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The shape of the filling-systole subspace in surface moduli space and critical points of the systole function

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We study the space $X_g \subset \mathcal{M}_g$ consisting of surfaces with filling systoles and its subset, critical points of the systole function. In the first part we obtain a surface with Teichmüller distance $\frac{1}{5}\log\log g$ to X_g, and in the second and third parts prove that most points in $\mathcal{M}_g$ have Teichmüller distance $\frac{1}{5}\log\log g$ and Weil–Petersson distance $0.6521(\sqrt{\log g} - \sqrt{7\log\log g})$ to X_g. So the radius-r neighborhood of X_g cannot cover the thick part of $\mathcal{M}_g$ for any fixed $r > 0$. In the last two parts, we get critical points with small and large (comparable to the diameter of the thick part of $\mathcal{M}_g$) distances.

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1 Introduction

1.A Motivations

A long-standing and difficult question on the moduli space of Riemann surfaces of genus g (denoted by $\mathcal{M}_g$) is to construct a spine of $\mathcal{M}_g$ (the deformation retract of $\mathcal{M}_g$ with minimal dimension.)[1] This question is equivalent to constructing a mapping class group equivariant deformation retract with the minimal dimension of the Teichmüller space $\mathcal{T}_g$. In an unpublished manuscript, Thurston [1986b] proposed a candidate for the spine of $\mathcal{M}_g$; see Anderson, Parlier and Pettet [Anderson et al. 2016]. This candidate consists of surfaces whose shortest geodesics are filling, and is denoted by X_g (A finite set of

[1] In some papers a deformation retract of $\mathcal{M}_g$ is called a spine of $\mathcal{M}_g$, and the ones with minimal dimension are called minimal (or optimal) spines

essential curves on a surface is filling if the curves cut the surface into polygonal disks.) Thurston outlined a proof that X_g is a deformation retract of $\mathcal{M}_g$, but the proof seems difficult to complete. Recently, some progress on the dimension of X_g has been made; for example, a codimension-2 deformation retract of $\mathcal{M}_g$ containing X_g (see Ji [2014]) and a $(4g-5)$–cell contained in X_g (see Fortier Bourque [2020]). But determining the dimension of X_g still seems very difficult.

Our work mainly concerns the shape of X_g with respect to the Teichmüller and Weil–Petersson metrics on $\mathcal{M}_g$. The shape of X_g was first studied by Anderson, Parlier and Pettet [Anderson et al. 2016], and our work is partly inspired by the notion of the sparseness of subsets in $\mathcal{M}_g$ they raised. Our question is:

Question 1.1 Does there exist a number $R = R(g) > 0$ such that, for most points $p \in \mathcal{M}_g$, $d_{\mathcal{T}}(p, X_g)$ (or $d_{\mathrm{WP}}(p, X_g)$) is larger than $R(g)$?

In other words: is X_g in some sense "sparse" in $\mathcal{M}_g$?

Another motivation to study the shape of X_g is to understand the shape of the critical-point set of the systole function. On each surface $p \in \mathcal{M}_g$, the systole is the length of the shortest geodesics on p. Therefore it can be treated as a function on $\mathcal{M}_g$. Akrout [2003] showed that this function is a topological Morse function; hence the systole function has regular and critical points. The critical-point set of this function is denoted by $\mathrm{Crit}(\mathrm{sys}_g)$. By Schmutz Schaller [1999, Corollary 20], $\mathrm{Crit}(\mathrm{sys}_g) \subset X_g$. Therefore conclusions on the shape of X_g imply corollaries on the shape of $\mathrm{Crit}(\mathrm{sys}_g)$. On the other hand, a natural question is to compare the shape difference between X_g and $\mathrm{Crit}(\mathrm{sys}_g)$. This program is closely related to the question of Mirzakhani as to whether long fingers exist. Details are in the following subsection.

1.B Results and perspectives

Our first result is the construction of an example of a surface in the thick part of $\mathcal{M}_g$ that is distant from X_g.

Proposition 3.6 *When $g \geq 3$ there is a surface S_g with $\mathrm{sys}(S_g) = \operatorname{arccosh} 2$ whose distance to X_g is at least $\frac{1}{4}\log(\log g - K)$, where $K = \log 12$.*

Remark 1.2 If a surface's systole is sufficiently small, then its Teichmüller distance to X_g could be arbitrarily large. But our example has constant systole while it is distant from X_g.

Before stating Theorem 4.3, we make "most points" in Question 1.1 precise.

The Weil–Petersson metric is a mapping class group equivariant Riemannian metric on the Teichmüller space. Therefore the volume of $\mathcal{M}_g$ and Borel subsets of $\mathcal{M}_g$ with respect to this metric is well defined. Mirzakhani [2007] invented the integration formula for geometric functions on $\mathcal{M}_g$ with respect to this volume and then calculated the volume of $\mathcal{M}_g$. She initiated a fast-growing area: random surfaces with respect to the Weil–Petersson metric; see Mirzakhani [2007; 2013].

The random surface theory is based on the probability of Borel sets in $\mathcal{M}_g$. Mirzakhani defined the probability of a Borel set $B \subset \mathcal{M}_g$ as

$$P_{\mathrm{WP}}(B) = \frac{\mathrm{vol}_{\mathrm{WP}}(B)}{\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g)}.$$

Theorem 4.3 $$P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid d_{\mathcal{T}}(S, X_g) < \tfrac{1}{5}\log\log g\} \to 0 \quad \text{as } g \to \infty.$$

Remark 1.3 The distance $\frac{1}{5}\log\log g$ is calculated from (3-1) in Lemma 3.2 and the width by Nie, Wu and Xue [Nie et al. 2023, Theorem 2]. Actually, if we replace $\frac{1}{5}$ by any number smaller than $\frac{1}{4}$, this theorem still holds. Besides Lemma 3.2 and [Nie et al. 2023, Theorem 2], Theorem 4.3 also depends on Mirzakhani's Theorem 2.8 in [Mirzakhani and Petri 2019].

Theorem 4.3 gives a positive answer to Question 1.1 with respect to Teichmüller distance. When g is sufficiently large, most points in $\mathcal{M}_g$ have Teichmüller distance at least $\frac{1}{5}\log\log g$ to X_g.

The moduli space $\mathcal{M}_g$ is divided into two parts. The thick part consists of surfaces with systole larger than or equal to ε for some fixed $\varepsilon > 0$, denoted by $\mathcal{M}_g^{\geq\varepsilon}$. This part is compact in $\mathcal{M}_g$, and its diameter with respect to the Teichmüller metric is $C\log(g/\varepsilon)$ for some $C > 0$ by Rafi and Tao [2013]. The complementary part of the thick part is the thin part.

By the collar lemma (see for example Buser [1992, Chapter 4]), X_g is contained in the thick part of $\mathcal{M}_g$ and we have:

Corollary 4.4 $P_{\mathrm{WP}}\{d_{\mathcal{T}}(S, X_g) < \frac{1}{5}\log\log g \mid S$ *lies in the thick part of* $\mathcal{M}_g\} \to 0$ *as* $g \to \infty$.

From Proposition 3.6 or Corollary 4.4, the Hausdorff distance between the thick part of $\mathcal{M}_g$ and X_g is at least $\frac{1}{5}\log\log g$.

The study of the shape of X_g with respect to the Teichmüller metric was pioneered by Anderson, Parlier and Pettet [Anderson et al. 2016]. By comparing X_g with Y_g, the subset of $\mathcal{M}_g$ with Bers' constant bounded above and below by constants, they obtained the following two results: the diameter of X_g is comparable with the thick part of $\mathcal{M}_g$ [Anderson et al. 2016, Theorem 1.1], and the sparseness of $X_g \cap Y_g$ in Y_g, that is, most points in Y_g have distance at least $\log g$ to $X_g \cap Y_g$ [Anderson et al. 2016, Theorem 1.3].[2]

The distance in Proposition 3.6 and Theorem 4.3 is smaller than that of [Anderson et al. 2016, Theorem 1.3], but we remove the restriction to Y_g and obtain the sparseness of X_g in $\mathcal{M}_g$ and thick part of $\mathcal{M}_g$.

An immediate corollary to Proposition 3.6 or Corollary 4.4 is:

Corollary 1.4 *For any* $R > 0$, *when* g *is sufficiently large, the* R*–neighborhood of* X_g *does not cover the thick part of* $\mathcal{M}_g$. *Hence the* R*–neighborhood of* $\mathrm{Crit}(\mathrm{sys}_g)$ *does not cover the thick part of* $\mathcal{M}_g$.

For the thick part of $\mathcal{M}_g$, Fletcher, Kahn and Markovic [Fletcher et al. 2013] determined the minimal size of a point set in $\mathcal{M}_g^{\geq\varepsilon}$ whose R neighborhood covers the whole thick part for any $R > 0$. The size

[2] For the meaning of the "most points" and the definition of the distance, see [Anderson et al. 2016].

is $(Cg)^{2g}$ for $C = C(\varepsilon, R) > 0$. Currently the size of $\mathrm{Crit}(\mathrm{sys}_g)$ is not determined, but a known lower bound for $|\mathrm{Crit}(\mathrm{sys}_g)|$ given by the Euler characteristic of $\mathcal{M}_g$ (see [Harer and Zagier 1986]) is quite close to this number. However, by Corollaries 4.4 and 1.4, $\mathrm{Crit}(\mathrm{sys}_g)$ is sparse in $\mathcal{M}_g^{\geq\varepsilon}$.

We also answer Question 1.1 with respect to the Weil–Petersson metric:

Theorem 5.7 $P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid d_{\mathrm{WP}}(S, X_g) < 0.6521(\sqrt{\log g} - \sqrt{7 \log\log g})\} \to 0$ *as* $g \to \infty$.

Besides the tools used in the proof of Theorem 4.3, to prove this theorem we also use Wu's estimate [2022] of lower bounds of Weil–Petersson distance. Using this estimate, Wu [2022, Theorem 1.4] has obtained that the probability of the Weil–Petersson $\sqrt{\log g}$–neighborhood of all surfaces with $o(\log g)$ Bers' constant tends to 0 as g tends to infinity.

After answering Question 1.1, a further question is:

Question 1.5 Is there a critical point $p \in \mathrm{Crit}(\mathrm{sys}_g)$ and a large number $R(g)$ such that $B(p, R(g))$ contains no critical point except p?

This question concerns the distances between the elements of $\mathrm{Crit}(\mathrm{sys}_g)$ and X_g. The radius gives a lower bound for the Hausdorff distance between X_g and $\mathrm{Crit}(\mathrm{sys}_g)$. Moreover, Question 1.5 is very close to but slightly weaker than Mirzakhani's question of whether there exists a long finger (see Fortier Bourque and Rafi [2022]) when the systole has a large local maximum at p.

For such a point p, a component of the level set $\{q \mid \mathrm{sys}(q) > L\}$ that contains p but does not contain any other critical point of the systole function is called a finger. The length of a finger is $\mathrm{sys}(p) - L$. If a finger is long, then the Teichmüller distance from p to other critical points is large (at least $\frac{1}{2}\log(\mathrm{sys}(p)/L)$).

We make the first attempt to compare the difference between X_g and $\mathrm{Crit}(\mathrm{sys}_g)$.

For any $g \geq 2$, we take three surfaces S_g^1, S_g^2 and S_g^3 that were originally constructed by Anderson, Parlier and Pettet [Anderson et al. 2011], Gao and Wang [2023] and Fortier Bourque and Rafi [2022], respectively. The surfaces S_g^1 and S_g^3 are known critical points, and we prove S_g^2 is a critical point by our Proposition 6.3. Then we calculate the distance between the critical points.

Theorem 8.3 *For the surfaces* $S_g^1, S_g^3 \in \mathrm{Crit}(\mathrm{sys}_g)$, *when* $g \geq 13$,

$$d_{\mathcal{T}}(S_g^1, S_g^3) > \tfrac{1}{2}\log(g-6) - K,$$

where $K = \frac{1}{2}\log\bigl(\frac{40}{3}\log((4g+4)/\pi)\bigr)$.

Hence the diameter of $\mathrm{Crit}(\mathrm{sys}_g)$ is comparable with the diameter of X_g and the diameter of the thick part of $\mathcal{M}_g$.

On the other hand, the distance between S_g^1 and S_g^2 is small.

Theorem 7.10 *For any* $g \geq 2$ *and* $S_g^1, S_g^2 \in \mathrm{Crit}(\mathrm{sys}_g)$,

$$d_{\mathcal{T}}(S_g^1, S_g^2) \leq 2.3.$$

It is worth mentioning that to prove the surface Σ_g^2 is a critical point, we use a conclusion (Proposition 6.3) that among all surfaces with a specific symmetry, the surface with maximal systole is a critical point. This proposition is a generalization of Schmutz Schaller [1999, Theorem 37] and Fortier Bourque [2020, Proposition 6.3]. The key point of this generalization is to construct a domain in $\mathcal{M}_g$ containing the point p we consider, and p is the maximal point of the systole function in the domain.

1.C Methods

To prove "most surfaces" are distant from X_g, we avail ourselves of lower bounds of Teichmüller and Weil–Petersson distance (Lemma 3.2 and Wu [2022, Theorem 1.1], respectively). For "most surfaces" there is an embedded cylinder with a large length and large width by Nie, Wu and Xue [Nie et al. 2023] and the systoles of the surfaces are relatively small by a theorem of Mirzakhani [Mirzakhani and Petri 2019, Theorem 2.8]. By the lower bound estimates, surfaces containing such a cylinder are distant from X_g.

Theorem 8.3 is obtained by comparing the diameter of the two surfaces. This method is from Rafi and Tao [2013, Lemma 5.1].

The shapes of S_g^1 and S_g^2 are similar. Then we can construct the deformation from S_g^1 to S_g^2 explicitly. From the deformation we describe in Section 7, we calculate the distance and get Theorem 7.10.

Organization In Section 2, we provide some preliminary knowledge on Teichmüller theory and the systole. Then we prove Proposition 3.6 in Section 3 and Theorem 4.3 in Section 4. On the Weil–Petersson distance, we prove Theorem 5.7 in Section 5. In Section 6, Proposition 6.3 is proved. Then using Proposition 6.3, Theorem 7.10 is proved in Section 7. Finally, Theorem 8.3 is proved in Section 8.

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2 Preliminaries

2.A Teichmüller space

We denote by $\mathcal{T}_g$ the Teichmüller space consisting of marked hyperbolic surfaces with genus g, and by $\mathcal{M}_g$ the moduli space consisting of hyperbolic surfaces with genus g. It is known that

$$\mathcal{M}_g \cong \mathcal{T}_g / \Gamma_g.$$

Here Γ_g is the mapping class group of a closed orientable surface of genus g.

The Teichmüller metric is a complete mapping class group equivariant metric on $\mathcal{T}_g$ defined using the dilatation of quasiconformal maps. For $X, Y \in \mathcal{T}_g$, the distance between X and Y is denoted by $d_{\mathcal{T}}(X, Y)$. The formal definition of this metric is deferred to Section 7.C.1 since it is not needed for most of this paper.

2.B Thurston's metric

Thurston [1986a] defined an asymmetric metric on the Teichmüller space. For $X, Y \in \mathcal{T}_g$ and $f : X \to Y$ a Lipschitz homeomorphism between X and Y, we let

$$L(f) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d(f(x), f(y))}{d(x, y)}.$$

Then this metric is defined as

$$d_L(X, Y) = \inf_f \{\log L(f) \mid f : X \to Y \text{ is a Lipschitz homeomorphism}\}.$$

Theorem 2.1 [Thurston 1986a] *For $X, Y \in \mathcal{M}_g$,*

$$d_L(X, Y) = \sup_{\alpha \in C(X)} \inf_{f : X \to Y} \log \frac{l_{f(\alpha)}(Y)}{l_\alpha(X)}.$$

Here f is a Lipschitz homeomorphism and $C(X)$ is the set of simple closed curves in X.

For $X, Y \in \mathcal{T}_g$, Rafi and Tao [2013, (2)] have shown that

$$\tfrac{1}{2} d_L(X, Y) \leq d_{\mathcal{T}}(X, Y). \tag{2-1}$$

2.C The topological Morse function and generalized systole

Definition 2.2 On a topological manifold M^n, a function $f : M^n \to \mathbb{R}$ is a topological Morse function if, at each point $p \in M$, there is a neighborhood U of p and a map $\psi : U \to \mathbb{R}^n$. Here ψ is a homeomorphism between U and its image such that $f \circ \psi^{-1}$ is either a linear function or

$$f \circ \psi^{-1}((x_1, x_2, \dots, x_n)) = f(p) - x_1^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_n^2.$$

In the former case the point p is called a regular point of f, while in the latter case the point p is called a singular point with index j.

On a Riemannian manifold M, $l_\alpha : M \to \mathbb{R}^+$ is a family of smooth functions on M indexed by $\alpha \in I$, called the (*generalized*) *length function*. The length function family is required to satisfy the following condition: for every $p \in M$ there exists a neighborhood U of p and a number $K > 0$ such that the set $\{\alpha \mid l_\alpha(q) \leq K \text{ for all } q \in U\}$ is a nonempty finite set. The (*generalized*) *systole function* is defined as

$$\operatorname{sys}(p) := \inf_{\alpha \in I} l_\alpha(p) \quad \text{for all } p \in M.$$

Theorem 2.3 [Akrout 2003] *If, for any $\alpha \in I$, the Hessian of l_α is positively definite, then the generalized systole function is a topological Morse function.*

The critical point of the systole function is also characterized in [Akrout 2003]. A $p \in M$ is a eutactic point if and only if it is a critical point of the systole function.

We assume that, for $p \in M$,

$$S(p) := \{\alpha \in I \mid l_\alpha(p) = \mathrm{sys}(p)\}.$$

Definition 2.4 For $p \in M$, p is eutactic if and only if 0 is contained in the interior of the convex hull of $\{\mathrm{d}l_\alpha|_p \mid \alpha \in S(p)\}$.

An equivalent definition is:

Definition 2.5 $p \in M$ is eutactic if and only if for $v \in T_pM$, if $\mathrm{d}l_\alpha(v) \geq 0$ for all $\alpha \in S(p)$, then $\mathrm{d}l_\alpha(v) = 0$ for all $\alpha \in S(p)$.

2.D Teichmüller space and length function

For a marked hyperbolic surface Σ in the Teichmüller space $\mathcal{T}_g$, $\alpha \subset \Sigma$ is an essential simple closed geodesic. Its length is denoted by $l_\alpha(\Sigma)$. In another point of view, l_α is a function on $\mathcal{T}_g$:

$$l_\alpha : \mathcal{T}_g \to \mathbb{R}^+, \quad \Sigma \mapsto l_\alpha(\Sigma).$$

The set of all the shortest geodesics on Σ is denoted by $S(\Sigma)$. For $\alpha \in S(\Sigma)$,

$$l_\alpha(\Sigma) \leq l_\beta(\Sigma) \quad \text{for all simple closed geodesics } \beta \subset \Sigma.$$

The length of the shortest geodesics of Σ is called *systole* of Σ.

Similarly, the systole can be treated as a function on $\mathcal{T}_g$, and we denote it by sys_g or shortly sys. Obviously

$$\mathrm{sys}(\Sigma) = l_\alpha(\Sigma) = \inf_{\text{simple closed geodesics } \beta \subset \Sigma} l_\beta(\Sigma).$$

Remark 2.6 In a small neighborhood U of Σ in $\mathcal{T}_g$, the systole function is realized by the minimum lengths of finitely many simple closed geodesics.

Remark 2.7 Systole function can also be defined as a function on $\mathcal{M}_g$:

$$\mathrm{sys} : \mathcal{M}_g \to \mathbb{R}^+, \quad \Sigma \mapsto \mathrm{sys}(\Sigma).$$

However, the length function l_α is not well-defined on $\mathcal{M}_g$ because of the monodromy.

By [Wolpert 1987], the Hessian of l_α is always positive definite for any simple closed geodesic $\alpha \subset \Sigma$ with respect to the Weil–Petersson metric. Therefore:

Corollary 2.8 [Akrout 2003, corollaire, page 2] *The systole function is a topological Morse function on $\mathcal{T}_g$.*

The systole function is also a topological Morse function on $\mathcal{M}_g$, because the systole function is an invariant function on Teichmüller space.

The set of all the critical points of sys_g in $\mathcal{T}_g$ is denoted by $\mathrm{Crit}(\mathrm{sys}_g)$.

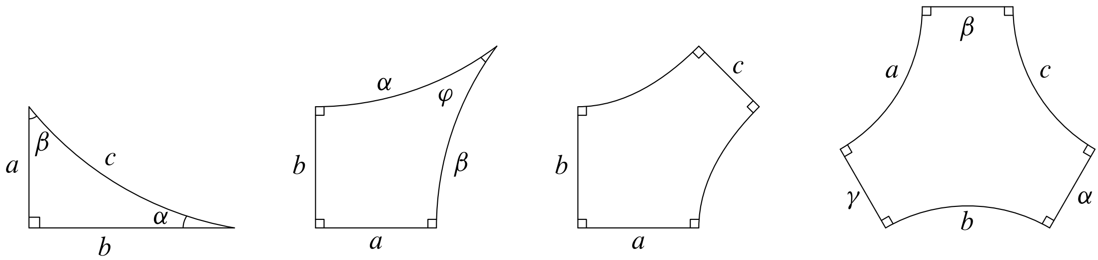

Figure 1: Hyperbolic polygons. The right-angled triangle (left), trirectangle (middle left), right-angled pentagon (middle right) and right-angled hexagon (right).

2.E Hyperbolic trigonometric formulae

The following are from [Buser 1992, page 454] and are pictured in Figure 1:

(2-2) $\cosh c = \cot\alpha \cot\beta.$ (right-angled triangles),

(2-3) $\cos\varphi = \sinh a \sinh b$ (trirectangles),

(2-4) $\cosh c = \sinh a \sinh b$ (right-angled pentagons),

(2-5) $\cosh c = \sinh a \sinh b \cosh\gamma - \cosh a \cosh b$ (right-angled hexagons).

3 The surface S_g

In this section we construct a surface S_g whose Teichmüller distance to X_g is at least $\frac{1}{4}\log(\log g - \log 12)$.

3.A Construction of the surface S_g when $g = 3 \cdot 2^{n-1}$

To construct a surface S_g, we first construct a tree $T(n)$ with m vertices. The tree's diameter is required to be comparable with $\log m$.

We define the tree $T(n)$ by the following two properties:

(1) Every vertex, except the leaves of $T(n)$, has degree 3.

(2) There is a vertex O of $T(n)$ such that the combinatorial distance from every leaf of $T(n)$ to O is n.

The tree $T(2)$ is shown in Figure 2.

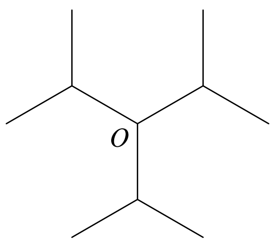

Figure 2: The tree $T(2)$.

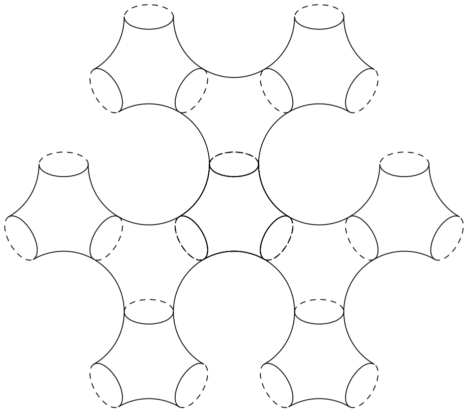

Figure 3: The sphere with $6 \cdot 2^n$ boundary components.

Now we construct the surface S_g from the tree $T(n)$. We pick several isometric pairs of pants as building blocks of S_g. Each pair consists of two regular right-angled hexagons. A boundary component of the pants is called a *cuff*, and an edge of the hexagons in the interior of the pants is called a seam. We glue the pants together according to the tree $T(n)$.

Then we glue together the pants. A vertex of $T(n)$ corresponds to a pair of pants; two pairs of pants are glued together at a cuff if there is an edge that connects the corresponding vertices. Now we get a sphere with $3 \cdot 2^n$ boundary components (Figure 3). For each pair of pants corresponding to a leaf in the tree, we glue together the two cuffs of the pair that are not glued with the other pants. Then we get a closed surface with genus g, where $g = 3 \cdot 2^{n-1}$. At each cuff, we require the gluing to have "no twist". In other words, when gluing two pairs of pants together at a cuff, endpoints of seams from one pair of pants are required to be glued with the endpoints of seams from the other; when gluing two cuffs in the same pair of pants, ends of seams from the two sides of the cuff are required to be glued together. Therefore we construct a unique hyperbolic surface, denoted by S_g.

In S_g, in each one-holed torus (glued from a pair of pants) corresponding to a leaf of the tree, there is a unique simple closed curve consisting of one seam of the pants. We denote this curve by α_k, where $k = 1, 2, \ldots, g$. Now we prove that this curve is the shortest in S_g.

Lemma 3.1 *The shortest closed geodesics on S_g are exactly the curves $\alpha_1, \alpha_2, \ldots, \alpha_g$, and therefore the systole of S_g is* $\operatorname{arccosh} 2$.

Proof By (2-5), the edge length of regular right-angled hexagons is $\operatorname{arccosh} 2$, and hence the cuff length of the pants is $2 \operatorname{arccosh} 2$ and the seam length is $\operatorname{arccosh} 2$. Therefore the length of α_k is the seam length of the pants, $\operatorname{arccosh} 2$. If a curve in S_g intersects at least three pairs of pants, then this curve is longer than α_k because this curve must pass through two cuffs that belong to one of the three pants.

In a pair of pants, the only simple closed geodesics are the cuffs. The cuff length of the pants is exactly twice the length of α_k.

If a curve is contained in two neighboring pairs of pants, then it intersects the two pants' shared cuff and the seams opposite the cuff. However, by (2-4), the distance between the cuff and the seam is larger than the length of α_k.

Therefore $\{\alpha_k\}_{k=1}^{g}$ is the set of shortest geodesics of S_g. □

3.B Distance between S_g and X_g

The distance between a surface and X_g is estimated below by the following lemma:

Lemma 3.2 *For a surface $S \in \mathcal{M}_g$, let $L > 0$. If, for any filling curve set F in which each pair of curves intersect at most once, F contains a curve longer than L, then*

$$d_{\mathcal{T}}(S, X_g) \geq \tfrac{1}{4} \log \frac{L}{\operatorname{sys}(S)}. \tag{3-1}$$

Proof We let $S \in \mathcal{M}_g$. For any filling curve set $F \subset S$ in which each pair of curves intersects at most once, F contains a curve longer than L.

For any $S' \in X_g$, we assume $F' \subset S'$ is the set of shortest geodesics in S'. Since $S' \in X_g$, F' is filling in S'.

For any Lipschitz homeomorphisms $f: S \to S'$ and $g: S' \to S$, we let $\alpha \subset S$ be a shortest geodesic in S and $\beta \subset S'$ be a shortest geodesic with $l_{g(\beta)(S)} > L$. Then by Theorem 2.1,

$$\exp(d_L(S, S')) \geq \frac{l_{f(\alpha)}(S')}{l_\alpha(S)} \geq \frac{\operatorname{sys}(S')}{\operatorname{sys}(S)}.$$

On the other hand,

$$\exp(d_L(S', S)) \geq \frac{l_{g(\beta)}(S)}{l_\beta(S')} \geq \frac{L}{\operatorname{sys}(S')}.$$

Then, by (2-1), $d_{\mathcal{T}}(S, S') \geq \frac{1}{2} d_L(S, S')$ and $d_{\mathcal{T}}(S, S') \geq \frac{1}{2} d_L(S', S)$. For any $\operatorname{sys}(S') > 0$,

$$\max\left(\frac{\operatorname{sys}(S')}{\operatorname{sys}(S)}, \frac{L}{\operatorname{sys}(S')}\right) \geq \sqrt{\frac{L}{\operatorname{sys}(S)}}.$$

Therefore,

$$d_{\mathcal{T}}(S, S') \geq \tfrac{1}{2} \log \sqrt{\frac{L}{\operatorname{sys}(S)}} = \tfrac{1}{4} \log \frac{L}{\operatorname{sys}(S)}.$$ □

Now we estimate the distance between S_g and X_g using Lemma 3.2.

We let P_k, $k = 1, \dots, g$ be the one-holed tori corresponding to leaves of the tree $T(n)$. An observation is that $S_g \backslash \{P_k\}_{k=1}^{g}$ is a g–holed sphere.

Immediately we have:

Lemma 3.3 *In S_g, for any filling curve set F in which each pair of curves intersects at most once, any curve in F intersects at least one P_k in $\{P_k\}_{k=1}^{g}$.*

Proof If a curve does not intersect any P_k for $k = 1, 2, \ldots, g$, then it is contained in the g–holed sphere $S_g \backslash \{P_k\}_{k=1}^{g}$, and hence is a separating curve. A separating curve cannot intersect any curve once. On the other hand, a curve in a filling set F always intersects other curves in F. □

Lemma 3.4 *In S_g, for any filling curve set F in which each pair of curves intersects at most once, F contains a curve β such that*

$$l_\beta(S_g) > n \operatorname{arccosh} 2,$$

where $g = 3 \cdot 2^{n-1}$.

Proof The construction of S_g gives a natural pants decomposition on S_g. A filling curve set must intersect every pair of pants in this decomposition because filling curve sets cut the surface into disks.

For the pants corresponding to the center vertex O shown in Figure 2, we let β be a curve in F passing through this pair of pants. Then by Lemma 3.3, β intersects some one-holed sphere corresponding to a leaf in the tree $T(n)$. The combinatorial distance between the vertex O and any leaf of the tree is at least n. Then by the construction of S_g, the distance between the corresponding two pairs of pants is at least $n \operatorname{arccosh} 2$, where $\operatorname{arccosh} 2$ is the length of seams of the pairs of pants used to construct S_g.

Therefore $l_\beta(S_g) > n \operatorname{arccosh} 2$. □

By Lemmas 3.4 and 3.2, immediately we have:

Proposition 3.5 *When $g = 3 \cdot 2^{n-1}$ for any positive integer n, the distance between S_g and X_g is larger than*

$$d_{\mathcal{T}}(S_g, X_g) > \tfrac{1}{4} \log n.$$

3.C Construction in general genus

We have proved Proposition 3.6 when $g = 3 \cdot 2^{n-1}$. Now we construct S_g when $3 \cdot 2^{n-1} < g < 3 \cdot 2^n$.

Take a tree T with g leaves, such that $T(n) \subset T \subset T(n+1)$. By the embedding $T(n) \to T$, we define the vertex of O in T as the image of vertex O in $T(n)$. Then in the tree T, the combinatorial distance from O to any leaf of T is larger than n.

Similarly to the construction at the beginning of this section, we can construct a genus-g surface S_g from the tree T. By Lemma 3.2, the distance between S_g and X_g is larger than $\frac{1}{4} \log n$. Since $g < 3 \cdot 2^n$, we have:

Proposition 3.6 *For any $g \geq 3$, the distance from the surface S_g with $\operatorname{sys}(S_g) = \operatorname{arccosh} 2$ to the space X_g is larger than*

$$d_{\mathcal{T}}(S_g, X_g) > \tfrac{1}{4} \log(\log g - \log 12).$$

4 Sparseness of X_g

4.A Two theorems on random surfaces

We list two theorems on random surfaces we need for the proof of Theorem 4.3.

Theorem 4.1 [Mirzakhani and Petri 2019, Theorem 2.8] *There exist $A, B > 0$ such that, for any sequence $\{c_g\}$ of positive numbers with $c_g < A \log g$, we have*

$$P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid \mathrm{sys}(S) > c_g\} < B c_g e^{-c_g}.$$

In a hyperbolic surface, the *half collar* of a simple closed geodesic γ with width w is an embedded cylinder in the surface. One of the boundary curves of the cylinder is the geodesic γ, and this cylinder consists of points with distance at most w to γ on one side of γ.

Theorem 4.2 [Nie et al. 2023, Theorems 1 and 2] *For any $\varepsilon > 0$, consider the following conditions*:

(a) *There is a simple closed curve γ in S that has a half collar with width $\frac{1}{2}\log g - \left(\frac{3}{2}+\varepsilon\right)\log\log g$.*

(b) *The length of the curve γ in* (a) *is larger than $2\log g - 5\log\log g$.*

Then

$$P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid S \text{ satisfies (a) and (b)}\} \to 1$$

as $g \to \infty$.

4.B The sparseness of X_g

Theorem 4.3 $P_{\mathrm{WP}}\left\{S \in \mathcal{M}_g \mid d_{\mathcal{T}}(S, X_g) < \frac{1}{5}\log\log g\right\} \to 0$ *as* $g \to \infty$.

Proof By Theorem 4.1, if we let $c_g = \frac{1}{5}\log\log g$, then

$$P_{\mathrm{WP}}\left\{S \in \mathcal{M}_g \mid \mathrm{sys}(S) > \tfrac{1}{5}\log\log g\right\} < B\frac{\frac{1}{5}\log\log g}{(\log g)^{1/5}}.$$

For $S \in \mathcal{M}_g$ and $\mathrm{sys}(S) \le \frac{1}{5}\log\log g$, if S satisfies Theorem 4.2(a), then for any filling curve set F in S, F contains a curve of length at least $\log g - 2\log\log g$ since in F there must be a curve intersecting the separating curve γ in condition (a). Then by Lemma 3.2, the distance between S and X_g is bounded below by

$$\frac{1}{4}\log\frac{\log g - 2\log\log g}{\frac{1}{5}\log\log g} > \frac{1}{5}\log\log g.$$

By Theorem 4.2, $P_{\mathrm{WP}}\left\{S \in \mathcal{M}_g \mid d_{\mathcal{T}}(S, X_g) > \frac{1}{5}\log\log g\right\} \to 1$ as $g \to \infty$ and so the theorem holds. □

Recall that X_g is contained in the thick part $\mathcal{M}_g^{\geq\varepsilon}$ in $\mathcal{M}_g$. The thick part $\mathcal{M}_g^{\geq\varepsilon}$ has positive probability in $\mathcal{M}_g$ by [Mirzakhani and Petri 2019, Theorem 4.1]; immediately we have:

Corollary 4.4 $P_{\mathrm{WP}}\left\{d_{\mathcal{T}}(S, X_g) < \frac{1}{5}\log\log g \mid S \text{ lies in the thick part of } \mathcal{M}_g\right\} \to 0$ *as* $g \to \infty$.

5 The Weil–Petersson distance version of Theorem 4.3

Besides the Teichmüller distance, if we consider the Weil–Petersson distance to X_g, we can prove Theorem 5.7.

5.A Lower bounds on Weil–Petersson distance

The main tools to prove Theorem 5.7 are Theorems 4.1 and 4.2, and the lower bounds on Weil–Petersson distance of Wu [2022].

Before stating Wu's result, we prepare some definitions; for details, see [Wu 2022].

We let $\mathcal{M}$ be the space of complete Riemannian metrics on the topological surface S_g with constant curvature -1. Then by the definition of Teichmüller space, $\mathcal{T}_g = \mathcal{M}/\mathrm{Diff}_0(S_g)$ where $\mathrm{Diff}_0(S_g)$ is the group of diffeomorphism of S_g isotopic to the identity. Let $\pi\colon \mathcal{M} \to \mathcal{T}_g$ be the natural projection. We recall from Rupflin and Topping [2018] that a smooth path $c(t) \subset \mathcal{M}$ is a *horizontal curve* if there exists a holomorphic quadratic differential $q(t)$ on $c(t)$ such that $\partial c(t)/\partial t = \operatorname{Re} q(t)$.[3]

On a surface $X \in \mathcal{M}$ for $p \in X$, we let $\mathrm{inj}_X(p)$ be the *injectivity radius* of X at p, namely the half length of shortest essential loop on X passing through p. Then we define

Definition 5.1 On a topological surface $\Sigma_g\, (g \geq 2)$, fix $p \in \Sigma_g$. For any $X, Y \in \mathcal{T}_g$, we define

$$|\sqrt{\mathrm{inj}_X(p)} - \sqrt{\mathrm{inj}_Y(p)}| := \sup_c |\sqrt{\mathrm{inj}_{c(0)}(p)} - \sqrt{\mathrm{inj}_{c(1)}(p)}|,$$

where $c\colon [0,1] \to \mathcal{M}$ runs over all smooth horizontal curves, with $\pi(c(0)) = X$, $\pi(c(1)) = Y$ and $\pi(c([0,1)) \subset \mathcal{T}_g$ the Weil–Petersson geodesic connecting X and Y.

Theorem 5.2 [Wu 2022, Theorem 1.1] *For a topological surface Σ_g with $g \geq 2$, fix a point $p \in S_g$. Then, for any $X, Y \in \mathcal{T}_g$,*

$$|\sqrt{\mathrm{inj}_X(p)} - \sqrt{\mathrm{inj}_Y(p)}| \leq 0.3884 d_{\mathrm{WP}}(X, Y),$$

where $d_{\mathrm{WP}}(X,Y)$ is the Weil–Petersson distance.

A corollary to this theorem is also needed:

Corollary 5.3 [Wu 2022, Corollary 1.2] *For $X, Y \in \mathcal{T}_g$,*

$$|\sqrt{\mathrm{sys}(X)} - \sqrt{\mathrm{sys}(Y)}| \leq 0.5492\, d_{\mathrm{WP}}(X, Y)$$

Remark 5.4 Before this corollary, the function $\sqrt{\mathrm{sys}(\cdot)}$ was proved to be uniformly Lipschitz on $\mathcal{T}_g$ endowed with the Weil–Petersson metric by Wu [2019].

5.B The theorem with respect to Weil–Petersson distance

Now we begin to prove Theorem 5.7. First, we prove the following two lemmas:

Lemma 5.5 *If $S \in \mathcal{T}_g$ satisfies Theorem 4.2*(a)–(b), *then there is a curve $\alpha \subset S$, freely homotopic to the geodesic γ in the conditions* (a) *and* (b), *such that, for any point $p \in \alpha$,*

$$\mathrm{inj}_S(p) \geq \tfrac{1}{4} \log g - \left(\tfrac{3}{4} + \tfrac{\varepsilon}{2}\right) \log\log g.$$

[3] For a hyperbolic metric $g \in \mathcal{M}$, the tangent space of $\mathcal{M}$ can be decomposed as $\{\operatorname{Re} q \mid q \text{ is a quadratic differential on } (S,g)\} \oplus \{\mathcal{L}_g \mid X \in \Gamma(TS)\}$. For details, see [Rupflin and Topping 2018].

Proof By conditions (a) and (b), $\gamma \subset S$ is a simple closed geodesic of length $2\log g - 5\log\log g$, having a half collar of width $\frac{1}{2}\log g - \left(\frac{3}{2}+\varepsilon\right)\log\log g$. Then let α be the curve in the half collar of γ consisting of points whose distance to γ is $\frac{1}{4}\log g - \left(\frac{3}{4}+\frac{\varepsilon}{2}\right)\log\log g$. The lemma follows immediately. $\square$

Lemma 5.6 *For any surface $S' \in X_g$, on any essential curve $\alpha' \subset S'$ there is at least one point $p' \in \alpha'$ such that*

$$\mathrm{inj}_{S'}(p') \le \tfrac{1}{2}\,\mathrm{sys}(S').$$

Proof Recall that $S' \in X_g$ means that the shortest geodesics on S' form a filling set of curves. Then any essential curve α' intersects at least one shortest closed geodesic. We pick one of the shortest geodesics that intersects α' and denote it by β'. We let p' be a point in $\alpha' \cap \beta'$. Then $\mathrm{inj}_{S'}(p') \le \frac{1}{2}\, l_{\beta'}(S') = \frac{1}{2}\,\mathrm{sys}(S')$. $\square$

Theorem 5.7 $P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid d_{\mathrm{WP}}(S, X_g) < 0.6521(\sqrt{\log g} - \sqrt{7\log\log g})\} \to 0$ *as* $g \to \infty$.

Proof By Theorem 4.1, if we let $c_g = \log\log g$, then

$$P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid \mathrm{sys}(S) > \log\log g\} < B\frac{\log\log g}{\log g}. \tag{5-1}$$

Let $S \in \mathcal{M}_g$ satisfy Theorem 4.2(a) and (b) and $\mathrm{sys}(S) \le \log\log g$. For any $S' \in X_g$, by Corollary 5.3,

$$0.5492\, d_{\mathrm{WP}}(S, S') \ge |\sqrt{\mathrm{sys}(S')} - \sqrt{\mathrm{sys}(S)}| \ge \sqrt{\mathrm{sys}(S')} - \sqrt{\mathrm{sys}(S)} \ge \sqrt{\mathrm{sys}(S')} - \sqrt{\log\log g}. \tag{5-2}$$

On the other hand, since S satisfies conditions (a) and (b), by Lemma 5.5 there is a curve $\alpha \subset S$ such that, for any $p \in \alpha$,

$$\mathrm{inj}_S(p) \ge \tfrac{1}{4}\log g - \left(\tfrac{3}{4}+\tfrac{\varepsilon}{2}\right)\log\log g. \tag{5-3}$$

We choose an arbitrary horizontal curve $c(t)\colon [0,1] \to \mathcal{M}_{-1}$ with $\pi(c(0)) = S$, $\pi(c(1)) = S'$ and $\pi(c([0,1))$ a Weil–Petersson geodesic connecting S and S'. Then by deforming the metric of S along $c(t)$ to the metric of S', α is also a well-defined essential simple closed curve on S'. By Lemma 5.6, there is a point $p \in \alpha \subset S'$ such that

$$\mathrm{inj}_{S'}(p) \le \tfrac{1}{2}\,\mathrm{sys}(S'). \tag{5-4}$$

Therefore, by Definition 5.1, (5-3) and (5-4),

$$\begin{aligned} 0.3884\, d_{\mathrm{WP}}(S, S') &\ge |\sqrt{\mathrm{inj}_S(p)} - \sqrt{\mathrm{inj}_{S'}(p)}| \ge \sqrt{\mathrm{inj}_S(p)} - \sqrt{\mathrm{inj}_{S'}(p)} \\ &\ge \sqrt{\tfrac{1}{4}\log g - \left(\tfrac{3}{4}+\tfrac{\varepsilon}{2}\right)\log\log g} - \sqrt{\tfrac{1}{2}\,\mathrm{sys}(S')}. \end{aligned} \tag{5-5}$$

Combining (5-2) and (5-5), then eliminating $\mathrm{sys}(S')$, we have

$$d_{\mathrm{WP}}(S, S') \ge 0.6521(\sqrt{\log g} - \sqrt{7\log\log g}).$$

Hence, for any S satisfying (a), (b) and $\mathrm{sys}(S) \le \log\log g$,

$$d_{\mathrm{WP}}(S, X_g) \ge 0.6521(\sqrt{\log g} - \sqrt{7\log\log g}).$$

On the other hand, by Theorem 4.2 and (5-1),

$$P_{\mathrm{WP}}\{S \mid S \text{ satisfies (a), (b) and } \operatorname{sys}(S) \le \log\log g\} \to 1$$

as $g \to \infty$. Therefore,

$$P_{\mathrm{WP}}\{S \mid d_{\mathrm{WP}}(S, X_g) \ge 0.6521(\sqrt{\log g} - \sqrt{7\log\log g})\} \to 1$$

as $g \to \infty$, and the theorem holds. □

6 A criterion for the critical points

This section aims to prove Proposition 6.3: the surface with maximal systole among all the surfaces admitting a specific group action must be a critical point of the systole function.

In Section 6.A, some required knowledge on the tangent space of $\mathcal{T}_g$ for the proof is provided. In Section 6.B, we prove lemmas on local properties of the subspace consisting of surfaces admitting a specific group action. At last, in Section 6.C, we prove the proposition.

6.A Tangent space of the Teichmüller space

This subsection contains some required definitions and conclusions on the tangent space of $\mathcal{T}_g$ for the proof of Proposition 6.3. One may refer to [Imayoshi and Taniguchi 1992; Wolpert 1987; Liu 2023] for details.

For $S \in \mathcal{T}_g$, let Γ be the Fuchsian group that uniformizes S; hence $S \cong \mathbb{H}^2/\Gamma$. The tangent space of $\mathcal{T}_g$ is identified with the space of harmonic Beltrami differentials with respect to Γ, denoted by $\mathrm{HB}(\mathbb{H}^2, \Gamma)$.

Here $B(\mathbb{H}^2, \Gamma)$ consists of a Γ–invariant $(-1,1)$–tensor $\mu \in L^\infty(\mathbb{H}^2)$ with $|\mu| < 1$. A Γ–invariant $(-1,1)$–tensor μ satisfies that for any $\gamma \in \Gamma$,

$$\mu = (\mu \circ \gamma)\frac{\overline{\gamma'}}{\gamma'} \quad \text{almost everywhere on } \mathbb{H}^2. \tag{6-1}$$

The map H is a projection from $B(\mathbb{H}^2, \Gamma)$ to itself, depending only on the complex structure of $\mathcal{T}_g$, and $\mathrm{HB}(\mathbb{H}^2, \Gamma)$ is the image of this projection.

There is an exponential map $\Phi\colon \mathrm{HB}(\mathbb{H}^2, \gamma) \to \mathcal{T}_g$, given by associating to $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma)$ the (equivalence class of the marked) surface $\mathbb{H}^2/f^\mu\Gamma(f^\mu)^{-1}$, where f^μ is the quasiconformal map on $\mathbb{H}^2$ satisfying $f^\mu_{\bar z} = \mu f^\mu_z$ and fixing 0, 1 and ∞. Note that Φ is a holomorphic homeomorphism; see [Wolpert 1987].

6.B Symmetric surfaces

For genus-g surface S_g, we assume G is a finite subgroup of $\mathrm{MCG}(S_g)$, and ρ is a marked hyperbolic structure on S_g such that $\Sigma_g = (S_g, \rho) \in \mathcal{T}_g$. Then we define $X_g^G \subset \mathcal{T}_g$, the hyperbolic surfaces admitting a G action:

$$X_g^G = \{\Sigma_g = (S_g, \rho) \in \mathcal{T}_g \mid G \le \mathrm{Aut}(\Sigma_g)\}.$$

Here $\mathrm{Aut}(\Sigma_g)$ is the automorphism group of the hyperbolic surface Σ_g.

The following lemma says that the set of G–invariant tangent vectors at $S \in X_g^G$ is $\mathrm{HB}(\mathbb{H}^2, \Gamma')$ for the Fuchsian group Γ' that uniformizes the orbifold S/G.

Lemma 6.1 *For $S \in X_g^g$, we let S be uniformized by the Fuchsian group Γ, and the orbifold S/G be uniformized by a Fuchsian group denoted by Γ'. Hence $\Gamma \trianglelefteq \Gamma'$ and $G \cong \Gamma'/\Gamma$. Then $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma)$ is a G–invariant tangent vector to $\mathcal{T}_g$ if and only if $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma')$.*

Proof For $g \in \mathrm{Aut}(S)$, since $\mathrm{HB}(\mathbb{H}^2, \Gamma)$ consists of $(-1, 1)$–tensors we know g acts on $\mathrm{HB}(\mathbb{H}^2, \Gamma)$ by

$$g_*(\mu) = (\mu \circ \tilde{g}^{-1}) \frac{\overline{(\tilde{g}^{-1})'}}{(\tilde{g}^{-1})'}, \tag{6-2}$$

where $\tilde{g}$ is a lift of g onto $\mathbb{H}^2$.

Since a lift of g is contained in Γ' and $G \cong \Gamma'/\Gamma$, by (6-2), $\mu = g_*(\mu)$ is equivalent to $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma')$. □

For the exponential map Φ, we have:

Lemma 6.2 *For the G–invariant tangent vector $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma')$, $\Phi(\mu) \in X_g^G$.*

Proof The group G, as a subgroup of the mapping class group MCG_g, acts on $\mathcal{T}_g$. To prove $\Phi(\mu) \in X_g^G$ is to prove $\Phi(\mu)$ is a fixed point of this action.

For $g \in G$ and $\Phi(\mu) = \mathbb{H}^2/f^\mu \Gamma (f^\mu)^{-1}$, g acts on $\Phi(\mu)$ by

$$\mathbb{H}^2/f^\mu \Gamma (f^\mu)^{-1} \mapsto \mathbb{H}^2/(\tilde{g})^{-1} f^\mu \Gamma (f^\mu)^{-1} \tilde{g},$$

where $\tilde{g}$ is a lift of g onto $\mathbb{H}^2$.

By the definition of f^μ, $f^\mu \circ (\tilde{g})^{-1} = f^\mu$ if and only if $\mu = (\mu \circ \tilde{g}^{-1}) \overline{(\tilde{g}^{-1})'}/(\tilde{g}^{-1})'$; namely, $\mu = g_*(\mu)$. Therefore, $\Phi(\mu)$ is G–invariant if μ is G–invariant. □

6.C The criterion

Proposition 6.3 *If $R \in X_g^G$ realizes the maximum of the systole function on X_g^G, namely*

$$\operatorname{sys} R \geq \operatorname{sys} S \quad \text{for all } S \in X_g^G,$$

then R is a critical point of the systole function in $\mathcal{T}_g$.

Proof We assume that R realizes the maximum of sys on X_g^G, $S(R)$ is the set of systoles of R, R is uniformized by the Fuchsian group Γ, and the orbifold R/G is uniformized by the Fuchsian group Γ'.

For $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma)$, if for any $\alpha \in S(R)$ we have $\mathrm{d}l_\alpha(\mu) \geq 0$, we consider $\nu = \sum_{g \in G} g_* \mu$; then by [Fortier Bourque 2020, (6.1)],

$$\mathrm{d}l_\alpha(\nu) = \mathrm{d}l_\alpha\left(\sum_{g \in G} g_* \mu\right) = \sum_{g \in G} \mathrm{d}l_\alpha(g_* \mu) = \sum_{g \in G} \mathrm{d}l_{g(\alpha)}(\mu) \geq \mathrm{d}l_\alpha(\mu) \geq 0. \tag{6-3}$$

The vector $\nu = \sum_{g \in G} g_* \mu$ is in $\mathrm{HB}(\mathbb{H}^2, \Gamma')$. We let ε_0 be a small positive number and consider $U = \{\nu \mid \nu \in \mathrm{HB}(\mathbb{H}^2, \Gamma') \text{ and } \|\nu\|_\infty < \varepsilon_0\}$. Since U is an open neighborhood of 0 in $\mathrm{HB}(\mathbb{H}^2, \Gamma')$, $\Phi(U)$ is an open neighborhood of R in X_g^G. If ε_0 is small enough, then for any $S \in \Phi(U)$ there is at least one curve $\alpha \in S(R)$ such that α is a systole of S. The Hessian of $l_\alpha|_{\Phi(U)}$ is positive definite since the Hessian of l_α is positive definite. Then by Theorem 2.3, $\mathrm{sys}|_{\Phi(U)}$ is a topological Morse function.

Since R realizes the maximum of $\mathrm{sys}|_{X_g^G}$, R realizes the maximum of $\mathrm{sys}|_{\Phi(U)}$ and R is a critical point of $\mathrm{sys}|_{\Phi(U)}$. $\mathrm{HB}(\mathbb{H}^2, \Gamma')$ is the tangent space of $\Phi(U)$ at the basepoint. By Definition 2.5, for $\nu \in \mathrm{HB}(\mathbb{H}^2, \Gamma')$, if $\mathrm{d}l_\alpha(\nu) \ge 0$ for all $\alpha \in S(R)$, then $\mathrm{d}l_\alpha(\nu) = 0$ for all $\alpha \in S(R)$.

Therefore by (6-3), for $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma)$, if $\mathrm{d}l_\alpha(\mu) \ge 0$ for all $\alpha \in S(R)$, then $\mathrm{d}l_\alpha(\mu) = 0$ for all $\alpha \in S(R)$. By Definition 2.5 R is a eutactic surface, and therefore a critical point of the systole function. □

7 Small distance

7.A Construction of S_g^1 and S_g^2

The surface S_g^1 was initially constructed in [Anderson et al. 2011], while S_g^2 was initially constructed in [Gao and Wang 2023]. We briefly construct these two surfaces for completeness, which implies how to obtain the Teichmüller distance between the two surfaces.

We first construct a family of genus-g hyperbolic surfaces denoted by $\{S_g(c,t)\}$; each surface in this family is determined by two parameters, c and t for $c > 0$ and $0 \le t \le \frac{1}{2}c$. The example S_g^1 is a $S_g(c_1, 0)$–surface for some $c_1 > 0$, while the example S_g^2 is a $S_g(c_2, t_2)$–surface for some $c_2, t_2 > 0$.

Let $n \ge 3$ and pick two isometric right-angled hyperbolic polygons with $2n$ edges admitting an order-n rotation. Two such polygons can be glued to an n–holed sphere admitting the order-n rotation extended from the polygons. By this rotation, all boundary curves of this n–holed sphere have equal length. The geometry of the n–holed sphere is determined by its boundary curves' length (denoted by c), and we denote the corresponding n–holed sphere by $S(c)$. We call the boundary curves of $S(c)$ *cuffs* and the edges of the polygons contained in the interior of $S(c)$ *seams*. By rotational symmetry, all seams also have equal length.

We pick two isometric n–holed spheres and glue them along their cuffs, getting a closed surface. As shown in Figure 4, when gluing the two n–holed spheres, we require that every cuff of one of the n–holed spheres is identified with a cuff in the other n–holed sphere, and every seam of one n–holed sphere is half of a closed curve (denoted by α_k for $k = 1, 2, \dots, n$) while the other half of α_k is a seam in the other n–holed sphere. This constructed surface has genus $g = n - 1$, and the geometry of this closed surface is determined by the cuff length c. We denote this surface by $S_g(c, 0)$.

For $t > 0$, the surface $S_g(c,t)$ is constructed from $S_g(c,0)$ by conducting a Fenchel–Nielsen deformation of length t simultaneously along each cuff γ_k. Here a *Fenchel–Nielsen deformation* on $X \in \mathcal{T}_g$ along a simple closed geodesic $\alpha \subset X$ with length t is constructed by cutting X along α and then regluing the boundary curves with a left twist of length t.

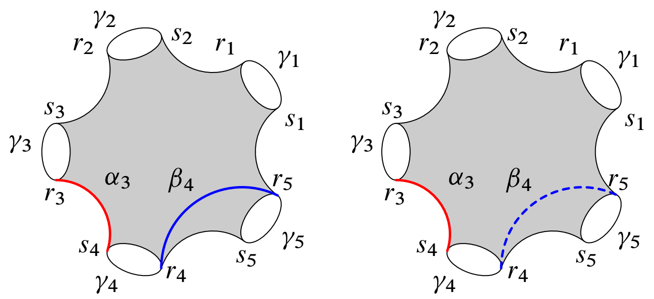

Figure 4

There is a $c_1 > 0$ such that on the surface $S_g(c_1, 0)$, $l(\alpha_k) = l(\gamma_k)$. This surface is the surface S_g^1. The shortest geodesics of S_g^1 consist of α_k and γ_k for $k = 1, 2, \dots, g+1$ by the proof of [Anderson et al. 2011, Theorem 3].

In a surface $S_g(c, t)$, we let β_k be the image of α_k by a Dehn twist along γ_{k+1} (Figure 4). The orientation of this Dehn twist is required to be opposite to the Fenchel–Nielsen deformation.

There is a pair (c_2, t_2) such that on the surface $S_g(c_2, t_2)$, $l(\alpha_k) = l(\beta_k) = l(\gamma_k)$. This surface is the surface S_g^2. The shortest geodesics of S_g^2 consist of α_k, β_k and γ_k for $k = 1, 2, \dots, g+1$ by [Gao and Wang 2023, Proposition 4].

7.B Symmetry on $S(c, t)$

We consider a group G acting isometrically on $S_g(c, t)$, generated by three elements, σ, τ and ς. Here σ is the rotation of order n, τ is the order-2 rotation that exchanges the two n–holed spheres, and ς is the order-2 rotation that is invariant on each n–holed sphere and when restricted to one of the two n–holed spheres exchanges the two $2n$–gons.

On the surface $S_g(c, 0)$, there is a reflection ρ extended from the reflection on one of the n–holed spheres exchanging the two polygons of the n–holed sphere. The symmetric group generated by σ, τ, ς and ρ is denoted by $\overline{G}$.

Remark 7.1 A reflection on the n–holed sphere can be extended to the whole surface $S_g(c, t)$ only if $t = 0$ or $t = \frac{1}{2}c$.

The reflection on $S_g\left(c, \frac{1}{2}c\right)$, denoted by $\rho_{\frac{1}{2}c}$, is not conjugate to ρ. This is because their fixed-point sets are different. The fixed points of ρ on $S_g(c, 0)$ consist of $g+1$ curves (the β_k curves), while fixed points of $\rho_{\frac{1}{2}c}$ consist of one curve (when g is even) or two curves (when g is odd).

The surface S_g^1 has been proved to be a critical point of the systole function; see [Fortier Bourque 2020, Example 4.2 and Proposition 6.3].

On the other hand, it is proved in [Gao and Wang 2023] that the surface S_g^2 is the surface with the maximal systole among the surfaces admitting the action of G. Then immediately by Proposition 6.3, S_g^2 is a critical point of the systole function.

Hence we have:

Proposition 7.2 *The surfaces S_g^1 and S_g^2 are critical points of the systole function.*

7.C Distance

This subsection aims to bound the Teichmüller distance between S_g^1 and S_g^2.

Recall the parameter of the surfaces $S_g^1 = S_g(c_1, 0)$ and $S_g^2 = S_g(c_2, t_2)$. To get an upper bound of $d_{\mathcal{T}}(S_g^1, S_g^2)$, we need an intermediate surface $S_g(c_2, 0)$. Distance between S_g^1 and S_g^2 is bounded from above by the sum of $d_{\mathcal{T}}(S_g^1, S(c_2, 0))$ and $d_{\mathcal{T}}(S(c_2, 0), S_g^2)$.

7.C.1 Quadratic differential and Teichmüller geodesics Before the calculation, we need some preparations; for details, see [Masur 2009].

For a quasiconformal map $f: X \to Y$ for $X, Y \in \mathcal{T}_g$, the $(-1,1)$–tensor $\mu_f(z) = f_{\bar{z}}/f_z$ is called the *Beltrami differential* of f, where z is a local coordinate of X. We let

$$K(f) = \sup_{z \in X} \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

Here μ_f is the complex dilatation of f defined in the last subsection.

The Teichmüller distance on $\mathcal{T}_g$ is defined to be

$$d_{\mathcal{T}}(X) = \tfrac{1}{2} \inf_{f \sim \mathrm{id}} \{\log K(f) \mid f: X \to Y\}.$$

A Teichmüller geodesic ray with respect to Teichmüller distance from $X \in \mathcal{T}_g$ can be induced from a holomorphic quadratic differential q on X. A *holomorphic quadratic differential* is a tensor locally written as $\psi(z)\mathrm{d}z^2$, where $\psi(z)$ is a holomorphic function. We denote the space of quadratic differentials on X by $\mathrm{QD}(X)$. The bundle of quadratic differentials over $\mathcal{T}_g$ is denoted by QD_g.

For $X \in \mathcal{T}_g$ and $q \in \mathrm{QD}(X)$, for any $0 < k < 1$, $\mu_k = k\bar{q}/q$ is a Beltrami coefficient on X. We let f_k be the quasiconformal map induced by μ_k, $f_k: X \to X^{(k)}$. Then f_k is the Teichmüller map from X to $X^{(k)}$, and the Teichmüller geodesic ray induced by (X, q) consists of all the $X^{(k)}$ for all $k \in (0, 1)$.

A nonzero $q \in \mathrm{QD}(X)$ has a canonical coordinate. In this coordinate, q can be locally written as $\mathrm{d}z^2$ in the neighborhood of any nonzero point of q, and q has only finitely many zero points.

The quadratic differential q determines a pair of transverse measured foliations on X, called *horizontal and vertical foliations* for q and denoted by $F_h(q)$ and $F_v(q)$, respectively. In the canonical coordinate of q, the leaves of $F_h(q)$ are given by $y = \text{const}$ and the leaves of $F_v(q)$ are given by $x = \text{const}$. Here $z = x + iy$ is the coordinate. The measures of $F_h(q)$ and $F_v(q)$ are given by $|\mathrm{d}y|$ and $|\mathrm{d}x|$, respectively.

For X_t on the geodesic induced by (X, q) with $d_{\mathcal{T}}(X, X_t) = t$, there is a quadratic differential $q_t \in \mathrm{QD}(X_t)$ as the pushforward of q by f_t. We let $z = x + iy$ be the canonical coordinate of (X, q) and $w = u + iv$ be the canonical coordinate of (X, q). Then

$$u = e^t x \quad \text{and} \quad v = e^{-t} y. \tag{7-1}$$

7.C.2 Extremal length and the Jenkins–Strebel differential A quadratic differential $q \in \mathrm{QD}(X)$ is called a *Jenkins–Strebel differential* if any leaf of $F_h(q)$ and $F_v(q)$ is a simple closed curve, except finitely many leaves that connect zeros of q.

For a Jenkins–Strebel differential $q \in \mathrm{QD}(X)$ and a simple closed leaf α of $F_h(q)$, all simple closed leaves of $F_h(q)$ parallel to α form a cylinder in X. This cylinder is called the *characteristic ring domain* of α and, with respect to the metric $|q|$, is isometric to a Euclidean cylinder

$$R = [0,a] \times (0,b)/((0,t) \sim (a,t), 0 < t < b).$$

We call a the *length* of R and b the *height* of R.

We need the following theorem on the Jenkins–Strebel differential:

Theorem 7.3 [Strebel 1984, Theorem 21.1] *Let $(\gamma_1, \dots, \gamma_p)$ be a finite pairwise-disjoint essential curve system in $X \in \mathcal{T}_g$. For each γ_i, there is a regular neighborhood R_i' of γ_i in X and $R_1', \dots, R_p'$ are pairwise disjoint. Then for any $(b_1, \dots, b_p) \in \mathbb{R}_+^p$, there is a unique Jenkins–Strebel differential $q \in \mathrm{QD}(X)$ such that*:

- *γ_i is a leaf of $F_h(q)$ and any simple closed leaf of $F_h(q)$ is freely homotopic to a γ_i. Here $i = 1, 2, \dots, p$.*
- *The height of the characteristic ring domain of γ_i is b_i.*

The definition of the *extremal length* of an essential curve α in a Riemann surface X is given by

$$\mathrm{Ext}_\alpha(X) = \sup_\rho \frac{l_\alpha(\rho)^2}{\mathrm{Area}(X,\rho)}.$$

Here the supremum is taken over all metrics ρ conformal to the metric on X, $l_\alpha(\rho)$ is the length of α in the metric ρ and $\mathrm{Area}(X,\rho)$ is the area of X in the metric ρ.

For a Euclidean cylinder with length a and height b, the extremal length of its core curve in the cylinder is a/b; see for example [Ahlfors 1966].

Distance between points on a Teichmüller geodesic can be expressed by extremal lengths of horizontal foliation leaves in their characteristic ring domains. For a Jenkins–Strebel differential $q \in \mathrm{QD}(X)$, we let α be a simple closed leaf of $F_h(q)$ and R be the characteristic ring domain of α with length a and height b. For X_t on the Teichmüller geodesic induced by (X,q) with $d_{\mathcal{T}}(X, X_t) = t$, the characteristic ring $R_t \subset X_t$ corresponding to $R \subset X$ has length $e^t a$ and height $e^{-t} b$. Hence for the simple closed curve α_t corresponding to α, $\mathrm{Ext}_{\alpha_t}(R_t) = e^{2t} a/b$ and

$$d_{\mathcal{T}}(X, X_t) = \frac{1}{2}\left|\log \frac{\mathrm{Ext}_{\alpha_t}(R_t)}{\mathrm{Ext}_\alpha(R)}\right|. \tag{7-2}$$

The last necessary tool for estimating the distance is the comparison between hyperbolic length and extremal length by Maskit.

For a simple closed geodesic α in a hyperbolic surface X, the *collar* of α with width w is an embedded cylinder in X consisting of points with distance at most w to α.

Theorem 7.4 [Maskit 1985] *In hyperbolic surface X, if a simple closed geodesic α has collar C with width* $\operatorname{arccosh}(1/\cos\theta)$ *then*

$$\frac{1}{\pi} l_\alpha(X) \le \operatorname{Ext}_\alpha(X) \le \operatorname{Ext}_\alpha(C) \le \frac{1}{2\theta} l_\alpha(X). \tag{7-3}$$

7.C.3 The distance between $S_g^1 = S(c_1, 0)$ and $S(c_2, 0)$ We estimate this distance in two steps:

(1) Prove $\{S(c,0) \mid c > 0\}$ is a Teichmüller geodesic induced by a Jenkins–Strebel differential on some surface $S(c,0)$.

(2) Estimate distance between two points by (7-2) and (7-3).

For $c > 0$, on the surface $S(c,0)$ we consider the cuffs of the n–holed spheres in $S(c,0)$, namely $\{\gamma_k\}_{k=1}^{g+1}$, and assign to each γ_k a positive number b. Then by Theorem 7.3, $\{(\gamma_k, b)\}_{k=1}^{g+1}$ induces a quadratic differential q on $S(c,0)$.

Lemma 7.5 *The quadratic differential $q \in \operatorname{QD}(S_g(c,0))$ is invariant under the action of $\overline{G}$.*

Proof For $g \in \overline{G}$, the quadratic form g^*q is induced by the set $\{(g^{-1}(\gamma_k), b)\}_{k=1}^{g+1}$. By the action of $\overline{G}$ on $S_g(c,0)$, $\{(g^{-1}(\gamma_k), b)\}_{k=1}^{g+1} = \{(\gamma_k, b)\}_{k=1}^{g+1}$. Therefore $g^*q = q$ and q is invariant. □

We consider the Teichmüller geodesic induced by $(S(c,0), q)$.

Lemma 7.6 *We write the Teichmüller geodesic induced by $(S(c,0), q)$ as l. Then the Teichmüller geodesic l coincides with the curve $\{S_g(c,0) \mid c > 0\}$.*

Proof Since q is $\overline{G}$–invariant by Lemma 7.5, for any surface $S' \in l$ the Beltrami coefficient of the Teichmüller map $f\colon S(c,0) \to S'$ is $t\bar{q}/q$ for some $t \in (0,1)$. Hence this Beltrami coefficient is $\overline{G}$–invariant. Then, by Lemma 6.2, $\overline{G}$ isometrically acts on S' by

$$f \circ g \circ f^{-1}\colon S' \to S'$$

for any $g \in \overline{G}$.

Consider the set of cuffs of the n–holed spheres on $S_g(c,0)$, denoted by $\{\gamma_k\}_{k=1}^{g+1}$. Its image $\{f(\gamma_k)\}_{k=1}^{g+1}$ in S' cuts S' into two n–holed spheres. Then $\overline{G}$ isometrically acts on these two n–holed spheres as $\overline{G}$ acts on the two n–holed spheres in $S(c,0)$. Hence S' is a $S(c',0)$–surface, where c' is the length of $f(\gamma_k)$ on S'.

Therefore the Teichmüller geodesic l is contained in the curve $\{S_g(c,0) \mid c > 0\}$. Then by the completeness of Teichmüller geodesics, $\{S_g(c,0) \mid c > 0\}$ coincides with l. □

Now we are ready to estimate:

Proposition 7.7 *For $S_g^1 = S_g(c_1, 0)$ and $S_g(c_2, 0)$, we have*

$$d_{\mathcal{T}}(S_g^1, S_g(c_2, 0)) \le 0.65.$$

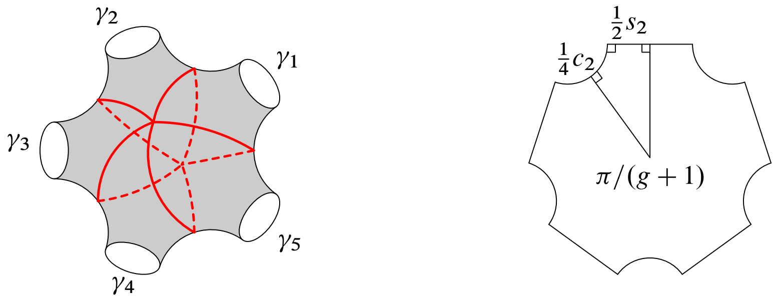

Figure 5: Left: characteristic ring domains. Right: calculate $\frac{1}{2}s_2$.

Proof Recall that c_1 and c_2 are the systoles of S_g^1 and S_g^2, respectively. Then by [Anderson et al. 2011] $c_1 = 4\operatorname{arcsinh}\sqrt{\cos(\pi/(g+1))}$, and c_2 is given by the formula in [Gao and Wang 2023, Theorem 1]. Then we use the following lemma to get the Teichmüller distance:

Lemma 7.8 *The Teichmüller distance between the hyperbolic surfaces $S_g(c_1,0)$ and $S_g(c_2,0)$ with $c_1 < c_2$ is bounded above by*

$$\tfrac{1}{2}\log\frac{\pi c_2}{2\theta c_1},$$

where

$$\cos\theta = \left(1 + \frac{\cos^2(\pi/(g+1))}{\sinh^2(c_2/4)}\right)^{-\frac{1}{2}}.$$

Proof For $i = 1, 2$, we let $\{\gamma_k^{(i)}\}_{k=1}^{g+1}$ be the cuffs in $S_g(c_i,0)$, $q_i \in \mathrm{QD}(S_g(c_i,0))$ be the quadratic differential induced by $\{(\gamma_k^{(i)}, b)\}_{k=1}^{g+1}$ for some $b > 0$, and $R_k^{(i)}$ be the characteristic ring domain of $\gamma_k^{(i)}$. Then, by Theorem 7.4,

$$\mathrm{Ext}_{\gamma_k^{(1)}}(R_k^{(1)}) \geq \mathrm{Ext}_{\gamma_k^{(1)}}(S_g(c_1,0)) \geq \frac{l(\gamma_k^{(1)})}{\pi} = \frac{c_1}{\pi}. \tag{7-4}$$

The set of characteristic ring domains $\{R_k^{(2)}\}_{k=1}^{g+1}$ is invariant under the $\overline{G}$–action. Then by the symmetry of $\overline{G}$, in $S_g(c_2,0)$ the ring domains $R_k^{(2)}$ for $k = 1, \dots, g+1$ are bounded by the hyperbolic geodesics connecting a center of the $2n$–gons and a middle point of the seams (Figure 5, left); otherwise, $\{R_k^{(2)}\}_{k=1}^{g+1}$ is not $\overline{G}$–invariant.

Therefore, if the seam length of n–holed spheres of $S_g(c_2,0)$ is s_2, then the collar C_k of $\gamma_k^{(2)}$ with width s_2/s is contained in the characteristic ring domain $R_k^{(2)}$.

The seam length s_2 is given by the trirectangle formula (2-3):

$$\sinh\left(\tfrac{1}{2}s_2\right)\sinh\left(\tfrac{1}{4}c_2\right) = \cos\frac{\pi}{g+1}. \tag{7-5}$$

See Figure 5, right. Therefore, by Theorem 7.4,

$$\mathrm{Ext}_{\gamma_k^{(2)}}(R_k^{(2)}) \leq \mathrm{Ext}_{\gamma_k^{(2)}}(C_k) \leq \frac{l(\gamma_k^{(2)})}{2\arccos\left(1/\cosh\left(\frac{1}{2}s_2\right)\right)} = \frac{c_2}{2\arccos\left(1/\cosh\left(\frac{1}{2}s_2\right)\right)}. \tag{7-6}$$

By combining (7-2), (7-4), (7-6) and (7-5), this lemma holds. □

Proposition 7.7 follows immediately by Lemma 7.8. □

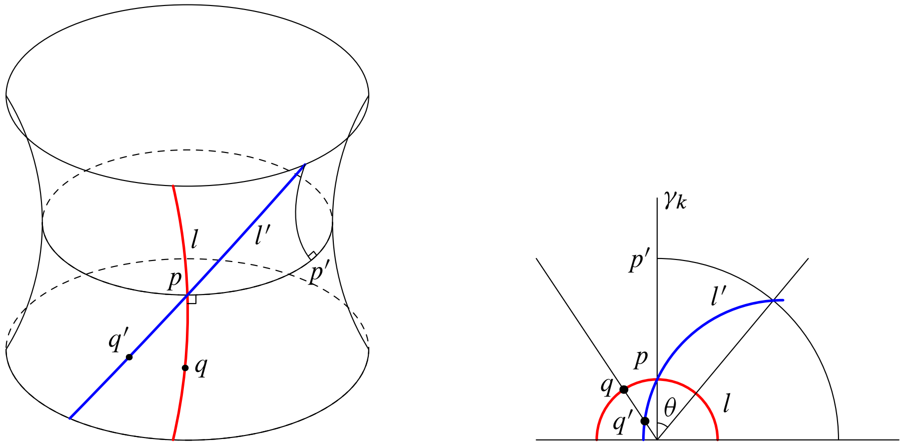

Figure 6: Left: the homeomorphism $h|_{C_k}$. Right: the lift of C_k to $\mathbb{H}^2$.

7.C.4 The distance between $S_g(c_2,0)$ and $S_g^2 = S_g(c_2,t_2)$ Recall that $S_g(c_2,t_2)$ is obtained from $S_g(c_2,0)$ by a Fenchel–Nielsen deformation along the cuffs $\{\gamma_k\}_{k=1}^{g+1}$ in $S(c_2,0)$ with time t_2. For the collar C_k of γ_k, we construct a homeomorphism $h\colon S_g(c_2,0) \to S_g(c_2,t_2)$ such that h is an isometry outside all these collars. Hence the dilatation $K(h)$ is reduced to the dilation restricted to a collar $K(h|_{C_k})$, and the Teichmüller distance between the two surfaces is bounded from above by $\frac{1}{2}\log K(h|_{C_k})$.

Proposition 7.9 *For Σ_g^2 and $\Sigma_g^{1,2}$, we have*

$$d_{\mathcal{T}}(S_g^2, S_g(c_2,0)) \le 1.6450.$$

Proof We proceed by constructing the homeomorphism h and calculating its dilatation on the largest collar of γ_k.

We let C_k be the collar of γ_k with the width $\frac{1}{2}s_2$, where s_2 is the seam length of the n–holed spheres as in Lemma 7.8. The homeomorphism h on C_k is described in Figure 6, left. A geodesic l orthogonal to the core curve γ_k is always mapped to a geodesic l'. The line l is required to intersect l' at a point p on γ_k. The projection of one of the endpoints of l' (denoted by p') is required to have distance $\frac{1}{2}t_2$ to p.

We let h outside the collars be an isometry on this surface of $S_g(c_2,0)$; then the homeomorphism h maps $S_g(c_2,0)$ to $S_g(c_2,t_2)$ by the construction on the collars.

The rest of the proof consists of the calculation of $K(h)$ on the collar C_k. To calculate this dilatation, we lift C_k on the upper half-plane $\mathbb{H}^2$ (Figure 6, right).

We lift γ_k to the y–axis, assuming $p = i$ and $p' = i\,e^{t_2/2}$. The collar of γ_k with width $\frac{1}{2}s_2$ is lifted to a strip $\left\{re^{i\varphi} \in \mathbb{H}^2 \mid -\theta + \frac{1}{2}\pi < \varphi < \theta + \frac{1}{2}\pi\right\}$, where

(7-7) $$\cos\theta = \frac{1}{\cosh\frac{1}{2}s_2}.$$

In this strip, l is the unit circle, and l' is the geodesic connecting i and $\exp\left(\frac{1}{2}t_2 + i\sin\theta\right)$.

The homeomorphism h can be expressed in the form

$$h(re^{i\varphi}) = r\Phi(\varphi)e^{i\varphi}.$$

When $r = 1$, h maps l to l' in Figure 6, right. By this requirement, we can calculate that

$$\Phi(\varphi) = \sinh\bigl(\tfrac{1}{2}t_2\bigr)\frac{\cos\varphi}{\sin\theta} + \sqrt{\sinh^2\bigl(\tfrac{1}{2}t_2\bigr)\frac{\cos^2\varphi}{\sin^2\theta} + 1}. \tag{7-8}$$

The dilatation $K(h)$ is given by

$$K(h) = \frac{|h_z| + |h_{\bar{z}}|}{|h_z| - |h_{\bar{z}}|} = \frac{\sqrt{\Phi^2 + \frac{1}{4}\Phi'^2} + \frac{1}{2}|\Phi'|}{\sqrt{\Phi^2 + \frac{1}{4}\Phi'^2} - \frac{1}{2}|\Phi'|}. \tag{7-9}$$

Here $z = re^{i\varphi}$ and $\bar{z} = re^{-i\varphi}$.

Combining (7-9), (7-8), (7-7), (7-5) and the formula for (c_2, t_2) in [Gao and Wang 2023, Theorem 1], we obtain $d_{\mathcal{T}}(S_g(c_2, 0), S_g(c_2, t_2)) \le \frac{1}{2}\log K(h) \le 1.6450$. □

Hence by Propositions 7.7 and 7.9, we have:

Theorem 7.10 *For any $g \ge 2$,*

$$d_{\mathcal{T}}(S_g^1, S_g^2) \le 2.3.$$

8 Large distance

8.A The S_g^3 surface

We take the $X(\Gamma)$–surface in [Fortier Bourque and Rafi 2022] when $n = 2$ as the surface S_g^3. We briefly describe this surface for completeness.

We consider the four-holed sphere admitting the order-4 rotation. We pick infinitely many copies of the four-holed sphere $\{P_k\}_{k=-\infty}^{+\infty}$ and glue them together into a surface S_∞ with infinite genus, as shown in Figure 7.

The surface S_∞ admits an isometric action $\psi\colon S_\infty \to S_\infty$ which takes every P_k to P_{k+1}. The surface S_g^3 is the quotient $S_\infty/\langle\psi^{g-1}\rangle$. When $g \ge 13$, S_g^3 is a local maximal point of the systole function.

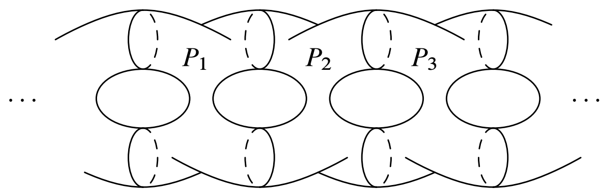

Figure 7: The surface S_∞.

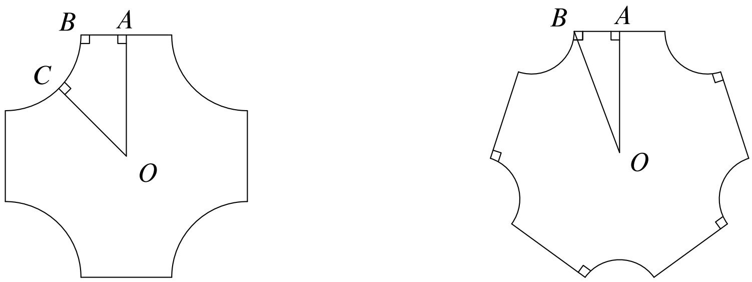

Figure 8: Left: the right-angled octagon. Right: the polygon Q.

8.B The distance between S_g^1 and S_g^3

This distance is obtained from diameter comparison. The diameter of S_g^3 is comparable with g while the diameter of S_g^1 is comparable with $\log g$. Then the distance between these two surfaces is comparable with $\log g$ by the method in the proof of [Rafi and Tao 2013, Lemma 5.1].

Proposition 8.1 *For the diameter of the surface S_g^3, we have*

$$\operatorname{diam}(S_g^3) \geq 0.6\left\lfloor \tfrac{1}{2}(g-5)\right\rfloor.$$

Proof By the construction, the surface S_g^3 consists of $g-1$ four-holed spheres, P_k for $k = 1, 2, \dots, g-1$. When $g \geq 5$, for any $x \in P_k$ and $y \in P_{k+2}$ for some k, a curve connecting x and y must pass through at least one of the four-holed spheres other than P_k or P_{k+2}. Without loss of generality, we assume this curve passes through P_{k+1}; then this curve, if given an orientation, enters P_{k+1} at one cuff and leaves P_{k+1} at another cuff. Therefore, $d(x, y)$ is bounded from below by the distance between neighboring cuffs of P_{k+1}. We denote this distance by d. Then inductively, when $k \leq \frac{1}{2}(g-1)$, distance between $x \in P_1$ and $y \in P_k$ is at least $d\left\lfloor \frac{1}{2}(g-1)-2\right\rfloor$. Hence

$$\operatorname{diam}(S_g^3) \geq d\left\lfloor \tfrac{1}{2}(g-1)-2\right\rfloor.$$

The rest of this proof is to calculate d. The distance d is the seam length of the four-holed spheres. The seam length d is determined by the cuff length (denoted by c) of the four-holed sphere by (8-1). In Figure 8, left, one of the two octagons forming the four-holed sphere, we have

(8-1) $$\sinh|AB|\sinh|BC| = \cos\angle O, \quad \text{which gives} \quad \sinh\left(\tfrac{1}{4}c\right)\sinh\left(\tfrac{1}{2}d\right) = \cos\left(\tfrac{1}{4}\pi\right).$$

According to [Fortier Bourque and Rafi 2022, Lemma 2.5], the cuff length of the four-holed spheres is approximately 6.980. Then by (8-1), this proposition holds. □

For the surface S_g^1, we have:

Proposition 8.2 *The diameter of the surface S_g^1 satisfies*

$$\operatorname{diam}(S_g^1) < 4\log\left(\frac{4g+4}{\pi}\right).$$

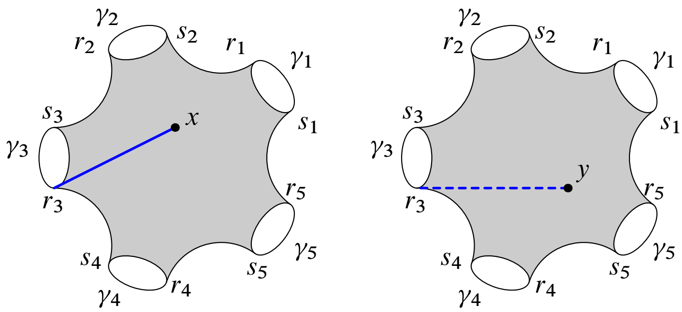

Figure 9: The path between x and y.

Proof Recall that the surface S_g^1 consists of two $(g+1)$–holed spheres, and each of the $(g+1)$–holed spheres consists of two right-angled regular $(2g+2)$–gons. For any $x, y \in S_g^1$, for the two (possibly coinciding) regular $(2g+2)$–gons containing x and y, there is a curve connecting x and y, contained in the union of these two polygons (see Figure 9). Therefore, if we denote one of the four regular $(2g+2)$–gons by Q,

$$\operatorname{diam}(S_g^1) \le 2 \operatorname{diam}(Q),$$

The diameter of Q is realized by $2|OB|$ in Figure 8, right. In the triangle $\triangle OAB$, by (2-2),

$$\cosh|OB| = \cot\angle O \cot\angle B,$$

and so

$$\cosh|OB| = \cot\left(\tfrac{1}{4}\pi\right) \cot\frac{\pi}{2g+2} = \cot\frac{\pi}{2g+2} < \frac{2g+2}{\pi}.$$

Therefore,

$$\operatorname{diam}(S_g^1) \le 2\operatorname{diam}(Q) \le 4|OB| < 4\operatorname{arccosh}\left(\frac{2g+2}{\pi}\right) < 4\log\left(\frac{4g+4}{\pi}\right). \qquad \square$$

Theorem 8.3 *When* $g \ge 13$,

$$d_{\mathcal{T}}(S_g^1, S_g^3) > \tfrac{1}{2}\log(g-6) - \tfrac{1}{2}\log\left(\tfrac{40}{3}\log\left(\frac{4g+4}{\pi}\right)\right).$$

Proof The proof here is similar to the proof of [Rafi and Tao 2013, Lemma 5.1].

We let $f\colon S_g^1 \to S_g^3$ be a Lipschitz homeomorphism with $L(f) = d_L(S_g^1, S_g^3)$. (The existence of this homeomorphism is verified in [Thurston 1986a].) By Proposition 8.1, we pick $x, y \in S_g^3$ with $d(x,y) \ge 0.6\left\lfloor \frac{1}{2}(g-5)\right\rfloor$. By Proposition 8.2, $d(f^{-1}(x), f^{-1}(y)) < 4\log((4g+4)/\pi)$. Then

$$L(f) \ge \frac{d(x,y)}{d(f^{-1}(x), f^{-1}(y))} > \frac{0.6\left\lfloor \frac{1}{2}(g-5)\right\rfloor}{4\log((4g+4)/\pi)} > \frac{3(g-6)}{40\log((4g+4)/\pi)}.$$

Hence,

$$d_L(S_g^1, S_g^3) = \log L(f) > \log(g-6) - \log\left(\tfrac{40}{3}\log\left(\frac{4g+4}{\pi}\right)\right).$$

By (2-1),

$$d_{\mathcal{T}}(S_g^1, S_g^3) \ge \tfrac{1}{2} d_L(\Sigma_g^1, \Sigma_g^3) > \tfrac{1}{2}\log(g-6) - \tfrac{1}{2}\log\left(\tfrac{40}{3}\log\left(\frac{4g+4}{\pi}\right)\right). \qquad \square$$

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