translation surfaces leaves any compact set if and only if $\mathrm{Sys}(S_n) \to 0$; see Kerckhoff, Masur and Smillie [5, Proposition 1]. The set of translation surfaces with short relative systole and compactification issues of strata are related to dynamics and counting problems on translation surfaces, and have been widely studied in the last 30 years; see for instance Eskin, Kontsevich and Zorich [2], Eskin, Masur and Zorich [3] and Kerckhoff, Masur and Smillie [5].

Here we are interested in the opposite problem: we study surfaces that are "far" from the boundary. In [1], we have characterized global maxima for Sys, and we have shown that each stratum of genus greater than or equal to 3 contains local but nonglobal maxima for the function Sys.

We prove that there are no such local maxima in hyperelliptic connected components (Theorem 3.1), while they exist in every other connected component (Theorem 4.1). This gives us the following characterization:

Main Theorem *Let $\mathcal{C}$ be a connected component of a stratum of area-1 surfaces with no marked points. The relative systole function on $\mathcal{C}$ admits a local maximum that is not a global maximum if and only if $\mathcal{C}$ is not a hyperelliptic connected component.*

Note that our notion of relative systole is different from the "true systole" (ie shortest closed curve) that has been studied by Judge and Parlier in [4]. Henceforth, for simplicity, if not mentioned otherwise the term "systole" will mean "relative systole".

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2 Background

2.1 Translation surfaces

A *translation surface* is a (real compact connected) genus-g surface S with a translation atlas, ie a triple $(S, \mathcal{U}, \Sigma)$ such that Σ (whose elements are called *singularities*) is a finite subset of S and $\mathcal{U} = \{(U_i, z_i)\}$ is an atlas of $S \setminus \Sigma$ whose transition maps are translations of $\mathbb{C} \simeq \mathbb{R}^2$. We will require that for each $s \in \Sigma$ there is a neighborhood of s isometric to a Euclidean cone whose total angle is a multiple of 2π. One can show that the holomorphic structure on $S \setminus \Sigma$ extends to S and that the holomorphic 1–form $\omega = dz_i$ extends to a holomorphic 1–form on S where Σ corresponds to the zeroes of ω and maybe some marked points. We usually call ω an *abelian differential*. A zero of ω of order k corresponds to a singularity of angle $(k+1)2\pi$. By a slight abuse of notation, we allow the order of a zero to be 0, and in this case it corresponds to a (regular) marked point.

A *saddle connection* is a geodesic segment joining two singularities (possibly the same) and with no singularity in its interior. Integrating ω along the saddle connection we get a complex number. Considered as a planar vector, this complex number represents the affine holonomy vector of the saddle connection. In particular, its Euclidean length is the modulus of its holonomy vector.

For $g \geq 1$, we define the moduli space of abelian differentials $\mathcal{H}_g$ as the moduli space of pairs (X, ω) where X is a genus-g (compact connected) Riemann surface and ω a nonzero holomorphic 1–form defined on X. The term moduli space means that we identify the points (X, ω) and (X', ω') if there exists an analytic isomorphism $f : X \to X'$ such that $f^*\omega' = \omega$.

One can also see a translation surface obtained from a polygon (or a finite union of polygons) whose sides come by pairs, and for each pair, the corresponding segments are parallel and of the same length. These parallel sides are glued together by translation and we assume that this identification preserves the natural orientation of the polygons. In this context, two translation surfaces are identified in the moduli space of abelian differentials if and only if the corresponding polygons can be obtained from each other by cutting and gluing, and preserving the identifications.

The moduli space of abelian differentials is stratified by the combinatorics of the zeroes; we will denote by $\mathcal{H}(k_1,\dots,k_r)$ the stratum of $\mathcal{H}_g$ consisting of (classes of) pairs (X,ω) such that ω has exactly r zeroes, of order $k_1,\dots,k_r$. This space is (Hausdorff) complex analytic, and local coordinates for a stratum of abelian differentials are obtained by integrating the holomorphic 1–form along a basis of the relative homology $H_1(S,\Sigma;\mathbb{Z})$, where Σ denotes the set of conical singularities of S; see for instance [7; 8; 9]. We have the classical Gauss–Bonnet formula $\sum_i k_i = 2g-2$, where g is the genus of the underlying surfaces. We often restrict to the subset $\mathcal{H}_1(k_1,\dots,k_r)$ of *area*-1 surfaces.

2.2 Connected component of strata

Here we recall the Kontsevich–Zorich classification of the connected components of the strata of abelian differentials [6].

A translation surface (X,ω) is *hyperelliptic* if the underlying Riemann surface is hyperelliptic, ie there is an involution τ such that X/τ is the Riemann sphere. In this case ω satisfies $\tau^*\omega = -\omega$. A connected component of a stratum is said to be *hyperelliptic* if it consists only of hyperelliptic translation surfaces (note that a connected component which is not hyperelliptic may contain some hyperelliptic translation surfaces).

Let γ be a simple closed smooth curve parametrized by the arc length on a translation surface that avoids the singularities. Then $t \to \gamma'(t)$ defines a map from $\mathbb{S}^1$ to $\mathbb{S}^1$. We denote by $\mathrm{Ind}(\gamma)$ the index of this map. Assume that the translation surface S has only even-degree singularities $S \in \mathcal{H}(2k_1,\dots,2k_r)$. Let $(a_i,b_i)_{i\in\{1,\dots,g\}}$ be a collection of simple closed curves as above that represents a symplectic basis of the homology of S. Then

$$\sum_{i=1}^{g} (\mathrm{Ind}(a_i)+1)(\mathrm{Ind}(b_i)+1) \mod 2$$

is an invariant of connected components and is called the *parity of the spin structure* (see [6] for details).

Here is a reformulation of the classification of connected components of strata by Kontsevich and Zorich:

Theorem 2.1 [6, Theorems 1 and 2] *Let $\mathcal{H}=\mathcal{H}(k_1,\dots,k_r)$ be a stratum of genus $g\geq 2$ translation surfaces, without marked points.*

- *The stratum $\mathcal{H}$ contains a hyperelliptic connected component if and only if $\mathcal{H}=\mathcal{H}(2g-2)$ or $\mathcal{H}=\mathcal{H}(g-1,g-1)$. In this case there is only one hyperelliptic component. In genus 2, any stratum is connected (and hyperelliptic).*
- *If there exists i such that k_i is odd, or if $g=3$, then there exists a unique nonhyperelliptic connected component.*
- *If $g\geq 4$ and, for all i, k_i is even, then there are exactly two nonhyperelliptic connected components distinguished by the parity of the spin structure.*

The following lemma is classical and will be useful in the next section.

Lemma 2.2 *Let S be a translation surface in a hyperelliptic connected component and let γ be a saddle connection. Then $[\gamma] = -[\tau(\gamma)]$ in $H_1(S, \Sigma; \mathbb{Z})$.*

Proof If the images in S of γ and $\tau(\gamma)$ coincide, $[\gamma] = -[\tau(\gamma)]$ since they have opposite orientation.

Otherwise the images in S of γ and $\tau(\gamma)$ intersect at most at the ends of the curves. We note that in the case of the stratum $\mathcal{H}(g-1, g-1)$, the two singularities are interchanged by the involution; see [6, Section 2.1].

Hence, the image of γ (and $\tau(\gamma)$) in the quotient sphere S/τ is always a simple closed curve. Therefore it is the boundary of a subsurface that contains ramification points of the covering. Considering its preimage, we obtain that $\gamma \cup \tau(\gamma)$ is the boundary of a subsurface of S. □

3 Hyperelliptic connected component

In this section, we prove the first part of the Main Theorem.

Theorem 3.1 *Let $\mathcal{C}$ be a hyperelliptic connected component of the moduli space of abelian differentials. Let $S \in \mathcal{C}$ be a local maximum of the relative systole function* Sys*. Then S is a global maximum for* Sys *in $\mathcal{C}$.*

The proof uses the following technical lemma. We postpone its proof to the end of the section.

Lemma 3.2 *Let D be a translation surface that is topologically a disk and whose boundary consists of n–saddle connections (an "n–gon") with $n \geq 4$. We assume that all boundary saddle connections are of length greater than or equal to 1. Then we can continuously deform D so that its area decreases and the boundary saddle connections of length 1 remain of length 1.*

Proof of Theorem 3.1 Let $S \in \mathcal{C}$ be a translation surface that such that $\mathrm{Sys}(S)$ is not a global maximum. We use the same normalization as in [1]: after rescaling the surface we assume that $\mathrm{Sys}(S)$ equals 1, and then continuously deform S so that $\mathrm{Sys}(S)$ remains 1 and $\mathrm{Area}(S)$ decreases.

Let $\{\gamma_1, \dots, \gamma_r\}$ be the set of saddle connections realizing the systole. Recall that $\gamma_1, \dots, \gamma_r$ are sides of the Delaunay triangulation and that global maxima correspond to surfaces whose Delaunay cells are only equilateral triangles; see [1, Lemma 3.1 and Theorem 3.3]. Let $C_1, \dots, C_k$ be the connected components of $S \setminus \bigcup_i \gamma_i$. Up to renumbering we can assume that C_1 is not a triangle. We consider $\tau(C_1)$, where τ is the hyperelliptic involution. We study the two possible cases: whether $\tau(C_1)$ equals C_1 or not. Note that C_1 does not contain any singularity in its interior, since there are at most two singularities in S and if there are two singularities P_1 and P_2, we must have $\tau(P_1) = P_2$.

Case 1 We first assume that $\tau(C_1) \neq C_1$. Since the hyperelliptic involution preserves $\bigcup_i \gamma_i$, up to renumbering, $\tau(C_1) = C_2$.

We observe that C_1 has only one boundary component. Indeed, suppose that there are more than one such components and consider a saddle connection η in C_1 that joins a singularity of one boundary component to a singularity of another boundary component. Then $\tau(\eta)$ is a curve in C_2 and $[\tau(\eta)] = -[\eta]$ by Lemma 2.2. But $C_1 \setminus \eta$ is connected, and hence $S \setminus (\eta \cup \tau(\eta))$ is connected, which is a contradiction. Therefore C_1 is a disk because it embeds in S/τ, which is a sphere.

Since the boundary of C_1 consists of at least four saddle connections of length 1, by Lemma 3.2 we can continuously decrease its area while keeping the boundary saddle connections of length 1.

This continuous deformation of C_1 leads to the following area-decreasing continuous deformation of S:

The component C_2 is deformed in a symmetric way as C_1.

For each saddle connection γ in the boundary of C_1, the components of $S \setminus (\gamma \cup \tau(\gamma))$ correspond to components of the complement of $[\gamma]$ in the quotient sphere S/τ. Since $[C_1] = [C_2]$, we have that C_1 and C_2 are in the same connected component of $S \setminus (\gamma \cup \tau(\gamma))$. We denote by D_γ the other component. By construction, the boundary of D_γ consists of γ and $\tau(\gamma)$. Note that D_γ is empty if γ and $\tau(\gamma)$ have the same image in S. We observe that if γ and γ' are two distinct saddle connections in the boundary of C_1, then D_γ and $D_{\gamma'}$ are disjoint.

We denote by $\gamma_1, \dots, \gamma_k$ the boundary saddle connections of C_1. When continuously deforming C_1, each γ_i is rotated by an angle θ_i (with $\theta_1, \dots, \theta_k$ continuous functions) and $\tau(\gamma_i)$ is also rotated by θ_i since C_2 is deformed in a symmetric way. Since the components $D_{\gamma_1}, \dots, D_{\gamma_r}$ are disjoint, for each i we can glue by translation the component D_{γ_i} rotated by θ_i with the boundary saddle connections corresponding to γ_i and $\tau(\gamma_i)$.

Since the identifications are done by translation, we get a continuous family of translation surfaces and they are in the same stratum.

Case 2 Now we assume that $\tau(C_1) = C_1$.

We claim that we can cut C_1 along saddle connections and obtain two discs A and B such that $\tau(A) = B$ and for each saddle connection γ in the boundary of A either γ is of length 1 or $\tau(\gamma) = \gamma$ (equivalently, γ is also a boundary saddle connection of B).

To prove the claim, we first consider the Delaunay cells of S. Recall that the shortest geodesics (and hence the boundary saddle connections of C_1) are sides of the Delaunay cells; see [1, Lemma 3.1]. This induces a decomposition of C_1 into Delaunay cells, and this decomposition is preserved by the involution τ because of the uniqueness of the Delaunay cell decomposition. We define a Delaunay subdivision $\mathcal{D}$ in the following way: For each Delaunay cell d, if $\tau(d) \neq d$ then $d, \tau(d) \in \mathcal{D}$. If $\tau(d) = d$ (and since d is cyclic) it can be cut by a diagonal into two polygons d' and $d'' = \tau(d')$. Then $d', d'' \in \mathcal{D}$.

Now we use the following algorithm:

- We start from a pair $(d_0, \tau(d_0))$ in $\mathcal{D}^2$ and let $A_0 = d_0$ and $B_0 = \tau(d_0)$.
- Suppose we have constructed the disks A_k and B_k such that $\tau(A_k) = B_k$, and A_k and B_k are unions of elements in $\mathcal{D}$.

If $A_k \cup B_k \neq C_1$, there exists an element $d_{k+1} \in \mathcal{D}$ adjacent to A_k along a saddle connection γ_k (and $\tau(d_{k+1}) \in \mathcal{D}$ is adjacent to B_k along $\tau(\gamma_k)$). We define A_{k+1} by gluing A_k and d_{k+1} along γ_k. Note that γ_k is the only saddle connection in the common boundary of A_k and d_{k+1}, because otherwise $S \setminus (\gamma_k \cup \tau(\gamma_k))$ would be connected, which is impossible in the hyperelliptic connected component.

If $A_k \cup B_k = C_1$ we define $A = A_k$ and $B = B_k$.

The boundary of the disk A consists of $n \geq 3$ saddle connections of lengths at least 1.

If $n \geq 4$, then from Lemma 3.2 it can be continuously deformed so that the area decreases and the boundary saddle connections of length 1 remain of length 1.

If $n = 3$ then A is a triangle. Two of its sides are boundary saddle connections of C_1, and hence of length 1. The third side of A is a saddle connection inside C_1 and is, by construction of C_1, of length greater than 1 (recall that $C_1, \dots, C_r$ are obtained after removing all saddle connections of length 1). Such a triangle can be deformed so that the area decreases and the boundary saddle connections of length 1 remain of length 1.

We deform B in a symmetric way. Note that A and B are directly glued together in C_1 along the boundary saddle connections of lengths greater than 1. Therefore the possible changes of these saddle connections are not a problem. The deformation of $S \setminus C_1$ is treated as in the previous case. □

Proof of Lemma 3.2 The sum of the boundary angles (coming from the intersection of two consecutive boundary saddle connections) of D equals $(n-2)\pi$. Therefore D has boundary angles smaller than π. If such a boundary angle has a corresponding boundary saddle connection which is of length greater than 1, then by slightly changing its length we can decrease the area of the corresponding triangle and hence of D.

So we can assume that for each boundary angle smaller than π the two adjacent saddle connections are of length 1. We claim that we can find two consecutive angles such that one is smaller than π and the other is smaller than 2π (note that since D is not necessarily embedded in the plane, it can have boundary angles greater than 2π). Indeed, consider the sequence of consecutive boundary angles of D. If each time an angle is smaller than π the following one is greater than or equal to 2π, then the global sum will be greater than $n\pi$, which is not possible.

Now we consider the three consecutive saddle connections corresponding to these two angles, and see them as a broken line on the plane. We close this line by adding a segment t to obtain a quadrilateral $\mathcal{Q}$ (that can be also crossed). Without loss of generality, we can assume that t is horizontal. We have

$$\mathrm{Area}(D) = \mathrm{Area}(D_0) + \mathrm{Area}_{\mathrm{alg}}(\mathcal{Q}),$$

where D_0 is the translation surface obtained by "replacing" the broken line by t (see Figure 1). Here $\mathrm{Area}_{\mathrm{alg}}(\mathcal{Q})$ means that the part of $\mathcal{Q}$ below the segment t is counted negatively.

Claim *We can continuously deform $\mathcal{Q}$ without changing the lengths of its sides so that* $\mathrm{Area}_{\mathrm{alg}}(\mathcal{Q})$ *decreases.*

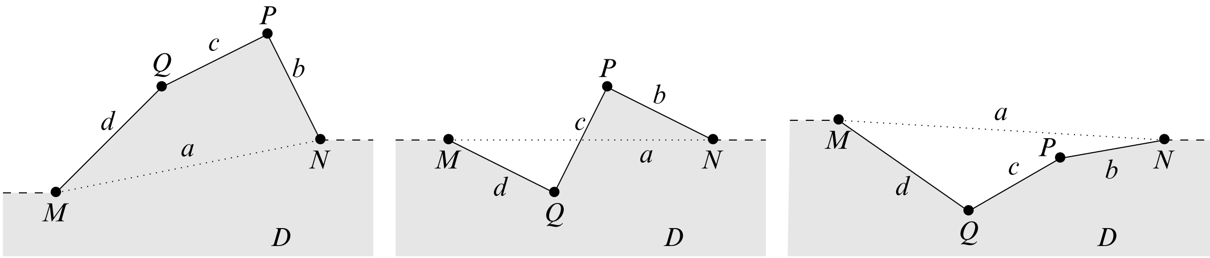


Figure 1: The disk D and the quadrilateral $\mathcal{Q}$ in three configurations.

Denote by $MNPQ$ the quadrilateral $\mathcal{Q}$, and by a, b, c and d the lengths of the sides of $\mathcal{Q}$ with a being the length of the segment $t = MN$. Denote by α the oriented angle from MN to MQ, and by γ its opposite angle in $\mathcal{Q}$ (ie the angle from PQ to PN). Without loss of generality we assume that $b = c = 1$, $d \geq 1$ and $0 < \gamma < \pi$ (in fact we must have $\gamma > \frac{1}{3}\pi$, otherwise there would be a smallest saddle connection). We also have $-\pi < \alpha < \pi$. Further, the sides NP and QM do not intersect since it would imply intersecting boundary saddle connections in D (see Figure 1).

Write $K = \mathrm{Area}_{\mathrm{alg}}(\mathcal{Q})$. We compute K by adding the (algebraic) area of the triangles MNQ and NPQ. We obtain

$$K = \tfrac{1}{2}(ad\sin(\alpha) + bc\sin(\gamma)). \tag{1}$$

The expression of the length of NQ gives the second equality:

$$a^2 + d^2 - 2ad\cos(\alpha) = b^2 + c^2 - 2bc\cos(\gamma). \tag{2}$$

These two equations imply Bretschneider's formula for $\mathcal{Q}$:

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos\big(\tfrac{1}{2}(\alpha+\gamma)\big). \tag{3}$$

Here $s = \frac{1}{2}(a+b+c+d)$.

From now on we fix a, b, c and d and study the variations of the area with respect to α and γ. Equation (2) implies that γ depends differentially on α. Hence we can write $K = K(\alpha)$. We need to prove that either $K'(\alpha) \neq 0$ or $K(\alpha)$ is a strict local maximum (note that α varies in an open set). We have

$$(K^2)'(\alpha) = abcd(1+\gamma'(\alpha))\sin\big(\tfrac{1}{2}(\alpha+\gamma)\big)\cos\big(\tfrac{1}{2}(\alpha+\gamma)\big).$$

We assume that $K'(\alpha) = 0$, and hence $(K^2)'(\alpha) = 0$, so we are in one of the following three cases:

(i) $\mathbf{\big(\sin\big(\tfrac{1}{2}(\alpha+\gamma)\big) = 0\big)}$ The conditions $-\pi < \alpha < \pi$ and $0 < \gamma < \pi$ imply $\alpha = -\gamma < 0$. Hence the quadrilateral $\mathcal{Q}$ has self-intersections. Since the sides NP and QM do not intersect, the sides MN and PQ intersect. The condition $\alpha = -\gamma$ implies that the points M, N, P and Q are cocyclic, and since $b = c = 1$ we must have $d < 1$, which is a contradiction.

(ii) $\mathbf{\big(\cos\big(\tfrac{1}{2}(\alpha+\gamma)\big) = 0\big)}$ Then $\alpha + \gamma = \pi$, and therefore $\alpha > 0$, and hence $K > 0$. From (2) and (3) we have a strict local maximum for K^2, and therefore for K.

(iii) **($\gamma'(\alpha) = -1$)** By differentiating (2) and using (1), we see that $K = 0$, and hence Q has a self-intersection $I = MN \cap PQ$. By differentiating (1) and using (2), we obtain $K'(\alpha) = 0 = \frac{1}{2}\left(\frac{1}{2}(a^2+d^2)-1\right)$, and hence $a^2 + d^2 = 2$. Since $d \geq 1$, we have $a \leq 1 \leq d$. However, triangle inequalities for INP and IMQ give $a + c > d + b$, and hence $a > d$, which is a contradiction. □

4 Nonhyperelliptic connected components

In this section, we prove the second part of the Main Theorem.

Theorem 4.1 *Each nonhyperelliptic connected component of each stratum of area-1 surfaces with no marked points contains local maxima of the function* Sys *that are not global.*

We will need the following lemma, which is a refinement of [1, Lemma 3.2(2)].

Lemma 4.2 *Let $\mathcal{C} \subset \mathcal{H}(2k_1, \dots, 2k_r)$ be a connected component of a stratum of abelian differentials with $2k_1, \dots, 2k_r \geq 0$. There exists a surface $S \in \mathcal{C}$ realizing the global maximum for the systole function, and such that there exists a shortest saddle connection γ joining a singularity of degree $2k_1$ to itself and* $\mathrm{Ind}([\gamma]) = 0$.

Proof We do as in the proof of [1, Lemma 3.2]. There exists a square-tiled surface in $\mathcal{C}$ with singularities on each corner of the squares as in Figure 2, and we can assume that the top left horizontal segment identifies with the bottom left horizontal segment (see Figure 2). After a suitable transformation as in the figure, we obtain the required surface. □

Proof of Theorem 4.1 In [1, Theorem 4.7] we have already constructed examples in each genus $g \geq 3$ stratum. By Theorem 3.1 each such example is in a nonhyperelliptic component. So it remains to construct new examples only in strata with more than one nonhyperelliptic connected component.

From the theorem of Kontsevich and Zorich (Theorem 2.1) there is more than one nonhyperelliptic connected component only for genus $g \geq 4$ strata with only even-degree singularities, and in this case there are two nonhyperelliptic components distinguished by the parity of the spin structure.

In Figure 3 we give surfaces $S_{2,0} \in \mathcal{H}(2,0)$ and $S_{2,0,0} \in \mathcal{H}(2,0,0)$ that are local but nonglobal maxima for the systole function.

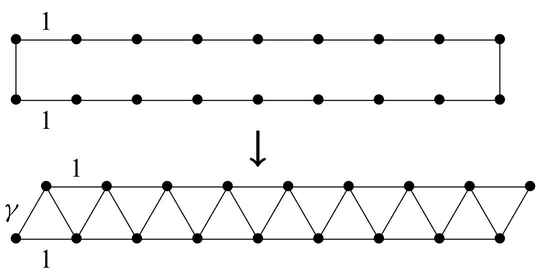
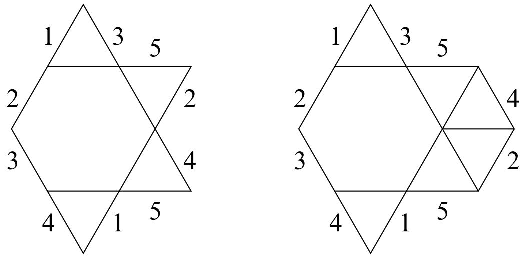


Figure 2: A global maximum with a closed shortest saddle connection γ satisfying $\mathrm{Ind}([\gamma]) = 0$.



$S_{2,0} \in \mathcal{H}(2,0)$ $S_{2,0,0} \in \mathcal{H}(2,0,0)$

Figure 3: Local but nonglobal maxima in $\mathcal{H}(2,0)$ and $\mathcal{H}(2,0,0)$.

We consider the following construction: Start from the surface $S_{2,0}$ and a surface M that is a global maximum for Sys in $\mathcal{H}(2k_1,\dots,2k_r)$. There exists a shortest saddle connection γ_1 in $S_{2,0}$ joining the two singularities. By Lemma 4.2, we can assume that there exists a shortest saddle connection γ_2 in M joining the singularity of degree $2k_1$ to itself and such that $\mathrm{Ind}([\gamma_2])=0$. We can further assume that γ_1 and γ_2 are vertical and of the same length. Now we glue the two surfaces by the following classical surgery: cut the two surfaces along γ_1 and γ_2, and glue the left side of γ_1 with the right side of γ_2 and the right side of γ_1 with the left side of γ_2. We get a surface S in $\mathcal{H}(2k_1+4,2k_2,\dots,2k_r)$ that satisfies the hypothesis of [1, Theorem 4.1], and hence is a local but nonglobal maximum. By Theorem 3.1, the surface S is necessarily in a nonhyperelliptic component.

We compute $\mathrm{Spin}(S)$: Choose a symplectic basis $(a_i,b_i)_i$ of $H_1(M,\mathbb{Z})$ such that $[\gamma_2]=a_1$. Then a simple computation gives

(4) $$\mathrm{Spin}(S)=\mathrm{Spin}(S_{0,2})+\mathrm{Spin}(M)+\mathrm{Ind}(a_1)+1 \mod 2.$$

Since $\mathrm{Ind}(a_1)=0$,

$$\mathrm{Spin}(S)=\mathrm{Spin}(S_{0,2})+\mathrm{Spin}(M)+1 \mod 2.$$

When $\sum_i 2k_i \geq 4$, we can prescribe any value of $\mathrm{Spin}(M)$ by choosing M in a suitable component, and in this way we can obtain any possible value for $\mathrm{Spin}(S)$. Note that this is also true for $M\in\mathcal{H}(4)$ or $M\in\mathcal{H}(2,2)$. Indeed, in these strata there are two components, hyperelliptic and nonhyperelliptic, and the spin structure distinguishes them; see [6, Theorem 2 and Corollary 5].

By this construction, we obtain a local but nonglobal maximum for Sys in any (nonhyperelliptic) connected component of any stratum $\mathcal{H}(2n_1,\dots,2n_r)$ for $r\geq 1$, as soon as $\sum_i 2n_i \geq 8$ and $2n_j\geq 4$ for at least one $j\in\{1,\dots,r\}$.

We do an analogous construction as above starting from $S_{2,0,0}$ (see Figure 3) and $M\in\mathcal{H}(0,2^r)$, with $\gamma_1\in S_{2,0,0}$ joining the two marked points and $\gamma_2\in M$ joining the marked point to itself. We obtain a local but nonglobal maximum in $\mathcal{H}(2^{r+2})$. For $r\geq 2$ we can choose the spin structure of M and thus get S in any nonhyperelliptic component of $\mathcal{H}(2^{r+2})$. Note that for $r=1$ we get $S\in\mathcal{H}(2,2,2)$ with odd spin structure.

There remain the following cases:

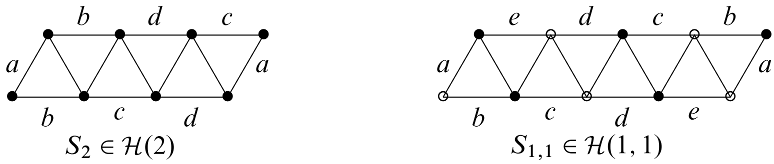


Figure 4: Global maxima in $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$.

- **$\mathcal{H}(6)$** We do the same construction as above, starting from $S_{2,0}$ and $M \in \mathcal{H}(2)$. We consider for $M \in \mathcal{H}(2)$ the surface S_2 in Figure 4. We see that $[a]$ and $[b]$ in this figure have different indices mod 2. Hence choosing $\gamma_2 = a$ or $\gamma_2 = b$ gives surfaces with different Spin structure; see (4).
- **$\mathcal{H}(4,2)$** We do the same as for $\mathcal{H}(6)$, starting from $S_{2,0,0}$ and $M = S_2$.
- **The even component of $\mathcal{H}(2,2,2)$** We do the same construction but starting from $S_{2,0,0}$ and $M \in \mathcal{H}(1,1)$, the surface $S_{1,1}$ in Figure 4. We consider $\gamma_2 = a$ (joining the two singularities of degree 1). By a direct computation, the above construction gives a surface $S \in \mathcal{H}(2,2,2)$ with $\mathrm{Spin}(S) = 0 \mod 2$. □

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