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Classical shadows of stated skein representations at roots of unity

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We extend some results of Bonahon, Wong, Bullock and Turaev concerning the skein algebras of closed surfaces to Lê's stated skein algebras associated to open surfaces. We prove that the stated skein algebra with deforming parameter $+1$ embeds canonically into the center of the stated skein algebra whose deforming parameter is an odd root unity. We also construct an isomorphism between the stated skein algebra at $+1$ and the algebra of regular functions of the relative SL_2–character variety of the surface. As a result, we associate to each isomorphism class of irreducible or local representations of the stated skein algebra an invariant which is a point in the relative character variety.

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1 Introduction

A *punctured surface* is a pair $\underline{\Sigma} = (\Sigma, \mathcal{P})$, where Σ is a compact oriented surface and $\mathcal{P}$ is a (possibly empty) finite subset of Σ which intersects nontrivially each boundary component. We write $\Sigma_{\mathcal{P}} := \Sigma \setminus \mathcal{P}$. The set $\partial\Sigma \setminus \mathcal{P}$ consists of a disjoint union of open arcs which we call *boundary arcs*.

Warning In this paper, the punctured surface $\underline{\Sigma}$ will be called open if the surface Σ has nonempty boundary and closed if Σ is closed. This convention differs from the traditional one, where some authors refer to an open punctured surface as a punctured surface $\underline{\Sigma} = (\Sigma, \mathcal{P})$ with Σ closed and $\mathcal{P} \neq \varnothing$ (in which case $\Sigma_{\mathcal{P}}$ is not closed).

We will consider two related objects associated to a punctured surface, namely the Kauffman-bracket skein algebra and the $\mathrm{SL}_2(\mathbb{C})$–character variety. These objects have been well studied in the case where the punctured surface is closed. They were recently generalized to open punctured surfaces in such a way that they have a nice behavior relative to the operation of gluing two boundary arcs together. The goal of this paper is to extend some classical results concerning skein algebras and character varieties to the case of open punctured surfaces. Before we state the main results, let us give a brief historical background.

Historical background *Closed surfaces*: Culler and Shalen [1983] defined the $\mathrm{SL}_2(\mathbb{C})$ character variety $\mathcal{X}_{\mathrm{SL}_2}(M)$ of a manifold M whose fundamental group is finitely generated. This affine variety is closely related to the moduli space of flat connections on a trivial $\mathrm{SL}_2(\mathbb{C})$ bundle over M and, therefore, it is related to Chern–Simons topological quantum field theory, gauge theory and low-dimensional topology; see [Labourie 2013; Marché 2012; 2016] for surveys. If Σ is a closed oriented surface, the smooth part of $\mathscr{X}_{\mathrm{SL}_2}(\Sigma)$ carries a symplectic form, first defined in [Atiyah and Bott 1983] in the context of gauge theory. This symplectic structure was used by Goldman [1986] to equip the algebra of regular functions $\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ with a Poisson bracket. A similar Poisson structure for character varieties of punctured closed surfaces was introduced by Fock and Rosly [1999] (see also [Alekseev et al. 2002] for an alternative construction) in the differential geometric context.

Turaev [1988] and Hoste and Przytycki [1992] independently defined the *Kauffman-bracket skein algebra* $\mathscr{S}_A(\underline{\Sigma})$ as a tool to study the Jones polynomial and the SU(2) Witten–Reshetikhin–Turaev TQFTs. Skein algebras are defined for any commutative unital ring $\mathscr{R}$ together with an invertible element $A \in \mathscr{R}^\times$ and a closed punctured surface $\underline{\Sigma}$.

Skein algebras are deformations of the algebra of regular functions of character varieties of closed punctured surfaces. In particular, this means that there is an isomorphism of Poisson algebras between $\mathscr{S}_{+1}(\underline{\Sigma})$ and $\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$. In more detail, this relies on a (noncanonical) isomorphism from $\mathscr{S}_{+1}(\underline{\Sigma})$ to $\mathscr{S}_{-1}(\underline{\Sigma})$ [Barrett 1999]. The latter algebra carries a natural Poisson bracket (see Section 2.5). An isomorphism of algebras between $\mathscr{S}_{-1}(\underline{\Sigma})$ and $\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ was defined by Bullock [1997], assuming that the skein algebra is reduced, ie that its nilradical is null. This latter fact was later proved independently in [Przytycki and Sikora 2000] and [Charles and Marché 2012]. Turaev [1991] showed that Bullock's isomorphism is Poisson.

In TQFT, skein algebras appear through their nontrivial finite-dimensional representations. Skein algebras admit such representations if and only if the parameter A is a root of unity. A recent result of Bonahon and Wong [2016] states, in particular, that when A has odd order, there exists an embedding of $\mathscr{S}_{+1}(\underline{\Sigma})$ into the center of $\mathscr{S}_A(\underline{\Sigma})$. Since each simple representation induces a character on the center of the skein algebra, using Bullock's isomorphism, one can associate to each isomorphism class of simple representation a point in the character variety. This invariant is called *the classical shadow* of the representation.

Open surfaces: Lê [2018] generalized the Kauffman-bracket skein algebras to open punctured surfaces based on the original work of Bonahon and Wong [2011]. We call it *stated skein algebra* and denote it by $\mathscr{S}_\omega(\underline{\Sigma})$. It depends on an invertible element $\omega \in \mathscr{R}^\times$. When the surface is closed, it coincides with the classical skein algebra with parameter $A = \omega^{-2}$. An important feature of the stated skein algebra is its behavior under gluing of surfaces. More precisely, let a and b be two boundary arcs of an open punctured surface $\underline{\Sigma}$, and let us denote by $\underline{\Sigma}|_{a\#b}$ the surface obtained from $\underline{\Sigma}$ by gluing a and b. Lê showed that there is an injective algebra morphism

$$i|_{a\#b} : \mathscr{S}_\omega(\underline{\Sigma}|_{a\#b}) \hookrightarrow \mathscr{S}_\omega(\underline{\Sigma}) \tag{1}$$

which is coassociative in that it does not depend on the order we glue the arcs, ie for four distinct boundary arcs a, b, c and d, one has $i|_{a\#b} \circ i|_{c\#d} = i|_{c\#d} \circ i|_{a\#b}$. In particular, for each topological triangulation Δ of $\underline{\Sigma}$, one has an injective morphism of algebras

$$i^\Delta : \mathscr{S}_\omega(\underline{\Sigma}) \hookrightarrow \bigotimes_{\mathbb{T} \in F(\Delta)} \mathscr{S}_\omega(\mathbb{T}). \tag{2}$$

Here $\mathbb{T}$ denotes the triangle, ie a disc with three punctures on its boundary. A punctured surface is *triangulable* if it can be obtained from a disjoint union of triangles by gluing some pair of boundary arcs (ie faces of triangles) together. A *topological triangulation* is the data of such a union of triangles together with the pairs of glued boundary arcs. In (2), the tensor product runs over the faces of the triangulation; see Section 2 for precise definitions.

As applications, Lê provided a simple proof that the algebra $\mathcal{S}_\omega(\mathbf{\Sigma})$ has no zero divisor (the case of closed triangulable punctured surfaces was proved earlier in [Bonahon and Wong 2011] using the quantum trace and the case of closed unpunctured surfaces was proved in [Przytycki and Sikora 2019]) and he obtained a simpler formulation of Bonahon and Wong's [2011] quantum trace map.

Motivated by Lê's construction, Korinman [2019] defined a generalization of character varieties to open punctured surfaces. We denote it by $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$. This (relative) character variety is a Poisson affine variety which coincides with the classical one when the surface is closed. It shares a similar gluing property to the stated skein algebra; namely, there exist injective Poisson morphisms $i|_{a\#b}\colon \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}|_{a\#b})] \hookrightarrow \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ and $i^\Delta\colon \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})] \hookrightarrow \bigotimes_{\mathbb{T}\in F(\Delta)} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]$ between the Poisson algebras of regular functions. However, the Poisson structure on $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ depends on a choice of an orientation $\mathfrak{o}$ of the boundary arcs of the punctured surface. We denote by $\{\cdot,\cdot\}^{\mathfrak{o}}$ its Poisson bracket.

Main results Let $\mathbf{\Sigma}$ be a punctured surface. Lê's morphism (2) embeds the skein algebra of a triangulated surface into a tensor product of the skein algebras of the triangle. However, it does not provide a full description of the stated skein algebra in terms of these smaller pieces. In a first result we provide such a description; it goes as follows. Note that (1) endows the skein algebra of the bigon $\mathbb{B}$ (ie a disc with two punctures on its boundary) with a bialgebra structure. It is in fact a Hopf algebra and one can show that it is canonically isomorphic to the classical quantum SL_2–algebra $\mathcal{O}_q[\mathrm{SL}_2]$ described in [Chari and Pressley 1994; Kassel 1995] (with $q=\omega^{-4}$). Note also that (1) induces Hopf comodule maps $\Delta_a^L\colon \mathcal{S}_\omega(\mathbf{\Sigma}) \to \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\mathbf{\Sigma})$ and $\Delta_b^R\colon \mathcal{S}_\omega(\mathbf{\Sigma}) \to \mathcal{S}_\omega(\mathbf{\Sigma}) \otimes \mathcal{S}_\omega(\mathbb{B})$ obtained by gluing a bigon on a boundary arc, a or b, of $\mathbf{\Sigma}$; see Section 2.2 for details.

Theorem 1.1 *The sequence*

$$0 \to \mathcal{S}_\omega(\mathbf{\Sigma}|_{a\#b}) \xrightarrow{i|_{a\#b}} \mathcal{S}_\omega(\mathbf{\Sigma}) \xrightarrow{\Delta_a^L - \sigma\circ\Delta_b^R} \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\mathbf{\Sigma})$$

is exact, where $\sigma(x\otimes y) = y\otimes x$.

Theorem 1.1 can be reformulated using co-Hochschild cohomology, whose 0^{th} group (see Definition 2.26 and [Hess et al. 2009]) computes the skein algebra

$$\mathcal{S}_\omega(\mathbf{\Sigma}|_{a\#b}) \cong \mathrm{coHH}^0(\mathcal{O}_q[\mathrm{SL}_2], {}_a\mathcal{S}_\omega(\mathbf{\Sigma})_b),$$

where ${}_a\mathcal{S}_\omega(\mathbf{\Sigma})_b$ is seen as a bicomodule over $\mathcal{O}_q[\mathrm{SL}_2]$ via the comodule maps Δ_a^L and Δ_b^R.

Theorem 1.1 provides, for any topological triangulation Δ of $\underline{\Sigma}$, an isomorphism of algebras

$$\mathcal{S}_\omega(\underline{\Sigma}) \cong \mathrm{coHH}^0\Big(\bigotimes_{e \in \mathring{\mathcal{E}}(\Delta)} \mathcal{O}_q[\mathrm{SL}_2], \bigotimes_{\mathbb{T} \in F(\Delta)} \mathcal{S}_\omega(\mathbb{T}) \Big),$$

where the first tensor product runs over the inner edges of the triangulation and the second over the faces of the triangulation. Hence $\mathcal{S}_\omega(\underline{\Sigma})$ is completely determined by the combinatoric of the triangulation together with $\mathcal{S}_\omega(\mathbb{T})$ and its appropriated structures of comodule over $\mathcal{O}_q[\mathrm{SL}_2]$. This is a key feature in the proofs of the next two theorems.

Our second result is a generalization to open punctured surfaces of Bonahon and Wong's [2016] main theorem in the case where the root of unity has odd order. Given $N \geq 1$, denote by $T_N(X)$ the N^{th} Chebyshev polynomial of first kind.

Theorem 1.2 *Suppose that ω is a root of unity of odd order $N \geq 1$. There exists an embedding*

$$j_{\underline{\Sigma}} : \mathcal{S}_{+1}(\underline{\Sigma}) \hookrightarrow \mathcal{Z}(\mathcal{S}_\omega(\underline{\Sigma}))$$

of the (commutative) stated skein algebra with parameter $+1$ into the center of the stated skein algebra with parameter ω. Moreover, the morphism $j_{\underline{\Sigma}}$ is characterized by the property that it sends a closed curve γ to $T_N(\gamma)$ and a stated arc $\alpha_{\varepsilon\varepsilon'}$ to $\alpha^{(N)}_{\varepsilon\varepsilon'}$, where $\alpha^{(N)}_{\varepsilon\varepsilon'}$ is the tangle made by stacking N parallel copies of $\alpha_{\varepsilon\varepsilon'}$ on top of the others.

In Theorem 1.2 we restrict ourselves to roots of unity of odd order for simplicity. Theorem 1.2 should be compared to [Lê and Paprocki 2019, Theorem 8.1]. A marked 3–manifold is a pair $(M, \mathcal{N})$ where M is an oriented 3–manifold and $\mathcal{N} \subset \partial M$ is an oriented submanifold whose connected components are diffeomorphic to $[0,1]$. To such a pair and $\zeta \in \mathbb{C}^*$, Lê and Paprocki [2019] associate a vector space $\mathcal{S}_\zeta(M, \mathcal{N})$, which generalizes the Muller algebra. And for a root of unity ζ such that ζ^4 has arbitrary order $N > 1$ (not necessary odd), Lê and Paprocki [2019, Theorem 8.1] defined an injective linear map $\Phi_\zeta : \mathcal{S}_{(\zeta)^{N^2}}(M, \mathcal{N}) \hookrightarrow \mathcal{S}_\zeta(M, \mathcal{N})$. If $(\Sigma, \mathcal{P})$ is a punctured surface with no inner punctures and nontrivial boundary, $(M, \mathcal{N}) := (\Sigma \times (0,1), \mathcal{P} \times (0,1))$ is a marked 3–manifold and $\mathcal{S}_\zeta(M, \mathcal{N})$ is a subalgebra of the stated skein algebra $\mathcal{S}_\zeta(\Sigma, \mathcal{P})$. If ζ has odd order $N > 1$, the embedding $j_{\underline{\Sigma}}$ of Theorem 1.2 restricts to the embedding Φ_ζ of [Lê and Paprocki 2019, Theorem 8.1]. A generalization of Theorem 1.2 for roots of unity of even order has been recently proved by Bloomquist and Lê [2022, Theorem 1.2] though in this case the source of $j_{\underline{\Sigma}}$ is the skein algebra at $\eta := \omega^{N^2}$ and the image is not always central but rather spanned by $(-1)^{1+N'}$–transparent elements, where $N' := \mathrm{ord}(\omega^4)$ (see [Bloomquist and Lê 2022, Theorem 4.10] for details). Also a generalization of Theorem 1.2 for skein algebras of arbitrary connected reductive groups G and for marked surfaces having 0 or 1 boundary arc was found by Ganev, Jordan and Safronov [Ganev et al. 2024].

In the last result we generalize to open punctured surfaces Bullock's isomorphism [1997] and Turaev's theorem [1991]; we prove that the stated skein algebra is a deformation of the relative character variety. The fundamental result in this direction is as follows.

The $\mathbb{C}[\![\hbar]\!]$–module $\mathscr{S}_{+1}(\underline{\Sigma})[\![\hbar]\!] := \mathscr{S}_{+1}(\underline{\Sigma}) \otimes_{\mathbb{C}} \mathbb{C}[\![\hbar]\!]$ is endowed with a star product $\star_\hbar$. The latter is obtained by pulling back the product of $\mathscr{S}_{+1}(\underline{\Sigma})$ along an isomorphism $\mathscr{S}_{+1}(\underline{\Sigma})[\![\hbar]\!] \xrightarrow{\cong} \mathscr{S}_{\omega_\hbar}(\underline{\Sigma})$ of vector spaces, where $\omega_\hbar := \exp\bigl(-\frac{1}{4}\hbar\bigr)$ (see Section 2.7 for details). This equips $\mathscr{S}_{+1}(\underline{\Sigma})$ with a Poisson algebra structure; its Poisson bracket $\{\cdot,\cdot\}^s$ is defined by

$$f \star_\hbar g - g \star_\hbar f = \hbar\{f,g\}^s \pmod{\hbar^2} \quad \text{for all } f,g \in \mathscr{S}_{+1}(\underline{\Sigma}).$$

The superscript s stands for "skein". See Section 2.7.3 for an explicit description.

Theorem 1.3 *Suppose that $\underline{\Sigma}$ has a topological triangulation Δ. Let $\mathfrak{o}_\Delta$ be an orientation of the edges of Δ and $\mathfrak{o}$ be the induced orientation of the boundary arcs of $\underline{\Sigma}$. There exists an isomorphism of Poisson algebras*

$$\Psi^{(\Delta,\mathfrak{o}_\Delta)} : (\mathscr{S}_{+1}(\underline{\Sigma}), \{\cdot,\cdot\}^s) \xrightarrow{\cong} (\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})], \{\cdot,\cdot\}^{\mathfrak{o}}).$$

Moreover, the above isomorphism exists for small punctured surfaces (*see Definition 2.8*), *for which it only depends on $\mathfrak{o}$.*

The isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ induces, by tensoring with $\mathbb{C}[\![\hbar]\!]$, an isomorphism of vector spaces

$$\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})][\![\hbar]\!] \xrightarrow{\cong} \mathscr{S}_{+1}(\underline{\Sigma})[\![\hbar]\!].$$

Denote by $\star_{(\Delta,\mathfrak{o}_\Delta)}$ the product on $\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})][\![\hbar]\!]$ obtained by pulling back the product $\star_\hbar$ by this isomorphism.

Corollary 1.4 *For any triangulable punctured surface $\underline{\Sigma}$, the algebra $\bigl(\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})][\![\hbar]\!], \star_{(\Delta,\mathfrak{o}_\Delta)}\bigr)$ is a deformation quantization of the character variety with Poisson structure given by $\mathfrak{o}$.*

Theorems 1.2 and 1.3 allow us to extend Bonahon and Wong's [2016] *classical shadow* to open punctured surfaces. Indeed, suppose that ω is a root of unity of odd order. A finite-dimensional representation $\mathscr{S}_\omega(\underline{\Sigma}) \to \mathrm{End}(V)$ that sends each element of the image of $j_{\underline{\Sigma}} : \mathscr{S}_{+1}(\underline{\Sigma}) \hookrightarrow \mathscr{S}_\omega(\underline{\Sigma})$ to scalar operators, induces a character on the algebra $\mathscr{S}_{+1}(\underline{\Sigma}) \cong \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$, hence defines a point in $\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})$. To sum up, and calling these representations *central*, one has the following.

Corollary 1.5 *When ω is a root of unity of odd order and $\underline{\Sigma}$ is triangulable, to each isomorphism class of central representations of the stated skein algebra $\mathscr{S}_\omega(\underline{\Sigma})$, one can associate an invariant which is a point in the relative character variety $\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})$.*

Central representations include the families of irreducible representations, local representations and representations induced by simple modules of the balanced Chekhov–Fock algebras using the quantum trace map (see Section 3.3 for details).

Soon after the prepublication of this paper on arXiv, Costantino and Lê [2022] prepublished independently some results similar to Theorems 1.1 and 1.3. More precisely, [Costantino and Lê 2022, Theorem 4.7] is identical to Theorem 1.1, and [Costantino and Lê 2022, Theorem 8.12] is closely related, though different, to our Theorem 1.3. Instead of using the generalized character variety $\mathcal{X}_{\mathrm{SL}_2}(\Sigma)$ defined in [Korinman 2019], the authors defined a twisted character variety $\chi(\Sigma)$ (without Poisson structure) and constructed a canonical algebra isomorphism between the stated skein algebra in $+1$ and the algebra of regular functions of $\chi(\Sigma)$, whereas our isomorphism in Theorem 1.3 depends on the noncanonical choice $(\Delta, \mathfrak{o}_\Delta)$ of a triangulation and an orientation of the edges (and is Poisson). Inspired by their enlightening approach, in this new version of the paper we add the following clarification of the isomorphism in Theorem 1.3. As explained before, when the punctured surface is closed, the "standard" isomorphisms between $\mathcal{S}_{+1}(\Sigma)$ and $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\Sigma)]$ are indexed by spin structures. In Section 3.3, we define the notion of *relative spin structure* for punctured surfaces, which coincides with the standard definition when the punctured surface is closed. The motivation for this definition is its good behavior for the operation of gluing boundary arcs together. In particular we associate to each combinatorial data $(\Delta, \mathfrak{o}_\Delta)$, appearing in Theorem 1.3, a relative spin structure and prove:

Theorem 1.6 *The isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ of Theorem 1.3 only depends on the relative spin structure associated to* $(\Delta, \mathfrak{o})$.

In fact, in Theorem 3.20, we provide explicit formulas for the value of $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ on stated arcs and closed curves in terms of the relative spin structure. When the punctured surface is closed, we show that our isomorphism coincides with the standard isomorphism associated to classical spin structures. We also give, in Section 3.3.5, a detailed comparison between the isomorphism in Theorem 1.3 and Costantino and Lê's isomorphism [2022, Theorem 8.12].

Even though our proof of Theorem 1.2 makes uses of triangulations, the theorem is proved for arbitrary punctured surfaces, including (nontriangulable) closed surfaces without punctures, thus providing an alternative proof of the results in [Bonahon and Wong 2016]. However, our proof of Theorem 1.3 only works for triangulable punctured surfaces (and for the bigon), so it does not provide an alternative proof of the result of [Bullock 1997] for closed unpunctured surfaces.

Plan of the paper In the second section we briefly recall from [Lê 2018] the definition and general properties of the stated skein algebra and prove Theorem 1.1. We then use the triangular decomposition to reduce the proof of Theorem 1.2 to the cases of the bigon and the triangle for which the proof is a simple computation. We eventually characterize the Poisson bracket arising in skein theory. In the third section, we briefly recall from [Korinman 2019] the definition of character varieties for open surfaces. Again, using triangular decompositions, we reduce the proof of Theorem 1.3 to the cases of the bigon and the triangles for which the proof is elementary. We then introduce and study the notion of relative spin structure and give in Theorem 3.20 an explicit description of the isomorphism of Theorem 1.3, from which Theorem 1.6 is a straightforward consequence. In the appendix, we prove a technical result needed in the proof of Theorem 1.2 and derive a generalization of the main theorem of [Bonahon 2019].

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Notation Throughout the paper we reserve the notation $A := \omega^{-2}$ and $q := \omega^{-4}$.

2 Stated skein algebras

2.1 Definitions and general properties of the stated skein algebras

We briefly review from [Lê 2018] the definition and main properties of the stated skein algebras.

Definition 2.1 A *punctured surface* is a pair $\underline{\Sigma} = (\Sigma, \mathcal{P})$ where Σ is a compact oriented surface and $\mathcal{P}$ is a finite subset of Σ which intersects nontrivially each boundary component. A *boundary arc* is a connected component of $\partial\Sigma \setminus \mathcal{P}$. The punctured surface is *open* when $\partial\Sigma \neq \varnothing$ and *closed* otherwise.

Definition of stated skein algebras Let $\underline{\Sigma} = (\Sigma, \mathcal{P})$ be a punctured surface and write $\Sigma_{\mathcal{P}} := \Sigma \setminus \mathcal{P}$. A *tangle* in $\Sigma_{\mathcal{P}} \times (0, 1)$ is a compact framed, properly embedded 1–dimensional manifold $T \subset \Sigma_{\mathcal{P}} \times (0, 1)$ such that for every point of $\partial T \subset \partial\Sigma_{\mathcal{P}} \times (0, 1)$ the framing is parallel to the $(0, 1)$ factor and points in the direction of 1. Here, by framing, we refer to a thickening of T to an oriented surface. Define the *height* of a point $(v, h) \in \Sigma_{\mathcal{P}} \times (0, 1)$ to be h. If b is a boundary arc and T a tangle, the points of $\partial_b T := \partial T \cap b \times (0, 1)$ are totally ordered by their height and we impose that no two points in $\partial_b T$ have the same height. A tangle has *vertical framing* if for each of its points, the framing is parallel to the $(0, 1)$ factor and points in the direction of 1. Two tangles are isotopic if they are isotopic through the class of tangles that preserves the partial boundary height orders. By convention, the empty set is a tangle only isotopic to itself.

Every tangle is isotopic to a tangle with vertical framing. We can further isotope a tangle such that it is in general position with the standard projection $\pi : \Sigma_{\mathcal{P}} \times (0, 1) \to \Sigma_{\mathcal{P}}$ with $\pi(v, h) = v$, that is such that $\pi|_T : T \to \Sigma_{\mathcal{P}}$ is an immersion with at most transversal double points in the interior of $\Sigma_{\mathcal{P}}$. We call a *diagram* of T the image $D = \pi(T)$ together with the over/undercrossing information at each double point. An isotopy class of diagram D together with a total order of $\partial_b D = \partial D \cap b$ for each boundary arc b define uniquely an isotopy class of tangle. Here isotopy of diagrams refers to isotopies where endpoints of diagrams are not allowed to cross. When choosing an orientation $\mathfrak{o}(b)$ of a boundary arc b and a diagram D, the set $\partial_b D$ receives a natural total order $\le_{\mathfrak{o}}$ by setting that the points are increasing when going in the direction of $\mathfrak{o}(b)$. We will represent tangles by drawing a diagram and an orientation (an arrow) for each boundary arc. When a boundary arc b is oriented, $\partial_b D$ is ordered by $\le_{\mathfrak{o}}$ according to the orientation. The data of an isotopy class of diagram D and a choice $\mathfrak{o}$ of orientations of the boundary arcs define uniquely an isotopy class of tangle T by imposing that for every boundary arc a, for $v, w \in \partial_a D$ such that $v \le_{\mathfrak{o}} w$, the endpoint of $\partial_a T$ corresponding to w has higher height than the endpoint corresponding to v. A *state* of a tangle is a map $s\colon \partial T \to \{-,+\}$. A pair (T, s) is called a *stated tangle*. We define a *stated diagram* (D, s) in a similar manner.

Let $\mathcal{R}$ be a commutative unital ring and $\omega \in \mathcal{R}^\times$ an invertible element.

Definition 2.2 The *stated skein algebra* $\mathcal{S}_\omega(\Sigma)$ is the free $\mathcal{R}$–module generated by isotopy classes of stated tangles in $\Sigma_{\mathcal{P}} \times (0,1)$ modulo the relations (3) and (4), which are

- the Kauffman bracket relations

(3) 

- the boundary relations

(4) 

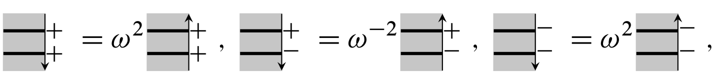
According to our graphical conventions, in these skein relations, the boundary points are consecutive in the height order. The product of two classes of stated tangles $[T_1, s_1]$ and $[T_2, s_2]$ is defined by isotoping T_1 and T_2 in $\Sigma_{\mathcal{P}} \times \left(\frac{1}{2}, 1\right)$ and $\Sigma_{\mathcal{P}} \times \left(0, \frac{1}{2}\right)$, respectively, and then setting $[T_1, s_1] \cdot [T_2, s_2] = [T_1 \cup T_2, s_1 \cup s_2]$.

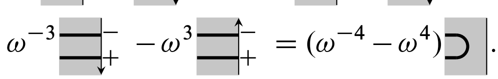
Bases for stated skein algebras A closed component of a diagram D is trivial if it bounds an embedded disc in $\Sigma_{\mathcal{P}}$. An open component of D is trivial if it can be isotoped, relatively to its boundary, inside some boundary arc. A diagram is *simple* if it has neither double points nor trivial component. The empty set is considered as a simple diagram. Let $\mathfrak{o}$ be an orientation of the boundary arcs of Σ and denote by $\le_{\mathfrak{o}}$ the total orders induced on each boundary arc. A state $s\colon \partial D \to \{-,+\}$ is $\mathfrak{o}$*–increasing* if for any boundary arc b and any points $x, y \in \partial_b D$, $x <_{\mathfrak{o}} y$ implies $s(x) < s(y)$. Here we choose the convention $- < +$. We denote by $[D, s] \in \mathcal{S}_\omega(\Sigma)$ the class of the stated tangle associated to (D, s) (note that $[D, s]$ depends on the orientation $\mathfrak{o}$).

Definition 2.3 We denote by $\mathcal{B}^{\mathfrak{o}} \subset \mathcal{S}_\omega(\Sigma)$ the set of classes $[D, s]$ such that D is simple and s is $\mathfrak{o}$–increasing.

Theorem 2.4 [Lê 2018, Theorem 2.11] *The set $\mathcal{B}^{\mathfrak{o}}$ is an $\mathcal{R}$–module basis of $\mathcal{S}_\omega(\Sigma)$.*

Height exchange moves Important properties that we will use throughout the paper are the following *height exchange moves* (5) and (6) proved in [Lê 2018, Lemma 2.4]. Note that the formula (20) of Lemma 2.4 of [loc. cit.] contains a misprint. It is corrected here in (6):

(5) 

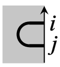
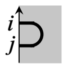
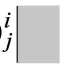
(6) 

Remark 2.5 An important case that we will be led to consider is the stated skein algebra at parameter $\omega = +1$. As shown in [Lê 2018, Corollary 2.5] it is commutative; this is a direct consequence of (3) and the height exchange formulas (5) and (6).

Trivial arcs relations We will also use the following *trivial arcs relations*. Set

$$C = \begin{pmatrix} C_+^+ & C_-^+ \\ C_+^- & C_-^- \end{pmatrix} := \begin{pmatrix} 0 & \omega \\ -\omega^5 & 0 \end{pmatrix} \quad \text{and} \quad C^{-1} = -A^3 C = \begin{pmatrix} 0 & -\omega^{-5} \\ \omega^{-1} & 0 \end{pmatrix}.$$

Lemma 2.6 [Lê 2018, Lemma 2.3] *One has the following **trivial arcs relations**:*

(7)  $= C^i_j$  *and*  $= (C^{-1})^i_j$  .

Splitting morphisms Suppose that $\underline{\Sigma}$ has two boundary arcs, say a and b. Let $\underline{\Sigma}|_{a\#b}$ be the punctured surface obtained from $\underline{\Sigma}$ by gluing a and b. Denote by $\pi\colon \Sigma_{\mathcal{P}} \to (\Sigma|_{a\#b})_{\mathcal{P}|_{a\#b}}$ the projection and $c := \pi(a) = \pi(b)$. Let (T_0, s_0) be a stated framed tangle of $\Sigma|_{a\#b\mathcal{P}|_{a\#b}} \times (0,1)$ transversed to $c \times (0,1)$ and such that the heights of the points of $T_0 \cap c \times (0,1)$ are pairwise distinct and such that framings of the points of $c \times (0,1)$ are vertical. Let $T \subset \Sigma_{\mathcal{P}} \times (0,1)$ be the framed tangle obtained by cutting T_0 along c. Using the partition $\partial T = \partial_a T \sqcup \pi^{-1}(\partial T_0) \sqcup \partial_b T$, a state on T can be written (s_a, s, s_b) where s_a, s and s_b are states on $\partial_a T$, ∂T_0 and $\partial_b T$, respectively. Both the sets $\partial_a T$ and $\partial_b T$ are in canonical bijection with the set $T_0 \cap c$ by the map π. Hence the two sets of states s_a and s_b are both in canonical bijection with the set $\mathrm{St}(c) := \{s\colon c \cap T_0 \to \{-,+\}\}$. Let $i|_{a\#b}\colon \mathcal{S}_\omega(\underline{\Sigma}|_{a\#b}) \to \mathcal{S}_\omega(\underline{\Sigma})$ be the linear map given, for any (T_0, s_0) as above, by

$$i|_{a\#b}([T_0, s_0]) := \sum_{s \in \mathrm{St}(c)} [T, (s, s_0, s)].$$

Theorem 2.7 [Lê 2018, Theorem 3.1] *The linear map $i|_{a\#b}$ is an injective morphism of algebras. Moreover the gluing operation is coassociative in the sense that if a, b, c and d are four distinct boundary arcs, then $i|_{a\#b} \circ i|_{c\#d} = i|_{c\#d} \circ i|_{a\#b}$.*

Note that the splitting morphism $i|_{a\#b}$ does not depend on any choice of the boundary arcs.

Triangulations

Definition 2.8 A *small* punctured surface is one of the following four connected punctured surfaces: the sphere with one or two punctures; the disc with only one puncture (on its boundary); and the bigon (disc with two punctures on its boundary).

Definition 2.9 A punctured surface is said to *admit a triangulation* if each of its connected components has at least one puncture and is not small.

Definition 2.10 Suppose $\underline{\Sigma} = (\Sigma, \mathcal{P})$ admits a triangulation. A *topological triangulation* Δ of $\underline{\Sigma}$ is a collection $\mathcal{E}(\Delta)$ of arcs in Σ (named edges) which satisfy the following conditions: the endpoints of the edges belong to $\mathcal{P}$; the interior of the edges are pairwise disjoint and do not intersect $\mathcal{P}$; the edges are not contractible and are pairwise nonisotopic in $\Sigma_{\mathcal{P}}$, if fixed their endpoints; and the boundary arcs of $\underline{\Sigma}$ belong to $\mathcal{E}(\Delta)$. Moreover, the collection $\mathcal{E}(\Delta)$ is required to be maximal for these properties.

Each connected component of $\Sigma \setminus \mathcal{E}(\Delta)$ is called a *face* and the set of faces is denoted by $F(\Delta)$. Given a topological triangulation Δ, the punctured surface is obtained from the disjoint union $\bigsqcup_{\mathbb{T} \in F(\Delta)} \mathbb{T}$ of triangles by gluing the triangles along the boundary arcs corresponding to the edges of the triangulation. Very often, we will let $\mathbb{T}$ be both a face (which is an open contractible space) and the triangle (which is a disc with exactly three punctures on its boundary). We hope that this abuse of notation is harmless. By composing the associated splitting morphisms, one obtains an injective morphism of algebras

$$i^{\Delta}\colon \mathcal{S}_{\omega}(\underline{\Sigma}) \hookrightarrow \bigotimes_{\mathbb{T} \in F(\Delta)} \mathcal{S}_{\omega}(\mathbb{T}).$$

Filtrations The stated skein algebra has natural filtrations defined as follows. Let $S = \{a_1, \dots, a_n\}$ be a set of boundary arcs of $\underline{\Sigma}$ and fix an orientation $\mathfrak{o}$ of the boundary arcs of $\underline{\Sigma}$. For a basis element $[D, s]$ of $\mathcal{B}^{\mathfrak{o}}$, write $d([D, s]) := \sum_{a \in S} |\partial_a D|$. The map d extends to a map $d\colon \mathcal{S}_{\omega}(\underline{\Sigma}) \to \mathbb{Z}^{\geq 0}$ by the formula $d\big(\sum_i x_i [D_i, s_i]\big) := \max_{i \mid x_i \neq 0} d([D_i, s_i])$. It follows from the relations (3) and (4) that for each $x, y \in \mathcal{S}_{\omega}(\underline{\Sigma})$, we have $d(xy) \leq d(x) + d(y)$. Given $m \geq 0$, denote by $\mathcal{F}_m \subset \mathcal{S}_{\omega}(\underline{\Sigma})$ the subvector space of those vectors x satisfying $d(x) \leq m$. These subspaces satisfy $\mathcal{F}_m \subset \mathcal{F}_{m+1}$, $\mathcal{S}_{\omega}(\underline{\Sigma}) = \bigcup_{m \geq 0} \mathcal{F}_m$ and $\mathcal{F}_{m_1} \cdot \mathcal{F}_{m_2} \subset \mathcal{F}_{m_1+m_2}$; hence they form an algebra filtration of the stated skein algebra.

Definition 2.11 The sequence $(\mathcal{F}_m)_{m \geq 0}$ is called the *filtration* of $\mathcal{S}_{\omega}(\underline{\Sigma})$ associated to the orientation $\mathfrak{o}$ and the set S of boundary arcs. For an element $X = \sum_{i \in I} x_i [D_i, s_i] \in \mathcal{S}_{\omega}(\underline{\Sigma})$, developed in the basis $\mathcal{B}^{\mathfrak{o}}$, we call the *leading term* of X the element

$$\mathrm{lt}(X) := \sum_{\substack{j \in I \\ d([D_j, s_j]) = d(X)}} x_j [D_j, s_j].$$

2.2 Alternative bases

In the next subsection, we will need alternative bases of $\mathcal{S}_{\omega}(\underline{\Sigma})$ which we now introduce. We fix an arbitrary orientation $\mathfrak{o}$ for each boundary arc. Recall that $\mathfrak{o}$ induces a total order $\leq_{\mathfrak{o}}$ on each boundary arc that we use to associate a tangle to a diagram.

Notation 2.12 Let $\mathcal{D}(\underline{\Sigma})$ be the set of isotopy classes of simple diagrams and $\mathcal{CD}(\underline{\Sigma})$ be its subset of classes of connected diagrams. Fix an arbitrary total order $\prec$ on $\mathcal{CD}(\underline{\Sigma})$ and fix an orientation $\mathfrak{o}$ of the boundary arcs of $\underline{\Sigma}$ as before. For $[D] \in \mathcal{CD}(\underline{\Sigma})$, we denote by $[T(D)]$ the isotopy class of the tangle $T(D)$ with vertical framing whose projection is D and such that if $\partial T(D) = \{v_1, v_2\}$ with v_1 and v_2 in

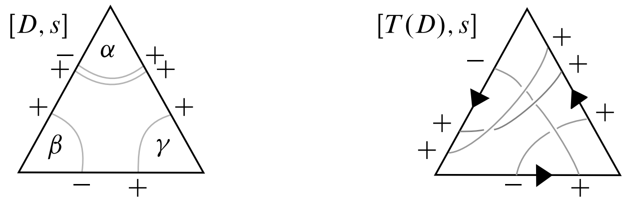


Figure 1: A stated diagram $[D,s]$ in the triangle and its associated stated tangle $[T(D),s]$. Here, we use the order $\gamma \prec \beta \prec \alpha$. Here s is $\mathfrak{o}$–increasing so $[T(D),s] \in \mathcal{TB}^{\mathfrak{o}}$.

the same boundary arc a with $v_1 \leq_{\mathfrak{o}} v_2$, then $h(v_1) < h(v_2)$. For a general class of diagram $[D] \in \mathcal{D}(\underline{\Sigma})$ with connected components $D = \bigsqcup_{i=1}^n D_i$, where $[D_i] \preceq [D_{i+1}]$, we denote by $[T(D)]$ the class of the tangle $T(D) := \bigsqcup_{i=1}^n T(D_i)$ in $\Sigma_{\mathcal{P}} \times (0,1)$, where $T(D_{i+1})$ is on the top of $T(D_i)$ in the height direction. See Figure 1 for an illustration. Let $\nu \colon \partial D \xrightarrow{\cong} \partial T(D)$ be the unique bijection such that, for a a boundary arc, ν restricts to a bijection $\nu|_a \colon \partial_a D \to \partial_a T(D)$ which preserves the order $\leq_{\mathfrak{o}}$ on $\partial_a D$ and the height order on $\partial_a T(D)$. Recall that $\partial_a D = D \cap a$ and that $\partial_a T(D) = T(D) \cap a \times (0,1)$. A state s on D defines a state $s \circ \nu^{-1}$ on $T(D)$ and we denote by $[T(D),s]$ the class of the stated tangle $(T(D), s \circ \nu^{-1})$.

Definition 2.13 We denote by $\mathcal{TB}^{\mathfrak{o}} \subset \mathcal{S}_\omega(\underline{\Sigma})$ the set of classes $[T(D),s]$ with $[D] \in \mathcal{D}(\underline{\Sigma})$ and s an $\mathfrak{o}$–increasing state.

Note that in our pictures the orientation $\mathfrak{o}$ is never represented, the arrows always refer to the height order and not to $\mathfrak{o}$. The following lemma was proved in [Lê 2018], during the proof of Theorem 4.6, in the particular case where $\underline{\Sigma}$ is a triangle.

Proposition 2.14 *The set $\mathcal{TB}^{\mathfrak{o}}$ is a basis of $\mathcal{S}_\omega(\underline{\Sigma})$.*

As an immediate consequence of Proposition 2.14, we get:

Corollary 2.15 *The stated skein algebra is algebraically generated by the classes of closed curves and stated arcs.*

Here by closed curves and stated arcs we mean connected stated diagrams with no crossing which are closed and open, respectively. Obviously, it is sufficient to prove Proposition 2.14 in the case where Σ is connected. If $\partial\Sigma = \varnothing$ or if $\underline{\Sigma}$ is a disc with one puncture on its boundary or a bigon whose boundary arcs points towards the same puncture, then $\mathcal{TB}^{\mathfrak{o}} = \mathcal{B}^{\mathfrak{o}}$ so the proposition follows from Theorem 2.4 in those cases. For the bigon whose boundary arcs point towards distinct punctures, Proposition 2.14 was proved in [Lê 2018, Step 1 of the proof of Theorem 4.1]. So we now assume that $\underline{\Sigma}$ admits a topological triangulation Δ that we fix. The proof of Proposition 2.14 is an easy adaption of Lê's argument from the case of the triangle to the case of a triangulable punctured surface. The key feature is to consider a suitable filtration that we now introduce.

For a diagram D and an edge $e \in \mathscr{E}(\Delta)$, we denote by $i(D,e) \in \mathbb{N}$ the geometric intersection of D with e; that is, the minimal number of intersection points when isotoping D in such a way that it intersects e transversally. We write

$$|D| := \sum_{e \in \mathscr{E}(\Delta)} i(D,e),$$

and, for $i \in \mathbb{N}$, we set

$$\mathscr{F}_i := \mathrm{Span}\{[D,s] \text{ such that } |D| \le i\}.$$

Lemma 2.16 (1) *One has* $\mathscr{F}_i \cdot \mathscr{F}_j \subset \mathscr{F}_{i+j}$.

(2) *The submodule* $\mathscr{F}_i$ *has basis the set* B_i *of elements* $[D,s] \in \mathscr{B}^o$ *such that* $|D| \le i$.

(3) *For* $[D,s] \in \mathscr{B}^o$, *there exists* $n \in \mathbb{Z}$ *such that*

$$[T(D),s] \equiv A^n[D,s] \pmod{\mathscr{F}_{|D|-2}}.$$

Proof (1) Let $[D_1,s_1]$ and $[D_2,s_2]$ be two classes such that

(i) $D_1 \cup D_2$ has only transversed double intersection points in the interior of $\Sigma_{\mathscr{P}}$ away from the edges of Δ, and

(ii) D_1 and D_2 are transversed to the edges of $\mathscr{E}(\Delta)$ with minimal intersection such that

$$|D_i| = |D_i \cap \mathscr{E}(\Delta)|, \quad i = 1,2.$$

Let D denote the diagram obtained by stacking D_1 on top of D_2 and s the state corresponding to s_1 and s_2 such that $[D,s] = [D_1,s_1][D_2,s_2]$. Then $|D| \le |D \cap \mathscr{E}(\Delta)| = |D_1| + |D_2|$. Therefore, $[D_1,s_1][D_2,s_2] \in \mathscr{F}_{|D_1|+|D_2|}$ and the first assertion is proved.

(2) To prove the second assertion, first note that since B_i is a subset of $\mathscr{B}^o$, it is free. We need to show that B_i generates $\mathscr{F}_i$. We proceed in two steps:

Step 1 We first prove that any class of stated diagram $[D,s]$ is a linear combination of elements $[D_i,s_i]$ with $|D_i| = |D|$ and such that D_i has no crossing.

Step 2 We then prove that any $[D,s]$, where D has no crossing, is a linear combination of elements of $B_{|D|}$.

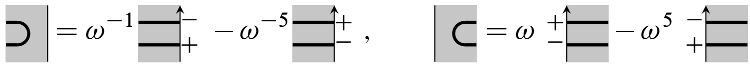
The two steps imply that B_i generates $\mathscr{F}_i$ and conclude the proof of the second assertion.

To prove the first step, fix an arbitrary stated diagram (D,s). A *resolution* of D is a diagram obtained from D by replacing each crossing  by either  (positive resolution of the crossing) or  (negative resolution). Write $\mathrm{Res}(D)$ the set of resolutions and for $D_0 \in \mathrm{Res}(D)$, denote by $n(D_0)$ the difference between the numbers of positive and negative resolution crossings in D_0. Then, by the Kauffman-bracket skein relation (3), one has

$$[D,s] = \sum_{D_i \in \mathrm{Res}(D)} A^{n(D_i)}[D_i,s],$$

where for each resolution D_i, one has $|D_i \cap \mathscr{E}(\Delta)| = |D \cap \mathscr{E}(\Delta)| = |D|$, so $|D_i| = |D|$ and Step 1 is proved.

To prove the second step, consider a stated diagram (D, s) where D has no crossing. If s is $\mathfrak{o}$–increasing, let (D', s) be the stated diagram obtained from (D, s) by removing its trivial components, so $|D'| \le |D|$. Then there exists a scalar c such that $[D, s] = c[D', s]$ and $[D', s] \in B_{|D|}$. Otherwise, we show by induction on the number $m(D, s)$ of pairs of points $v <_{\mathfrak{o}} w$ in ∂D lying in the same boundary arc such that $(s(v), s(w)) = (+, -)$, that (D, s) is a linear combination of elements of $B_{|D|}$. Consider such a pair (v, w) of points which are consecutive for $<_{\mathfrak{o}}$, and let s' be the state on D which agrees with s on $\partial D \setminus \{v, w\}$ and such that $(s'(v), s'(w)) = (-, +)$. The skein relations



show that there exists $n \in \mathbb{Z}$ such that $[D, s] \equiv \omega^n [D, s'] \pmod{\mathcal{F}_{|D|-1}}$ (because the stated diagram representing either the term  or  is in $\mathcal{F}_{|D|-1}$). Since $m(D, s') < m(D, s)$, we conclude by decreasing induction on m that $[D, s]$ is a linear combination of elements $[D_i, s_i]$ where D_i has no crossing and s_i is $\mathfrak{o}$–increasing. Now write $[D_i, s_i] = c_i [D'_i, s_i]$, where c_i is a scalar and (D'_i, s_i) is obtained from (D_i, s_i) by removing its trivial components so that $[D'_i, s_i] \in B_{|D|}$. This concludes Step 2 and the proof of the second item.

(3) Let us first make an obvious but useful remark. Let D be a diagram transversed to $\mathcal{E}(\Delta)$. We say that *D contains a returning arc* if there exists a face $\mathbb{T}$ such that $D \cap \mathbb{T}$ contains a connected component that is an arc with both endpoints in the same edge. If D contains a returning arc, then D is not in minimal intersection position with respect to $\mathcal{E}(\Delta)$ so for all states s, $[D, s] \in \mathcal{F}_{|D|-2}$.

Now consider $[D, s] \in \mathcal{B}^{\mathfrak{o}}$ and denote by TD the projection diagram of the tangle $T(D)$ so that $[T(D), s] = [TD, s]$ (think of Figure 1). We further suppose that TD is transversed to $\mathcal{E}(\Delta)$ in minimal position and has its crossings outside $\mathcal{E}(\Delta)$. In the decomposition

$$[TD, s] = \sum_{D_i \in \mathrm{Res}(TD)} A^{n(D_i)} [D_i, s],$$

we claim that there is exactly one resolution $D_0 \in \mathrm{Res}(TD)$ such that $D_0 = D$ and that any other resolution $D_i \neq D_0$ contains a returning arc, so satisfies $[D_i, s_i] \in \mathcal{F}_{|D|-2}$. Since resolving a crossing is a local operation, it is sufficient to prove the claim in the case of the triangle; this was done by Lê [2018, Lemma 4.7]. Recall that Lê's proof consists noting that if $[T(D), s]$ has two connected components, it has 0 or 1 crossing (after eventually isotoping TD) and when there is one crossing in TD, exactly one of the two resolutions does not contain returning arc. The results then follows by induction on the number of components of $T(D)$ using the fact that the arcs in $T(D)$ are stacked on top of each other.

So we have $[T(D), s] \equiv A^{n(D)} [D, s] \pmod{\mathcal{F}_{|D|-2}}$ and the proof is completed. □

Obviously one has $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ and $\bigcup_{i \ge 0} \mathcal{F}_i = \mathcal{S}_\omega(\Sigma)$. The first assertion of Lemma 2.16 implies that $(\mathcal{F}_i)_{i \ge 0}$ forms an algebra filtration of $\mathcal{S}_\omega(\Sigma)$. Consider the graded algebra $\mathbf{Gr}_\bullet$ associated to the filtration. In other words, we set $\mathbf{Gr}_0 := \mathcal{F}_0$, $\mathbf{Gr}_i := \mathcal{F}_i \,/\, \mathcal{F}_{i-1}$ for $i \ge 1$ and $\mathbf{Gr}_\bullet := \bigoplus_{i \ge 0} \mathbf{Gr}_i$. It follows from the second assertion of Lemma 2.16 that $\mathbf{Gr}_i$ has basis the set $\mathcal{B}^{\mathfrak{o}}_i$ of classes $[D, s] \in \mathcal{B}^{\mathfrak{o}}$ such that $|D| = i$.

Since the set $\{\mathcal{B}_i^o\}_{i\geq 0}$ forms a partition of $\mathcal{B}^o$, the natural graded morphism $\psi\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathbf{Gr}_\bullet$ is an isomorphism. To prove Proposition 2.14, we will derive from the third assertion of Lemma 2.16 that the image of $\mathcal{T}\mathcal{B}^o$ through ψ is a basis of $\mathbf{Gr}_\bullet$.

Proof of Proposition 2.14 As noted previously, if $\underline{\Sigma}$ is closed or if $\underline{\Sigma}$ is bigon or a disc with one puncture on its boundary, then $\mathcal{T}\mathcal{B}^o = \mathcal{B}^o$ so the lemma follows from Theorem 2.4. Otherwise, we can consider a topological triangulation and consider the associated graded isomorphism $\psi\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathbf{Gr}_\bullet$. Let $\mathcal{T}\mathcal{B}_i^o \subset \mathcal{T}\mathcal{B}^o$ be the subset of elements $[T(D), s]$ such that $|D| = i$. Since $\psi(\mathcal{B}_i^o)$ is a basis of $\mathbf{Gr}_i$, the third assertion of Lemma 2.16 implies that the image $\psi(\mathcal{T}\mathcal{B}_i^o)$ is also a basis of $\mathbf{Gr}_i$. Therefore $\psi(\mathcal{T}\mathcal{B}^o)$ is a basis of $\mathbf{Gr}_\bullet$, so $\mathcal{T}\mathcal{B}^o$ is a basis of $\mathcal{S}_\omega(\underline{\Sigma})$. □

2.3 Removing a puncture

Let $\underline{\Sigma} = (\Sigma, \mathcal{P})$ and consider a punctured surface $\underline{\Sigma}' := (\Sigma, \mathcal{P} \cup \{p_0\})$ obtained from $\underline{\Sigma}$ by adding a puncture $p_0 \in \Sigma_{\mathcal{P}}$ to $\mathcal{P}$. The goal of this subsection is to define and study a map $\varphi\colon \mathcal{S}_\omega(\underline{\Sigma}') \to \mathcal{S}_\omega(\underline{\Sigma})$. Let $\mathcal{T}(\underline{\Sigma})$ denote the set of stated tangles in $\Sigma_{\mathcal{P}} \times (0,1)$ and denote by $\mathcal{J}(\underline{\Sigma}) \subset \mathcal{R}[\mathcal{T}(\underline{\Sigma})]$ the ideal generated by the skein relations (3) and (4) and by the elements $(T, s) - (T', s)$, where T and T' are isotopic; so by definition, one has $\mathcal{S}_\omega(\underline{\Sigma}) := \mathcal{R}[\mathcal{T}(\underline{\Sigma})] \,/\, \mathcal{J}(\underline{\Sigma})$. The inclusion map $\iota\colon \Sigma_{\mathcal{P}\cup\{p_0\}} \times (0,1) \hookrightarrow \Sigma_{\mathcal{P}} \times (0,1)$ induces a linear map $\underline{\varphi}\colon \mathcal{R}[\mathcal{T}(\underline{\Sigma}')] \to \mathcal{R}[\mathcal{T}(\underline{\Sigma})]$ sending a stated tangle (T, s) to $(\iota(T), s \circ \iota^{-1})$.

First suppose that p_0 is in the interior of $\Sigma_{\mathcal{P}}$. In this case, $\underline{\varphi}$ obviously sends isotopic stated tangles to isotopic stated tangles and skein relations to skein relations, so it sends $\mathcal{J}(\underline{\Sigma}')$ to $\mathcal{J}(\underline{\Sigma})$ and it induces a linear map $\varphi\colon \mathcal{S}_\omega(\underline{\Sigma}') \to \mathcal{S}_\omega(\underline{\Sigma})$ by passing to the quotient. It is clear that φ is a morphism of algebras. Moreover, since any tangle in $\Sigma_{\mathcal{P}} \times (0,1)$ can be isotoped in $\Sigma_{\mathcal{P}\cup\{p_0\}} \times (0,1)$, the map φ is surjective.

When p_0 lies in some boundary arc, say a, of $\underline{\Sigma}$, the construction is more subtle. Denote by b and c the two boundary arcs of $\underline{\Sigma}'$ which are the connected components of $a \setminus \{p_0\}$. The linear map φ still sends skein relations to skein relations; however if (T, s) and (T', s') are two isotopic stated tangles, then $\underline{\varphi}(T, s)$ and $\underline{\varphi}(T', s')$ are no longer necessarily isotopic. Indeed, recall that in our definition of isotopy, for any boundary arc d, the height order of $\partial_d T$ should be preserved. Now if we choose T and T' isotopic in $\Sigma_{\mathcal{P}\cup\{p_0\}} \times (0,1)$, the isotopy relating T to T' preserves the height orders of $\partial_b T$ and $\partial_c T$, but not necessarily the height order of $\partial_a T$, so $\underline{\varphi}(T, s)$ and $\underline{\varphi}(T', s')$ might not be isotopic.

Even worse, T might have two endpoints in $\partial_b T$ and $\partial_c T$ with the same height, so $\iota(T)$ is not a tangle in our sense since it would have two points in $\partial_a \iota(T)$ with the same height.

To remedy this problem, we introduce the subset $\mathcal{T}^0(\underline{\Sigma}') \subset \mathcal{T}(\underline{\Sigma}')$ of stated tangles (T, s) in $\Sigma_{\mathcal{P}\cup\{p_0\}}$ such that for any two points $v \in \partial_b(T)$ and $v' \in \partial_c(T)$, one has $h(v) < h(v')$ (h is the height function). Since any stated tangle $(T, s) \in \mathcal{T}(\underline{\Sigma}')$ is isotopic to a stated tangle in $\mathcal{T}^0(\underline{\Sigma}')$, one has

$$\mathcal{S}_\omega(\underline{\Sigma}') = \mathcal{R}[\mathcal{T}^0(\underline{\Sigma}')] \,/\, \mathcal{J}(\underline{\Sigma}') \cap \mathcal{R}[\mathcal{T}^0(\underline{\Sigma}')].$$

Now, the restriction $\underline{\varphi}^0\colon \mathcal{R}[\mathcal{T}^0(\underline{\Sigma}')] \to \mathcal{R}[\mathcal{T}(\underline{\Sigma})]$ of $\underline{\varphi}^0$ preserves skein relations and (T,s) is isotopic to (T',s') implies that $\underline{\varphi}^0(T,s)$ is isotopic to $\underline{\varphi}^0(T',s')$; therefore φ^0 induces a linear map $\varphi\colon \mathcal{S}_\omega(\underline{\Sigma}') \to \mathcal{S}_\omega(\underline{\Sigma})$ which is obviously an algebra morphism and is surjective.

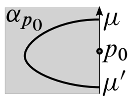
Definition 2.17 The *off-puncture ideal* $\mathcal{I}_{p_0} \subset \mathcal{S}_\omega(\underline{\Sigma}')$ is the ideal generated by

(1) the elements $\gamma - \gamma'$, where γ and γ' are noncontractible simple closed curves in $\Sigma_{\mathcal{P}\cup\{p_0\}}$ which are isotopic in $\Sigma_{\mathcal{P}}$;

(2) the elements $\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}$, where $\alpha_{\varepsilon\varepsilon'}$ and $\alpha'_{\varepsilon\varepsilon'}$ are nontrivial simple stated arcs in $\Sigma_{\mathcal{P}\cup\{p_0\}}$ which are isotopic in $\Sigma_{\mathcal{P}}$;

(3) when p_0 is an inner puncture, the element $\gamma_{p_0} + q + q^{-1}$, where γ_{p_0} is a peripheral curve encircling p_0 (recall that $q = \omega^{-4}$);

(4) when p_0 is on the boundary of $\Sigma_{\mathcal{P}}$, the elements $\alpha_{p_0\,\mu\mu'} - C^{\mu}_{\mu'}$, where α_{p_0} is the trivial arc encircling p_0 as

$\alpha_{p_0\,\mu\mu'} =$ 

such that the endpoint with state μ has bigger height than the endpoint with state μ'.

The purpose of this subsection it to prove:

Proposition 2.18 *The following sequence is exact*:

$$0 \to \mathcal{I}_{p_0} \to \mathcal{S}_\omega(\underline{\Sigma}') \xrightarrow{\varphi} \mathcal{S}_\omega(\underline{\Sigma}) \to 0. \tag{8}$$

The surjectivity of φ follows from the preceding discussion and the inclusion $\mathcal{I}_{p_0} \subset \ker(\varphi)$ is an immediate consequence of the definitions and the trivial arcs relations (7) (where the equalities $\varphi(\alpha_{p_0\,\mu\mu'}) = C^{\mu}_{\mu'}$ are proved), so we need to prove the inclusion $\ker(\varphi) \subset \mathcal{I}_{p_0}$.

Notation 2.19

- Let (D,s) be a connected simple stated diagram in $\Sigma_{\mathcal{P}\cup\{p_0\}}$ (so either a closed curve or a stated arc or the empty diagram) and define a scalar $c(D,s) \in \mathcal{R}$ as follows. If $\iota(D)$ is simple in $\Sigma_{\mathcal{P}}$, set $c(D,s) = 1$. If p_0 is an inner puncture and $(D,s) = \gamma_{p_0}$ is a peripheral curve around p_0, write $c(\gamma_{p_0}) = -q - q^{-1}$. If p_0 is on the boundary of $\Sigma_{\mathcal{P}}$ and $\iota(D)$ is a trivial arc encircling p_0, let $c(D,s)$ be the unique element $C^{\mu}_{\mu'}$ or $(C^{-1})^{\mu}_{\mu'}$ such that $\varphi(D,s) = c(D,s)$ (using the trivial arcs relations (7)).
- For a not necessarily connected stated diagram $(D,s) = \bigsqcup_{i\in I}(D_i,s_i)$, where the (D_i,s_i) are its connected components, write $c(D,s) = \prod_{i\in I} c(D_i,s_i)$. Let $J \subset I$ be the subset of indices $j \in I$ such that $\iota(D_j)$ is simple. The *reduction of* D is the simple diagram $D^{\mathrm{red}} := \bigsqcup_{j\in J} D_j$. By definition, one has

$$\varphi\big([T(D),s]\big) = c(D,s)\varphi\big([(T(D^{\mathrm{red}}),s)]\big). \tag{9}$$

Lemma 2.20 *Let M and M' be two free $\mathcal{R}$–modules with respective bases $\mathcal{B}$ and $\mathcal{B}'$. Let $\pi:\mathcal{B}'\to\mathcal{B}$ and $c:\mathcal{B}'\to\mathcal{R}$ two maps and suppose that there exists $\mathcal{B}'^{\mathrm{red}}\subset\mathcal{B}'$ such that the restriction $\pi|_{\mathcal{B}'^{\mathrm{red}}}\mathcal{B}'^{\mathrm{red}}\to\mathcal{B}$ is surjective and such that $c(b'^{\mathrm{red}})=1$ for all $b'^{\mathrm{red}}\in\mathcal{B}'^{\mathrm{red}}$. Consider the linear morphism $\varphi\colon M'\to M$ defined by $\varphi(b'):=c(b')\pi(b')$, for $b'\in\mathcal{B}'$. Then*

$$\ker(\varphi)=\mathrm{Span}\{b'-c(b')b'^{\mathrm{red}} \text{ such that } \pi(b'^{\mathrm{red}})=\pi(b'),\, b'^{\mathrm{red}}\in\mathcal{B}'^{\mathrm{red}},\, b'\in\mathcal{B}'\}.$$

Proof Let $V\subset M'$ be the submodule linearly spanned by the elements $b'-c(b')b'^{\mathrm{red}}$ with $\pi(b'^{\mathrm{red}})=\pi(b')$ for $b'^{\mathrm{red}}\in\mathcal{B}'^{\mathrm{red}}$ and $b'\in\mathcal{B}'$. By definition, $\varphi(b'-c(b')b'^{\mathrm{red}})=c(b')(\pi(b')-\pi(b'^{\mathrm{red}}))=0$ so the inclusion $V\subset\ker(\varphi)$ is obvious. Conversely, consider an arbitrary element $x=\sum_{b'\in\mathcal{B}'}\alpha_{b'}b'\in\ker(\varphi)$. Fix a right inverse $\iota\colon\mathcal{B}\to\mathcal{B}'^{\mathrm{red}}$ to π; that is a map such that $\pi\circ\iota=\mathrm{id}$. For $b\in\mathcal{B}$, write $x_b:=\sum_{b'\in\pi^{-1}(b)}\alpha_{b'}b'$ so that $x=\sum_{b\in\mathcal{B}}x_b$. Since $\mathcal{B}$ is a basis, the elements $\varphi(x_b)$ are linearly independent so $\varphi(x)=0$ implies that $\varphi(x_b)=0$ for all $b\in\mathcal{B}$. Let $b\in\mathcal{B}$ be such that $x_b\neq 0$ and let us prove that $x_b\in V$. Let $b'^{\mathrm{red}}:=\iota(b)\in\mathcal{B}'^{\mathrm{red}}$. Since $\varphi(x_b)=0$, one has $\sum_{b'\in\pi^{-1}(b)}\alpha_{b'}c(b')=0$. Now

$$\begin{aligned}x_b=\sum_{b'\in\pi^{-1}(b)}\alpha_{b'}b'&=\sum_{b'\in\pi^{-1}(b)}\alpha_{b'}(b'-c(b')b'^{\mathrm{red}})+\Bigl(\sum_{b'\in\pi^{-1}(b)}\alpha_{b'}c(b')\Bigr)b'^{\mathrm{red}}\\&=\sum_{b'\in\pi^{-1}(b)}\alpha_{b'}(b'-c(b')b'^{\mathrm{red}})\in V.\end{aligned}$$ □

Proof of Proposition 2.18 Applying Lemma 2.20 to $M=\mathcal{S}_\omega(\underline{\Sigma})$, $M'=\mathcal{S}_\omega(\underline{\Sigma}')$, $\mathcal{B}=\mathcal{T}\mathcal{B}^{\mathfrak{o}}(\underline{\Sigma})$, $\mathcal{B}'=\mathcal{T}\mathcal{B}^{\mathfrak{o}}(\underline{\Sigma}')$ and $\mathcal{B}'^{\mathrm{red}}$ the subset of $\mathcal{B}'$ of diagrams $(T(D),s)$ such that $D^{\mathrm{red}}=D$ and π the reduction map, we obtain that $\ker(\varphi)$ is spanned by elements of the form $[T(D),s]-c(D,s)[T(D^{\mathrm{red}}),s]$. By definition, the off-puncture ideal is the ideal generated by the elements $[T(D),s]-c(D,s)[T(D^{\mathrm{red}}),s]$, where D is connected. Let us prove by induction on the number of connected components of D that $[T(D),s]-c(D,s)[T(D^{\mathrm{red}}),s]\in\mathcal{I}_{p_0}$. If D is connected or reduced, this is immediate. Otherwise, (D,s) contains a connected component (D_0,s_0) such that $\iota(D_0)$ is either contractible or a trivial arc. Decompose $(D,s)=(D_1,s_1)\sqcup(D_0,s_0)\sqcup(D_2,s_2)$ so that for any connected component $C_1\subset D_1$, one has $C_1\preceq D_0$ and for any connected component $C_2\subset D_2$ one has $D_0\preceq C_2$ (recall that $\preceq$ was defined in Section 2.2). By definition, $[T(D),s]=[T(D_2),s_2][T(D_0),s_0][T(D_1),s_1]$ in $\mathcal{S}_\omega(\underline{\Sigma}')$ (this is where working with the basis $\mathcal{T}\mathcal{B}^{\mathfrak{o}}$ is important), where s_i are the restriction of s to D_i. Therefore

$$\begin{aligned}&[T(D),s]-c(D,s)[T(D^{\mathrm{red}}),s]\\&\quad=[T(D_2),s_2]\bigl([T(D_0),s_0]-c(D_0,s_0)\bigr)[T(D_1),s_1]\\&\qquad\qquad+c(D_0,s_0)\bigl([T(D_2\cup D_1),s_2\cup s_1]-c(D_2\cup D_1,s_2\cup s_1)[T((D_2\cup D_1)^{\mathrm{red}}),s]\bigr)\\&\quad\equiv c\bigl([T(D',s')]-c(D',s')[T(D'^{\mathrm{red}}),s]\bigr)\pmod{\mathcal{I}_{p_0}},\end{aligned}$$

where $c=c(D_0,s_0)$ and $D'=D_2\cup D_1$ has one connected component less than D, so we can apply the induction hypothesis to prove that $[T(D),s]-c(D,s)[T(D^{\mathrm{red}}),s]\in\mathcal{I}_{p_0}$. □

2.4 Hopf comodule maps

Recall that the bigon $\mathbb{B}$ is a disc with two punctures on its boundary. It has two boundary arcs, say b_L and b_R. Consider the simple diagram α made of a single arc joining b_L and b_R. For $n \geq 0$, denote by $\alpha^{(n)}$ the diagram made of n parallel copies of α. Denote by $\alpha_{\varepsilon\varepsilon'}$ the class in $\mathcal{S}_\omega(\mathbb{B})$ of the stated diagram (α, s) where $s(\alpha \cap b_L) = \varepsilon$ and $s(\alpha \cap b_R) = \varepsilon'$. It is proved in [Lê 2018, Theorem 4.1] that the stated skein algebra $\mathcal{S}_\omega(\mathbb{B})$ is presented by the four generators $\alpha_{\varepsilon\varepsilon'}$, with $\varepsilon, \varepsilon' = \pm$, and the following relations, where we put $q := \omega^{-4}$:

$$\begin{gathered} \alpha_{++}\alpha_{+-} = q^{-1}\alpha_{+-}\alpha_{++}, \quad \alpha_{++}\alpha_{-+} = q^{-1}\alpha_{-+}\alpha_{++}, \quad \alpha_{++}\alpha_{--} = 1 + q^{-1}\alpha_{+-}\alpha_{-+}, \\ \alpha_{--}\alpha_{+-} = q\alpha_{+-}\alpha_{--}, \quad \alpha_{--}\alpha_{-+} = q\alpha_{-+}\alpha_{--}, \quad \alpha_{--}\alpha_{++} = 1 + q\alpha_{+-}\alpha_{-+}, \\ \alpha_{-+}\alpha_{+-} = \alpha_{+-}\alpha_{-+}. \end{gathered}$$

Consider a disjoint union $\mathbb{B} \sqcup \mathbb{B}'$ of two bigons. When gluing the boundary arcs b_R with b'_L, we obtain another bigon. Denote by $\Delta\colon \mathcal{S}_\omega(\mathbb{B}) \to \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\mathbb{B})$ the composition

$$\Delta\colon \mathcal{S}_\omega(\mathbb{B}) \xrightarrow{i|_{b_R \# b'_L}} \mathcal{S}_\omega(\mathbb{B} \sqcup \mathbb{B}') \xrightarrow{\cong} \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\mathbb{B}).$$

The map Δ is characterized by the formula $\Delta(\alpha_{\varepsilon\varepsilon'}) = (\alpha_{\varepsilon+} \otimes \alpha_{+\varepsilon'}) + (\alpha_{\varepsilon-} \otimes \alpha_{-\varepsilon'})$. Define an algebra morphism $\epsilon\colon \mathcal{S}_\omega(\mathbb{B}) \to \mathcal{R}$ and an antialgebra morphism (that is S is linear and $S(xy) = S(y)S(x)$) $S\colon \mathcal{S}_\omega(\mathbb{B}) \to \mathcal{S}_\omega(\mathbb{B})$ by the formulas $\epsilon(\alpha_{\varepsilon\varepsilon'}) = \delta_{\varepsilon\varepsilon'}$, $S(\alpha_{++}) = \alpha_{--}$, $S(\alpha_{--}) = \alpha_{++}$, $S(\alpha_{+-}) = -q\alpha_{+-}$ and $S(\alpha_{-+}) = -q^{-1}\alpha_{-+}$. The coproduct Δ, the counit ϵ and the antipode S endow $\mathcal{S}_\omega(\mathbb{B})$ with the structure of a Hopf algebra. This Hopf algebra is canonically isomorphic to the so-called *quantum* SL_2 Hopf algebra $\mathbb{O}_q[\mathrm{SL}_2]$ as defined in [Brown and Goodearl 2002, Definition I.1.10; Chari and Pressley 1994, Definition 7.1.1; Kassel 1995, Chapter IV Section 6; Manin 1988] where the generators α_{++}, α_{-+}, α_{+-} and α_{--} are denoted by a, b, c and d.

For later use, let us write the coproduct, counit and antipode by the more compact form

$$\begin{pmatrix} \Delta(\alpha_{++}) & \Delta(\alpha_{+-}) \\ \Delta(\alpha_{-+}) & \Delta(\alpha_{--}) \end{pmatrix} = \begin{pmatrix} \alpha_{++} & \alpha_{+-} \\ \alpha_{-+} & \alpha_{--} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{++} & \alpha_{+-} \\ \alpha_{-+} & \alpha_{--} \end{pmatrix},$$

$$\begin{pmatrix} \epsilon(\alpha_{++}) & \epsilon(\alpha_{+-}) \\ \epsilon(\alpha_{-+}) & \epsilon(\alpha_{--}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} S(\alpha_{++}) & S(\alpha_{+-}) \\ S(\alpha_{-+}) & S(\alpha_{--}) \end{pmatrix} = \begin{pmatrix} \alpha_{--} & -q\alpha_{+-} \\ -q^{-1}\alpha_{-+} & \alpha_{++} \end{pmatrix}.$$

Note that when $q = +1$, we recover the Hopf algebra of regular functions of $\mathrm{SL}_2(\mathbb{C})$.

Consider a punctured surface $\underline{\Sigma}$ with boundary arc a. When gluing the boundary a of $\underline{\Sigma}$ with the boundary arc b_L of $\mathbb{B}$ we obtain the same punctured surface $\underline{\Sigma}$. Define a left Hopf comodule map (see eg [Kassel 1995, Definition III.7.1]) $\Delta^L_a\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma})$ as the composition

$$\Delta^L_a\colon \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{i|_{a\# b_L}} \mathcal{S}_\omega(\mathbb{B} \sqcup \underline{\Sigma}) \xrightarrow{\cong} \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma}).$$

Similarly, define a right Hopf comodule map $\Delta^R_a\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathcal{S}_\omega(\underline{\Sigma}) \otimes \mathcal{S}_\omega(\mathbb{B})$ as the composition

$$\Delta^R_a\colon \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{i|_{b_R\# a}} \mathcal{S}_\omega(\underline{\Sigma} \sqcup \mathbb{B}) \xrightarrow{\cong} \mathcal{S}_\omega(\underline{\Sigma}) \otimes \mathcal{S}_\omega(\mathbb{B}).$$

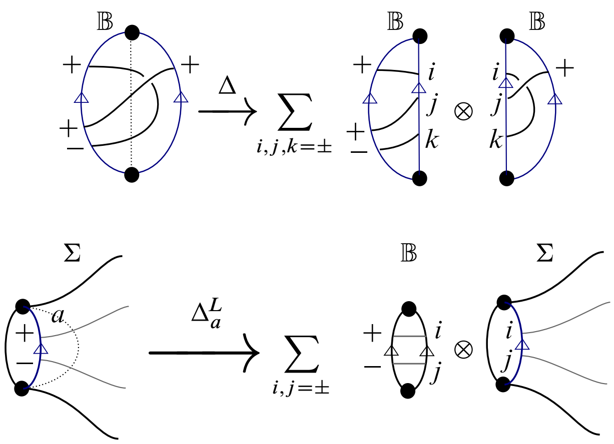


Figure 2: Top: the coproduct in $\mathcal{S}_\omega(\mathbb{B})$. Bottom: the comodule map.

The coassociativity of Δ_a^L and Δ_a^R follows from the coassociativity of the splitting morphisms. Figure 2 illustrates the coproduct and the (left) comodule map.

2.5 The image of the splitting morphism

The goal of this subsection is to prove Theorem 1.1 that we rewrite here for convenience of the reader:

Theorem 2.21 *Let $\underline{\Sigma}$ be a punctured surface, and a and b two distinct boundary arcs. Then the sequence*

$$0 \to \mathcal{S}_\omega(\underline{\Sigma}|_{a\#b}) \xrightarrow{i|_{a\#b}} \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{\Delta_a^L - \sigma\circ\Delta_b^R} \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma})$$

is exact, where $\sigma(x \otimes y) = y \otimes x$.

Throughout this subsection, we fix an orientation $\mathfrak{o}$ of its boundary arcs (though Theorem 2.21 is obviously independent of this choice).

Notation 2.22 For a boundary arc a and a diagram D, we write $n_a(D) := |\partial_a D|$. Given $n \geq 1$, define the set $\mathrm{St}(n) := \{-,+\}^n$ and the subset $\mathrm{St}^\uparrow(n) \subset \mathrm{St}(n)$ which consists of n–tuples $(\varepsilon_1, \dots, \varepsilon_n)$ such that $i < j$ implies $\varepsilon_i \leq \varepsilon_j$. If $s = (\varepsilon_1, \dots, \varepsilon_n) \in \mathrm{St}(n)$, denote by $s^\uparrow = (\varepsilon'_1, \dots, \varepsilon'_n) \in \mathrm{St}^\uparrow(n)$ the unique element such that the number of indices i such that $\varepsilon_i = +$ is equal to the number of indices j such that $\varepsilon'_j = +$. Given $s = (\varepsilon_1, \dots, \varepsilon_n) \in \mathrm{St}(n)$, denote by $k(s)$ the number of pairs (i, j) such that $i < j$ and $\varepsilon_i > \varepsilon_j$. For $s \in \mathrm{St}^\uparrow(n)$, let

$$H_s(q) := \sum_{\substack{s' \in \mathrm{St}(n) \\ s'^\uparrow = s}} q^{2k(s')}.$$

Let a and b be two boundary arcs of $\underline{\Sigma}$ and consider the filtration associated to $S := \{a, b\}$ and $\mathfrak{o}$ of Definition 2.11.

Lemma 2.23 *Let (D,s) be an $\mathfrak{o}$–oriented simple stated diagram and consider v_1 and v_2 two points which both belong either to $\partial_a D$ or to $\partial_b D$. Suppose that $v_1 <_{\mathfrak{o}} v_2$ and that there is no $v \in \partial D$ such that $v_1 <_{\mathfrak{o}} v <_{\mathfrak{o}} v_2$. Further assume that $s(v_1) = +$ and $s(v_2) = -$. Let s' be the state of D such that $s'(v_1) = -$, $s'(v_2) = +$ and $s'(v) = s(v)$ if $v \in \partial D \setminus \{v_1, v_2\}$. Then one has $\operatorname{lt}([D,s]) = q \operatorname{lt}([D,s'])$, where the leading term* lt *is defined in Definition 2.11.*

Proof This is a straightforward consequence of the boundary relations (4) and the height exchange formulas (5) and (6). □

Let (D,s) be an $\mathfrak{o}$–oriented simple stated diagram of $\underline{\Sigma}$ and write $s = (s_a, s_0, s_b)$ as in the definition of the gluing map before Theorem 2.7. By Lemma 2.23 we have the equality

$$\operatorname{lt}\big([D,(s_a,s_0,s_b)]\big) = q^{k(s_a)+k(s_b)} \operatorname{lt}\big([D,(s_a^{\uparrow}, s_0, s_b^{\uparrow})]\big).$$

Fix an orientation $\mathfrak{o}_{\mathbb{B}}$ of the left and right boundary arcs of the bigon. Consider the filtration of

$$\mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma}) \cong \mathcal{S}_\omega(\mathbb{B} \sqcup \underline{\Sigma})$$

associated to the set of boundary arcs $S' := \{b_L, b_R, a, b\}$ and the orientations $\mathfrak{o}$ and $\mathfrak{o}_{\mathbb{B}}$, as in Definition 2.11. Given $X' \in \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma})$, we denote by $\operatorname{lt}'(X')$ the associated leading term. By definition of the left comodule map, we have the formula

$$\Delta_a^L\big([D,(s_a,s_0,s_b)]\big) = \sum_{s \in \mathrm{St}(n_a(D))} [\alpha^{(n_a(D))}, (s_a, s)] \otimes [D, (s, s_0, s_b)].$$

Lemma 2.24 *Let $[D,(s_a,s_0,s_b)]$ be an element of the basis $\mathcal{B}^{\mathfrak{o}}$. Then*

$$\operatorname{lt}'\big(\Delta_a^L\big([D,(s_a,s_0,s_b)]\big)\big) = \sum_{s \in \mathrm{St}^{\uparrow}(n_a(D))} H_s(q) [\alpha^{(|\partial_a(D)|)}, (s_a, s)] \otimes [D, (s, s_0, s_b)],$$

$$\operatorname{lt}'\big(\sigma \circ \Delta_b^R\big([D,(s_a,s_0,s_b)]\big)\big) = \sum_{s \in \mathrm{St}^{\uparrow}(n_b(D))} H_s(q) [\alpha^{(|\partial_b(D)|)}, (s, s_b)] \otimes [D, (s_a, s_0, s)],$$

where the summands are written in the basis associated to $(\mathfrak{o}, \mathfrak{o}_{\mathbb{B}})$ of $\mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma})$.

Proof This is a straightforward consequence of Lemma 2.23. □

Proof of Theorems 1.1 and 2.21 We want to show that the sequence

$$0 \to \mathcal{S}_\omega(\underline{\Sigma}|_{a\#b}) \xrightarrow{i|_{a\#b}} \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{\Delta_a^L - \sigma \circ \Delta_b^R} \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma})$$

is exact, where $\sigma(x \otimes y) = y \otimes x$. The injectivity of $i|_{a\#b}$ was proved in [Lê 2018]. The inclusion $\operatorname{Im}(i|_{a\#b}) \subset \ker(\Delta_a^L - \sigma \circ \Delta_b^R)$ follows from the coassociativity of the comodule maps. To prove the reverse inclusion, consider an element $X := \sum_{i \in I} x_i [D_i, s_i] \in \ker(\Delta_a^L - \sigma \circ \Delta_b^R)$ developed in the basis $\mathcal{B}^{\mathfrak{o}}$. If $\operatorname{lt}(X) = 0$, then X is a linear combination of diagrams which do not intersect a and b; hence X belongs to the image of $i|_{a\#b}$. Suppose that $\operatorname{lt}(X) > 0$. We will find an element $Y \in \mathcal{S}_\omega(\underline{\Sigma}|_{a\#b})$ such that $\operatorname{lt}(i|_{a\#b}(Y)) = \operatorname{lt}(X)$. Now X belongs to the image of $i|_{a\#b}$ if and only if $Z := X - i|_{a\#b}(Y)$ belongs to this image. Since $d(Z) < d(X)$, the proof will follow by induction on $d(X)$.

Consider the set $\widetilde{\mathfrak{D}}$ of pairs (D, s_0) for which there exists some states s_a and s_b such that the basis element $[D, (s_a, s_0, s_b)]$ appears in the expression of X. Given $\widetilde{D} = (D, s_0) \in \widetilde{\mathfrak{D}}$, denote by $\mathrm{St}_X(\widetilde{D})$ the set of couples (s_a, s_b) such that $[D, (s_a, s_0, s_b)]$ appears in the expression of X. We rewrite the development of X in the basis as

$$X = \sum_{\widetilde{D}=(D,s_0)\in\widetilde{\mathfrak{D}}} \sum_{(s_a,s_b)\in \mathrm{St}_X(\widetilde{D})} x_{[D,(s_a,s_0,s_b)]}[D, (s_a, s_0, s_b)].$$

Consider the subset $\widetilde{\mathfrak{D}}_{\max} \subset \widetilde{\mathfrak{D}}$ of pairs (D, s_0) such that $d(X) = n_a(D) + n_b(D)$. By Lemma 2.24,

$$\begin{aligned}
&\mathrm{lt}'(\Delta_a^L(X)) \\
&\quad= \sum_{(D,s_0)\in\widetilde{\mathfrak{D}}_{\max}} \sum_{(s_a,s_b)\in \mathrm{St}_X((D,s_0))} x_{[D,(s_a,s_0,s_b)]} \sum_{s\in \mathrm{St}^{\uparrow}(n_a(D))} H_s(q)[\alpha^{(n_a(D))}, (s_a, s)] \otimes [D, (s, s_0, s_b)],\\
&\mathrm{lt}'(\sigma \circ \Delta_b^R(X)) \\
&\quad= \sum_{(D,s_0)\in\widetilde{\mathfrak{D}}_{\max}} \sum_{(s_a,s_b)\in \mathrm{St}_X((D,s_0))} x_{[D,(s_a,s_0,s_b)]} \sum_{s'\in \mathrm{St}^{\uparrow}(n_b(D))} H_{s'}(q)[\alpha^{(n_b(D))}, (s', s_b)] \otimes [D, (s_a, s_0, s')].
\end{aligned}$$

From the equality $\mathrm{lt}'(\Delta_a^L(X)) = \mathrm{lt}'(\sigma \circ \Delta_b^R(X))$, we find that for any pair $(D, s_0) \in \widetilde{\mathfrak{D}}_{\max}$, for any pair $(s_a, s_b) \in \mathrm{St}_X((D, s_0))$ and for any state $s \in \mathrm{St}^{\uparrow}(n_a(D))$, there exists a unique pair $(s'_a, s'_b) \in \mathrm{St}_X((D, s_0))$ and a unique state $s' \in \mathrm{St}^{\uparrow}(n_b(D))$ such that

$$\begin{aligned}
&x_{[D,(s_a,s_0,s_b)]} H_s(q)[\alpha^{(n_a(D))}, (s_a, s)] \otimes [D, (s, s_0, s_b)]\\
&\qquad = x_{[D,(s'_a,s_0,s'_b)]} H_{s'}(q)[\alpha^{(n_b(D))}, (s', s'_b)] \otimes [D, (s'_a, s_0, s')].
\end{aligned}$$

We deduce the following:

- For any $(D, s_0) \in \widetilde{\mathfrak{D}}_{\max}$, we have $n_a(D) = n_b(D) = \frac{1}{2} d(X)$. We will denote by n this integer.
- We have the equalities $s' = s_a = s_b$ and $s = s'_a = s'_b$. Hence for any $(D, s_0) \in \widetilde{\mathfrak{D}}_{\max}$, we have $\mathrm{St}_X((D, s_0)) = \{(s, s), s \in \mathrm{St}^{\uparrow}(n)\}$.
- For any $(D, s_0) \in \widetilde{\mathfrak{D}}_{\max}$ and $s \in \mathrm{St}^{\uparrow}(n)$, the coefficient $x_{[D,(s,s_0,s)]} H_s(q)$ is independent of s. We will denote this coefficient by $x_{(D,s_0)}$.

With the above notation, we rewrite the leading term of X as

$$\mathrm{lt}(X) = \sum_{(D,s_0)\in\widetilde{\mathfrak{D}}_{\max}} x_{(D,s_0)} \sum_{s\in \mathrm{St}^{\uparrow}(n)} [D, (s, s_0, s)].$$

Given $(D, s_0) \in \widetilde{\mathfrak{D}}_{\max}$, since $n_a(D) = n_b(D) = n$, there exists a diagram D_0 of $\underline{\Sigma}|_{a\#b}$ such that D is obtained from D_0 by cutting along the common image in $\Sigma|_{a\#b}$ of a and b by the projection. Define the element

$$Y := \sum_{(D,s_0)\in\widetilde{\mathfrak{D}}_{\max}} x_{(D,s_0)}[D_0, s_0] \in \mathcal{S}_\omega(\underline{\Sigma}).$$

By the above expression, $\mathrm{lt}(X) = \mathrm{lt}(i|_{a\#b}(Y))$. $\square$

Consider a topological triangulation Δ of Σ. The punctured surface $\underline{\Sigma}$ is obtained from the disjoint union $\Sigma_\Delta := \bigsqcup_{\mathbb{T}\in F(\Delta)} \mathbb{T}$ by gluing the triangles along their common edges. Denote by $\mathring{\mathcal{E}}(\Delta) \subset \mathcal{E}(\Delta)$ the subset of edges which are not boundary arcs. Each edge $e \in \mathring{\mathcal{E}}(\Delta)$ lifts in Σ_Δ to two boundary arcs e_L and e_R. By composing all the left comodule maps $\Delta^L_{e_L}$ together (the order does not matter thanks to the coassociativity property in Theorem 2.7) one gets a Hopf comodule map

$$\Delta^L\colon \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_\omega(\mathbb{T}) \to \Big(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)} \mathcal{S}_\omega(\mathbb{B})\Big) \otimes \Big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_\omega(\mathbb{T})\Big).$$

Similarly, composing all the right comodule maps $\Delta^R_{e_R}$ together gives

$$\Delta^R\colon \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_\omega(\mathbb{T}) \to \Big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_\omega(\mathbb{T})\Big) \otimes \Big(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)} \mathcal{S}_\omega(\mathbb{B})\Big).$$

Recall the definition of i^Δ in Section 2.1.

Corollary 2.25 *The following sequence is exact*:

$$0 \to \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{i^\Delta} \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_\omega(\mathbb{T}) \xrightarrow{\Delta^L - \sigma\circ\Delta^R} \Big(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)} \mathcal{S}_\omega(\mathbb{B})\Big) \otimes \Big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_\omega(\mathbb{T})\Big).$$

Proof Theorem 1.1 applied to each inner edge provides an isomorphism between $\mathcal{S}_\omega(\underline{\Sigma})$ and the intersection, over the inner edges e, of $\mathrm{Ker}(\Delta^L_{e_L} - \sigma\circ\Delta^R_{e_R})$. We conclude by observing that the latter intersection is $\mathrm{Ker}(\Delta^L - \sigma\circ\Delta^R)$. □

We can reformulate the above exact sequence in terms of co-Hochschild cohomology.

Definition 2.26 Given a coalgebra C with a bicomodule M, with comodules maps $\Delta^L\colon M \to C\otimes M$ and $\Delta^R\colon M \to M\otimes C$, the 0^{th} co-Hochschild cohomology group is $\mathrm{coHH}^0(C, M) := \ker(\Delta^L - \sigma\circ\Delta^R)$.

We refer to [Hess et al. 2009] for a self-contained introduction to co-Hochschild (co)homology. The above triangular decomposition of skein algebra can be rewritten as

$$\mathcal{S}_\omega(\underline{\Sigma}) \cong \mathrm{coHH}^0\Big(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)} \mathcal{O}_q[\mathrm{SL}_2], \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_\omega(\mathbb{T})\Big).$$

2.6 The center of stated skein algebras at odd roots of unity

Here we prove Theorem 1.2. We prove it for the bigon, then for the triangle, and we conclude with the general case. Let us start by the following classical result.

Lemma 2.27 *Let $\mathcal{R}$ be a ring and $q \in \mathcal{R}^\times$ a root of unity of order $N > 1$. Suppose that $\mathcal{A}$ is an $\mathcal{R}$–algebra and $x, y \in \mathcal{A}$ are such that $yx = qxy$. Then $(x+y)^N = x^N + y^N$.*

Proof By [Kassel 1995, Proposition IV.2.2],

$$(x+y)^N = \sum_{k=0}^{N} \binom{N}{k}_q x^k y^{N-k},$$

where

$$\binom{N}{k}_q := \prod_{i=0}^{k-1} \left(\frac{1-q^{N-i}}{1-q^{i+1}} \right).$$

Since $q^N = 1$, the coefficients $\binom{N}{k}_q$ vanish for $1 \le k \le N-1$, and we get the desired formula. □

2.6.1 The case of the bigon Recall from Section 2.2 that the Hopf algebra $\mathcal{S}_\omega(\mathbb{B})$ is canonically isomorphic to $\mathbb{O}_q[\mathrm{SL}_2]$. In this case, Theorem 1.2 is a well-known theorem of Lusztig. More precisely, it is proved in [Lusztig 1990] (see also [Lusztig 1993, Theorem 3.5.1]) that there exists a morphism of braided Hopf algebras $\mathrm{Fr}_* : \dot{U}_q\mathfrak{sl}_2 \to \dot{U}_{+1}\mathfrak{sl}_2$ which induces a braided functor $\mathrm{Fr} : \mathrm{Rep}(\mathrm{SL}_2) \to \mathrm{Rep}_q(\mathrm{SL}_2)$ between the category of finite-rank representations of SL_2 and the category $\mathrm{Rep}_q(\mathrm{SL}_2)$ of finite-rank $\dot{U}_q\mathfrak{sl}_2$ modules. Since $\mathbb{O}_q[\mathrm{SL}_2]$ (resp. $\mathbb{O}[\mathrm{SL}_2]$) is isomorphic to the coend of the forgetful functor $F : \mathrm{Rep}_q(\mathrm{SL}_2) \to \mathrm{Mod}_{\mathcal{R}}$ (resp. of the forgetful functor $\mathrm{Rep}(\mathrm{SL}_2) \to \mathrm{Mod}_{\mathcal{R}}$) the Frobenius functor Fr induces a morphism $j : \mathbb{O}[\mathrm{SL}_2] \to \mathbb{O}_q[\mathrm{SL}_2]$. Moreover, as noticed in [Negron 2021], the image of Fr lies in the Mügen center of $\mathrm{Rep}_q(\mathrm{SL}_2)$ so the image of j is central. We refer to [Negron 2021, Section 5.1] for details on this approach. A down-to-earth construction of j, based on elementary computations using the definition of $\mathbb{O}_q[\mathrm{SL}_2]$ by generators and relations, was described by Brown and Goodearl and goes as follows:

Lemma 2.28 [Brown and Goodearl 2002, Proposition III.3.1] *Suppose that $q := \omega^{-4}$ is a root of unity of odd order $N \ge 1$. There exists a injective morphism of Hopf algebras $j_{\mathbb{B}} : \mathcal{S}_{+1}(\mathbb{B}) \to \mathcal{S}_\omega(\mathbb{B})$ characterized by $j_{\mathbb{B}}(\alpha_{\varepsilon\varepsilon'}) := (\alpha_{\varepsilon\varepsilon'})^N$ whose image lies in the center of $\mathcal{S}_\omega(\mathbb{B})$.*

2.6.2 The case of the triangle Denote by α, β and γ the three arcs of Figure 3 and τ the automorphism of $\mathcal{S}_\omega(\mathbb{T})$ induced by the rotation sending α, β and γ to β, γ and α, respectively. In [Lê 2018, Theorem 4.6], it was proved that the stated skein algebra $\mathcal{S}_\omega(\mathbb{T})$ is presented by the generators $\alpha_{\varepsilon\varepsilon'}$, $\beta_{\varepsilon\varepsilon'}$ and $\gamma_{\varepsilon\varepsilon'}$, and the following relations together with their images through τ and τ^2:

$$\alpha_{-\varepsilon}\alpha_{+\varepsilon'} = A^2\alpha_{+\varepsilon}\alpha_{-\varepsilon'} - \omega^{-5}C^{\varepsilon}_{\varepsilon'}, \tag{10}$$

$$\alpha_{\varepsilon-}\alpha_{\varepsilon'+} = A^2\alpha_{\varepsilon+}\alpha_{\varepsilon'-} - \omega^{-5}C^{\varepsilon}_{\varepsilon'}, \tag{11}$$

$$\beta_{\mu\varepsilon}\alpha_{\mu'\varepsilon'} = A\alpha_{\varepsilon\varepsilon'}\beta_{\mu\mu'} - A^2C^{\varepsilon}_{\mu'}\gamma_{\varepsilon'\mu}, \tag{12}$$

$$\alpha_{-\varepsilon}\beta_{\varepsilon'+} = A^2\alpha_{+\varepsilon}\beta_{\varepsilon'-} - \omega^{-5}\gamma_{\varepsilon\varepsilon'}, \tag{13}$$

$$\alpha_{\varepsilon-}\gamma_{+\varepsilon'} = A^2\alpha_{\varepsilon+}\gamma_{-\varepsilon'} + \omega\beta_{\varepsilon'\varepsilon}. \tag{14}$$

Here we use the notation $A := \omega^{-2}$, $C^-_- = C^+_+ := 0$, $C^-_+ := -\omega^5$ and $C^+_- := \omega$.

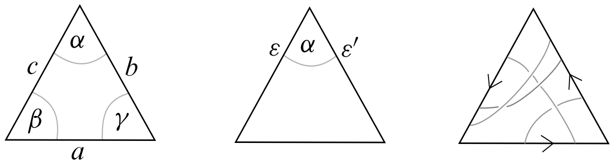


Figure 3: Left: the three diagrams α, β and γ. Middle: the stated diagram representing $\alpha_{\varepsilon\varepsilon'}$. Right: the diagram $\theta^{(2,1,1)}$.

When $\omega = +1$, the algebra $\mathcal{S}_{+1}(\mathbb{T})$ has the following simpler presentation. Consider the commutative unital polynomial algebra $\mathcal{A} := \mathcal{R}[\alpha_{\varepsilon\varepsilon'}, \beta_{\varepsilon\varepsilon'}, \gamma_{\varepsilon\varepsilon'} | \varepsilon, \varepsilon' = \pm]$. Given $\delta \in \{\alpha, \beta, \gamma\}$, denote by M_δ the 2×2 matrix with coefficients in $\mathcal{A}$ defined by

$$M_\delta := \begin{pmatrix} \delta_{++} & \delta_{+-} \\ \delta_{-+} & \delta_{--} \end{pmatrix}$$

and write $C := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mathbb{1} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Lemma 2.29 *The algebra $\mathcal{S}_{+1}(\mathbb{T})$ is isomorphic to*

$$\mathcal{R}[\alpha_{\varepsilon\varepsilon'}, \beta_{\varepsilon\varepsilon'}, \gamma_{\varepsilon\varepsilon'} \mid \varepsilon, \varepsilon' = \pm]/(\det(M_\alpha) = \det(M_\beta) = \det(M_\gamma) = 1,\ M_\gamma C M_\beta C M_\alpha C = \mathbb{1}).$$

Proof That $\mathcal{S}_{+1}(\mathbb{T})$ commutative is a particular case of [Lê 2018, Corollary 2.5]. After setting $\omega = +1$ we see that (10) and (11) coincide; (14) is the image of (13) by rotation, and the latter is a particular case of (12). Moreover, a direct inspection shows that the other part of (10) and of (12) correspond to $\det(M_\alpha) = 1$ and $(M_\gamma C)^{-1} = M_\beta C M_\alpha C$, respectively. □

Lemma 2.30 *Suppose that ω is a root of unity of odd order $N \geq 1$. For every $\varepsilon, \varepsilon', \mu, \mu' \in \{-,+\}$ with $\varepsilon \neq \mu'$, one has*

$$\alpha^N_{\mu'\varepsilon'} \beta^N_{\mu\varepsilon} - \alpha^N_{\varepsilon\varepsilon'} \beta^N_{\mu\mu'} = \gamma^N_{\varepsilon',\mu}.$$

Proof We suppose that $(\varepsilon, \mu') = (-,+)$. The proof in the case where $(\varepsilon, \mu') = (+,-)$ is similar and left to the reader. For $n \geq 0$, let D_n be the simple diagram made of n parallel copies of α and n parallel copies of β and consider the orientation $\mathfrak{o}$ depicted in Figure 4. For $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n) \in \{-,+\}^n$ let $\boldsymbol{\eta}^\vee := \{-\eta_n, \dots, -\eta_1\}$. For $\boldsymbol{\eta}, \boldsymbol{\eta}' \in \{-,+\}^n$, let $s_{\boldsymbol{\eta},\boldsymbol{\eta}'}$ be the state of D_n sending all points of $\partial_b D_n$ to ε', all points of $\partial_a D_n$ to μ and the points $(p_1, \dots, p_n, p'_1, \dots, p'_n)$ of $\partial_c D_n$ ordered by $\mathfrak{o}$, to the states $(\eta_1, \dots, \eta_n, \eta'_1, \dots, \eta'_n)$. Write $X_{\boldsymbol{\eta},\boldsymbol{\eta}'} := [D_n, s_{\boldsymbol{\eta},\boldsymbol{\eta}'}]$.

Using the skein relations (4), as illustrated in Figure 4, we find that

$$X_{\boldsymbol{\eta},\boldsymbol{\eta}'} \gamma_{\varepsilon',\mu} = \omega^{-1} X_{(\boldsymbol{\eta},+),(-,\boldsymbol{\eta}')} - \omega^{-5} X_{(\boldsymbol{\eta},-),(+,\boldsymbol{\eta}')}, \tag{15}$$

where $(\boldsymbol{\eta}, +) := (\eta_1, \dots, \eta_n, +)$ and $(-, \boldsymbol{\eta}') := (-, \eta'_1, \dots, \eta'_n)$. Let $n_+(\eta)$ be the number of indices $i \in \{1, \dots, n\}$ such that $\eta_i = +$. Using (15), we prove by induction of n that

$$(\gamma_{\varepsilon'\mu})^n = \sum_{\eta \in \{-,+\}^n} (\omega^{-1})^{n_+(\eta)} (-\omega^{-5})^{n-n_+(\eta)} X_{\eta,\eta^\vee}. \tag{16}$$

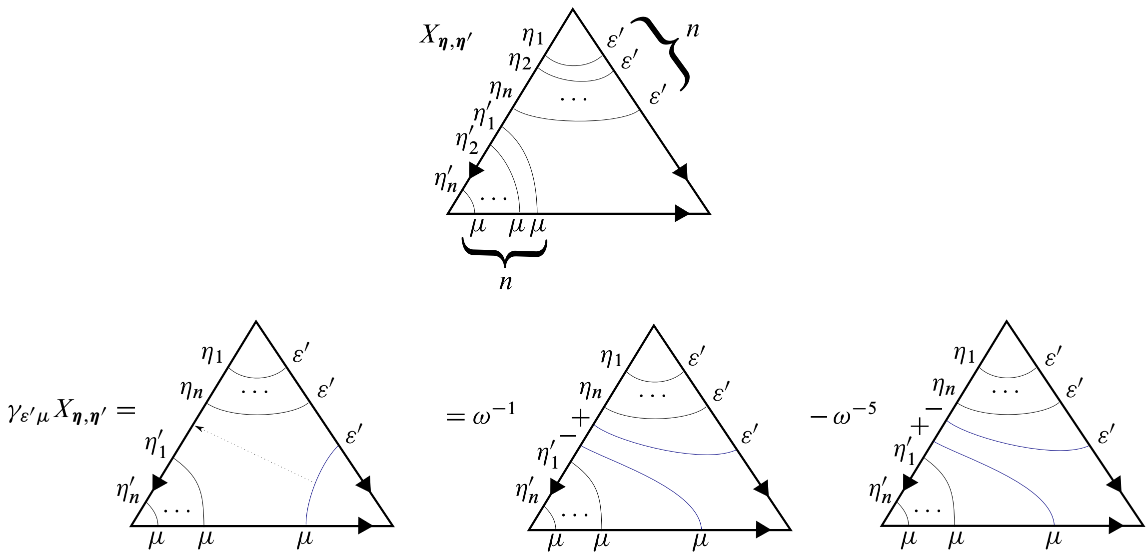


Figure 4: Top: the element $X_{\boldsymbol{\eta},\boldsymbol{\eta}'}$. Bottom: an illustration of (15).

Let $m(\eta) := \#\{1 \le i < j \le n \mid (\eta_i, \eta_j) = (+,-)\}$ and denote by η_+ the unique element of $\{-,+\}^n$ such that $n_+(\eta) = n_+(\eta_+)$ and $m(\eta_+) = 0$. Note that $m(\eta) = m(\eta^\vee)$. Using the skein relation (4), we find that for any $\eta, \eta' \in \{-,+\}^n$,

$$X_{\eta,\eta'} = q^{m(\eta)+m(\eta')} X_{\eta_+,\eta'_+}. \tag{17}$$

For $1 \le k \le N$, let $\eta_+^{(k)} \in \{-,+\}^N$ be the unique element such that $m(\eta_+^{(k)}) = 0$ and $n_+(\eta_+^{(k)}) = k$, ie

$$\eta_{+\,i}^{(k)} = \begin{cases} - & \text{for } 1 \le i \le N-k, \\ + & \text{for } i > N-k. \end{cases}$$

Putting (16) and (17) together, one finds that

$$(\gamma_{\varepsilon'\mu})^N = \sum_{k=0}^{N} (\omega^{-1})^k (-\omega^{-5})^{N-k} \Biggl(\sum_{\substack{\eta \in \{-,+\}^N \\ n_+(\eta)=k}} q^{2m(\eta)} \Biggr) X_{\eta_+^{(k)},\eta_+^{(k)\vee}}.$$

Now, a simple computation shows that

$$\Biggl(\sum_{\substack{\eta \in \{-,+\}^N \\ n_+(\eta)=k}} q^{2m(\eta)} \Biggr) = q^{2nN-n(n-1)} \sum_{1 \le i_1 < i_2 < \cdots < i_n \le N} q^{2(i_1+\cdots+i_n)} = \begin{cases} 1 & \text{if } k=0 \text{ or } k=N, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$(\gamma_{\varepsilon'\mu})^N = X_{\eta_+^{(N)},\eta_-^{(N)}} - X_{\eta_-^{(N)},\eta_+^{(N)}} = \alpha_{+\varepsilon'}^N \beta_{\mu-}^N - \alpha_{-\varepsilon'}^N \beta_{\mu+}^N.$$

Note that we used that $(-1)^N = -1$, so that N is odd. $\square$

Lemma 2.31 *Suppose that ω is a root of unity of odd order $N \geq 1$. There exists an injective morphism of algebras $j_{\mathbb{T}} : \mathcal{S}_{+1}(\mathbb{T}) \to \mathcal{S}_{\omega}(\mathbb{T})$, whose image lies in the center of $\mathcal{S}_{\omega}(\mathbb{T})$, characterized by $j_{\mathbb{T}}(\delta_{\varepsilon\varepsilon'}) := (\delta_{\varepsilon\varepsilon'})^N$ for $\delta \in \{\alpha, \beta, \gamma\}$ and $\varepsilon, \varepsilon' = \pm$. Moreover, if a is a boundary arc of $\mathbb{T}$, the following diagrams commute*:

$$\begin{tikzcd} \mathcal{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta_a^L"] \arrow[d, hook, "j_{\mathbb{T}}"] & \mathcal{S}_{+1}(\mathbb{B}) \otimes \mathcal{S}_{+1}(\mathbb{T}) \arrow[d, hook, "j_{\mathbb{B}} \otimes j_{\mathbb{T}}"] \\ \mathcal{S}_{\omega}(\mathbb{T}) \arrow[r, "\Delta_a^L"] & \mathcal{S}_{\omega}(\mathbb{B}) \otimes \mathcal{S}_{\omega}(\mathbb{T}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta_a^R"] \arrow[d, hook, "j_{\mathbb{T}}"] & \mathcal{S}_{+1}(\mathbb{T}) \otimes \mathcal{S}_{+1}(\mathbb{B}) \arrow[d, hook, "j_{\mathbb{T}} \otimes j_{\mathbb{B}}"] \\ \mathcal{S}_{\omega}(\mathbb{T}) \arrow[r, "\Delta_a^R"] & \mathcal{S}_{\omega}(\mathbb{T}) \otimes \mathcal{S}_{\omega}(\mathbb{B}) \end{tikzcd}$$

Proof We proceed in a similar way to Lemma 2.28, by showing first that the extension of the assignment $j_{\mathbb{T}}(\delta_{\varepsilon\varepsilon'}) := \delta^N_{\varepsilon\varepsilon'}$ to a morphism of algebras is well defined. In virtue of Lemma 2.29 and by the rotation automorphism, it is enough to show that $\alpha^N_{\varepsilon\varepsilon'}$ lies in the center of $\mathcal{S}_{\omega}(\mathbb{T})$ and that $j_{\mathbb{T}}$ sends $\det(M_\alpha) - 1$ and $M_\gamma C M_\beta C M_\alpha C - \mathbb{1}$ to zero.

First note that the relations (10) and (11) put together coincide with the defining relations of $\mathcal{S}_{\omega}(\mathbb{B})$; hence one has an inclusion of algebras $\phi : \mathcal{S}_{\omega}(\mathbb{B}) \hookrightarrow \mathcal{S}_{\omega}(\mathbb{T})$ defined by $\phi(\alpha_{\varepsilon\varepsilon'}) = \alpha_{\varepsilon\varepsilon'}$. By applying Lemma 2.28, one obtains an inclusion $\phi \circ j_{\mathbb{B}} : \mathcal{S}_{+1}(\mathbb{B}) \hookrightarrow \mathcal{S}_{\omega}(\mathbb{T})$ which coincides with $j_{\mathbb{T}}$ on the $\alpha_{\varepsilon\varepsilon'}$'s. It remains to show that the $\alpha^N_{\varepsilon\varepsilon'}$'s commute with the $\beta_{\mu\mu'}$'s and the $\gamma_{\mu\mu'}$'s, and that $j_{\mathbb{T}}$ vanishes on $M_\gamma C M_\beta C M_\alpha C - \mathbb{1}$.

We have $\alpha^N_{\varepsilon\varepsilon'} \beta_{\mu\varepsilon} = A^{-N} \beta_{\mu\varepsilon} \alpha^N_{\varepsilon\varepsilon'} = \beta_{\mu\varepsilon} \alpha^N_{\varepsilon\varepsilon'}$. From

$$\alpha^N_{+\varepsilon} \beta_{\varepsilon'-} = \alpha^{N-1}_{+\varepsilon}(A^{-2}\alpha_{-\varepsilon}\beta_{\varepsilon'+} + \omega^{-1}\gamma_{\varepsilon\varepsilon'}) = (A^{-3N+1}\alpha_{-\varepsilon}\beta_{\varepsilon'+} + \omega^{-1}A^{N-1}\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{+\varepsilon}$$

and

$$\beta_{\varepsilon'-}\alpha^N_{+\varepsilon} = (A\alpha_{-\varepsilon}\beta_{\varepsilon'+} + \omega\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{+\varepsilon},$$

one obtains

$$\alpha^N_{+\varepsilon}\beta_{\varepsilon'-} - \beta_{\varepsilon'-}\alpha^N_{+\varepsilon} = (A(A^{-3N} - 1)\alpha_{-\varepsilon}\beta_{\varepsilon'+} + \omega(A^N - 1)\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{+\varepsilon} = 0.$$

Similarly, we compute

$$\begin{aligned} \alpha^N_{-\varepsilon}\beta_{\varepsilon'+} &= \alpha^{N-1}_{-\varepsilon}(A^2\alpha_{+\varepsilon}\beta_{\varepsilon'-} - \omega^{-5}\gamma_{\varepsilon\varepsilon'}) = (A^{N+1}\alpha_{+\varepsilon}\beta_{\varepsilon'-} - \omega^{-3}A^N\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{-\varepsilon}, \\ \beta_{\varepsilon'+}\alpha^N_{-\varepsilon} &= (A\alpha_{+\varepsilon}\beta_{\varepsilon'-} - \omega^{-3}\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{-\varepsilon}. \end{aligned}$$

Thus we find

$$\alpha^N_{-\varepsilon}\beta_{\varepsilon'+} - \beta_{\varepsilon'+}\alpha^N_{-\varepsilon} = (A(A^N - 1)\alpha_{+\varepsilon}\beta_{\varepsilon'-} - \omega^{-3}(A^N - 1)\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{-\varepsilon} = 0.$$

So we have proven that $\alpha^N_{\varepsilon\varepsilon'}$ commutes with every elements $\beta_{\mu\mu'}$. The commutativity of $\alpha^N_{\varepsilon\varepsilon'}$ with each element $\gamma_{\mu\mu'}$ is shown in a very similar way.

Next, showing that $j_{\mathbb{T}}$ vanishes on $M_\gamma C M_\beta C M_\alpha C - \mathbb{1}$ amounts to showing that

$$\beta^N_{\mu\varepsilon}\alpha^N_{\mu'\varepsilon'} - \alpha^N_{\varepsilon\varepsilon'}\beta^N_{\mu\mu'} = \gamma^N_{\varepsilon',\mu} \quad \text{for } \varepsilon \neq \mu'.$$

This was proved in Lemma 2.30.

Now let us prove that $j_{\mathbb{T}}$ is injective. To this end, let us consider the following basis of $\mathcal{S}_\omega(\mathbb{T})$.

Consider the counterclockwise orientation $\mathfrak{o}$ of the boundary arcs of $\mathbb{T}$ as in Figure 3. Given

$$\boldsymbol{k} = (k_\alpha, k_\beta, k_\gamma) \in (\mathbb{Z}^{\geq 0})^3,$$

denote by $\theta^{\boldsymbol{k}}$ the (not simple) diagram $\alpha^{k_\alpha}\beta^{k_\beta}\gamma^{k_\gamma}$; see Figure 3 for an example. By Proposition 2.14 the set of classes $[\theta^{\boldsymbol{k}}, s]$, where s is $\mathfrak{o}$–increasing, forms a basis of $\mathcal{S}_\omega(\mathbb{T})$.

By construction, $j_{\mathbb{T}}$ sends the elements $[\theta^{\boldsymbol{k}}, s]$ of $\mathcal{S}_{+1}(\mathbb{T})$, where s is $\mathfrak{o}$–increasing, to some basis elements $[\theta^{N\boldsymbol{k}}, s']$, where s' is also $\mathfrak{o}$ increasing, therefore $j_{\mathbb{T}}$ is injective.

It remains to prove that $j_{\mathbb{T}}$ is a morphism of Hopf comodules. To avoid confusion, let us denote by $x_{\varepsilon\varepsilon'}$ the generators of $\mathcal{S}_\omega(\mathbb{B})$ and reserve the notation $\alpha_{\varepsilon\varepsilon'}$ for the element of $\mathcal{S}_\omega(\mathbb{T})$. By definition, we have $\Delta_c^L(\alpha_{\varepsilon\varepsilon'}) = x_{\varepsilon+} \otimes \alpha_{+\varepsilon'} + x_{\varepsilon-} \otimes \alpha_{-\varepsilon'}$. Write $u := x_{\varepsilon+} \otimes \alpha_{+\varepsilon'}$ and $v := x_{\varepsilon-} \otimes \alpha_{-\varepsilon'}$. Since $uv = q^{-2}vu$, by Lemma 2.27 we have $(u+v)^N = u^N + v^N$, so

$$\begin{aligned}\Delta_c^L(j_{\mathbb{B}}(\alpha_{\varepsilon\varepsilon'})) = (\Delta_c^L(\alpha_{\varepsilon\varepsilon'}))^N = (u+v)^N = u^N + v^N \\ = x_{\varepsilon+}^N \otimes \alpha_{+\varepsilon'}^N + x_{\varepsilon-}^N \otimes \alpha_{-\varepsilon'}^N = j_{\mathbb{B}} \otimes j_{\mathbb{T}}(\Delta_c^L(\alpha_{\varepsilon\varepsilon'})).\end{aligned}$$

The proof that $\Delta_b^L(j_{\mathbb{B}}(\alpha_{\varepsilon\varepsilon'})) = j_{\mathbb{B}} \otimes j_{\mathbb{T}}(\Delta_b^L(\alpha_{\varepsilon\varepsilon'}))$ is done using a similar computation and the equality $\Delta_a^L(j_{\mathbb{B}}(\alpha_{\varepsilon\varepsilon'})) = j_{\mathbb{B}} \otimes j_{\mathbb{T}}(\Delta_a^L(\alpha_{\varepsilon\varepsilon'}))$ holds since both sides are equal to $1 \otimes \alpha_{\varepsilon\varepsilon'}^N$. By symmetry in the generators α, β, γ, we have proved that $j_{\mathbb{B}}$ commutes with the left comodule maps. That it commutes with the right comodule maps is proved similarly. □

2.6.3 The general case: proof of Theorem 1.2 We restate Theorem 1.2 here for the convenience of the reader:

Theorem 2.32 *Suppose that ω is a root of unity of odd order $N \geq 1$ and Σ a punctured surface. There exists an embedding*

$$j_{\Sigma}\colon \mathcal{S}_{+1}(\Sigma) \hookrightarrow \mathcal{Z}(\mathcal{S}_\omega(\Sigma))$$

of the (commutative) stated skein algebra with parameter $+1$ into the center of the stated skein algebra with parameter ω. Moreover, the morphism j_Σ is characterized by the property that it sends a closed curve γ to $T_N(\gamma)$ and a stated arc $\alpha_{\varepsilon\varepsilon'}$ to $\alpha_{\varepsilon\varepsilon'}^{(N)}$, where $\alpha_{\varepsilon\varepsilon'}^{(N)}$ is the tangle made by stacking N parallel copies of $\alpha_{\varepsilon\varepsilon'}$ on top of the others.

Recall from Section 2.2 that closed curves and arcs do not have self-intersection points by definition. We divide the proof in five steps.

In Step 1, we show that the decomposition Theorem 1.1 together with the two previous sections provide an injective morphism of algebras

$$j_{(\Sigma,\Delta)}\colon \mathcal{S}_{+1}(\Sigma) \hookrightarrow \mathcal{S}_\omega(\Sigma), \tag{18}$$

which is central. We study further properties of $j_{(\underline{\Sigma},\Delta)}$ and we show that it is does *not* depend on a topological triangulation Δ. The other steps are devoted to making explicit the morphism $j_{(\underline{\Sigma},\Delta)}$ on arcs and loops. In Steps 2–4, we suppose that the punctured surface has a nondegenerated triangulation (see below); in Step 5 we treat the other punctured surfaces.

In Step 2, we prove that $j_{(\underline{\Sigma},\Delta)}$ sends the stated arcs that have their endpoints on *two different* boundary arcs of Σ, to their N^{th} power.

In Step 3, we prove that $j_{(\underline{\Sigma},\Delta)}$ sends some particular closed curves of $\Sigma_{\mathcal{P}}$ to their N^{th} Chebyshev polynomial of first kind.

Step 4 is more involved. We first prove a structural result. Adding a puncture on a surface $\underline{\Sigma}$ gives rise to a surjective map φ from the skein algebra of the new punctured surface to that of the initial one defined in Section 2.3. We show that $j_{(\underline{\Sigma},\Delta)}$ commutes with these surjections (see Lemma 2.40). From this, we deduce the image by $j_{(\underline{\Sigma},\Delta)}$ of stated arcs that have both their endpoints on the *same* boundary arc of Σ and of *any* closed curve of $\Sigma_{\mathcal{P}}$.

In Step 5, we treat the remaining cases of connected punctured surfaces that do not admit a nondegenerate topological triangulation (including those with no puncture). The proof consists, again, in adding a puncture and using the previous study.

These five steps prove Theorem 1.2.

Throughout this section, $\underline{\Sigma}$ is a punctured surface, Δ a topological triangulation $\underline{\Sigma}$ and ω a root of unity of odd order $N \geq 1$. Except for Steps 1 and 5, the triangulation Δ is required to be *nondegenerate*, that is, such that each of its inner edges separates two distinct faces.

Step 1: formal definition Assume that $\underline{\Sigma}$ admits a (possibly degenerate) triangulation Δ. Consider the diagram

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, "i^\Delta"] \arrow[d, dotted, hook, "j_{(\underline{\Sigma},\Delta)}\ \exists!"'] & \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta^L-\sigma\circ\Delta^R"] \arrow[d, hook, "\otimes_{\mathbb{T}} j_{\mathbb{T}}"] & \big(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)} \mathcal{S}_{+1}(\mathbb{B})\big)\otimes\big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_{+1}(\mathbb{T})\big) \arrow[d, hook, "(\otimes_e j_{\mathbb{B}})\otimes(\otimes_{\mathbb{T}} j_{\mathbb{T}})"] \\ 0 \arrow[r] & \mathcal{S}_{\omega}(\underline{\Sigma}) \arrow[r, "i^\Delta"] & \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_{\omega}(\mathbb{T}) \arrow[r, "\Delta^L-\sigma\circ\Delta^R"] & \big(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)} \mathcal{S}_{\omega}(\mathbb{B})\big)\otimes\big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_{\omega}(\mathbb{T})\big) \end{tikzcd} \tag{19}$$

where both lines are exact by Theorem 1.1 and the vertical maps are given by Lemmas 2.28 and 2.31.

The existence of an injective morphism $j_{(\underline{\Sigma},\Delta)} \colon \mathcal{S}_{+1}(\underline{\Sigma}) \hookrightarrow \mathcal{S}_{\omega}(\underline{\Sigma})$ follows from the exactness of the lines and the injectivity of $\bigotimes_{\mathbb{T}\in F(\Delta)} j_{\mathbb{T}}$ (and the fact that all maps involved in the diagram are algebra morphisms). Moreover, since $j_{\mathbb{T}}$ is central, so is $j_{(\underline{\Sigma},\Delta)}$.

Let us show that $j_{(\underline{\Sigma},\Delta)}$ is compatible with the gluing maps.

Lemma 2.33 *If a and b are two boundary arcs of $\underline{\Sigma}$, the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{S}_{+1}(\underline{\Sigma}|_{a\#b}) \arrow[r, hook, "j_{\underline{\Sigma}_{a\#b}}"] \arrow[d, hook, "i|_{a\#b}"] & \mathcal{S}_{\omega}(\underline{\Sigma}|_{a\#b}) \arrow[d, hook, "i|_{a\#b}"] \\
\mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, hook, "j_{\underline{\Sigma}}"] & \mathcal{S}_{\omega}(\underline{\Sigma})
\end{tikzcd}$$

Proof Let $\Delta_{a\#b}$ the topological triangulation of $\underline{\Sigma}|_{a\#b}$ that is induced by Δ. Let us consider the diagram

$$\begin{tikzcd}
\mathcal{S}_{+1}(\underline{\Sigma}|_{a\#b}) \arrow[r, hook, "i|_{a\#b}"] \arrow[d, hook, "j_{(\underline{\Sigma}|_{a\#b},\Delta_{a\#b})}"] \arrow[rr, bend left, "i^{\Delta_{a\#b}}"] & \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, hook, "i^{\Delta}"] \arrow[d, hook, "j_{(\underline{\Sigma},\Delta)}"] & \otimes_{\mathbb{T}} \mathcal{S}_{+1}(\mathbb{T}) \arrow[d, hook, "\otimes_{\mathbb{T}} j_{\mathbb{T}}"] \\
\mathcal{S}_{\omega}(\underline{\Sigma}|_{a\#b}) \arrow[r, hook, "i|_{a\#b}"] \arrow[rr, bend right, "i^{\Delta_{a\#b}}"'] & \mathcal{S}_{\omega}(\underline{\Sigma}) \arrow[r, hook, "i^{\Delta}"] & \otimes_{\mathbb{T}} \mathcal{S}_{\omega}(\mathbb{T})
\end{tikzcd}$$

The outer triangles commute by coassociativity of the gluing maps. Two of the three squares commute by diagram (19). Since i^{Δ} is injective, the remaining (left-hand side) square commutes. □

We now prove that the morphism $j_{(\underline{\Sigma},\Delta)}$ does not depend on Δ. We first need a preliminary result.

Lemma 2.34 *Let Q be a square (ie a disc with four punctures on its boundary) and Δ_Q a topological triangulation of Q. If $\alpha_{\varepsilon\varepsilon'} \in \mathcal{S}_{\omega}(Q)$ is the class of a stated arc, then $j_{(Q,\Delta_Q)}(\alpha_{\varepsilon\varepsilon'}) = \alpha^N_{\varepsilon\varepsilon'}$. In particular, $j_{(Q,\Delta_Q)}$ does not depend on Δ_Q.*

Proof Let e be the inner edge of Δ_Q which is a common boundary arc of two triangles $\mathbb{T}_1$ and $\mathbb{T}_2$. Make the intersection $\alpha \cap e$ transversal and minimal via an isotopy on α. If the intersection is empty, then α is included in one of the triangles and the lemma follows from Lemma 2.31. If $\alpha \cap e$ is not empty, then it has only one element. Therefore, by letting $\alpha^{\mathbb{T}_i} := \alpha \cap \mathbb{T}_i$ for $i = 1, 2$, one has

$$i^{\Delta_Q}(\alpha_{\varepsilon\varepsilon'}) = \alpha^{\mathbb{T}_1}_{\varepsilon+} \otimes \alpha^{\mathbb{T}_2}_{+\varepsilon'} + \alpha^{\mathbb{T}_1}_{\varepsilon-} \otimes \alpha^{\mathbb{T}_2}_{-\varepsilon'}.$$

Write $x := \alpha^{\mathbb{T}_1}_{\varepsilon+} \otimes \alpha^{\mathbb{T}_2}_{+\varepsilon'}$ and $y := \alpha^{\mathbb{T}_1}_{\varepsilon-} \otimes \alpha^{\mathbb{T}_2}_{-\varepsilon'}$ and note that $xy = q^{-2}yx$. By Lemma 2.27,

$$i^{\Delta_Q}(\alpha^N_{\varepsilon\varepsilon'}) = i^{\Delta_Q}(\alpha_{\varepsilon\varepsilon'})^N = (x+y)^N = x^N + y^N = (j_{\mathbb{T}_1} \otimes j_{\mathbb{T}_2}) \circ i^{\Delta_Q}(\alpha_{\varepsilon\varepsilon'}).$$

Hence, $j_{(Q,\Delta_Q)}(\alpha_{\varepsilon\varepsilon'}) = \alpha^N_{\varepsilon\varepsilon'}$. □

Lemma 2.35 *The morphism $j_{(\underline{\Sigma},\Delta)}$ does not depend on Δ.*

Proof Every two triangulations can be related by a finite sequence of flips on the edges. Therefore, it is enough to prove that if Δ' differs from Δ by a flip of one edge, then $j_{(\underline{\Sigma},\Delta)} = j_{(\underline{\Sigma},\Delta')}$.

Let e be an inner edge of Δ that bounds two distinct faces $\mathbb{T}_1$ and $\mathbb{T}_2$. Consider the topological triangulation Δ' obtained from Δ by flipping the edge e inside the square $Q = \mathbb{T}_1 \cup \mathbb{T}_2$. Let

$$i : \mathcal{S}_{\omega}(\underline{\Sigma}) \hookrightarrow \mathcal{S}_{\omega}(\underline{\Sigma} \setminus Q) \otimes \mathcal{S}_{\omega}(Q)$$

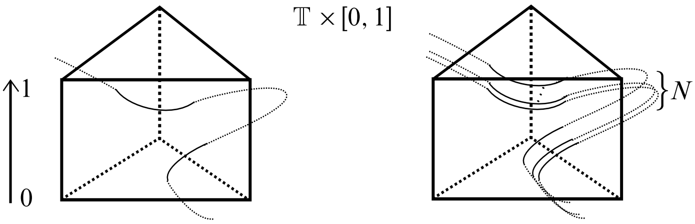


Figure 5: Instance of tangles $T_{\mathbb{T}}$ and $T_{\mathbb{T}}^{(N)}$.

be the gluing morphism. By Lemma 2.34, the morphism $j_Q\colon \mathscr{S}_{+1}(Q)\hookrightarrow\mathscr{S}_\omega(Q)$ does not depend on the triangulation of Q. Therefore, by Lemma 2.33, both the morphisms $j_{(\underline{\Sigma},\Delta)}$ and $j_{(\underline{\Sigma},\Delta')}$ make the diagram

$$\begin{tikzcd} \mathscr{S}_{+1}(\underline{\Sigma}) \arrow[r, hook, "i"] \arrow[d, hook, shift right, "j_{(\underline{\Sigma},\Delta')}"'] \arrow[d, hook, shift left, "j_{(\underline{\Sigma},\Delta)}"] & \mathscr{S}_{+1}(\underline{\Sigma}\setminus Q)\otimes\mathscr{S}_{+1}(Q) \arrow[d, hook, "j_{(\underline{\Sigma}\setminus Q,\Delta_{\underline{\Sigma}\setminus Q})}\otimes j_Q"] \\ \mathscr{S}_\omega(\underline{\Sigma}) \arrow[r, hook, "i"] & \mathscr{S}_\omega(\underline{\Sigma}\setminus Q)\otimes\mathscr{S}_\omega(Q) \end{tikzcd}$$

commutative. This proves that $j_{(\underline{\Sigma},\Delta)}=j_{(\underline{\Sigma},\Delta')}$. □

Step 2: arcs with endpoints in distinct boundary arcs We now assume that the triangulation Δ is nondegenerate.

Lemma 2.36 *If $\alpha_{\varepsilon\varepsilon'}\in\mathscr{S}_\omega(\underline{\Sigma})$ is the class of a stated arc such that its endpoints lie on two different boundary arcs, then $j_{\underline{\Sigma}}(\alpha_{\varepsilon\varepsilon'})=\alpha^N_{\varepsilon\varepsilon'}$.*

Proof By the defining property of $j_{\underline{\Sigma}}$, as depicted in diagram (19), it is enough to prove that

$$i^\Delta(\alpha^N_{\varepsilon\varepsilon'})=\Big(\bigotimes_{\mathbb{T}\in F(\Delta)} j_{\mathbb{T}}\Big)i^\Delta(\alpha_{\varepsilon\varepsilon'}). \tag{20}$$

Without lost of generality, we suppose that the arc α is in minimal and transverse position with the edges of Δ. Let T be a (vertical framed) tangle of $\Sigma_{\mathcal{P}}\times(0,1)$ that projects on α and such that its height projection is an injective map (this is possible since α is an arc). Note that for each $\mathbb{T}\in F(\Delta)$, the tangle $T_{\mathbb{T}}:=T\cap(\mathbb{T}\times(0,1))$ may have various connected components; since the height projection is injective, these components are ordered by height. Let $T^{(N)}$ be a tangle of N parallel copies of T obtained by stacking N copies of T, but close enough to have the following property. For each $\mathbb{T}\in F(\Delta)$, if T_1 and T_2 are two connected components of $T_{\mathbb{T}}$ such that T_1 is below T_2, then, in $T^{(N)}_{\mathbb{T}}:=T^{(N)}\cap(\mathbb{T}\times(0,1))$, each copy of T_1 is below all the copies of T_2. See Figure 5 for an illustration. Note that since α is an arc with boundary points at two distinct boundary arcs, the tangle $T^{(N)}$ is a representative of the N^{th} product of $\alpha_{\varepsilon\varepsilon'}$ in $\mathscr{S}_\omega(\underline{\Sigma})$; otherwise it may not be true.

The left-hand term of (20) can be described as the cutting of $T^{(N)}$ along each edge of the triangulation, and summing the result over all possible states at each edge. More formally, it is described as follows.

Let K be a subset of edges of Δ that intersect α. We let $\mathrm{St}_K(\alpha)$ be the set of maps

$$s\colon T\cap(K\times(0,1))\to\{-,+\}.$$

We identify $\mathrm{St}_K(\alpha)$ with $\bigsqcup_{e\in K}\mathrm{St}_{\{e\}}(\alpha)$, which allows us to write $s\in\mathrm{St}_K(\alpha)$ as $\sqcup s_e$. We will only consider the two sets K: the set E of all the *internal* edges of Δ that intersect α, and the set $K=\{e\}$ for an edge e.

For $s\in\mathrm{St}_E(\alpha)$, write $s^{(N)}:=(s,\dots,s)\in\mathrm{St}_E(\alpha)^{\times N}$. We denote by s_0 the state of $\alpha_{\varepsilon\varepsilon'}$ (so $\alpha_{\varepsilon\varepsilon'}=[T,s_0]$).

For $\boldsymbol{s}=(s_1,\dots,s_N)\in\mathrm{St}_E(\alpha)^{\times N}$, we let

$$\alpha(\boldsymbol{s}):=\bigotimes_{\mathbb{T}\in F(\Delta)}[T_{\mathbb{T}}^{(N)},(\boldsymbol{s}\sqcup s_0^{(N)})|_{\partial\mathbb{T}}]\in\bigotimes_{\mathbb{T}\in F(\Delta)}\mathcal{S}_\omega(\mathbb{T}),$$

where we associate, to the k^{th} copy of $T_{\mathbb{T}}^{(N)}$, the restriction of the state s_k. With this notation, the left-hand term of (20) can be written as

$$i^\Delta(\alpha_{\varepsilon\varepsilon'}^N)=\sum_{\boldsymbol{s}\in\mathrm{St}_E(\alpha)^{\times N}}\alpha(\boldsymbol{s}). \tag{21}$$

Now, let us describe the right-hand term of (20). Note that the construction of $T^{(N)}$ ensures that, for each triangle $\mathbb{T}$ and each state s of $T_{\mathbb{T}}$, one has $j_{\mathbb{T}}([T_{\mathbb{T}},s])=[T_{\mathbb{T}}^{(N)},s^{(N)}]$. Therefore, using that $j_{\mathbb{T}}$ is an algebra morphism,

$$\Big(\bigotimes_{\mathbb{T}\in F(\Delta)}j_{\mathbb{T}}\Big)i^\Delta(\alpha_{\varepsilon\varepsilon'})=\sum_{s\in\mathrm{St}_E(\alpha)}\alpha(s^{(N)}). \tag{22}$$

Let Y be the set of nondiagonal states $\mathrm{St}_E(\alpha)^{\times N}\setminus\{(s,\dots,s)\mid s\in\mathrm{St}_E(\alpha)\}$. The sum in (21) and in (22) differ by the sum of $\alpha(\boldsymbol{s})$ for $\boldsymbol{s}\in Y$.

Let us fix an edge e of E and let us split Y into $J\sqcup Y_e$ where Y_e is the set of N–tuples of states at e, that is, $Y_e=\{\boldsymbol{s}\in Y\mid \boldsymbol{s}\colon T^{(N)}\cap(e\times(0,1))\to\{-,+\}\}$. Therefore, showing (20) amounts to showing that

$$\sum_{\boldsymbol{s}'\in J}\sum_{\boldsymbol{s}\in Y_e}\alpha(\boldsymbol{s}'\sqcup\boldsymbol{s})=0.$$

In fact, let us show that, for each $\boldsymbol{s}'\in J$, one has $\sum_{\boldsymbol{s}\in Y_e}\alpha(\boldsymbol{s}'\sqcup\boldsymbol{s})=0$.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be the two triangles adjoining e (they are distinct since Δ is assumed nondegenerate) and let $Q\subset\Sigma_{\mathcal{P}}$ be the resulting square. Denote by $i_Q\colon\mathcal{S}_\omega(Q)\hookrightarrow\bigotimes_{\mathbb{T}\in F(\Delta)}\mathcal{S}_\omega(\mathbb{T})$ the corresponding embedding and write $T_Q:=T\cap(Q\times(0,1))$. For each $\boldsymbol{s}'\in J$,

$$\sum_{\boldsymbol{s}\in Y_e}\alpha(\boldsymbol{s}'\sqcup\boldsymbol{s})=\Big(\bigotimes_{\mathbb{T}\neq\mathbb{T}_1,\mathbb{T}_2}[T_{\mathbb{T}}^{(N)},\boldsymbol{s}'|_{\partial\mathbb{T}}]\Big)\otimes\big(i_Q([T_Q^{(N)},\boldsymbol{s}'|_{\partial Q}])-(j_{\mathbb{T}_1}\otimes j_{\mathbb{T}_2})\circ i_Q([T_Q^{(N)},\boldsymbol{s}'|_Q])\big).$$

The last term is zero by Lemma 2.34 and the commutativity of the diagrams in Lemma 2.31. $\square$

Step 3: closed curves that intersect Δ nicely

Definition 2.37 The N^{th} Chebyshev polynomial of first kind is the polynomial $T_N(X) \in \mathbb{Z}[X]$ defined by the recursive formulas $T_0(X) = 2$, $T_1(X) = X$ and $T_{n+2}(X) = X T_{n+1}(X) - T_n(X)$ for $n \geq 0$.

The following proposition is at the heart of (our proof of) the so-called "miraculous cancellations" from [Bonahon and Wong 2016]. We postpone its proof to the appendix.

Proposition 2.38 *If ω is a root of unity of odd order $N \geq 1$, then in $\mathcal{S}_\omega(\mathbb{B})$,*

$$T_N(\alpha_{++} + \alpha_{--}) = \alpha_{++}^N + \alpha_{--}^N.$$

Recall that we suppose that the triangulation is nondegenerate.

Lemma 2.39 *Let $\gamma \in \mathcal{S}_\omega(\underline{\Sigma})$ be the class of a closed curve. If the closed curve can be chosen such that it intersects an edge of Δ once and only once, then $j_{\underline{\Sigma}}(\gamma) = T_N(\gamma)$.*

Proof Consider the punctured surface $\underline{\Sigma}(e)$ obtained from $\underline{\Sigma}$ by replacing e by two arcs e' and e'' parallel to e with the same endpoints and removing the bigon between e' and e''. Consider the injective morphism $i|_{e'\#e''} : \mathcal{S}_\omega(\underline{\Sigma}) \hookrightarrow \mathcal{S}_\omega(\underline{\Sigma}(e))$. By Lemma 2.33, the following diagram commutes:

$$\begin{tikzcd} \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, hook, "j_{\underline{\Sigma}}"] \arrow[d, hook, "i|_{e'\#e''}"] & \mathcal{S}_\omega(\underline{\Sigma}) \arrow[d, hook, "i|_{e'\#e''}"] \\ \mathcal{S}_{+1}(\underline{\Sigma}(e)) \arrow[r, hook, "j_{\underline{\Sigma}(e)}"] & \mathcal{S}_\omega(\underline{\Sigma}(e)) \end{tikzcd}$$

By cutting γ along e, we get an arc $\beta \subset \Sigma(e)$ such that, by the hypothesis, $i|_{e'\#e''}(\gamma) = \beta_{++} + \beta_{--}$. Consider the algebra morphism $\varphi : \mathcal{S}_\omega(\mathbb{B}) \to \mathcal{S}_\omega(\underline{\Sigma}(e))$ sending $\alpha_{\varepsilon\varepsilon'}$ to $\beta_{\varepsilon\varepsilon'}$. One has

$$\begin{aligned} j_{\underline{\Sigma}(e)} \circ i|_{e'\#e''}(\gamma) &= j_{\underline{\Sigma}(e)}(\beta_{++} + \beta_{--}) \\ &= \varphi(\alpha_{++}^N + \alpha_{--}^N) && \text{(by Lemma 2.36)} \\ &= \varphi(T_N(\alpha_{++} + \alpha_{--})) && \text{(by Proposition 2.38)} \\ &= i|_{e'\#e''}(T_N(\gamma)). \end{aligned}$$

Hence, by the above diagram, $j_{\underline{\Sigma}}(\gamma) = T_N(\gamma)$. □

Step 4: adding a puncture Let $\underline{\Sigma}' = (\Sigma, \mathcal{P} \cup \{p_0\})$ be a punctured surface obtained from $\underline{\Sigma} = (\Sigma, \mathcal{P})$ by adding one puncture $p_0 \in \Sigma_{\mathcal{P}}$ and consider the algebra morphism $\varphi : \mathcal{S}_\omega(\underline{\Sigma}') \to \mathcal{S}_\omega(\underline{\Sigma})$ of Section 2.3. We assume that $\underline{\Sigma}$ is equipped with a nondegenerated triangulation.

Lemma 2.40 *The following diagram is commutative:*

$$\begin{tikzcd} \mathcal{S}_{+1}(\underline{\Sigma}') \arrow[r, hook, "j_{\underline{\Sigma}'}"] \arrow[d, two heads, "\varphi"] & \mathcal{S}_\omega(\underline{\Sigma}') \arrow[d, two heads, "\varphi"] \\ \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, hook, "j_{\underline{\Sigma}}"] & \mathcal{S}_\omega(\underline{\Sigma}) \end{tikzcd}$$

Proof First consider the diagram

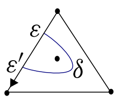
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}^{+1}_{p_0} \arrow[r] \arrow[d, hook, "j_{\underline{\Sigma}'}"] & \mathcal{S}_{+1}(\underline{\Sigma}') \arrow[r, "\varphi"] \arrow[d, hook, "j_{\underline{\Sigma}'}"] & \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r] \arrow[d, hook, "j_{\underline{\Sigma}}"] & 0 \\ 0 \arrow[r] & \mathcal{I}_{p_0} \arrow[r] & \mathcal{S}_{\omega}(\underline{\Sigma}') \arrow[r, "\varphi"] & \mathcal{S}_{\omega}(\underline{\Sigma}) \arrow[r] & 0 \end{tikzcd} \tag{23}$$

where $\mathcal{I}^{+1}_{p_0} \subset \mathcal{S}_{+1}(\underline{\Sigma}')$ and $\mathcal{I}_{p_0} \subset \mathcal{S}_{\omega}(\underline{\Sigma}')$ denote the off-puncture ideals in $\mathcal{S}_{+1}(\underline{\Sigma}')$ and $\mathcal{S}_{\omega}(\underline{\Sigma}')$, respectively (see Definition 2.17). By Proposition 2.18, both lines are exact so we need to prove the inclusion $j_{\underline{\Sigma}'}(\mathcal{I}^{+1}_{p_0}) \subset \mathcal{I}_{p_0}$ to conclude. We divide the proof in two steps.

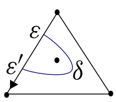
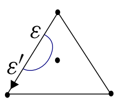
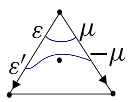
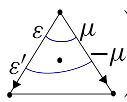
Step 1 We first suppose that $\underline{\Sigma} = \mathbb{T}_0$ is a triangle. In this case, $\mathbb{T}'_0$ is a punctured triangle and we have two possibilities depending whether p_0 is in the boundary or the interior of $\mathbb{T}_0$. Some nondegenerate triangulations Δ'_0 of $\mathbb{T}'_0$ are drawn in Figure 6.

Claim *The off-kernel ideal $\mathcal{I}_{p_0}$ is generated by elements $\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}$ and $\gamma - \gamma'$, where α and α' are arcs isotopic in $\mathbb{T}_0$ whose endpoints lie in distinct boundary arcs and γ and γ' are curves isotopic in $\mathbb{T}_0$ which intersect each edge of Δ'_0 once.*

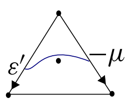
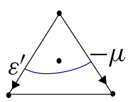
If the claim is proved, then for $\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}$ and $\gamma - \gamma'$ some generators of $\mathcal{I}_{p_0}$, Lemma 2.36 implies that $j_{\mathbb{T}'_0}(\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}) \subset \mathcal{I}_{p_0}$ and Lemma 2.39 implies that $j_{\mathbb{T}'_0}(\gamma - \gamma') = T_N(\gamma) - T_N(\gamma') \in \mathcal{I}_{p_0}$. The claim implies the inclusion $j_{\mathbb{T}'_0}(\mathcal{I}^{+1}_{p_0}) \subset \mathcal{I}_{p_0}$, which concludes the proof in the case of the triangle. To prove the claim, recall from Proposition 2.18 that $\mathcal{I}_{p_0}$ is generated by elements $\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}$ and $\gamma - \gamma'$ with α and α' isotopic in $\mathbb{T}_0$ and γ and γ' isotopic in $\mathbb{T}_0$. First note that when p_0 lies in the boundary of $\mathbb{T}_0$, then $\mathbb{T}'_0$ does not contain any noncontractible simple closed curve and the nontrivial arcs of $\mathbb{T}'_0$ have endpoints in distinct boundary arcs, so the claim is immediate in this case. When p_0 lies in the interior of $\mathbb{T}_0$, there is only one nontrivial simple closed curve (which encircles p_0 once) and this curves intersects each edges of Δ'_0 once. However $\mathbb{T}'_0$ contains three nontrivial arcs with endpoints in the same boundary arcs which are related by a $\frac{2}{3}\pi$ radian rotation. Let δ be one of these arcs and

$\delta_{\varepsilon\varepsilon'} =$ 

Since $x := \delta_{\varepsilon\varepsilon'} - C^{\varepsilon'}_{\varepsilon} \in \mathcal{I}_{p_0}$, we need to show that x belongs to the ideal $\mathcal{J}_{p_0}$ generated by elements $\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}$ with α and α' isotopic in $\mathbb{T}_0$ with distinct endpoints. This is done by a simple application of the skein relation (4):

$x =$  $-$  $= \sum_{\mu=+,-} C^{-\mu}_{\mu} \Big($  $-$  $\Big).$

Therefore x belongs to the ideal generated by elements

 $-$ 

This proves the claim and concludes the proof of the lemma in the case where $\underline{\Sigma} = \mathbb{T}_0$.

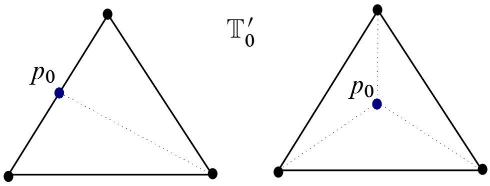


Figure 6: Punctured triangles $\mathbb{T}_0'$ and their nondegenerated triangulations.

Step 2 We consider the general case. Recall that $\underline{\Sigma}$ is equipped with a nondegenerate triangulation Δ and let $\mathbb{T}_0$ be the face containing the point p_0. Let $\underline{\Sigma}_0$ be the (possibly empty) punctured surface made of the faces of Δ distinct from $\mathbb{T}_0$ so that $\underline{\Sigma}$ is obtained from $\mathbb{T}_0 \sqcup \underline{\Sigma}_0$ by gluing some pairs of boundary arcs together and let $i\colon \mathscr{S}_\omega(\underline{\Sigma}) \hookrightarrow \mathscr{S}_\omega(\mathbb{T}_0) \otimes \mathscr{S}_\omega(\underline{\Sigma}_0)$ denote the gluing map. Similarly, let $i'\colon \mathscr{S}_\omega(\underline{\Sigma}') \hookrightarrow \mathscr{S}_\omega(\mathbb{T}_0') \otimes \mathscr{S}_\omega(\underline{\Sigma}_0)$ be the gluing map of $\underline{\Sigma}'$. Consider the diagram

$$\begin{tikzcd}
\mathscr{S}_{+1}(\mathbb{T}_0') \otimes \mathscr{S}_{+1}(\underline{\Sigma}_0) \arrow[rrr, hook, "j_{\mathbb{T}_0'} \otimes j_{\underline{\Sigma}_0}"] \arrow[ddd, two heads, "\varphi_0 \otimes \mathrm{id}"] & & & \mathscr{S}_\omega(\mathbb{T}_0') \otimes \mathscr{S}_\omega(\underline{\Sigma}_0) \arrow[ddd, two heads, "\varphi_0 \otimes \mathrm{id}"] \\
& \mathscr{S}_{+1}(\underline{\Sigma}') \arrow[ul, hook, "i'"] \arrow[r, hook, "j_{\underline{\Sigma}'}"] \arrow[d, two heads, "\varphi"] & \mathscr{S}_\omega(\underline{\Sigma}') \arrow[ur, hook, "i'"] \arrow[d, two heads, "\varphi"] & \\
& \mathscr{S}_\omega(\underline{\Sigma}') \arrow[dl, hook, "i"] \arrow[r, hook, "j_{\underline{\Sigma}}"] & \mathscr{S}_\omega(\underline{\Sigma}) \arrow[dr, hook, "i"] & \\
\mathscr{S}_{+1}(\mathbb{T}_0) \otimes \mathscr{S}_{+1}(\underline{\Sigma}_0) \arrow[rrr, hook, "j_{\mathbb{T}_0} \otimes j_{\underline{\Sigma}_0}"] & & & \mathscr{S}_\omega(\mathbb{T}_0) \otimes \mathscr{S}_\omega(\underline{\Sigma}_0)
\end{tikzcd}$$

In this diagram,

- the outer square commutes by Step 1;
- the squares on the top and bottom commute by Lemma 2.33;
- the squares on the left and right sides commute by definition of φ.

Therefore the innermost square commutes. □

Notation 2.41 For $\alpha_{\varepsilon\varepsilon'} \in \mathscr{S}_\omega(\underline{\Sigma})$ the class of a stated arc, we denote by $\alpha^{(N)}_{\varepsilon\varepsilon'}$ be the class of the stated tangle made by stacking N parallel copies of $\alpha_{\varepsilon\varepsilon'}$ on top of the others in the framing direction. More precisely, if both endpoints of α lie in different boundary arcs, then $\alpha^{(N)}_{\varepsilon\varepsilon'} = (\alpha_{\varepsilon\varepsilon'})^N$. If α has its two endpoints, say v and w, in the same boundary arc with $h(v) < h(w)$ such that v has state ε and w has state ε', then $\alpha^{(N)}_{\varepsilon\varepsilon'}$ is the class of the stated tangle $(\alpha^{(N)}, s^{(N)})$ defined as follows. The tangle $\alpha^{(N)}$ is made of N parallel copies $\alpha^{(N)} = \alpha_1 \cup \cdots \cup \alpha_N$ of α such that the height order is given by $h(v_1) < h(v_2) < \cdots < h(v_N) < h(w_1) < \cdots < h(w_N)$. The state $s^{(N)}$ sends the points v_i to ε and the points w_j to ε'.

Lemma 2.42 *If $\alpha_{\varepsilon\varepsilon'} \in \mathscr{S}_\omega(\underline{\Sigma})$ is the class of a stated arc such that its endpoints lie on the same boundary arcs, then $j_{\underline{\Sigma}}(\alpha_{\varepsilon\varepsilon'}) = \alpha^{(N)}_{\varepsilon\varepsilon'}$.*

Proof Since the two endpoints of α lie on the same boundary arc a, we can pick a puncture $p_0 \in a$ that lies between these two endpoints. Denote by $\underline{\Sigma}' = (\Sigma, \mathcal{P} \cup \{p_0\})$ the punctured surface obtained by adding this puncture, and $\varphi\colon \mathcal{S}_\omega(\underline{\Sigma}') \to \mathcal{S}_\omega(\underline{\Sigma})$ the morphism of Section 2.3. With the notation of Section 2.3, the two components of $a \setminus \{p_0\}$ are two boundary arcs b and c of $\underline{\Sigma}'$ and we choose the convention such that $\alpha \in \mathcal{T}^{(0)}(\underline{\Sigma})$. Note that $\alpha^{(N)}$ is in $\mathcal{T}^{(0)}(\underline{\Sigma})$ as well. To avoid confusion, we denote by α' the arc α seen as an arc in $\Sigma_{\mathcal{P}\cup\{p_0\}}$, so that $\iota(\alpha') = \alpha$. By Lemma 2.36, $j_{\underline{\Sigma}'}(\alpha'_{\varepsilon\varepsilon'}) = (\alpha'_{\varepsilon\varepsilon'})^N = \alpha'^{(N)}_{\varepsilon\varepsilon'}$. By commutativity of the diagram in Lemma 2.40 and by definition of φ, the image $j_{\underline{\Sigma}}(\alpha_{\varepsilon\varepsilon'})$ is the class in $\mathcal{S}_\omega(\underline{\Sigma})$ of the unique stated tangle in $\mathcal{T}^{(0)}(\underline{\Sigma})$ which is isotopic to $\alpha'^{(N)}_{\varepsilon\varepsilon'}$: this is $\alpha^{(N)}_{\varepsilon\varepsilon'}$. $\square$

Lemma 2.43 *If $\gamma \in \mathcal{S}_\omega(\underline{\Sigma})$ is the class of a closed curve, then $j_{\underline{\Sigma}}(\gamma) = T_N(\gamma)$.*

Proof If the closed curve can be chosen such that it intersects an edge of Δ once and only once, then this is Lemma 2.39. Otherwise, we can refine the triangulation by adding an inner puncture in order to have this property. Denote by $\underline{\Sigma}'$ the resulting punctured surface and let $\gamma' \in \mathcal{S}_{+1}(\underline{\Sigma}')$ be such that $\iota(\gamma') = \gamma$. Lemma 2.39 implies that $j_{\underline{\Sigma}'}(\gamma') = T_N(\gamma')$ and Lemma 2.40 implies that $j_{\underline{\Sigma}}(\gamma) = T_N(\gamma)$. $\square$

Step 5: punctured surfaces which do not admit nondegenerate triangulations It remains to prove Theorem 1.2 for connected punctured surfaces which do not admit nondegenerate topological triangulations; that is, for the small punctured surfaces, for the disc with one inner puncture and one puncture on its boundary and for the unpunctured surfaces $\underline{\Sigma} = (\Sigma, \varnothing)$ with empty set of puncture.

The disc with only one puncture (on its boundary) and the sphere with zero or one puncture both have trivial skein algebra, while the sphere with two punctures has a commutative skein algebra. Therefore, Theorem 1.2 holds trivially for them. It remains to prove:

Lemma 2.44 *Theorem 1.2 holds when $\underline{\Sigma}$ is either a disc with one inner puncture and one puncture on its boundary or an unpunctured surface $\underline{\Sigma} = (\Sigma, \varnothing)$ of genus at least one.*

Proof Choose an inner puncture $p_0 \in \mathring{\Sigma}_{\mathcal{P}}$ and consider the punctured surface $\underline{\Sigma}' := (\Sigma, \mathcal{P} \cup \{p_0\})$. Since $\underline{\Sigma}'$ admits a nondegenerate triangulation, our previous study shows the existence of the Chebyshev morphism $j_{\underline{\Sigma}'}\colon \mathcal{S}_{+1}(\underline{\Sigma}') \hookrightarrow \mathcal{Z}(\mathcal{S}_\omega(\underline{\Sigma}'))$. Consider the off-puncture ideals $\mathcal{I}^{+1}_{p_0} \subset \mathcal{S}_{+1}(\underline{\Sigma}')$ and $\mathcal{I}_{p_0} \subset \mathcal{S}_\omega(\underline{\Sigma}')$. Exactly the same argument used in the proof of Lemma 2.40 shows the inclusion $j_{\underline{\Sigma}'}(\mathcal{I}^{+1}_{p_0}) \subset \mathcal{I}_{p_0}$. By Proposition 2.18, both lines in the following diagram are exact:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}^{+1}_{p_0} \arrow[r] \arrow[d, hook, "j_{\underline{\Sigma}'}"] & \mathcal{S}_{+1}(\underline{\Sigma}') \arrow[r, "\varphi"] \arrow[d, hook, "j_{\underline{\Sigma}'}"] & \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r] \arrow[d, dotted, "\exists ! j_{\underline{\Sigma}}"] & 0 \\ 0 \arrow[r] & \mathcal{I}_{p_0} \arrow[r] & \mathcal{S}_\omega(\underline{\Sigma}') \arrow[r, "\varphi"] & \mathcal{S}_\omega(\underline{\Sigma}) \arrow[r] & 0 \end{tikzcd}$$

Therefore there exists a unique algebra morphism $j_{\underline{\Sigma}}\colon \mathcal{S}_{+1}(\underline{\Sigma}) \to \mathcal{S}_\omega(\underline{\Sigma})$ which makes the diagram commute. Since $j_{\underline{\Sigma}}$ is obtained from $j_{\underline{\Sigma}'}$ by passing to the quotient, its image is also central and one has the equalities $j_{\underline{\Sigma}}([\gamma]) = T_N([\gamma])$ and $j_{\underline{\Sigma}}(\alpha_{\varepsilon\varepsilon'}) = \alpha^{(N)}_{\varepsilon\varepsilon'}$ for any closed curve γ and any stated arc $\alpha_{\varepsilon\varepsilon'}$. $\square$

2.7 A Poisson bracket on $\mathcal{S}_{+1}(\underline{\Sigma})$

In this section, we define and make explicit a Poisson structure on $\mathcal{S}_{+1}(\underline{\Sigma})$.

2.7.1 Preliminaries We briefly recall some general facts concerning deformation quantization.

Let $\mathcal{A}$ be a complex commutative unital algebra, $\mathbb{C}[\![\hbar]\!]$ be the ring of formal series in a parameter $\hbar$ and $\mathcal{A}[\![\hbar]\!] := \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[\![\hbar]\!]$. A *star product* $\star$ on $\mathcal{A}$ is an associative product on $\mathcal{A}[\![\hbar]\!]$ such that if $f = \sum_i f_i \hbar^i$ and $g = \sum_i g_i \hbar^i$ are elements of $\mathcal{A}[\![\hbar]\!]$, then

$$f \star g = f_0 g_0 \mod \hbar,$$

where $f_0 g_0$ denotes the product of f_0 and g_0 in $\mathcal{A}$. A star product induces a Poisson structure on $\mathcal{A}$ by the formula

$$f \star g - g \star f = \hbar\{f, g\} \mod \hbar^2, \tag{24}$$

for all $f, g \in \mathcal{A}$. The algebra $(\mathcal{A}[\![\hbar]\!], \star)$ is called a *deformation quantization* of the commutative Poisson algebra $(\mathcal{A}, \{\cdot, \cdot\})$. We refer to [Kontsevich 2003; Gutt et al. 2005, II.2] for detailed discussions. A *morphism of star products* between $(\mathcal{A}, \star_{\mathcal{A}})$ and $(\mathcal{B}, \star_{\mathcal{B}})$ is an algebra morphism $\psi : \mathcal{A}[\![\hbar]\!] \to \mathcal{B}[\![\hbar]\!]$ whose restriction to $\mathcal{A} \subset \mathcal{A}[\![\hbar]\!]$ induces a morphism $\phi : \mathcal{A} \to \mathcal{B}$. Note that such a ϕ is, in fact, a morphism of Poisson algebras for the induced Poisson algebra structures. An isomorphism

$$\psi : (\mathcal{A}[\![\hbar]\!], \star_1) \xrightarrow{\cong} (\mathcal{A}[\![\hbar]\!], \star_2)$$

of star products is called a *gauge equivalence* if $\psi(f) = f \pmod{\hbar}$. If two star products are gauge equivalent, they induce the same Poisson bracket on $\mathcal{A}$.

To end this preamble, let us mention that deformation quantization is well behaved with respect to the tensor product. Indeed, if $\mathcal{A}[\![\hbar]\!]$ and $\mathcal{B}[\![\hbar]\!]$ are deformation quantizations of $\mathcal{A}$ and $\mathcal{B}$, respectively, then $\mathcal{A}[\![\hbar]\!] \otimes \mathcal{B}[\![\hbar]\!] \cong (\mathcal{A} \otimes \mathcal{B})[\![\hbar]\!]$ is a deformation quantization of $\mathcal{A} \otimes \mathcal{B}$. Note also that the Poisson structure on $\mathcal{A} \otimes \mathcal{B}$ given by (24) is

$$\{f \otimes g, f' \otimes g'\} = ff' \otimes \{g, g'\} + \{f, f'\} \otimes gg' \tag{25}$$

for $f, f' \in \mathcal{A}$ and $g, g' \in \mathcal{B}$.

2.7.2 Formal definition Let $\underline{\Sigma}$ be a punctured surface and $\mathfrak{o}$ an orientation of its boundary arc. Denote by $\mathcal{S}_{+1}(\underline{\Sigma})$ the stated skein algebra associated to the ring $\mathbb{C}$ with $\omega = +1$ and denote by $\mathcal{S}_{\omega_\hbar}(\underline{\Sigma})$ the stated skein algebra associated to the ring $\mathbb{C}[\![\hbar]\!]$ with $\omega_\hbar := \exp\left(-\frac{1}{4}\hbar\right)$. The convention is chosen so that $q = \exp(\hbar)$. Recall the basis $\mathcal{B}^{\mathfrak{o}}$ from Definition 2.3. Since $\mathcal{B}^{\mathfrak{o}}$ is independent of ω, one has an isomorphism of $\mathbb{C}[\![\hbar]\!]$–modules

$$\psi^{\mathfrak{o}} : \mathcal{S}_{+1}(\underline{\Sigma})[\![\hbar]\!] \xrightarrow{\cong} \mathcal{S}_{\omega_\hbar}(\underline{\Sigma}). \tag{26}$$

Note that $\mathfrak{o}$ tells us how to lift the basis elements $[D, s]$ of $\mathcal{S}_{+1}(\underline{\Sigma})$ (which are independent of the height order) in $\mathcal{S}_{\omega_\hbar}(\underline{\Sigma})$. We emphasize that $\psi^{\mathfrak{o}}$ is not an algebra morphism.

Definition 2.45 Pulling back the product of $\mathcal{S}_{\omega_\hbar}(\underline{\Sigma})$ along $\psi^{\mathfrak{o}}$ gives a star product $\star_\hbar$ on $\mathcal{S}_{+1}(\underline{\Sigma})$. We denote by $\{\cdot,\cdot\}^s$ the resulting Poisson bracket on $\mathcal{S}_{+1}(\underline{\Sigma})$ given by (24).

Here the superscript s stands for "skein".

Remark 2.46 For any two orientations $\mathfrak{o}_1$ and $\mathfrak{o}_2$ of the boundary arcs of $\underline{\Sigma}$, the automorphism $(\psi^{\mathfrak{o}_2})^{-1}\circ \psi^{\mathfrak{o}_1}:\mathcal{S}_{+1}(\underline{\Sigma})[\![\hbar]\!]\xrightarrow{\cong}\mathcal{S}_{+1}(\underline{\Sigma})[\![\hbar]\!]$ is a gauge equivalence; hence the Poisson bracket $\{\cdot,\cdot\}^s$ does not depend on $\mathfrak{o}$.

By definition, $(\mathcal{S}_{+1}(\underline{\Sigma})[\![\hbar]\!],\star_\hbar)$ is a quantization deformation of the Poisson algebra $(\mathcal{S}_{+1}(\underline{\Sigma}),\{\cdot,\cdot\}^s)$. Moreover, this structure of Poisson algebra is compatible with decompositions of surfaces. More precisely, one has the following.

Lemma 2.47 *The gluing maps* $i|_{a\#b}:\mathcal{S}_{+1}(\underline{\Sigma}|_{a\#b})\hookrightarrow\mathcal{S}_{+1}(\underline{\Sigma})$*, the maps*

$$i^\Delta:\mathcal{S}_{+1}(\underline{\Sigma})\hookrightarrow\bigotimes_{\mathbb{T}\in F(\Delta)}\mathcal{S}_{+1}(\mathbb{T})$$

and the coproduct maps Δ^L *and* Δ^R *are Poisson morphisms.*

Proof This follows from the fact that each of these morphisms arises from a morphism of star products. □

2.7.3 Explicit formula This section is devoted to making explicit the Poisson bracket $\{\cdot,\cdot\}^s$ on stated diagrams. It will be expressed in terms of *resolutions* of stated diagrams, which are defined at crossings and at points on the boundary arcs.

Throughout this section, $\underline{\Sigma}$ is a punctured surface.

Resolution at a crossing Let (D,s) be a stated diagram and c a crossing of D. Denote by D_+ and D_- the diagrams obtained from D by replacing the crossing c  by its positive  and negative  resolution, respectively. The resolution of (D,s) at the crossing c is defined by

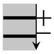
$$\mathrm{Res}_c(D,s):=[D_+,s]-[D_-,s]\in\mathcal{S}_{+1}(\underline{\Sigma}).$$

Resolution at boundary points Let $b_1,\dots,b_k$ be the boundary arcs of $\Sigma_{\mathcal{P}}$.

Definition 2.48 A *height order* on a stated diagram (D,s) of $\Sigma_{\mathcal{P}}$ is a k–tuple $\mathfrak{o}=(\mathfrak{o}_1,\dots,\mathfrak{o}_k)$ of bijections of sets $\mathfrak{o}_i:\partial_{b_i}D\to\{1,\dots,|\partial_{b_i}D|\}$.

Note that the product of symmetric groups $\mathbb{S}_{n_1}\times\cdots\times\mathbb{S}_{n_k}$ acts freely and transitively on the set of height orders by left composition.

To a height order $\mathfrak{o}$ on (D,s) corresponds a stated tangle with same height order and which projects to (D,s). Therefore, one can consider the class of $(D,s,\mathfrak{o})$ in $\mathcal{S}_\omega(\underline{\Sigma})$. If $\omega=+1$, the class $[D,s,\mathfrak{o}]\in\mathcal{S}_{+1}(\underline{\Sigma})$ is independent of $\mathfrak{o}$, and we denote it simply by $[D,s]$.

Let us choose a boundary arc b_i and suppose there are two points p_H and p_L of $\partial_{b_i}D$ such that $\mathfrak{o}_i(p_H)=\mathfrak{o}_i(p_L)+1$ (ie p_H is the $\mathfrak{o}_i$–successor of p_L). Let $\tilde{\mathfrak{o}}$ be the order on b_i that is induced by the orientation of Σ. To alleviate notation, we write $p <_{\tilde{\mathfrak{o}}} q$ for $\tilde{\mathfrak{o}}(p) < \tilde{\mathfrak{o}}(q)$. For instance, in the stated diagram , if p_L is the endpoint with $s(p_L) = +$, p_H the endpoint with $s(p_H) = -$ and $\mathfrak{o}$ is the orientation given by the arrow, then $p_L >_{\tilde{\mathfrak{o}}} p_H$ whereas $p_L <_{\mathfrak{o}} p_H$ (because the $\mathfrak{o}$ and $\tilde{\mathfrak{o}}$ orientation of the boundary arc where live p_L and p_H are opposite).

Let $\tau \in \mathbb{S}_{n_i}$ be the transposition that exchanges the $\mathfrak{o}_i$ order of p_H and p_L. The resolution of (D, s) along τ, denoted by $\mathrm{Res}_\tau(D, s, \mathfrak{o}) \in \mathcal{S}_{+1}(\underline{\Sigma})$, is given by

$$\begin{cases} \frac{1}{2}[D,s] & \text{if } s(p_H)=s(p_L) \text{ and } p_L <_{\tilde{\mathfrak{o}}} p_H \text{ or } (s(p_H),s(p_L))=(-,+) \text{ and } p_H <_{\tilde{\mathfrak{o}}} p_L, \\ -\frac{1}{2}[D,s] & \text{if } s(p_H)=s(p_L) \text{ and } p_H <_{\tilde{\mathfrak{o}}} p_L \text{ or } (s(p_H),s(p_L))=(+,-) \text{ and } p_L <_{\tilde{\mathfrak{o}}} p_H, \\ \frac{1}{2}[D,s]-2[D,\tau s] & \text{if } (s(p_H),s(p_L))=(+,-) \text{ and } p_H <_{\tilde{\mathfrak{o}}} p_L, \\ -\frac{1}{2}[D,s]+2[D,\tau s] & \text{if } (s(p_H),s(p_L))=(-,+) \text{ and } p_L <_{\tilde{\mathfrak{o}}} p_H, \end{cases}$$

where τs is the state that differs from s only by exchanging the states of p_H and p_L.

Let us extend the resolution to several points, namely any permutation of the boundary heights on a given boundary component. For two transpositions σ_1 and σ_2 of $\mathfrak{o}$–consecutive points, let

$$\mathrm{Res}_{\sigma_1 \circ \sigma_2}(D, s, \mathfrak{o}) = \mathrm{Res}_{\sigma_1}(D, s, \sigma_2 \circ \mathfrak{o}) + \mathrm{Res}_{\sigma_2}(D, s, \mathfrak{o}). \tag{27}$$

Definition 2.49 For a permutation $\sigma \in \mathbb{S}_{n_1} \times \cdots \times \mathbb{S}_{n_k}$, the resolution $\mathrm{Res}_\sigma(D, s, \mathfrak{o})$ is defined via (27), by considering the decomposition of σ into transpositions of $\mathfrak{o}$–consecutive points. This is clearly independent of the choice of decomposition into transpositions.

Remark 2.50 The resolution $\mathrm{Res}_\sigma(D, s, \mathfrak{o})$ is invariant under isotopy of (D, s). Also, $\mathrm{Res}_{\mathrm{id}}(D, s, \mathfrak{o}) = 0$.

Lemma 2.51 *In the skein algebra $\mathcal{S}_{\omega_\hbar}(\underline{\Sigma})$, the following two statements hold.*

(1) *Let $D_{\diagup}$ and $D_{\diagdown}$ be two diagrams that differ from each other only by a change of a crossing c. Then*

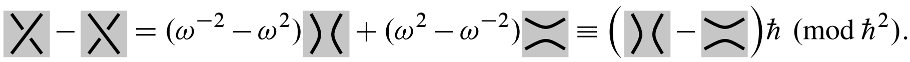
$$[D_{\diagup}, s, \mathfrak{o}] - [D_{\diagdown}, s, \mathfrak{o}] = \hbar\, \mathrm{Res}_c(D_{\diagup}, s) \mod \hbar^2.$$

(2) *Let $(D, s, \mathfrak{o})$ be an $\mathfrak{o}$–ordered stated diagram. For $\pi \in \mathbb{S}_{n_1} \times \cdots \times \mathbb{S}_{n_k}$,*

$$[D, s, \mathfrak{o}] - [D, s, \pi \circ \mathfrak{o}] = \hbar\, \mathrm{Res}_\pi(D, s, \mathfrak{o}) \mod \hbar^2.$$

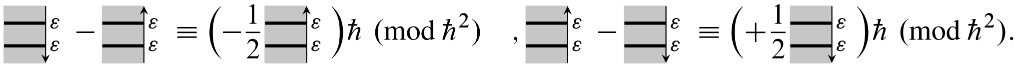
In the two statements, the resolutions Res *are seen in $\mathcal{S}_{\omega_\hbar}(\underline{\Sigma})$ via the isomorphism $\psi^{\tilde{\mathfrak{o}}}$ of (26).*

Proof Recall that $\omega_\hbar = \exp\bigl(-\frac{1}{4}\hbar\bigr) \equiv 1 - \frac{1}{4}\hbar \pmod{\hbar^2}$. The first equality follows from (3):

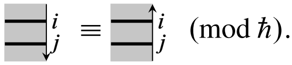


Let us prove the second equality when π a transposition of two consecutive points p_H, p_L with $p_H >_{\mathfrak{o}} p_L$. If $s(p_H) = s(p_L) = \varepsilon$, then (5) gives

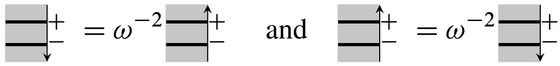
from which we deduce



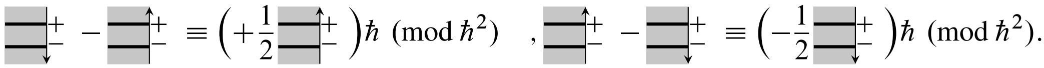
Note that in the stated skein algebra at $\omega = +1$, the height order is irrelevant; said differently, at $\omega_\hbar = \exp\left(-\frac{1}{4}\hbar\right)$, we have the skein relation



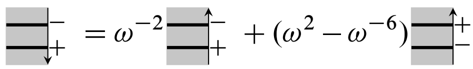
Now, if either $p_H <_{\tilde{\mathfrak{o}}} p_L$ and $(s(p_H), s(p_L)) = (-,+)$ or if $p_L <_{\tilde{\mathfrak{o}}} p_H$ and $(s(p_H), s(p_L)) = (+,-)$ then, using (5),



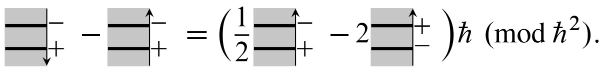
from which we deduce



If $p_H <_{\tilde{\mathfrak{o}}} p_L$ and $(s(p_H), s(p_L)) = (+,-)$, then (6) and (4) imply that



from which we deduce



Eventually the case where $p_L <_{\tilde{\mathfrak{o}}} p_H$ and $(s(p_H), s(p_L)) = (-,+)$ is deduced from this case by taking the opposite of the preceding equality. This concludes the proof of the second equality of the lemma when τ is a transposition. The case of a general permutation π follows by induction on the number of transpositions in a decomposition of π. $\square$

Proposition 2.52 *Let $(D_1, s_2, \mathfrak{o}_1)$ and $(D_2, s_2, \mathfrak{o}_2)$ be two height ordered stated diagrams such that D_1 and D_2 intersect transversally in the interior of $\Sigma_{\mathcal{P}}$. Let $(D_1 D_2, s_1 s_2)$ be the stated diagram obtained by staking D_1 on top of D_2, $\mathfrak{o}_1\mathfrak{o}_2$ the resulting height order and π the permutation sending $\mathfrak{o}_2\mathfrak{o}_1$ to $\mathfrak{o}_1\mathfrak{o}_2$. In $\mathcal{S}_{+1}(\Sigma)$, the Poisson bracket from Definition 2.45 satisfies*

$$\{[D_1, s_1], [D_2, s_2]\}^s = \sum_{c \in D_1 \cap D_2} \mathrm{Res}_c(D_1 D_2, s_1 s_2) + \mathrm{Res}_\pi(D_1 D_2, s_1 s_2, \mathfrak{o}_1\mathfrak{o}_2).$$

Proof In the algebra $\mathcal{S}_{\omega_\hbar}(\Sigma)$, the product gives $[D_1, s_1, \mathfrak{o}_1] \cdot [D_2, s_2, \mathfrak{o}_2] = [D_1 D_2, s_1 s_2, \mathfrak{o}_1\mathfrak{o}_2]$ and $[D_2, s_2, \mathfrak{o}_2] \cdot [D_1, s_1, \mathfrak{o}_1] = [D_2 D_1, s_2 s_1, \mathfrak{o}_2\mathfrak{o}_1]$. We pass from the diagram $D_1 D_2$ to $D_2 D_1$ by changing each crossing in the intersection of the diagrams and changing the height order using π, so the formula is a consequence of Lemma 2.51. $\square$

Remark 2.53 Neither $\{\cdot,\cdot\}^s$ nor the formula in Proposition 2.52 depend on a choice of orientation of the boundary arcs by Remark 2.46. When Σ is a closed surface, we recover Goldman's formula [1986]. When Σ has nontrivial boundary and no inner punctures, the subalgebra of the stated skein algebra generated by tangles with states having only value $+$ is isomorphic to the Muller algebra defined in [Muller 2016] (see also [Lê 2018, Section 6]). The Poisson bracket restricts to the corresponding subalgebra of $\mathcal{S}_{+1}(\Sigma)$ and the resulting Poisson algebra is isomorphic to Yuasa's Poisson algebra [2015].

Example 2.54 The Poisson bracket $\{-,-\}^s$ on the commutative algebra $\mathcal{S}_{+1}(\mathbb{B})$ is given by

$$\begin{aligned}
\{\alpha_{++},\alpha_{+-}\}^s &= -\alpha_{+-}\alpha_{++}, & \{\alpha_{++},\alpha_{-+}\}^s &= -\alpha_{-+}\alpha_{++},\\
\{\alpha_{--},\alpha_{+-}\}^s &= \alpha_{+-}\alpha_{--}, & \{\alpha_{--},\alpha_{-+}\}^s &= \alpha_{-+}\alpha_{--},\\
\{\alpha_{+-},\alpha_{-+}\}^s &= 0, & \{\alpha_{++},\alpha_{--}\}^s &= -2\alpha_{+-}\alpha_{-+}.
\end{aligned}$$

Example 2.55 For the triangle $\mathbb{T}$, the Poisson structure is described by the formulas in Example 2.54 by replacing α by each of the three arcs α, β and γ, together with the following relations and their images through the automorphisms τ and τ^2:

$$\{\gamma_{\varepsilon\mu},\alpha_{\mu'\varepsilon}\}^s = -\tfrac{1}{2}\gamma_{\varepsilon\mu}\alpha_{\mu'\varepsilon},\quad \{\gamma_{-\mu},\alpha_{\mu'+}\}^s = \tfrac{1}{2}\gamma_{-\mu}\alpha_{\mu'+},\quad \{\gamma_{+\mu},\alpha_{\mu'-}\}^s = -\tfrac{3}{2}\gamma_{+\mu}\alpha_{\mu'-}+2\beta_{\mu\mu'}.$$

3 Relative character varieties

3.1 Relative character varieties for open surfaces

In this subsection we briefly recall from [Korinman 2019] the definition and main properties of character varieties for open surfaces.

The character variety of a closed punctured connected surface $\underline{\Sigma}$ is the algebraic quotient (familiar in geometric invariant theory)

$$\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma}) := \mathrm{Hom}(\pi_1(\Sigma_{\mathcal{P}}),\mathrm{SL}_2(\mathbb{C}))/\!\!/\mathrm{SL}_2(\mathbb{C})$$

under the action by conjugation of $\mathrm{SL}_2(\mathbb{C})$. Recall that by "closed", we mean that Σ is closed though in this case $\Sigma_{\mathcal{P}}$ is not closed when $\mathcal{P}\neq\varnothing$. Goldman [1986] defined a Poisson structure on its algebra of regular functions. It follows from [Barrett 1999; Bullock 1997; Przytycki and Sikora 2000; Turaev 1991] that, given a spin structure S on Σ with quadratic form ω_S, there is a Poisson isomorphism

$$\phi^S\colon (\mathcal{S}_{+1}(\underline{\Sigma}),\{\cdot,\cdot\}^s) \xrightarrow{\cong} \big(\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})],\{\cdot,\cdot\}\big).$$

For each noncontractible closed curve γ, it is given by $\phi^S(\gamma)=(-1)^{\omega_S([\gamma])+1}\tau_\gamma$, where τ_γ is the regular function $\tau_\gamma([\rho]) := \mathrm{Tr}(\rho(\gamma))$.

Korinman [2019] introduced a generalization of the character varieties to punctured surfaces which are not necessarily closed and which is closely related to the construction of Fock and Rosly [1999] and specifies to the constructions in [Alekseev and Malkin 1995; Alekseev et al. 1998; 2002; Guruprasad et al. 1997] when the marked surface is connected and has exactly one boundary arc (see [Korinman 2019] for a precise comparison). We will also denote it by $\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})$.

Notation 3.1 For a topological space X, we let $\Pi_1(X)$ denote its fundamental groupoid: objects are the points in X and morphisms are homotopy classes of oriented paths. We let s and t denote the source and target maps, which for a morphism $\alpha\colon v_1\to v_2$ are given by $s(\alpha)=v_1$ and $t(\alpha)=v_2$. By convention, we compose the morphisms from left to right, ie if $\alpha_1\colon v_1 \to v_2$ and $\alpha_2\colon v_2 \to v_3$ are two paths, their composition is a path $\alpha_1\alpha_2\colon v_1 \to v_3$. For $S \subset X$, we denote by $\Pi_1(X,S)$ the full subcategory of $\Pi_1(X)$ whose objects are points in S. For a group G, the set $\mathrm{Hom}(\Pi_1(X,S),G)$ denotes the set of functors $\rho\colon \Pi_1(X,S) \to G$, where G is seen as a category with one element. With our conventions, if $t(\alpha_1) = s(\alpha_2)$, then $\rho(\alpha_1\alpha_2) = \rho(\alpha_1)\rho(\alpha_2)$.

Let $\mathcal{R}_{\mathrm{SL}_2}(\underline{\Sigma})$ be the set of functors $\rho\colon \Pi_1(\Sigma_{\mathcal{P}}) \to \mathrm{SL}_2$ whose restriction to $\Pi_1(\partial\Sigma_{\mathcal{P}}) \subset \Pi_1(\Sigma_{\mathcal{P}})$ is the constant map with value the neutral element $\mathbb{1}_2 \in \mathrm{SL}_2$. Let $\mathcal{G}$ be the group of maps $g\colon \Sigma_{\mathcal{P}} \to \mathrm{SL}_2$ whose restriction to $\partial\Sigma_{\mathcal{P}}$ is constant with value $\mathbb{1}_2$ and with finite support. It acts on $\mathcal{R}_{\mathrm{SL}_2}(\underline{\Sigma})$ by the formula

$$g\cdot\rho(\alpha) := g(s(\alpha))^{-1}\rho(\alpha)g(t(\alpha)), \quad \rho \in \mathcal{R}_{\mathrm{SL}_2}(\underline{\Sigma}), g \in \mathcal{G}, \alpha \in \Pi_1(\Sigma_{\mathcal{P}}).$$

Both $\mathcal{R}_{\mathrm{SL}_2}(\underline{\Sigma})$ and $\mathcal{G}$ have a structure of affine scheme and the action is algebraic so we can define the GIT quotient

$$\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma}) := \mathcal{R}_{\mathrm{SL}_2}(\underline{\Sigma}) /\!\!/ \mathcal{G}. \tag{28}$$

The character variety turns out to be an affine Poisson variety whose Poisson structure (given by a generalized Goldman formula) depends on a choice of orientation of the boundary arcs. It is proved in [Korinman 2019, Theorem 1.1] that its algebra of regular functions $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ is well behaved under triangular decompositions: for a topological triangulation Δ, there are an injective Poisson morphism $i^\Delta\colon \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})] \hookrightarrow \bigotimes_{\mathbb{T}\in F(\Delta)} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]$ and Poisson Hopf comodule maps Δ^L and Δ^R such that the following sequence is exact:

$$0 \to \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})] \xrightarrow{i^\Delta} \bigotimes_{\mathbb{T}\in F(\Delta)} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \xrightarrow{\Delta^L - \sigma\circ\Delta^R} \Big(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)} \mathbb{C}[\mathrm{SL}_2]\Big) \otimes \Big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]\Big). \tag{29}$$

In the present paper, we proceed by describing the character variety for the bigon and the triangle, together with the Hopf comodule maps Δ^L and Δ^R. Then, in virtue of the above property, we characterize the Poisson structure of the relative character variety for any triangulated punctured surface as the kernel of $\Delta^L - \sigma \circ \Delta^R$.

First, recall that $\mathfrak{sl}_2$ denotes the Lie algebra of the 2×2 traceless matrices. It has a basis formed by

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In order to define the Poisson structure, we will need the following.

Definition 3.2 The *classical r–matrices* $r^\pm \in \mathfrak{sl}_2^{\otimes 2}$ are the bivectors $r^+ := \frac{1}{2}H\otimes H + 2E\otimes F$ and $r^- := \frac{1}{2}H\otimes H + 2F\otimes E$. Their symmetric part $\tau = \frac{1}{2}H\otimes H + E\otimes F + F\otimes E$ is the invariant bivector associated to the (suitably normalized) Killing form and we denote by $\bar{r}^+ := E\otimes F - F\otimes E =: -\bar{r}^-$ their skew-symmetric part.

The classical r–matrices satisfy the classical Yang–Baxter equation (see [Chari and Pressley 1994, Section 2.1; Drinfeld 1983] for details).

Notation 3.3 Given a a boundary arc of Σ, we write $\mathfrak{o}(a) = +$ if the $\mathfrak{o}$–orientation of a coincides with the orientation induced by the orientation of $\Sigma_{\mathcal{P}}$, and write $\mathfrak{o}(a) = -$ if the orientation are opposite.

3.1.1 The bigon Consider the bigon $\mathbb{B}$ and write $\mathfrak{o}(b_L) = \varepsilon_1$ and $\mathfrak{o}(b_R) = \varepsilon_2$.

Definition 3.4 The relative character variety of the bigon is $\mathcal{X}_{\mathrm{SL}_2}(\mathbb{B}) := \mathrm{SL}_2(\mathbb{C})$. Denote by

$$N = \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix}$$

the 2×2 matrix with coefficients in $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{B})]$. The Poisson bracket associated to $\mathfrak{o}$ is defined by

$$\{N \otimes N\}^{\varepsilon_1, \varepsilon_2} := \bar{r}^{\varepsilon_1}(N \otimes N) + (N \otimes N)\bar{r}^{\varepsilon_2}.$$

Here we used the classical notation $\{N \otimes N\}$ to denote the matrix defined by $\{N \otimes N\}_{\varepsilon\varepsilon'\mu\mu'} = \{x_{\varepsilon\varepsilon'}, x_{\mu\mu'}\}$ (see for instance [Chari and Pressley 1994, Section 2.2.A] for details on this notation).

Denote the Poisson variety $(\mathbb{C}[\mathrm{SL}_2], \{\cdot,\cdot\}^{\varepsilon_1,\varepsilon_2})$ by $\mathbb{C}[\mathrm{SL}_2]^{\varepsilon_1,\varepsilon_2}$. Note that $\{\cdot,\cdot\}^{\varepsilon_1,\varepsilon_2} = -\{\cdot,\cdot\}^{-\varepsilon_1,-\varepsilon_2}$. By [Korinman 2019, Lemma 4.1], the coproduct $\Delta\colon \mathbb{C}[\mathrm{SL}_2]^{\varepsilon_1,\varepsilon_2} \to \mathbb{C}[\mathrm{SL}_2]^{\varepsilon_1,\varepsilon} \otimes \mathbb{C}[\mathrm{SL}_2]^{-\varepsilon,\varepsilon_2}$ and the antipode $S\colon \mathbb{C}[\mathrm{SL}_2]^{\varepsilon_1,\varepsilon_2} \to \mathbb{C}[\mathrm{SL}_2]^{-\varepsilon_1,-\varepsilon_2}$ are Poisson morphisms. In particular, the Poisson brackets $\{\cdot,\cdot\}^{-,+}$ and $\{\cdot,\cdot\}^{+,-}$ are the only ones which endow $\mathrm{SL}_2(\mathbb{C})$ with a Poisson–Lie structure.

3.1.2 The triangle Consider the triangle $\mathbb{T}$ and fix an orientation $\mathfrak{o}$ of each of its three boundary arcs a, b and c. We will use the notation $s(\alpha) = t(\beta) := c$, $s(\gamma) = t(\alpha) := b$ and $s(\beta) = t(\gamma) := a$. Here, for instance, we think of α as an oriented path joining a point in $c = s(\alpha)$ (source) to a point in $b = t(\alpha)$ (target).

Definition 3.5 The relative character variety of the triangle is the affine variety

$$\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T}) := \{(M_\alpha, M_\beta, M_\gamma) \in \mathrm{SL}_2(\mathbb{C})^3 \mid M_\gamma M_\beta M_\alpha = \mathbb{1}\}.$$

Given $\delta \in \{\alpha, \beta, \gamma\}$, denote by

$$N_\delta := \begin{pmatrix} \delta(+,+) & \delta(+,-) \\ \delta(-,+) & \delta(-,-) \end{pmatrix}$$

the 2×2 matrix with coefficients in $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]$. The Poisson bracket $\{\cdot,\cdot\}^{\mathfrak{o}}$ is defined by the formulas

$$\begin{aligned}
\{N_\delta \otimes N_\delta\}^{\mathfrak{o}} &:= \bar{r}^{\mathfrak{o}(s(\delta))}(N_\delta \otimes N_\delta) + (N_\delta \otimes N_\delta)\bar{r}^{\mathfrak{o}(t(\delta))}, \quad \delta \in \{\alpha, \beta, \gamma\}, \\
\{N_\alpha \otimes N_\gamma\}^{\mathfrak{o}} &:= -(N_\alpha \otimes \mathbb{1}) r^{\mathfrak{o}(b)} (\mathbb{1} \otimes N_\gamma), \\
\{N_\gamma \otimes N_\beta\}^{\mathfrak{o}} &:= -(N_\gamma \otimes \mathbb{1}) r^{\mathfrak{o}(a)} (\mathbb{1} \otimes N_\beta), \\
\{N_\beta \otimes N_\alpha\}^{\mathfrak{o}} &:= -(N_\beta \otimes \mathbb{1}) r^{\mathfrak{o}(c)} (\mathbb{1} \otimes N_\alpha).
\end{aligned}$$

Note that, writing

$$S(N_\delta) := \begin{pmatrix} \delta(-,-) & -\delta(+,-) \\ -\delta(-,+) & \delta(+,+) \end{pmatrix},$$

the last expressions can be rewritten in the form

$$\{N_\alpha \otimes S(N_\gamma)\}^{\mathfrak{o}} = (N_\alpha \otimes S(N_\gamma)) r^{\mathfrak{o}(b)},$$
$$\{N_\gamma \otimes S(N_\beta)\}^{\mathfrak{o}} = (N_\gamma \otimes S(N_\beta)) r^{\mathfrak{o}(a)},$$
$$\{N_\beta \otimes S(N_\alpha)\}^{\mathfrak{o}} = (N_\beta \otimes S(N_\alpha)) r^{\mathfrak{o}(c)}.$$

Given a boundary arc $d \in \{a, b, c\}$, we define a left Hopf-comodule

$$\Delta_d^L \colon \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \to \mathbb{C}[\mathrm{SL}_2]^{(+\mathfrak{o}(d),-\mathfrak{o}(d))} \otimes \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})],$$
$$\begin{pmatrix} \Delta_d^L(\delta(+,+)) & \Delta_d^L(\delta(+,-)) \\ \Delta_d^L(\delta(-,+)) & \Delta_d^L(\delta(-,-)) \end{pmatrix} := \begin{cases} \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix} \otimes N_\delta & \text{if } s(\delta) = d, \\ \mathbb{1} \otimes N_\delta & \text{otherwise.} \end{cases}$$

Similarly, define a right Hopf-comodule $\Delta_d^R \colon \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \to \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \otimes \mathbb{C}[\mathrm{SL}_2]^{(-\mathfrak{o}(d),+\mathfrak{o}(d))}$ by

$$\begin{pmatrix} \Delta_d^R(\delta(+,+)) & \Delta_d^R(\delta(+,-)) \\ \Delta_d^R(\delta(-,+)) & \Delta_d^R(\delta(-,-)) \end{pmatrix} := \begin{cases} N_\delta \otimes \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix} & \text{if } t(\delta) = d, \\ N_\delta \otimes \mathbb{1} & \text{otherwise.} \end{cases}$$

By [Korinman 2019, Lemma 4.6], both Δ_d^L and Δ_d^R are Poisson morphisms.

3.1.3 The general case Let Σ be a punctured surface, Δ a topological triangulation of Σ, and $\mathfrak{o}_\Delta$ an orientation of each edge of Δ. For a face $\mathbb{T} \in F(\Delta)$, let $\mathfrak{o}_{\mathbb{T}}$ be the orientation of its boundary arcs given by $\mathfrak{o}_\Delta$. Equip the algebra $\bigotimes_{\mathbb{T} \in F(\Delta)} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]^{\mathfrak{o}_{\mathbb{T}}}$ with the Poisson bracket defined in Definition 3.5. Each inner edge $e \in \mathring{\mathcal{E}}(\Delta)$ lifts to two oriented boundary arcs in $\Sigma_\Delta := \bigsqcup_{\mathbb{T} \in F(\Delta)} \mathbb{T}$. We denote by e_L the lift of e whose orientation coincides with the induced orientation of Σ_Δ and by e_R the other lift. The comodule maps $\Delta_{e_L}^L$ and $\Delta_{e_R}^R$ induce the comodule maps

$$\Delta^L \colon \bigotimes_{\mathbb{T} \in F(\Delta)} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]^{\mathfrak{o}_{\mathbb{T}}} \to \Big(\bigotimes_{e \in \mathring{\mathcal{E}}(\Delta)} \mathbb{C}[\mathrm{SL}_2]^{-,+} \Big) \otimes \Big(\bigotimes_{\mathbb{T} \in F(\Delta)} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]^{\mathfrak{o}_{\mathbb{T}}} \Big),$$
$$\Delta^R \colon \bigotimes_{\mathbb{T} \in F(\Delta)} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]^{\mathfrak{o}_{\mathbb{T}}} \to \Big(\bigotimes_{\mathbb{T} \in F(\Delta)} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]^{\mathfrak{o}_{\mathbb{T}}} \Big) \otimes \Big(\bigotimes_{e \in \mathring{\mathcal{E}}(\Delta)} \mathbb{C}[\mathrm{SL}_2]^{-,+} \Big).$$

Definition 3.6 The relative character variety $\mathcal{X}_{\mathrm{SL}_2}(\Sigma)$ is the affine variety whose algebra of regular functions is the kernel of $\Delta^L - \sigma \circ \Delta^R$.

Lemma 3.7 [Korinman 2019, Theorem 1.4] *As a Poisson variety, $\mathcal{X}_{\mathrm{SL}_2}(\Sigma)$ only depends, up to canonical isomorphism, on the marked surface Σ and the orientation $\mathfrak{o}$ of the boundary arcs (so does not depend on the triangulation Δ or on $\mathfrak{o}_\Delta$).*

We denote by $\{\cdot,\cdot\}^{\mathfrak{o}}$ the Poisson bracket on $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\Sigma)]$. More precisely, in [Korinman 2019], we endow the variety $\mathcal{X}_{\mathrm{SL}_2}(\Sigma) := \mathcal{R}_{\mathrm{SL}_2}(\Sigma) /\!\!/ \mathcal{G}$ (which only depends on Σ) with a Poisson structure, given by a generalization of Goldman formula, which only depends on $\mathfrak{o}$. We then construct a splitting morphism i^Δ and prove in [Korinman 2019, Theorem 1.4] that we have the exact sequence (29), thus $\mathcal{X}_{\mathrm{SL}_2}(\Sigma)$ can be alternatively defined using Definition 3.6.

Moreover when Σ is closed, the Poisson variety $\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})$ is canonically isomorphic to the "classical" (Culler–Shalen) character variety with its Goldman Poisson structure [Korinman 2019, Theorem 1.1].

3.2 Relation between relative character varieties and stated skein algebras

The goal of this subsection is to prove Theorem 1.3 which we recall here for the reader's convenience:

Theorem 3.8 *Suppose that $\underline{\Sigma}$ has a topological triangulation Δ. Let $\mathfrak{o}_\Delta$ be an orientation of the edges of Δ and $\mathfrak{o}$ be the induced orientation of the boundary arcs of $\underline{\Sigma}$. There exists an isomorphism of Poisson algebras*

$$\Psi^{(\Delta,\mathfrak{o}_\Delta)}\colon (\mathcal{S}_{+1}(\underline{\Sigma}), \{\cdot,\cdot\}^s) \xrightarrow{\cong} (\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})], \{\cdot,\cdot\}^{\mathfrak{o}}).$$

Moreover, the above isomorphism exists for small punctured surfaces (see Definition 2.8), for which it only depends on $\mathfrak{o}$.

We first prove this theorem for the bigon and the triangle, then we prove the general case using a topological triangulation.

3.2.1 The case of the bigon

Let

$$M := \begin{pmatrix} \alpha_{++} & \alpha_{+-} \\ \alpha_{-+} & \alpha_{--} \end{pmatrix}, \quad N := \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be three matrices with coefficients in $\mathcal{S}_{+1}(\mathbb{B})$, $\mathbb{C}[\mathrm{SL}_2]$ and $\mathbb{C}$, respectively.

Lemma 3.9 *For $\varepsilon_1, \varepsilon_2 \in \{-,+\}$, there is a Poisson isomorphism*

$$\Psi^{\varepsilon_1,\varepsilon_2}\colon (\mathcal{S}_{+1}(\mathbb{B}), \{\cdot,\cdot\}^s) \xrightarrow{\cong} \mathbb{C}[\mathrm{SL}_2]^{\varepsilon_1,\varepsilon_2}$$

defined by

$$\Psi^{\varepsilon_1,\varepsilon_2}(M) := \begin{cases} N & \text{if } (\varepsilon_1,\varepsilon_2) = (-,+), \\ CNC & \text{if } (\varepsilon_1,\varepsilon_2) = (+,-), \\ -CN & \text{if } (\varepsilon_1,\varepsilon_2) = (+,+), \\ -NC & \text{if } (\varepsilon_1,\varepsilon_2) = (-,-). \end{cases}$$

Proof That $\Psi^{\varepsilon_1,\varepsilon_2}$ is an isomorphism of algebras follows from the fact that $\det(C) = 1$. Let us see the compatibility of $\Psi^{\varepsilon_1,\varepsilon_2}$ with the Poisson structures. For $(\varepsilon_1,\varepsilon_2) = (-,+)$, this follows from a direct comparison of Definition 3.4 and Example 2.54. Indeed,

$$\begin{aligned}
\{N \otimes N\}^{-,+} &= \bar{r}^-(N\otimes N) + (N\otimes N)\bar{r}^+ \\
&= (F\otimes E - E\otimes F)(N\otimes N) + (N\otimes N)(E\otimes F - F\otimes E) \\
&= \begin{pmatrix} 0 & x_{++} \\ 0 & x_{-+} \end{pmatrix} \otimes \begin{pmatrix} x_{+-} & 0 \\ x_{--} & 0 \end{pmatrix} - \begin{pmatrix} x_{+-} & 0 \\ x_{--} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x_{++} \\ 0 & x_{-+} \end{pmatrix} \\
&\qquad + \begin{pmatrix} 0 & 0 \\ x_{++} & x_{+-} \end{pmatrix} \otimes \begin{pmatrix} x_{-+} & x_{--} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} x_{-+} & x_{--} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ x_{++} & x_{+-} \end{pmatrix}.
\end{aligned}$$

We recover the formulas computed in Example 2.54. For $(\varepsilon_1,\varepsilon_2)=(+,+)$, we prove that the isomorphism $\varphi\colon \mathbb{C}[\mathrm{SL}_2]^{-,+} \xrightarrow{\cong} \mathbb{C}[\mathrm{SL}_2]^{+,+}$ given by $\varphi := \Psi^{+,+}\circ(\Psi^{-,+})^{-1}$, is a Poisson morphism. Note that $\varphi(N)=-CN$ and that $(C\otimes C)\bar{r}^{\varepsilon}=\bar{r}^{-\varepsilon}(C\otimes C)$. It follows that

$$\begin{aligned}\{\varphi(N)\otimes\varphi(N)\}^{+,+} &= \bar{r}^{+}(CN\otimes CN)+(CN\otimes CN)\bar{r}^{+}\\ &= (C\otimes C)(\bar{r}^{-}(N\otimes N)+(N\otimes N)\bar{r}^{+}) = \varphi^{\otimes 2}(\{N\otimes N\}^{-,+}),\end{aligned}$$

which proves the claim. The two remaining cases for $(\varepsilon_1,\varepsilon_2)$ are proved similarly. □

3.2.2 The case of the triangle For $\delta\in\{\alpha,\beta,\gamma\}$, let

$$M_\delta := \begin{pmatrix}\delta_{++} & \delta_{+-}\\ \delta_{-+} & \delta_{--}\end{pmatrix} \quad\text{and}\quad N_\delta := \begin{pmatrix}\delta(+,+) & \delta(+,-)\\ \delta(-,+) & \delta(-,-)\end{pmatrix}$$

be two matrices with coefficients in $\mathcal{S}_{+1}(\mathbb{T})$ and $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]$, respectively.

Lemma 3.10 *There is a Poisson isomorphism $\Psi^{\mathfrak{o}}\colon (\mathcal{S}_{+1}(\mathbb{T}),\{\cdot,\cdot\}^s) \xrightarrow{\cong} \big(\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})],\{\cdot,\cdot\}^{\mathfrak{o}}\big)$ defined by*

$$\Psi^{\mathfrak{o}}(M_\delta) := \begin{cases} N_\delta & \text{if } (\mathfrak{o}(s(\alpha)),\mathfrak{o}(t(\alpha)))=(-,+),\\ CN_\delta C & \text{if } (\mathfrak{o}(s(\alpha)),\mathfrak{o}(t(\alpha)))=(+,-),\\ -CN_\delta & \text{if } (\mathfrak{o}(s(\alpha)),\mathfrak{o}(t(\alpha)))=(+,+),\\ -N_\delta C & \text{if } (\mathfrak{o}(s(\alpha)),\mathfrak{o}(t(\alpha)))=(-,-),\end{cases}$$

for each $\delta\in\{\alpha,\beta,\gamma\}$. Moreover, if $d\in\{a,b,c\}$ is a boundary arc of $\mathbb{T}$, the following diagrams commute:

$$\begin{tikzcd} \mathcal{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta^L_d"] \arrow[d, "\Psi^{\mathfrak{o}}", "\cong"'] & \mathcal{S}_{+1}(\mathbb{B})\otimes\mathcal{S}_{+1}(\mathbb{T}) \arrow[d, "\Psi^{\mathfrak{o}(d),-\mathfrak{o}(d)}\otimes\Psi^{\mathfrak{o}}", "\cong"'] \\ \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \arrow[r, "\Delta^L_d"] & \mathbb{C}[\mathrm{SL}_2]\otimes\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta^R_d"] \arrow[d, "\Psi^{\mathfrak{o}}", "\cong"'] & \mathcal{S}_{+1}(\mathbb{T})\otimes\mathcal{S}_{+1}(\mathbb{B}) \arrow[d, "\Psi^{\mathfrak{o}}\otimes\Psi^{-\mathfrak{o}(d),\mathfrak{o}(d)}", "\cong"'] \\ \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \arrow[r, "\Delta^R_d"] & \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]\otimes\mathbb{C}[\mathrm{SL}_2] \end{tikzcd}$$

Proof That $\Psi^{\mathfrak{o}}$ is an algebra morphism follows from Lemma 2.29. For $\delta\in\{\alpha,\beta,\gamma\}$, the equality $(\Psi^{\mathfrak{o}})^{\otimes 2}(\{\delta_{\varepsilon\varepsilon'},\delta_{\mu\mu'}\}^{\mathfrak{o}}) = \{\Psi^{\mathfrak{o}}(\delta_{\varepsilon\varepsilon'}),\Psi^{\mathfrak{o}}(\delta_{\mu\mu'})\}^s$ follows from the same computation that the proof of Lemma 3.9. For $\mathfrak{o}(a)=\mathfrak{o}(b)=\mathfrak{o}(c)=+$,

$$\begin{aligned}\{N_\alpha\otimes N_\gamma\}^{\mathfrak{o}} &= -(N_\alpha\otimes\mathbb{1})\Big(\frac{1}{2}H\otimes H+2E\otimes F\Big)(\mathbb{1}\otimes N_\gamma)\\ &= -\frac{1}{2}\begin{pmatrix}\alpha(+,+) & -\alpha(+,-)\\ \alpha(-,+) & -\alpha(-,-)\end{pmatrix}\otimes\begin{pmatrix}\gamma(+,+) & \gamma(+,-)\\ -\gamma(-,+) & -\gamma(-,-)\end{pmatrix}\\ &\qquad -2\begin{pmatrix}0 & \alpha(+,+)\\ 0 & \alpha(-,+)\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ \gamma(+,+) & \gamma(+,-)\end{pmatrix}.\end{aligned}$$

We recover the formulas of Example 2.55; hence $(\Psi^{\mathfrak{o}})^{\otimes 2}(\{\alpha_{\varepsilon\varepsilon'},\gamma_{\mu\mu'}\}^{\mathfrak{o}}) = \{\Psi^{\mathfrak{o}}(\alpha_{\varepsilon\varepsilon'}),\Psi^{\mathfrak{o}}(\gamma_{\mu\mu'})\}^s$. We get similar formulas by permuting cyclically the arcs γ, β and α. This proves that $\Psi^{\mathfrak{o}}$ is a Poisson morphism when $\mathfrak{o}(a)=\mathfrak{o}(b)=\mathfrak{o}(c)=+$. For another choice $\mathfrak{o}'$ of orientation of the boundary arcs, we prove that $\Psi^{\mathfrak{o}'}$ is Poisson by showing that the isomorphism $\Psi^{\mathfrak{o}'}\circ(\Psi^{\mathfrak{o}})^{-1}$ is Poisson. This follows from a similar computation to the one in the proof of Lemma 3.9 by using the fact that $(C\otimes C)r^{\varepsilon}=r^{-\varepsilon}(C\otimes C)$. The fact that the two diagrams in the lemma commute follows from a straightforward computation. $\square$

3.2.3 The general case: proof of Theorem 1.3 Consider a topological triangulation Δ of a punctured surface $\underline{\Sigma}$, together with a choice $\mathfrak{o}_\Delta$ of orientation of its edges. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, "i^\Delta"] \arrow[d, dotted, "\exists!\Psi^{(\Delta,\mathfrak{o}_\Delta)}"', "\cong"] & \otimes_{\mathbb{T}}\mathcal{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta^L-\sigma\circ\Delta^R"] \arrow[d, "\otimes_{\mathbb{T}}\Psi^{\mathfrak{o}\mathbb{T}}"', "\cong"] & (\otimes_e\mathcal{S}_{+1}(\mathbb{B}))\otimes(\otimes_{\mathbb{T}}\mathcal{S}_{+1}(\mathbb{T})) \arrow[d, "(\otimes_e\Psi^{-,+})\otimes(\otimes_{\mathbb{T}}\Psi^{\mathfrak{o}(\mathbb{T})})"', "\cong"] \\
0 \arrow[r] & \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})] \arrow[r, "i^\Delta"] & \otimes_{\mathbb{T}}\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \arrow[r, "\Delta^L-\sigma\circ\Delta^R"] & (\otimes_e\mathbb{C}[\mathrm{SL}_2]^{-,+})\otimes(\otimes_{\mathbb{T}}\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})])
\end{tikzcd}$$

In this diagram, both lines are exact and all morphisms are Poisson by Lemma 2.47 and [Korinman 2019]; hence there exists a unique Poisson isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}\colon(\mathcal{S}_{+1}(\underline{\Sigma}),\{\cdot,\cdot\}^s)\xrightarrow{\cong}(\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})],\{\cdot,\cdot\}^{\mathfrak{o}})$ induced by restriction of $\otimes_{\mathbb{T}}\Psi^{\mathfrak{o}(\mathbb{T})}$. This concludes the proof.

3.3 Relative spin structures and explicit formulas

The goal of this subsection is to give an explicit formula for the morphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$, when evaluated on the generators of $\mathcal{S}_{+1}(\underline{\Sigma})$. A key point is to have a global method to compute some signs that depend on the combinatorial data $(\Delta,\mathfrak{o}_\Delta)$. We provide such a method by introducing the notion of relative spin structure, which gives a geometric interpretation these signs. We end the section by relating the $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ with the morphism of [Costantino and Lê 2022, Theorem 8.12].

3.3.1 Relative spin structures Since the classical identifications between skein algebras of closed punctured surfaces and character varieties are indexed by spin structures, it is natural to expect that the combinatorial data $(\Delta,\mathfrak{o}_\Delta)$ indexing the isomorphism of Theorem 1.3 encode a generalization of the notion of spin structures which would have a good behavior for the operation of gluing boundary arcs together. Before defining this notion, we introduce some notation.

Notation 3.11 (1) In this subsection, $\underline{\Sigma}=(\Sigma,\mathcal{P})$ will denote a triangulable punctured surface, $\mathfrak{o}$ an orientation of its boundary arcs and $(\Delta,\mathfrak{o}_\Delta)$ a combinatorial data, and we equip $\Sigma_{\mathcal{P}}$ with a Riemannian structure compatible with the orientation. For each boundary arc a, we fix a point $v_a\in a$. If $\partial\Sigma\neq\varnothing$, we write $\mathbb{V}:=\{v_a\}_a$ where a runs through the set of boundary arcs. If Σ is closed, we fix an arbitrarily point v_a in each connected component a of $\Sigma_{\mathcal{P}}$ and write $\mathbb{V}:=\{v_a\}_a$.

(2) Let $\pi\colon U\Sigma_{\mathcal{P}}\to\Sigma_{\mathcal{P}}$ denote the unitary tangent bundle. For $\vec{v}=(v,u)\in U\Sigma_{\mathcal{P}}$, we denote by $-\vec{v}=(v,-u)$ the vector with opposite orientation. Let $\theta_{\vec{v}}^{1/2}\colon\vec{v}\to-\vec{v}$ be the class in $\Pi_1(U\Sigma_{\mathcal{P}})$ of a path making a half-twist in the fiber over $\pi(\vec{v})$ in the direction given by the orientation and write $\theta_{\vec{v}}:=\theta_{\vec{v}}^{1/2}\theta_{-\vec{v}}^{1/2}$. For simplicity, for a path $\alpha\colon\vec{v}_1\to\vec{v}_2$, we will write $\theta^{1/2}\alpha$ and $\alpha\theta^{1/2}$ instead of $\theta_{-\vec{v}_1}^{1/2}\alpha$ and $\alpha\theta_{\vec{v}_2}^{1/2}$ with no confusion possible. When $\partial\Sigma\neq\varnothing$, for each boundary arc a, we denote by $\vec{v}_a\in U\Sigma_{\mathcal{P}}$ the lift of v_a pointing in the direction of $\mathfrak{o}$. When Σ is closed, we fix an arbitrarily lift $\vec{v}_a$ of each v_a. We write $\widehat{\mathbb{V}}_+:=\{\vec{v}_a\}_a$ and $\widehat{\mathbb{V}}:=\{\vec{v}_a,-\vec{v}_a\}_a$.

Definition 3.12 A *relative spin structure* on $\underline{\Sigma}$ is a functor $W \in \mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+), \mathbb{Z}/2\mathbb{Z})$ such that $W(\theta_{\vec{v}}) = 1$ for all $\vec{v} \in \widehat{\mathbb{V}}_+$. We denote by $\mathrm{Spin}(\underline{\Sigma})$ the set of relative spin structures on $\underline{\Sigma}$.

Remark 3.13 When Σ is closed and connected, an element $W \in \mathrm{Spin}(\underline{\Sigma})$ is a group morphism $W\colon \pi_1(U\Sigma_{\mathcal{P}}, \vec{v}_0^{\,+}) \to \mathbb{Z}/2\mathbb{Z}$ such that $W(\theta_{\vec{v}_0^{\,+}}) = 1$. Since $\mathbb{Z}/2\mathbb{Z}$ is abelian, such a morphism is equivalent to a group morphism $\underline{W}\colon \mathrm{H}_1(U\Sigma_{\mathcal{P}}, \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ satisfying $W([\theta]) = 1$. Such a morphism $\underline{W}$ defines a regular double covering $\widetilde{U}$ of $U\Sigma_{\mathcal{P}}$ such that the covering on each fiber is nontrivial. Since $\mathrm{Spin}(2)$ is the only nontrivial double covering of $\mathrm{SO}(2)$, the space $\widetilde{U}$ is the total space of a $\mathrm{Spin}(2)$ fiber bundle over $\Sigma_{\mathcal{P}}$ lifting the bundle of orthogonal frames induced by the metric; hence it defines a spin structure. There is actually a one-to-one correspondence between isomorphism classes of spin structures and such morphisms $\underline{W}$ (see [Milnor 1963] for details). Therefore a relative spin structure is the same as a "standard" spin structure in the closed case. When the surface has nonempty boundary, an element $W \in \mathrm{Spin}(\underline{\Sigma})$ still induces a group morphism $\underline{W}$, thus a spin structure. However, the functor W contains more information than $\underline{W}$ which permits to "glue" relative spin structures together.

Let a and b be two distinct boundary arcs of $\underline{\Sigma}$ and denote by $p\colon \Sigma_{\mathcal{P}} \to \Sigma_{\mathcal{P}}|_{a\#b}$ the projection. Write $c := p(a) = p(b)$. We assume that

(1) the restriction $p\colon \Sigma_{\mathcal{P}} \setminus (a \cup b) \to \Sigma_{\mathcal{P}}|_{a\#b} \setminus c$ is an isometry,

(2) the restriction $p\colon a \to c$ and $p\colon b \to c$ are isometries, and

(3) the orientations $\mathfrak{o}$ of a and b coincide when gluing the arcs and $p(v_a) = p(v_b) =: v_c$.

The projection induces a lift $\vec{v}_c \in U\Sigma_{\mathcal{P}}|_{a\#b}$ of v_c and a functor

$$p_*\colon \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+) \to \Pi_1(U\Sigma_{\mathcal{P}}|_{a\#b}, \widehat{\mathbb{V}}_+^{a\#b} \cup \{\vec{v}_c\}).$$

Lemma 3.14 *For $W \in \mathrm{Spin}(\Sigma)$, there exists a unique $W|_{a\#b} \in \mathrm{Spin}(\Sigma|_{a\#b})$ such that*

$$W|_{a\#b}(p_*(\alpha)) = W(\alpha)$$

for all $\alpha \in \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+)$.

Proof Note that the image of p_* generates the groupoid $\Pi_1(U\Sigma_{\mathcal{P}}|_{a\#b}, \widehat{\mathbb{V}}_+^{a\#b} \cup \{\vec{v}_c\})$ in the sense that any path $\alpha \in \Pi_1(U\Sigma_{\mathcal{P}}|_{a\#b}, \widehat{\mathbb{V}}_+^{a\#b} \cup \{\vec{v}_c\})$ can be written as a composition $\alpha = p_*(\alpha_1) \cdots p_*(\alpha_n)$ for some $\alpha_i \in \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+)$. Hence for $W \in \mathrm{Spin}(\underline{\Sigma})$, there exists a unique functor

$$\widetilde{W}\colon \Pi_1(U\Sigma_{\mathcal{P}}|_{a\#b}, \widehat{\mathbb{V}}_+^{a\#b} \cup \{\vec{v}_c\}) \to \mathbb{Z}/2\mathbb{Z}$$

such that $\widetilde{W}(\pi_*(\alpha)) = W(\alpha)$ for all $\alpha \in \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+)$, and $W|_{a\#b}$ has to be the restriction of $\widetilde{W}$ to the full subcategory $\Pi_1(U\Sigma_{\mathcal{P}}|_{a\#b}, \widehat{\mathbb{V}}_+^{a\#b})$. □

Note that the map $r_{a\#b}\colon \mathrm{Spin}(\underline{\Sigma}) \to \mathrm{Spin}(\underline{\Sigma}|_{a\#b})$ sending W to $W|_{a\#b}$ is surjective but not injective. Indeed when lifting a functor in $\mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+), \mathbb{Z}/2\mathbb{Z})$ to a functor in $\mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+ \cup \{\vec{v}_c\}), \mathbb{Z}/2\mathbb{Z})$ there is a $\mathbb{Z}/2\mathbb{Z}$ ambiguity. Note also that if a, b, c and d are four distinct boundary arcs, one obviously has $r_{a\#b} \circ r_{c\#d} = r_{c\#d} \circ r_{a\#b}$. In particular, once some combinatorial data $(\Delta, \mathfrak{o}_\Delta)$ of $\underline{\Sigma}$ are fixed, any relative spin structure on $\underline{\Sigma}$ can be obtained by gluing some relative spin structure on each face of the triangulation.

3.3.2 Lifts of embedded curves and the function w Let us call *embedded arc* a smooth embedding $\alpha\colon [0,1] \to \Sigma_{\mathcal{P}}$ such that $\alpha(0), \alpha(1) \in \partial\Sigma_{\mathcal{P}}$. To any embedded arc and any simple closed curve, we associate two lifts in $U\Sigma_{\mathcal{P}}$ as follows.

For α an embedded arc oriented from the boundary arc a to the boundary arc b, we isotope α (in the class of embedded arc) such that $\alpha(0) = v_a$, $\alpha(1) = v_b$, the vectors $\alpha'(0)$ and $\alpha'(1)$ are tangent to a and b, and such that $\alpha'(0)$ points in the direction of a opposite to the orientation induced by the orientation of $\Sigma_{\mathcal{P}}$ and $\alpha'(1)$ points in the direction of b induced by the orientation of $\Sigma_{\mathcal{P}}$. The *positive lift* of α is the homotopy class $\hat{\alpha}^+ \in \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}})$ of the continuous map $t \mapsto (\alpha(t), \alpha'(t)/\|\alpha'(t)\|)$.

For v a point in a boundary arc a, we write $\mathfrak{o}(v) = 0$ if the orientation of a agrees with the induced orientation of $\Sigma_{\mathcal{P}}$ and $\mathfrak{o}(v) = 1$ otherwise. The $\mathfrak{o}$–*lift* $\hat{\alpha}^{\mathfrak{o}} \in \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+)$ is defined by the formula

$$\hat{\alpha}^+ = (\theta^{1/2})^{1-\mathfrak{o}(s(\alpha))}\hat{\alpha}^{\mathfrak{o}}(\theta^{1/2})^{\mathfrak{o}(t(\alpha))}. \tag{30}$$

Let γ be a smooth embedded curve and $v \in \mathbb{V}$. We define $\hat{\gamma}_v^+$ as the as the homotopy class of a map $t \mapsto \big(\beta(t), \beta'(t)/\|\beta'(t)\|\big)$ where β is a smooth immersion $\beta\colon [0,1] \to \Sigma_{\mathcal{P}}$ which is isotopic to γ such that $\beta(0) = v = \beta(1)$ and $\beta'(0)$ points in the direction induced by the orientation of the surface for $\hat{\gamma}_v^+$. Similarly, we define $\hat{\gamma}_v^{\mathfrak{o}}$ as the homotopy class of a map $t \mapsto \big(\beta(t), \beta'(t)/\|\beta'(t)\|\big)$ where this time $\beta'(0)$ points in the direction of $\mathfrak{o}$ for $\hat{\gamma}_v^{\mathfrak{o}}$. If Σ is closed and γ is in a connected component b, we impose that $\hat{\gamma}_v^+ = \hat{\gamma}_v^{\mathfrak{o}}$ is defined from an immersion β such that $(\beta(0), \beta'(0)) = v_b$.

Notation 3.15 For $W \in \mathrm{Spin}(\underline{\Sigma})$ and α an embedded arc, we write $w(\alpha) := W(\hat{\alpha}^{\mathfrak{o}}) \in \mathbb{Z}/2\mathbb{Z}$. For γ a closed curve we write $w(\gamma) := W(\hat{\gamma}_v^{\mathfrak{o}})$.

Remark 3.16 The value $w(\gamma)$ associated to a closed curve is obviously independent of the choice of the point v. Moreover, as noted in Remark 3.13, the value $W(\hat{\gamma})$ only depends on the homology class $[\hat{\gamma}^{\mathfrak{o}}] \in \mathrm{H}_1(U\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z})$ and is closely related to the Johnson quadratic form as follows. Let $\{\gamma_i\}_{i=1,\dots,n}$ be a collection of simple closed curves. Johnson [1980, Theorem 1.A] proved that the class

$$y := \sum_{i=1}^{n} [\hat{\gamma}_i^{\mathfrak{o}}] + n[\theta] \in \mathrm{H}_1(U\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z})$$

only depends on the homology class of $x := \sum_{i=1}^n [\gamma_i] \in \mathrm{H}_1(\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z})$; hence the assignation $x \mapsto y$ defines a map (not a morphism) $\mathrm{H}_1(\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}_1(U\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z})$. Moreover, for a (relative) spin structure W, Johnson [1980, Theorem 1.B] proved that the map $\omega\colon \mathrm{H}_1(\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ defined by $\omega\big(\sum_{i=1}^n [\gamma_i]\big) := n + \sum_{i=1}^n w([\gamma_i]) \pmod 2$ satisfies the relation

$$\omega([\alpha + \beta]) = \omega([\alpha]) + \omega([\beta]) + \langle [\alpha], [\beta] \rangle;$$

hence ω is a quadratic form for $(\mathrm{H}_1(\Sigma_{\mathcal{P}};\mathbb{Z}/2\mathbb{Z}),\langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ represents the intersection form. Thus the values $w(\gamma)$ in Notation 3.15 are related to the Johnson quadratic form of the underlying spin structure by $\omega([\gamma]) = w(\gamma)+1 \pmod 2$.

3.3.3 Relative spin structures associated to combinatorial data In order to assign a relative spin structure to some combinatorial data $(\Delta,\mathfrak{o}_\Delta)$ in a canonical way, we need to assign to each triangle $\mathbb{T}$, equipped with an orientation $\mathfrak{o}_\mathbb{T}$ of its boundary arcs, a canonical relative spin structure and then glue the triangles along their faces. Let α, β and γ be the three paths in Figure 3 which generate the groupoid $\Pi_1(\mathbb{T},\mathbb{V})$ with relation $\gamma\beta\alpha=1$. Note that for any choice of $\mathfrak{o}_\mathbb{T}$, one has the relation $\hat{\gamma}^{\mathfrak{o}_\mathbb{T}}\hat{\beta}^{\mathfrak{o}_\mathbb{T}}\hat{\alpha}^{\mathfrak{o}_\mathbb{T}}=\theta^{-2}$. Hence a relative spin structure W on $\mathbb{T}$ is described by three elements $W(\hat{\alpha}^{\mathfrak{o}_\mathbb{T}})$, $W(\hat{\beta}^{\mathfrak{o}_\mathbb{T}})$, $W(\hat{\gamma}^{\mathfrak{o}_\mathbb{T}})\in\mathbb{Z}/2\mathbb{Z}$ such that $W(\hat{\alpha}^{\mathfrak{o}_\mathbb{T}})+W(\hat{\beta}^{\mathfrak{o}_\mathbb{T}})+W(\hat{\gamma}^{\mathfrak{o}_\mathbb{T}})=0$. Therefore there exist four different relative spin structures on $\mathbb{T}$.

Definition 3.17 The *distinguished* relative spin structure on $\mathbb{T}$ is the relative spin structure W such that $W(\hat{\alpha}^{\mathfrak{o}_\mathbb{T}})=W(\hat{\beta}^{\mathfrak{o}_\mathbb{T}})=W(\hat{\gamma}^{\mathfrak{o}_\mathbb{T}})=0$. For $\underline{\Sigma}$ a punctured surface with combinatorial data $(\Delta,\mathfrak{o}_\Delta)$, we associate a relative spin structure $W^{(\Delta,\mathfrak{o}_\Delta)}\in\mathrm{Spin}(\underline{\Sigma})$ by gluing together the distinguished spin structures on the faces of the triangulation.

Note that the distinguished relative spin structure W on $\mathbb{T}$ satisfies $w(\alpha)=w(\beta)=w(\gamma)=0$ and $w(\alpha^{-1})=w(\beta^{-1})=w(\gamma^{-1})=1$.

Remark 3.18 Since we associate to each face a specific (named distinguished) relative spin structure, there is no reason to believe that every spin structure on $\Sigma_{\mathcal{P}}$ can be associated to some combinatorial data. Moreover we will not investigate under which condition two combinatorial data induce the same relative spin structure. Novak and Runkel [2015] showed that any spin structure on a surface can be encoded by the combinatorial data consisting in a triangulation (with no degenerate face), an orientation of the edges and a choice of distinguished vertex for each face. Moreover they proved that two such combinatorial data induce the same spin structure if and only if they can be related by a sequence of elementary moves. It would be interesting to compare their approach to Definition 3.17.

We now state an explicit formula for the values $w(\alpha)$ associated to a relative spin structure $W^{(\Delta,\mathfrak{o}_\Delta)}$. For each edge $e\in\mathcal{E}(\Delta)$, fix a point $v_e\in e$ and let $\mathbb{V}^\Delta=\{v_e\}_{e\in\mathcal{E}(\Delta)}$. When $\partial\Sigma\neq\varnothing$, we assume that $\mathbb{V}^\Delta\cap\partial\Sigma_{\mathcal{P}}=\mathbb{V}$. When Σ is closed, we assume that $\mathbb{V}\subset\mathbb{V}^\Delta$. Let $\vec{v}_e\in U\Sigma_{\mathcal{P}}$ be the lift of v_e oriented in the direction of $\mathfrak{o}_\Delta$ and set $\widehat{\mathbb{V}}^\Delta_+ := \{\vec{v}_e \mid e\in\mathcal{E}(\Delta)\}$ and $\widehat{\mathbb{V}}^\Delta := \{\vec{v}_e,-\vec{v}_e \mid e\in\mathcal{E}(\Delta)\}$. Note that the set

$$\widehat{\mathbb{G}}^\Delta := \{(\hat{\alpha}^{\mathfrak{o}}_\mathbb{T})^{\pm1},(\hat{\beta}^{\mathfrak{o}}_\mathbb{T})^{\pm1},(\hat{\gamma}^{\mathfrak{o}}_\mathbb{T})^{\pm1} \mid \mathbb{T}\in F(\Delta)\}$$

generates the groupoid $\Pi_1(U\Sigma_{\mathcal{P}},\widehat{\mathbb{V}}^\Delta_+)$. By definition of the gluing operation, the functor $W^{(\Delta,\mathfrak{o}_\Delta)}$ is the restriction of the functor $\widetilde{W}\in\mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}},\widehat{\mathbb{V}}^\Delta_+),\mathbb{Z}/2\mathbb{Z})$ characterized by

$$\widetilde{W}(\hat{\alpha}^{\mathfrak{o}_\mathbb{T}}_\mathbb{T})=\widetilde{W}(\hat{\beta}^{\mathfrak{o}_\mathbb{T}}_\mathbb{T})=\widetilde{W}(\hat{\gamma}^{\mathfrak{o}_\mathbb{T}}_\mathbb{T})=0$$

for every face $\mathbb{T}$ and $\widetilde{W}(\theta_{\vec{v}})=1$ for any $\vec{v}\in\hat{\mathbb{V}}_+^{\Delta}$. Set $\mathbb{G}^{\Delta}:=\pi(\hat{\mathbb{G}}_+^{\Delta})=\{\alpha_{\mathbb{T}}^{\pm 1},\beta_{\mathbb{T}}^{\pm 1},\gamma_{\mathbb{T}}^{\pm 1};\mathbb{T}\in F(\Delta)\}$ and for $\delta\in\mathbb{G}^{\Delta}$ a path in $\mathbb{T}$, write $w(\delta):=\widetilde{W}(\hat{\delta}^{\mathfrak{o}_{\mathbb{T}}})$. Hence $w(\delta)=0$ if $\delta=\alpha_{\mathbb{T}}$, $\beta_{\mathbb{T}}$ or $\gamma_{\mathbb{T}}$ and $w(\delta)=1$ if $\delta=\alpha_{\mathbb{T}}^{-1}$, $\beta_{\mathbb{T}}^{-1}$ or $\gamma_{\mathbb{T}}^{-1}$.

Let α be either an embedded arc or a closed curve and choose a decomposition

$$\alpha=\alpha_1\cdots\alpha_n,\quad \alpha_i\in\mathbb{G}^{\Delta}, \tag{31}$$

such that either α_i and α_{i+1} lie in different faces $\mathbb{T}_i\neq\mathbb{T}_{i+1}$ of Δ, or $\mathbb{T}_i=\mathbb{T}_{i+1}$ is a degenerate triangle, with two boundary arcs glued together to give an arc c in $\Sigma_{\mathcal{P}}$, and $\alpha_i\alpha_{i+1}$ crosses $c=t(\alpha_i)=s(\alpha_{i+1})$ transversally. In the above statement, the indices i are taken in $\mathbb{Z}/n\mathbb{Z}$ when α is a closed curve. Note that such a decomposition is obtained by isotoping α transversally with minimal intersection to the edges of the triangulation, and then cutting α along the edges. For $(\mathbb{T},\mathfrak{o}_{\mathbb{T}})$ a triangle with oriented edges, a an edge and $v_a\in a$, recall that we write $\mathfrak{o}_{\mathbb{T}}(v_a)=0$ if the orientation of a corresponds to the orientation induced by the orientation of $\mathbb{T}$ and write $\mathfrak{o}_{\mathbb{T}}(v_a)=+1$ otherwise.

Lemma 3.19 *The function w associated to the relative spin structure $W^{(\Delta,\mathfrak{o}_\Delta)}$ is characterized by the formula*

$$w(\alpha)=\begin{cases}\sum_{i=1}^n w(\alpha_i)+\sum_{i=1}^{n-1}\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i)) \pmod 2 & \text{if } \alpha \text{ is an embedded arc},\\ \sum_{i=1}^n w(\alpha_i)+\sum_{i=1}^{n}\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i)) \pmod 2 & \text{if } \alpha \text{ is a closed curve}.\end{cases}$$

Proof First note that for the positive lifts,

$$\hat{\alpha}^+=\hat{\alpha}_1^+\cdots\hat{\alpha}_n^+.$$

This equality follows from the fact that the embedded curve chosen to represent $\hat{\alpha}^+$ can be isotoped such that it crosses tangentially the edges of Δ in such a way that, when cutting along the edges, one obtains the composition $\hat{\alpha}_1^+\cdots\hat{\alpha}_n^+$. Note also that this equality is essentially [Costantino and Lê 2022, Proposition 8.11]. Recall from (30) that $\hat{\alpha}_i^+=(\theta^{1/2})^{1-\mathfrak{o}(s(\alpha_i))}\hat{\alpha}_i^{\mathfrak{o}}(\theta^{1/2})^{\mathfrak{o}(t(\alpha_i))}$ and note that, since we assume that the faces $\mathbb{T}_i$ and $\mathbb{T}_{i+1}$ are distinct,

$$(1-\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i)))+\mathfrak{o}_{\mathbb{T}_{i+1}}(s(\alpha_{i+1}))=2\mathfrak{o}_{\mathbb{T}_{i+1}}(s(\alpha_i))$$

(where indices are understood modulo n when α is a closed curve). When α is an arc, we thus obtain the equality

$$\hat{\alpha}_1^{\mathfrak{o}_{\mathbb{T}_1}}\cdots\hat{\alpha}_n^{\mathfrak{o}_{\mathbb{T}_n}}=\theta^{\sum_{i=1}^{n-1}\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i))}(\theta^{1/2})^{1-\mathfrak{o}(s(\alpha))}\hat{\alpha}^+(\theta^{1/2})^{\mathfrak{o}(t(\alpha))},$$

from which we deduce that

$$\begin{aligned}w(\alpha):=W(\hat{\alpha}^{\mathfrak{o}})&=W\big((\theta^{-1/2})^{1-\mathfrak{o}(s(\alpha))}\hat{\alpha}^+(\theta^{-1/2})^{\mathfrak{o}(t(\alpha))}\big)\\&=W\big(\theta^{-\sum_{i=1}^{n-1}\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i))}\hat{\alpha}_1^{\mathfrak{o}_{\mathbb{T}_1}}\cdots\hat{\alpha}_n^{\mathfrak{o}_{\mathbb{T}_n}}\big)\\&=\sum_{i=1}^{n-1}\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i))+\sum_{i=1}^{n}w(\alpha_i)\pmod 2.\end{aligned}$$

The computation when α is a closed curve is done in the same manner. □

3.3.4 Explicit formulas for the isomorphism In order to describe the isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ of Theorem 1.3 more explicitly, let us recall from [Korinman 2019] a set of generators for the ring of regular functions of the relative character varieties.

For α an embedded arc, seen as a path in the fundamental groupoid, and $\varepsilon,\varepsilon'=\pm$, the regular function $F_{\alpha_{\varepsilon\varepsilon'}}\in\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ is defined on the class $[\rho]$ of a functor $\rho\in\mathscr{R}_{\mathrm{SL}_2}(\underline{\Sigma}_{\mathscr{P}})$ by

$$\rho(\alpha)=\begin{pmatrix}F_{\alpha_{++}}(\rho) & F_{\alpha_{+-}}(\rho)\\ F_{\alpha_{-+}}(\rho) & F_{\alpha_{--}}(\rho)\end{pmatrix}.$$

For γ a closed curve, represented by an arbitrary path $\gamma_v\in\Pi_1(\Sigma_{\mathscr{P}},\mathbb{V})$, one defines $F_\gamma\in\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ by $F_\gamma([\rho]):=\mathrm{Tr}(\rho(\gamma_v))$. Since the trace is invariant by conjugacy, the value $\mathrm{Tr}(\rho(\gamma_v))$ does not depend on the choice of base point v nor on the representative ρ in the class $[\rho]$. The functions $F_{\alpha_{\varepsilon\varepsilon'}}$ and F_γ generate the algebra $\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$. For α an arc, we set

$$N_\alpha:=\begin{pmatrix}F_{\alpha_{++}} & F_{\alpha_{+-}}\\ F_{\alpha_{-+}} & F_{\alpha_{--}}\end{pmatrix}$$

the 2×2 matrix with coefficients in $\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$. Note that

$$N_{\alpha^{-1}}=\begin{pmatrix}F_{\alpha_{--}} & -F_{\alpha_{+-}}\\ -F_{\alpha_{-+}} & F_{\alpha_{++}}\end{pmatrix}.$$

For α an embedded arc and $\varepsilon,\varepsilon'=\pm$, we denote by $\alpha_{\varepsilon\varepsilon'}\in\mathscr{S}_{+1}(\underline{\Sigma})$ the class of the arc α with state ε at $s(\alpha)$ and ε' at $t(\alpha)$. We write

$$M_\alpha:=\begin{pmatrix}\alpha_{++} & \alpha_{+-}\\ \alpha_{-+} & \alpha_{--}\end{pmatrix}$$

the 2×2 matrix with coefficients in $\mathscr{S}_{+1}(\underline{\Sigma})$. Note that

$$M_{\alpha^{-1}}=(M_\alpha)^{\mathsf{T}}=\begin{pmatrix}\alpha_{++} & \alpha_{-+}\\ \alpha_{+-} & \alpha_{--}\end{pmatrix}.$$

Recall the isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ of Theorem 1.3 and recall that $C^{-1}=\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$.

Theorem 3.20 *For each embedded arc α,*

$$\Psi^{(\Delta,\mathfrak{o}_\Delta)}(M_\alpha)=(-1)^{w(\alpha)}(C^{-1})^{1-\mathfrak{o}(\alpha(0))}N_\alpha(C^{-1})^{\mathfrak{o}(\alpha(1))}. \tag{32}$$

For each closed curve γ,

$$\Psi^{(\Delta,\mathfrak{o}_\Delta)}(\gamma)=(-1)^{w(\gamma)}F_\gamma. \tag{33}$$

Remark 3.21 When Σ is closed, recall from Remarks 3.13 and 3.16 that $W^{(\Delta,\mathfrak{o}_\Delta)}$ is a standard spin structure associated to a quadratic form ω such that $w(\gamma)=\omega([\gamma])+1$. Hence in the closed case, the isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ coincides with the "standard" isomorphisms described at the beginning of Section 3.1.

Recall that $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ is defined by the diagram

$$\begin{tikzcd}
\mathscr{S}_{+1}(\Sigma) \arrow[r, hook, "i^\Delta"] \arrow[d, "\Psi^{(\Delta,\mathfrak{o}_\Delta)}"', "\cong"] & \otimes_{\mathbb{T}} \mathscr{S}_{+1}(\mathbb{T}) \arrow[d, "\otimes_{\mathbb{T}} \Psi^{\mathfrak{o}_{\mathbb{T}}}"', "\cong"] \\
\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\Sigma)] \arrow[r, "i^\Delta"] & \otimes_{\mathbb{T}} \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})]
\end{tikzcd} \tag{34}$$

For $x \in \mathscr{S}_{+1}(\mathbb{T})$, we still denote by x the element in $\otimes_{\mathbb{T}} \mathscr{S}_{+1}(\mathbb{T})$ having 1 in the factors $\mathscr{S}_{+1}(\mathbb{T}')$ for $\mathbb{T}' \neq \mathbb{T}$ and x in the factor $\mathscr{S}_{+1}(\mathbb{T})$. Hence for $\delta \in \mathbb{G}^\Delta$ a path in $\mathbb{T}$, the matrix M_δ is considered as a 2×2 matrix with coefficients in $\otimes_{\mathbb{T}} \mathscr{S}_{+1}(\mathbb{T})$. Similarly, the matrix N_δ is considered as a 2×2 matrix with coefficients in $\otimes_{\mathbb{T}} \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})]$.

Proof We first show that if (32) holds for an arc α, then it holds for α^{-1}. This follows from the fact that $w(\alpha^{-1}) = w(\alpha)+1$, from the equalities $(C^{-1})^\intercal = C$ and $A^{-1} = -C^{-1}A^\intercal C^{-1}$ for $A \in \mathrm{SL}_2(\mathbb{C})$, and from the computation

$$\begin{aligned}
\Psi(M_{\alpha^{-1}}) = \Psi(M_\alpha^\intercal) &= (-1)^{w(\alpha)} C^{\mathfrak{o}(t(\alpha))} (N_\alpha)^\intercal C^{1-\mathfrak{o}(s(\alpha))} \\
&= (-1)^{w(\alpha)+1} (C^{-1})^{1-\mathfrak{o}(s(\alpha^{-1}))} (-C^{-1} N_\alpha^\intercal C^{-1}) (C^{-1})^{\mathfrak{o}(t(\alpha^{-1}))} \\
&= (-1)^{w(\alpha^{-1})} (C^{-1})^{1-\mathfrak{o}(s(\alpha^{-1}))} N_{\alpha^{-1}} (C^{-1})^{\mathfrak{o}(t(\alpha^{-1}))}.
\end{aligned}$$

Next let us prove the theorem for the triangle $\mathbb{T}$. The fact that (32) holds for the arcs $\alpha_{\mathbb{T}}$, $\beta_{\mathbb{T}}$ and $\gamma_{\mathbb{T}}$ is an immediate consequence of the definition of $\Psi^{\mathfrak{o}_{\mathbb{T}}}$ in Lemma 3.10 and from the definition of the canonical spin structure in $\mathbb{T}$. By the preceding arguments, (32) also holds for the arcs $\alpha_{\mathbb{T}}^{-1}$, $\beta_{\mathbb{T}}^{-1}$ and $\gamma_{\mathbb{T}}^{-1}$, and the theorem is proved for $\mathbb{T}$.

In the general case, consider an arc α and choose a decomposition

$$\alpha = \alpha_1 \cdots \alpha_n, \quad \alpha_i \in \mathbb{G}^\Delta,$$

as before. By the gluing formula for stated skein algebras [Lê 2018, Theorem 3.1], $i^\Delta(M_\alpha) = M_{\alpha_1} \cdots M_{\alpha_n}$. By definition of the morphism i^Δ in (29), $i^\Delta(N_\alpha) = N_{\alpha_1} \cdots N_{\alpha_n}$. By the preceding case of the triangle,

$$(\otimes_{\mathbb{T}} \Psi^{\mathfrak{o}_{\mathbb{T}}})(M_{\alpha_i}) = (-1)^{w(\alpha_i)} (C^{-1})^{1-\mathfrak{o}_{\mathbb{T}_i}(s(\alpha_i))} N_{\alpha_i} (C^{-1})^{\mathfrak{o}_{\mathbb{T}_i}(t(\alpha))}.$$

Hence, by Lemma 3.19,

$$(\otimes_{\mathbb{T}} \Psi^{\mathfrak{o}_{\mathbb{T}}}) \circ i^\Delta(M_\alpha) = i^\Delta\big((-1)^{w(\alpha)} (C^{-1})^{1-\mathfrak{o}(s(\alpha))} N_\alpha (C^{-1})^{\mathfrak{o}(t(\alpha))}\big),$$

and (32) follows from the commutativity of the diagram (34). The proof for a closed curved is done similarly by taking the trace of the above equality. □

3.3.5 Comparison with Costantino and Lê's isomorphism Let Σ be a connected punctured surface with nontrivial boundary. Costantino and Lê [2022] defined the twisted character variety $\chi(\Sigma)$ as the space of functors $\hat{\rho} \in \mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}), \mathrm{SL}_2(\mathbb{C}))$ such that $\hat{\rho}(\theta_{\vec{v}}^{1/2}) = C^{-1}$ for any $\vec{v} \in \widehat{\mathbb{V}}$. Let $\mathscr{S}$

denote the maximal spectrum of $\mathcal{S}_{+1}(\underline{\Sigma})$. For $\chi \in \mathcal{S}$, seen as a character $\chi\colon \mathcal{S}_{+1}(\underline{\Sigma}) \to \mathbb{C}^*$, and for α an oriented arc, write

$$\chi(\alpha) := \begin{pmatrix} \chi(\alpha_{++}) & \chi(\alpha_{+-}) \\ \chi(\alpha_{+-}) & \chi(\alpha_{--}) \end{pmatrix}.$$

Costantino and Lê [2022, Theorem 8.12] defined an affine isomorphism $\Theta\colon \mathcal{S} \xrightarrow{\cong} \chi(\underline{\Sigma})$ sending a character χ to a functor $\hat{\rho}$ such that $\chi(\alpha) = \hat{\rho}(\hat{\alpha}^+)$ for any embedded (even immersed) arc and such that $\chi(\gamma) = \mathrm{Tr}(\hat{\rho}(\hat{\gamma}^+))$ for any closed curve. Composing Θ with the isomorphism induced by $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$, one obtains an isomorphism $\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma}) \cong \chi(\underline{\Sigma})$. By Theorem 3.20, this isomorphism sends a functor $\rho \in \mathrm{Hom}(\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V}), \mathrm{SL}_2(\mathbb{C}))$ to a functor $\hat{\rho} \in \mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}), \mathrm{SL}_2(\mathbb{C}))$ characterized by the formulas $\hat{\rho}(\hat{\alpha}^{\mathfrak{o}}) = (-1)^{w(\alpha)}\rho(\alpha)$ for any arc α, $\mathrm{Tr}(\hat{\rho}(\hat{\gamma}^{\mathfrak{o}}) = (-1)^{w(\gamma)}\,\mathrm{Tr}(\rho(\gamma))$ for any closed curve γ and $\hat{\rho}(\theta_{\vec{v}}^{1/2}) = C^{-1}$ for any $\vec{v} \in \widehat{\mathbb{V}}$.

3.4 Classical Shadows

Suppose that $\omega \in \mathbb{C}$ is a root of unity of odd order $N > 1$. A *central representation* of the stated skein algebra is a finite-dimensional representation $r\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathrm{End}(V)$ which sends each element of the image of the morphism j of Theorem 1.2 to scalar operators. Fix a topological triangulation Δ of $\underline{\Sigma}$ and an orientation $\mathfrak{o}_\Delta$ of its edges. Then r induces a character on $\mathcal{S}_{+1}(\underline{\Sigma}) \xrightarrow[\cong]{\Psi^{(\Delta,\mathfrak{o}_\Delta)}} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ and this character induces a point in the relative character variety $\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})$ that we call the *classical shadow* of r, as in [Bonahon and Wong 2016] in the closed case. By definition, the classical shadow only depends on the isomorphism class of r.

To motivate the results of this paper, we list three families of central representations. First, irreducible representations are obviously central. Then choose for each triangle $\mathbb{T} \in F(\Delta)$ an irreducible representation $r^{\mathbb{T}}: \mathcal{S}_\omega(\mathbb{T}) \to \mathrm{End}(V_{\mathbb{T}})$ and consider the composition

$$r : \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{i^\Delta} \bigotimes_{\mathbb{T} \in F(\Delta)} \mathcal{S}_\omega(\mathbb{T}) \xrightarrow{\otimes_{\mathbb{T}} r^{\mathbb{T}}} \mathrm{End}(\otimes_{\mathbb{T}} V_{\mathbb{T}}).$$

Such a representation is central and were called *local representations* in [Bai et al. 2007]. Eventually, consider the balanced Chekhov–Fock algebra $\mathcal{Z}_\omega(\underline{\Sigma}, \Delta)$ defined in [Bonahon and Wong 2011] after the original construction of [Fock and Chekhov 1999]. Given a triangulated marked surface, Bonahon and Wong [2011] defined an algebra morphism (the quantum trace) $\mathrm{Tr}\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathcal{Z}_\omega(\underline{\Sigma}, \Delta)$ (see also [Lê 2018]). One motivation is the fact that the representation theory of the balanced Chekhov–Fock algebra is easier to study than the one of the skein algebras (see [Bonahon and Liu 2007; Bonahon and Wong 2017]). For an irreducible representation $\pi : \mathcal{Z}_\omega(\underline{\Sigma}, \Delta) \to \mathrm{End}\, V$ of the balanced Chekhov–Fock algebra, we call the *quantum Teichmüller representation*, the composition

$$r : \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{\mathrm{Tr}} \mathcal{Z}_\omega(\underline{\Sigma}, \Delta) \xrightarrow{\pi} \mathrm{End}(V).$$

Quantum Teichmüller representations are central.

Appendix Proof of Proposition 2.38 and an application

A.1 Proof of Proposition 2.38

We divide the proof of Proposition 2.38 into five lemmas.

Throughout this section, we write $A := \omega^{-2}$. Denote by $\mathbb{A} = ([0,1] \times S^1, \{p, p'\})$ the annulus with punctures $p = \{0\} \times \{1\}$ and $p' = \{1\} \times \{1\}$ in each of its boundary components and let $b = \{0\} \times S^1 \setminus \{p\}$ and $b' = \{1\} \times S^1 \setminus \{p'\}$ be its boundary arcs. Let $\gamma \subset [0,1] \times S^1$ be the curve $\{\frac{1}{2}\} \times S^1$. Let $\delta^{(n)}, \eta^{(n)} \subset [0,1] \times S^1$ be the arcs with endpoints b and b' such that $\delta^{(n)}$ spirals n times in the counterclockwise direction and $\eta^{(n)}$ spirals n times in the clockwise direction while oriented from b' to b. The arcs are drawn in Figure 7. By convention, $\delta^{(0)}$ and $\eta^{(0)}$ represent the empty diagram. Denote by β the arc $[0,1] \times \{-1\}$. By convention, if α is one of the arcs β, $\delta^{(n)}$ or $\eta^{(n)}$, we denote by $\alpha_{\varepsilon\varepsilon'} \in \mathcal{S}_\omega(\mathbb{A})$ the class of the corresponding stated tangle with sign ε in b and ε' in b'. The following lemma and its proof are quite similar, though stated in a different skein algebra, to [Lê 2015, Proposition 2.2].

Lemma A.1 *In $\mathcal{S}_\omega(\mathbb{A})$, the elements $T_N(\gamma)$ and $\beta_{\varepsilon\varepsilon'}$ commute.*

Proof First note that a direct application of the Kauffman bracket skein relations implies that

$$\gamma \cdot \delta^{(n)}_{\varepsilon\varepsilon'} = A\delta^{(n+1)}_{\varepsilon\varepsilon'} + A^{-1}\delta^{(n-1)}_{\varepsilon\varepsilon'} \quad \text{and} \quad \gamma \cdot \eta^{(n)}_{\varepsilon\varepsilon'} = A\eta^{(n-1)}_{\varepsilon\varepsilon'} + A^{-1}\eta^{(n+1)}_{\varepsilon\varepsilon'}$$

when $n \geq 1$. Next we show by induction on $n \geq 0$ that $T_n(\gamma) \cdot \beta_{\varepsilon\varepsilon'} = A^n\delta^{(n)}_{\varepsilon\varepsilon'} + A^{-n}\eta^{(n)}_{\varepsilon\varepsilon'}$. The statements is an immediate consequence of the definitions when $n = 0$ and a direct application of the Kauffman bracket relations when $n = 1$. Suppose that the results holds for n and $n+1$. Then

$$\begin{aligned} T_{n+2}(\gamma)\beta_{\varepsilon\varepsilon'} &= \gamma \cdot T_{n+1}(\gamma) \cdot \beta_{\varepsilon\varepsilon'} - T_n(\gamma) \cdot \beta_{\varepsilon\varepsilon'} \\ &= \gamma \cdot (A^{n+1}\delta^{(n+1)}_{\varepsilon\varepsilon'} + A^{-(n+1)}\eta^{(n+1)}_{\varepsilon\varepsilon'}) - (A^n\delta^{(n)}_{\varepsilon\varepsilon'} + A^{-n}\eta^{(n)}_{\varepsilon\varepsilon'}) \\ &= A^{n+2}\delta^{(n+2)}_{\varepsilon\varepsilon'} + A^{-(n+2)}\eta^{n+2}_{\varepsilon\varepsilon'}, \end{aligned}$$

and the statement follows by induction. Similarly, we show that $\beta_{\varepsilon\varepsilon'} \cdot T_n(\gamma) = A^{-n}\delta^{(n)}_{\varepsilon\varepsilon'} + A^n\eta^{(n)}_{\varepsilon\varepsilon'}$. Hence,

$$T_N(\gamma) \cdot \beta_{\varepsilon\varepsilon'} - \beta_{\varepsilon\varepsilon'} \cdot T_N(\gamma) = (A^N - A^{-N})(\delta^{(N)}_{\varepsilon\varepsilon'} - \eta^{(N)}_{\varepsilon\varepsilon'}) = 0. \qquad \square$$

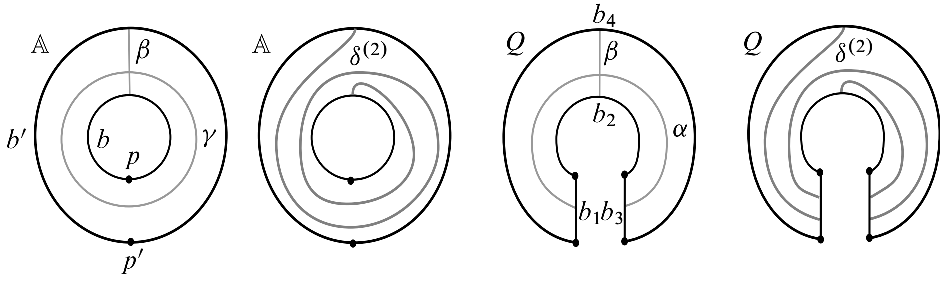


Figure 7: The annulus $\mathbb{A}$, the square Q and some arcs and curves.

Denote by Q the square, ie a disc with four punctures on its boundary. Let $b_1, \dots, b_4$ be its four boundary arcs labeled in the counterclockwise order. When gluing b_1 along b_3, we obtain the annulus with b_2 sent to b and b_4 sent to b'. We denote by $i|_{b_1 \# b_3} \colon \mathscr{S}_\omega(\mathbb{A}) \hookrightarrow \mathscr{S}_\omega(Q)$ the gluing morphism. Let $\alpha, \beta, \delta^{(n)}, \eta^{(n)} \subset Q$ be the arcs which are glued together to form γ, β, $\delta^{(n)}$ and $\eta^{(n)}$, respectively, as in Figure 7. Fix an arbitrary orientation $\mathfrak{o}$ of the boundary arcs of Q and consider the filtration $(\mathscr{F}_m)_{m \geq 0}$ associated to $S = \{b_1, b_3\}$ of Definition 2.11. Write $d \colon \mathscr{S}_\omega(Q) \to \mathbb{Z}^{\geq 0}$ the corresponding map and $\mathscr{G}_m := \mathscr{F}_m \,/\, \mathscr{F}_{m-1}$ the corresponding graduation.

Lemma A.2 $\quad \mathrm{lt}((\alpha_{++} + \alpha_{--})^N) = \mathrm{lt}(T_N(\alpha_{++} + \alpha_{--})) = \alpha_{++}^N + \alpha_{--}^N.$

Proof First note that in $\mathscr{G}_4$, we have $\alpha_{--}\alpha_{++} = q^2 \alpha_{++}\alpha_{--}$. So it follows from Lemma 2.27 that in $\mathscr{G}_{2N}$, we have $\mathrm{lt}((\alpha_{++} + \alpha_{--})^N) = \alpha_{++}^N + \alpha_{--}^N$. Since $T_N(X) - X^N$ is a polynomial of degree strictly smaller that N, the degree of $T_N(\alpha_{++} + \alpha_{--}) - (\alpha_{++} + \alpha_{--})^N$ is strictly smaller than $2N$; thus $\mathrm{lt}(T_N(\alpha_{++} + \alpha_{--})) = \mathrm{lt}((\alpha_{++} + \alpha_{--})^N)$. □

Let $\alpha^{(n)}$ be the diagram made of n parallel copies of α. Using the identifications $\partial\delta^{(n)} = \partial\eta^{(n)} = \partial\alpha^{(n)} \cup \partial\beta$, we denote by $\delta^{(n)}_{(s,\varepsilon,\varepsilon')}, \eta^{(n)}_{(s,\varepsilon,\varepsilon')} \in \mathscr{S}_\omega(Q)$ the classes of the tangles $\delta^{(n)}$ and $\eta^{(n)}$ with states given by a state s of $\alpha^{(n)}$ and a state $(\varepsilon, \varepsilon')$ of β.

Lemma A.3 *For $0 < n < N$ and s a state of $\alpha^{(n)}$,*

$$\mathrm{lt}\big([[\alpha^{(n)}, s], \beta_{\varepsilon\varepsilon'}]\big) = (A^n - A^{-n})(\delta^{(n)}_{(s,\varepsilon,\varepsilon')} - \eta^{(n)}_{(s,\varepsilon,\varepsilon')}),$$

where we used the notation $[x, y] = xy - yx$.

Proof The diagram obtained by stacking $\alpha^{(n)}$ on top of β has n crossings and thus 2^n resolutions using the Kauffman bracket relation. We remark that the resolution obtained by replacing each crossing by $\genfrac{}{}{0pt}{}{\smile}{\frown}$ is $A^n \delta^{(n)}_{(s,\varepsilon,\varepsilon')}$ while the resolution obtained by replacing each crossing by $)($ is $A^{-n}\eta^{(n)}_{(s,\varepsilon,\varepsilon')}$. These two resolutions have degree $2n$ and all the others resolutions have degrees strictly smaller; thus

$$\mathrm{lt}([\alpha^{(n)}, s] \cdot \beta_{\varepsilon\varepsilon'}) = A^n \delta^{(n)}_{(s,\varepsilon,\varepsilon')} + A^{-n}\eta^{(n)}_{(s,\varepsilon,\varepsilon')}.$$

We similarly prove $\mathrm{lt}(\beta_{\varepsilon\varepsilon'} \cdot [\alpha^{(n)}, s]) = A^{-n}\delta^{(n)}_{(s,\varepsilon,\varepsilon')} + A^n \eta^{(n)}_{(s,\varepsilon,\varepsilon')}$ and conclude by taking the difference. □

Lemma A.4 *If $x \in \mathscr{S}_\omega(Q)$ is a polynomial in $\mathscr{S}_\omega(Q)$ in the elements $\alpha_{\varepsilon\varepsilon'}$ such that $d(x) < 2N$ and such that x commutes with all elements $\beta_{\mu,\mu'}$, then x is a constant.*

Proof Let $x = \sum_{i \in I} x_i [\alpha^{n_i}, s_i]$ be the decomposition in the basis of stated tangles with increasing states s_i and denote by $2n < 2N$ its degree. Suppose, for the sake of contradiction, that $n \neq 0$. Let $J = \{j \in I \mid n_i = n\} \subset I$, so $\mathrm{lt}(x) = \sum_{j \in J} x_j [\alpha^n, s_j]$. The hypothesis on x and Lemma A.3 imply that

$$0 = \mathrm{lt}([x, \beta_{\varepsilon\varepsilon'}]) = \sum_{j \in J} x_j (A^n - A^{-n})(\delta^{(n)}_{(s_j,\varepsilon,\varepsilon')} - \eta^{(n)}_{(s_j,\varepsilon,\varepsilon')}).$$

Since the elements $\delta^{(n)}_{(s_j,\varepsilon,\varepsilon')}$ and $\eta^{(n)}_{(s_j,\varepsilon,\varepsilon')}$ are linearly independent for $n \geq 1$, we conclude that

$$x_j(A^n - A^{-n}) = 0$$

for all $j \in J$. Since $0 < n < N$ and N is odd, we obtain that $x_j = 0$ for all $j \in J$ thus $\mathrm{lt}(x) = 0$. This gives the contradiction. $\square$

The set $\mathcal{B}' := \{\alpha^a_{-+}\alpha^b_{++}\alpha^c_{+-}, a, b, c \geq 0\} \cup \{\alpha^a_{-+}\alpha^b_{--}\alpha^c_{+-}, a, b, c \geq 0\}$ forms a basis of the algebra $\mathcal{S}_\omega(\mathbb{B})$. This fact is Exercise 7 in Chapter IV, Section 6 of [Kassel 1995], and is proved as follows. Choose an orientation $\mathfrak{o}$ of the boundary arcs of $\mathbb{B}$ such that b_L and b_R points towards different punctures and consider the filtration associated to $S = \{b_L, b_R\}$. For each element of the basis $\mathcal{B}^{\mathfrak{o}}$, there exists exactly one element of $\mathcal{B}'$ which has the same leading term. For $x \in \mathcal{S}_\omega(\mathbb{B})$, denote by $c(x) \in \mathcal{R}$ the coefficient of 1 in the decomposition of the basis $\mathcal{B}'$.

Lemma A.5 $$c(T_N(\alpha_{++} + \alpha_{--})) = 0.$$

Proof Let $n \geq 1$ be an odd integer and let us show that $c((\alpha_{++} + \alpha_{--})^n) = 0$. The proof will then follow from the fact that $T_N(X)$ is an odd polynomial, thus is a linear combination of such elements, and the fact that c is linear. The product $((\alpha_{++} + \alpha_{--})^n)$ develops as a sum of terms of the form $x = x_1 \cdots x_n$ where x_i is either α_{++} or α_{--}. Using the defining relations of $\mathcal{S}_\omega(\mathbb{B})$, we can further develop each term x as a linear combination of terms of the form $\alpha^a_{-+}\alpha^b_{++}\alpha^a_{+-}$ and $\alpha^a_{-+}\alpha^b_{--}\alpha^a_{+-}$ where $2a + b$ has the same parity as n. Since n is odd, each of these summands satisfies $b \neq 0$ so $c(x) = 0$. $\square$

Proof of Proposition 2.38 Consider the element $x := T_N(\alpha_{++} + \alpha_{--}) - \alpha^N_{++} - \alpha^N_{--} \in \mathcal{S}_\omega(Q)$. By Lemma A.2, its degree is strictly smaller that $2N$. By Lemma A.1, in $\mathcal{S}_\omega(\mathbb{A})$ the elements $T_N(\gamma)$ and $\beta_{\varepsilon\varepsilon'}$ commute. The image through the algebra morphism $i|_{b_1\#b_3}\colon \mathcal{S}_\omega(\mathbb{A}) \hookrightarrow \mathcal{S}_\omega(Q)$ of $T_N(\gamma)$ and $\beta_{\varepsilon\varepsilon'}$ are respectively $T_N(\alpha_{++} + \alpha_{--})$ and $\beta_{\varepsilon\varepsilon'}$, thus they commute. By Lemma 2.36, the elements α^N_{++} and α^N_{--} also commute with $\beta_{\varepsilon\varepsilon'}$ so x commutes with each element $\beta_{\varepsilon\varepsilon'}$. Lemma A.4 implies that x is a constant and Lemma A.5 implies that this constant is null. $\square$

A.2 A generalization of a theorem of Bonahon

Proposition 2.38 provides the following generalization of the main theorem of [Bonahon 2019]. Let $\mathcal{A}$ be an $\mathcal{R}$–algebra and $\rho\colon \mathbb{C}_q[\mathrm{SL}_2]^{\otimes k} \to \mathcal{A}$ be a morphism of algebras. Let ρ_i be the i^{th} component of ρ. For $1 \leq i \leq k$, consider the following two matrices with coefficients in $\mathcal{A}$:

$$A_i := \begin{pmatrix} \rho_i(\alpha_{++}) & \rho_i(\alpha_{+-}) \\ \rho_i(\alpha_{-+}) & \rho_i(\alpha_{--}) \end{pmatrix}, \quad A_i^{(N)} := \begin{pmatrix} \rho_i(\alpha_{++})^N & \rho_i(\alpha_{+-})^N \\ \rho_i(\alpha_{-+})^N & \rho_i(\alpha_{--})^N \end{pmatrix}.$$

The following proposition was proved in [Bonahon 2019, Theorem 1] in the particular case where $\rho_i(\alpha_{+-})\rho_i(\alpha_{-+}) = 0$ for each $i \in \{1, \dots, k\}$.

Proposition A.6 *If q is a root of unity of odd order $N > 1$, then*

$$T_N(\mathrm{Tr}(A_1 \cdots A_k)) = \mathrm{Tr}(A_1^{(N)} \cdots A_k^{(N)}).$$

Proof By Proposition 2.38 and using that both ρ and the $(k-1)^{\text{st}}$ coproduct

$$\Delta^{(k-1)}\colon \mathbb{C}_q[\mathrm{SL}_2] \to \mathbb{C}_q[\mathrm{SL}_2]^{\otimes k}$$

are morphisms of algebras,

$$T_N \circ \rho \circ \Delta^{(k-1)}(\alpha_{++} + \alpha_{--}) = \rho \circ \Delta^{(k-1)}(\alpha_{++}^N + \alpha_{--}^N).$$

We conclude by remarking that

$$\rho \circ \Delta^{(k-1)}(\alpha_{++} + \alpha_{--}) = \mathrm{Tr}(A_1 \cdots A_k) \quad \text{and} \quad \rho \circ \Delta^{(k-1)}(\alpha_{++}^N + \alpha_{--}^N) = \mathrm{Tr}(A_1^{(N)} \cdots A_k^{(N)}),$$

where the second equality follows from the fact that $j_{\mathbb{B}}$ is a morphism of Hopf algebras (Lemma 2.28), hence commutes with $\Delta^{(k-1)}$. □

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