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Hiroshi Kihara

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Smooth singular complexes and diffeological principal bundles

HIROSHI KIHARA

In previous papers, we used the standard simplices Δ^p ($p \geq 0$) endowed with diffeologies having several "good" properties to introduce the singular complex $S^{\mathcal{D}}(X)$ of a diffeological space X. (Here, $\mathcal{D}$ denotes the category of diffeological spaces.) On the other hand, Hector and Christensen–Wu used the standard simplices Δ^p_{sub} ($p \geq 0$) endowed with the subdiffeology of $\mathbb{R}^{p+1}$ and the standard affine p–spaces $\mathbb{A}^p$ ($p \geq 0$) to introduce the singular complexes $S^{\mathcal{D}}_{\text{sub}}(X)$ and $S^{\mathcal{D}}_{\text{aff}}(X)$, respectively, of a diffeological space X. We prove that $S^{\mathcal{D}}(X)$ is a fibrant approximation of both $S^{\mathcal{D}}_{\text{sub}}(X)$ and $S^{\mathcal{D}}_{\text{aff}}(X)$. This result immediately implies that the homotopy groups of $S^{\mathcal{D}}_{\text{sub}}(X)$ and $S^{\mathcal{D}}_{\text{aff}}(X)$ are isomorphic to the smooth homotopy groups of X, which enables us to give a positive answer to a conjecture of Christensen and Wu. Further, we characterize diffeological principal bundles (ie principal bundles in the sense of Iglesias-Zemmour) using the singular functor $S^{\mathcal{D}}_{\text{aff}}$. By using these results, we extend the characteristic classes for $\mathcal{D}$–numerable principal bundles to those for diffeological principal bundles.

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1 Introduction

Let $\mathcal{D}$ denote the category of diffeological spaces. In [Kihara 2019], we constructed diffeologies on $\Delta^p = \{(x_0, \dots, x_p) \in \mathbb{R}^{p+1} \mid \sum x_i = 1,\ x_i \geq 0 \text{ for any } i\}$ ($p \geq 0$). We called them *good* because they allowed us to define the singular complex $S^{\mathcal{D}}(X)$ of a diffeological space X, which enables us to introduce a model structure on the category $\mathcal{D}$ (see Section 2.2). Further, in [Kihara 2023], we also used the singular functor $S^{\mathcal{D}}$ to introduce a simplicial category structure on $\mathcal{D}$, and developed a smooth homotopy theory based on the simplicial and model category structures on $\mathcal{D}$.

On the other hand, Hector [1995] used the sets Δ^p endowed with the subdiffeology of $\mathbb{R}^{p+1}$ ($p \geq 0$) to define the singular complex $S^{\mathcal{D}}_{\text{sub}}(X)$ of a diffeological space X. His singular complex is also used in [Kuribayashi 2020]. Christensen and Wu [2014] also used the affine spaces

$$\mathbb{A}^p = \left\{(x_0, \dots, x_p) \in \mathbb{R}^{p+1} \;\middle|\; \sum x_i = 1\right\}$$

endowed with the subdiffeology of $\mathbb{R}^{p+1}$ ($p \geq 0$) to define the singular complex $S^{\mathcal{D}}_{\text{aff}}(X)$ in an attempt to construct a model structure on $\mathcal{D}$. Their singular complex is also used in [Bunk 2022; Kuribayashi 2020; 2021].

As is described in the references cited above, the singular complexes $S^{\mathcal{D}}(X)$, $S^{\mathcal{D}}_{\text{sub}}(X)$, and $S^{\mathcal{D}}_{\text{aff}}(X)$ are playing crucial roles in the smooth homotopical study of diffeological spaces. However, the natural weak equivalences between them have not yet been established.

In this paper, we show that the singular complexes $S^{\mathcal{D}}(X)$, $S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ are weakly equivalent (Theorem 1.1). As a corollary of this result, we identify the homotopy groups of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ and $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ with the smooth homotopy groups of X, proving a conjecture of Christensen and Wu (Corollary 1.2). Though we mainly use the singular functor $S^{\mathcal{D}}$, we also use the singular functor $S^{\mathcal{D}}_{\mathrm{aff}}$ to characterize diffeological principal bundles (ie principal bundles in the sense of Iglesias-Zemmour) (Theorem 1.3). This theorem, along with the weak equivalence between $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ and $S^{\mathcal{D}}(X)$, is used to extend the characteristic classes for $\mathcal{D}$–numerable principal G–bundles to those for diffeological principal G–bundles (Proposition 1.4).

Throughout this paper, $\mathcal{D}$ and $\mathcal{S}$ denote the category of diffeological spaces and the category of simplicial sets, respectively. (See [Goerss and Jardine 1999; May 1992; Kihara 2014] for the basics of simplicial homotopy theory.)

Weak equivalences between $S^{\mathcal{D}}(X)$, $S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $S^{\mathcal{D}}_{\mathrm{aff}}(X)$

The following theorem is the main result of this paper. Note that the canonical maps $\Delta^p \xrightarrow{\mathrm{id}} \Delta^p_{\mathrm{sub}} \hookrightarrow \mathbb{A}^p$ ($p \geq 0$) induce natural morphisms of simplicial sets $S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathcal{D}}(X)$ (see Lemma 3.1(3) and Proposition 3.4); note that the first and second canonical maps induce the second and first morphisms of singular complexes, respectively. Recall that $S^{\mathcal{D}}(X)$ is always Kan (ie fibrant in the category $\mathcal{S}$); see Corollary 2.6(1) (cf Remark 3.2(2)).

Theorem 1.1 *The natural morphisms of simplicial sets*

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathcal{D}}(X)$$

are weak equivalences. In particular, $S^{\mathcal{D}}(X)$ is a fibrant approximation of both $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ and $S^{\mathcal{D}}_{\mathrm{sub}}(X)$.

That $S^{\mathcal{D}}(X)$ is a fibrant approximation of $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ was announced in [Kihara 2019, Remark A.5].

Next we recall that $\pi_i(S^{\mathcal{D}}(X), x)$ is isomorphic to the smooth homotopy group $\pi_i^{\mathcal{D}}(X, x)$ (Theorem 2.7), and use Theorem 1.1 to identify the homotopy groups of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ and $S^{\mathcal{D}}_{\mathrm{sub}}(X)$; see Section 4.4 for the homotopy groups of a simplicial set which need not satisfy the Kan condition.

Corollary 1.2 *Let (X, x) be a pointed diffeological space. Then both $\pi_i(S^{\mathcal{D}}_{\mathrm{aff}}(X), x)$ and $\pi_i(S^{\mathcal{D}}_{\mathrm{sub}}(X), x)$ are naturally isomorphic to the smooth homotopy group $\pi_i^{\mathcal{D}}(X, x)$ for $i \geq 0$.*

Christensen and Wu [2014, Theorem 4.11] showed that if $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ is fibrant, then $\pi_i(S^{\mathcal{D}}_{\mathrm{aff}}(X), x)$ is isomorphic to the smooth homotopy group $\pi_i^{\mathcal{D}}(X, x)$ for $i \geq 0$, and conjectured that for every diffeological space X, $\pi_i(S^{\mathcal{D}}_{\mathrm{aff}}(X), x)$ is isomorphic to $\pi_i^{\mathcal{D}}(X, x)$ for $i \geq 0$ [Christensen and Wu 2014, page 1272]. Corollary 1.2 contains their conjecture.

(Co)homology of diffeological spaces Following [Kihara 2023, Section 3.1], we define the homology $H_*(X; A)$ and the cohomology $H^*(X; A)$ of a diffeological space X with coefficients in an abelian group A by

$$H_*(X; A) = H_*(\mathbb{Z}S^{\mathcal{D}}(X) \otimes A), \quad H^*(X; A) = H^*\big(\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}(X), A)\big),$$

where the simplicial abelian group $\mathbb{Z}K$ freely generated by a simplicial set K is regarded as a chain complex by setting $\partial = \sum(-1)^i d_i$. It follows from Theorem 1.1 that the (co)homology of X is naturally isomorphic to the (co)homologies defined using $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ and $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ instead of $S^{\mathcal{D}}(X)$. However, this fact is actually proved in Section 3.2 as a key to proving Theorem 1.1; the (co)homology of X is also naturally isomorphic to the cubic (co)homology introduced in [Iglesias-Zemmour 2013, pages 176–186] (Remark 3.6).

Application to diffeological principal bundles

Let G be a diffeological group. A $\mathcal{D}$–numerable principal G–bundle $\pi : P \to X$ is a principal G–bundle which admits a trivialization open cover $\{U_i\}$ of X and a smooth partition of unity subordinate to it. On the other hand, Iglesias-Zemmour introduced a weaker notion of a principal G–bundle; such a principal G–bundle, referred to as a diffeological principal G–bundle, is defined by local triviality of the pullback along any plot (Definition 5.1(2)).

Though we mainly use the singular complexes $S^{\mathcal{D}}(X)$ in smooth homotopy theory, the singular complexes $S^{\mathcal{D}}_{\mathrm{aff}}(X)$, along with Theorem 1.1 play an essential role in the study of diffeological principal bundles, as explained below.

Characterization of diffeological principal G–bundles Let $\mathscr{C}$ be a category with finite products, and G a group in $\mathscr{C}$. Then $\mathscr{C}G$ denotes the category of right G–objects of $\mathscr{C}$ (ie objects of $\mathscr{C}$ endowed with a right G–action). For $B \in \mathscr{C}$, $\mathscr{C}G/B$ denotes the category of objects of $\mathscr{C}G$ over B, where B is regarded as an object of $\mathscr{C}G$ with trivial G–action.

Since $S^{\mathcal{D}}_{\mathrm{aff}} : \mathcal{D} \to \mathcal{S}$ is a right adjoint (Remark 3.2(1)), $S^{\mathcal{D}}_{\mathrm{aff}}$ induces the functor $\mathcal{D}G/X$ to $\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)/S^{\mathcal{D}}_{\mathrm{aff}}(X)$. We then have the following characterization theorem for diffeological principal G–bundles (the notion of a simplicial principal bundle is introduced in Definition 5.3).

Theorem 1.3 (1) *Let $\pi : P \to X$ be an object of $\mathcal{D}G/X$. Then $\pi : P \to X$ is a diffeological principal G–bundle if and only if*

$$S^{\mathcal{D}}_{\mathrm{aff}}(\pi) : S^{\mathcal{D}}_{\mathrm{aff}}(P) \to S^{\mathcal{D}}_{\mathrm{aff}}(X)$$

is a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle.

(2) *The functor $S^{\mathcal{D}}_{\mathrm{aff}} : \mathcal{D} \to \mathcal{S}$ induces a faithful functor from the category $\mathsf{P}\mathcal{D}G_{\mathrm{diff}}$ of diffeological principal G–bundles to the category $\mathsf{P}\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)$ of principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundles.*

The essential reason why $S^{\mathcal{D}}_{\mathrm{aff}}$ is useful in the study of diffeological principal G–bundles is because $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ can be regarded as the set of global plots of X. We can use Theorem 1.3 to calculate the (co)homology of exceptional diffeological spaces such as irrational tori and $\mathbb{R}/\mathbb{Q}$ (see Section 2.3 and Example 6.7); other cohomology theories of irrational tori were calculated by Iglesias-Zemmour and Kuribayashi (see Remark 6.8).

Characteristic classes of diffeological principal G–bundles We apply Theorem 1.3 to construct characteristic classes for diffeological principal G–bundles.

A *characteristic class* for a class $\mathcal{P}$ of smooth principal G–bundles is a rule assigning to a principal G–bundle $\pi : P \to X$ in $\mathcal{P}$ a cohomology class $\alpha(P)$ of X such that $\alpha(f^*P) = f^*\alpha(P)$. Christensen and Wu [2021, Theorem 5.10] constructed the universal $\mathcal{D}$–numerable principal G–bundle $\pi_G : EG \to BG$ and proved that the set of isomorphism classes of $\mathcal{D}$–numerable principal G–bundles over X bijectively corresponds to the smooth homotopy set $[X, BG]_{\mathcal{D}}$. Thus, a cohomology class $\alpha \in H^k(BG; A)$ defines the characteristic class $\alpha(\cdot)$ for the class of $\mathcal{D}$–numerable principal G–bundles. More precisely, the characteristic class $\alpha(P) \in H^k(X; A)$ of a $\mathcal{D}$–numerable principal G–bundle $\pi : P \to X$ is defined by

$$\alpha(P) = f_P^*\alpha,$$

where $f_P : X \to BG$ is a classifying map of P.

We would like to extend the characteristic class $\alpha(\cdot)$ to the class of diffeological principal G–bundles. Since pullbacks of EG are necessarily $\mathcal{D}$–numerable, the above definition of the characteristic class $\alpha(\cdot)$ does not apply to the class of diffeological principal G–bundles. Further, since the class of diffeological principal G–bundles does not have the homotopy invariance property with respect to pullback, it has no classifying space; see [Christensen and Wu 2021, Section 3].

Nevertheless, we can prove the following result.

Proposition 1.4 *Let G be a diffeological group and α an element of $H^k(BG; A)$. Then the characteristic class $\alpha(\cdot)$ for $\mathcal{D}$–numerable principal G–bundles extends to a characteristic class for diffeological principal G–bundles.*

This paper is organized as follows. In Section 2, we recall the basic notions and results on diffeological spaces and the singular functor $S^{\mathcal{D}}$. In Section 3, we briefly review the singular functors $S^{\mathcal{D}}_{\mathrm{sub}}$ and $S^{\mathcal{D}}_{\mathrm{aff}}$, and show that there exist natural morphisms between $S^{\mathcal{D}}_{\mathrm{aff}}(X)$, $S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $S^{\mathcal{D}}(X)$ which induce isomorphisms on (co)homology. We prove Theorem 1.1 and Corollary 1.2 in Section 4. In Section 5, we recall the notions of a diffeological principal bundle and a simplicial principal bundle, and prove Theorem 1.3. In Section 6, we prove Proposition 1.4 and discuss the sets of characteristic classes for the three classes $\mathsf{P}\mathcal{D}G$, $\mathsf{P}\mathcal{D}G_{\mathrm{num}}$, and $\mathsf{P}\mathcal{D}G_{\mathrm{diff}}$ of smooth principal G–bundles (see Definition 5.1(3) for these three classes).

2 Diffeological spaces

In this section, we first recall the convenient properties of the category $\mathcal{D}$ of diffeological spaces, along with the adjoint pair $\tilde{\cdot} : \mathcal{D} \rightleftarrows \mathcal{C}^0 : R$ of the underlying topological space functor and its right adjoint (Section 2.1). Then we recall the standard simplices Δ^p $(p \geq 0)$ and the adjoint pair $|\cdot|_{\mathcal{D}} : \mathcal{S} \rightleftarrows \mathcal{D} : S^{\mathcal{D}}$ of the realization and singular functors (see Section 2.2). Last, we make a brief review of some results of [Kihara 2023], in which the adjoint pairs $(\tilde{\cdot}, R)$ and $(|\cdot|_{\mathcal{D}}, S^{\mathcal{D}})$ play an essential role (Section 2.3).

2.1 Categories $\mathscr{D}$ and $\mathscr{C}^0$

In this subsection, we summarize the convenient properties of the category $\mathscr{D}$ of diffeological spaces, recalling the adjoint pair $\tilde{\cdot} : \mathscr{D} \rightleftarrows \mathscr{C}^0 : R$ of the underlying topological space functor and its right adjoint; see [Iglesias-Zemmour 2013; Kihara 2019] for full details.

Let us begin with the definition of a diffeological space. A *parametrization* of a set X is a (set-theoretic) map $p\colon U \to X$, where U is an open subset of $\mathbb{R}^n$ for some n.

Definition 2.1 (1) A *diffeological space* is a set X together with a specified set D_X of parametrizations of X satisfying the following conditions:

(i) **Covering** Every constant parametrization $p\colon U \to X$ is in D_X.

(ii) **Locality** Let $p\colon U \to X$ be a parametrization such that there exists an open cover $\{U_i\}$ of U satisfying $p|_{U_i} \in D_X$. Then p is in D_X.

(iii) **Smooth compatibility** Let $p\colon U \to X$ be in D_X. Then for every $n \geq 0$, every open set V of $\mathbb{R}^n$, and every smooth map $F\colon V \to U$, $p \circ F$ is in D_X.

The set D_X is called the *diffeology* of X, and its elements are called *plots*.

(2) Let $X = (X, D_X)$ and $Y = (Y, D_Y)$ be diffeological spaces, and let $f\colon X \to Y$ be a (set-theoretic) map. We say that f is *smooth* if $f \circ p \in D_Y$ for every $p \in D_X$.

The convenient properties of $\mathscr{D}$ are summarized in the following proposition. Recall that a topological space X is called *arc-generated* if its topology is final for the continuous curves from $\mathbb{R}$ to X, and let $\mathscr{C}^0$ denote the category of arc-generated spaces and continuous maps. See [Frölicher and Kriegl 1988, pages 230–233] for initial and final structures with respect to the underlying set functor.

Proposition 2.2 (1) *The category $\mathscr{D}$ has initial and final structures with respect to the underlying set functor. In particular, $\mathscr{D}$ is complete and cocomplete.*

(2) *The category $\mathscr{D}$ is cartesian closed.*

(3) *The underlying set functor $\mathscr{D} \to$ Set is factored as the underlying topological space functor $\tilde{\cdot} : \mathscr{D} \to \mathscr{C}^0$ followed by the underlying set functor $\mathscr{C}^0 \to$ Set. Further, the functor $\tilde{\cdot} : \mathscr{D} \to \mathscr{C}^0$ has a right adjoint $R\colon \mathscr{C}^0 \to \mathscr{D}$.*

Proof See [Christensen et al. 2014, page 90; Iglesias-Zemmour 2013, pages 35–36; Kihara 2019, Propositions 2.1 and 2.10]. $\square$

The following remark relates to Proposition 2.2.

Remark 2.3 (1) Let $\mathfrak{X}$ be a concrete category (ie a category equipped with a faithful functor to Set); the faithful functor $\mathfrak{X} \to$ Set is called the underlying set functor. See [Frölicher and Kriegl 1988, Section 8.8] for the notions of an $\mathfrak{X}$–embedding, an $\mathfrak{X}$–subspace, an $\mathfrak{X}$–quotient map, and an

$\mathfrak{X}$–quotient space. $\mathcal{D}$–subspaces and $\mathcal{D}$–quotient spaces are usually called diffeological subspaces and diffeological quotient spaces, respectively.

(2) For Proposition 2.2(3), recall that the underlying topological space $\widetilde{A}$ of a diffeological space $A = (A, D_A)$ is defined to be the set A endowed with the final topology for D_A [Iglesias-Zemmour 2013, 2.8] and that R assigns to an arc-generated space X the set X endowed with the diffeology

$$D_{RX} = \{\text{continuous parametrizations in } X\}.$$

Then we can easily see that $\tilde{\cdot} \circ R = \mathrm{Id}_{\mathcal{C}^0}$ and that the unit $A \to R\widetilde{A}$ of the adjoint pair $(\tilde{\cdot}, R)$ is set-theoretically the identity map.

(3) The notion of an arc-generated space is equivalent to that of a Δ–generated space (see [Christensen et al. 2014; Kihara 2019, Section 2.2]). The categories $\mathcal{D}$ and $\mathcal{C}^0$ share convenient properties (1) and (2) in Proposition 2.2, which often enables us to deal with $\mathcal{D}$ and $\mathcal{C}^0$ simultaneously (see [Kihara 2023]). See [Kihara 2023, Remark 2.4] for the reason why $\mathcal{C}^0$ is the most suitable category as a target category of the underlying topological space functor for diffeological spaces.

2.2 Standard simplices Δ^p

In this subsection, we recall the standard simplices Δ^p $(p \geq 0)$, along with the adjoint pair $|\cdot|_{\mathcal{D}} : \mathcal{S} \rightleftarrows \mathcal{D} : S^{\mathcal{D}}$ of the realization and singular functors.

In [Kihara 2019], we introduced a model structure on the category $\mathcal{D}$. The principal part of our construction of a model structure on $\mathcal{D}$ is the construction of so-called good diffeologies on the sets

$$\Delta^p = \left\{(x_0, \ldots, x_p) \in \mathbb{R}^{p+1} \;\middle|\; \sum x_i = 1,\, x_i \geq 0 \text{ for any } i\right\} \quad (p \geq 0)$$

which enable us to define weak equivalences, fibrations, and cofibrations and to verify the model axioms (see Remark 2.8). The required properties of the diffeologies on Δ^p $(p \geq 0)$ are expressed in the following four axioms:

Axiom 1 *The underlying topological space of Δ^p is the topological standard p–simplex Δ^p_{top} for $p \geq 0$.*

Recall that $f : \Delta^p \to \Delta^q$ is an *affine map* if f preserves convex combinations.

Axiom 2 *Any affine map $f : \Delta^p \to \Delta^q$ is smooth.*

For $K \in \mathcal{S}$, the *simplex category* $\Delta \downarrow K$ is defined to be the full subcategory of the overcategory $\mathcal{S} \downarrow K$ consisting of maps $\sigma : \Delta[n] \to K$. By Axiom 2, we can consider the diagram $\Delta \downarrow K \to \mathcal{D}$ sending $\sigma : \Delta[n] \to K$ to Δ^n. Thus, we define the *realization functor*

$$|\cdot|_{\mathcal{D}} : \mathcal{S} \to \mathcal{D}$$

by $|K|_{\mathcal{D}} = \mathrm{colim}_{\Delta \downarrow K}\, \Delta^n$.

Consider the smooth map $|\dot{\Delta}[p]|_{\mathcal{D}} \hookrightarrow |\Delta[p]|_{\mathcal{D}} = \Delta^p$ induced by the inclusion of the boundary $\dot{\Delta}[p]$ into $\Delta[p]$.

Axiom 3 *The canonical smooth injection*

$$|\dot{\Delta}[p]|_{\mathfrak{D}} \hookrightarrow \Delta^p$$

is a $\mathfrak{D}$–embedding.

The $\mathfrak{D}$–homotopical notions, especially the notion of a $\mathfrak{D}$–deformation retract, are defined in the same manner as in the category of topological spaces by using the unit interval $I = [0,1]$ endowed with a diffeology via the canonical bijection with Δ^1 [Kihara 2019, Section 2.4]. The k^{th} *horn* of Δ^p is a diffeological subspace of Δ^p defined by

$$\Lambda^p_k = \{(x_0, \dots, x_p) \in \Delta^p \mid x_i = 0 \text{ for some } i \neq k\}.$$

Axiom 4 *The k^{th} horn Λ^p_k is a $\mathfrak{D}$–deformation retract of Δ^p for $p \geq 1$ and $0 \leq k \leq p$.*

For a subset A of the affine p–space $\mathbb{A}^p = \{(x_0, \dots, x_p) \in \mathbb{R}^{p+1} \mid \sum x_i = 1\}$, A_{sub} denotes the set A endowed with the subdiffeology of $\mathbb{A}^p$ (and hence of $\mathbb{R}^{p+1}$). The diffeological spaces Δ^p_{sub} $(p \geq 0)$ satisfy Axioms 1 and 2, but Δ^p_{sub} satisfies neither Axiom 3 nor 4 for $p \geq 2$ [Kihara 2019, Proposition A.2]. Thus, we must construct a new diffeology on Δ^p, at least for $p \geq 2$.

Let (i) denote the vertex $(0, \dots, 1_{(i)}, \dots, 0)$ of Δ^p, and let d^i denote the affine map from Δ^{p-1} to Δ^p, defined by

$$d^i((k)) = \begin{cases} (k) & \text{if } k < i, \\ (k+1) & \text{if } k \geq i. \end{cases}$$

Definition 2.4 We define the *standard p–simplices* Δ^p $(p \geq 0)$ inductively. Set $\Delta^p = \Delta^p_{\text{sub}}$ for $p \leq 1$. Suppose that the diffeologies on Δ^k $(k < p)$ are defined. We define the map

$$\varphi_i \colon \Delta^{p-1} \times [0,1) \to \Delta^p$$

by $\varphi_i(x,t) = (1-t)(i) + t d^i(x)$, and endow Δ^p with the final structure for the maps $\varphi_0, \dots, \varphi_p$.

The following result is established in [Kihara 2019, Propositions 3.2, 5.1, 7.1, and 8.1].

Proposition 2.5 *The standard p–simplices Δ^p $(p \geq 0)$ in Definition 2.4 satisfy Axioms 1–4.*

Without explicit mention, the symbol Δ^p denotes the standard p–simplex defined in Definition 2.4 and a subset of Δ^p is endowed with the subdiffeology of Δ^p. Since the diffeology of Δ^p is the subdiffeology of $\mathbb{A}^p$ for $p \leq 1$, the $\mathfrak{D}$–homotopical notions, especially the notion of a $\mathfrak{D}$–deformation retract, coincide with the ordinary smooth homotopical notions in the theory of diffeological spaces [Iglesias-Zemmour 2013, page 108; Kihara 2019, Remark 2.14].

Since $\Delta^\bullet = \{\Delta^p\}$ is a cosimplicial diffeological space by Axiom 2, the singular complex $S^{\mathfrak{D}}(X)$ is defined by

$$S^{\mathfrak{D}}(X) = \mathfrak{D}(\Delta^\bullet, X).$$

We can easily see that $|\cdot|_{\mathscr{D}} : \mathscr{S} \rightleftarrows \mathscr{D} : S^{\mathscr{D}}$ is an adjoint pair [Kihara 2019, Proposition 9.1]. Further, we can derive the following result from Proposition 2.5.

Corollary 2.6 (1) *The natural isomorphisms*

$$|\Delta[p]|_{\mathscr{D}} = \Delta^p, \quad |\dot{\Delta}[p]| = \dot{\Delta}^p \quad \text{and} \quad |\Lambda_k[p]|_{\mathscr{D}} = \Lambda^p_k$$

exist.

(2) *$S^{\mathscr{D}}X$ is a Kan complex for any diffeological space X.*

Proof (1) See [Kihara 2019, Proposition 9.2].

(2) See [Kihara 2019, Lemma 9.4(1)]. □

See [Christensen and Wu 2014, Section 3.1] or [Iglesias-Zemmour 2013, Chapter 5] for the smooth homotopy groups $\pi^{\mathscr{D}}_p(X,x)$ of a pointed diffeological space (X,x). Note that $S^{\mathscr{D}}X$ is always a Kan complex (Corollary 2.6(2)) and see [Goerss and Jardine 1999, page 25] for the homotopy groups $\pi_p(K,x)$ of a pointed Kan complex (K,x).

Theorem 2.7 *Let (X,x) be a pointed diffeological space. Then there exists a natural bijection*

$$\Theta_X : \pi^{\mathscr{D}}_p(X,x) \to \pi_p(S^{\mathscr{D}}X,x) \quad \text{for } p \geq 0,$$

that is an isomorphism of groups for $p > 0$.

Proof See [Kihara 2019, Theorem 1.4]. □

Remark 2.8 (1) Define a map $f : X \to Y$ in $\mathscr{D}$ to be

(i) a *weak equivalence* if $S^{\mathscr{D}}f : S^{\mathscr{D}}X \to S^{\mathscr{D}}Y$ is a weak equivalence in the category of simplicial sets,

(ii) a *fibration* if the map f has the right lifting property with respect to the inclusions $\Lambda^p_k \hookrightarrow \Delta^p$ for all $p > 0$ and $0 \leq k \leq p$, and

(iii) a *cofibration* if the map f has the left lifting property with respect to all maps that are both fibrations and weak equivalences.

Then $\mathscr{D}$ is a compactly generated model category whose object is always fibrant. In fact, the sets of morphisms of $\mathscr{D}$,

$$\mathscr{I} = \{\dot{\Delta}^p \hookrightarrow \Delta^p \mid p \geq 0\},$$
$$\mathscr{J} = \{\Lambda^p_k \hookrightarrow \Delta^p \mid p > 0,\ 0 \leq k \leq p\},$$

are the sets of generating cofibrations and generating trivial cofibrations, respectively [Kihara 2019, Theorem 1.3]. See [May and Ponto 2012, Definition 15.2.1] for a compactly generated model category.

By Theorem 2.7, weak equivalences in $\mathscr{D}$ are just smooth maps inducing isomorphisms on smooth homotopy groups.

(2) The adjoint pairs

$$|\cdot|_{\mathscr{D}} : \mathscr{S} \rightleftarrows \mathscr{D} : S^{\mathscr{D}} \quad \text{and} \quad \tilde{\cdot} : \mathscr{D} \rightleftarrows \mathscr{C}^0 : R$$

are pairs of Quillen equivalences [Kihara 2023, Theorem 1.5]. Note that the composite of these adjoint pairs is just the adjoint pair

$$|\cdot| : \mathscr{S} \rightleftarrows \mathscr{C}^0 : S$$

of the topological realization and singular functors.

2.3 Homotopy type of $S^{\mathscr{D}}(X)$

In this subsection, we recall from [Kihara 2023] the basic results on the homotopy type of $S^{\mathscr{D}}(X)$; they are not essential in the later sections, but they are related to a few results in Section 6.

For a diffeological space X, consider the unit id: $X \to R\widetilde{X}$ of the adjoint pair $\tilde{\cdot} : \mathscr{D} \rightleftarrows \mathscr{C}^0 : R$. By applying $S^{\mathscr{D}} (= \mathscr{D}(\Delta^{\bullet}, \cdot))$, we have the natural inclusion

$$S^{\mathscr{D}}(X) \hookrightarrow S(\widetilde{X})$$

(see Proposition 2.5, in particular Axiom 1).

If X is a nice diffeological space such as a cofibrant object or a C^{∞}–manifold in the sense of [Kriegl and Michor 1997, Section 27], then $S^{\mathscr{D}}(X) \hookrightarrow S(\widetilde{X})$ is a weak equivalence [Kihara 2023, Corollary 1.6, Proposition 2.6, and Theorem 11.2]. Hence, we can calculate the homotopy groups and the (co)homology groups of such nice diffeological spaces as those of the underlying topological spaces.

Conversely, if X is an exceptional diffeological space such as an irrational torus, then $S^{\mathscr{D}}(X) \hookrightarrow S(\widetilde{X})$ is not a weak equivalence; see [Kihara 2023, Appendix A]. See Section 6.2 for an approach to the homotopy type of $S^{\mathscr{D}}(X)$ of exceptional diffeological spaces X such as irrational tori and $\mathbb{R}/\mathbb{Q}$.

Remark 2.9 The (co)homology and homotopy groups of diffeological spaces have the same desirable properties as those of topological spaces. Further, the (co)homology and homotopy groups of a diffeological space are just those of its singular complex. Thus, we can apply various algebraic topological and simplicial homotopical tools to the calculation of the (co)homology and homotopy groups of a diffeological space X whether or not X is a nice diffeological space; see [Kihara 2023, Section 3.1], Theorem 2.7, and Remark 5.8.

3 Smooth singular complexes

In this section, we summarize the basic notions and results on the smooth singular complexes $S^{\mathscr{D}}_{\text{sub}}(X)$ and $S^{\mathscr{D}}_{\text{aff}}(X)$ (Section 3.1), and then show that there exist natural morphisms between $S^{\mathscr{D}}_{\text{aff}}(X)$, $S^{\mathscr{D}}_{\text{sub}}(X)$, and $S^{\mathcal{D}}(X)$ which induce chain homotopy equivalences, and hence isomorphisms on (co)homology (Section 3.2). We also show that the singular functors $S^{\mathcal{D}}_{\mathrm{aff}}$, $S^{\mathcal{D}}_{\mathrm{sub}}$, and $S^{\mathcal{D}}$ transform diffeological coverings to simplicial coverings (Section 3.3); this result is used to reduce the proof of Theorem 1.1.

3.1 Smooth singular complexes $S^{\mathcal{D}}(X)$, $S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $S^{\mathcal{D}}_{\mathrm{aff}}(X)$

By using the cosimplicial diffeological space $\Delta^\bullet = \{\Delta^p\}$, the singular complex $S^{\mathcal{D}}(X)$ is defined by

$$S^{\mathcal{D}}(X) = \mathcal{D}(\Delta^\bullet, X),$$

which is intensively studied in [Kihara 2019; 2023] (see Section 2.2).

Let $\mathbb{A}^p$ denote the affine p–space $\{(x_0, \dots, x_p) \in \mathbb{R}^{p+1} \mid \sum x_i = 1\}$ endowed with the subdiffeology of $\mathbb{R}^{p+1}$. Since $\mathbb{A}^\bullet = \{\mathbb{A}^p\}$ is a cosimplicial diffeological space, the singular complex $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ is defined by

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) = \mathcal{D}(\mathbb{A}^\bullet, X).$$

The singular complex $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ was introduced by Christensen and Wu [2014]; they used the singular functor $S^{\mathcal{D}}_{\mathrm{aff}}$ to define the classes of weak equivalences, fibrations, and cofibrations in $\mathcal{D}$, but the model axioms are not yet verified.

Let Δ^p_{sub} denote the set Δ^p endowed with the subdiffeology of $\mathbb{A}^p$. Since $\Delta^\bullet_{\mathrm{sub}} = \{\Delta^p_{\mathrm{sub}}\}$ is a cosimplicial diffeological space, the singular complex $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ is defined by

$$S^{\mathcal{D}}_{\mathrm{sub}}(X) = \mathcal{D}(\Delta^\bullet_{\mathrm{sub}}, X).$$

The singular complex $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ was used by Hector [1995] to study diffeological spaces by homotopical means such as singular (co)homology.

Now, we summarize the basic properties of Δ^p, Δ^p_{sub}, and $\mathbb{A}^p$, and the relations among them, which are needed later. A subset A of $\mathbb{A}^p$ endowed with the subdiffeology of $\mathbb{A}^p$ is denoted by A_{sub}. The notion of $\mathcal{D}$–contractibility (or smooth contractibility) is defined in the obvious manner (a $\mathcal{D}$–contractible diffeological space is often called simply a contractible diffeological space if there is no confusion in context).

Lemma 3.1 (1) *The diffeological spaces Δ^p, Δ^p_{sub}, and $\mathbb{A}^p$ are smoothly contractible.*

(2) *The underlying topological space of Δ^p and Δ^p_{sub} is just the standard topological p–simplex. The underlying topological space of $\mathbb{A}^p$ is just the set $\mathbb{A}^p$ endowed with the usual topology.*

(3) *The map $\mathrm{id}\colon \Delta^p \to \Delta^p_{\mathrm{sub}}$ is smooth, which restricts to the diffeomorphism*

$$\mathrm{id}\colon \Delta^p - \mathrm{sk}_{p-2}\,\Delta^p \xrightarrow[\cong]{} (\Delta^p - \mathrm{sk}_{p-2}\,\Delta^p)_{\mathrm{sub}},$$

where $\mathrm{sk}_{p-2}\,\Delta^p$ denotes the $(p-2)$–skeleton of Δ^p.

Proof (1) The smooth contractibility of Δ^p_{sub} and $\mathbb{A}^p$ are obvious. See [Kihara 2019, Remark 9.3] for the smooth contractibility of Δ^p.

(2) The result for Δ^p follows from Proposition 2.5. The results for Δ^p_{sub} and $\mathbb{A}^p$ follow from [Kihara 2019, Lemma 2.12].

(3) See [Kihara 2019, Lemmas 3.1 and 4.2]. □

Remark 3.2 In this remark, we recall the left adjoints of $S^{\mathcal{D}}_{\mathrm{sub}}$ and $S^{\mathcal{D}}_{\mathrm{aff}}$, and see that $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ and $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ need not be Kan.

(1) As mentioned above, the realization functor $|\cdot|_{\mathcal{D}}\colon \mathcal{S} \to \mathcal{D}$ is a left adjoint of the singular functor $S^{\mathcal{D}}\colon \mathcal{D} \to \mathcal{S}$, and the composite of the adjoint pairs $|\cdot|_{\mathcal{D}}\colon \mathcal{S} \rightleftarrows \mathcal{D} : S^{\mathcal{D}}$ and $\tilde{\cdot}\colon \mathcal{D} \rightleftarrows \mathcal{C}^0 : R$ is just the adjoint pair $|\cdot|\colon \mathcal{S} \rightleftarrows \mathcal{C}^0 : S$ (see Remark 2.8(2)).

Similarly, we can define the realization functor $|\cdot|'_{\mathcal{D}}\colon \mathcal{S} \to \mathcal{D}$ by

$$|K|'_{\mathcal{D}} = \operatorname*{colim}_{\Delta\downarrow K} \Delta^n_{\mathrm{sub}},$$

which is a left adjoint of the singular functor $S^{\mathcal{D}}_{\mathrm{sub}}\colon \mathcal{D} \to \mathcal{S}$. The composite of the adjoint pairs $|\cdot|'_{\mathcal{D}}\colon \mathcal{S} \rightleftarrows \mathcal{D} : S^{\mathcal{D}}_{\mathrm{sub}}$ and $\tilde{\cdot}\colon \mathcal{D} \rightleftarrows \mathcal{C}^0 : R$ is also just the adjoint pair $|\cdot|\colon \mathcal{S} \rightleftarrows \mathcal{C}^0 : S$ (see Lemma 3.1(2)).

The realizations $|K|_{\mathcal{D}}$ and $|K|'_{\mathcal{D}}$ of a simplicial complex K viewed as a simplicial set [May 1992, Example 1.4] are just the diffeological polyhedra $|K|_{\mathcal{D}}$ and $|K|'_{\mathcal{D}}$ respectively [Kihara 2023, Section 8.1]; they played an essential role in the proof of the homotopy cofibrancy theorem [Kihara 2023, Theorem 1.10].

Christensen and Wu [2014] defined the realization functor $|\cdot|''_{\mathcal{D}}\colon \mathcal{S} \to \mathcal{D}$ by

$$|K|''_{\mathcal{D}} = \operatorname*{colim}_{\Delta\downarrow K} \mathbb{A}^n,$$

which is a left adjoint of the singular functor $S^{\mathcal{D}}_{\mathrm{aff}}\colon \mathcal{D} \to \mathcal{S}$.

(2) Let us see that $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ need not be Kan. For this, we consider the extension problem in $\mathcal{S}$

$$\begin{tikzcd} \Lambda_0[2] \arrow[r, "d^1+d^2"] \arrow[d, hook] & S^{\mathcal{D}}_{\mathrm{sub}}(\Lambda^2_{0\,\mathrm{sub}}) \\ \Delta[2] \arrow[ur, dashed] & \end{tikzcd}$$

where $\Lambda_0[2] \xrightarrow{d^1+d^2} S^{\mathcal{D}}_{\mathrm{sub}}(\Lambda^2_{0\,\mathrm{sub}})$ is the simplicial map whose restriction to the i^{th} face corresponds to (the corestriction of) $d^i\colon \Delta^1 \to \Delta^2$ for $i = 1, 2$. Suppose that this extension problem has a solution r. Then we have the commutative diagram in $\mathcal{D}$

$$\begin{tikzcd} |\Lambda_0[2]|'_{\mathcal{D}} \arrow[r, "d^1+d^2"] \arrow[d, hook] & \Lambda^2_{0\,\mathrm{sub}} \\ \Delta^2_{\mathrm{sub}} \arrow[ur, "r"'] & \end{tikzcd}$$

(see part (1)). Noticing that $|\Lambda_0[2]|'_{\mathfrak{D}}$ can be set-theoretically identified with Λ^2_0, we see that r is a $\mathfrak{D}$–retraction of Δ^2_{sub} onto $\Lambda^2_{0\,\mathrm{sub}}$, which is a contradiction [Kihara 2019, Proposition A.2(2)]; see also [Kihara 2023, Remark 8.2].

Similarly, we can use [Bröcker and Jänich 1982, Theorem 5.13] to see that $S^{\mathfrak{D}}_{\mathrm{aff}}((d^1\mathbb{A}^1 \cup d^2\mathbb{A}^1)_{\mathrm{sub}})$ is not Kan; however, it has already been shown that $S^{\mathfrak{D}}_{\mathrm{aff}}(X)$ need not be Kan [Christensen and Wu 2014, Section 4.3].

3.2 Natural transformations between $S^{\mathfrak{D}}_{\mathrm{aff}}$, $S^{\mathfrak{D}}_{\mathrm{sub}}$, and $S^{\mathfrak{D}}$

In this subsection, we construct natural morphisms between $S^{\mathfrak{D}}_{\mathrm{aff}}(X)$, $S^{\mathfrak{D}}_{\mathrm{sub}}(X)$, and $S^{\mathfrak{D}}(X)$, and show that they induce chain homotopy equivalences between $\mathbb{Z}S^{\mathfrak{D}}_{\mathrm{aff}}(X)$, $\mathbb{Z}S^{\mathfrak{D}}_{\mathrm{sub}}(X)$, and $\mathbb{Z}S^{\mathfrak{D}}(X)$, and hence isomorphisms on the (co)homology with arbitrary coefficients.

First, we show that the singular functors $S^{\mathfrak{D}}$, $S^{\mathfrak{D}}_{\mathrm{sub}}$, and $S^{\mathfrak{D}}_{\mathrm{aff}}$ preserve homotopy. Recall the $\mathfrak{D}$–homotopical notions from Section 2.2 and let $\simeq_{\mathfrak{D}}$ denote the $\mathfrak{D}$–homotopy relation.

Lemma 3.3 *For smooth maps $f, g\colon X \to Y$, consider the conditions*

(i) $f \simeq_{\mathfrak{D}} g\colon X \to Y$,

(ii) $S^{\mathfrak{D}} f \simeq S^{\mathfrak{D}} g\colon S^{\mathfrak{D}}(X) \to S^{\mathfrak{D}}(Y)$,

(iii) $H_*(f;\mathbb{Z}) = H_*(g;\mathbb{Z})\colon H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$.

The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) *hold. The same conclusion applies to the functors $S^{\mathfrak{D}}_{\mathrm{sub}}$ and $S^{\mathfrak{D}}_{\mathrm{aff}}$, and their homologies.*

Proof For $S^{\mathfrak{D}}$: see [Kihara 2019, Lemma 9.4(2)] for (i) $\Longrightarrow$ (ii), and [May 1992, pages 12–13] for (ii) $\Longrightarrow$ (iii).

For $S^{\mathfrak{D}}_{\mathrm{sub}}$: recall that $\Delta^1 = \Delta^1_{\mathrm{sub}}$; then a similar argument applies.

For $S^{\mathfrak{D}}_{\mathrm{aff}}$: observe that $f \simeq_{\mathfrak{D}} g$ if and only if there exists a smooth map $H\colon X \times \mathbb{A}^1 \to Y$ such that $H(\cdot,(0)) = f$ and $H(\cdot,(1)) = g$; then a similar argument applies. □

Using Lemmas 3.1 and 3.3, we can prove the following result.

Proposition 3.4 *There exist natural morphisms of simplicial sets*

$$S^{\mathfrak{D}}_{\mathrm{aff}}(X) \to S^{\mathfrak{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathfrak{D}}(X)$$

which induce chain homotopy equivalences

$$\mathbb{Z}S^{\mathfrak{D}}_{\mathrm{aff}}(X) \to \mathbb{Z}S^{\mathfrak{D}}_{\mathrm{sub}}(X) \to \mathbb{Z}S^{\mathfrak{D}}(X).$$

Proof We prove the result in three steps.

Step 1: construction of natural morphisms By Lemma 3.1(3), we have the canonical morphisms of cosimplicial diffeological spaces

$$\Delta^\bullet \xrightarrow{\mathrm{id}} \Delta^\bullet_{\mathrm{sub}} \hookrightarrow \mathbb{A}^\bullet,$$

which induce natural morphisms

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) \xrightarrow{\kappa} S^{\mathcal{D}}_{\mathrm{sub}}(X) \xhookrightarrow{\iota} S^{\mathcal{D}}(X).$$

(Note that the first and second morphisms of cosimplicial diffeological spaces induce the second and first morphisms of singular complexes, respectively.)

Step 2 We show that for $p \geq 0$,

$$H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(\Delta^p)) \cong H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(\Delta^p)) \cong H_*(\mathbb{Z}S^{\mathcal{D}}(\Delta^p)) \cong \mathbb{Z}[0],$$

where $\mathbb{Z}[0]$ denotes the graded module with $\mathbb{Z}[0]_0 = \mathbb{Z}$ and $\mathbb{Z}[0]_i = 0$ ($i \neq 0$). It is easily seen that these isomorphisms hold for $p = 0$. Thus, they hold for any $p \geq 0$ by Lemmas 3.3 and 3.1(1).

Step 3 To prove the rest of the statement, we "augment" the singular chain complexes $\mathbb{Z}S^{\mathcal{D}}(X)$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)$ in a canonical manner (see [Eilenberg and Mac Lane 1953, page 194]); the augmented singular chain complexes are denoted by $\mathbb{Z}S^{\mathcal{D}}(X)\tilde{}$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)\tilde{}$, and $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)\tilde{}$. Then

$$H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(\Delta^p)\tilde{}) = H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(\Delta^p)\tilde{}) = H_*(\mathbb{Z}S^{\mathcal{D}}(\Delta^p)\tilde{}) = 0$$

(by Step 2). Since each component of degree ≥ 0 of $\mathbb{Z}S^{\mathcal{D}}(X)\tilde{}$ (resp. $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)\tilde{}$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)\tilde{}$) is representable for the set of model objects $\{\Delta^p\}_{p\geq 0}$ (resp. $\{\Delta^p_{\mathrm{sub}}\}_{p\geq 0}$, $\{\mathbb{A}^p\}_{p\geq 0}$) in the sense of [Eilenberg and Mac Lane 1953, page 189], we can use [Eilenberg and Mac Lane 1953, Theorem II] to construct chain homotopy inverses of the augmented natural chain maps

$$\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)\tilde{} \xrightarrow{\mathbb{Z}\kappa\tilde{}} \mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)\tilde{} \xrightarrow{\mathbb{Z}\iota\tilde{}} \mathbb{Z}S^{\mathcal{D}}(X)\tilde{}$$

such that they restrict to chain homotopy inverses of the natural chain maps

$$\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X) \xrightarrow{\mathbb{Z}\kappa} \mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X) \xrightarrow{\mathbb{Z}\iota} \mathbb{Z}S^{\mathcal{D}}(X)$$

(see Step 1). □

Recall the definitions of $H_*(X; A)$ and $H^*(X; A)$ from Section 1.

Corollary 3.5 *Let A be an abelian group.*

(1) *The natural morphisms of simplicial sets*

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) \xrightarrow{\kappa} S^{\mathcal{D}}_{\mathrm{sub}}(X) \xhookrightarrow{\iota} S^{\mathcal{D}}(X)$$

induce isomorphisms of graded modules

$$H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X) \otimes A) \xrightarrow[\cong]{\kappa_*} H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X) \otimes A) \xrightarrow[\cong]{\iota_*} H_*(X; A),$$
$$H^*\big(\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X), A)\big) \xleftarrow[\cong]{\kappa^*} H^*\big(\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X), A)\big) \xleftarrow[\cong]{\iota^*} H^*(X; A).$$

(2) *If A is a commutative associative ring with unit, then $H^*(X;A)$, $H^*\big(\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X),A)\big)$, and $H^*\big(\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X),A)\big)$ have natural commutative graded A–algebra structures and the isomorphisms between them are isomorphisms of graded A–algebras.*

Proof (1) The result is immediate from Proposition 3.4.

(2) See [Kihara 2023, Remark 3.8(2)] for $H^*(X;A)$. The argument there can also be applied to $H^*\,\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X),A)$ and $H^*\,\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X),A)$. Since the cohomology isomorphisms in part (1) are induced by the natural simplicial maps

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathcal{D}}(X),$$

they are isomorphisms of graded A–algebras. □

Remark 3.6 In the study of differential forms and de Rham cohomology of diffeological spaces, Iglesias-Zemmour [2013, pages 182–183] introduced the complex $\mathsf{C}_\star(X)$ of reduced groups of cubic chains for a diffeological space X, and called its homology $\mathsf{H}_*(X)$ the cubic homology of X.

We can easily see that $\mathsf{H}_*(X)$ is a smooth homotopy invariant. In fact, given a smooth homotopy $H:\mathbb{R}\times X\to Y$ between f and g, a chain homotopy $H_\sharp:\mathsf{C}_*(X)\to\mathsf{C}_{*+1}(Y)$ between $\mathsf{C}_*(f)$ and $\mathsf{C}_*(g)$ is defined by

$$\mathbb{R}^p \xrightarrow{\sigma} X \mapsto \mathbb{R}^{p+1} = \mathbb{R}\times\mathbb{R}^p \xrightarrow{1\times\sigma} \mathbb{R}\times X \xrightarrow{H} Y.$$

Thus, by an argument similar to that in the proof of Proposition 3.4, we can use [Eilenberg and Mac Lane 1953, Theorem II] to construct a natural chain homotopy equivalence between $\mathsf{C}_*(X)$ and $\mathbb{Z}S^{\mathcal{D}}(X)$, showing that $\mathsf{H}_*(X)$ is naturally isomorphic to $H_*(X)$.

The basic idea of the proof that $\mathbb{Z}S^{\mathcal{D}}(X)$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $\mathsf{C}_*(X)$ are chain homotopy equivalent was briefly discussed in [Kihara 2023, Remark 3.9]. It is also shown in [Kuribayashi 2020, Section 4.1] that $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $\mathsf{C}_*(X)$ are chain homotopy equivalent.

3.3 Diffeological coverings

The notion of a diffeological fiber bundle is a generalization of that of a locally trivial fiber bundle, and is defined by local triviality of the pullback along any plot; see [Iglesias-Zemmour 2013, 8.9]. A diffeological fiber bundle with discrete fiber is called a *diffeological covering*.

Similarly, a simplicial fiber bundle is defined by triviality of the pullback along any map from $\Delta[p]$ $(p\ge 0)$; see [May 1992, Definition 11.8]. A simplicial fiber bundle with discrete fiber is called a *simplicial covering*.

We prove the following result, which is used in the proof of Theorem 1.1.

Proposition 3.7 *The singular functors $S^{\mathscr{D}}$, $S^{\mathscr{D}}_{\mathrm{sub}}$, and $S^{\mathscr{D}}_{\mathrm{aff}}$ transform diffeological coverings with fiber F to simplicial coverings with fiber F. Hence, a diffeological covering $\pi : E \to X$ with fiber F defines the natural morphisms of simplicial coverings with fiber F*

$$\begin{tikzcd} S^{\mathscr{D}}_{\mathrm{aff}}(E) \arrow[r] \arrow[d, "S^{\mathscr{D}}_{\mathrm{aff}}(\pi)"'] & S^{\mathscr{D}}_{\mathrm{sub}}(E) \arrow[r, hook] \arrow[d, "S^{\mathscr{D}}_{\mathrm{sub}}(\pi)"'] & S^{\mathscr{D}}(E) \arrow[d, "S^{\mathscr{D}}(\pi)"'] \\ S^{\mathscr{D}}_{\mathrm{aff}}(X) \arrow[r] & S^{\mathscr{D}}_{\mathrm{sub}}(X) \arrow[r, hook] & S^{\mathscr{D}}(X). \end{tikzcd}$$

Proof We prove the result in three steps.

Step 1 We show that $S^{\mathscr{D}}(\pi) \colon S^{\mathscr{D}}(E) \to S^{\mathscr{D}}(X)$ is a simplicial covering with fiber F.

Assume given a map $k \colon \Delta[p] \to S^{\mathscr{D}}(X)$ and let $\kappa \colon \Delta^p \to X$ be the smooth map corresponding to k. Noticing that Δ^p is smoothly contractible (Lemma 3.1(1)), we then have a pullback diagram in $\mathscr{D}$

$$\begin{tikzcd} \Delta^p \times F \arrow[r] \arrow[d, "\mathrm{proj}"'] & E \arrow[d, "\pi"] \\ \Delta^p \arrow[r, "\kappa"] & X \end{tikzcd}$$

(see [Iglesias-Zemmour 2013, page 264]). Note that $S^{\mathscr{D}}$ is a right adjoint and consider the commutative diagram in $\mathscr{S}$ consisting of two pullback squares

$$\begin{tikzcd} \Delta[p] \times S^{\mathscr{D}}(F) \arrow[r] \arrow[d, "\mathrm{proj}"'] & S^{\mathscr{D}}(\Delta^p) \times S^{\mathscr{D}}(F) \arrow[r] \arrow[d, "\mathrm{proj}"'] & S^{\mathscr{D}}(E) \arrow[d, "S^{\mathscr{D}}(\pi)"'] \\ \Delta[p] \arrow[r] & S^{\mathscr{D}}(\Delta^p) \arrow[r, "S^{\mathscr{D}}(\kappa)"] & S^{\mathscr{D}}(X) \end{tikzcd}$$

where $\Delta[p] \to S^{\mathscr{D}}(\Delta^p)$ is the map corresponding to the p–simplex 1_{Δ^p} of $S^{\mathscr{D}}(\Delta^p)$. Then the outer rectangle gives the desired local triviality of $S^{\mathscr{D}}(\pi)$; see [Mac Lane 1998, Exercise 8 on page 72].

Step 2 Note that Δ^p_{sub} and $\mathbb{A}^p$ are smoothly contractible (Lemma 3.1(1)) and that $S^{\mathscr{D}}_{\mathrm{sub}}$ and $S^{\mathscr{D}}_{\mathrm{aff}}$ are right adjoints (Remark 3.2(1)). Then, by an argument similar to that in Step 1, we can see that $S^{\mathscr{D}}_{\mathrm{sub}}(\pi)$ and $S^{\mathscr{D}}_{\mathrm{aff}}(\pi)$ are also simplicial coverings with fiber F.

Step 3 The natural morphisms of simplicial coverings are defined by Proposition 3.4. □

4 Weak equivalences between smooth singular complexes

In this section, we prove Theorem 1.1 and Corollary 1.2, using results of Section 3.

The main statement of Theorem 1.1 is divided into the following two parts:

(I) The natural map $S^{\mathscr{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathscr{D}}(X)$ is a weak equivalence in $\mathscr{S}$.

(II) The natural map $S^{\mathscr{D}}_{\mathrm{aff}}(X) \to S^{\mathscr{D}}(X)$ is a weak equivalence in $\mathscr{S}$.

After constructing a fibrant approximation functor for the category of simplicial sets in Section 4.1, we prove parts (I) and (II) in Sections 4.2 and 4.3, respectively. We complete the proofs of Theorem 1.1 and Corollary 1.2 in Section 4.4.

4.1 Fibrant approximation to a simplicial set

The category $\mathcal{S}$ of simplicial sets is a cofibrantly generated model category having

$$\mathcal{J}_{\mathcal{S}} = \{\Lambda_k[p] \hookrightarrow \Delta[p] \mid p > 0, 0 \le k \le p\}$$

as a set of generating trivial cofibrations. Applying the infinite gluing construction [Dwyer and Spaliński 1995, pages 104–105] for $\mathcal{J}_S$ to a simplicial map $\varphi\colon K \to L$, we obtain the factorization

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[dr, "\varphi"'] & K' \arrow[d, "p"] \\ & L \end{tikzcd}$$

where i is a trivial cofibration and p is a fibration. However, since every simplicial map to the terminal object $*$ has a right lifting property for $\Lambda_k[1] \hookrightarrow \Delta[1]$ $(k = 0, 1)$, we can construct a fibrant approximation $K\hat{}$ of K by applying the infinite gluing construction for

$$\mathcal{J}'_{\mathcal{S}} = \{\Lambda_k[p] \hookrightarrow \Delta[p] \mid p > 1, 0 \le k \le p\}$$

to $K \to *$. Let $\mathcal{S}_f$ denote the full subcategory of $\mathcal{S}$ consisting of fibrant objects (ie Kan complexes). Then the functor $\cdot\hat{}\colon \mathcal{S} \to \mathcal{S}_f$ is a fibrant approximation functor, for which $K\hat{}_0 = K_0$ holds. An attachment of $\Delta[2]$ along $\Lambda_k[2]$ adds one nondegenerate 2–simplex and one nondegenerate 1–simplex, which correspond to the basic 2–simplex of $\Delta[2]$ and its k^{th} face respectively.

4.2 Proof of part (I)

We prove part (I) of Theorem 1.1 (see the introduction of this section). Let us begin by reducing the proof to simpler cases. First, consider the decomposition $X = \coprod X_\alpha$ into connected components; see [Iglesias-Zemmour 2013, pages 105–107]. Since

$$S^{\mathcal{D}}_{\text{sub}}(X) = \coprod S^{\mathcal{D}}_{\text{sub}}(X_\alpha) \quad \text{and} \quad S^{\mathcal{D}}(X) = \coprod S^{\mathcal{D}}(X_\alpha),$$

we may assume that X is connected.

Next consider the universal covering $\varpi : Z \to X$; see [Iglesias-Zemmour 2013, page 264]. By Proposition 3.7, we then have the morphism of simplicial coverings with fiber $\pi_1^{\mathcal{D}}(X)$

$$\begin{tikzcd} \pi_1^{\mathcal{D}}(X) \arrow[r, equal] \arrow[d] & \pi_1^{\mathcal{D}}(X) \arrow[d] \\ S^{\mathcal{D}}_{\text{sub}}(Z) \arrow[r, hook] \arrow[d] & S^{\mathcal{D}}(Z) \arrow[d] \\ S^{\mathcal{D}}_{\text{sub}}(X) \arrow[r, hook] & S^{\mathcal{D}}(X) \end{tikzcd}$$

Hence, we may assume that X is 1–connected (note that $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ need not be a Kan complex and use [Gabriel and Zisman 1967, Chapter III, Theorem 4.2]).

Since $S^{\mathcal{D}}(X)$ is Kan (Corollary 2.6(2)), the inclusion $S^{\mathcal{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathcal{D}}(X)$ extends to a map

$$S^{\mathcal{D}}_{\mathrm{sub}}(X)\hat{} \to S^{\mathcal{D}}(X),$$

which induces an isomorphism on the homology (Corollary 3.5). Thus, we have only to show that

$$\pi_1(S^{\mathcal{D}}_{\mathrm{sub}}(X)\hat{}, x_0) = 0$$

for any fixed $x_0 \in X$ (see Theorem 2.7 and [May 1992, Theorem 13.9]).

Recall from [Goerss and Jardine 1999, page 8; May 1992, Lemma 16.3] the following facts concerning the topological realization functor $|\cdot|\colon \mathcal{S} \to \mathcal{C}^0$:

- The topological realization $|K|$ of a simplicial set K is a CW–complex having one n–cell for each nondegenerate n–simplex of K.
- For a pointed Kan complex (K, k_0), the simplicial fundamental group $\pi_1(K, k_0)$ is naturally isomorphic to the topological fundamental group $\pi_1(|K|, k_0)$.

For a simplicial set K, NK_n denotes the set of nondegenerate n–simplices of K. The n–cell of $|K|$ corresponding to $\sigma \in NK_n$ is also denoted by σ. The 1–cell σ of $|K|$ is endowed with the canonical orientation; the 1–cell σ endowed with the reverse orientation is denoted by $\bar{\sigma}$. We also use the standard notation $\mathrm{sk}_n\, K$ for the n–skeleton of K.

From these facts and the construction of the fibrant approximation $K\hat{}$ of K, we see the following:

- $\pi_1(S^{\mathcal{D}}_{\mathrm{sub}}(X)\hat{}, x_0) \cong \pi_1(|\mathrm{sk}_2\, S^{\mathcal{D}}_{\mathrm{sub}}(X)\hat{}|, x_0)$.
- Every element of $\pi_1(|\mathrm{sk}_2\, S^{\mathcal{D}}_{\mathrm{sub}}(X)\hat{}|, x_0)$ can be represented by a continuous map

  $$\omega\colon (\Delta^1_{\mathrm{top}}, \dot{\Delta}^1_{\mathrm{top}}) \to (|\mathrm{sk}_1\, S^{\mathcal{D}}_{\mathrm{sub}}(X)\hat{}|, x_0).$$

  Further, ω can be chosen as the concatenation of finitely many 1–cells $\tau_1, \dots, \tau_l$, where $\tau_j = \sigma_j$ or $\bar{\sigma}_j$ for some $\sigma_j \in NS^{\mathcal{D}}_{\mathrm{sub}}(X)_1$.

We would like to simplify the expression $\tau_1 \cdots \tau_l$ for ω and show that ω is null homotopic rel $\dot{\Delta}^1_{\mathrm{top}}$.

A smooth 1–simplex $\sigma\colon \Delta^1_{\mathrm{sub}} \to X$ of a diffeological space X is called *tame* if σ is constant near each vertex. By the following lemma, we may assume that each σ_j is tame.

Lemma 4.1 *Let X be a diffeological space and σ a 1–simplex of $S^{\mathcal{D}}_{\mathrm{sub}}(X)$. Then there exists a 2–simplex Σ of $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ such that $d_0\Sigma$ is the constant map to $\sigma((1))$, $d_1\Sigma$ is tame, and $d_2\Sigma = \sigma$.*

Proof We choose a nondecreasing smooth function $\mu\colon [0,1] \to [0,1]$ such that $\mu \equiv 0$ near 0 and $\mu \equiv 1$ near 1, and construct the desired 2–simplex Σ of $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ in two steps.

Step 1: construction of $F:\Delta^2_{\mathrm{sub}}\to\Delta^2_{\mathrm{sub}}$ We construct a smooth map $F:\Delta^2_{\mathrm{sub}}\to\Delta^2_{\mathrm{sub}}$ (ie a 2–simplex F of $S^{\mathfrak{D}}_{\mathrm{sub}}(\Delta^2_{\mathrm{sub}})$) satisfying the following condition:

- Each d_iF corestricts to the i^{th} face of Δ^2_{sub} and the corestriction of d_iF is identified with
$$\begin{cases}\mathrm{id} & \text{if } i=0,2,\\ \mu & \text{if } i=1,\end{cases}$$
in a canonical manner.

Set $U=\{(x_0,x_1,x_2)\in\Delta^2_{\mathrm{sub}}\mid 0\le x_1<\frac12\}$. Choose a nonincreasing smooth function $\phi:[0,\frac12]\to[0,1]$ such that $\phi\equiv1$ near 0 and $\phi\equiv0$ near $\frac12$. Under the identification

$$U\xrightarrow{\cong}[0,1]\times\left[0,\tfrac12\right),\quad (x_0,x_1,x_2)\mapsto\left(\frac{x_2}{1-x_1},x_1\right),$$

define the self-map $U\xrightarrow{f}U$ by

$$f(x,y)=\big(\phi(y)\mu(x)+(1-\phi(y))x,y\big).$$

Then the desired map $\Delta^2_{\mathrm{sub}}\xrightarrow{F}\Delta^2_{\mathrm{sub}}$ is defined by

$$F=\begin{cases}f & \text{on } U,\\ \mathrm{id} & \text{outside } U.\end{cases}$$

Step 2: construction of $\Sigma:\Delta^2_{\mathrm{sub}}\to X$ The desired 2–simplex Σ of $S^{\mathfrak{D}}_{\mathrm{sub}}(X)$ is defined to be the composite

$$\Delta^2_{\mathrm{sub}}\xrightarrow{F}\Delta^2_{\mathrm{sub}}\xrightarrow{s^1}\Delta^1_{\mathrm{sub}}\xrightarrow{\sigma}X,$$

where s^1 is defined by $s^1(x_0,x_1,x_2)=(x_0,x_1+x_2)$. □

Second, let us see that ω can be chosen as the concatenation of $\sigma_1,\dots,\sigma_l$ for some $\sigma_1,\dots,\sigma_l\in NS^{\mathfrak{D}}_{\mathrm{sub}}(X)_1$. For this, consider $\Sigma_j\in S^{\mathfrak{D}}_{\mathrm{sub}}(X)_2$ defined to be the composite

$$\Delta^2_{\mathrm{sub}}\xrightarrow{s}\Delta^1_{\mathrm{sub}}\xrightarrow{\sigma_j}X,$$

where $s(x_0,x_1,x_2)=(x_0+x_2,x_1)$. Then $d_2\Sigma_j=\sigma_j$, $d_1\Sigma_j$ is constant, and $\sigma'_j:=d_0\Sigma_j$ satisfies $\sigma'_j(t)=\sigma_j(1-t)$. Thus, if $\tau_j=\bar{\sigma}_j$, then we can replace τ_j with σ'_j. Hence, we may assume that ω is the concatenation of $\sigma_1,\dots,\sigma_l$.

Third, let us see that ω can be chosen as the continuous map corresponding to a single tame 1–simplex σ of $S^{\mathfrak{D}}_{\mathrm{sub}}(X)$. For this, we first consider the extension problem in $\mathfrak{D}$

$$\begin{tikzcd}\Lambda^2_{1\,\mathrm{sub}} \arrow[r, "\sigma_2+\sigma_1"] \arrow[d, hook] & X \\ \Delta^2_{\mathrm{sub}} \arrow[ur, dashed] & \end{tikzcd}$$

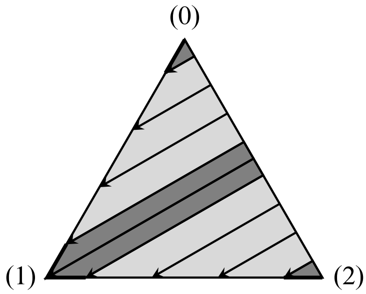

Figure 1: The retraction r.

where $\sigma_2 + \sigma_1 : \Lambda^2_{1\,\mathrm{sub}} \to X$ is defined to be σ_2 on the 0^{th} face and σ_1 on the 2^{nd} face. (the smoothness of $\sigma_2 + \sigma_1$ follows from the tameness of σ_1 and σ_2). We define the map $\Sigma\colon \Delta^2_{\mathrm{sub}} \to X$ to be the composite

$$\Delta^2_{\mathrm{sub}} \xrightarrow{r} \Lambda^2_{1\,\mathrm{sub}} \xrightarrow{\sigma_2+\sigma_1} X,$$

where r is the continuous retraction onto $\Lambda^2_{1\,\mathrm{sub}}$ depicted in Figure 1. Noticing that σ_1 and σ_2 are tame, we can easily see that Σ is a solution of the extension problem in $\mathscr{D}$ such that $\eta := d_1\Sigma$ is also tame. Thus, ω can be chosen as the concatenation of $\eta, \sigma_3, \dots, \sigma_l$. By iterating this procedure, we may assume that ω is the continuous map corresponding to a single tame 1–simplex σ of $S^{\mathscr{D}}_{\mathrm{sub}}(X)$.

Last, let us see that ω is null homotopic rel $\dot{\Delta}^1_{\mathrm{top}}$. Since X is 1–connected, the extension problem in $\mathscr{D}$

$$\begin{tikzcd} \dot{\Delta}^2 \arrow[r, "\sigma+0+0"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

has a solution Σ, where 0 denotes the constant map to the base point x_0 (see Theorem 2.7).

Now, we recall the smooth map $\psi^2_0 \colon \Delta^2 \to \Delta^2$ from [Kihara 2023, Steps 1–3 in the proof of Theorem 8.6]. For $0 < \epsilon < \frac{1}{2}$, set $V_i(\epsilon) = \{(x_0, x_1, x_2) \in \Delta^2 \mid x_i > 1 - \epsilon\}$. For a given ϵ_0 with $0 < \epsilon_0 < \frac{1}{2}$, the smooth map

$$\psi^2_0 \colon \Delta^2 \to \Delta^2$$

is constructed such that

- ψ^2_0 preserves each closed simplex of Δ^2,
- ψ^2_0 maps each $V_i(\epsilon_0/2)$ to the vertex (i),
- ψ^2_0 coincides with 1_{Δ^2} on $\Delta^2 \backslash \bigcup V_i(\epsilon_0)$.

Thus, we see from Lemma 3.1(3) that $\psi^2_0 \colon \Delta^2_{\mathrm{sub}} \to \Delta^2$ is smooth.

Consider the smooth map $\psi^2_0 \colon \Delta^2_{\mathrm{sub}} \to \Delta^2$ defined for sufficiently small $\epsilon_0 > 0$, and define the 2–simplex Σ' of $S^{\mathscr{D}}_{\mathrm{sub}}(X)$ to be the composite

$$\Delta^2_{\mathrm{sub}} \xrightarrow{\psi^2_0} \Delta^2 \xrightarrow{\Sigma} X.$$

Since $\Sigma'|_{\dot{\Delta}^2_{\mathrm{sub}}} = \sigma + 0 + 0$, Σ' yields a homotopy (rel $\dot{\Delta}^1_{\mathrm{top}}$) between ω and 0, which completes the proof.

4.3 Proof of part (II)

We prove part (II) of Theorem 1.1 (see the introduction of this section). By Proposition 3.7 and an argument similar to that in Section 4.2, we may assume that X is 1–connected.

Since $S^{\mathcal{D}}(X)$ is Kan (Corollary 2.6(2)), the canonical map $S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}(X)$ extends to a map

$$S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{\ } \to S^{\mathcal{D}}(X),$$

which induces an isomorphism on the homology (Corollary 3.5). Thus, we have only to show that

$$\pi_1(S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{\ }, x_0) = 0$$

for any fixed $x_0 \in X$ (see Theorem 2.7 and [May 1992, Theorem 13.9]).

Similarly to the proof of part (I), we have the following:

- $\pi_1(S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{\ }, x_0) \cong \pi_1(|\mathrm{sk}_2\, S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{\ }|, x_0)$.
- Every element of $\pi_1(|\mathrm{sk}_2\, S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{\ }|, x_0)$ can be represented by a continuous map

  $$\omega\colon (\Delta^1_{\mathrm{top}}, \dot{\Delta}^1_{\mathrm{top}}) \to (|\mathrm{sk}_1\, S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{\ }|, x_0).$$

  Further, ω can be chosen as the concatenation of finitely many 1–cells $\tau_1, \dots, \tau_l$, where $\tau_j = \sigma_j$ or $\bar{\sigma}_j$ for some $\sigma_j \in NS^{\mathcal{D}}_{\mathrm{aff}}(X)_1$.

We would like to simplify the expression $\tau_1 \cdots \tau_l$ for ω and show that ω is null homotopic rel $\dot{\Delta}^1_{\mathrm{top}}$.

A smooth 1–simplex $\sigma\colon \mathbb{A}^1 \to X$ of a diffeological space X is called *tame* if σ is constant near $(-\infty, 0]$ and near $[1, \infty)$, where $\mathbb{A}^1$ is identified with $\mathbb{R}$ in a canonical manner. By the following analogue of Lemma 4.1, we may assume that each σ_j is tame.

Lemma 4.2 *Let X be a diffeological space and σ a 1–simplex of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$. Then there exists a 2–simplex Σ of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ such that $d_0\Sigma$ is the constant map to $\sigma((1))$, $d_1\Sigma$ is tame, and $d_2\Sigma = \sigma$.*

Proof Set $U = \{(x_0, x_1, x_2) \in \mathbb{A}^2 \mid -\frac{1}{2} < x_1 < \frac{1}{2}\}$. Choose a nondecreasing smooth function $\mu\colon \mathbb{R} \to [0,1]$ such that $\mu \equiv 0$ near $(-\infty, 0]$ and $\mu \equiv 1$ near $[1, \infty)$, and a smooth function $\phi\colon [-\frac{1}{2}, \frac{1}{2}] \to [0,1]$ such that $\phi \equiv 1$ near 0 and $\phi \equiv 0$ near $\{-\frac{1}{2}, \frac{1}{2}\}$. Then we can construct the desired 2–simplex Σ in a manner similar to that in the proof of Lemma 4.1. □

Second, let us see that ω can be chosen as the concatenation of $\sigma_1, \dots, \sigma_l$ for some $\sigma_1, \dots, \sigma_l \in NS^{\mathcal{D}}_{\mathrm{aff}}(X)_1$. For this, consider $\Sigma_j \in S^{\mathcal{D}}_{\mathrm{aff}}(X)_2$ defined to be the composite

$$\mathbb{A}^2 \xrightarrow{s} \mathbb{A}^1 \xrightarrow{\sigma_j} X,$$

where $s(x_0, x_1, x_2) = (x_0 + x_2, x_1)$. Then $d_2\Sigma_j = \sigma_j$, $d_1\Sigma_j$ is constant, and $\sigma'_j := d_0\Sigma_j$ satisfies $\sigma'_j(t) = \sigma_j(1-t)$. Thus, if $\tau_j = \bar{\sigma}_j$, then we can replace τ_j with σ'_j. Hence, we may assume that ω is the concatenation of $\sigma_1, \dots, \sigma_l$.

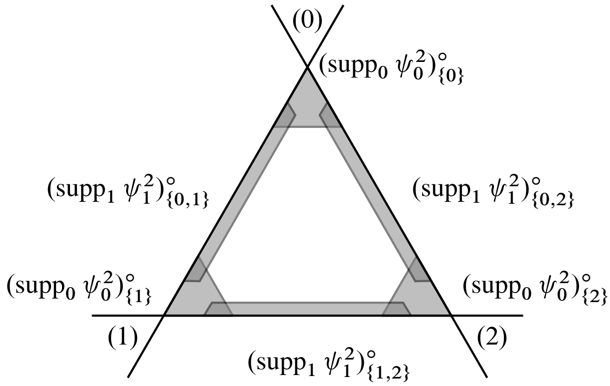

Figure 2

Next we show the following lemma. For $i = 0, 1, 2$, $d^i : \mathbb{A}^1 \to \mathbb{A}^2$ denotes the obvious affine extension of $d^i : \Delta^1 \to \Delta^2$ (see Section 2.2).

Lemma 4.3 *Let X be a diffeological space and γ_0, γ_1, and γ_2 tame 1–simplices of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ such that $d_0\gamma_2 = d_1\gamma_0$, $d_0\gamma_0 = d_0\gamma_1$, and $d_1\gamma_1 = d_1\gamma_2$. If the extension problem in $\mathcal{D}$*

$$\begin{tikzcd} \dot{\Delta}^2 \arrow[r, "\sum \gamma_i|_{\Delta^1}"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

has a solution, then the extension problem in $\mathcal{D}$

$$\begin{tikzcd} \bigcup d^i \mathbb{A}^1 \arrow[r, "\sum \gamma_i"] \arrow[d, hook] & X \\ \mathbb{A}^2 \arrow[ur, dashed] & \end{tikzcd}$$

also has a solution.

Proof We choose a solution Σ of the first extension problem, and use the smooth map $\psi^2 : \Delta^2 \to \Delta^2$ constructed in [Kihara 2023, Steps 1–3 in the proof of Theorem 8.6] to modify and extend Σ.

To describe the basic properties of ψ^2, we adopt the following notation. For a continuous self-map f of Δ^p, we set

$$\mathrm{carr}_k\ f = \{x \in \Delta^p \mid f(x) \neq x,\ f(x) \in \mathrm{sk}_k\ \Delta^p\} \quad \text{and} \quad \mathrm{supp}_k\ f = \overline{\mathrm{carr}_k\ f}.$$

Further, for a subset $\{i_0, \dots, i_k\}$ of $\{0, \dots, p\}$, we set

$$V_{\{i_0,\dots,i_k\}} = \{(x_0, \dots, x_p) \in \Delta^p \mid x_i > x_j \text{ for } i \in \{i_0, \dots, i_k\} \text{ and } j \notin \{i_0, \dots, i_k\}\},$$
$$(\mathrm{supp}_k\ f)^\circ_{\{i_0,\dots,i_k\}} = (\mathrm{supp}_k\ f)^\circ \cap V_{\{i_0,\dots,i_k\}}.$$

For a given ϵ_0 with $0 < \epsilon_0 < \frac{1}{2}$, the smooth maps $\psi_k^2 : \Delta^2 \to \Delta^2$ $(k = 0, 1)$ are defined such that they satisfy the following conditions (see Figure 2):

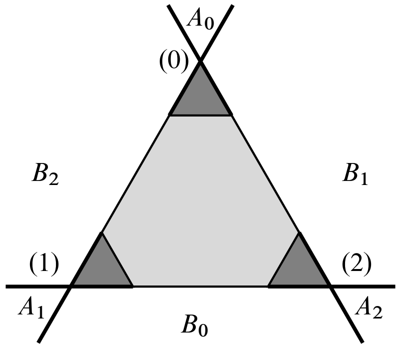

Figure 3

- ψ_k^2 preserves each closed simplex of Δ^2 and $\psi_k^2 = \mathrm{id}$ on $\mathrm{sk}_k\, \Delta^2$ (note that $\mathrm{sk}_0\, \Delta^2 = \{(0), (1), (2)\}$ and that $\mathrm{sk}_1\, \Delta^2 = \dot{\Delta}^2$).
- $(\mathrm{supp}_0\, \psi_0^2)^\circ_{\{i\}} = V_i(\epsilon_0/2)$ and $\psi_0^2 = \mathrm{id}$ on $\Delta^2 \backslash \bigcup V_i(\epsilon_0)$ (see Section 4.2).
- $(\mathrm{supp}_0\, \psi_0^2)^\circ \cup (\mathrm{supp}_1\, \psi_1^2)^\circ$ is a neighborhood of $\dot{\Delta}^2$.
- $(\mathrm{supp}_1 \psi_1^2)^\circ = (\mathrm{supp}_1 \psi_1^2)^\circ_{\{1,2\}} \coprod (\mathrm{supp}_1 \psi_1^2)^\circ_{\{0,2\}} \coprod (\mathrm{supp}_1 \psi_1^2)^\circ_{\{0,1\}}$, and ψ_1^2 preserves each $V_i(\epsilon_0/2)$ and maps a point x of $(\mathrm{supp}_1\, \psi_1^2)^\circ_{\{i_0,i_1\}}$ to the intersection of the i^{th} face of Δ^2 and the line through the vertex (i) and x, where $i \neq i_0, i_1$.

The map $\psi^2 \colon \Delta^2 \to \Delta^2$ is defined to be the composite

$$\Delta^2 \xrightarrow{\psi_1^2} \Delta^2 \xrightarrow{\psi_0^2} \Delta^2.$$

Consider the smooth map $\psi^2 \colon \Delta^2 \to \Delta^2$ for a sufficiently small $\epsilon_0 > 0$ and define Σ' to be the composite

$$\Delta^2 \xrightarrow{\psi^2} \Delta^2 \xrightarrow{\Sigma} X.$$

Then Σ' has the following properties:

- $\Sigma'|_{\dot{\Delta}^2} = \Sigma|_{\dot{\Delta}^2}$.
- $\Sigma'|_{(\mathrm{supp}_0\, \psi_0^2)^\circ_{\{i\}}}$ is constant.
- $\Sigma'|_{(\mathrm{supp}_1\, \psi_1^2)^\circ_{\{i_0,i_1\}}}$ is constant along any ray from the vertex (i) with $i \neq i_0, i_1$.

We thus extend Σ' to $\mathbb{A}^2$ as follows. Define $\Sigma'|_{A_i}$ to be constant for $i = 0, 1, 2$, and define $\Sigma'|_{B_i}$ to be constant along any ray from the vertex (i) (see Figure 3). Then we can easily see that $\Sigma' \colon \mathbb{A}^2 \to X$ is the desired solution of the second extension problem. □

Let us see that ω can be chosen as the continuous map corresponding to a single tame 1–simplex σ of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$. For this, we first consider the extension problem in $\mathcal{D}$

$$\begin{tikzcd} \Lambda_1^2 \arrow[r, "\sigma_2|_{\Delta^1} + \sigma_1|_{\Delta^1}"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

Then we can use the continuous retraction $r\colon \Delta^2 \to \Lambda^2_1$ depicted in Figure 1 to construct a solution $\Sigma\colon \Delta^2 \to X$ such that the composite $\Delta^1 \xrightarrow{d^1} \Delta^2 \xrightarrow{\Sigma} X$ is constant near each vertex. Define the tame 1–simplex η of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ by $\eta|_{\Delta^1} = \Sigma \circ d^1$ and consider the extension problem in $\mathcal{D}$

$$\begin{tikzcd} \bigcup d^i \mathbb{A}^1 \arrow[r, "\sigma_2+\eta+\sigma_1"] \arrow[d, hook] & X \\ \mathbb{A}^2 \arrow[ur, dashed] & \end{tikzcd}$$

Since this extension problem has a solution (see Lemma 4.3), ω can be chosen as the concatenation of $\eta, \sigma_3, \dots, \sigma_l$. By iterating this procedure, we may assume that ω is the continuous map corresponding to a single tame 1–simplex σ of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$.

Last, let us see that ω is null homotopic rel $\dot{\Delta}^1_{\mathrm{top}}$. Since X is 1–connected, the extension problem in $\mathcal{D}$

$$\begin{tikzcd} \dot{\Delta}^2 \arrow[r, "\sigma|_{\Delta^1}+0+0"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

has a solution (see Theorem 2.7). Hence, the extension problem in $\mathcal{D}$

$$\begin{tikzcd} \bigcup d^i \mathbb{A}^1 \arrow[r, "\sigma+0+0"] \arrow[d, hook] & X \\ \mathbb{A}^2 \arrow[ur, dashed] & \end{tikzcd}$$

also has a solution (Lemma 4.3), which shows that ω is null homotopic rel $\dot{\Delta}^1_{\mathrm{top}}$.

4.4 Proofs of Theorem 1.1 and Corollary 1.2

In this subsection, we complete the proofs of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1 The proof of the main statement is given in Sections 4.2 and 4.3. Since $S^{\mathcal{D}}(X)$ is always fibrant (Corollary 2.6(2)), the last statement is obvious. □

Let $\mathcal{S}_*$ denote the category of pointed simplicial sets, and let $\mathcal{S}_{*f}$ denote the full subcategory of $\mathcal{S}_*$ consisting of fibrant objects (ie pointed Kan complexes). Choosing a fibrant approximation functor $R\colon \mathcal{S}_* \to \mathcal{S}_{*f}$, we define the i^{th} homotopy group functor $\pi_i : \mathcal{S}_* \to \mathrm{Gr}$ to be the composite

$$\mathcal{S}_* \xrightarrow{R} \mathcal{S}_{*f} \xrightarrow{\pi_i} \mathrm{Gr}.$$

(Strictly speaking, π_0 is defined as a Set_*–valued functor, where Set_* denotes the category of pointed sets.) Then, up to natural isomorphisms, the functor $\pi_i : \mathcal{S}_* \to \mathrm{Gr}$ extends the original homotopy group functor $\pi_i : \mathcal{S}_{*f} \to \mathrm{Gr}$ and the extension $\pi_i : \mathcal{S}_* \to \mathrm{Gr}$ is independent of the choice of R. Further, we can see that if a fibrant approximation $K \to K'$ and a point k of K are given, then $\pi_i(K,k)$ is canonically isomorphic to the i^{th} homotopy group of the pointed Kan complex (K',k).

Proof of Corollary 1.2 The result follows immediately from Theorems 1.1 and 2.7. $\square$

5 Diffeological principal bundles

In this section, we recall the notions of a diffeological principal bundle and a simplicial principal bundle (Section 5.1) and establish Theorem 1.3, which characterizes diffeological principal bundles using the singular functor $S^{\mathscr{D}}_{\text{aff}}$ (Section 5.2).

5.1 Diffeological and simplicial principal bundles

In this subsection, we recall the three notions of principal bundles in $\mathscr{D}$; the weakest notion is due to Iglesias-Zemmour (see Definition 5.1(2)). We also make a brief review on simplicial principal bundles.

Let $\mathscr{C}$ be a category with finite products, and G a group in $\mathscr{C}$. Then $\mathscr{C}G$ denotes the category of right G–objects of $\mathscr{C}$. For $B \in \mathscr{C}$, $\mathscr{C}G/B$ denotes the category of objects of $\mathscr{C}G$ over B, where B is regarded as an object of $\mathscr{C}G$ with trivial G–action.

Definition 5.1 Let G be a diffeological group, and X a diffeological space.

(1) An object $\pi: E \to X$ of $\mathscr{D}G/X$ is called a *locally trivial principal G–bundle* if there exists an open cover $\{U_i\}$ of X such that for each i, a pullback diagram in $\mathscr{D}$

$$\begin{tikzcd} U_i \times G \arrow[r, hook] \arrow[d, "\text{proj}"'] & E \arrow[d, "\pi"] \\ U_i \arrow[r, hook] & X \end{tikzcd}$$

with equivariant upper arrow exists; such an open cover $\{U_i\}$ is called a *trivialization open cover* of $\pi: E \to X$. An object $\pi: E \to X$ of $\mathscr{D}G/X$ is called a *$\mathscr{D}$–numerable principal G–bundle* if π admits a $\mathscr{D}$–numerable trivialization open cover (ie a trivialization open cover $\{U_i\}$ which admits a smooth partition of unity subordinate to it).

(2) An object $\pi: E \to X$ of $\mathscr{D}G/X$ is called a *diffeological principal G–bundle* if for any plot $p: U \to X$, the pullback $p^*E \to U$ is a locally trivial principal G–bundle.

(3) A morphism between locally trivial (or diffeological) principal G–bundles $\pi: E \to X$ and $\pi': E' \to X'$ is a commutative diagram in $\mathscr{D}G$ of the form

$$\begin{tikzcd} E \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \tag{5-1}$$

Note that (5-1) is necessarily a pullback diagram in $\mathscr{D}$; see [Iglesias-Zemmour 2013, 8.13 Note 2]. The categories of locally trivial principal G–bundles, $\mathscr{D}$–numerable principal G–bundles, and diffeological principal G–bundles are denoted by $\mathsf{P}\mathscr{D}G$, $\mathsf{P}\mathscr{D}G_{\mathrm{num}}$, and $\mathsf{P}\mathscr{D}G_{\mathrm{diff}}$, respectively.

We have the obvious fully faithful embeddings

$$\mathsf{P}\mathscr{D}G_{\mathrm{num}} \hookrightarrow \mathsf{P}\mathscr{D}G \hookrightarrow \mathsf{P}\mathscr{D}G_{\mathrm{diff}}.$$

We see from the following examples that the two inclusions are proper (or strict). Recall from [Iglesias-Zemmour 2013, 8.15] that for a diffeological group G and its diffeological subgroup H, the quotient map $\pi\colon G \to G/H$ is a diffeological principal H–bundle.

Example 5.2 (1) Let $\gamma\colon \mathbb{Z}^m \to \mathbb{R}^n$ be a monomorphism of abelian groups with $\Gamma := \gamma(\mathbb{Z}^m)$ dense. Then the quotient diffeological group $T_\Gamma = \mathbb{R}^n/\Gamma$ is called an *irrational torus*. Since the underling topology of T_Γ is indiscrete, the diffeological principal $\mathbb{Z}^m$–bundle $\pi\colon \mathbb{R}^n \to T_\Gamma$ is not locally trivial.

(2) Christensen and Wu constructed a nontrivial locally trivial principal $\mathbb{R}^{>0}$–bundle $\pi\colon P \to X$ with $X \simeq_{\mathscr{D}} *$; see [Christensen and Wu 2021, Example 3.12]. By [Christensen and Wu 2021, Theorem 5.10], the locally trivial principal $\mathbb{R}^{>0}$–bundle π is not $\mathscr{D}$–numerable.

To study diffeological principal bundles, we also need the notion of a simplicial principal bundle [May 1992, Chapter IV].

Definition 5.3 Let H be a simplicial group, and K a simplicial set.

(1) An object $\pi\colon E \to K$ of $\mathscr{S}H/K$ is called a *principal H–bundle* if for any map $k\colon \Delta[p] \to K$, there exists a pullback diagram

$$\begin{array}{ccc} \Delta[p]\times H & \xrightarrow{\hat{k}} & E \\ {\scriptstyle \mathrm{proj}}\downarrow & & \downarrow \\ \Delta[p] & \xrightarrow{k} & K \end{array}$$

with $\hat{k}$ equivariant.

(2) A morphism between principal H–bundles $\pi\colon E \to K$ and $\pi'\colon E' \to K'$ is a commutative diagram in $\mathscr{S}H$ of the form

$$\begin{array}{ccc} E & \xrightarrow{\hat{f}} & E' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ K & \xrightarrow{f} & K' \end{array} \tag{5-2}$$

Note that (5-2) is necessarily a pullback diagram in $\mathscr{S}$. The category of principal H–bundles are denoted by $\mathsf{P}\mathscr{S}H$.

Remark 5.4 An object $\pi\colon E \to K$ of $\mathcal{S}H/K$ is a principal H–bundle if and only if the action of H on E is free and π induces the isomorphism $E/H \xrightarrow[\cong]{\bar{\pi}} K$; see [May 1992, Definition 18.1].

Let $\cdot_0\colon \mathcal{S} \to \mathrm{Set}$ denote the 0^{th} component functor, which is naturally isomorphic to the functor $\mathcal{S}(\Delta[0], \cdot)$. The following simple result is used in the proof of Theorem 1.3.

Lemma 5.5 (1) *The composite*

$$\mathcal{D} \xrightarrow{S^{\mathcal{D}}_{\mathrm{aff}}} \mathcal{S} \xrightarrow{\cdot_0} \mathrm{Set}$$

is naturally isomorphic to the underlying set functor for $\mathcal{D}$.

(2) *The functor* $\cdot_0\colon \mathcal{S} \to \mathrm{Set}$ *is a right adjoint.*

Proof (1) Obvious.

(2) Define the functor $d\colon \mathrm{Set} \to \mathcal{S}$ to assign to a set A the discrete simplicial set whose 0^{th} component is A. Then we can easily see that $(d, \cdot_0)$ is an adjoint pair. □

For a given set A, the discrete simplicial set dA is usually denoted by A.

5.2 Proof of Theorem 1.3

In this subsection, we prove Theorem 1.3; we begin by proving the "only if" part of (1) and (2), and then prove the "if" part of (1).

Recall that $S^{\mathcal{D}}_{\mathrm{aff}}$ is a right adjoint (Remark 3.2(1)). Then we see that $S^{\mathcal{D}}_{\mathrm{aff}}(G)$ is a simplicial group and that $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)\colon S^{\mathcal{D}}_{\mathrm{aff}}(P) \to S^{\mathcal{D}}_{\mathrm{aff}}(X)$ is an object of $\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)/S^{\mathcal{D}}_{\mathrm{aff}}(X)$.

Proof of the "only if" part of Theorem 1.3(1) Assume given a map $k\colon \Delta[p] \to S^{\mathcal{D}}_{\mathrm{aff}}(X)$ and let $\kappa\colon \mathbb{A}^p \to X$ be the smooth map corresponding to k. Then we have a pullback diagram in $\mathcal{D}$

$$\begin{tikzcd} \mathbb{A}^p \times G \arrow[r] \arrow[d, "\mathrm{proj}"'] & P \arrow[d, "\pi"] \\ \mathbb{A}^p \arrow[r, "\kappa"] & X \end{tikzcd}$$

with equivariant upper arrow; see [Iglesias-Zemmour 2013, 8.19]. Note that $S^{\mathcal{D}}_{\mathrm{aff}}$ is a right adjoint and consider the commutative diagram in $\mathcal{S}$ consisting of two pullback squares with equivariant upper arrows

$$\begin{tikzcd} \Delta[p] \times S^{\mathcal{D}}_{\mathrm{aff}}(G) \arrow[r] \arrow[d, "\mathrm{proj}"'] & S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{A}^p) \times S^{\mathcal{D}}_{\mathrm{aff}}(G) \arrow[r] \arrow[d, "\mathrm{proj}"'] & S^{\mathcal{D}}_{\mathrm{aff}}(P) \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi)"'] \\ \Delta[p] \arrow[r] & S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{A}^p) \arrow[r, "S^{\mathcal{D}}_{\mathrm{aff}}(\kappa)"] & S^{\mathcal{D}}_{\mathrm{aff}}(X) \end{tikzcd}$$

where $\Delta[p] \to S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{A}^p)$ is the map corresponding to the p–simplex $1_{\mathbb{A}^p}$ of $S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{A}^p)$. Then the outer rectangle gives the desired local triviality of $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)$; see [Mac Lane 1998, Exercise 8 on page 72]. □

Proof of Theorem 1.3(2) Noting that $S^{\mathcal{D}}_{\mathrm{aff}}$ is a right adjoint, we see from part (1) that $S^{\mathcal{D}}_{\mathrm{aff}}$ induces a functor from $\mathsf{P}\mathcal{D}G_{\mathrm{diff}}$ to $\mathsf{P}\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)$. The faithfulness of the functor follows from Lemma 5.5(1). $\square$

Remark 5.6 The functor $S^{\mathcal{D}}_{\mathrm{aff}}\colon \mathsf{P}\mathcal{D}G_{\mathrm{diff}} \to \mathsf{P}\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)$ need not be fully faithful. In fact, let $\pi\colon P \to X$ be the locally trivial principal $\mathbb{R}^{>0}$–bundle in Example 5.2(2), and let $\pi'\colon P' \to X$ be the trivial principal $\mathbb{R}^{>0}$–bundle. Since $X \simeq_{\mathcal{D}} *$, the diagram in $\mathcal{D}$

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "1_X"'] & * \arrow[r, hook, "x"] & X \end{tikzcd}$$

is commutative up to homotopy for $x \in X$. Thus, by Lemma 3.3, the diagram in $\mathcal{S}$

$$\begin{tikzcd} S^{\mathcal{D}}_{\mathrm{aff}}(X) \arrow[r] \arrow[rr, bend right, "1_{S^{\mathcal{D}}_{\mathrm{aff}}(X)}"'] & * \arrow[r, hook, "x"] & S^{\mathcal{D}}_{\mathrm{aff}}(X) \end{tikzcd}$$

is also commutative up to homotopy. Hence, both $S^{\mathcal{D}}_{\mathrm{aff}}(P)$ and $S^{\mathcal{D}}_{\mathrm{aff}}(P')$ are trivial principal $S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{R}^{>0})$–bundles (see [May 1992, Corollary 20.6]), which shows that $S^{\mathcal{D}}_{\mathrm{aff}}\colon \mathsf{P}\mathcal{D}\mathbb{R}^{>0} \to \mathsf{P}\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{R}^{>0})$ is not fully faithful. (From this argument, we also see that the faithful functor $S^{\mathcal{D}}_{\mathrm{aff}}\colon \mathcal{D} \to \mathcal{S}$ is not fully faithful.)

Next we prove the following lemma, which is used in the proof of "if" part of Theorem 1.3(1).

Lemma 5.7 *Let $\pi\colon P \to X$ be an object of $\mathcal{D}G/X$. Then $\pi\colon P \to X$ is a diffeological principal G–bundle if and only if π satisfies the following conditions*:

(i) *G acts on P freely and $\pi\colon P \to X$ induces a bijection $P/G \to X$.*

(ii) *Given a solid arrow diagram in $\mathcal{D}$*

$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ \mathbb{A}^p \arrow[ur, dashed] \arrow[r, "\kappa"] & X \end{tikzcd}$$

there exists a dotted arrow, making the diagram commute.

(iii) *The translation function $\tau\colon P \times_X P \to G$, defined by $u \cdot \tau(u, v) = v$, is smooth.*

Proof We begin with the forward direction.

(i) Obvious.

(ii) By [Iglesias-Zemmour 2013, 8.19],

$$\kappa^* P \cong \mathbb{A}^p \times G \quad \text{in } \mathcal{D}G/\mathbb{A}^p. \tag{5-3}$$

Hence, π satisfies condition (ii).

(iii) We have only to show that $\tau\colon P\times_X P\to G$ preserves global plots.
Assume given a global plot $f:\mathbb{A}^p\to P\times_X P$. Since the components f_1 and f_2 of f are global plots of P with $\pi\circ f_1=\pi\circ f_2$, we only have to show that the composite

$$\kappa^*P\underset{\mathbb{A}^p}{\times}\kappa^*P\to P\underset{X}{\times}P\xrightarrow{\tau}G$$

is smooth, where $\kappa:=\pi\circ f_1=\pi\circ f_2$. By (5-3), we have the identifications

$$\kappa^*P\underset{\mathbb{A}^p}{\times}\kappa^*P\cong(\mathbb{A}^p\times G)\underset{\mathbb{A}^p}{\times}(\mathbb{A}^p\times G)\cong\mathbb{A}^p\times G\times G,$$

under which the composite $\kappa^*P\times_{\mathbb{A}^p}\kappa^*P\to P\times_X P\xrightarrow{\tau}G$ is just the smooth map

$$\mathbb{A}^p\times G\times G\to G$$

sending (x,g,h) to $g^{-1}h$.

For the reverse direction, assume we are given a smooth map $\kappa\colon\mathbb{A}^p\to X$. By condition (ii), we can choose a section σ of the pullback $\kappa^*P\xrightarrow{\hat{\pi}}\mathbb{A}^p$ of $P\xrightarrow{\pi}X$ along κ. Define the maps

$$\mathbb{A}^p\times G\underset{\psi_\kappa}{\overset{\phi_\kappa}{\rightleftarrows}}\kappa^*P$$

by $\phi_\kappa(x,g)=\sigma(x)\cdot g$ and $\psi_\kappa(u)=\big(\hat{\pi}(u),\tau\big(\sigma(\hat{\pi}(u)),u\big)\big)$, respectively. Then we see that ϕ_κ and ψ_κ are mutually inverses in $\mathscr{D}G/\mathbb{A}^p$. □

We give a proof of the "if" part of Theorem 1.3(1), completing the proof of Theorem 1.3.

Proof of the "if" part of Theorem 1.3(1) We only have to show that $\pi\colon P\to X$ satisfies conditions (i)–(iii) in Lemma 5.7. Throughout this proof, bear the following in mind: for a diffeological space Z,

- $S^{\mathscr{D}}_{\mathrm{aff}}(Z)_0$ is just the set Z,
- $S^{\mathscr{D}}_{\mathrm{aff}}(Z)$ can be regarded as the set of global plots of Z.

Recall also that $S^{\mathscr{D}}_{\mathrm{aff}}$ is a right adjoint (see Remark 3.2(1)).

(i) Consider the pullback diagram in $\mathscr{D}$

$$\begin{tikzcd} P_x \arrow[r] \arrow[d] & P \arrow[d, "\pi"] \\ \{x\} \arrow[r] & X \end{tikzcd}$$

for $x\in X$. By applying the singular functor $S^{\mathscr{D}}_{\mathrm{aff}}$, we have the pullback diagram in $\mathscr{S}$

$$\begin{tikzcd} S^{\mathscr{D}}_{\mathrm{aff}}(P_x) \arrow[r] \arrow[d] & S^{\mathscr{D}}_{\mathrm{aff}}(P) \arrow[d, "S^{\mathscr{D}}_{\mathrm{aff}}(\pi)"] \\ \Delta[0] \arrow[r] & S^{\mathscr{D}}_{\mathrm{aff}}(X) \end{tikzcd}$$

Since $S^{\mathscr{D}}_{\mathrm{aff}}(\pi)$ is a principal $S^{\mathscr{D}}_{\mathrm{aff}}(G)$–bundle, $S^{\mathscr{D}}_{\mathrm{aff}}(P_x)\cong S^{\mathscr{D}}_{\mathrm{aff}}(G)$ in $\mathscr{S}S^{\mathscr{D}}_{\mathrm{aff}}(G)$, and hence $P_x\cong G$ in $\mathrm{Set}\,G$ holds (see Lemma 5.5), which shows that π satisfies condition (i).

(ii) Consider the pullback diagram in $\mathscr{D}$

$$\begin{tikzcd}
\kappa^* P \arrow[r] \arrow[d] & P \arrow[d, "\pi"] \\
\mathbb{A}^p \arrow[r, "\kappa"] & X
\end{tikzcd}$$

and let k denote the simplicial map $\Delta[p] \to S^{\mathscr{D}}_{\mathrm{aff}}(X)$ corresponding to κ. Then we have the commutative diagram in $\mathscr{S}$ consisting of two pullback squares

$$\begin{tikzcd}
k^* S^{\mathscr{D}}_{\mathrm{aff}}(P) \arrow[r] \arrow[d] & S^{\mathscr{D}}_{\mathrm{aff}}(\kappa^* P) \arrow[r] \arrow[d] & S^{\mathscr{D}}_{\mathrm{aff}}(P) \arrow[d, "S^{\mathscr{D}}_{\mathrm{aff}}(\pi)"] \\
\Delta[p] \arrow[r] & S^{\mathscr{D}}_{\mathrm{aff}}(\mathbb{A}^p) \arrow[r, "S^{\mathscr{D}}_{\mathrm{aff}}(\kappa)"] & S^{\mathscr{D}}_{\mathrm{aff}}(X)
\end{tikzcd}$$

where $\Delta[p] \to S^{\mathscr{D}}_{\mathrm{aff}}(\mathbb{A}^p)$ is the simplicial map corresponding to the p–simplex $1_{\mathbb{A}^p}$ of $S^{\mathscr{D}}_{\mathrm{aff}}(\mathbb{A}^p)$; see [Mac Lane 1998, Exercise 8 on page 72]. Since $S^{\mathscr{D}}_{\mathrm{aff}}(\pi)$ is a simplicial $S^{\mathscr{D}}_{\mathrm{aff}}(G)$–bundle, $k^* S^{\mathscr{D}}_{\mathrm{aff}}(P) \to \Delta[p]$ has a section s. Then the composite

$$\Delta[p] \xrightarrow{s} k^* S^{\mathscr{D}}_{\mathrm{aff}}(P) \to S^{\mathscr{D}}_{\mathrm{aff}}(P)$$

defines the desired lifting of κ along π.

(iii) We show that the map $\tau\colon P \times_X P \to G$ preserves global plots. Assume given a global plot $f = (f_1, f_2)\colon \mathbb{A}^p \to P \times_X P$. Since f_1 and f_2 are global plots of P with $\pi \circ f_1 = \pi \circ f_2$, we set $\kappa = \pi \circ f_1 = \pi \circ f_2$ and let σ_1 and σ_2 denote the sections of $\kappa^* P \to \mathbb{A}^p$ corresponding to f_1 and f_2, respectively. Then σ_1 and σ_2 correspond to sections of $k^* S^{\mathscr{D}}_{\mathrm{aff}}(P) \to \Delta[p]$, which are denoted by s_1 and s_2, respectively (see the verification of condition (ii)). Since the principal $S^{\mathscr{D}}_{\mathrm{aff}}(G)$–bundle $k^* S^{\mathscr{D}}_{\mathrm{aff}}(P) \to \Delta[p]$ is trivial, there exists a unique p–simplex g of $S^{\mathscr{D}}_{\mathrm{aff}}(G)$ such that $s_1 \cdot g = s_2$. We thus see that the composite

$$\mathbb{A}^p \xrightarrow{f} P \underset{X}{\times} P \xrightarrow{\tau} G$$

is just the global plot g. □

Remark 5.8 (1) Recall the notion of a diffeological fiber bundle and that of a simplicial fiber bundle from Section 3.3. We can then use the argument in the proof of the "only if" part of Theorem 1.3(1) to prove the following: If $\pi : E \to X$ is a diffeological fiber bundle, then $S^{\mathscr{D}}_{\mathrm{aff}}(\pi) : S^{\mathscr{D}}_{\mathrm{aff}}(E) \to S^{\mathscr{D}}_{\mathrm{aff}}(X)$ is a simplicial fiber bundle.

This result along with Theorem 1.1 enables us to apply the Serre spectral sequence [May 1992, Section 32] to diffeological fiber bundles (cf [Kihara 2023, Remark 3.8(3)]).

(2) If we restrict ourselves to locally trivial principal G–bundles (resp. locally trivial fiber bundles), then the "only if" part of Theorem 1.3(1) (resp. the result stated in part (1)) remains true for the functor $S^{\mathscr{D}}$ (instead of $S^{\mathscr{D}}_{\mathrm{aff}}$); see [Kihara 2023, Corollary 5.15(1)].

(3) If we restrict ourselves to diffeological coverings, then the result stated in part (1) remains true for the functor $S^{\mathcal{D}}$ (instead of $S^{\mathcal{D}}_{\mathrm{aff}}$); see Proposition 3.7. Similarly, if G is discrete, then Theorem 1.3 remains true for the functor $S^{\mathcal{D}}$ (instead of $S^{\mathcal{D}}_{\mathrm{aff}}$).

6 Characteristic classes of diffeological principal bundles

In this section, we first give a criterion for a simplicial principal bundle to be universal (Section 6.1). We then use this criterion to determine the homotopy type of $S^{\mathcal{D}}(X)$ for a diffeological space X which admits a diffeological principal bundle with contractible total space (Proposition 6.3), applying it to the classifying space BG of a diffeological group G and exceptional diffeological spaces such as irrational tori and $\mathbb{R}/\mathbb{Q}$ (Section 6.2). We use the proof of Proposition 6.3 along with Theorems 1.1 and 1.3 to prove Proposition 1.4 (Section 6.3). We end this section by discussing the sets of characteristic classes for various classes of principal bundles and their relation (Section 6.4).

6.1 Universal simplicial principal bundles

In this subsection, we recall the basics of universal simplicial principal bundles and give a criterion for a simplicial principal bundle to be universal.

Let H be a simplicial group. A principal H–bundle $\varpi\colon E \to L$ is called *universal* if L is Kan (ie fibrant in $\mathcal{S}$) and the natural map

$$[K, L] \to \{\text{isomorphism classes of principal } H\text{–bundles over } K\}, \qquad [f] \mapsto [f^*E],$$

is bijective; the base L of a universal principal H–bundle $\varpi\colon E \to L$ is called a *classifying complex* of H. By a simple argument, a classifying complex of H is unique up to homotopy. Recall that the W–construction $q\colon WH \to \overline{W}H$ is a universal principal H–bundle [Goerss and Jardine 1999, Chapter V, Section 4; May 1992, Section 21] and that WH is contractible [May 1992, Proposition 21.5].

Lemma 6.1 *Let H be a simplicial group, and $\varpi\colon E \to L$ be a principal H–bundle. Then the following are equivalent*:

(i) *$\varpi\colon E \to L$ is universal.*

(ii) *L is Kan and the canonical map $E \to *$ is a weak equivalence.*

(iii) *E is a contractible Kan complex.*

Proof (ii) $\Longleftrightarrow$ (iii) Noticing that H is Kan [May 1992, Theorem 17.1], we see that L is Kan if and only if E is Kan (see [May 1992, Proposition 7.5]), and hence that (ii) $\Longleftrightarrow$ (iii).

(i) $\Longleftrightarrow$ (iii) We have only to prove that under the assumption that L is Kan,

$$\varpi\colon E \to L \text{ is universal} \iff E \text{ is contractible}$$

(see [May 1992, Proposition 7.5]).

Since $q\colon WH \to \overline{W}H$ is universal, we have a morphism of principal H–bundles

$$\begin{tikzcd} E \arrow[r] \arrow[d, "\varpi"'] & WH \arrow[d, "q"] \\ L \arrow[r, "\varphi"] & \overline{W}H \end{tikzcd} \tag{6-1}$$

Note that H and the four simplicial sets in (6-1) are Kan and consider the morphism between the homotopy exact sequences induced by (6-1). Then we have the equivalences

$$\varpi : E \to L \text{ is universal} \iff \varphi\colon L \to \overline{W}H \text{ is a homotopy equivalence} \iff E \text{ is contractible.} \qquad \square$$

Remark 6.2 Lemma 6.1 can be regarded as a variant of [Goerss and Jardine 1999, Chapter V, Theorem 3.9]. However, we record this lemma along with its proof for the following two reasons: one reason is to avoid using the model structure on $\mathcal{S}G$ (see [Goerss and Jardine 1999, Section V.2]) and the other reason is to emphasize the importance of the fibrancy of the base (cf the proof of Proposition 6.3).

6.2 Diffeological principal bundles with contractible total space

In this subsection, we determine the homotopy type of $S^{\mathcal{D}}(X)$ for a diffeological space X which admits a diffeological principal bundle $\pi : E \to X$ with E weakly contractible. Here, a diffeological space Z is called *weakly contractible* if the canonical map $Z \to *$ is a weak equivalence. We can easily see that

$$Z \text{ is weakly contractible} \quad \iff \quad S^{\mathcal{D}}(Z) \simeq * \quad \iff \quad \pi_*^{\mathcal{D}}(Z, z) = 0 \quad \text{for any } z \in Z$$

(see Remark 2.8(1), Corollary 2.6(2), and Theorem 2.7).

Proposition 6.3 *Let G be a diffeological group and $\pi : E \to X$ a diffeological principal G–bundle with E weakly contractible. Then $S^{\mathcal{D}}(X)$ is a classifying complex of the simplicial group $S^{\mathcal{D}}(G)$.*

Proof By Theorem 1.3, $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)\colon S^{\mathcal{D}}_{\mathrm{aff}}(E) \to S^{\mathcal{D}}_{\mathrm{aff}}(X)$ is a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle. Let us construct a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)'\colon S^{\mathcal{D}}_{\mathrm{aff}}(E)' \to S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{}$ (see Section 4.1) and a morphism of principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundles

$$\begin{tikzcd} S^{\mathcal{D}}_{\mathrm{aff}}(E) \arrow[r, hook] \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi)"'] & S^{\mathcal{D}}_{\mathrm{aff}}(E)' \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi)'"] \\ S^{\mathcal{D}}_{\mathrm{aff}}(X) \arrow[r, hook] & S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{} \end{tikzcd}$$

First, choose a classifying map $\varphi_E : S^{\mathcal{D}}_{\mathrm{aff}}(X) \to \overline{W} S^{\mathcal{D}}_{\mathrm{aff}}(G)$. Then note that $\overline{W} S^{\mathcal{D}}_{\mathrm{aff}}(G)$ is Kan and choose an extension $\varphi'_E : S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{} \to \overline{W} S^{\mathcal{D}}_{\mathrm{aff}}(G)$. By setting $S^{\mathcal{D}}_{\mathrm{aff}}(E)' = {\varphi'_E}^* W S^{\mathcal{D}}_{\mathrm{aff}}(G)$, we then obtain the desired diagram.

Thus, we can use [Gabriel and Zisman 1967, Chapter III, Theorem 4.2] to see that $S^{\mathcal{D}}_{\mathrm{aff}}(E) \hookrightarrow S^{\mathcal{D}}_{\mathrm{aff}}(E)'$ is a weak equivalence. Noticing that $S^{\mathcal{D}}_{\mathrm{aff}}(E) \to *$ is a weak equivalence (see Theorem 1.1), we see from Lemma 6.1 that $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)'\colon S^{\mathcal{D}}_{\mathrm{aff}}(E)' \to S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{}$ is a universal principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle. Hence, $S^{\mathcal{D}}(X)$ is a classifying complex of $S^{\mathcal{D}}_{\mathrm{aff}}(G)$, and hence of $S^{\mathcal{D}}(G)$ (see Theorem 1.1). $\square$

Corollary 6.4 *Let G be a diffeological group. Then the singular complex $S^{\mathcal{D}}(BG)$ of the classifying space BG is a classifying complex of the simplicial group $S^{\mathcal{D}}(G)$.*

Proof Recall from [Christensen and Wu 2021, Corollary 5.5] that EG is smoothly contractible. Then the result is immediate from Proposition 6.3. □

Corollary 6.5 *Suppose that X is a pointed diffeological space which has the weakly contractible universal covering. Then the singular complex $S^{\mathcal{D}}(X)$ is the Eilenberg–Mac Lane complex $K(\pi_1^{\mathcal{D}}(X), 1)$. In particular, the (co)homology of X is just the (co)homology of the group $\pi_1^{\mathcal{D}}(X)$.*

Proof Recall from [Iglesias-Zemmour 2013, 8.26] that the universal covering $\pi: Z \to X$ is a diffeological principal $\pi_1^{\mathcal{D}}(X)$–bundle. Then the result follows from Proposition 6.3. □

Remark 6.6 (1) We can prove Corollary 6.4, using neither the functor $S^{\mathcal{D}}_{\mathrm{aff}}$ nor Theorem 1.1. In fact, by Remark 5.8(2) and Lemma 6.1, $S^{\mathcal{D}}(\pi_G): S^{\mathcal{D}}(EG) \to S^{\mathcal{D}}(BG)$ is a universal principal $S^{\mathcal{D}}(G)$–bundle. However, the construction in the proof of Proposition 6.3 is useful in the proof of Proposition 1.4.

(2) We can also prove Corollary 6.5, using neither the functor $S^{\mathcal{D}}_{\mathrm{aff}}$ nor Theorem 1.1. In fact, Corollary 6.5 follows from Proposition 3.7. Alternatively, Corollary 6.5 follows from [Iglesias-Zemmour 2013, 8.24] and Theorem 2.7.

Corollary 6.5 determines the homotopy type of $S^{\mathcal{D}}(X)$, and hence the (co)homology of X for well-known homogeneous diffeological spaces X such as irrational tori and $\mathbb{R}/\mathbb{Q}$.

Example 6.7 (1) Let $\gamma: \mathbb{Z}^m \to \mathbb{R}^n$ be a monomorphism of abelian groups with $\Gamma := \gamma(\mathbb{Z}^m)$ dense, and consider the irrational torus $T_\Gamma = \mathbb{R}^n/\Gamma$. By Corollary 6.5, the singular complex $S^{\mathcal{D}}(T_\Gamma)$ of T_Γ is just the m–dimensional torus $K(\mathbb{Z}^m, 1)$. Hence, $H^*(T_\Gamma; \mathbb{Z}) \cong \Lambda(\mathbb{Z}^m)$ holds.

(2) The singular complex $S^{\mathcal{D}}(\mathbb{R}/\mathbb{Q})$ of the quotient diffeological group $\mathbb{R}/\mathbb{Q}$ is just the rationalized circle $K(\mathbb{Q}, 1)$, and hence $\widetilde{H}_*(\mathbb{R}/\mathbb{Q}; \mathbb{Z}) = H_1(\mathbb{R}/\mathbb{Q}; \mathbb{Z}) = \mathbb{Q}$. More generally, let A be a countable subgroup of $\mathbb{F}$ $(= \mathbb{R}, \mathbb{C})$. Then the singular complex $S^{\mathcal{D}}(\mathbb{F}/A)$ of the quotient diffeological group $\mathbb{F}/A$ is just $K(A, 1)$.

Remark 6.8 Iglesias-Zemmour [2024, Corollary, page 253] and Kuribayashi [2020, Remark 2.9; 2021, Proposition 3.2] obtained calculational results similar to Example 6.7(1) for other cohomology theories of irrational tori. On the other hand, the de Rham cohomology $H^*_{dR}(T_\Gamma)$ is isomorphic to $\Lambda(\mathbb{R}^n)$ [Iglesias-Zemmour 2013, Exercise 119], which along with Example 6.7(1), shows that the de Rham theorem does not hold for irrational tori. This motivates the study of a forthcoming paper [Kihara $\geq$ 2024].

Next we introduce new aspherical homogeneous diffeological spaces, using Corollary 6.5.

Example 6.9 Let k be a countable subfield of $\mathbb{F}$ $(= \mathbb{R}, \mathbb{C})$ (eg an algebraic number field or a countable extension of $\mathbb{Q}$ such as $\overline{\mathbb{Q}} \cap \mathbb{R}$ or $\overline{\mathbb{Q}}$). For an algebraic group G over k, we can consider the homogeneous diffeological space $G(\mathbb{F})/G(k)$.

If G is a unipotent algebraic group over k, then the exponential map $\exp\colon \mathfrak{g} \to G$ is an isomorphism of algebraic varieties, where $\mathfrak{g}$ is the Lie algebra of G; see [Milne 2017, page 289]. Thus, we have the diffeomorphism

$$\mathfrak{g}(\mathbb{F}) \xrightarrow[\cong]{\exp} G(\mathbb{F})$$

and the universal covering

$$G(\mathbb{F}) \to G(\mathbb{F})/G(k)$$

of $G(\mathbb{F})/G(k)$. Hence, by Corollary 6.5,

$$S^{\mathcal{D}}(G(\mathbb{F})/G(k)) = K(G(k), 1),$$

so the (co)homology of $G(\mathbb{F})/G(k)$ is that of the group $G(k)$. The group $U_n(k)$ of upper triangular unipotent matrices and the Hisenberg group $H_n(k)$ (see [Onishchik and Vinberg 1994, page 54]) are typical examples of unipotent algebraic groups.

Further if G is defined over a subring k_0 of k, then

$$S^{\mathcal{D}}(G(\mathbb{F})/G(k_0)) = K(G(k_0), 1).$$

We are interested in the case where k_0 is the ring $\mathbb{O}_k$ of integers of an algebraic number field k. If k is an algebraic number field of degree n with $\mathbb{Q} \subsetneqq k \subsetneqq \mathbb{R}$, then k_0 $(= \mathbb{O}_k)$ is a finitely generated free $\mathbb{Z}$–module of rank n, and hence is dense in $\mathbb{R}$.

6.3 Proof of Proposition 1.4

In this subsection, we prove Proposition 1.4.

Proof of Proposition 1.4 Let $\pi_G\colon EG \to BG$ denote the universal $\mathcal{D}$–numerable principal G–bundle constructed in [Christensen and Wu 2021]. Then by Theorem 1.3(1), $S^{\mathcal{D}}_{\mathrm{aff}}(\pi_G)\colon S^{\mathcal{D}}_{\mathrm{aff}}(EG) \to S^{\mathcal{D}}_{\mathrm{aff}}(BG)$ is a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle.

We prove the result in two steps.

Step 1: construction of a universal principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle which is an extension of $S^{\mathcal{D}}_{\mathrm{aff}}(\pi_G)$ Recall from [Christensen and Wu 2021, Corollary 5.5] that EG is smoothly contractible. Then, by the proof of Proposition 6.3, we have a universal principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle $S^{\mathcal{D}}_{\mathrm{aff}}(\pi_G)'\colon S^{\mathcal{D}}_{\mathrm{aff}}(EG)' \to S^{\mathcal{D}}_{\mathrm{aff}}(BG)^{\hat{}}$ and a morphism of principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundles

$$\begin{tikzcd}
S^{\mathcal{D}}_{\mathrm{aff}}(EG) \arrow[r, hook] \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi_G)"'] & S^{\mathcal{D}}_{\mathrm{aff}}(EG)' \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi_G)'"] \\
S^{\mathcal{D}}_{\mathrm{aff}}(BG) \arrow[r, hook] & S^{\mathcal{D}}_{\mathrm{aff}}(BG)^{\hat{}}
\end{tikzcd}$$

Step 2: definition of $\alpha(P)$ Let $\pi\colon P \to X$ be a diffeological principal G–bundle. Since

$$S^{\mathcal{D}}_{\mathrm{aff}}(\pi)\colon S^{\mathcal{D}}_{\mathrm{aff}}(P) \to S^{\mathcal{D}}_{\mathrm{aff}}(X)$$

is a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle (Theorem 1.3(1)), we have a classifying map $\varphi_P\colon S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}_{\mathrm{aff}}(BG)^{\hat{}}$.

Note that $H^*(Z;A) := H^*\operatorname{Hom}(\mathbb{Z}S^{\mathcal{D}}(Z),A) \cong H^*\operatorname{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(Z),A)$ (see Corollary 3.5) and that $H^*\operatorname{Hom}(\mathbb{Z}K,A) \cong H^*\operatorname{Hom}(\mathbb{Z}K\hat{\ },A)$. Then we can define $\alpha(P) \in H^k(X;A)$ by $\alpha(P) = \varphi_P^*\alpha$. We can use Theorem 1.3 to show that $\alpha(f^*P) = f^*\alpha(P)$, and hence that $\alpha(\cdot)$ defines a characteristic class for diffeological principal G–bundles.

Similarly, we can use Theorem 1.3 to show that $\alpha(\cdot)$ extends the characteristic class $\alpha(\cdot)$ for $\mathcal{D}$–numerable principal G–bundles (see Section 1 for the definition). □

Remark 6.10 The author does not know whether $S^{\mathcal{D}}_{\mathrm{aff}}(BG)$ is always Kan. If $S^{\mathcal{D}}_{\mathrm{aff}}(BG)$ is always Kan, the proof of Proposition 1.4 becomes simpler (see Lemma 6.1).

Let us apply Proposition 1.4 to special cases.

Example 6.11 (1) Let $\pi\colon Z \to X$ be a Galois covering with structure group Γ; see [Iglesias-Zemmour 2013, page 262]. Then for a given class $\alpha \in H^k(\Gamma;A)$ $(\cong H^k(B\Gamma;A))$, the class $\alpha(Z) \in H^k(X;A)$ is defined by Proposition 1.4.

(2) Let G be a diffeological group and H a diffeological subgroup of G. Then for a given class $\alpha \in H^k(BH;A)$, the class $\alpha(G) \in H^k(G/H;A)$ is defined by Proposition 1.4; see [Iglesias-Zemmour 2013, 8.15].

If a relevant diffeological principal bundle in Example 6.11 happens to be $\mathcal{D}$–numerable, then the class at issue is just the image of α under the homomorphism induced by the classifying map. However, this is not the case in general. See the following example, which specializes both parts (1) and (2) of Example 6.11.

Example 6.12 Let $\gamma\colon \mathbb{Z}^m \to \mathbb{R}^n$ be a monomorphism of abelian groups with $\Gamma := \gamma(\mathbb{Z}^m)$ dense, and consider the diffeological principal $\mathbb{Z}^m$–bundle $P := \mathbb{R}^n \xrightarrow{\pi} T_\Gamma$ over the irrational torus T_Γ (see Examples 6.7(1) and 6.11(2)); note that T_Γ is a diffeological group and that π is the universal covering of T_Γ.

Since $S^{\mathcal{D}}_{\mathrm{aff}}(T_\Gamma)$ is already Kan (see [Christensen and Wu 2014, Proposition 4.30 or Theorem 4.34]), $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)\colon S^{\mathcal{D}}_{\mathrm{aff}}(P) \to S^{\mathcal{D}}_{\mathrm{aff}}(T_\Gamma)$ is a universal principal $\mathbb{Z}^m$–bundle (see Step 1 in the proof of Proposition 1.4), and hence, we have a classifying map $\varphi_P\colon S^{\mathcal{D}}_{\mathrm{aff}}(T_\Gamma) \to S^{\mathcal{D}}_{\mathrm{aff}}(B\mathbb{Z}^m)\hat{\ }$ which is obviously a homotopy equivalence in $\mathcal{S}$.

Since $S^{\mathcal{D}}_{\mathrm{aff}}(B\mathbb{Z}^m)\hat{\ }$ is just the Eilenberg–Mac Lane complex $K(\mathbb{Z}^m,1)$, $H^*(B\mathbb{Z}^m;A) \cong (\Lambda\mathbb{Z}^m)\otimes A$. Thus, for any $\alpha \in H^*(B\mathbb{Z}^m;A)$, the characteristic class $\alpha(P) \in H^*(T_\Gamma;A)$ is just the image $\varphi_P^*(\alpha)$ under the isomorphism $H^*(T_\Gamma;A) \xleftarrow[\cong]{\varphi_P^*} H^*(B\mathbb{Z}^m;A)$.

On the other hand, since $\pi\colon P \to T_\Gamma$ is not locally trivial (see Example 5.2(1)), P has no classifying map to $B\mathbb{Z}^m$. Further, every nonzero element $\beta \in \widetilde{H}^*(T_\Gamma;A)$ is not contained in the image of the homomorphism induced by any smooth map $f\colon T_\Gamma \to B\mathbb{Z}^m$. In fact, we have the commutative diagram

$$\begin{tikzcd} S^{\mathcal{D}}(T_\Gamma) \arrow[r, "S^{\mathcal{D}}(f)"] \arrow[d] & S^{\mathcal{D}}(B\mathbb{Z}^m) \arrow[d] \\ S(\widetilde{T}_\Gamma) \arrow[r, "S(\tilde{f})"] & S(\widetilde{B\mathbb{Z}^m}) \end{tikzcd}$$

(see Section 2.3). Since $S^{\mathcal{D}}(B\mathbb{Z}^m) \to S(\widetilde{B\mathbb{Z}^m})$ is a homotopy equivalence (see [Kihara 2023, Corollary 5.16]) and $S(\widetilde{T}_\Gamma) \simeq *$, $S^{\mathcal{D}}(f)$ is homotopic to a constant map. (We actually show that $B\mathbb{Z}^m$ is smoothly homotopy equivalent to the torus T^m, and hence that f is smoothly homotopic to a constant map; see a forthcoming paper.)

6.4 Sets of characteristic classes for the classes $\mathsf{P}\mathcal{D}G$, $\mathsf{P}\mathcal{D}G_{\mathrm{num}}$, and $\mathsf{P}\mathcal{D}G_{\mathrm{diff}}$

In this subsection, we discuss the sets of characteristic classes for the classes (or categories) $\mathsf{P}\mathcal{D}G$, $\mathsf{P}\mathcal{D}G_{\mathrm{num}}$, and $\mathsf{P}\mathcal{D}G_{\mathrm{diff}}$ (see Definition 5.1) and their relation.

Let $\mathcal{P}$ denote one of the categories $\mathsf{P}\mathcal{D}G$, $\mathsf{P}\mathcal{D}G_{\mathrm{num}}$, and $\mathsf{P}\mathcal{D}G_{\mathrm{diff}}$. For an abelian group A, $\mathrm{char}(\mathcal{P}; A)$ denotes the set of characteristic classes with coefficients in A for the class $\mathcal{P}$. Then, by [Christensen and Wu 2021, Theorem 5.10] and Proposition 1.4, we have the natural bijection

$$\mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{num}}; A) \cong H^*(BG; A)$$

and the retract diagram

$$\begin{tikzcd} \mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{num}}; A) \arrow[r, "\mathrm{ext}"] \arrow[rr, bend right, "1"'] & \mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{diff}}; A) \arrow[r, "\mathrm{res}"] & \mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{num}}; A) \end{tikzcd}$$

where res is the obvious restriction map and ext is the extension map introduced in Proposition 1.4.

We can also show that $\mathrm{char}(\mathsf{P}\mathcal{D}G; A) \cong \mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{num}}; A)$. To prove this, we define the map

$$\mathrm{ext}\colon \mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{num}}; A) \to \mathrm{char}(\mathsf{P}\mathcal{D}G; A)$$

as follows. Let $\alpha(\,\cdot\,)$ be an element of $\mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{num}}; A)$ corresponding to $\alpha \in H^*(BG; A)$. For a given locally trivial principal G–bundle $\pi\colon P \to X$, consider the CW–approximation $|S^{\mathcal{D}}(X)|_{\mathcal{D}} \xrightarrow{p_X} X$ in $\mathcal{D}$, which is the counit of the adjoint pair $(|\cdot|_{\mathcal{D}}, S^{\mathcal{D}})$; see Remark 2.8(2) and [Kihara 2023, Section 3]. Since we can prove that every CW–complex in $\mathcal{D}$ is smoothly paracompact (see [Kihara $\geq$ 2024]), the pullback $p_X^* P$ is a $\mathcal{D}$–numerable principal G–bundle. Thus, we can define the characteristic class $\alpha(P)$ of P by $\alpha(P) = \alpha(p_X^* P)$ under the identification $H^*(X; A) \cong H^*(|S^{\mathcal{D}}(X)|_{\mathcal{D}}, A)$. Then it is clear that the map $\mathrm{ext}\colon \mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{num}}; A) \to \mathrm{char}(\mathsf{P}\mathcal{D}G; A)$ and the obvious restriction map

$$\mathrm{res}\colon \mathrm{char}(\mathsf{P}\mathcal{D}G; A) \to \mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{num}}; A)$$

are mutually inverses. We can easily see from Theorem 1.3 that $\mathrm{ext}\colon \mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{num}}; A) \to \mathrm{char}(\mathsf{P}\mathcal{D}G; A)$ is just the corestriction of $\mathrm{ext}\colon \mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{num}}; A) \to \mathrm{char}(\mathsf{P}\mathcal{D}G_{\mathrm{diff}}; A)$. (Recall that the class of locally trivial principal G–bundles also does not have the homotopy invariance property with respect to pullback and hence that it has no classifying space; see [Christensen and Wu 2021, Section 3].)

We end this section by raising a problem on diffeological principal bundles.

Problem Let X be a CW–complex in $\mathscr{D}$ (or more generally, a cofibrant diffeological space); see [Kihara 2023, Section 3.1]. Is every diffeological principal G–bundle over X locally trivial?

This problem asks whether there exists a non-locally-trivial diffeological principal bundle over a nice diffeological space; all the non-locally-trivial diffeological principal bundles the author knows are ones over bad diffeological spaces.

If the problem is solved affirmatively, we can use the CW–approximation $|S^{\mathscr{D}}(X)|_{\mathscr{D}} \xrightarrow{p_X} X$ to directly construct the map

$$\mathrm{char}(\mathsf{P}\mathscr{D}G_{\mathrm{num}}; A) \xrightarrow{\mathrm{ext}} \mathrm{char}(\mathsf{P}\mathscr{D}G_{\mathrm{diff}}; A)$$

which is the inverse of $\mathrm{char}(\mathsf{P}\mathscr{D}G_{\mathrm{diff}}; A) \xrightarrow{\mathrm{res}} \mathrm{char}(\mathsf{P}\mathscr{D}G_{\mathrm{num}}; A)$.

Further, if the problem is solved affirmatively, then we can replace the singular functor $S^{\mathscr{D}}_{\mathrm{aff}}$ with $S^{\mathscr{D}}$ in Theorem 1.3 and Remark 5.8(1).

Remark 6.13 (1) Results similar to those mentioned above hold in the category $\mathscr{T}$ of topological spaces. More precisely, the homotopy invariance property with respect to pullback need not hold for topological principal G–bundles which are not numerable, and hence the class of topological principal G–bundles does not have a classifying space; see [Andrade 2013; Christensen and Wu 2021, Section 3; Goodwillie 2012]. However, we have two ways of extending the characteristic class associated to a cohomology class α of the (topological) classifying space BG; one uses the CW–approximation $|S(X)| \xrightarrow{p_X} X$ of the base and the other uses the theory of simplicial principal bundles. We can easily see that they define the same extension; the resulting map is denoted by

$$\mathrm{char}(\mathsf{P}\mathscr{T}G_{\mathrm{num}}; A) \xrightarrow{\mathrm{ext}} \mathrm{char}(\mathsf{P}\mathscr{T}G; A),$$

where $\mathrm{char}(\mathsf{P}\mathscr{T}G_{\mathrm{num}}; A)$ and $\mathrm{char}(\mathsf{P}\mathscr{T}G; A)$ are defined in a way similar to the diffeological case. We then see that

$$\mathrm{char}(\mathsf{P}\mathscr{T}G_{\mathrm{num}}; A) \cong H^*(BG; A)$$

and that

$$\mathrm{char}(\mathsf{P}\mathscr{T}G_{\mathrm{num}}; A) \underset{\mathrm{res}}{\overset{\mathrm{ext}}{\rightleftarrows}} \mathrm{char}(\mathsf{P}\mathscr{T}G; A)$$

are mutually inverses.

The results here remain true even if $\mathscr{T}$ is replaced with the category $\mathscr{C}^0$ of arc-generated spaces; see [Kihara 2023, Proposition 5.14(1)].

(2) Since the underlying topological space functor $\tilde{\cdot} : \mathscr{D} \to \mathscr{C}^0$ preserves finite products [Kihara 2019, Proposition 2.13], it induces the functor

$$\tilde{\cdot} : \mathsf{P}\mathscr{D}G \to \mathsf{P}\mathscr{C}^0\widetilde{G}$$

(see [Kihara 2023, Lemma 5.7 and Remark 5.8]). Thus, we use this functor to study the relation between characteristic classes of smooth principal G–bundles and ones of continuous principal G–bundles.

The natural inclusion $S^{\mathcal{D}}X \hookrightarrow S\widetilde{X}$ (see Section 2.3) induces the natural homomorphism

$$H^*(X;A) \xleftarrow{\psi_X} H^*(\widetilde{X};A),$$

which along with [Kihara 2023, Proposition 5.14], defines the horizontal arrows in the commutative diagram

$$\begin{tikzcd} H^*(BG;A) \arrow[d, "\wr|"] & H^*(B\widetilde{G};A) \arrow[l, "\psi_{BG}"'] \arrow[d, "\wr|"'] \\ \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}};A) \arrow[d, "\mathrm{ext}"', "\wr|"] & \mathrm{char}(\mathsf{P}\mathscr{C}^0\widetilde{G}_{\mathrm{num}};A) \arrow[l] \arrow[d, "\wr|"', "\mathrm{ext}"] \\ \mathrm{char}(\mathsf{P}\mathfrak{D}G;A) & \mathrm{char}(\mathsf{P}\mathscr{C}^0\widetilde{G};A) \arrow[l] \end{tikzcd}$$

We can easily see that the equality

$$(\psi_{BG}\alpha)(P) = \psi_X(\alpha(\widetilde{P}))$$

holds for $P \in \mathsf{P}\mathfrak{D}G$.

If G is a Lie group (or more generally, in the class $\mathscr{V}_{\mathfrak{D}}$), then $H^*(BG;A) \xleftarrow{\psi_{BG}} H^*(B\widetilde{G};A)$ is an isomorphism (see [Kihara 2023, Theorem 11.2, and Corollaries 1.6 and 5.16]), and hence all the arrows in the above commutative diagram are bijective. (Here, a Lie group is defined to be a group in the category C^∞ of C^∞–manifolds in the sense of [Kriegl and Michor 1997, Section 27]; see [Kihara 2023, Section 2.2].)

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Center for Mathematical Sciences, University of Aizu
Aizuwakamatsu, Japan

kihara@u-aizu.ac.jp

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