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**Moduli spaces of geometric graphs**

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We define a new family of graph invariants, studying the topology of the moduli space of their geometric realizations in Euclidean spaces, using a limiting procedure reminiscent of Floer homology.

Given a labeled graph  $G$  on  $n$  vertices and  $d \geq 1$ , let  $W_{G,d} \subseteq \mathbb{R}^{d \times n}$  denote the space of nondegenerate realizations of  $G$  in  $\mathbb{R}^d$ . For example, if  $G$  is the empty graph, then  $W_{G,d}$  is homotopy equivalent to the configuration space of  $n$  points in  $\mathbb{R}^d$ . Questions about when a certain graph  $G$  exists as a geometric graph in  $\mathbb{R}^d$  have been considered in the literature and in our notation have to do with deciding when  $W_{G,d}$  is nonempty. However,  $W_{G,d}$  need not be connected, even when it is nonempty, and we refer to the connected components of  $W_{G,d}$  as *rigid isotopy classes* of  $G$  in  $\mathbb{R}^d$ . We study the topology of these rigid isotopy classes. First, regarding the connectivity of  $W_{G,d}$ , we generalize a result of Maehara that  $W_{G,d}$  is nonempty for  $d \geq n$  to show that  $W_{G,d}$  is  $k$ -connected for  $d \geq n + k + 1$ , and so  $W_{G,\infty}$  is always contractible.

While  $\pi_k(W_{G,d}) = 0$  for  $G, k$  fixed and  $d$  large enough, we also prove that, in spite of this, when  $d \rightarrow \infty$  the structure of the nonvanishing homology of  $W_{G,d}$  exhibits a stabilization phenomenon. The nonzero part of its homology is concentrated in at most  $n - 1$  equally spaced clusters in degrees between  $d - n$  and  $(n - 1)(d - 1)$ , and whose structure does not depend on  $d$ , for  $d$  large enough. This leads to the definition of a family of graph invariants, capturing the asymptotic structure of the homology of the rigid isotopy class. For instance, the sum of the Betti numbers of  $W_{G,d}$  does not depend on  $d$  for  $d$  large enough; we call this number the *Floer number* of the graph  $G$ . This terminology comes by analogy with Floer theory, because of the shifting phenomenon in the degrees of positive Betti numbers of  $W_{G,d}$  as  $d$  tends to infinity.

Finally, we give asymptotic estimates on the number of rigid isotopy classes of  $\mathbb{R}^d$ -geometric graphs on  $n$  vertices for  $d$  fixed and  $n$  tending to infinity. When  $d = 1$  we show that asymptotically as  $n \rightarrow \infty$ , each isomorphism class corresponds to a constant number of rigid isotopy classes, on average. For  $d > 1$  we prove a similar statement at the logarithmic scale.

14P10, 55R80; 05C30

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# 1 Introduction

Let  $P = (p_1, \dots, p_n)$  be a point in  $\mathbb{R}^{d \times n}$ . The geometric graph associated to  $P$  is the labeled graph<sup>1</sup>  $G(P)$  whose vertices and edges are, respectively,

$$V(G(P)) = \{(1, p_1), \dots, (n, p_n)\}, \quad E(G(P)) = \{((i, p_i), (j, p_j)) \mid i < j, \|p_i - p_j\|^2 < 1\}.$$

Some readers might be more familiar with the following equivalent definition:  $G(P)$  is the 1–skeleton of the Čech complex associated to the covering consisting of open balls of radius  $1/\sqrt{2}$  centered at the points  $p_1, \dots, p_n$  in  $\mathbb{R}^d$ . We notice that  $G(P)$  is not an embedded graph, but it is sometimes useful to visualize it as embedded.

If a graph  $G$  on  $n$  vertices is isomorphic as a labeled graph to a geometric graph  $G(P)$  for some  $P \in \mathbb{R}^{d \times n}$ , we say it is realizable as an  $\mathbb{R}^d$ –geometric graph on  $n$  vertices. Maehara [23] proved that when  $d \geq n$ , every graph on  $n$  vertices is realizable as an  $\mathbb{R}^d$ –geometric graph. In particular, denoting by  $\#_{d,n}$  the number of isomorphism classes of labeled  $\mathbb{R}^d$ –geometric graphs on  $n$  vertices, for  $d \geq n$  we have

$$(1-1) \quad \#_{d,n} = 2^{\binom{n}{2}}.$$

This statement can be rephrased using the theory of *discriminants* from real algebraic geometry. To explain this idea let us first introduce the notion of nondegenerate geometric graph: the  $\mathbb{R}^d$ –geometric graph  $G(P)$  is called *nondegenerate* if there is no pair of indices  $1 \leq i < j \leq n$  such that  $\|p_i - p_j\|^2 = 1$ . Studying nondegenerate graphs is not an actual restriction, since the set of isomorphism classes of labeled nondegenerate  $\mathbb{R}^d$ –geometric graphs coincides with the set of all possible isomorphism classes of labeled  $\mathbb{R}^d$ –geometric graphs; see Lemma 16 below. Moreover, nondegenerate geometric graphs are simpler to study, because of their stability under small perturbations of the defining points.

In this setting the discriminant consists of the set of *degenerate*  $\mathbb{R}^d$ –geometric graphs

$$\Delta_{d,n} = \{P \in \mathbb{R}^{d \times n} \mid \text{there exist } 1 \leq i < j \leq n \text{ such that } \|p_i - p_j\|^2 = 1\} \subset \mathbb{R}^{d \times n}.$$

This discriminant partitions  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  into many disjoint, connected open sets, which we will call *chambers*. If two points  $P_0$  and  $P_1$  belong to the same chamber in  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  then clearly  $G(P_0)$  and  $G(P_1)$  are isomorphic, but the reverse implication does not hold in general, leading to the following definition.

**Definition 1** If two points  $P_0, P_1 \in \mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  belong to the same chamber—that is, if there is a continuous curve  $P: [0, 1] \rightarrow \mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  with  $P(0) = P_0$  and  $P(1) = P_1$ —we will say that the geometric graphs  $G(P_0)$  and  $G(P_1)$  are *rigidly isotopic*. We will call the curve  $P: [0, 1] \rightarrow \mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  a *rigid isotopy*.

As an example of  $\mathbb{R}^d$ –geometric graphs which are isomorphic but not rigidly isotopic, consider points  $P_0 = (-2, 0)$  and  $P_1 = (0, -2)$ : the  $\mathbb{R}$ –geometric graphs  $G(P_0)$  and  $G(P_1)$  are isomorphic; they are

<sup>1</sup>From now on, unless differently specified, the word “graph” stands for “labeled graph”.

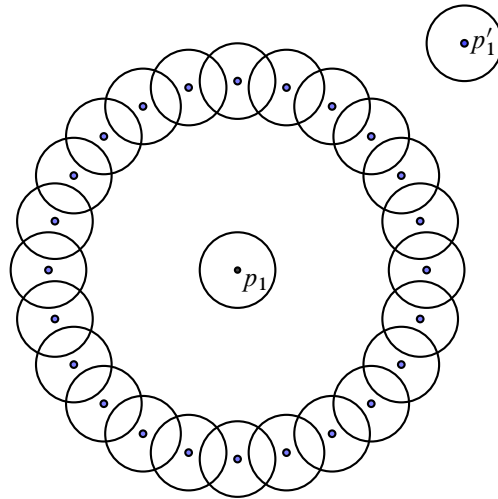


Figure 1: Here we are drawing points in  $\mathbb{R}^2$  together with the circles centered at those points with radius  $1/\sqrt{2}$ . Let us define points  $P$  and  $P'$  in  $\mathbb{R}^{2 \times 25}$  in such a way that  $p_1$  is the point inside the big circle and  $p'_1$  is the point outside the big circle, while  $p_i = p'_i$  for  $i > 1$  and they are the points on the big circle. Then the two geometric graphs  $G(P)$  and  $G(P')$  are isomorphic, but not rigidly isotopic.

both the graph on 2 vertices with no edges, but they are not rigidly isotopic since any curve  $P(t) \in \mathbb{R}^{1 \times 2}$  with  $P(0) = P_0$  and  $P(1) = P_1$  must intersect the discriminant. Another example is depicted in Figure 1.

For  $n$  and  $d$  the number of rigid isotopy classes of geometric graphs on  $n$  vertices in  $\mathbb{R}^d$  is exactly given by  $b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n})$ , and we always clearly have

$$b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}) \geq \#_{d,n}.$$

One natural question therefore is: for what values of  $n$  and  $d$  do the two notions coincide? Moreover, one could consider higher-dimensional notions of connectivity of  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  and study the higher homology and the homotopy groups of the space of its connected components. As we will see, this study will lead us to a definition of a new graph invariant which reminds of Floer homology, as well as precise asymptotics for the enumeration of rigid isotopy and isomorphism classes of geometric graphs.

**Remark 2** Graphs which are realizable as  $\mathbb{R}^d$ -geometric graphs are often called  $d$ -sphere graphs, while the minimal dimension  $d$  such that a given graph is a  $d$ -sphere graph is called its *sphericity*. The result of Maehara [23] mentioned above tells that every graph has finite sphericity; Kang and Müller [20] prove that the problem of deciding, given a graph  $G$ , whether  $G$  is a  $d$ -sphere graph is NP-hard for all  $d > 1$ . Note that, for every  $d > 0$ , there are graphs that are not  $d$ -sphere graphs. In the particular case of  $d = 1$ , the 1-sphere graphs are also called *indifference graphs* or *unit interval graphs*; there are many characterizations of such graphs, see for instance Lekkerkerker and Boland [21], Roberts [29], Jackowski [19], Gutierrez and Oubiña [14] and Mertzios [25].

## 1.1 The case $d \rightarrow \infty$

As we will prove in [Corollary 47](#) below, for  $d \geq n + 1$  the two notions of isomorphic and rigidly isotopic coincide, and  $b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}) = \#_{d,n}$ . Therefore, adopting this language we can reformulate<sup>2</sup> the identity in (1-1) as:

$$b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}) = 2^{\binom{n}{2}},$$

which is true for  $d \geq n + 1$ . The realizability result of Maehara [23] and the fact that for large  $d$  “rigid isotopy” and “isomorphism” are the same notion, seem to settle all relevant questions related to the study of the asymptotics for the number of chambers of  $b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n})$  for fixed  $n$  and large  $d$ . However, as we will see, the topology of the chambers of the complement of the discriminant is extremely rich and some unexpected structure emerges as  $d \rightarrow \infty$ .

In order to explain this phenomenon, let us label the chambers of  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  with the corresponding isomorphism class of labeled geometric graphs: given a graph  $G$  on  $n$  vertices we define

$$W_{G,d} = \{P \in \mathbb{R}^{d \times n} \setminus \Delta_{d,n} \mid G(P) \cong G\} \subset \mathbb{R}^{d \times n}.$$

In other words,  $W_{G,d}$  consists of all the points  $P \in \mathbb{R}^{d \times n}$  which are not on the discriminant and whose associated geometric graph is isomorphic to  $G$ . For small  $d$  this set could be a union of several chambers, but for large  $d$  it is an actual chamber, that is, a connected open set. This can be rephrased by saying that for every graph  $G$  on  $n$  vertices and for large enough  $d$ , the homotopy group  $\pi_0(W_{G,d})$  consists of a single element. In fact, as we will show, the same statement is true for all the homotopy groups, once the group is fixed and  $d$  becomes large enough.

**Theorem 3** *For every  $k \geq 0$  and for  $d \geq k + n + 1$ , we have  $\pi_k(W_{G,d}) = 0$ .*

[Theorem 3](#) in fact generalizes the result of Maehara [23]. Taking the standard convention that a topological space is said to be  $(-1)$ -connected provided it is nonempty, [Theorem 3](#) for  $k = -1$  is Maehara’s result that every graph on  $n$  vertices can be realized as a geometric graph in  $\mathbb{R}^d$  for  $d \geq n$ . [Theorem 3](#) is most likely not sharp for any value of  $k$ . Indeed even in the case of  $k = -1$ , Maehara also shows in [23] that any graph  $G$  which is not a clique can be realized as a geometric graph in  $(|G| - \omega(G))$ -dimensional space. Here  $\omega(G)$  denotes the clique number of  $G$ , the maximum number of vertices in  $G$  that form a complete subgraph. A clique can be realized (with  $n$  points in distinct positions) in  $\mathbb{R}$  and so for any graph  $G$  on at least two vertices,  $\pi_{-1}(W_{G,d}) = 0$  for  $d \geq n - 1$ . An interesting open question would be to improve the general lower bound on  $d$  in [Theorem 3](#).

Notice that there is a natural sequence of inclusions

$$(1-2) \quad \cdots \hookrightarrow W_{G,d} \hookrightarrow W_{G,d+1} \hookrightarrow \cdots$$

<sup>2</sup>Here and later, for a topological space  $X$  we will denote its  $k^{\text{th}}$  Betti number by  $b_k(X) = \dim_{\mathbb{Z}_2}(H^k(X; \mathbb{Z}_2))$  and its total Betti number by  $b(X) = \sum_{k=0}^{\infty} b_k(X)$ , whenever these numbers are defined. This will happen for all the spaces that we will consider in this paper: they will all be homotopy equivalent to finite CW-complexes.

obtained by simply including  $\mathbb{R}^{d \times n}$  into  $\mathbb{R}^{(d+1) \times n}$  by appending a list of zeros to the coordinates of  $P$ . The proof of [Theorem 3](#) goes through two intermediate steps, which are of independent interest: we first prove that for every  $k \geq 0$  and for  $d \geq k + n + 1$ , the inclusion  $W_{G,d} \hookrightarrow W_{G,d+1}$  induces an injection on the homotopy classes of maps, then we prove that the inclusion  $W_{G,d} \hookrightarrow W_{G,d+n}$  is homotopic to a constant map.

**Example 4** (homotopy groups of the configuration space of  $n$  points in  $\mathbb{R}^d$ ) Let us consider the graph  $G$  consisting of  $n$  vertices and no edges. It is easy to see that the corresponding chamber  $W_{G,d}$  is homotopy equivalent<sup>3</sup> to the configuration space of  $n$  distinct points in  $\mathbb{R}^d$ :

$$W_{G,d} \sim \text{Conf}_n(\mathbb{R}^d).$$

In this case one can compute exactly the homotopy groups of  $W_{G,d}$ : for every  $k \geq 0$  and for  $d \geq 3$  we have (see [[11](#), Chapter 2, Theorem 1.1])

$$\pi_k(W_{G,d}) \simeq \pi_k(\text{Conf}_n(\mathbb{R}^d)) \simeq \bigoplus_{j=1}^{n-1} \pi_k(\underbrace{S^{d-1} \vee \dots \vee S^{d-1}}_{\text{bouquet of } j \text{ spheres}}).$$

Since  $\pi_k(S^{d-1} \vee \dots \vee S^{d-1}) = 0$  for  $d \geq k + 2$ , in this case we immediately see that also  $\pi_k(W_{G,d}) = 0$  for  $d \geq k + 2$ .

It is natural at this point to put the sequence of inclusions (1-2) into the infinite-dimensional space

$$\mathbb{R}^{\infty \times n} = \varinjlim \mathbb{R}^{d \times n}$$

consisting of  $n$ -tuples of sequences  $(p_1, \dots, p_n)$  such that for every  $j = 1, \dots, n$  all but finitely many elements in the sequence  $p_j$  are zero. The definition of geometric graph and discriminant also makes sense in this infinite-dimensional space; see [Section 4.1](#). The chambers are now defined as follows: for a given graph  $G$  on  $n$  vertices, we set

$$W_{G,\infty} = \{P = (p_1, \dots, p_n) \in \mathbb{R}^{\infty \times n} \setminus \Delta_{\infty,n} \mid G(P) \cong G\}.$$

From [Theorem 3](#) we deduce the following.

**Theorem 5** For every graph  $G$ , the set  $W_{G,\infty} = \varinjlim W_{G,d}$  is contractible.

## 1.2 Floer homology of a graph

Summarizing the picture so far: as  $d \rightarrow \infty$ , each  $W_{G,d}$  eventually becomes  $k$ -connected and its direct limit  $W_{G,\infty}$  has no homotopy. Moreover, by the Hurewicz theorem, each fixed reduced Betti number of  $W_{G,d}$  vanishes for  $d$  large enough: more precisely, for every  $k > 0$  there exists  $d(k) > 0$  such that

$$b_k(W_{G,d}) = 0 \quad \text{for all } d \geq d(k).$$

<sup>3</sup>Here and below, for two topological spaces  $X$  and  $Y$  we use the symbol  $X \simeq Y$  to denote that they are homeomorphic and  $X \sim Y$  to denote that they are homotopy equivalent.

But this is not the whole story. Before we continue let us discuss one more, likely familiar, example.

**Example 6** (the infinite-dimensional sphere) Let  $G$  be the graph consisting of two disjoint points. Then  $W_{G,d} \simeq \mathbb{R}^d \times S^{d-1} \sim S^{d-1}$  and  $W_{G,\infty} \simeq S^\infty \times \mathbb{R}^{d \times (n-1)} \sim S^\infty$ . In this case (1-2) becomes

$$\dots \hookrightarrow S^{d-1} \hookrightarrow S^d \hookrightarrow \dots \hookrightarrow S^\infty = \varinjlim S^d.$$

If we now look at the Betti numbers of  $S^d$ , we see that there is a hole in dimension  $d$  that moves to infinity as  $d \rightarrow \infty$ , and it disappears when  $d = \infty$ . The sphere  $S^\infty$  has no cohomology except in dimension zero, but it still has cohomology every time we cut it with a finite-dimensional space.

The phenomenon described in Example 6 can be interpreted using some extraordinary cohomology theory, in the context of the Leray–Schauder degree and, more generally, of Floer homology theories; see Szulkin [31], Gęba and Granas [13] and Abbondandolo [1]. This behavior has also been observed for nonholonomic loop spaces in Carnot groups; see Agrachëv, Gentile and Lerario [4]. In all these examples we are dealing with a sequence of spaces  $X_d$  whose direct limit  $X_\infty$  is contractible, but for every  $d$  large enough, each space carries the same amount of cohomology, just shifted in its dimension. A more general family of examples where this occurs is the iterated suspension.

**Example 7** (the iterated suspension) Let  $X_0$  be a CW-complex and define  $X_d = SX_{d-1}$ , where

$$SX = (X \times I)/\sim$$

is the suspension of  $X$ , and the equivalence relation  $\sim$  is given by  $(x_1, 0) \sim (x_2, 0)$  and  $(x_1, 1) \sim (x_2, 1)$  for all  $x_1, x_2 \in X$ . We have a natural sequence of inclusions

$$(1-3) \quad \dots \hookrightarrow X_d \hookrightarrow X_{d+1} \hookrightarrow \dots$$

given by mapping  $X_d \rightarrow SX_d$  homeomorphically to  $X_d \times \{\frac{1}{2}\}$ . We denote by  $X_\infty = \varinjlim X_d$  the direct limit of the sequence of inclusions (1-3). If  $X_0 = \{x_1, x_2\}$ , then  $X_d = S^d$  and  $X_\infty = S^\infty$ . For every  $k$  the space  $X_d$  becomes eventually  $k$ -connected when  $d$  is large enough, and  $X_\infty$  is contractible. If one looks at the homology of  $X_d$ , this is made by a cluster of holes that shifts to infinity as  $d \rightarrow \infty$ . This can be expressed, for example, by looking at the Poincaré polynomial of  $X_d$ ,

$$P_{X_d}(t) = 1 + t^d (P_{X_0}(t) - 1).$$

These holes are not present when  $d = \infty$ , but the sum of the Betti numbers of  $X_d$  is constant,

$$b(X_d) = P_{X_d}(1) \equiv P_{X_0}(1) = b(X_0).$$

We will prove that a similar phenomenon happens for all the spaces  $W_{G,d}$ : their reduced cohomology is made of “clusters of holes” that “shift” to infinity as  $d \rightarrow \infty$ ; see Figure 2. In fact we show also that for  $G$  on  $n$  vertices there are at most  $n - 1$  such clusters. More precisely, we have the following result.

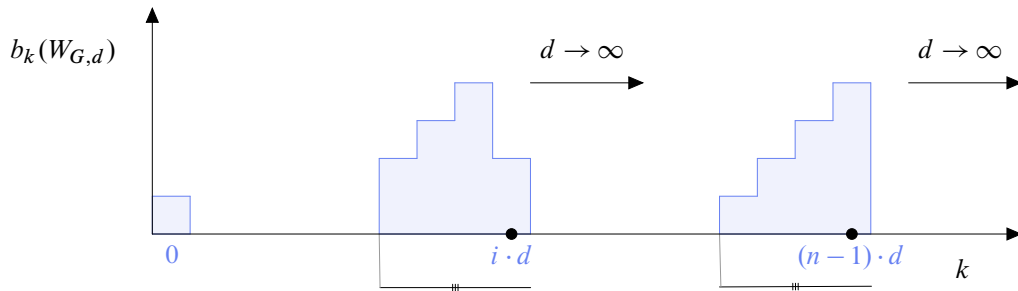


Figure 2: A plot of the Betti numbers of  $W_{G,d}$ . The width of each nonzero cluster of holes is  $\binom{n}{2} + 1$ , which is a constant. Each of these clusters is placed at a multiple of  $d$ , and as  $d \rightarrow \infty$  they shift to infinity. The total Betti number of  $W_{G,d}$ , ie the shaded area, becomes constant for  $d$  large enough.

**Theorem 8** For every graph  $G$  on  $n$  vertices there exist polynomials<sup>4</sup>  $Q_{G,1}, \dots, Q_{G,n-1}$  each of degree at most  $\binom{n}{2} + 1$  such that for  $d \geq \binom{n}{2} + 2$ , the Poincaré polynomial of  $W_{G,d}$  is

$$P_{W_{G,d}}(t) = 1 + t^{d-\binom{n}{2}-1} Q_{G,1}(t) + \dots + t^{md-\binom{n}{2}-1} Q_{G,m}(t) + \dots + t^{(n-1)d-\binom{n}{2}-1} Q_{G,n-1}(t).$$

In particular, there exists  $\beta(G) > 0$  such that

$$b(W_{G,d}) = P_{W_{G,d}}(1) \equiv 1 + \sum_{\ell=1}^{n-1} Q_{G,\ell}(1) = \beta(G) \quad \text{for } d \text{ large enough,}$$

ie the sum of the Betti numbers of  $W_{G,d}$  becomes a constant, which depends on  $G$  only.

Each polynomial  $Q_{G,m}$  corresponds to one of the clusters above and keeps track of the Betti numbers  $b_k(W_{G,d})$  with  $0 \leq dm - k \leq \binom{n}{2}$ , ie with index  $k$  located “near”  $dm$  — remember that  $\binom{n}{2}$  is a constant in this asymptotic regime. These clusters are the “Floer homologies” of the graph.

While we show that nonvanishing homology clusters around multiples of  $d$ , we also prove the following result that holds for any  $n$  and  $d$ , in part to determine which multiples of  $d$  have to be considered.

**Theorem 9** For every  $d$  and  $n$ ,  $\widehat{\Delta}_{d,n}$  is  $(n+d-3)$ -connected, but not  $(n+d-2)$ -connected. By Alexander–Pontryagin duality this implies that  $H^{nd-n-d+1}(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}) \neq 0$ , but all higher cohomology groups of  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  vanish.

The proof of **Theorem 8** uses a spectral sequence argument: each  $W_{G,d}$  can be described as a system of quadratic inequalities and one can use the technique developed in Agrachëv [3] and Agrachëv and Lerario [5] for the study of its Betti numbers. In this case, as  $d \rightarrow \infty$ , the spectral sequence that we need to consider converges at the second step, ie  $E_2 = E_\infty$ . One can prove that both  $E_2$  and the second differential  $d_2$  have an asymptotic stable shape: as  $d \rightarrow \infty$  the nonzero part of the spectral sequence looks like a skinny table where there is no room for higher differentials and the nonzero elements of this table are located near the rows which are labeled by indices which are multiples of  $d$ ; see **Figure 3** later.

<sup>4</sup>These polynomials only depend on  $G$  and not on  $d$ .



graph	Poincaré polynomial	$\beta(G)$	labeled copies
	$1 + 6t^{d-1} + 11t^{2d-2} + 6t^{3d-3}$	24	1
	$1 + 3t^{d-1} + 2t^{2d-2}$	6	6
	$1 + t^{d-1}$	2	3
	$1 + 2t^{d-1} + t^{2d-2}$	4	12
	$1 + t^{d-1}$	2	12
	$1 + t^{d-2} + t^{d-1} + t^{2d-3}$	4	4
	$1 + t^{d-1}$	2	4
	$1 + t^{d-1}$	2	12
	$1 + t^{d-2} + t^{d-1} + t^{2d-3}$	4	3
	$1 + t^{d-1}$	2	6
	1	1	1

Table 1: Poincaré polynomials and Floer numbers for  $W_{G,d}$  for  $G$  on four vertices. For every graph, “labeled copies” refers to the number of isomorphism classes of labeled graphs with the same unlabeled graph.

An interesting question arising from [Theorem 8](#) is, for a given graph  $G$  on  $n$  vertices, how to compute

$$\beta(G) = \lim_{d \rightarrow \infty} b(W_{G,d}).$$

We call this number the *Floer number* of the graph  $G$ . This number is a graph invariant, as well as the polynomials from [Theorem 8](#). [Table 1](#) shows the value of this number for all the possible graphs on four vertices, but the general case is still mysterious. There is more discussion about some of the details of [Table 1](#) in [Example 67](#). Note that the polynomials  $Q_{G,m}$  are also invariants of  $G$ , but their meaning is even more mysterious.

We can make some observations in a few cases that suggest conjectures about  $\beta(G)$  for general graphs. In the case of graphs on four or fewer vertices, we see that the Poincaré polynomial of  $W_{G,d}$  has a general form with exponents given in terms of  $d$ . Once  $d$  is large enough that  $G$  has a realization as a geometric graph in  $\mathbb{R}^d$ , we see that the Poincaré polynomial is determined by the general form. From this we conjecture that for every graph  $G$  there is a general form of the Poincaré polynomial of  $W_{G,d}$  with exponents given in terms of  $d$ , which is valid as long as  $d$  is large enough that  $W_{G,d}$  is nonempty. This would imply that as soon as  $G$  can be realized as a geometric graph in  $\mathbb{R}^d$ ,  $b(W_{G,d}) = \beta(G)$ . In the case of graphs realizable in  $\mathbb{R}$ , we would have that  $\beta(G)$  counts the number of chambers of  $W_{G,1}$ . To see this recall that in the  $d = 1$  case,  $\mathbb{R}^{1 \times n} \setminus \Delta_{1,n}$  is a disjoint union of polyhedra, so every chamber

is contractible; therefore homology can only exist in degree zero. A first step toward the proof of this conjecture would be to prove that all the differentials of the spectral sequence that we use in the proof of [Theorem 8](#) are zero for all  $d \geq n$ .

**Example 10** (Betti numbers of the configuration space of  $n$  points in  $\mathbb{R}^d$ ) The Poincaré polynomial of  $\text{Conf}_n(\mathbb{R}^d)$  for  $d \geq 1$  is given (see [\[11, Chapter V, Corollary 1.4\]](#)) by

$$P_{\text{Conf}_n(\mathbb{R}^d)}(t) = \prod_{j=1}^{n-1} (1 + jt^{d-1}).$$

Consequently,  $b(\text{Conf}_n(\mathbb{R}^d)) = P_{\text{Conf}_n(\mathbb{R}^d)}(t)(1) \equiv n!$ . In other words, for the graph  $G$  consisting of  $n$  disjoint points, we have  $\beta(G) = n!$ .

### 1.3 The case $n \rightarrow \infty$

Concerning the other asymptotic regime, the first case of interest is when  $d = 1$ : here  $\Delta_{1,n}$  is a hyperplane arrangement, since each quadric  $\{|p_i - p_j|^2 = 1\} \subset \mathbb{R}^{1 \times n}$  is the union of the two hyperplanes  $\{p_i - p_j = 1\}$  and  $\{p_i - p_j = -1\}$ . It turns out that the number of chambers of the complement of such a hyperplane arrangement, that is, the number of rigid isotopy classes of  $\mathbb{R}$ -geometric graphs on  $n$  vertices, equals the number of *labeled semiorders* of  $[n]$ . Using techniques from analytic combinatorics, we will prove the following theorem.

**Theorem 11** *The number of rigid isotopy classes of  $\mathbb{R}$ -geometric graphs on  $n$  vertices equals*

$$b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n}) = \frac{1}{n} \cdot \sqrt{6 \log \frac{4}{3}} \cdot \left( \frac{n}{e \log \frac{4}{3}} \right)^n (1 + O(n^{-1/2})).$$

It is in fact possible also to compute the asymptotics of  $\#_{1,n}$  as  $n \rightarrow \infty$ ; we do this in [Theorem 52](#). It is remarkable that the two numbers  $b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n})$  and  $\#_{1,n}$  have the same asymptotic, up to a multiplicative constant<sup>5</sup>

$$(1-4) \quad b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n}) = \frac{8}{e^{1/12}} \cdot \#_{1,n} (1 + O(n^{-1/2})).$$

The case when  $d \geq 2$  is more delicate to handle. This is in large part because the discriminant in higher dimensions is an arrangement of quadrics rather than an arrangement of hyperplanes. For this general case we will prove the following upper and lower bounds for the number of rigid isotopy classes and isomorphism classes.

**Theorem 12** *For  $d \geq 2$  fixed and for  $n \geq 4d + 1$ , one has the bounds*

$$\left( \frac{1}{(d+1)e^2} \right)^{dn} n^{dn} \leq \#_{d,n} \leq b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}) \leq 2dn \left( \frac{3e}{2d} \right)^{dn} n^{dn}$$

<sup>5</sup>The constant  $8/e^{1/12}$  is approximately 7.36.

Note that these bounds imply that  $\#_{d,n}$  and  $b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n})$  become equivalent at the logarithmic scale, giving the following analogue of (1-4):

$$\log b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}) = (\log \#_{d,n})(1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

For the proof of the upper bound we will use the fact that  $\Delta_{d,n}$  is a real algebraic set: using Alexander–Pontryagin duality, we bound the topology of the complement of the discriminant  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  by studying the topology of the one-point compactification of  $\Delta_{d,n}$ , which we denote by  $\widehat{\Delta}_{d,n}$ . This one-point compactification can also be described in an algebraic way and studied using Milnor [27]. Along the way to proving this upper bound we also develop the notation for the spectral sequence that we use to prove Theorem 9.

For the lower bound, our proof is a higher-dimensional version of work of McDiarmid and Müller [24] which showed that, in the case  $d = 2$ , there exists a constant  $\alpha > 0$  such that  $\#_{2,n} \geq \alpha^n n^{2n}$ .

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## 2 Preliminaries

### 2.1 Geometric graphs

There are several different notions of geometric graphs in the literature. The type of geometric graph we consider here are also sometimes called intersection graphs or space graphs. Formally the definition for geometric graph we use here is the following.

**Definition 13** (geometric graph) Given a point  $P \in \mathbb{R}^{d \times n}$  we denote by  $G(P)$  the labeled graph whose vertices and edges are, respectively,

$$V(G(P)) = \{(1, p_1), \dots, (n, p_n)\}, \quad E(G(P)) = \{((i, p_i), (j, p_j)) \mid i < j, \|p_i - p_j\|^2 < 1\}.$$

We say that  $G(P)$  is an  $\mathbb{R}^d$ -geometric graph. If a labeled graph  $G$  is isomorphic to  $G(P)$  for some  $P \in \mathbb{R}^{d \times n}$ , we say that  $G$  is *realizable* as an  $\mathbb{R}^d$ -geometric graph.

We say that the geometric graph  $G(P)$  is nondegenerate if  $P \notin \Delta_{d,n}$ , where

$$\Delta_{d,n} = \{P \in \mathbb{R}^{d \times n} \mid \text{there exist } 1 \leq i < j \leq n \text{ such that } \|p_i - p_j\|^2 = 1\} \subset \mathbb{R}^{d \times n}.$$

**Remark 14** The reason for considering in our definition the list of pairs  $\{(1, p_1), \dots, (n, p_n)\}$  as the set of vertices of  $G(P)$ , instead of the list  $\{p_1, \dots, p_n\}$ , is just formal. In other settings it may be more natural to take  $p_1, \dots, p_n$  to be *distinct* points in  $\mathbb{R}^d$  and then to define a graph with vertex set  $\{p_1, \dots, p_n\}$  and edges  $(p_i, p_j)$  provided that  $\|p_i - p_j\|^2 < 1$ . This is the approach taken by Maehara [23], who studies

the *sphericity* of graphs, the minimum dimension  $d$  in which a graph may be realized as a geometric graph in  $\mathbb{R}^d$  with vertices given by distinct points. For us it makes sense to associate graphs on  $n$  vertices in  $\mathbb{R}^d$  to points of  $\mathbb{R}^{d \times n}$ , therefore the actual points in  $\mathbb{R}^d$  may not all be unique from one another. The two-coordinate approach to describe the vertices allows for such repetition and naturally associates each point in  $\mathbb{R}^d \setminus \Delta_{d,n}$  to unique labeled graph.

We will consider the following notions of equivalence of geometric graphs.

**Definition 15** Let  $G(P_0)$  and  $G(P_1)$  be two  $\mathbb{R}^d$ -geometric graphs on  $n$  vertices, with  $P_0, P_1 \in \mathbb{R}^{d \times n}$ . We will say that they are *isomorphic* if they are isomorphic as labeled geometric graphs. Moreover, if they are both nondegenerate, we will say that they are *rigidly isotopic* if there exists a continuous curve  $P : [0, 1] \rightarrow \mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  such that  $P(0) = P_0$  and  $P(1) = P_1$ .

Since  $\Delta_{d,n}$  is an algebraic set, its complement is a semialgebraic set and its path components are the same as its connected components. Therefore two nondegenerate geometric graphs  $G(P_0)$  and  $G(P_1)$  are rigidly isotopic if and only if  $P_0$  and  $P_1$  belong to the same connected component of  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$ .

Let us introduce the notation

$$(2-1) \quad \#_{d,n} := \#\{\text{isomorphism classes of geometric graphs on } n \text{ vertices in } \mathbb{R}^d\}.$$

In the definition of  $\#_{d,n}$  we did not assume the nondegeneracy of the graphs. However, the following lemma proves that isomorphism classes of *nondegenerate* graphs are the same as all isomorphism classes as in (2-1); see also [6, Lemma 2.2] for an analogous statement in the more general context of geometric complexes.

**Lemma 16** For every  $P \in \Delta_{d,n}$  there exists  $\tilde{P} \in \mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  such that  $G(P)$  and  $G(\tilde{P})$  are isomorphic as labeled graphs.

**Proof** Take  $P = (p_1, \dots, p_n) \in \Delta_{d,n}$  and for  $\epsilon > 0$  small enough, consider

$$\tilde{P} := (1 + \epsilon)P.$$

Then, for  $\epsilon$  small enough, we have

$$((i, p_i), (j, p_j)) \in E(G(P)) \iff ((i, (1 + \epsilon)p_i), (j, (1 + \epsilon)p_j)) \in E(G(\tilde{P})),$$

which proves that  $G(P)$  and  $G(\tilde{P})$  are isomorphic as labeled graphs. Moreover, again for  $\epsilon$  small enough, we have that  $\tilde{P} \notin \Delta_{d,n}$ . □

By definition, for nondegenerate graphs we have

$$\text{rigidly isotopic} \implies \text{isomorphic},$$

which means that isomorphism classes are union of rigid isotopy classes. The number of rigid isotopy classes of geometric graphs on  $n$  vertices in  $\mathbb{R}^d$  is given  $b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n})$ , and Lemma 16 implies we

can compare the number of rigid isotopy classes with the number of isomorphism classes,

$$\#_{d,n} \leq b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}).$$

Below we will prove, as [Corollary 47](#), that for  $d \geq n + 1$ , two  $\mathbb{R}^d$ -geometric graphs on  $n$  vertices are isomorphic if and only if they are rigidly isotopic, ie that

$$\#_{d,n} = b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}) \quad \text{for } d \geq n + 1.$$

We will deal with the asymptotic of  $\#_{d,n}$  and  $b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n})$  in the case  $d$  fixed and  $n \rightarrow \infty$  in [Section 5](#).

## 2.2 Alexander duality and the discriminant

As we are interested in the topology of  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$ , the topology of  $\Delta_{d,n}$  should play an important role as well, and in some cases it will be easier to study. The key tool for connecting the topology of the two is Alexander duality. Given a compact, locally contractible, nonempty and proper subspace  $X$  of the  $N$ -dimensional sphere  $S^N$ , Alexander duality [[16](#), Corollary 3.45] provides a way to study the topology of  $X$  from the topology of  $S^N \setminus X$ . Namely, for every  $k$  we have the following isomorphisms between the homology of  $X$  and the cohomology of  $S^N \setminus X$ :

$$\tilde{H}_k(X) \cong \tilde{H}^{N-k-1}(S^N \setminus X).$$

**Remark 17** If we are working with  $\mathbb{Z}_2$ -coefficients, as we will throughout, and with a space  $X \subset S^N$  with finitely generated homology, we can relate the Betti numbers of  $X$  with those of its complement in the sphere, ie we can freely identify homology and cohomology. When working with compact semialgebraic sets in the sphere, this last requirement will be satisfied thanks to [[9](#), Theorem 9.4.1].

In order to use this duality in the present setting, we work in the one-point compactification of  $\Delta_{d,n} \subset \mathbb{R}^{d \times n}$ , denoted by  $\hat{\Delta}_{d,n} \subset S^{d \times n}$ . Now  $\hat{\Delta}_{d,n}$  contains the point at infinity so  $S^{d \times n} \setminus \hat{\Delta}_{d,n} = \mathbb{R}^{d \times n} \setminus \Delta_{d,n}$ .

The discriminant itself is a union of quadratic hypersurfaces of the form

$$(2-2) \quad \Delta_{d,n}^{i,j} = \{(x_1, \dots, x_n) \in \mathbb{R}^{d \times n} \mid \|x_i - x_j\|^2 = 1\}.$$

Each of these quadrics is topologically  $S^{d-1} \times \mathbb{R}^{d \times (n-1)}$  and establishing bounds on the top Betti number of  $\hat{\Delta}_{d,n}$  establishes bounds on the number of rigid isotopy classes of  $\mathbb{R}^d$ -geometric graphs on  $n$  vertices. We take such an approach in [Section 5.3](#).

## 2.3 Semialgebraic triviality

A most useful technical tool that we will use in the paper is [Theorem 19](#), which relates the structure of semialgebraic families and their homotopy.

**Definition 18** Let  $S, T$  and  $T'$  be semialgebraic sets such that  $T' \subset T$ , and let  $f : S \rightarrow T$  be a continuous semialgebraic mapping. A semialgebraic trivialization of  $f$  over  $T'$ , with fiber  $F$ , is a semialgebraic homeomorphism  $\theta : T' \times F \rightarrow f^{-1}(T')$  such that  $f \circ \theta$  is the projection mapping  $\pi : T' \times F \rightarrow T'$ . We say that the semialgebraic trivialization  $\theta$  is compatible with a subset  $S'$  of  $S$  if there is a subset  $F'$  of  $F$  such that  $\theta(T' \times F') = S' \cap f^{-1}(T')$ .

**Theorem 19** (semialgebraic triviality) *Let  $S$  and  $T$  be two semialgebraic sets,  $f : S \rightarrow T$  a semialgebraic mapping, and  $(S_j)_{j=1, \dots, q}$  a finite family of semialgebraic subsets of  $S$ . There exist a finite partition of  $T$  into semialgebraic sets  $T = \bigcup_{l=1}^r T_l$  and, for each  $l$ , a semialgebraic trivialization  $\theta_l : T_l \times F_l \rightarrow f^{-1}(T_l)$  of  $f$  over  $T_l$  compatible with  $S_j$  for  $j = 1, \dots, q$ , ie there exists  $F_l^j \subset F_l$  such that  $\theta_l(T_l \times F_l^j) = S_j \cap f^{-1}(T_l)$ .*

**Proof** This is [9, Theorem 9.3.2]. □

**Corollary 20** *Let  $S$  be a semialgebraic set and  $f : S \rightarrow \mathbb{R}$  be a continuous semialgebraic function. Then for  $\epsilon > 0$  small enough, the inclusion*

$$\{x \in S \mid f \geq \epsilon\} \hookrightarrow \{x \in S \mid f > 0\}$$

*is a homotopy equivalence.*

**Proof** Thanks to semialgebraic triviality, we know that for  $\epsilon$  sufficiently small there exists  $T_l$  such that  $(0, \epsilon] \subset T_l$ . Then we define a map

$$H : \{f > 0\} \times [0, 1] \rightarrow \{f > 0\}, \quad H(x, t) = \begin{cases} x & \text{if } f(x) \notin (0, \epsilon), \\ \theta_l((1-t) \cdot (\pi_1 \circ \theta_l^{-1}) + t\epsilon, \pi_2 \circ \theta_l^{-1}) & \text{if } f(x) \in (0, \epsilon]. \end{cases}$$

This is a continuous function because the two expressions agree on  $f^{-1}(\epsilon) \times [0, 1]$  and both of them are continuous on closed subsets. Thus the map  $H$  is a deformation retraction of  $\{f > 0\}$  onto  $f \geq \epsilon$ . □

**Corollary 21** *Let  $Y$  be the set of solutions in  $\mathbb{R}^d$  of the system*

$$\begin{cases} q_1(x) > 0, \\ \vdots \\ q_r(x) > 0, \end{cases}$$

*where the  $q_i$  are polynomial functions. There exists  $\delta > 0$  such that for all  $\epsilon$  such that  $0 < \epsilon < \delta$ , the inclusion of  $Y_\epsilon \setminus \{0\}$  in the set  $Y \setminus \{0\}$ , where  $Y_\epsilon$  is the set of the solutions in  $\mathbb{R}^d$  of the system*

$$\begin{cases} q_1(x) \geq \epsilon, \\ \vdots \\ q_r(x) \geq \epsilon, \\ \|x\|^2 \leq \frac{1}{\epsilon}, \end{cases}$$

*is a homotopy equivalence.*

**Proof** Define  $\alpha: \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$  by

$$\alpha = \min \left\{ \delta, q_1, \dots, q_r, \frac{1}{\|x\|^2} \right\}.$$

Then for  $0 < \epsilon < \delta$  we have  $Y_\epsilon \setminus \{0\} = \{\alpha \geq \epsilon\}$  and  $\alpha$  is a continuous semialgebraic function. By the previous corollary, for  $\epsilon$  small enough we get that the inclusion  $Y_\epsilon = \{\alpha \geq \epsilon\} \hookrightarrow \{\alpha > 0\} = Y \setminus \{0\}$  is a homotopy equivalence.  $\square$

## 2.4 Systems of quadratic inequalities

In this section we recall a general construction from [3; 5] for computing the Betti numbers of the set of solutions of a system of quadratic inequalities.

To start with, let  $h: \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{k+1}$  be a *quadratic map*, ie a map whose components  $h = (h_0, \dots, h_k)$  are homogeneous quadratic forms. Let also  $K \subseteq \mathbb{R}^{k+1}$  be a closed convex polyhedral cone (centered at the origin). We are interested in the Betti numbers of

$$(2-3) \quad V = h^{-1}(K) \cap S^N \subset \mathbb{R}^{N+1}.$$

Such a set  $V$  can be seen as the set of solutions of a system of homogeneous quadratic inequalities on the sphere  $S^N$ : in fact, since  $K$  is polyhedral, we have

$$K = \{\eta_1 \leq 0, \dots, \eta_\ell \leq 0\}$$

for some linear forms  $\eta_1, \dots, \eta_\ell \in (\mathbb{R}^{k+1})^*$  and

$$V = \{\eta_1 h \leq 0, \dots, \eta_\ell h \leq 0\} \cap S^N,$$

which is a system of quadratic inequalities. (Here given a linear form  $\eta \in (\mathbb{R}^{k+1})^*$  and a quadratic map  $h: \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{k+1}$ , we simply denote by  $\eta h$  the composition of the two.) Every homogeneous system can be written in this way.

We denote by  $K^\circ$  the polar of  $K$ , ie  $K^\circ = \{\eta \in (\mathbb{R}^{k+1})^* \mid \eta(y) \leq 0 \text{ for all } y \in K\}$ , and we set

$$\Omega = K^\circ \cap S^k,$$

where  $S^k$  denotes the unit sphere in  $(\mathbb{R}^{k+1})^*$  with respect to a fixed scalar product. The scalar product on  $\mathbb{R}^{k+1}$  plays no role, but we will also use a scalar product on  $\mathbb{R}^{N+1}$  by choosing a positive definite quadratic form  $g$  on  $\mathbb{R}^{N+1}$ . This scalar product will play a role, and we denote it by  $\langle \cdot, \cdot \rangle_g$ . It is defined by  $\langle x, x \rangle_g = g(x)$  for all  $x \in \mathbb{R}^{N+1}$ . For practical purposes we will omit the  $g$  subscripts when not needed.

Once the scalar product on  $\mathbb{R}^{N+1}$  has been fixed, we can associate to a quadratic form  $q: \mathbb{R}^{N+1} \rightarrow \mathbb{R}$  a real symmetric matrix, via the equation

$$(2-4) \quad q(x) = \langle x, Qx \rangle_g \quad \text{for all } x \in \mathbb{R}^{N+1}.$$

We will often use small letters for the quadratic form and capital letters for the associated matrices.

Accordingly we can define the eigenvalues of  $q$  (with respect to  $g$ ) as those of  $Q$ :

$$\lambda_1(q) \geq \dots \geq \lambda_{N+1}(q).$$

The eigenvalues (and the eigenvectors) of  $q$  depend therefore on the chosen scalar product, but again we will omit this dependence in the notation if not needed.

**2.4.1 The index function** Using the above notation, we will denote by  $\text{ind}^+(q) = \text{ind}^+(Q)$  the positive inertia index, ie the number of positive eigenvalues of the symmetric matrix  $Q$ . Note that the index of a quadratic form *does not* depend on the chosen scalar product.

When we are in the situation as above, ie when given a homogeneous quadratic map  $h: \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{k+1}$ , for every covector  $\eta \in (\mathbb{R}^{k+1})^*$  we can consider the composition  $\eta h$ , which is a quadratic form. For every natural number  $j \geq 0$  we define the sets

$$\Omega^j = \{\omega \in \Omega \mid \text{ind}^+(\omega h) \geq j\}.$$

These sets are open and semialgebraic, as it is easily verified. Moreover, these sets *do not* depend on the choice of the scalar product  $g$ .

Over each set  $\Omega^j \setminus \Omega^{j+1}$  the function  $\text{ind}^+ \equiv j$  is constant, ie the number of positive eigenvalues of the corresponding matrices is  $j$  and there exists a natural vector bundle  $P^j \subseteq \Omega^j \setminus \Omega^{j+1} \times \mathbb{R}^{N+1}$

$$(2-5) \quad \begin{array}{ccc} \mathbb{R}^j & \hookrightarrow & P^j \\ & & \downarrow \\ & & \Omega^j \setminus \Omega^{j+1} \end{array}$$

whose fiber over a point  $\omega$  is the positive eigenspace of  $\omega h = \omega_0 h_0 + \dots + \omega_k h_k$ . In fact this bundle is the restriction of a more general bundle over the set

$$D_j = \{\omega \mid \lambda_j(\omega h) \neq \lambda_{j+1}(\omega h)\},$$

ie the set where the  $j^{\text{th}}$  eigenvalue of  $\omega h$  is distinct from the  $(j+1)^{\text{st}}$ . We still denote this bundle by  $P_j \subset D_j \times \mathbb{R}^{N+1}$ :

$$(2-6) \quad \begin{array}{ccc} \mathbb{R}^j & \hookrightarrow & P^j \\ & & \downarrow \\ & & D_j \end{array}$$

Here the fiber over a point  $\omega \in D_j$  consists of the eigenspace of  $\omega h$  associated to the first  $j$  eigenvalues (this is well defined); note however that the bundle over  $D_j$  *depends* on the choice of the scalar product (since  $D_j$  itself depends on this choice).

We denote the first Stiefel–Whitney class of this bundle by

$$(2-7) \quad v_j \in H^1(D_j).$$

The following lemma will be useful for us.



**Lemma 22** *The cup product with the class  $v_j$  defines a map*

$$(2-8) \quad (\cdot) \smile v_j : H^*(\Omega^j, \Omega^{j+1}) \rightarrow H^{*+1}(\Omega^j, \Omega^{j+1}).$$

**Proof** To see that the previous cup product is well defined, observe that we can write

$$\Omega^j = A \cup B, \quad A = \Omega^{j+1}, \quad B = \Omega^j \cap D_j.$$

In fact, if a point  $\omega$  belongs to  $\Omega^j$ , then either  $\omega \in \Omega^{j+1}$ , or  $\text{ind}^+(\omega h) = j$  and consequently  $\omega \in \Omega^j \cap D_j$ . Since both  $A$  and  $B$  are open, by excision we get

$$H^*(\Omega^j, \Omega^{j+1}) \simeq H^*(\Omega^j \cap D_j, \Omega^{j+1} \cap D_j).$$

In particular, in order to see that (2-8) is well defined, it is enough to see that

$$(\cdot) \smile v_j : H^*(\Omega^j \cap D_j, \Omega^{j+1} \cap D_j) \rightarrow H^{*+1}(\Omega^j \cap D_j, \Omega^{j+1} \cap D_j)$$

is well defined. Suppose that  $\psi \in C^k(\Omega^j)$  is a singular cochain representing a cohomology class in  $H^k(\Omega^j \cap D_j, \Omega^{j+1} \cap D_j)$  and  $\phi_j \in C^1(D_j)$  is a cochain representing  $v_j$ . The cup product of  $\psi$  and  $\phi_j$  is defined on a singular chain  $\sigma : [v_0, \dots, v_{k+1}] \rightarrow \Omega^j \cap D_j$  in the usual way,

$$(\psi \smile \phi_j)(\sigma) = \psi(\sigma|_{[v_0, \dots, v_k]})\phi_j(\sigma|_{[v_k, v_{k+1}]}),$$

from which we see that  $\psi \smile \phi_j$  vanishes on  $C_{k+1}(\Omega^{j+1} \cap D_j)$  and consequently defines an element of  $H^{k+1}(\Omega^j \cap D_j, \Omega^{j+1} \cap D_j)$ . □

Since we have inclusions  $\Omega^j \supseteq \Omega^{j+1} \supseteq \Omega^{j+2}$ , we also consider the connecting homomorphisms

$$\partial : H^*(\Omega^{j+1}, \Omega^{j+2}) \rightarrow H^{*+1}(\Omega^j, \Omega^{j+1})$$

of the long exact sequence for the triple  $(\Omega^j, \Omega^{j+1}, \Omega^{j+2})$ .

We summarize the directions of these homomorphisms in the *noncommutative* diagram of maps

$$\begin{array}{ccc} H^i(\Omega^{j+1}, \Omega^{j+2}) & \xrightarrow{\partial} & H^{i+1}(\Omega^j, \Omega^{j+1}) \\ (\cdot) \smile v_{j+1} \downarrow & & \downarrow (\cdot) \smile v_j \\ H^{i+1}(\Omega^{j+1}, \Omega^{j+2}) & \xrightarrow{\partial} & H^{i+2}(\Omega^j, \Omega^{j+1}) \end{array}$$

**Remark 23** Let  $\partial : H^i(X, Y) \rightarrow H^{i+1}(Z, X)$  be the boundary operator in the exact sequence of the triple  $(Z, X, Y)$ , where all spaces are open. Following [16, page 201], thanks to the fact that we are working with  $\mathbb{Z}_2$ -coefficients, we have that  $\partial([\phi]) = [\phi \circ \pi \circ \delta]$ , where  $\delta : C_{i+1}(Z, X) \rightarrow C_i(X)$  is the boundary operator and  $\pi : C_i(X) \rightarrow C_i(X, Y)$  is the projection operator. Let us also consider  $\tilde{X} \subset A$  both open and such that  $(X \cap \tilde{X}) \cup Y$  is open. If we take the relative cup product

$$H^i(X, Y) \times H^1(\tilde{X}) \rightarrow H^1(X, Y)$$

as defined in Lemma 22, then this coincides with the cup product

$$H^i(X, Y) \times H^1(A) \rightarrow H^1(X, Y)$$

as defined in Lemma 22, meaning that given  $a \in H^i(X, Y)$  and  $b \in H^1(A)$ , then  $a \smile b = a \smile r^*(b)$ , where  $r^*$  is just the restriction. In the same way, if we suppose that  $\tilde{Z} \subset A$ , we can repeat a similar reasoning for the cup product  $H^i(Z, X) \times H^1(\tilde{Z}) \rightarrow H^{i+1}(Z, X)$ . Now, given  $[\gamma] \in H^1(A)$ , we claim that

$$(2-9) \quad \partial([a] \smile [\gamma]) = \partial[a] \smile [\gamma].$$

At the level of cochains, if we take  $c_{i+1} \in C_{i+1}(Z, X)$  to be a singular chain then  $\partial(a \smile \gamma)(c_{i+1}) = (a \smile \gamma)(\pi \circ \delta c_{i+1})$ . Reasoning as in [16, proof of Lemma 3.6, page 206] shows  $(a \smile \gamma)(\pi \circ \delta c_{i+1}) = \partial a \smile \gamma + (-1)^i a \smile \delta^*(\gamma)$  and (2-9) follows from  $\delta^*(\gamma) = 0$ .

**2.4.2 The spectral sequence** For the computation of the Betti numbers of  $V$ , defined in (2-3) we will need the following result, which is an adaptation from [3; 5]. Clearly the computation of the cohomology of  $V$  is equivalent to that of  $S^N \setminus V$ , by Alexander duality, and in [3; 5] a spectral sequence is introduced for computing the latter.

The delicate part here is that in [3], the spectral sequence is defined for *nondegenerate* systems of quadrics, ie for systems such that the map  $h$  is transversal to the cone  $K$ , in the sense of [5]; in [22] the spectral sequence is defined also for degenerate systems, but the second differential is not computed explicitly, and in [5] it is defined also for degenerate systems, and the second differential is explicitly computed, but the solutions are studied in the projective space rather than the sphere. Since in our case the system of quadratic inequalities is always degenerate<sup>6</sup> for  $m \geq 2$ , we will need to prove the existence of such a spectral sequence and to compute its second differential.

**Theorem 24** *Let  $V = h^{-1}(K) \cap S^N$  be defined by a system of quadratic inequalities, as above. There exists a cohomology spectral sequence  $(E_r, d_r)_{r \geq 1}$  converging to  $H^*(S^N \setminus V; \mathbb{Z}_2)$  such that:*

(1) *The second page of the spectral sequence is given, for  $j > 0$ , by*

$$E_2^{i,j} = H^i(\Omega^{j+1}, \Omega^{j+2}; \mathbb{Z}_2).$$

*For  $j = 0$ , the elements of the second page of the spectral sequence fit into a long exact sequence*

$$(2-10) \quad \dots \rightarrow H^i(\Omega^1; \mathbb{Z}_2) \rightarrow E_2^{i,0} \rightarrow H^i(\Omega^1, \Omega^2; \mathbb{Z}_2) \xrightarrow{(\cdot) \smile v_1} H^{i+1}(\Omega^1; \mathbb{Z}_2) \rightarrow \dots$$

(2) *For  $j \geq 1$  the second differential  $d_2^{i,j} : H^i(\Omega^{j+1}, \Omega^{j+2}) \rightarrow H^{i+2}(\Omega^j, \Omega^{j+1})$  is given by*

$$d_2^{i,j} \xi = \partial(\xi \smile v_{j+1}) + \partial \xi \smile v_j.$$

**Proof** The proof proceeds similarly to [5, Theorems 25 and 28], using a regularization process and taking the limit over the regularizing parameter. More precisely, let  $q_0$  be a positive definite quadratic form, chosen as in [5, Lemma 13], and for  $t > 0$  consider the set

$$B(t) = \{(\omega, x) \in \Omega \times S^N \mid \omega h(x) - tq_0(x) \geq 0\}.$$

<sup>6</sup>This is a consequence of Lemma 33: in fact, for nondegenerate systems the difference of two nearby values of the index function is  $\pm 1$ , whereas here it is always greater than 1.

The choice of  $q_0$  as in [5, Lemma 13] makes the map  $\omega \mapsto \omega h - tq_0$  nondegenerate with respect to  $K$  and will allow us to compute the second differential of our spectral sequence. By semialgebraic triviality, for  $t > 0$  small enough the set  $B(t)$  is homotopy equivalent to

$$B = \{(\omega, x) \in \Omega \times S^N \mid \omega h(x) > 0\}.$$

Moreover, the projection onto the second factor (ie  $p_2: \Omega \times S^N \rightarrow S^N$ ) restricts to a homotopy equivalence  $B \sim p_2(B) = S^N \setminus V$ ; see [22, Section 3.2]. Therefore, for  $t > 0$  small enough,

$$H^*(S^N \setminus V) \simeq H^*(B(t)).$$

We consider now the Leray spectral sequence  $(E_r[t], d_r[t])_{r \geq 0}$  of the map

$$p_t := p_1|_{B(t)}: B(t) \rightarrow \Omega.$$

This spectral sequence converges to the cohomology of  $B(t)$ .

For the first part of the statement, the structure of  $E_2^{i,j}$  in the case  $j > 0$  is proved in [22, Section 3.2], as follows. If  $t_1 < t_2$  then  $B(t_2) \hookrightarrow B(t_1)$  is a homotopy equivalence and  $p_{t_1}|_{B(t_2)} = p_{t_2}$ . For  $0 < t_1 < t_2 < \delta$  the inclusion defines a morphism of filtered differential graded modules

$$\phi_0(t_1, t_2): (E_0[t_1], d_0[t_1]) \rightarrow (E_0[t_2], d_0[t_2])$$

turning  $\{E_0[t]\}_t$  into an inverse system and thus  $\{(E_r[t], d_r[t])\}_t$  into an inverse system of spectral sequences. Then, we can define a new spectral sequence

$$(E_r, d_r) := \varprojlim_t \{(E_r[t], d_r[t])\}.$$

The proof shows that for  $j > 0$  we have  $E_2^{i,j}[t] = H^i(\Omega_{n-j}[t], \Omega_{n-j-1}[t]; \mathbb{Z}_2)$ , where the sets  $\Omega_k[t]$  are defined by

$$\Omega_k[t] := \{w \in \Omega \mid i^-(w \cdot h - tg) \leq k\}.$$

Moreover we also have that for  $j > 0$  the isomorphism  $\phi_2(t_1, t_2)$  is just the homomorphism induced in cohomology by the inclusion  $\Omega_j[t_2] \subseteq \Omega_j[t_1]$ , and that

$$E_2^{i,j} = \varprojlim_t E_2^{i,j}[t] = H^i(\Omega^j, \Omega^{j+1}; \mathbb{Z}_2).$$

Thanks to this, by semialgebraic triviality,  $\phi_2(t_1, t_2)$  is an isomorphism for  $0 < t_1 < t_2 < \delta$  with  $\delta$  sufficiently small, and therefore also  $\phi_\infty(t_1, t_2)$  is an isomorphism, assuring the convergence of  $(E_r, d_r)$  to  $B(t)$ ; see also the proof of [5, Theorem 25] for more details on this point. Let us call

$$e_t^*: H^*(\Omega^j, \Omega^{j+1}; \mathbb{Z}_2) \rightarrow H^*(\Omega_{n-j}(t), \Omega_{n-j-1}(t); \mathbb{Z}_2)$$

the isomorphism induced by the inclusion. From now on we choose our scalar product on  $\mathbb{R}^{N+1}$  to be  $g = q_0$ , in such a way that the matrix associated to  $q_0$  through the polarization identity (2-4) is the identity matrix.

For the case  $j = 0$  we know, thanks to [2], that there exists a long exact sequence

$$(2-11) \quad \cdots \rightarrow H^i(\Omega_{n-1}[t]; \mathbb{Z}_2) \rightarrow E_2^{i,0}[t] \rightarrow H^i(\Omega_{n-1}[t], \Omega_{n-2}[t]; \mathbb{Z}_2) \xrightarrow{(\cdot) \smile \nu_1} H^{i+1}(\Omega_{n-1}[t]; \mathbb{Z}_2) \rightarrow \cdots .$$

We can pass to the inverse limit of these long exact sequences respect to  $t$  in the obvious way, obtaining a long exact sequence<sup>7</sup>

$$\cdots \rightarrow H^i(\Omega^1; \mathbb{Z}_2) \rightarrow E_2^{i,0} \rightarrow H^i(\Omega^1, \Omega^2; \mathbb{Z}_2) \xrightarrow{(\cdot) \smile \nu_1} H^{i+1}(\Omega^1; \mathbb{Z}_2) \rightarrow \cdots ,$$

where we have used the fact that  $(e^*)^{-1} \circ ((\cdot) \smile \nu_j) \circ e^* = (\cdot) \smile \nu_j$ . We will get back to this point later.

This proves point (1) of the statement. For the point concerning the differential, thanks to [2, Theorem 3] we know that the second differential  $d_2[t]$  of the spectral sequence  $(E_r[t], d_r[t])$  with

$$d_2^{i,j}[t]: H^i(\Omega_{n-j-1}[t], \Omega_{n-j-2}[t]; \mathbb{Z}_2) \rightarrow H^{i+2}(\Omega_{n-j}[t], \Omega_{n-j-1}[t]; \mathbb{Z}_2)$$

has the form

$$d_2^{i,j}[t]\xi = \partial_t(i_t^*)^{-1}(i_t^*\xi \smile \nu_{j+1}) + (i_t^*)^{-1}(i_t^*\partial_t\xi \smile \nu_j),$$

where

$$\partial_t: H^i(\Omega_{n-j-1}[t], \Omega_{n-j-2}[t]; \mathbb{Z}_2) \rightarrow H^{i+1}(\Omega_{n-j}[t], \Omega_{n-j-1}[t]; \mathbb{Z}_2)$$

is the connecting homomorphism in the exact sequence of the triple  $(\Omega_{n-j}[t], \Omega_{n-j-1}[t], \Omega_{n-j-2}[t])$ , and the map

$$i_t^*: H^i(\Omega_{n-j}[t], \Omega_{n-j-1}[t]; \mathbb{Z}_2) \rightarrow H^i(\Omega_{n-j}[t] \cap \mathcal{D}_j, \Omega_{n-j-1}(t) \cap \mathcal{D}_j; \mathbb{Z}_2)$$

is the map induced by the inclusion; this map is an isomorphism by excision.

The second differential for  $j > 1$  of our new spectral sequence  $(E_r^{i,j}, d_r)$  is  $d_2^{i,j} := (e_t^*)^{-1} \circ d_2^{i,j}(t) \circ e_t^*$ . More explicitly,

$$d_2^{i,j} = \partial((i_t^* \circ e_t^*)^{-1}((i_t^* \circ e_t^*)\xi \smile \nu_{j+1}) + (i_t^* \circ e_t^*)^{-1}((i_t^* \circ e_t^*)\partial\xi \smile \nu_j)),$$

thanks to the naturality of the connecting homomorphism.

Let us now consider the diagram

$$\begin{array}{ccc} & (\Omega^j, \Omega^{j+1}) & \\ e_t \circ i_t \nearrow & & \nwarrow i \\ (\Omega_{n-j-1}[t] \cap \mathcal{D}_j, \Omega_{n-j-2}[t] \cap \mathcal{D}_j) & \xrightarrow{j_t} & (\Omega^j \cap \mathcal{D}_j, \Omega^{j+1} \cap \mathcal{D}_j) \end{array}$$

where all the maps are inclusions, and all the induced homomorphisms in cohomology are isomorphisms.

<sup>7</sup>In this long exact sequence we are still using  $\nu_1$  because we chose our scalar product to be  $g_0$ . Same for the definition of  $d_2(t)$ , where we used  $\nu_j$ .

We can write

$$\begin{aligned}
 (2-12) \quad d_2^{i,j} &= \partial(j_t^* \circ i^*)^{-1}((j_t^* \circ i^*)\xi \smile \nu_{j+1}) + (j_t^* \circ i^*)^{-1}((j_t^* \circ i^*)\partial\xi \smile \nu_j) \\
 &= \partial(i^*)^{-1}(j^*)^{-1}(j_t^*(i^*\xi) \smile j_t^* \circ \nu_{j+1}) + (i^*)^{-1}(j_t^*)^{-1}(j_t^*(i^*\partial\xi) \smile j_t^* \circ \nu_j) \\
 &= \partial(i^*)^{-1}(i^*\xi \smile \nu_{j+1}) + (i^*)^{-1}(i^*\partial\xi \smile \nu_j),
 \end{aligned}$$

where the pullback property of the pullback in the third equality holds true because in that case it is just the standard cup product. Because of how we defined the cup product in [Lemma 22](#), we have the claim.  $\square$

### 2.5 Analytic combinatorics

In order to study the asymptotic of the number of isotopy classes of geometric graphs on the real line we will need some tools from analytic combinatorics. For a full introduction to the topic, see [\[12\]](#). Given a generating function  $G(x) = \sum_{n=0}^\infty a_n x^n$  of a sequence  $a_n$ , we want to study the asymptotics of such a sequence. There are various techniques to do this.

**Definition 25** We say that a sequence  $\{a_n\}$  is of exponential order  $K^n$ , which we abbreviate as  $a_n \asymp K^n$ , if and only if  $\limsup |a_n|^{1/n} = K$ .

If we have  $a_n \asymp K^n$  then  $a_n = K^n \theta(n)$  with  $\limsup |\theta(n)|^{1/n} = 1$ . The term  $\theta(n)$  is called the subexponential factor. In order to study the subexponential factor  $\theta(n)$  we should look at the singularities of the generating function.

**Definition 26** Given two numbers  $\phi$  and  $R$  with  $R > 1$  and  $0 < \phi < \pi/2$ , define an open domain

$$D(\phi, R) := \{z \in \mathbb{C} \mid |z| < R, z \neq 1, |\arg(z - 1)| > \phi\}.$$

A domain of this type is called  $D$ -domain.

Denoting by  $S$  the set of all meromorphic functions of the form

$$S := \{(1 - z)^{-\alpha} \mid \alpha \in \mathbb{R}, z \in \mathbb{C}\},$$

we recall the next result [\[12, Theorem VI.4\]](#), which we will need in the sequel.

**Theorem 27** Let  $G(z)$  be an analytic function at 0 with a singularity at  $\zeta$ , such that  $G(z)$  can be continued to a domain of the form  $\zeta \cdot D_0$  for a  $D$ -domain  $D_0$ , where  $\zeta \cdot D_0$  is the image of  $D_0$  by the mapping  $z \rightarrow \zeta z$ . Assume there exist two functions  $\sigma$  and  $\tau$ , where  $\sigma$  is a finite linear combination of elements in  $S$  and  $\tau \in S$ , such that

$$G(z) = \sigma\left(\frac{z}{\zeta}\right) + O\left(\tau\left(\frac{z}{\zeta}\right)\right) \quad \text{as } z \rightarrow \zeta \text{ in } \zeta \cdot D_0.$$

Then the coefficients of  $G(z)$  satisfy the asymptotic estimate

$$a_n = \zeta^{-n} \sigma_n + O(\zeta^{-n} \tau_n^*),$$

where  $\sigma(z) = \sum_{n=0}^\infty \sigma_n z^n$  and  $\tau_n^* = n^{\alpha-1}$  if  $\tau(z) = (1 - z)^{-\alpha}$ .

**Remark 28** For later use, we record the following. The Newton binomial series is defined by

$$(1 - z)^{-\alpha} = \sum_{n=0}^{\infty} b_n z^n, \quad \text{where } b_n = \binom{n + \alpha - 1}{n}.$$

Using

$$\binom{n + \alpha - 1}{n} = \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)\Gamma(n + 1)},$$

we get the following asymptotics for its coefficients:

$$b_n = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left( 1 + O\left(\frac{1}{n}\right) \right).$$

### 3 Homology of the chambers and the Floer number

#### 3.1 Graphs and sign conditions

Recall that given a graph  $G$  on  $n$  vertices we have defined

$$W_{G,d} = \{P \in \mathbb{R}^{d \times n} \setminus \Delta_{d,n} \mid G(P) \cong G\} \subset \mathbb{R}^{d \times n}.$$

In other words,  $W_{G,d}$  consists of all the points  $P \in \mathbb{R}^{d \times n}$  not on the discriminant whose corresponding graph is isomorphic to  $G$ . For small  $d$  this set could be a union of several chambers, but for large  $d$  it is an actual chamber (a connected open set).

Now we introduce an alternative notation for labeling the sets  $W_{G,d}$ . For every  $1 \leq i < j \leq n$  let us denote by  $q_{ij} : \mathbb{R}^{d \times n} \rightarrow \mathbb{R}$  the quadratic polynomial

$$(3-1) \quad q_{ij}(x_1, \dots, x_n) = \|x_i - x_j\|^2 - 1, \quad \text{where } (x_1, \dots, x_n) \in \mathbb{R}^{d \times n}.$$

Notice that the discriminant  $\Delta_{d,n}$  is given by

$$(3-2) \quad \Delta_{d,n} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^{d \times n} \mid \prod_{i < j} q_{ij}(x_1, \dots, x_n) = 0 \right\} = \bigcup_{i < j} \Delta_{d,n}^{(i,j)},$$

where the sets  $\Delta_{d,n}^{(i,j)}$  are as defined in (2-2). We denote by  $\binom{[n]}{2}$  the set of all possible pairs  $(i, j)$  with  $1 \leq i < j \leq n$ , and by  $2^{\binom{[n]}{2}}$  the set of all possible choices of signs  $\sigma_{ij} \in \{\pm\}$  for elements in  $\binom{[n]}{2}$ .

**Definition 29** (sign condition) For every  $\sigma \in 2^{\binom{[n]}{2}}$ , we denote by  $W_{\sigma,d} \in \mathbb{R}^{d \times n}$  the open set

$$W_{\sigma,d} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^{d \times n} \mid \text{sign}(q_{ij}(x)) = \sigma_{ij}\}.$$

At this point, what is clear from (3-2) is that  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  can be written as the union of all the possible sign conditions. The following lemma will be useful. It tells us that we can label the chambers of  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  either with a graph or with a sign condition — however, at this point we only prove that the sets  $\{W_{G,d}\}$  and  $\{W_{\sigma,d}\}$  coincide; the fact that the chambers of  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$ , for  $d \geq n + 1$ , are exactly the sign conditions will follow from Corollary 47.

**Lemma 30** For every  $G$  graph on  $n$  vertices, there is a sign condition  $\sigma = \sigma(G)$  such that  $W_{G,d} = W_{\sigma,d}$ . Conversely, for every  $\sigma$  there exists a  $G(\sigma)$  such that  $W_{\sigma,d} = W_{G,\sigma}$ . In other words, the signs of the family of quadrics  $\{q_{ij}\}_{1 \leq i < j \leq n}$  on a point  $P$  determine the isomorphism class of  $G(P)$  uniquely as a labeled graph.

**Proof** Given  $P \in \mathbb{R}^{d \times n} \setminus \Delta_{d,n}$ , it is clear from the definition of geometric graph that  $G(P) \cong G$  if and only if  $q_{i,j} < 0$  when  $(i, j) \in G$  and  $q_{i,j} > 0$  when  $(i, j) \notin G$ . From this it follows that  $W_{G,d} = W_{\sigma,d}$ , where  $\sigma_{i,j} = 1$  if  $(i, j) \in G$ , and  $\sigma_{i,j} = -1$  if  $(i, j) \notin G$ .  $\square$

## 3.2 Betti numbers of the chambers

In this section we study the asymptotic distribution of the Betti numbers of the chambers. Before giving the main result, we will need some intermediate steps.

**3.2.1 Some preliminary reductions** Using Lemma 30 we can immediately switch from the graph labeling to the sign condition, and given  $G$  there exists  $\sigma$  such that  $W_{G,d} = W_{\sigma,d}$ . In this way we describe the chamber we are interested in with a system of quadratic inequalities, and we will take advantage of this description.

From now on we will assume that  $G$  is not the complete graph, since for this case  $W_{G,d}$  is convex and therefore the study of its topology is complete. Our first step is to replace  $W_{\sigma,d}$  with another space which has the same homology and which is compact. To start with, we have

$$W_{\sigma,d} = \{x \in \mathbb{R}^{d \times n} \mid \text{sign}(q_{ij}(x)) = \sigma_{i,j} \text{ for all } 1 \leq i < j \leq n\},$$

with the quadrics  $q_{ij}: \mathbb{R}^{d \times n} \rightarrow \mathbb{R}$  defined above. Since  $\sigma$  is fixed, it will be convenient for us to define the new quadrics

$$s_{ij} = \sigma_{ij} q_{ij} \quad \text{and} \quad h_{ij}(x, z) = \sigma_{ij} (\|x_i - x_j\|^2 - z^2).$$

We set  $N = nd$ , and  $k = \binom{n}{2}$  and for every  $\epsilon > 0$  consider the set

$$W_{\sigma,d}(\epsilon) = \{[x : z] \in \mathbb{RP}^N \mid h_{ij}(x, z) \geq \epsilon z^2 \text{ for all } 1 \leq i < j \leq n, \|x\|^2 \leq \epsilon^{-1} z^2\} \subseteq \mathbb{RP}^N.$$

Notice that  $W_{\sigma,d}(\epsilon) \cap \{z = 0\} = \emptyset$ , because if  $z = 0$  then the last inequality defining  $W_{\sigma,d}(\epsilon)$  forces  $x = 0$ . Therefore, in the affine chart  $\{z \neq 0\}$  the set  $W_{\sigma,d}(\epsilon)$  can be described as

$$W_{\sigma,d}(\epsilon) \cap \{z \neq 0\} = \{x \in \mathbb{R}^N \mid s_{ij}(x) \geq \epsilon \text{ for all } 1 \leq i < j \leq n, \|x\|^2 \leq \epsilon^{-1}\},$$

and can be identified with a subset of  $W_{\sigma,d}$ .

**Proposition 31** For every  $d > 0$  there exists a  $\epsilon(d)$  such that for all  $\epsilon < \epsilon(d)$ , the inclusion

$$W_{\sigma,d}(\epsilon) \hookrightarrow W_{\sigma,d}$$

is a homotopy equivalence.

**Proof** First of all, let us notice that  $W_{\sigma,d}(\epsilon) \cap \{z = 0\} = \emptyset$ , and we can write

$$W_{\sigma,d}(\epsilon) = \left\{ (p_1, \dots, p_n) \in \mathbb{R}^{d \times n} \mid (\|p_i - p_j\|^2 - 1)\sigma_{i,j}^G \geq \epsilon, \|P\|^2 \leq \frac{1}{\epsilon} \right\}.$$

If the sign condition  $\sigma$  is not always negative for every  $(i, j)$  — ie the graph  $G$  is not complete — we conclude by [Corollary 21](#). □

The set  $W_{\sigma,d}(\epsilon)$  is now compact and for  $\epsilon < \epsilon(d)$  has the same homology of  $W_{\sigma,d}$ . For technical reasons, this is not yet the set we will work with. Instead we will work with its double cover

$$V_{\sigma,d}(\epsilon) = \{x \in S^N \mid h_{ij}(x, z) \geq \epsilon z^2 \text{ for all } 1 \leq i < j \leq n, \|x\|^2 \leq \epsilon^{-1} z^2\} \subset S^N.$$

This will not be an obstacle for computing the Betti numbers of  $W_{\sigma,d}(\epsilon)$ , because of the next lemma.

**Lemma 32** *For every  $\epsilon > 0$ , the set  $V_{d,\epsilon}(\epsilon) \subset S^N$  consists of two disjoint copies of  $W_{\sigma,d}(\epsilon)$ . In particular, for all  $k \geq 0$ ,*

$$b_k(W_{\sigma,d}(\epsilon)) = \frac{1}{2} b_k(V_{\sigma,d}(\epsilon)).$$

**Proof** Let  $\{z = 0\} \simeq S^{N-1}$  be the equator in  $S^N$  and observe that  $\{z = 0\} \cap V_{\sigma,d}(\epsilon) = \emptyset$ . This implies that

$$V_{\sigma,d}(\epsilon) = (V_{\sigma,d}(\epsilon) \cap \{z > 0\}) \sqcup (V_{\sigma,d}(\epsilon) \cap \{z < 0\}).$$

The involution  $(x, z) \mapsto (-x - z)$  on the sphere  $S^N$  restricts to a homeomorphism between  $V_{\sigma,d}(\epsilon) \cap \{z > 0\}$  and  $V_{\sigma,d}(\epsilon) \cap \{z < 0\}$ . Each of these sets is homeomorphic to its projection to the projective space  $\mathbb{R}P^N$ , which is the set  $W_{\sigma,d}(\epsilon)$ . □

**3.2.2 Systems of quadratic inequalities** The set  $V_{\sigma,d}(\epsilon)$  defined above is the set of solutions of a system of quadratic inequalities, and we will now use the spectral sequence from [Section 2.4](#) for computing its Betti numbers with  $\mathbb{Z}_2$ -coefficients.

Let us introduce homogeneous quadrics  $h_{ij,\epsilon}, h_{0,\epsilon}: \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{k+1}$  defined for all  $1 \leq i < j \leq n$  by

$$h_{ij,\epsilon}(x, z) = \sigma_{ij} \|x_i - x_j\|^2 - \sigma_{ij} z^2 - \epsilon z^2 \quad \text{and} \quad h_{0,\epsilon}(x, z) = \|x\|^2 - \epsilon^{-1} z^2,$$

in order to reduce to the framework of [Section 2.4](#). These quadrics can be put as the components of a quadratic *map* defined by

$$h_\epsilon = (h_{0,\epsilon}, h_{1,\epsilon}, h_{2,\epsilon}, \dots, h_{k,\epsilon}): \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{k+1},$$

where we are using the identification of sets of indices  $\{1, 2, \dots, k\} = \{(1, 2), (1, 3), \dots, (n - 1, n)\}$ . Inside the space  $\mathbb{R}^{k+1}$  we can consider the closed convex cone

$$K = \{y_0 \leq 0, y_1 \geq 0, \dots, y_k \geq 0\},$$

so that our original set can be written as

$$V_{\sigma,d}(\epsilon) = h_\epsilon^{-1}(K).$$

In this case the set  $\Omega \subset S^k$  is the set

$$\Omega = \{(\omega_0, \dots, \omega_k) \in S^k \mid \omega_0 \geq 0, \omega_1 \leq 0, \dots, \omega_k \leq 0\}.$$



For every point  $\omega = (\omega_0, \dots, \omega_k)$  we can consider the quadratic form  $\omega h_\epsilon$  defined by

$$\omega h_\epsilon = \omega_0 h_{0,\epsilon} + \dots + \omega_k h_{k,\epsilon}.$$

Using this notation, for every  $j \geq 0$  we define the sets

$$\Omega^j(\epsilon) = \{(\omega_0, \dots, \omega_k) \in \Omega \mid \text{ind}^+(\omega H_\epsilon) \geq j\}.$$

These are just the sets  $\Omega^j$  defined in Section 2.4, in the case of the quadratic map  $h_\epsilon$ .

For every  $1 \leq i < j \leq n$  let us also denote by  $U_{ij} \in \text{Sym}(n, \mathbb{R})$  the symmetric matrix representing the quadratic form  $u_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$u_{ij}(t_1, \dots, t_n) = \sigma_{ij}(t_i - t_j)^2.$$

Then, if  $H_{ij} \in \text{Sym}(dn, \mathbb{R})$  is the matrix representing the quadratic form  $x \mapsto \sigma_{ij} \|x_i - x_j\|^2$ , we have

$$H_{ij} = U_{ij} \otimes \mathbf{1}_d.$$

**Lemma 33** *The index function  $\text{ind}^+ : \Omega \rightarrow \mathbb{N}$  for our family of quadrics can be written as*

$$\text{ind}^+(\omega H_\epsilon) = d \cdot \text{ind}_1^+(\omega) + \text{ind}_{0,\epsilon}^+(\omega),$$

where

$$\text{ind}_1^+(\omega) = \text{ind}^+\left(\omega_0 \mathbf{1}_n + \sum_{i < j} \omega_{ij} U_{ij}\right) \quad \text{and} \quad \text{ind}_{0,\epsilon}^+(\omega) = \text{ind}^+\left(-\frac{\omega_0}{\epsilon} - \sum_{i < j} \omega_{ij} (\sigma_{ij} + \epsilon)\right).$$

Before giving the proof, observe that none of the functions  $\text{ind}_1^+, \text{ind}_{0,\epsilon}^+ : \Omega \rightarrow \mathbb{N}$  depends on  $d$  and that  $\text{ind}_1^+$  does not even depend on  $\epsilon$ .

**Proof** Observe that, for  $\omega = (\omega_0, \omega_{ij}) \in \Omega$ , the matrix  $\omega H_\epsilon$  is a block matrix:

$$\omega H_\epsilon = \left( \begin{array}{c|ccc} -\omega_0/\epsilon - \sum_{i < j} \omega_{ij} (\sigma_{ij} + \epsilon) & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & & \\ 0 & \omega_0 \mathbf{1}_{dn} + \sum_{i < j} \omega_{ij} H_{ij} & & \end{array} \right)$$

and, in particular,

$$\text{ind}^+(\omega H_\epsilon) = \text{ind}^+\left(-\frac{\omega_0}{\epsilon} - \sum_{i < j} \omega_{ij} (\sigma_{ij} + \epsilon)\right) + \text{ind}^+\left(\omega_0 \mathbf{1}_{dn} + \sum_{i < j} \omega_{ij} H_{ij}\right).$$

The matrix  $\omega_0 \mathbf{1}_{dn} + \sum_{i < j} \omega_{ij} H_{ij}$  is a tensor product of matrices,

$$\omega_0 \mathbf{1}_{dn} + \sum_{i < j} \omega_{ij} H_{ij} = \left(\omega_0 \mathbf{1}_d + \sum_{i < j} \omega_{ij} U_{ij}\right) \otimes \mathbf{1}_n.$$

If a matrix  $Q \in \text{Sym}(n, \mathbb{R})$  has eigenvalues  $\lambda_1(Q) \geq \dots \geq \lambda_n(Q)$  (possibly with repetitions), the matrix  $Q \otimes \mathbf{1}_d$  has eigenvalues

$$\lambda_{i,j}(Q \otimes \mathbf{1}_d) = \lambda_i(Q) \quad \text{for } i = 1, \dots, n, j = 1, \dots, d.$$

In particular,

$$\text{ind}^+(Q \otimes \mathbf{1}_d) = d \cdot \text{ind}^+(Q),$$

and the result now follows. □

**Corollary 34** For  $d \geq n + 1$ , the set  $\Omega^{nd}(\epsilon)$  is contractible and  $\Omega^{nd+1}(\epsilon)$  is empty.

**Proof** Let us first show that  $\Omega^{nd+1}(\epsilon) = \emptyset$ . To this end, consider the set

$$B(\epsilon) = \{(\omega, [x]) \in \Omega \times \mathbb{R}P^N \mid \omega h_\epsilon(x) \geq 0\}.$$

By [5, Lemma 24] the projection  $\pi = p_2|_{B(\epsilon)}$  on the second factor gives a homotopy equivalence between  $B(\epsilon)$  and its image

$$\pi(B(\epsilon)) = \mathbb{R}P^N \setminus W_{\sigma,d}(\epsilon).$$

Since  $W_{\sigma,d}(\epsilon)$  is nonempty, we know that

$$(3-3) \quad \pi(B(\epsilon)) \neq \mathbb{R}P^N.$$

If now there was  $\omega \in \Omega$  such that  $\text{ind}^+(\omega) = N + 1$ , then  $\omega h_\epsilon > 0$  and  $\{\omega\} \times \mathbb{R}P^N \subset B(\epsilon)$ . This would imply that  $\pi(B(\epsilon)) = \mathbb{R}P^N$ , which contradicts (3-3).

Let us now prove that  $\Omega^{nd}(\epsilon)$  is contractible. For  $d \geq n + 1$ , since  $\Omega^{nd+1}(\epsilon) = \emptyset$ , then the set  $\Omega^{nd}(\epsilon)$  can be described as

$$\Omega^{nd}(\epsilon) = \{\text{ind}^+ = nd\} = \{d \cdot \text{ind}_1^+ + \text{ind}_{0,\epsilon} \geq nd\} = \{\text{ind}_1^+ = n\} \cap \{\text{ind}_{0,\epsilon} = 0\}.$$

Observe that the point  $\omega = (1, 0, \dots, 0) \in \Omega$  belongs to both the sets  $\{\text{ind}_1^+ = n\}$  and  $\{\text{ind}_{0,\epsilon} = 0\}$ , and their intersection is nonempty.

Now,  $\{\text{ind}_1^+ = n\}$  and  $\{\text{ind}_{0,\epsilon} = 0\}$  are obtained by intersecting a convex set in  $\mathbb{R}^{k+1}$  with  $\Omega \cap S^k$ , as they coincide with the set of the points where the linear families of symmetric matrices  $\omega_0 \mathbf{1}_n + \sum_{i < j} \omega_{ij} U_{ij}$  and  $-\omega_0/\epsilon - \sum_{i < j} \omega_{ij}(\sigma_{ij} + \epsilon)$  are, respectively, positive definite and negative semidefinite. In other words,  $\{\text{ind}_1^+ = n\}$  is the preimage of the positive definite cone under the linear map  $\omega \mapsto \omega_0 \mathbf{1}_d + \sum_{i < j} \omega_{ij} U_{ij}$ , and  $\{\text{ind}_{0,\epsilon} = 0\}$  is the preimage of the negative semidefinite cone under the linear map  $\omega \mapsto -\omega_0/\epsilon - \sum_{i < j} \omega_{ij}(\sigma_{ij} + \epsilon)$ .

Therefore  $\Omega^{nd}(\epsilon)$  is the intersection in  $S^k \cap \Omega$  of convex sets, and being  $\Omega$  itself also convex, this intersection is contractible. □

Recalling the notation of Section 2.4, but making it dependent on  $\epsilon$ , we have the vector bundle  $P^j(\epsilon) \subseteq \Omega^j(\epsilon) \setminus \Omega^{j+1}(\epsilon) \times \mathbb{R}^{N+1}$

$$(3-4) \quad \begin{array}{ccc} \mathbb{R}^j & \hookrightarrow & P^j(\epsilon) \\ & & \downarrow \\ & & \Omega^j(\epsilon) \setminus \Omega^{j+1}(\epsilon) \end{array}$$

whose fiber over a point  $\omega$  is the positive eigenspace of  $\omega H_\epsilon$ . As above, this bundle is the restriction of a bundle over the set

$$D_j(\epsilon) = \{\omega \mid \lambda_j(\omega H_\epsilon) \neq \lambda_{j+1}(\omega H_\epsilon)\},$$

ie the set where the  $j$ th eigenvalue of  $\omega H_\epsilon$  is distinct from the  $(j+1)^{\text{st}}$ . We still denote this bundle by  $P_j(\epsilon) \subset D_j(\epsilon) \times \mathbb{R}^{N+1}$ :

$$(3-5) \quad \begin{array}{ccc} \mathbb{R}^j & \hookrightarrow & P^j(\epsilon) \\ & & \downarrow \\ & & D_j(\epsilon) \end{array}$$

Here the fiber over a point  $\omega \in D_j(\epsilon)$  consists of the eigenspace of  $\omega H_\epsilon$  associated to the first  $j$  eigenvalues. We denote the first Stiefel–Whitney class of this bundle by

$$(3-6) \quad v_j(\epsilon) \in H^1(D_j(\epsilon)).$$

Restating Theorem 24 in this setting, we get the following.

**Theorem 35** *There exists a cohomology spectral sequence  $(E_r(\epsilon), d_r(\epsilon)_{r \geq 1})$  which converges to  $H^*(S^N \setminus V_{\sigma,d}(\epsilon); \mathbb{Z}_2)$  and is such that:*

- (1) *The second page of the spectral sequence is given, for  $j > 0$ , by*

$$E_2^{i,j}(\epsilon) = H^i(\Omega^{j+1}(\epsilon), \Omega^{j+2}(\epsilon); \mathbb{Z}_2).$$

*For  $j = 0$ , the elements of the second page of the spectral sequence fit into a long exact sequence*

$$(3-7) \quad \dots \rightarrow H^i(\Omega^1(\epsilon); \mathbb{Z}_2) \rightarrow E_2^{i,0}(\epsilon) \rightarrow H^i(\Omega^1(\epsilon), \Omega^2(\epsilon); \mathbb{Z}_2) \xrightarrow{(\cdot) \smile v_1(\epsilon)} H^{i+1}(\Omega^1(\epsilon); \mathbb{Z}_2) \rightarrow \dots$$

- (2) *For  $j \geq 1$ , the second differential  $d_2^{i,j}(\epsilon): H^i(\Omega^{j+1}(\epsilon), \Omega^{j+2}(\epsilon)) \rightarrow H^{i+2}(\Omega^j(\epsilon), \Omega^{j+1}(\epsilon))$  is given by*

$$d_2^{i,j}(\epsilon)\xi = \partial(\xi \smile v_{j+1}(\epsilon)) + \partial\xi \smile v_j(\epsilon).$$

**Remark 36** As explained in [5, Introduction], the second differential only depends on the restriction of  $v_j(\epsilon)$  to the set  $\Omega^j(\epsilon) \setminus \Omega^{j+1}(\epsilon)$ .

**Remark 37** In the previous spectral sequence, the coefficient group for the various cohomologies is the field  $\mathbb{Z}_2$ . There is an analogous spectral sequence for coefficients in  $\mathbb{Z}$ , but the description of its differentials is less clear.

**3.2.3 The analysis of the spectral sequence and its asymptotic structure** We will start by proving the following proposition, which deals with the stabilization of entries of the second page of the spectral sequence of [Theorem 35](#).

**Proposition 38** *There exist semialgebraic topological spaces*

$$\Omega = A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \supseteq \dots \supseteq A_n \supseteq B_n = \emptyset,$$

vector spaces  $N^{0,0}, \dots, N^{k,0}$  and  $\epsilon_1 > 0$  such that for all  $\epsilon \leq \epsilon_1$ , the second page of the spectral sequence of [Theorem 35](#) has the structure

$$(3-8) \quad E_2^{i,j}(\epsilon) \simeq \begin{cases} H^i(B_\ell, A_{\ell+1}) & \text{if } j = \ell d, \\ H^i(A_\ell, B_\ell) & \text{if } j = \ell d - 1, \\ N^{i,0} & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** Observe first that the second page of the spectral sequence is zero in the region

$$\{(i, j) \mid i \geq k + 1, j > 0\},$$

because all the sets  $\Omega^j(\epsilon)$  are semialgebraic and of dimension at most  $k$  (since they are contained in  $\Omega \subset S^k$ ). The  $j = 0$  row of the spectral sequence is also zero for  $i \geq k + 2$ , since for the same reason all the groups in the exact sequence in [\(3-7\)](#) are zero.

Observe now that [Lemma 33](#) implies that the only possible values of the function  $\text{ind}^+ : \Omega \rightarrow \mathbb{N}$  are  $0, 1, d, d + 1, \dots, nd, nd + 1$  and in particular,

$$(3-9) \quad \Omega = \Omega^0(\epsilon) \supseteq \Omega^1(\epsilon) \supseteq \Omega^2(\epsilon) = \Omega^3(\epsilon) = \dots = \Omega^d(\epsilon) \supseteq \Omega^{d+1}(\epsilon) \supseteq \Omega^{d+2}(\epsilon) = \Omega^{d+3}(\epsilon) \\ = \dots = \Omega^{nd-1}(\epsilon) = \Omega^{nd}(\epsilon) \supseteq \Omega^{nd+1}(\epsilon) \supseteq \emptyset.$$

In particular, for every  $\ell = 0, \dots, n$ , we deduce the vanishing of the homology of all the relative pairs:

$$H^*(\Omega^{d\ell+2}(\epsilon), \Omega^{d\ell+3}(\epsilon)) = \dots = H^*(\Omega^{(\ell+1)d-1}(\epsilon), \Omega^{(\ell+1)d}(\epsilon)) = 0.$$

This proves the “otherwise” part of the claim in [\(3-8\)](#).

We now define the sets  $A_\ell(\epsilon) = \{\text{ind}^+ \geq d\ell\}$  and  $B_\ell(\epsilon) = \{\text{ind}^+ \geq d\ell + 1\}$  and observe that

$$A_\ell(\epsilon) = \{\text{ind}_1^+ \geq \ell\}, \quad B_\ell(\epsilon) = (\{\text{ind}_1^+ \geq \ell\} \cap \{\text{ind}_{0,\epsilon}^+ = 1\}) \cup \{\text{ind}_1^+ \geq \ell + 1\},$$

where the index functions  $\text{ind}_{0,\epsilon}^+, \text{ind}_1^+ : \Omega \rightarrow \mathbb{N}$  are defined in [Lemma 33](#). Since  $\text{ind}_1^+$  does not depend on  $d$  nor on  $\epsilon$  and  $\text{ind}_{0,\epsilon}$  does not depend on  $d$ , by semialgebraic triviality it follows that there exists  $\epsilon_1 > 0$  such that the homotopy of the sequence of inclusions

$$\Omega = A_0(\epsilon) \supseteq B_0(\epsilon) \supseteq A_1(\epsilon) \supseteq B_1(\epsilon) \supseteq \dots \supseteq A_n(\epsilon) \supseteq B_n(\epsilon) = \emptyset$$

stabilizes for  $\epsilon \leq \epsilon_1$ .

We define  $A_\ell = A_\ell(\epsilon_1)$  and  $B_\ell = B_\ell(\epsilon_1)$ . With this notation we have that the sequence of inclusions (3-9) for  $\epsilon \leq \epsilon_0$  becomes, up to natural homotopy equivalences,

$$\Omega = A_0 \supseteq B_0 \supseteq A_1 = A_1 = \cdots = A_1 \supseteq B_1 \supseteq A_2 = A_2 = \cdots = A_n = A_n \supseteq B_n \supseteq \emptyset.$$

This proves the statement for the term  $E_2^{i,j}(\epsilon)$  of the spectral sequence with  $j = \ell d - 1, \ell d$ .

In the case  $j = 0$ , we observe that the dimension of  $E_2^{i,0}(\epsilon)$  is determined by the exact sequence

$$0 \rightarrow \ker \rightarrow H^{i-1}(\Omega^1(\epsilon), \Omega^2(\epsilon)) \rightarrow H^i(\Omega^1(\epsilon)) \rightarrow E_2^{i,0}(\epsilon) \rightarrow H^i(\Omega^1(\epsilon), \Omega^2(\epsilon)) \rightarrow H^{i+1}(\Omega^1(\epsilon)) \rightarrow \text{coker} \rightarrow 0,$$

where  $\ker$  and  $\text{coker}$  refer to the map  $x \mapsto x \smile \nu_1(\epsilon)$ . The homotopy of the first, the third and the fourth element of the above sequence stabilizes for  $\epsilon \leq \epsilon_1$ ; moreover (possibly choosing a smaller  $\epsilon_1$ ) also the homotopy of the bundle  $P^1(\epsilon) \rightarrow D_1(\epsilon)$  from (3-5) stabilizes for  $\epsilon \leq \epsilon_1$  and therefore the map  $x \mapsto x \smile \nu_1(\epsilon)$  stabilizes as well, and consequently the ranks of  $\ker$  and  $\text{coker}$  stabilize. This gives the stabilization of  $\dim_{\mathbb{Z}_2}(E_2^{i,0}(\epsilon))$  to a finite number for  $\epsilon \leq \epsilon_1$ . We set

$$N^{i,0} := \mathbb{Z}_2^{\dim_{\mathbb{Z}_2}(E_2^{i,0}(\epsilon))} \quad \text{for all } \epsilon \leq \epsilon_1. \quad \square$$

Next we deal with the stabilization of the second differential.

**Proposition 39** *The second differential of the spectral sequence (3-8) is zero.*

**Proof** Observe that the only possible nonzero differential of the spectral sequence is, for  $d \geq 2$ ,

$$d_2^{*,\ell d}(\epsilon): E_2^{*,\ell d}(\epsilon) \rightarrow E_2^{*+2,\ell d-1}(\epsilon).$$

Let us recall that we have defined  $\omega H_\epsilon = \omega q_1 + \omega q_2$ , where  $\omega q_1 = (\omega_0 \mathbf{1}_{dn} + \sum_{i < j} \omega_{ij} H_{ij})$  and  $\omega q_2 = (-\omega_0/\epsilon - \sum_{i < j} \omega_{ij}(\sigma_{ij} + \epsilon))z^2$ . We introduce the vector bundles

$$\begin{array}{ccc} \mathbb{R}^{d \times l} & \hookrightarrow & N_{\ell d} \\ & & \downarrow \\ & & \mathcal{D}_{\ell d}^1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{R} & \hookrightarrow & E(\epsilon) \\ & & \downarrow \\ & & \Omega \end{array}$$

where  $\mathcal{D}_{\ell d}^1 := \{\omega \in \Omega \mid \lambda_{\ell d}(\omega q_1) \neq \lambda_{l d+1}(\omega q_1)\}$ ,  $N_{\ell d} \subset \mathcal{D}_{\ell d}^1 \times \mathbb{R}^{dn}$  is the bundle of the eigenspace of the first  $\ell d$  eigenvalues of the upper-left block of  $\omega H_\epsilon$ , and the bundle  $E(\epsilon)$  associates to every point of  $\Omega$  the unique eigenvector of  $\omega q_2$ .

Observe that  $\mathcal{D}_{\ell d}(\epsilon) \subset \mathcal{D}_{\ell d}^1$  and also  $\mathcal{D}_{\ell d+1}(\epsilon) \subset \mathcal{D}_{\ell d}^1$ .

The vector bundle  $P_{\ell d}(\epsilon)$  from (3-5) for  $j = \ell d$  has the property that

$$P_{\ell d}(\epsilon) = N_{\ell d}|_{\mathcal{D}_{\ell d}(\epsilon)}$$

when  $N_{\ell d}$  is thought as a subbundle of  $\mathcal{D}_{\ell d}^1 \times \mathbb{R}^{nd+1}$ . When  $j = \ell d + 1$  we have

$$P_{\ell d+1}(\epsilon) = N_{\ell d}|_{\mathcal{D}_{\ell d+1}(\epsilon)} \oplus E(\epsilon)|_{\mathcal{D}_{\ell d+1}(\epsilon)}$$

because the quadratic form  $\omega q$  has two diagonal blocks. In particular, denoting by  $\gamma_{\ell d}$  and by  $\eta(\epsilon)$  the first Stiefel–Whitney class of  $N_{\ell d}$  and  $E(\epsilon)$ , respectively, by naturality of characteristic classes we have the identities

$$v_{\ell d} = \gamma_{\ell d}|_{\mathcal{D}_{\ell d}(\epsilon)} \quad \text{and} \quad v_{\ell d+1} = \gamma_{\ell d}|_{\mathcal{D}_{\ell d+1}(\epsilon)} + \eta(\epsilon)|_{\mathcal{D}_{\ell d+1}(\epsilon)}.$$

Notice that both  $v_{\ell d}$  and  $v_{\ell d+1}$  contain the restriction of the same class  $\gamma_{\ell d}$  as a summand. Thanks to [Theorem 35](#), the second differential  $d_2^{*,\ell d}(\epsilon)$  can be written as

$$\begin{aligned} d_2^{*,\ell d}(\epsilon)\xi &= \partial(\xi \smile v_{\ell d+1}) + \partial\xi \smile v_{\ell d} = \partial(\xi \smile (\gamma_{\ell d}|_{\mathcal{D}_{\ell d+1}(\epsilon)} + \eta(\epsilon)|_{\mathcal{D}_{\ell d+1}(\epsilon)})) + \partial\xi \smile \gamma_{\ell d}|_{\mathcal{D}_{\ell d}(\epsilon)} \\ &= \partial(\xi \smile \eta(\epsilon)|_{\mathcal{D}_{\ell d+1}(\epsilon)}) = \partial(\xi \smile \eta(\epsilon)), \end{aligned}$$

where we have used [Remark 23](#) (taking  $(Z, X, Y) = (\Omega^{\ell d}(\epsilon), \Omega^{\ell d+1}(\epsilon), \Omega^{\ell d+2}(\epsilon))$  and  $(\tilde{X}, \tilde{Z}, A) = (\mathcal{D}_{\ell d+1}(\epsilon), \mathcal{D}_{\ell d}(\epsilon), \mathcal{D}_{\ell d}^1)$ ) and the fact that we are working with  $\mathbb{Z}_2$ -coefficients.

On the other hand, the bundle  $E(\epsilon)$  is trivial, because the space  $\Omega$  is contractible and the class  $\eta(\epsilon)$  is zero. Therefore the differential is zero and this concludes the proof.  $\square$

**Remark 40** It is actually possible to prove the stabilization of the second differential, up to subsequences, in a simpler way. In fact,  $\{d_2(\epsilon)^{*,\ell d} : H^*(A_\ell, B_\ell) \rightarrow H^{*+2}(B_\ell, A_{\ell+1})\}_{d \geq 0}$  is a sequence of maps between finite-dimensional  $\mathbb{Z}_2$ -vector spaces, ie

$$d_2(\epsilon)^{*,\ell d} \in \text{Hom}(\mathbb{Z}_2^a, \mathbb{Z}_2^b),$$

where  $a = \dim_{\mathbb{Z}_2}(H^*(A_\ell, B_\ell))$  and  $b = \dim_{\mathbb{Z}_2}(H^{*+2}(B_\ell, A_{\ell+1}))$ . Since  $\text{Hom}(\mathbb{Z}_2^a, \mathbb{Z}_2^b) \simeq \mathbb{Z}_2^{a \times b}$  is a finite set, then up to subsequences,  $d_2(\epsilon)^{*,\ell d}$  is eventually constant.

**3.2.4 The asymptotics for the Betti numbers of the chamber** We are now in the position of proving the main theorem of this section, namely [Theorem 8](#).

**Proof of Theorem 8** The proof of this theorem is based on the analysis of the structure of the spectral sequence and its last page. First observe that by [Proposition 31](#), for all  $k \geq 0$  and for all  $\epsilon < \epsilon(d)$  we have

$$b_k(W_{\sigma,d}) = b_k(W_{\sigma,d}(\epsilon)).$$

For the rest of the proof we will take  $\epsilon \leq \min\{\epsilon(d), \epsilon_2\}$ , where  $\epsilon_2 \leq \epsilon_1$  is given by [Proposition 39](#). [Lemma 32](#) implies now that

$$b_k(W_{\sigma,d}) = \frac{1}{2}b_k(V_{\sigma,d}(\epsilon)).$$

On the other hand, since the involved spaces are semialgebraic sets (hence triangulable), the Betti numbers of  $V_{\sigma,d}(\epsilon)$  are related to those of  $S^N \setminus V_{\sigma,d}(\epsilon)$  through Alexander duality:

$$\tilde{b}_k(V_{\sigma,d}(\epsilon)) = \tilde{b}_{N-k-1}(S^N \setminus V_{\sigma,d}(\epsilon)).$$

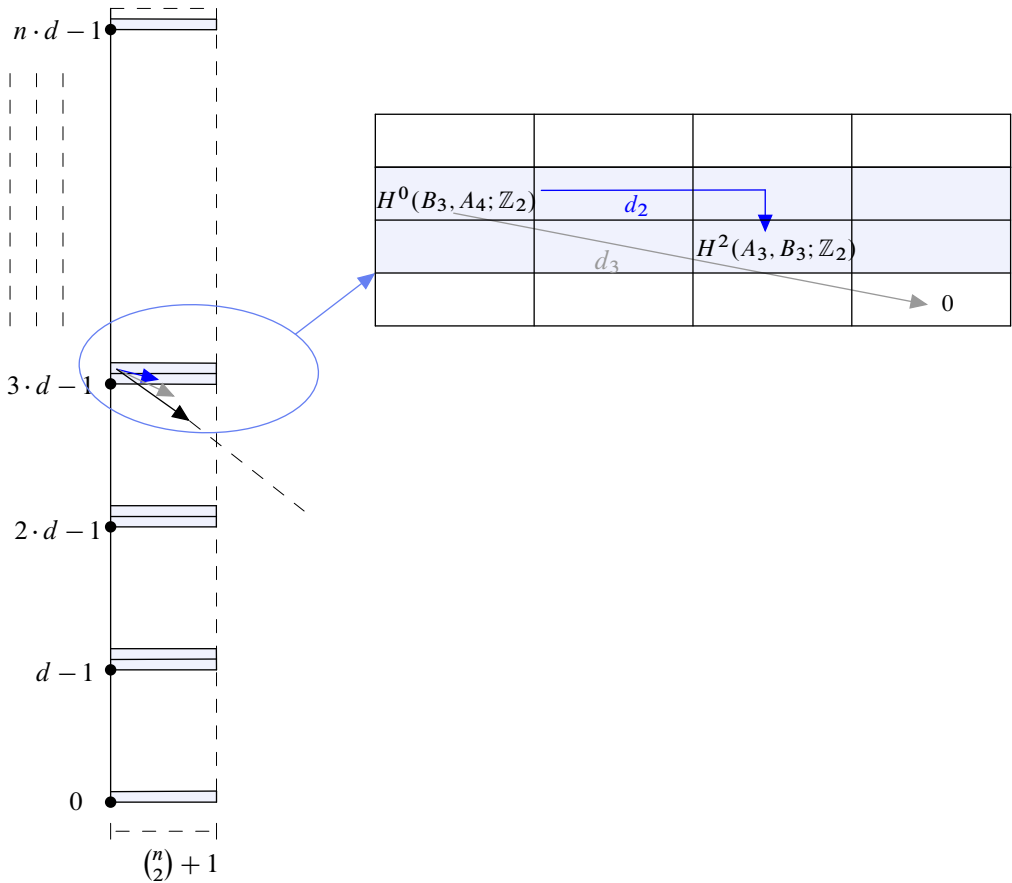


Figure 3: This is a schematic image of the  $E_2(\epsilon)$  term of the spectral sequence we are describing. The shaded parts correspond to the elements  $E_2^{i,j}(\epsilon)$  of the spectral sequence which are possibly nonzero.

Finally, denoting by  $e_\infty^{i,j}(\epsilon)$  the dimension of  $E_\infty^{i,j}(\epsilon)$ , where  $E_\infty(\epsilon)$  is the last page of the spectral sequence from [Theorem 35](#), we have

$$\tilde{b}_{N-k-1}(S^N \setminus V_{\sigma,d}(\epsilon)) = \sum_{i+j=N-k-1} e_\infty^{i,j}(\epsilon).$$

Collecting all this together, for  $\epsilon \leq \min\{\epsilon(d), \epsilon_2\}$  we have

$$(3-10) \quad b_k(W_{\sigma,d}) = \frac{1}{2} \begin{cases} 1 + \sum_{i+j=N-1} e_\infty^{i,j}(\epsilon) & \text{if } k = 0, \\ \sum_{i+j=N-k-1} e_\infty^{i,j}(\epsilon) & \text{if } 0 < k < N - 1, \\ -1 + e_\infty^{0,0}(\epsilon) & \text{if } k = N - 1. \end{cases}$$

Observe now that [Proposition 38](#) implies that in the second page of the spectral sequence only the first  $\binom{n}{2} + 1$  columns are nonzero (ie those with  $0 \leq i \leq \binom{n}{2}$ ); moreover, in the second page only the rows with  $j = \ell d$  and  $j = \ell d - 1$  are potentially nonzero, for  $\ell = 0, \dots, n$ . Therefore, for  $d \geq \binom{n}{2} + 2$  all

the higher differentials are zero, and

$$E_\infty(\epsilon) = E_3(\epsilon).$$

On the other hand Proposition 39 implies that  $E_2(\epsilon) = E_3(\epsilon) = E_\infty(\epsilon)$ , with the last equality for  $d \geq \binom{n}{2} + 2$ .

Looking now at the top two rows of  $E_\infty$ , by Corollary 34 we know that

$$E_2^{i,dn}(\epsilon) = 0 \quad \text{for all } i \geq 0, \quad E_2^{0,dn-1}(\epsilon) \simeq \mathbb{Z}_2, \quad E_2^{i,dn-1}(\epsilon) = 0 \quad \text{for all } i \geq 1.$$

Thus, for  $d \geq \binom{n}{2} + 2$ ,

$$e_\infty^{i,dn}(\epsilon) = 0 \quad \text{for all } i \geq 0, \quad e_\infty^{0,dn-1}(\epsilon) = 1, \quad e_\infty^{i,dn-1}(\epsilon) = 0 \quad \text{for all } i \geq 1.$$

From this we immediately see that for  $d \geq \binom{n}{2} + 2$  we have

$$b_0(V_{\sigma,d}(\epsilon)) = 2 \quad \text{and} \quad b_k(V_{\sigma,d}(\epsilon)) = 0 \quad \text{for all } 1 \leq k \leq \binom{n}{2}.$$

This already proves

$$(3-11) \quad b_0(W_{\sigma,d}) = 1 \quad \text{and} \quad b_k(W_{\sigma,d}) = 0 \quad \text{for all } 1 \leq k \leq \binom{n}{2}.$$

Observe now that the fact that the rows with  $j = \ell d$  and  $j = \ell d - 1$  in  $E_2 = E_\infty$  are the only possibly nonzero rows for  $\ell = 0, \dots, n$  influences the Betti numbers  $b_k(W_{\sigma,d})$  with

$$(3-12) \quad k = md, \dots, md - \binom{n}{2} - 1,$$

where  $m = n - \ell$ . We define now, for  $m = 1, \dots, n - 1$ ,

$$Q_{G,m}(t) = \frac{1}{2} \cdot \left( e_\infty^{0,(n-m)d-1} t^{\binom{n}{2}+1} + \sum_{i=1}^{\binom{n}{2}} (e_\infty^{\binom{n}{2}-i,(n-m)d} + e_\infty^{\binom{n}{2}-i+1,(n-m)d-1}) t^i + e_\infty^{\binom{n}{2},(n-m)d} \right).$$

The  $i^{\text{th}}$  coefficient of the polynomial  $Q_{G,m}$  is  $b_{md - \binom{n}{2} + i - 1}(W_{G,d})$ . In principle we would have to consider also the case  $m = n$ , but Theorem 9 guarantees that there is no homology in dimension greater than  $(n - 1)d - n + 1$ . The proof of Theorem 9 is proved in Section 5 as that is where we describe a *different* spectral sequence which we use to prove it via duality. By (3-12), the conclusion of the theorem follows.  $\square$

**Remark 41** As we noticed in the introduction, since the polynomials  $Q_{G,1}, \dots, Q_{G,n-1}$  do not depend on  $d$ , but only on the graph, and since these polynomials are the same for isomorphic graphs, they define a graph invariant. Similarly the same is true for the Floer number  $\beta(G)$ , which is just the sum of their coefficients. Of course the polynomials are finer invariants, however we do not have a clear interpretation of these quantities.

**Remark 42** From (3-11) it immediately follows that for  $d \geq \binom{n}{2} + 2$  each sign condition is connected. In particular, if  $d \geq \binom{n}{2} + 2$ , two  $\mathbb{R}^d$ -geometric graphs on  $n$  vertices are isomorphic if and only if they are rigidly isotopic. We will actually sharpen this in Corollary 47 below.



### 4 Homotopy groups of the chambers

We turn our attention now to proving [Theorem 3](#), and we start by introducing some notation. For every  $0 \leq r \leq \max\{n, d\}$  let us denote by  $(\mathbb{R}^{d \times n})_r$  the set of matrices of rank  $r$ ,

$$(\mathbb{R}^{d \times n})_r = \{P \in \mathbb{R}^{d \times n} \mid \text{rk}(P) = r\} \subseteq \mathbb{R}^{d \times n}.$$

When  $r = n$  we have that  $(\mathbb{R}^{d \times n})_n$  deformation retracts the Stiefel manifold of orthonormal  $n$ -frames in  $\mathbb{R}^n$  (the retraction is given by the Gram–Schmidt procedure; in the case  $d = n$  this is simply the deformation retraction of  $\text{GL}(n, \mathbb{R})$  onto  $O(n)$ ). We recall that for  $k \leq d - n - 1$  this Stiefel manifold is  $k$ -connected (see [\[16, Example 4.53\]](#)), ie

$$(4-1) \quad \pi_k((\mathbb{R}^{d \times n})_n) = 0 \quad \text{if } k \leq d - n - 1.$$

We will need the next elementary lemma.

**Lemma 43** *The complement of  $(\mathbb{R}^{d \times n})_n$  can be written as a finite union of smooth submanifolds of codimension at least  $d - n + 1$ .*

**Proof** Recall that for every  $0 \leq r \leq \max\{n, d\}$ , the codimension of  $(\mathbb{R}^{d \times n})_r$  in  $\mathbb{R}^{d \times n}$  equals  $(n - r)(d - r)$  (see [\[18, Chapter 3, Section 2, Exercise 4\]](#)) and, in particular, if  $r \leq n - 1$ ,

$$(4-2) \quad \text{codim}_{\mathbb{R}^{d \times n}}(\mathbb{R}^{d \times n})_r \geq d - n + 1.$$

Now, the complement of  $(\mathbb{R}^{d \times n})_n$  in  $\mathbb{R}^{d \times n}$  is a semialgebraic set that can be written as

$$\mathbb{R}^{d \times n} \setminus (\mathbb{R}^{d \times n})_n = \bigsqcup_{r=0}^{n-1} (\mathbb{R}^{d \times n})_r,$$

and it is therefore a semialgebraic set of codimension at least  $d - n + 1$ . □

We will now prove a sequence of results on the homotopy groups of the chambers. These results will imply [Theorem 3](#). Since  $W_{G,d}$  might not be connected if  $d \leq \binom{n}{2} + 1$ , part of these results are formulated using the set  $[S^k, W_{G,d}]$  of homotopy classes of continuous maps from  $S^k$  to  $W_{G,d}$ , instead of the homotopy group  $\pi_k(W_{G,d})$ . As soon as  $W_{G,d}$  becomes connected and simply connected, we can endow  $[S^k, W_{G,d}]$  with a group structure. To stress this subtlety we will keep both notations.

**Proposition 44** *If  $d \geq k + n + 1$ , the inclusion*

$$i : W_{G,d} \cap (\mathbb{R}^{d \times n})_n \hookrightarrow W_{G,d}$$

*induces a bijection between  $[S^k, W_{G,d} \cap (\mathbb{R}^{d \times n})_n]$  and  $[S^k, W_{G,d}]$ .*

**Proof** We need to prove that the map  $i_* : [S^k, W_{G,d} \cap (\mathbb{R}^{d \times n})_n] \rightarrow [S^k, W_{G,d}]$  induced by the inclusion is a bijection if  $k \leq d - n - 1$ .

We first prove the surjectivity of  $i_*$ . Let  $f_0: S^k \rightarrow W_{G,d}$  be a map representing an element of  $[S^k, W_{G,d}]$ . Since  $W_{G,d}$  is open, up to homotopies we can assume that the map  $f_0$  is smooth. Moreover, by [18, Chapter 3, Theorem 2.5], the map  $f_0$  is homotopic to a map  $f_1: S^k \rightarrow W_{G,d}$  which is transversal to all the strata of the complement of  $(\mathbb{R}^{d \times n})_n$ . If now  $k < d - n + 1$ , the transversality condition and Lemma 43 imply that the image of  $f_1$  does not intersect these strata; therefore  $f_1: S^k \rightarrow W_{G,d} \cap (\mathbb{R}^{d \times n})_n$  and  $i_*$  is surjective. (Notice that for the surjectivity we only need  $d \geq k + n$ .)

For the injectivity we argue similarly. Let  $f_0, f_1: S^k \rightarrow W_{G,d} \cap (\mathbb{R}^{d \times n})_n$  be two maps such that  $i \circ f_0: S^k \rightarrow W_{G,d}$  is homotopic to  $i \circ f_1: S^k \rightarrow W_{G,d}$ . This means there exists a map  $F: S^k \times I \rightarrow W_{G,d}$  such that  $F(\cdot, 0) = i \circ f_0$  and  $F(\cdot, 1) = i \circ f_1$ . Now, we can approximate  $F$  with a new map  $\tilde{F}: S^k \times I$  which is smooth, homotopic to  $F$  and  $C^0$  arbitrarily close to it, and transversal to all the strata of the complement of  $(\mathbb{R}^{d \times n})_n$ . By Lemma 43, if  $k + 1 < d - n + 1$  this implies that the image of  $\tilde{F}$  does not intersect the complement of  $(\mathbb{R}^{d \times n})_n$ . In particular we have a homotopy between  $\tilde{F}(\cdot, 0)$  and  $\tilde{F}(\cdot, 1)$  all contained in  $W_{G,d} \cap (\mathbb{R}^{d \times n})_n$ . On the other hand, since both  $F(\cdot, 0)$  and  $F(\cdot, 1)$  miss the complement of  $(\mathbb{R}^{d \times n})_n$ , which is closed, by compactness of  $S^k$ , any two maps  $C^0$  sufficiently close to these maps will be homotopic to them and will also miss this complement. In particular, if  $\tilde{F}$  is sufficiently close to  $F$ , then  $F(\cdot, 0)$  is homotopic to  $\tilde{F}(\cdot, 0)$ , and  $F(\cdot, 1)$  is homotopic to  $\tilde{F}(\cdot, 1)$ ; further, these homotopies miss the complement of  $(\mathbb{R}^{d \times n})_n$ . In this way we have build a homotopy between  $f_0$  and  $f_1$  already in  $W_{G,d} \cap (\mathbb{R}^{d \times n})_n$ , ie  $i_*$  is injective. □

**4.0.1 Some useful maps** We introduce now some useful maps. First recall the Gram–Schmidt map  $\sigma: \mathbb{R}^{d \times n} \rightarrow \mathbb{R}^{d \times n}$ , which orthonormalizes the columns of a matrix  $P \in \mathbb{R}^{d \times n}$  and is defined by

$$\sigma(P) = P(P^T P)^{-1/2}.$$

Since the columns of  $\sigma(P)$  are orthonormal, it follows that, if  $P \in \mathbb{R}_n^{d \times n}$ ,

$$(4-3) \quad \sigma(P)^T \sigma(P) = \mathbf{1}_n.$$

Moreover,  $\sigma(P)\sigma(P)^T$  is the orthogonal projection on the span of the columns of  $P$ .

Let now  $G$  be a geometric graph on  $n$  vertices and  $d \geq n$ . Our first useful map is

$$(4-4) \quad W_{G,n} \times (\mathbb{R}^{d \times n})_n \xrightarrow{\alpha_d} W_{G,d}, \quad (Q, P) \mapsto (\sigma(P)Q).$$

We need to verify that the isomorphism class of the labeled graph is unchanged, ie that  $G(\sigma(P)Q) \simeq G(Q) \simeq G$ . This is true because all the relative distances of the points in  $\sigma(P)Q$  are the same as the distances of the points in  $Q$ . More precisely, write  $Q = (q_1, \dots, q_n)$  and  $\sigma(P)Q = (p'_1, \dots, p'_n)$ , where  $p'_i = \sigma(P)q_i$ . Then, using (4-3), we have

$$\|p'_i - p'_j\|^2 = \|\sigma(P)(q_i - q_j)\|^2 = (q_i - q_j)^T \sigma(P)^T \sigma(P)(q_i - q_j) = (q_i - q_j)^T (q_i - q_j) = \|q_i - q_j\|^2.$$

The relative distances between the points are the same, so by definition it follows that  $G(\sigma(P)Q) \simeq G(Q)$ .

A second useful map is

$$W_{G,d} \cap (\mathbb{R}^{d \times n})_n \xrightarrow{\beta_d} W_{G,n} \times (\mathbb{R}^{d \times n})_n, \quad P \mapsto (\sigma(P)^T P, P).$$

Also for this map we need to check that its first component has target in  $W_{G,n}$ . Again this follows from the fact that the mutual distances of the corresponding points are preserved. Writing  $P = (p_1, \dots, p_n)$ , we have

$$\|\sigma(P)^T p_i - \sigma(P)^T p_j\|^2 = (p_i - p_j)^T \sigma(P) \sigma(P)^T (p_i - p_j) = (p_i - p_j)^T (p_i - p_j) = \|p_i - p_j\|^2,$$

where we have used the fact that  $\sigma(P) \sigma(P)^T$  is the orthogonal projection onto the span of the columns of  $P$ . The claim follows again from [Lemma 30](#).

Observe that it follows immediately from the definition of the maps  $\alpha$  and  $\beta$  that

$$(4-5) \quad \alpha_d \circ \beta_d = i: W_{G,d} \cap (\mathbb{R}^{d \times n})_n \hookrightarrow W_{G,d}.$$

### 4.0.2 Stabilization

**Proposition 45** *If  $d \geq k + n + 1$ , the map  $j_*: [S^k, W_{G,d}] \rightarrow [S^k, W_{G,d+1}]$  induced by the inclusion is injective.*

**Proof** Let  $g_0, g_1: S^k \rightarrow W_{G,d}$  be two continuous maps such that the compositions  $j \circ g_0: S^k \rightarrow W_{G,d+1}$  and  $j \circ g_1: S^k \rightarrow W_{G,d+1}$  are homotopic. Thanks to [Proposition 44](#) we can assume  $g_0$  and  $g_1$  to be elements of  $[S^k, W_{G,d} \cap (\mathbb{R}^{d \times n})_n]$ . We want to prove that  $g_0$  and  $g_1$  are homotopic. For a map  $f: S^k \rightarrow W_{G,d}$ , we consider the commutative diagram of maps

$$\begin{array}{ccccc}
 & & (W_{G,n} \cap (\mathbb{R}^{n^2})_n) \times (\mathbb{R}^{dn})_n & \xleftarrow{u} & (W_{G,n} \cap (\mathbb{R}^{n^2})_n) \times (\mathbb{R}^{(d+1)n})_n \\
 & \nearrow \beta_d \circ f = (f_1, f_2) & \downarrow \alpha_d & \curvearrowright \beta_d & \downarrow \alpha_{d+1} \\
 S^k & \xrightarrow{f} & W_{G,d} \cap (\mathbb{R}^{dn})_n & \xleftarrow{j} & W_{G,d+1} \cap (\mathbb{R}^{(d+1)n})_n
 \end{array}$$

Notice that here the maps  $\alpha_d$  and  $\alpha_{d+1}$ , defined in (4-4), are restricted to the set of pairs  $(Q, P)$  with  $\text{rk}(Q) = n$ ; the values of these maps are in the set of matrices of rank  $n$ .

Since  $\alpha_d \circ \beta_d = \text{id}$ , we can write the map  $f$  as

$$f = \alpha_d \circ (\beta_d \circ f) = \alpha_d \circ (f_1, f_2),$$

where  $(f_1, f_2)$  are the components of  $\beta_d \circ f$ . We apply now the diagram to the map  $f = g_0$  and  $f = g_1$ , writing them as

$$g_i = \alpha_d \circ (\beta_d \circ g_i) = \alpha_d \circ (g_{i,1}, g_{i,2}) \quad \text{for } i = 0, 1.$$

We will prove that both components are homotopic  $g_{0,1} \sim g_{1,1}$  and  $g_{0,2} \sim g_{1,2}$ , which implies that  $g_0$  is homotopic to  $g_1$ .

Since the map  $j \circ g_0$  is homotopic to  $j \circ g_1$ , then also the first component of  $\beta_{d+1} \circ j \circ g_0$  is homotopic to the first component of  $\beta_{d+1} \circ j \circ g_1$ . But the first component of  $\beta_{d+1} \circ j \circ g_0$  equals  $g_{0,1}$ , the first component of  $\beta_d \circ g_0$ , and similarly for the first component of  $\beta_{d+1} \circ j \circ g_1$ . Therefore  $g_{0,1} \sim g_{1,1}$ .

On the other hand the second components of  $\beta_d \circ g_0, \beta_d \circ g_1 : S^k \rightarrow (\mathbb{R}^{d \times n})_n$  are homotopic simply because  $\pi_k((\mathbb{R}^{d \times n})_n) = 0$  for  $k \leq d - n - 1$ . □

**Proposition 46** For every  $d \geq n$ , the inclusion  $W_{G,d} \hookrightarrow W_{G,d+n}$  is homotopic to a constant map.

Before we give the proof, let us observe that, since we do not know if  $W_{G,d}$  is path connected, there are several constant maps up to homotopy, one for each component; this proposition tells us that all the maps  $[S^k, W_{G,d}]$  are mapped to the same constant map in  $W_{G,d+n}$ . This also tells us that  $W_{G,d}$  is contained in just one connected component of  $W_{G,d+n}$ .

**Proof** Since for  $d \geq n$  every graph is realizable as a geometric graph, pick  $R = (r_1, \dots, r_n) \in W_{G,n}$  by the previously cited result of Maehara [23].

Consider the homotopy  $f_t : W_{G,d} \rightarrow \mathbb{R}^{(d+n) \times n}$  defined for  $t \in [0, 1]$  by

$$f_t(P) = \begin{pmatrix} \sqrt{1-t}P \\ \sqrt{t}R \end{pmatrix}.$$

With this choice,

$$f_0 = i : W_{G,d} \hookrightarrow W_{G,d+n} \subset \mathbb{R}^{(d+n) \times n} \quad \text{and} \quad f_1 \equiv \begin{pmatrix} 0 \\ R \end{pmatrix} \in W_{G,d+n}.$$

We only need to prove that  $f_t(W_{G,d}) \subseteq W_{G,d+n}$  for all  $t \in [0, 1]$ . To this end, let us write

$$f_t(P) = (p_1(t), \dots, p_n(t)) = \begin{pmatrix} \sqrt{1-t}p_1 & \dots & \sqrt{1-t}p_n \\ \sqrt{t}r_1 & \dots & \sqrt{t}r_n \end{pmatrix}.$$

Because of Lemma 30, in order to show that  $G(f_t(P)) \equiv G$  it is enough to show that the signs of the family of quadrics  $\{\|p_i - p_j\|^2 - 1 : \mathbb{R}^{(d+n) \times n} \rightarrow \mathbb{R}\}$  evaluated on  $f_t(P)$  are constants. We have

$$\|p_i(t) - p_j(t)\|^2 = (1-t)\|p_i - p_j\|^2 + t\|r_i - r_j\|^2,$$

and therefore, as

$$\text{sign}(\|p_i - p_j\|^2 - 1) = \text{sign}(\|r_i - r_j\|^2 - 1),$$

it must be the case that

$$\text{sign}(\|p_i(t) - p_j(t)\|^2 - 1) = \text{sign}(\|p_i - p_j\|^2 - 1) = \text{sign}(\|r_i - r_j\|^2 - 1). \quad \square$$

We are now ready to prove Theorem 3.

**Proof of Theorem 3** We first prove that for  $d \geq n + 1$  the set  $W_{G,d}$  is path connected. By Proposition 46 the map  $i_* : [S^0, W_{G,d}] \rightarrow [S^0, W_{G,d+n}]$  is the map that sends everything to the class of a constant map. On the other hand this map factors through the sequence of maps induced by the inclusions  $W_{G,d} \hookrightarrow W_{G,d+1}$

$$[S^0, W_{G,d}] \rightarrow [S^0, W_{G,d+1}] \rightarrow \dots \rightarrow [S^0, W_{G,d+n}].$$

Each map in the previous sequence is an injection for  $d \geq n + 1$  by Proposition 45, therefore also  $i_* : [S^0, W_{G,d}] \rightarrow [S^0, W_{G,d+n}]$  is an injection, and  $[S^0, W_{G,d}]$  consists of only one element. Therefore  $W_{G,d}$  is path connected.

We prove now that, for  $d \geq n + 2$ ,  $W_{G,d}$  is also simply connected. By Proposition 46 the map  $i_* : [S^1, W_{G,d}] \rightarrow [S^1, W_{G,d+n}]$  is the map that sends everything to the class of a constant map. On the other hand this map factors through the sequence of maps induced by the inclusions  $W_{G,d} \hookrightarrow W_{G,d+1}$ ,

$$[S^1, W_{G,d}] \rightarrow [S^1, W_{G,d+1}] \rightarrow \cdots \rightarrow [S^1, W_{G,d+n-1}] \rightarrow [S^1, W_{G,d+n}].$$

Each map in the previous sequence is an injection for  $d \geq n + 2$  by Proposition 45, therefore also  $i_* : [S^1, W_{G,d}] \rightarrow [S^1, W_{G,d+n}]$  is an injection, and  $[S^1, W_{G,d}]$  consists of only one element. Recall now that  $[S^1, W_{G,d}]$  consists of the set of conjugacy classes in  $\pi_1(W_{G,d})$  (we can omit the basepoint because  $W_{G,d}$  is path connected): the fact that  $[S^1, W_{G,d}]$  consists of one element implies that there is only one conjugacy class in  $\pi_1(W_{G,d})$ , which means that  $W_{G,d}$  is simply connected.

Let now  $k \geq 2$  and  $d \geq k + n + 1$ . Since  $\pi_1(W_{G,d}) = 0$  for  $d \geq n + 2$ , it follows that

$$[S^k, W_{G,d}] = \pi_k(W_{G,d}) / \pi_1(W_{G,d}) = \pi_k(W_{G,d}),$$

by [16, Proposition 4A.2]. By Proposition 46, the map  $i_* : \pi_k(W_{G,d}) \rightarrow \pi_k(W_{G,d+n})$  is the zero map. On the other hand, this map factors through the sequence of maps induced by the inclusions  $W_{G,d} \hookrightarrow W_{G,d+1}$ ,

$$\pi_k(W_{G,d}) \rightarrow \pi_k(W_{G,d+1}) \rightarrow \cdots \rightarrow \pi_k(W_{G,d+n-1}) \rightarrow \pi_k(W_{G,d+n}).$$

Each map in the previous sequence is an injection for  $d \geq k + n + 1$  by Proposition 45, therefore also  $i_* : \pi_k(W_{G,d}) \rightarrow \pi_k(W_{G,d+n})$  is an injection, and  $\pi_k(W_{G,d}) = 0$ . □

Notice that thanks to this theorem we have the following corollary.

**Corollary 47** *For  $d \geq n + 1$ , for each labeled graph  $G$  on  $[n]$ , the isomorphism class  $W_{G,d}$  is connected.*

### 4.1 The infinite-dimensional case

The space  $\mathbb{R}^{\infty \times n}$  is a pre-Hilbert space (since it is not complete) with respect to the natural scalar product.<sup>8</sup> The notion of geometric graph and discriminant also makes sense in this infinite-dimensional space. More precisely, given an element  $P = (p_1, \dots, p_n) \in \mathbb{R}^{\infty \times n}$ , we build the graph  $G(P)$  whose vertices and edges are defined as in Definition 13. The discriminant  $\Delta_{\infty,n}$  consists of points  $P = (p_1, \dots, p_n) \in \mathbb{R}^{\infty \times n}$  such that there exists a pair  $1 \leq i < j \leq n$  with  $\|p_i - p_j\|^2 = 1$ . The chambers are now defined as follows: for a given graph  $G$  on  $n$  vertices, we set

$$W_{G,\infty} = \{P = (p_1, \dots, p_n) \in \mathbb{R}^{\infty \times n} \setminus \Delta_{\infty,n} \mid G(P) \simeq G\}.$$

It is easy to see that  $W_{G,\infty}$  is the direct limit of the sequence of inclusions in (1-2). In particular, from Theorem 3 we deduce the following.

<sup>8</sup>The completion of  $\mathbb{R}^{\infty \times n}$  is  $(\ell^2(\mathbb{N}))^n = \{p = (x_1, x_2, \dots) \mid \sum_{k=1}^{\infty} x_k^2 < \infty\}$ .

**Theorem 48** For every graph  $G$ , the set  $W_{G,\infty} = \varinjlim W_{G,d}$  is contractible.

**Proof** We first observe that, by Lemma 30, for  $d \geq n$  each  $W_{G,d}$  is described by a list of quadratic inequalities and therefore it is semialgebraic and it has the homotopy type of a CW-complex. Since  $W_{G,\infty} = \varinjlim W_{G,d}$ , it follows by [26, Corollary on page 253] that also  $W_{G,\infty}$  has the homotopy type of a CW-complex. By Whitehead’s theorem [16, Theorem 4.5], in order to prove that  $W_{G,\infty}$  is contractible it is enough to prove that all its homotopy groups are zero.

But this is a consequence of Theorem 3: since  $S^k$  is compact and  $W_{G,d}$  comes with the final topology, any map  $f : S^k \rightarrow W_{G,\infty}$  will factor through a map  $f : S^k \rightarrow W_{G,d}$ . We can assume  $d$  large enough so that  $\pi_k(W_{G,d}) = 0$ . It follows that the map  $f$  is contractible.  $\square$

## 5 Increasing the number of points

### 5.1 Geometric graphs on the real line

We now want to study the number of possible isotopy classes of geometric graphs on the real line when the number of points is large: this is precisely the case  $d = 1$ , and  $n$  large. If we look at the discriminant  $\Delta_{1,n}$ , this is an arrangement of hyperplanes, namely

$$\Delta_{1,n} = \{(x_1, \dots, x_n) \in \mathbb{R}^{1 \times n} \mid \text{there exist } i, j \text{ such that } |x_i - x_j| = 1\}.$$

**Remark 49** There is a way to compute explicitly the number  $b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n})$  using a generalized version of the Mayer–Vietoris spectral sequence for semialgebraic sets. For  $n = 3$  this gives 19, for  $n = 4$  it gives 183, and for  $n = 5$  it gives 2371. The computations becomes tricky for larger  $n$ ; however, these numbers are the beginning of a known integer sequence, which is the sequence of *labeled semiorders on  $[n]$* .

The remark leads us to an obvious observation. An *interval order for intervals  $\{I_i\}_{i=1}^n$  of unit length* (=semiorder for  $[n]$ ) is the partial order corresponding to their left-to-right precedence relation, ie one interval  $I_i$  is considered less than another  $I_j$  if and only if  $I_i$  is completely to the left of  $I_j$ . In the case  $d = 1$ , the number of components of the complement of  $\Delta_{1,n}$  is exactly the number of possible semiorders for  $[n]$ . This is because, once we defined the intervals  $[p_i - 1, p_i]$  for all  $i$ , each component of  $\mathbb{R}^{1 \times n} \setminus \Delta_{1,n}$  is uniquely determined by whether  $p_i < p_j - 1$  or  $p_i \not< p_j - 1$ ; see [30, page 73].

**Remark 50** The type of semiorders introduced are usually addressed as semiorders on  $n$  labeled items. The number of distinct semiorders on  $n$  unlabeled items is given by the Catalan numbers  $\{C_n\}_n$ .

Let us define  $f(n) :=$  number of labeled semiorders of  $[n]$ . There is an explicit generating function for this sequence; see [30, page 78, Corollary 5.12]. We have

$$G(x) := \sum_{n \geq 0} f(n) \frac{x^n}{n!} = C(1 - e^{-x}),$$

where  $C$  is the generating function of the known sequence of Catalan numbers  $\{C_n\}$ . More explicitly,

$$\sum_{n \geq 0} C_n x^n = C(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$

**Theorem 51** *The number of rigid isotopy classes of  $\mathbb{R}$ -geometric graphs on  $n$  vertices is equal to*

$$b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n}) = \frac{1}{n} \cdot \sqrt{6 \log \frac{4}{3}} \cdot \left( \frac{n}{e \log \frac{4}{3}} \right)^n (1 + O(n^{-1/2})).$$

**Proof** First of all let us notice that [Theorem 27](#) can be applied to  $G(x)$  since we have only one singularity in  $\log \frac{4}{3}$  and we can extend the function to a  $\log \frac{4}{3} D$ -domain, and actually to the whole of  $\mathbb{C} \setminus [\log \frac{4}{3}, +\infty)$ . The function  $C(x)$  has a unique singularity at  $x = \frac{1}{4}$ . It is easy to see that

$$C(x) = 2 - 2\sqrt{1 - 4x} + O(1 - 4x).$$

By composition we get

$$G(x) = 2 - 2\sqrt{4e^{-x} - 3} + O(4e^{-x} - 3),$$

and, from this,

$$G(x) = 2 - 2\sqrt{3 \log \frac{4}{3}} \cdot \sqrt{1 - \frac{x}{\log \frac{4}{3}}} + O\left(1 - \frac{x}{\log \frac{4}{3}}\right) = F\left(\frac{x}{\log \frac{4}{3}}\right) + O\left(1 - \frac{x}{\log \frac{4}{3}}\right).$$

We can now apply [Theorem 27](#). We get

$$\frac{f(n)}{n!} = (\log \frac{4}{3})^{-n} \cdot \sigma_n + O\left((\log \frac{4}{3})^{-n} \frac{1}{n^2}\right),$$

where  $F(x) = \sum_{n=0}^{\infty} \sigma_n z^n$ . Using [Remark 28](#) and  $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$  we get

$$\frac{f(n)}{n!} = (\log \frac{4}{3})^{-n} \cdot \frac{1}{\sqrt{\pi n^3}} \cdot \sqrt{3 \log \frac{4}{3}} + O\left((\log \frac{4}{3})^{-n} \frac{1}{n^2}\right),$$

and, by Stirling's approximation,

$$f(n) = \left(\frac{n}{e \log \frac{4}{3}}\right)^n \cdot \frac{1}{n} \cdot \sqrt{6 \log \frac{4}{3}} + O\left(\left(\frac{n}{e \log \frac{4}{3}}\right)^n n^{-3/2}\right). \quad \square$$

With these computations we know asymptotically the number of isotopy classes of geometric graphs on the real line. However, as we discussed before, different isotopy classes can correspond to the same isomorphism class. It is therefore natural to ask for the number  $\#_{1,n}$  of isomorphism classes of  $\mathbb{R}$ -geometric graphs, for  $n$  large. In [\[15\]](#), Hanlon computes the exponential generating function for this sequence (and calls the corresponding graphs *labeled unit-interval graphs*).

The exponential generating function for  $\{\#_{1,n}\}_n$  is

$$\Lambda(x) = \exp(\Gamma(x)) - 1,$$

where  $\Gamma(x)$  is the generating function for the sequence  $\{b_n\}_n$  of isomorphism classes of *connected*  $\mathbb{R}$ -geometric graphs on  $n$  vertices. More explicitly, we have

$$\Gamma(x) = \frac{1}{4}(1 - 2z) - \frac{1}{4} \sqrt{\frac{1 - 3z}{1 + z}}, \quad \text{where } z = e^x - 1.$$

Reasoning as before, we prove the following theorem.

**Theorem 52** *The number of isomorphism classes of  $\mathbb{R}$ -geometric graphs on  $n$  vertices is equal to*

$$\#_{1,n} = \frac{e^{1/12}}{8} \cdot \frac{1}{n} \cdot \sqrt{6 \log \frac{4}{3}} \cdot \left( \frac{n}{e \log \frac{4}{3}} \right)^n (1 + O(n^{-1/2})).$$

**Proof** First of all let us notice that [Theorem 27](#) can be applied to  $\Lambda(x)$  since we have only one singularity at  $\log \frac{4}{3}$  and we can extend the function to a  $\log \frac{4}{3}D$ -domain, actually to  $\mathbb{C} \setminus [\log \frac{4}{3}, +\infty)$ . We start with

$$1 + \Lambda(x) = \exp\left(\frac{1}{4} \cdot (3 - 2e^x)\right) \cdot \exp\left(-\frac{1}{4} \sqrt{4e^{-x} - 3}\right).$$

Then

$$1 + \Lambda(x) = \left( e^{1/12} + O\left(1 - \frac{x}{\log \frac{4}{3}}\right) \right) \cdot \left( -\frac{1}{4} \sqrt{4e^{-x} - 3} + 1 + O(4e^{-x} - 3) \right),$$

$$1 + \Lambda(x) = e^{1/12} - \frac{1}{4} e^{1/12} \sqrt{3 \log \frac{4}{3}} \sqrt{1 - \frac{x}{\log \frac{4}{3}}} + O\left(1 - \frac{x}{\log \frac{4}{3}}\right).$$

Finally,

$$\frac{\#_{1,n}}{n!} = \frac{1}{8} e^{1/12} \cdot \left(\log \frac{4}{3}\right)^{-n} \cdot \frac{1}{\sqrt{\pi n^3}} \cdot \sqrt{3 \log \frac{4}{3}} + O\left(\left(\log \frac{4}{3}\right)^{-n} \frac{1}{n^2}\right),$$

$$\#_{1,n} = \frac{1}{8} e^{1/12} \cdot \left(\frac{n}{e \log \frac{4}{3}}\right)^n \cdot \frac{1}{n} \cdot \sqrt{6 \log \frac{4}{3}} + O\left(\left(\frac{n}{e \log \frac{4}{3}}\right)^n n^{-3/2}\right). \quad \square$$

Even though in the general case we still do not have a clear understanding of the relation between  $b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n})$  and  $\#_{1,n}$ , in the case  $d = 1$  we have the following corollary.

**Corollary 53** *We have*

$$b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n}) = \frac{8}{e^{1/12}} \cdot \#_{1,n} (1 + O(n^{-1/2})), \quad \text{where } \frac{8}{e^{1/12}} = 7.3603 \dots$$

The number  $8 / \sqrt[12]{e}$  can be roughly interpreted as the average number of rigid isotopy classes realizing a particular  $\mathbb{R}$ -geometric graph isomorphism type.

## 5.2 Asymptotic enumeration in higher dimensions

While the situation for isotopy classes of geometric graphs on the real line is given by the number of semiorders on  $[n]$ , such a closed-form description apparently does not exist for larger values of  $d$ . Nonetheless we are able to obtain reasonable bounds on the asymptotics following methods of McDiarmid and Müller [\[24\]](#), who study asymptotic enumeration of labeled disk graphs in  $\mathbb{R}^2$ . A disk graph in  $\mathbb{R}^2$  is a graph given by an arrangement of open disks in  $\mathbb{R}^2$  where the vertices are the disks and there is an edge between a pair of them if and only if the corresponding disks intersect one another. In the case that all the disks have the same radius, this is exactly the setting of our geometric graphs in the case  $d = 2$ . McDiarmid and Müller show that the number of labeled graphs on  $n$  vertices which are unit disk



graphs in  $\mathbb{R}^2$  — in our notation,  $\#_{2,n}$  — is order  $\exp(2n \log(n) + \Theta(n))$ , and adapting their method we prove the following theorem. In particular we prove that the right asymptotic rate of growth both for  $\#_{d,n}$  and for  $b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n})$  is  $\exp(dn \log(n) + \Theta(n))$ . We will prove the following theorem, which is a consequence of Theorems 56 and 58.

**Theorem 54** *For  $d \geq 2$  fixed and  $n \geq 4d + 1$ , one has the bounds*

$$\left(\frac{1}{(d+1)e^2}\right)^{dn} n^{dn} \leq \#_{d,n} \leq b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}) \leq 2dn \left(\frac{3e}{2d}\right)^{dn} n^{dn}.$$

In the  $d = 1$  case we saw that, on average, the number of rigid isotopy classes for an  $\mathbb{R}$ -geometric graph is a specific constant. Moreover, it is easy to see that there are  $n!$  rigid isotopy classes for the empty graph on  $n$  vertices in  $\mathbb{R}$ . In higher dimensions we leave the generalization of these as open questions: On average how many rigid isotopy classes correspond to a particular  $\mathbb{R}^d$ -geometric graph? Does this average depend on  $n$ ? What is the maximum for the number of rigid isotopy classes corresponding to an  $\mathbb{R}^d$ -geometric graph?

### 5.3 General case: the upper bound

While McDiarmid and Müller are primarily interested in enumerating labeled geometric graphs in the plane, in our notation the number  $\#_{2,n}$ , their upper bound holds for general  $d$ , as they point out in [24]. The key lemma in their proof of their upper bound is the following result of Warren [32].

**Theorem 55** [32] *If  $P_1, \dots, P_m$  are polynomials of degree at most  $t$  in real variables  $z_1, \dots, z_k$ , then the number of distinct sign patterns*

$$(\text{sign}(P_1(\bar{z})), \dots, \text{sign}(P_m(\bar{z}))) \in \{-1, 1\}^m$$

*that occur in  $\mathbb{R}^k \setminus \bigcup_{i=1}^m \{\bar{z} \mid P_i(\bar{z}) = 0\}$  is at most*

$$\left(\frac{4etm}{k}\right)^k.$$

Given  $n, d$  we take the  $\binom{n}{2}$  polynomials in variables  $(x_1, \dots, x_n) \in \mathbb{R}^{dn}$  given by  $q_{i,j}(x) = \|x_i - x_j\|^2 - 1$ , defined in (3-1). Then each sign pattern of these  $\binom{n}{2}$  degree 2 polynomials in  $dn$  variables corresponds to a unique isomorphism class of labeled geometric graphs on  $n$  vertices in  $\mathbb{R}^d$ . Therefore we have the bound

$$\#_{d,n} \leq \left(\frac{4e}{d}\right)^{nd} n^{nd}.$$

However, a single sign pattern could be a disjoint union of several rigid isotopy classes, so we need a different argument to bound  $b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n})$ . We prove the following theorem.

**Theorem 56** (upper bound) *For fixed  $d$  and for  $n \geq 4d + 1$ , we have the bound*

$$b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}) \leq 2dn \left(\frac{3e}{2d}\right)^{dn} n^{dn}.$$

Before the proof, we introduce some notation. One defines the unit-distance graph on  $\mathbb{R}^d$  to be the graph whose vertex set is all of  $\mathbb{R}^d$ , and two vertices  $x$  and  $y$  are connected by an edge if and only if  $\|x - y\| = 1$ . We remark that the unit-distance graph on  $\mathbb{R}^d$ , especially in the case  $d = 2$ , is studied in the case of the well-known Hadwiger–Nelson problem of establishing the chromatic number of the plane, that is, the chromatic number of the unit-distance graph on  $\mathbb{R}^2$ .

Given two graphs  $G$  and  $H$ , a *graph homomorphism* from  $G$  to  $H$  is a map  $\phi: V(G) \rightarrow V(H)$  such that if  $(u, v)$  is an edge of  $G$  then  $(\phi(u), \phi(v))$  is an edge of  $H$ . Thus each quadric  $\Delta_{d,n}^{i,j}$  is the space of images of homomorphisms from the graph on  $[n]$  with an edge between vertex  $i$  and vertex  $j$  into the unit-distance graph on  $\mathbb{R}^d$ . More generally, we denote an intersection of quadrics by  $\Delta_{d,n}^G$  for a graph  $G = ([n], E)$  by

$$(5-1) \quad \Delta_{d,n}^G := \bigcap_{(i,j) \in E} \Delta_{d,n}^{(i,j)}.$$

Then  $\Delta_{d,n}^G$  is the space of images of homomorphisms from the graph  $G$  to the unit-distance graph on  $\mathbb{R}^d$ . Putting all of this together, we have that  $\Delta_{d,n}$  itself is the set of all points in  $\mathbb{R}^{d \times n}$  that are the image of a graph homomorphism for a nonempty graph on  $n$  vertices, ie  $\Delta_{d,n}$  is the union of the  $\Delta_{d,n}^G$  across nonempty graphs  $G$  on  $n$  vertices.

**Remark 57** Let us denote with  $\widehat{\Delta}_{d,n}^G$  the one-point compactification of the set  $\Delta_{d,n}^G$  defined in (5-1), where  $G$  is any graph on  $[n]$ . This is an algebraic set  $X$  of  $\mathbb{R}^{nd+1} = (x_1, \dots, x_n, z)$ , defined by  $k + 1$  equations, which are

$$\|x_i - x_j\|^2 = (1 - z)^2$$

for  $(i, j)$  an edge of  $G$ , and the equation of the sphere  $\|x_1\|^2 + \dots + \|x_n\|^2 + z^2 = 1$ . In fact, if we look at the explicit expression of the stereographic projection we get an homeomorphism between  $\Delta_{d,n}^G$  and  $X \setminus \{(0, 1)\}$ , and from this the claim.

**Proof** By Alexander duality (Section 2.2)  $b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n}) = b_{dn-1}(\widehat{\Delta}_{d,n}) + 1$ , where  $\widehat{\Delta}_{d,n}$  is the one-point compactification of the discriminant. Therefore, it is sufficient to bound  $b_{dn-1}(\widehat{\Delta}_{d,n})$ . Let us consider the Mayer–Vietoris spectral sequence for simplicial complexes; see [7, Section 3.2] for a complete construction. Thanks to the previous remark,  $\widehat{\Delta}_{d,n}$  is an algebraic set and we can use the mentioned spectral sequence with respect to the algebraic covering  $\{\widehat{\Delta}_{d,n}^G\}_G$ , where  $G$  varies over nonempty labeled graphs on  $[n]$ . The  $E_1$  page of the spectral sequence has

$$(5-2) \quad E_1^{i,j} = \bigoplus_{\substack{G \text{ a graph on } [n] \\ \text{with exactly } i + 1 \text{ edges}}} H^j(\widehat{\Delta}_{d,n}^G),$$

and we have the bound

$$(5-3) \quad b_{dn-1}(\widehat{\Delta}_{d,n}) \leq \sum_{i=0}^{dn-1} \dim_{\mathbb{Z}_2}(E_1^{i,(dn-1-i)}).$$

Using [27, Theorem 2] and the fact that for any labeled graph  $G$  on  $[n]$  the topological space  $\widehat{\Delta}_{d,n}^G$  is an algebraic set defined by equations of degree 2 in  $\mathbb{R}^{dn}$ , we get that its total Betti number is at most  $2(3)^{dn-1}$ . Using this, we have

$$b_{dn-1}(\widehat{\Delta}_{d,n}) \leq \sum_{k=1}^{dn} \binom{n}{k} 2(3)^{dn} \leq 2(3)^{dn} \sum_{k=1}^{dn} \binom{n}{k} \leq 2(3)^{dn} dn \binom{n}{dn} \leq 2(3)^{dn} dn \left(\frac{n^2 e}{2dn}\right)^{dn},$$

where in the third inequality we used  $n \geq 4d + 1$ . □

### 5.4 General case: the lower bound

For the lower bound on the number of labeled disk graphs, McDiarmid and Müller give a procedure for inductively generating many distinct labeled disk graphs. Here we generalize this procedure to higher dimensions.

For each  $k \geq d + 1$  we construct a family  $U_{k,d}$  of nonisomorphic labeled geometric graphs on  $k$  vertices in  $\mathbb{R}^d$ . If we let  $u_{k,d}$  denote the number of graphs in  $U_{k,d}$ , we show that for  $k \geq d + 1$ ,

$$u_{k+1,d} \geq \left(\left\lfloor \frac{k}{d+1} \right\rfloor\right)^d u_{k,d}.$$

This recursion implies the following result, which we prove in Section 5.4.1.

**Theorem 58** (lower bound) *We have for  $n > d + 1$  that*

$$\left(\frac{n}{(d+1)e^2}\right)^{dn} \leq \#_{d,n}.$$

For the base of the recursion, we start with the regular  $d$ -simplex in  $\mathbb{R}^d$  with edges of length 1 and vertices given by  $P_1, P_2, \dots, P_{d+1}$ . The 1-skeleton of the  $d$ -simplex is a geometric graph in  $\mathbb{R}^d$ ; this will be the singleton element of  $U_{d+1,d}$ . Though this graph is degenerate, it will still contribute to  $\#_{d,n}$ , which is always a lower bound for  $b_0(\mathbb{R}^{d \times n} \setminus \Delta_{d,n})$ , by the discussion following Lemma 16.

To construct the families  $U_{k,d}$  for  $d + 1 < k \leq n$ , we need the following technical lemma, which generalizes [24, Lemma 4.1].

**Lemma 59** *There exist constants  $\epsilon_0 > 0$  and  $C > 0$  such that for all  $0 < \epsilon < \epsilon_0$  and all  $p_i \in B(P_i, \epsilon)$  for all  $i \in [d]$ , there exists a unique point*

$$q(p_1, \dots, p_d) \in B(P_{d+1}, C\epsilon)$$

with  $\|q - p_i\| = 1$  for all  $i \in \{1, \dots, d\}$ .

In other words, for  $\epsilon$  small enough, this lemma tells us that there is a well-defined Lipschitz-continuous function  $q$  on  $B(P_1, \epsilon) \times B(P_2, \epsilon) \times \dots \times B(P_d, \epsilon)$ , with Lipschitz constant  $C$ , mapping  $(x_1, \dots, x_d)$  to the unique point of the intersection of sphere  $S(x_1, 1) \cap S(x_2, 1) \cap \dots \cap S(x_d, 1)$  closest to  $P_{d+1}$ . The

$d = 2$  case is [24, Lemma 4.1], and is essentially proved directly via a closed form for  $q$  in terms of  $p_1, p_2 \in B(P_1, \epsilon) \times B(P_2, \epsilon)$ , which is well defined and Lipschitz continuous for  $\epsilon$  small enough. For larger values of  $d$ , writing down the closed form of  $q$  would be much more complicated. Therefore, we instead describe algorithmically how one would compute  $q$  given  $p_1, \dots, p_d$  sufficiently close to  $P_1, P_2, \dots, P_d$ , respectively, and show that  $q$  will ultimately be a combination of Lipschitz-continuous functions.

**Proof of Lemma 59** We show that for  $\epsilon$  small enough, the intersection  $S(p_1, 1) \cap \dots \cap S(p_d, 1)$  with  $p_i \in B(P_i, \epsilon)$  for all  $i$  is two points  $q^+(p_1, \dots, p_d)$  and  $q^-(p_1, \dots, p_d)$ , with  $q^+(p_1, \dots, p_d)$  the closer of the two to  $P_{d+1}$ , and that  $q^+ : B(P_1, \epsilon) \times \dots \times B(P_d, \epsilon) \rightarrow \mathbb{R}^d$  is Lipschitz continuous.

We will prove that  $q^+$  is well defined and Lipschitz-continuous close to  $P_1, P_2, \dots, P_d$  by describing the algorithm one would use to compute  $q^+$  and show that each step of the algorithm is given by composition or addition of Lipschitz-continuous functions. Given a tuple of points  $(p_1, p_2, \dots, p_d) \in B(P_1, \epsilon) \times B(P_2, \epsilon) \times \dots \times B(P_d, \epsilon)$ , with  $\epsilon$  sufficiently small, one could compute  $q^+$  via the following recursive procedure.

First find the  $(d-2)$ -dimensional sphere given by the intersection of  $S(p_1, 1)$  and  $S(p_2, 1)$ . Now for any  $0 \leq k \leq d-1$ , a  $k$ -dimensional sphere in  $\mathbb{R}^d$  may be described completely by its center, its radius, and the affine subspace of dimension  $k+1$  in which it is contained. In other words a  $k$ -dimensional sphere in  $\mathbb{R}^d$  is described by a point in  $\mathbb{R}^d$ , a positive real number, and an element of the Grassmannian  $\text{Gr}(k+1, d)$ . Given  $p_1$  and  $p_2$  in  $\mathbb{R}^d$  with the distance from  $p_1$  to  $p_2$  smaller than 2, the intersection  $S(p_1, 1) \cap S(p_2, 1)$  is a  $(d-2)$ -dimensional sphere. It follows that taking  $\epsilon$  small enough so that  $\|p_1 - p_2\|^2 < 4$  for any  $p_1, p_2 \in B(P_1, \epsilon) \times B(P_2, \epsilon)$ , we have a continuous function  $(C, R, G) : B(P_1, \epsilon) \times B(P_2, \epsilon) \rightarrow \mathbb{R}^d \times \mathbb{R}^+ \times \text{Gr}(d-1, d)$ . This map sends  $(p_1, p_2)$  to the  $(d-2)$ -dimensional sphere  $S(p_1, 1) \cap S(p_2, 1)$  with center  $C(p_1, p_2)$  and radius  $R(p_1, p_2)$  living in the affine hyperplane  $C(p_1, p_2) + G(p_1, p_2)$ .

Now given  $(p_1, \dots, p_d) \in B(P_1, \epsilon) \times \dots \times B(P_d, \epsilon)$  with  $\epsilon$  small enough, we have that the intersection of  $C(p_1, p_2) + G(p_1, p_2)$  with  $S(p_1, 1) \cup S(p_2, 1) \cup \dots \cup S(p_d, 1) \subseteq \mathbb{R}^d$  gives an arrangement of  $d-1$   $(d-2)$ -dimensional spheres in the affine hyperplane  $C(p_1, p_2) + G(p_1, p_2)$ . The center and radii of these spheres will be determined by how  $C(p_1, p_2) + G(p_1, p_2)$  intersects each  $S(p_i, 1)$ . By induction we find the two points of intersection of these  $(d-2)$ -dimensional spheres in the  $(d-1)$ -dimensional Euclidean space given by the affine hyperplane  $C(p_1, p_2) + G(p_1, p_2)$ . Once these two points of intersection have been found, we pick the one that is closest to  $P_{d+1}$  to be  $q^+(p_1, \dots, p_d)$ .

It can be verified routinely that  $(C, R, G)$ , as defined above, is Lipschitz continuous in each coordinate. Moreover, the intersection of  $C(p_1, p_2) + G(p_1, p_2)$  with  $S(p_1, 1) \cup S(p_2, 1) \cup \dots \cup S(p_d, 1) \subseteq \mathbb{R}^d$ , when  $p_i$  is sufficiently close to  $P_i$  for all  $i$ , can be described by a  $2(d-1)$ -tuple of points  $(c_2, r_2, \dots, c_d, r_d)$ , where each  $c_i$  belongs to  $C(p_1, p_2) + G(p_1, p_2)$  and  $r_i \in (0, 1]$ . Here  $c_i$  and  $r_i$  are, respectively, the

center and the radius of the  $(d-2)$ -dimensional sphere given by  $S(p_i, 1) \cap (C(p_1, p_2) + G(p_1, p_2))$  for  $i \geq 3$ , with  $c_2$  and  $r_2$ , respectively, the center and radius of the  $(d-2)$ -dimensional sphere given by the intersection  $S(p_1, 1) \cap S(p_2, 1)$ , ie  $c_2$  and  $r_2$  are  $C(p_1, p_2)$  and  $R(p_1, p_2)$ .

By continuity,  $C(p_1, p_2)$  can be made arbitrarily close to  $C(P_1, P_2)$ ,  $G(p_1, p_2)$  can be made arbitrarily close to  $G(P_1, P_2)$ , and  $R(p_1, p_2)$  can be made arbitrarily close to  $R(P_1, P_2)$ . From here one can verify that for  $\epsilon$  small enough, there is a Lipschitz-continuous function

$$\phi: B_{\text{Gr}(d-1,d)}(G(P_1, P_2), \epsilon) \times B_{\mathbb{R}^d}(C(P_1, P_2), \epsilon) \times B_{\mathbb{R}}(R(P_1, P_2), \epsilon) \times B_{\mathbb{R}^d}(P_3, \epsilon) \times \cdots \times B_{\mathbb{R}^d}(P_d, \epsilon) \rightarrow (\mathbb{R}^{d-1} \times \mathbb{R})^{d-1}$$

mapping an element of the domain to the arrangement of  $(d-1)$  many  $(d-2)$ -dimensional spheres in the affine hyperplane as described above. By induction we have that the  $S^0$  at the intersection of the arrangement is Lipschitz continuous on the image of  $\phi$ , and finally picking the closest of the two points to the fixed point  $P_{d+1}$  is Lipschitz continuous too. Note that the base case for the induction can simply be the  $d = 1$  case; given two points in  $\mathbb{R}$ , picking the one closest to a fixed point is always Lipschitz continuous when the center of the two points lives in some small enough interval around a second fixed point. □

Let us take the sequence  $0 < \epsilon_1 < \cdots < \epsilon_n$  defined by  $\epsilon_i = \epsilon_0 / C^{n-i}$ , where  $\epsilon_0$  and  $C$  as in the previous lemma and we are assuming  $C > 1$ . To construct elements of  $U_{k,d}$  recursively from  $U_{k-1,d}$ , we will also use as an inductive hypothesis that all the elements  $P = (p_1, \dots, p_k) \in U_{k,d}$  satisfy the properties

- (P1)  $\|p_i - p_j\| < \epsilon_i$  with  $i \equiv j \pmod{d+1}$ ,
- (P2)  $S(p_{i_1}, 1) \cap \cdots \cap S(p_{i_d}, 1) = \emptyset$  for all distinct  $\{i_1, \dots, i_{d+1}\}$ .

These two conditions hold for  $U_{d+1,d}$ , whose vertices are  $P_1, \dots, P_{d+1}$ .

Condition (P1) above naturally partitions the vertices of  $G \in U_{k,d}$  into  $d+1$  distinct classes given by the clustering of the vertices of  $G$  around the points  $P_1, \dots, P_d, P_{d+1}$ . The proof of the claimed recursive lower bound on  $u_{k,d}$  will be that if  $G \in U_{k-1,d}$  and, without loss of generality,  $k \equiv 0 \pmod{d+1}$  then picking a transversal  $\sigma = \{p_{i_1}, p_{i_2}, \dots, p_{i_d}\}$  of vertices of  $G$  where  $i_l \equiv l \pmod{d+1}$  for every  $i$ , we give a procedure to choose a position for a new vertex  $p_k$  to be added to  $G$  so that the neighborhood of  $p_k$  is unique for each choice of  $\sigma$  and so that (P1) and (P2) are satisfied still satisfied after adding  $p_k$ . As  $p_k$  will depend on  $\sigma$  and each choice of  $\sigma$  gives a distinct neighborhood for  $p_k$ , we have that there are at least as many combinatorially distinct ways to extend  $G$  as there are choices for  $\sigma$ .

If we denote by  $\mathcal{P}(G)$  for  $G \in U_{k-1,d}$  the set of all such transversals, that is, the number of  $d$ -tuples  $(i_1, \dots, i_d)$  with  $1 \leq i_j < k$  such that  $i_j \equiv j \pmod{d+1}$ , we have

$$|\mathcal{P}| \geq \lfloor (k-1)/(d+1) \rfloor^d.$$

Now let  $\mathcal{M}_\pi$  be defined as the intersection of the open balls

$$\mathcal{M}_\pi := B(p_{i_1}, 1) \cap \cdots \cap B(p_{i_d}, 1).$$

We have the following lemma, which in the 2–dimensional case is [24, Claim 4.3]. The proof, which we omit, is exactly analogous to the 2–dimensional case and relies on the fact that for each  $\pi = (i_1, \dots, i_d) \in \mathcal{P}$ ,  $S(p_{i_1}, 1) \cap \cdots \cap S(p_{i_d}, 1) \cap B(P_{d+1}, \epsilon_k)$  is a single point due to Lemma 59, and that single point is unique for each choice of  $\pi$  by condition (P2).

**Lemma 60** *There exist nonempty open sets  $O_\pi \subset \mathcal{M}_\pi$  such that for all  $\pi \neq \sigma \in \mathcal{P}$ , we have either  $O_\pi \cap \mathcal{M}_\sigma = \emptyset$ , or  $O_\sigma \cap \mathcal{M}_\pi = \emptyset$ .*

Now, for each  $\pi \in \mathcal{P}$ , let us pick an arbitrary

$$q_\pi = O_\pi \setminus \bigcup_{i=1}^{k-1} S(p_i, 1),$$

and we have that the  $\mathbb{R}^d$ –geometric graph obtained by adding the vertex  $q_\pi$  to  $G$ , which we will denote by  $(G, q_\pi)$ , satisfies conditions (P1) and (P2). Moreover, for every choice of  $\pi$  we obtain a unique way to extend  $G$ , by the following lemma.

**Lemma 61** *If  $\pi \neq \sigma \in \mathcal{P}(G)$  for  $G \in U_{k-1,d}$ , then the geometric graphs  $(G, q_\pi)$  and  $(G, q_\sigma)$  are not isomorphic.*

This holds because  $q_\pi$  and  $q_\sigma$  will have different sets of neighbors, generalizing [24, Claim 4.4].

**Lemma 62** *If  $\pi \neq \sigma \in \mathcal{P}(G)$  for  $G \in U_{k-1,d}$ , then  $N(q_\pi) \neq N(q_\sigma)$ , where  $N(v)$  denotes the neighbors of a point  $v$  in  $\mathbb{R}^d$  in the geometric graph  $(G, v)$ , that is, the vertices of  $G$  at distance less than 1 from  $v$ .*

**Proof** For  $\pi \neq \sigma$  we have that  $\sigma \subseteq N(q_\pi)$  if and only if  $q_\pi \in \mathcal{M}_\sigma$ . Clearly  $\sigma \subseteq N(q_\sigma)$  and  $\pi \subseteq N(q_\pi)$ , but by Lemma 60 it cannot be the case that both  $\sigma \subseteq N(q_\pi)$  and  $\pi \subseteq N(q_\sigma)$ .  $\square$

Now, for each  $G \in U_{k-1,d}$  and  $\pi \in \mathcal{P}(G)$  we construct a geometric graph  $(P, q_\pi)$  which satisfies conditions (P1) and (P2) and which satisfies Lemma 61. Then

$$u_{k,d} \geq |\mathcal{P}| \cdot u_{k-1,d} \geq \left\lfloor \frac{k-1}{d+1} \right\rfloor^d \cdot u_{k-1,d}.$$

**5.4.1 Proof of Theorem 58** We have

$$\#_{n,d} \geq u_{n,d} \geq \left( \prod_{i=d+1}^{n-1} \left\lfloor \frac{i}{d+1} \right\rfloor \right)^d \geq \left( \prod_{i=d+1}^{n-1} \frac{i-d}{d+1} \right)^d \geq \left( \frac{(n-d-1)!}{(d+1)^{n-d-1}} \right)^d.$$

Using the estimate

$$k! \geq \left(\frac{k}{e}\right)^k$$

we get

$$\#_{n,d} \geq \left( \frac{n-d-1}{(d+1)e} \right)^{d(n-d-1)} \geq \left( \frac{n}{(d+1)e} \right)^{dn} \left( \frac{n}{d+1} \right)^{-d(d+1)},$$

where for the last inequality we used

$$\left( 1 - \frac{d+1}{n} \right)^{d(n-d-1)} \geq e^{-d(d+1)},$$

which derives from

$$\left( 1 + \frac{d+1}{n-d-1} \right)^{d(n-d-1)} \leq e^{d(d+1)}.$$

We then use  $(n/(d+1))^{d(d+1)} \leq (e^d)^n$  to obtain the lower bound.

### 5.5 The top Betti numbers

The goal of this section will be to finally prove [Theorem 9](#) regarding the low-degree Betti numbers of the one-point compactification of the discriminant. We will prove it using the generalized nerve lemma of Björner.

**Theorem 63** (special case of [[8](#), Theorem 6]) *Let  $X$  be a regular CW-complex and  $(X_i)_{i \in I}$  a family of subcomplexes such that  $X = \bigcup_{i \in I} X_i$ . Suppose that every nonempty finite intersection  $X_{i_1} \cap \dots \cap X_{i_t}$  is  $(k-t+1)$ -connected. Then  $X$  is  $k$ -connected if and only if the nerve  $\mathcal{N}(X_i)$  is  $k$ -connected.*

Recall that given a CW-complex  $X$  and a covering by subcomplexes  $(X_i)_{i \in I}$  the nerve of the covering  $\mathcal{N}(X_i)$  is the simplicial complex on the vertex set  $I$ , where  $\sigma = [i_1, \dots, i_t]$  is a face of the nerve if and only if  $X_{i_1} \cap \dots \cap X_{i_t}$  is nonempty.

The covering that we will use for  $\widehat{\Delta}_{d,n}^{i,j}$  will be given by  $(\widehat{\Delta}_{d,n}^{i,j})_{1 \leq i < j \leq n}$ , where  $\widehat{\Delta}_{d,n}^{i,j}$  denotes the compactification in  $\mathbb{R}^{d \times n}$  of the space  $\Delta_{d,n}^{i,j} = \{(x_1, \dots, x_n) \in \mathbb{R}^{d \times n} \mid \|x_i - x_j\|^2 = 1\}$ .

Now each intersection of a set of  $\Delta_{d,n}^{i,j}$  spaces is naturally associated to a graph  $G$  and a space  $\Delta_{d,n}^G$  described in [\(5-1\)](#).

To be able to apply the generalized nerve lemma, a key step will be to establish the following about the topology of  $\widehat{\Delta}_{d,n}^G$ .

**Lemma 64** *For any graph  $G$  with  $\beta_0(G)$  connected components,  $\Delta_{d,n}^G$  is a direct product of a compact set  $K := K(G)$  of dimension at most  $(d-1)(n-\beta_0)$  and  $\mathbb{R}^{d \times \beta_0}$ . Therefore  $\widehat{\Delta}_{d,n}^G$  is  $(d\beta_0-1)$ -connected.*

This proof will use the connection to homomorphisms to unit-distance graphs defined in the discussion around [\(5-1\)](#).

**Proof of Lemma 64** Let  $H$  be a component of  $G$  and let  $T$  be a spanning tree of  $H$ . Clearly  $\Delta_{d,|V(H)|}^H \subseteq \Delta_{d,|V(H)|}^T$  as any homomorphism from  $H$  to the unit-distance graph on  $\mathbb{R}^d$  induces a homomorphism from  $T$  to the unit-distance graph on  $\mathbb{R}^d$ . Moreover, we have that  $\Delta_{d,|V(H)|}^T \sim \mathbb{R}^d \times (S^{d-1})^{|E(T)|}$ .

Indeed, we may regard  $T$  as being a rooted tree and we can map the root of  $T$  to any point of  $\mathbb{R}^d$  and from there every vertex may live anywhere on the sphere of radius 1 centered at the image of its parent vertex. Now taking  $\Delta_{d,|V(H)|}^H$  and modding out by the  $\mathbb{R}^d$  factor coming from the choice of image for the root in  $T$ , we have a closed subset of the compact set  $(S^{d-1})^{|E(T)|} = (S^{d-1})^{|V(H)|-1}$ . Thus, without fixing the image of the root, we have that  $\Delta_{d,|V(H)|}^H$  is the direct product of  $\mathbb{R}^d$  and the compact space  $K(H)$  given by the closed subset of  $(S^{d-1})^{|V(H)|-1}$ . It is clear that we may describe any homomorphism from  $G$  to the unit-distance graph on  $\mathbb{R}^d$  as a product of graph homomorphisms on the connected components. We have that

$$\Delta_{d,n}^G \cong K(G) \times \mathbb{R}^{d\beta_0},$$

where  $K(G)$  is the direct product of  $K(H)$  over all connected components  $H$ . Thus  $K(G)$  is contained in some  $(n-\beta_0)$ -fold product of  $(d-1)$ -dimensional spheres, so it is at most  $(d-1)(n-\beta_0)$ -dimensional.

We now turn our attention to the compactification of  $\Delta_{d,n}^G$ . By the description of  $\Delta_{d,n}^G$ , we have that  $\Delta_{d,n}^G$  is a  $d\beta_0$ -ranked vector bundle over a compact CW-complex (since we are working with semialgebraic sets), so its compactification is the Thom space of this vector bundle, which is  $(d\beta_0-1)$ -connected by [28, Lemma 18.1]. □

**Proof of Theorem 9** We apply Theorem 63 to prove that  $\widehat{\Delta}_{d,n}$  is  $(n+d-3)$ -connected. Consider the cover of  $\widehat{\Delta}_{d,n}$  by  $(\widehat{\Delta}_{d,n}^{i,j})_{1 \leq i < j \leq n}$  as discussed above, so that for any  $t$ , the  $t$ -fold intersection of complexes in the cover is of the form  $\widehat{\Delta}_{d,n}^G$  for some graph  $G$  on  $t$  edges. Moreover, we observe that the nerve of this covering is just the simplex on  $\binom{n}{2}$  vertices, as any intersection of the spaces in the cover at least contains the point at infinity, so in particular, the nerve is  $(n+d-3)$ -connected. Thus it suffices to check that  $\widehat{\Delta}_{d,n}^G$  is  $n+d-3-t+1 = n-t+d-2$  connected for any graph  $G$  on  $n$  vertices with  $t$  edges. By Lemma 64, it suffices to verify that, for such a graph,  $n-t+d-2 \leq d\beta_0(G)-1$ . This always holds for  $d \geq 1$  because  $\beta_0 \geq 1$ ,  $\beta_1 \geq 0$  and  $n-t = \beta_0 - \beta_1$ .

To show that  $\widehat{\Delta}_{n,d}$  is not  $(n+d-2)$ -connected, we use duality and verify that  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  has at least one  $dn-1-(n+d-2) = (n-1)(d-1)$  reduced homology class. The path components of  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  are the rigid isotopy classes of the graph on  $n$  vertices in  $\mathbb{R}^d$ . We consider the rigid isotopy classes of the empty graph on  $n$  vertices. The rigid isotopy classes of the empty graph on  $n$  vertices give all the configurations of  $n$  points in  $\mathbb{R}^d$  such that the distance between any pair of them is larger than 1. This is homotopy equivalent to  $\text{Conf}_n(\mathbb{R}^d)$  (see Examples 4 and 10), which has its top positive reduced Betti number in dimension  $(n-1)(d-1)$ . □

## 6 Examples

Here we work out a few examples for computing the Betti numbers of  $\mathbb{R}^{d \times n} \setminus \Delta_{d,n}$  for small values of  $d$  and  $n$ . We start with the case that  $d = 1$ . In the case that  $d = 1$ , computing the topology of  $\widehat{\Delta}_{1,n}^G$  across all graphs  $G$  on  $n$  vertices is sufficient to compute  $b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n})$ . While we have shown the



number of rigid isotopy classes of  $\mathbb{R}$ -geometric graphs on  $n$  vertices is given by the number of labeled semiorders on  $n$  elements, we work out a computation for  $n = 3$  here primarily to show how spectral sequences and Alexander–Pontryagin duality can be used to compute the exact number of rigid isotopy classes of  $\mathbb{R}$ -geometric graphs.

**Example 65** ( $n = 3, d = 1$ ) By Alexander–Pontryagin duality, it suffices to compute the Betti numbers of  $\widehat{\Delta}_{1,3}$ . This is a union of three compactified quadrics given by the solutions in  $\mathbb{R}$  to  $|x_i - x_j|^2 = 1$  for  $1 \leq i < j \leq 3$ . Each of these is simply the disjoint union of two hyperplanes in  $\mathbb{R}^{1 \times 3}$ . Thus  $\widehat{\Delta}_{1,3}$  is a 2-dimensional cell complex and we know by [Theorem 9](#) that this complex is 1-connected so only the second Betti number is interesting. This is not surprising, as the dual in  $S^3$  of  $\widehat{\Delta}_{1,3}$  is  $\mathbb{R}^3 \setminus \Delta_{1,3}$ , which is a disjoint union of finite intersections of halfspaces, so only its zeroth Betti number is interesting.

Now by computing the  $E_1$  page of the Mayer–Vietoris spectral sequence given by the covering of  $\widehat{\Delta}_{1,3}$  by  $(\widehat{\Delta}_{1,3}^{i,j})_{1 \leq i < j \leq n}$ , we have that the Euler characteristic of  $\widehat{\Delta}_{1,3}$  will be given by

$$\chi(\widehat{\Delta}_{1,3}) = \sum_{0 \leq i \leq 2, 0 \leq j \leq 2} (-1)^{i+j} \dim(E_1^{i,j}).$$

On the other hand,  $\chi(\widehat{\Delta}_{1,3}) = 1 + b_2(\widehat{\Delta}_{1,3})$ , so computing the first page is enough.

Now

$$E_1^{i,j} = \bigoplus_{\substack{G \text{ a graph on } \{1, 2, 3\} \\ \text{with exactly } i + 1 \text{ edges}}} H^j(\widehat{\Delta}_{1,3}^G).$$

So we can compute  $\dim(E_1^{i,j})$  for every value of  $i$  and  $j$ . If  $i = 0$  we are looking at  $\widehat{\Delta}_{1,3}^G$  for  $G$  a graph with three vertices and one edge. Suppose the edge is between vertex 1 and vertex 2. Then we may place vertex 1 anywhere on  $\mathbb{R}$  then vertex 2 at either point of the  $S^0$  centered at the location of vertex 1. Next vertex 3 may be mapped into  $\mathbb{R}$  arbitrarily. So  $\Delta_{1,3}^G$  is given by  $S^0 \times \mathbb{R} \times \mathbb{R}$ , which compactifies in  $\mathbb{R}^3$  to  $S^2 \vee S^2$ , and there are three graphs with exactly 1 edge. Next we look at graphs with two edges. If  $G$  is such a graph then it is easy to see that  $\Delta_{1,3}^G$  is  $\mathbb{R} \times S^0 \times S^0$ , which compactifies to  $S^1 \vee S^1 \vee S^1 \vee S^1$ ; there are three such choices for  $G$ . Finally, if  $G$  is the triangle, then there is no way to map  $G$  into the unit-distance graph on  $\mathbb{R}$ , so  $\Delta_{1,3}^G = \emptyset$ , which compactifies to the point at infinity. Therefore the following table stores the values of  $\dim(E_1^{i,j})$ :

2	6	0	0
1	0	12	0
0	3	3	1
$j \uparrow, i \rightarrow$	0	1	2

From this table we compute the Euler characteristic to be 19, so  $b_2(\widehat{\Delta}_{1,3}) = 18$ , from which it follows by duality that  $b_0(\mathbb{R}^3 \setminus \Delta_{1,3}) = 18 + 1 = 19$ , recovering the number of labeled semiorders on  $\{1, 2, 3\}$ .

We could take the same approach to compute  $b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n})$  for any  $n$ . For any graph  $G$ ,  $\Delta_{1,n}^G$  is always a product of a finite set  $X := X(G)$  (which is possibly empty) and  $\mathbb{R}^{\beta_0(G)}$ , so  $\widehat{\Delta}_{1,n}^G$  is a wedge of  $|X|$  spheres of dimension  $\beta_0(G)$ , or the point at infinity if  $X = \emptyset$ . Moreover,  $X(G)$  will be empty if and only if there is no way to map  $G$  into the unit-distance graph on the real line, but this means that  $X(G)$  is empty if and only if  $G$  is not bipartite.

On the other hand, if  $G$  is bipartite we can compute  $|X(G)|$  exactly, meaning that this approach could be used to compute the first page of the spectral sequence for  $d = 1$  and any value of  $n$ . However, we should not expect to be able to do so in a reasonable amount of time for large  $n$ , assuming  $P \neq NP$  as we explain below.

We have discussed  $\Delta_{d,n}^G$  as the space of graph homomorphisms from  $G$  into the unit-distance graph on  $\mathbb{R}^d$ , but it turns out that given two graphs  $G$  and  $H$ , it is in general NP-complete to decide if there are any graph homomorphisms from  $G$  to  $H$  by a result of Hell and Nešetřil [17]. Indeed, [17] shows that for any nonbipartite graph  $H$  it is NP-complete to decide whether a graph  $G$  admits any homomorphism into  $H$ . For us this means that we should not be able to even decide in general if  $\Delta_{d,n}^G$  is empty or not if  $d \geq 2$ , as the unit-distance graph on  $\mathbb{R}^d$  for  $d \geq 2$  is not bipartite. Now for  $d = 1$ ,  $\Delta_{d,n}^G$  will be computable, but a result Dyer and Greenhill [10] shows that the problem of enumerating graph homomorphism into a fixed bipartite graph  $H$  is #P-complete unless  $H$  is a special type of bipartite graph, which does not include the unit-distance graph on  $\mathbb{R}$ . Given these results, we should look for other ways to exactly count the number of isotopy classes given  $d$  and  $n$ , than computing the full Mayer–Vietoris spectral sequence. To have at least some explicit examples for larger  $d$  we work out the Betti numbers for  $n = 3, 4$  and any  $d \geq 2$ , though the methods we use are rather ad hoc and don't generalize to higher values of  $n$ .

**Example 66** ( $n = 3, d \geq 2$ ) Here we explicitly compute the topology of each rigid isotopy class on three vertices. The space  $W_{G,d}$  for  $G$  the complete graph on three vertices is contractible, in fact it is easy to see that it is convex.

For  $G$  on two edges we observe that  $W_{G,d}$  has the topology of  $S^{d-1}$ . To see this we consider  $G$  as a path on three vertices, the first vertex on the path can go anywhere, the second vertex can go anywhere in the punctured ball of radius 1 around the first vertex; it must be punctured since only vertices with identical closed neighborhoods can map to the same point. After mapping the first two vertices the final vertex can be moved freely in some contractible subspace of  $\mathbb{R}^d$  determined by the position of the first two vertices.

Now, for  $G$  on one edge we observe that  $W_{G,d}$  has the topology of the configuration space of two points in  $\mathbb{R}^d$ , which is just  $S^{d-1}$ . Finally, if  $G$  is the empty graph, then  $W_{G,d}$  is homotopy equivalent to the configuration space of three points in  $\mathbb{R}^d$ , which is known to have  $b_0 = 1, b_{d-1} = 3$  and  $b_{2(d-1)} = 2$ .

Putting this all together, we have that the empty graph contributes 1 to  $b_0$ , 3 to  $b_{d-1}$  and 2 to  $b_{2(d-1)}$ , the three graphs on one edge each contribute 1 to  $b_0$  and 1 to  $b_{d-1}$ , the three graphs on two edges also each contribute 1 to  $b_0$  and 1 to  $b_{d-1}$ , and the complete graph contributes 1 to  $b_0$ . So we have  $8 + 9x^{d-1} + 2x^{2d-2}$  as the Poincaré polynomial for  $\mathbb{R}^{d \times 3} \setminus \Delta_{d,3}$ .

**Example 67** ( $n = 4, d \geq 2$ ) The case  $n = 4$  is more interesting, but again the computation is rather ad hoc. We examine each graph  $G$  on four vertices and compute the Poincaré polynomial for  $W_{G,d}$  summarized in Table 1. These Poincaré polynomials are essentially computed by inspection; we don't give the full details for the computations. For example, the graph given by a path on three vertices and an isolated vertex has the homotopy type of  $S^{d-1} \times S^{d-1}$ . With one vertex of the path fixed, the next vertex is free to go anywhere in the punctured ball around the first vertex. Next the final vertex of the path may be placed freely inside some contractible set. So the path contributes a factor of  $S^{d-1}$ . Finally, after the path is placed in  $\mathbb{R}^d$ , the union of the balls around its vertices gives a contractible space in  $\mathbb{R}^d$  and the isolated vertex may be placed anywhere outside of this contractible space, so this contributes the other  $S^{d-1}$  factor. From Table 1 we can determine  $\beta(G)$  for any graph on four vertices, and determine that the Poincaré Polynomial of  $\mathbb{R}^{d \times 4} \setminus \Delta_{d,4}$  is

$$64 + 7x^{d-2} + 92x^{d-1} + 7x^{2d-3} + 35x^{2d-2} + 6x^{3d-3}.$$

In particular, in the case of the plane there are 71 rigid isotopy classes of graphs on four vertices, while there are 64 labeled graphs on four vertices, all of which can be realized as geometric graphs in the plane. We can also recover the number of rigid isotopy classes for four points on the real line with the observation that the 4-cycle and the star cannot be realized as geometric graphs in  $\mathbb{R}$ . Therefore in the case  $d = 1$ , the 211 coming from evaluating the Poincaré polynomial at  $x = 1$  overcounts by the contribution of 4 for each 4-cycle and for each star, so the overcount is 28, bringing the total number of chambers to  $211 - 28 = 183$ .

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