

AG
T

*Algebraic & Geometric
Topology*

Volume 24 (2024)

Commensurators of thin normal subgroups and abelian quotients

THOMAS KOBERDA

MAHAN MJ

Commensurators of thin normal subgroups and abelian quotients

THOMAS KOBERDA

MAHAN MJ

We give an affirmative answer to many cases of a question due to Shalom, which asks if the commensurator of a thin subgroup of a Lie group is discrete. Let $K < \Gamma < G$ be an infinite normal subgroup of an arithmetic lattice Γ in a rank-one simple Lie group G , such that the quotient $Q = \Gamma/K$ is infinite. We show that the commensurator of K in G is discrete, provided that Q admits a surjective homomorphism to \mathbb{Z} . In this case, we also show that the commensurator of K contains the normalizer of K with finite index. We thus vastly generalize a 2021 result of the authors, which showed that many natural normal subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ have discrete commensurator in $\mathrm{PSL}_2(\mathbb{R})$.

22E40; 20F65, 20F67, 57M50

1. Introduction	2149
2. Generalities on discrete subgroups of Lie groups	2153
3. Preservation of lines with holes	2155
4. Homological lines with holes and Hodge theory	2158
5. Abelian quotients and harmonic 1-forms	2163
Appendix. Discrete patterns of horoballs	2167
References	2169

1 Introduction

Let G be a semisimple \mathbb{Q} -algebraic group, and let $G(\mathbb{Z})$ denote its group of integer points. Roughly speaking, a subgroup Γ of G is called *arithmetic* if it is *commensurable in a wide sense* with $G(\mathbb{Z})$; see Witte Morris [36]. That is, there is an element $g \in G$ such that the group $G(\mathbb{Z}) \cap \Gamma^g$ has finite index in both $G(\mathbb{Z})$ and Γ^g . In general, if G is an algebraic group and $\Gamma < G$ is a subgroup, we write $\mathrm{Comm}_G(\Gamma)$ for the *commensurator* of Γ in G , ie the subgroup consisting of $g \in G$ such that $\Gamma \cap \Gamma^g$ has finite index in both Γ and Γ^g . The commensurability criterion for arithmeticity due to Margulis [24] (see also Witte Morris [36]) characterizes arithmetic subgroups of algebraic groups via their commensurators. A convention we shall follow throughout in this article: whenever we refer to a semisimple Lie group, we shall mean a connected semisimple real Lie group with no compact factors, unless noted otherwise.

Theorem 1.1 (Margulis) *Let G be a semisimple Lie group with no compact factors and let Γ be an irreducible lattice in G . Then Γ is arithmetic if and only if $\text{Comm}_G(\Gamma)$ is dense in G .*

Here, we are primarily concerned with the discreteness properties of commensurators of *thin groups*, a class of groups which has received a large amount of attention in recent years; see Sarnak [28]. A subgroup $K < G$ is *thin* if K is discrete and Zariski dense in G , and if G/K has infinite volume with respect to the Haar measure on G . Thus, K fails to be a lattice in G only by virtue of having infinite covolume in G . Natural examples of thin groups arise from infinite-index Zariski-dense subgroups of lattices in G .

In the present manuscript, we continue our previous investigations from [21] of the following question due to Shalom (see especially Shalom and Willis [33], wherein the problem has its genesis):

Question 1.2 [22] *Let K be a thin subgroup of a semisimple Lie group G .*

- (i) *Is the commensurator $\text{Comm}_G(K)$ of K in G discrete?*
- (ii) *In particular, is the normalizer of K in G of finite index in $\text{Comm}_G(K)$?*

For an infinite normal subgroup K of a lattice Γ , the two subquestions of [Question 1.2](#) are equivalent. Indeed, the commensurator of K contains its normalizer, which contains Γ . Since Γ is a lattice, we see that if $\text{Comm}_G(K)$ is discrete then it is a finite-index superlattice of Γ . For the other implication, any such K is discrete and Zariski dense, and thus has a discrete normalizer; cf [Lemma 2.1](#). Since the normalizer of K contains Γ and since Γ has finite covolume, we have that the normalizer of K is itself a lattice. Thus, if the commensurator of K contains the normalizer of K with finite index then the commensurator is discrete.

Positive answers to [Question 1.2](#) are known for all finitely generated thin subgroups K of $\text{PSL}_2(\mathbb{R})$ and $\text{PSL}_2(\mathbb{C})$ (see Greenberg [16], Leininger, Long and Reid [22] and Mj [26]), and for thin subgroups of a semisimple Lie groups with limit set a proper subset of the Furstenberg boundary [26]. Here, the *limit set* is a generalization of the limit set occurring in the theory of Kleinian groups, and is a minimal nonempty closed invariant subset of the Furstenberg boundary for a group acting on the corresponding symmetric space; see Benoist [4].

We were thus prompted in [21] to address [Question 1.2](#) when the ambient Lie group is the simplest possible, viz $\text{PSL}_2(\mathbb{R})$, for thin groups whose limit sets consist of the entire Furstenberg boundary, ie $S^1 = \partial\mathbb{H}^2$. More generally, natural examples of thin groups with limit set equal to the Furstenberg boundary come from normal subgroups of rank-one lattices. This general problem provides the context for this paper.

1.1 Main result

Since many rank-one arithmetic lattices surject onto nonabelian free groups, every finitely generated group can be realized as a quotient of an arithmetic lattice. Observe, in particular, that all finitely generated free groups arise as finite-index subgroups of $\Gamma(2)$, the level-two congruence subgroup of $\text{PSL}_2(\mathbb{Z})$, and

therefore all infinite, finitely generated groups arise as quotients of a rank-one arithmetic lattice by a thin normal subgroup. This level of generality has led us to impose some natural algebraic conditions on the quotient Q . We will establish the following result, which handles normal subgroups with “nice” quotients.

Theorem 1.3 *Let $\Gamma < G$ be an arithmetic lattice in a rank-one simple Lie group G and let $K < \Gamma$ be an infinite normal subgroup. Write $Q = \Gamma/K$ for the corresponding quotient group. Then the group $\text{Comm}_G(K)$ is discrete, provided that the group Q admits a surjective homomorphism to \mathbb{Z} . Under these hypotheses, the commensurator of K in G contains the normalizer of K with finite index.*

The reader is directed to [Theorem 5.1](#) for the context and proof surrounding the main result. Note that the hypotheses of [Theorem 1.3](#) are never satisfied for irreducible lattices in higher rank nor for lattices in the rank-one simple Lie groups $\text{Sp}(n, 1)$ for $n \geq 2$, nor in the exceptional group F_4^{-20} . This is because lattices in these Lie groups have Kazhdan’s Property (T). Thus, [Theorem 1.3](#) is vacuously true in these cases. Therefore in the course of establishing [Theorem 1.3](#), we pay exclusive attention to $G \in \{\text{SO}(n, 1), \text{SU}(n, 1)\}_{n \geq 2}$, which give rise to real and complex hyperbolic spaces, respectively, as the associated symmetric spaces of noncompact type.

In [\[21\]](#) we answered [Question 1.2](#) in the special case that K is the commutator subgroup of Γ , where $\Gamma < \text{PSL}_2(\mathbb{Z})$ is a finite-index normal subgroup of $\text{PSL}_2(\mathbb{Z})$ contained in a principal congruence subgroup $\Gamma(k)$ for some $k \geq 2$. We vastly generalize this result, since if $K = [\Gamma, \Gamma]$ has infinite index in Γ , then K falls under the purview of [Theorem 1.3](#).

1.2 Tools and techniques

The main theorems and techniques of [\[21\]](#) are the starting point of this paper.

Preserving lines with holes An important technical tool introduced in [\[21\]](#) was that of a homology pseudoaction. We adapt it here to the notion of *preservation of lines with holes*. Let Γ be a lattice in a rank-one simple Lie group G , let $K < \Gamma$ be a normal subgroup, and let $Q = \Gamma/K$. Quite generally, for $g \in G$ we say that g preserves Q -lines with holes if for all $\gamma \in \Gamma$, there exists $N > 0$ such that

$$\gamma^n \cong (\gamma^n)^g \pmod{K} \quad \text{for all } n \in N\mathbb{Z}.$$

The terminology arises from thinking of infinite cyclic groups as “lines” and a finite-index subgroup of an infinite cyclic group as a “line with holes”. We direct the reader to [Section 3](#) for a detailed discussion.

The usefulness of preserving lines with holes is illustrated by the following purely group-theoretic fact, which provides a rather general criterion for deciding noncommensurability (see [Theorem 3.4](#)):

Theorem 1.4 *Let $\Gamma < G$, let $K < \Gamma$ be normal, and let $Q = \Gamma/K$. If*

$$g \in \text{Comm}_G K \cap \text{Comm}_G \Gamma,$$

then $K^g := g^{-1}Kg$ preserves Q -lines with holes.

Harmonic forms and maps The other principal tool used in this paper comes from harmonic forms and harmonic maps via Hodge theory. These include classical Hodge theory and its L^2 analogue for noncompact manifolds. Preservation of Q -lines with holes, or equivalently, lines with holes in Γ modulo the normal subgroup K , can be promoted to something stronger: the harmonic form allows us to convert the “coarse” lines in Γ/K into actual maps to \mathbb{R} , ie it allows us to “fill the holes” of coarse lines in a canonical fashion, and thus find canonical G -invariant maps to \mathbb{R} .

Discrete patterns Harmonic maps are coupled with the notion of discrete patterns, an idea going back to Schwartz [30], and which was exploited in proving discreteness of commensurators in Leininger, Long and Reid [22] and Mj [26]. Throughout the paper, many of our ideas and methods are inspired by the basic example of arithmetic hyperbolic surfaces as well as the special case $K = [\Gamma, \Gamma]$, and in some places we explicate the underlying geometric intuition. In the context of $\mathrm{PSL}_2(\mathbb{R})$ and hyperbolic surfaces, Teichmüller-theoretic notions such as zeros and saddle connections of abelian differentials provide us the necessary discrete patterns that are preserved by the commensurator when the underlying surface has positive genus *and* lines with holes in the integral homology are preserved. Preservation of such discrete patterns finally ensures that the commensurator is discrete. With the notion of preserving homological lines with holes in place, the discussion for lattices in $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$ splits into uniform and nonuniform cases. For uniform lattices, we use Hodge theory coupled with a Lie-theoretic idea that we learned from Venkataramana [34] and Agol [1]. For nonuniform lattices, we use L^2 -Hodge theory along with the fact that preservation of homology lines with holes guarantees the preservation of a discrete pattern given by horoballs. Discreteness of a pattern-preserving subgroup is an essential ingredient in the nonvanishing cuspidal cases: see the proof of [Theorem 5.1](#), especially [Claim 5.2](#) therein.

Relationship with existing literature The previous works [22; 26] on discreteness of commensurators derived discreteness by showing that the commensurator preserves a “discrete geometric subobject” or “pattern” in the sense of Schwartz [29]. These may be regarded as a collection of geometrically defined subspaces of the domain symmetric space X . We refer the reader to the [appendix](#) for the material on patterns that will be used in this paper. There is a shift in focus in this paper, as we look at naturally defined dual objects. The canonical nature of harmonic maps ensures that they are preserved by the commensurator. We derive much of our inspiration from Shalom’s work [31; 32; 33].

1.3 Structure of the paper

[Section 2](#) contains an account of the general tools from the theory of lattices in Lie groups which we will need. [Section 3](#) describes preservation of lines with holes in detail. [Section 4](#) introduces the notion of a discrete invariant set as it arises from classical and L^2 -Hodge theory. In the same section, the commensurator of a form is introduced and the construction of an invariant harmonic form is carried out. [Section 5](#) proves [Theorem 1.3](#).

Remarks on notation Throughout this paper, we will use the notation K to denote a subgroup a discrete group. Usually, this will be a normal subgroup of an arithmetic lattice Γ . In particular, K will generally not denote a maximal compact subgroup of the ambient Lie group G . We will use N to denote a positive integer, as opposed to the more common notation of the unipotent subgroup in the Iwasawa decomposition of a semisimple Lie group. The Iwasawa decomposition will be used briefly in the proof of [Claim 5.2](#), but no confusion will arise. We will use the exponentiation shorthand for conjugation in groups, so that $K^g = g^{-1}Kg$, where K and g are contained in an ambient group. The group G will denote an ambient Lie group, which will typically be $\{\mathrm{SO}(n, 1), \mathrm{SU}(n, 1)\}_{n \geq 2}$ unless otherwise explicitly noted.

2 Generalities on discrete subgroups of Lie groups

In this section, we gather some general facts about Zariski-dense discrete subgroups of semisimple Lie groups which we will require in this article.

2.1 Zariski-dense subgroups and commensurators

We begin with the following general fact about normalizers of discrete groups. The statement and proof are contained as Lemma 2.1 in [\[21\]](#), and so we omit the proof.

Lemma 2.1 *Let G be a simple Lie group and let $\Gamma < G$ be a discrete Zariski-dense subgroup. Then the normalizer $N_G(\Gamma)$ is again discrete.*

The following well-known fact will be used throughout the paper.

Lemma 2.2 *Let G be a simple real group and let $H < G$ be a Zariski-dense subgroup. If H is not discrete then H is dense.*

Indeed, since H is not discrete, the topological closure \overline{H} of H has the property that the component H^0 of \overline{H} containing the identity is a Zariski-dense subgroup of G which has positive dimension, and therefore must be all of G ; indeed the tangent space to H^0 at the identity coincides with the tangent space to G , and so H^0 contains a neighborhood of the identity in G , which generates the identity component of G . We remark that if G is allowed to be a complex group then one must assume that H is not precompact, as can be seen from the Zariski density of the unit complex numbers in \mathbb{C} for instance.

The following lemma generalizes the corresponding statement in [\[21\]](#) for $\mathrm{PSL}_2(\mathbb{R})$.

Lemma 2.3 *Let Γ_0 be a lattice in a noncompact simple Lie group G . Let Γ be a subgroup of G containing Γ_0 such that there exists an $N > 0$ satisfying the property that for all $g \in \Gamma$, we have $g^N \in \Gamma_0$. Then Γ is also discrete.*

Proof We have that G acts by isometries on an associated symmetric space X of noncompact type. Since Γ_0 is a lattice, there exists $\epsilon > 0$ such that any semisimple element of Γ_0 has translation length on X at least ϵ . Since G is simple and Γ is Zariski dense, it follows that Γ is either discrete or dense in G . We argue by contradiction. If Γ is dense, then since the property of being semisimple is an open condition and since translation lengths of semisimple elements of G coincide with $\mathbb{R}_{>0}$, there exists a semisimple element $g \in \Gamma$ such that the translation length of g is less than $\epsilon/2N$. Hence g^N is a semisimple element with translation length at most $\epsilon/2$. In particular, $g^N \notin \Gamma_0$, which yields a contradiction. \square

We remark that [Lemma 2.3](#) is false for merely discrete subsets of G , since even the square roots of a fixed matrix can fail to be a discrete set. If G has rank one then one can allow Γ_0 to be a more general subset of G .

Let G be a semisimple Lie group and let $\Gamma < G$ be a subgroup. As usual, we write $\text{Comm}_G(\Gamma)$ to denote its commensurator in G . We shall need the following special case of a general theorem of Borel [[8](#), Theorem 2]; see Zimmer [[37](#), page 123]. This will be the only real use of arithmeticity of the ambient lattice Γ in [Theorem 1.3](#). Strictly speaking, the statement of Proposition 6.2.2 in [[37](#)] is for the full group of integral points in an ambient group. The reader will note however that the only salient feature of the group of integral points which is used is its Zariski density. Thus, we obtain the following conclusion:

Proposition 2.4 *Let $\Gamma < G$ be an arithmetic lattice in a semisimple algebraic \mathbb{Q} -group and let $K < \Gamma$ be a Zariski-dense subgroup. Then $\text{Comm}_G(K) < \text{Comm}_G(\Gamma)$. Suppose furthermore that the center of G is trivial. Then $\text{Comm}_G(\Gamma)$ coincides with the \mathbb{Q} -points of G .*

The hypothesis that G has trivial center in the second part of [Proposition 2.4](#) is crucial. For instance, the commensurator of $\text{SL}_2(\mathbb{Z})$ properly contains $\text{SL}_2(\mathbb{Q})$. The reader will observe that throughout this paper, we will implicitly assume that K is a Zariski-dense subgroup of an arithmetic lattice, though in the statement of [Theorem 1.3](#), we only assume that K is infinite and normal. This latter assumption implies that K is indeed Zariski dense:

Proposition 2.5 *Let $K < \Gamma$ be an infinite normal subgroup of an irreducible lattice in a semisimple algebraic group G . Then K is Zariski dense in G .*

Proof Let Λ denote the limit set of K . Since K is infinite, $\Lambda \neq \emptyset$, since the limit set consists of the limit points of K in the Furstenberg boundary of G . Let $p \in \Lambda$. If $\gamma \in \Gamma$ then $\gamma(p) \in \Lambda$, since K is normal in Γ . It follows that Λ is a closed, nonempty Γ -invariant subset of the Furstenberg boundary. It therefore contains all of the limit set of Γ by the lemma in Section 3.6 of [[4](#)]. It follows that Λ is equal to the limit set of Γ .

Since Γ is Zariski dense, so is K . Else, if K were contained in a proper Lie subgroup $H < G$, then Λ would be contained in the Furstenberg boundary of H , which in turn is not Zariski dense in the Furstenberg boundary of G . However, the limit set of Γ is Zariski dense in the boundary: see the remarks at the beginning of Section 3 of [[4](#)], especially the lemma in Section 3.6. This is a contradiction. \square

The following technical fact will be used several times in this paper, and we extract it for modularity.

Lemma 2.6 *Let $K < G$ be a Zariski-dense subgroup of a simple algebraic group G , and let*

$$K^G = \langle \{K^g \mid g \in \text{Comm}_G(K)\} \rangle$$

be the subgroup of G generated by the conjugates of K by $g \in \text{Comm}_G(K)$. If K^G is a discrete subgroup of G , then $\text{Comm}_G(K)$ is discrete.

It is a trivial though useful observation that $K^G < \text{Comm}_G(K)$.

Proof of Lemma 2.6 We have immediately that $K < \text{Comm}_G(K)$, since K normalizes itself. We therefore conclude that $\text{Comm}_G(K)$ is Zariski dense and hence is either discrete or dense in G . If $\text{Comm}_G(K)$ is dense then there is a sequence $g_i \rightarrow 1$ of nontrivial group elements in $\text{Comm}_G(K)$ converging to the identity. We write $K_i = K^{g_i}$, and we observe that $K_i < K^G$ for each i . Choosing finitely many elements $\{k_1, \dots, k_m\} \subset K$ which generate a Zariski-dense subgroup $K_0 < G$, we have that if g_i is nontrivial then it cannot fix the entire collection $\{k_1, \dots, k_m\}$, since then g_i would centralize K_0 , contradicting Zariski density of K_0 and the simplicity of G . However, as i tends to infinity, the conjugation action of g_i on $\{k_1, \dots, k_m\}$ tends to the identity. Thus, viewing G as a matrix group, we have that $\{k_1^{g_i}, \dots, k_m^{g_i}\}$ converges to $\{k_1, \dots, k_m\}$ in any matrix norm. Since $K_0^{g_i} < K_i < K^G$, the last of which is discrete, we have that $\{k_1^{g_i}, \dots, k_m^{g_i}\} = \{k_1, \dots, k_m\}$ elementwise for $i \gg 0$, and hence that g_i commutes with K_0 for $i \gg 0$. Again using the fact that K_0 is Zariski dense and G is simple and hence center-free, we conclude that g_i is the identity for $i \gg 0$. This is a contradiction, whence it follows that $\text{Comm}_G(K)$ is discrete. □

The argument in Lemma 2.6 even shows that only the set

$$\bigcup_{g \in \text{Comm}_G(K)} K_0^g$$

need be discrete in order to conclude the discreteness of $\text{Comm}_G(K)$, for an arbitrary Zariski-dense subgroup $K_0 < K$.

3 Preservation of lines with holes

In this section, we develop some ideas which originate in homological algebra and which play a central role in this paper, with the goal of producing a criterion for showing that a particular group element does not commensurate a given subgroup. The historical motivation comes from Chevalley–Weil theory — see Chevalley, Weil and Hecke [12] and Gaschütz [15] — and which we developed in [21] under the name of a *pseudoaction*.

Throughout this section, let $\Gamma < G$, let $K < \Gamma$ be a normal subgroup, and let $g \in \text{Comm}_G(\Gamma)$. We write $Q = \Gamma/K$ for the quotient group. Conjugating by $g \in G$, we obtain groups $K^g < \Gamma^g$ and a corresponding quotient $Q^g := \Gamma^g/K^g$.

For $\gamma \in \Gamma$, we shall refer to the cyclic group $\langle \gamma \rangle$ as a γ -line in Γ . Further, any finite-index subgroup $\langle \gamma^N \rangle$ of $\langle \gamma \rangle$ —considered for arbitrary $\gamma \in \Gamma$ and a positive integer N —will be referred to as a Γ -line with holes. For any $\gamma \in \Gamma$ and $g \in \text{Comm}_G(\Gamma)$, there exists a positive integer N such that $(\gamma^g)^N \in \Gamma$. Hence, for any $\gamma \in \Gamma$, and $g \in \text{Comm}_G(\Gamma)$, (the conjugation action by) g sends some γ -line with holes to a Γ -line with holes.

Definition 3.1 The element $g \in \text{Comm}_G(\Gamma)$ preserves Q -lines with holes if for all $\gamma \in \Gamma$ there exists an integer $N > 0$ such that

$$\gamma^n \equiv (\gamma^n)^g \pmod{K}$$

for all $n \in N\mathbb{Z}$. That is, there exists $N > 0$ such that $x_m = [\gamma^{mN}, g] \in K$ for all $m \in \mathbb{Z}$.

Thus if γ^N and $(\gamma^N)^g$ should both be elements of Γ (which they are after passing to multiples of a sufficiently large N , since g commensurates Γ), then one can compare their images in $Q = \Gamma/K$. If g preserves Q -lines with holes then they must represent the same element of Q . A special case of [Definition 3.1](#) is given by the following:

Definition 3.2 In [Definition 3.1](#), if we specialize to the case where K is the commutator subgroup $[\Gamma, \Gamma]$ (so that in particular $Q = H_1(\Gamma, \mathbb{Z})$), we say that g preserves homological lines with holes in Γ .

The usefulness of preservation of homological lines with holes will become apparent when one considers its cohomological consequences in [Section 4.1](#). For now, consider the set of all elements $g \in \text{Comm}_G(\Gamma)$ that preserve Q -lines with holes. It is not difficult to see that this subset of G is actually a monoid. Clearly the identity lies in this set. Moreover, if g and h preserve Q -lines with holes, then for all $\gamma \in \Gamma$, there is an $N = N(g, \gamma)$ such that $[\gamma^N, g] \in K$. Then, $(\gamma^N)^g = \gamma^N \cdot k \in \Gamma$, so there is an $M = M(h, \gamma^N \cdot k)$ such that $[(\gamma^N \cdot k)^M, h] \in K$. This shows that

$$\gamma^{NM} \equiv (\gamma^{NM})^{gh} \pmod{K},$$

which implies that the set of elements of $\text{Comm}_G(\Gamma)$ which preserve Q -lines with holes does in fact form a monoid. It is not clear that inversion of elements is possible within this set, however. We will not require this monoidal structure in the sequel, though we abstract out the following fact:

Observation 3.3 Consider the set $C \subset \text{Comm}_G(\Gamma)$ consisting of elements which preserve Q -lines with holes. Then C is closed under multiplication of group elements and contains the identity, and is therefore a monoid. In particular, if $K_1, K_2 \subset C$ are subgroups, then the group

$$\langle K_1, K_2 \rangle < \text{Comm}_G(\Gamma)$$

is contained in C .

The following is the basic result about preservation of Q -lines with holes.

Theorem 3.4 Let $\Gamma < G$, let K be a normal subgroup of Γ , and let $Q = \Gamma/K$. Suppose that

$$g \in \text{Comm}_G \Gamma \cap \text{Comm}_G K.$$

Then K^g preserves Q -lines with holes.

Proof Let $z \in K^g$ and let $\gamma \in \Gamma$ be arbitrary fixed elements. For $N \gg 0$ we have that $\gamma^N \in \Gamma \cap \Gamma^g$ and $(\gamma^N)^z \in \Gamma$. Let $a = (\gamma^N)^z$ and $b = \gamma^N$. We have that $a^m, b^m \in \Gamma$ for all $m \in \mathbb{Z}$.

Since $z \in K^g$ and since K^g is normal in Γ^g , we have that

$$a \equiv b \pmod{K^g}.$$

Hence, for all $m \in \mathbb{Z}$,

$$a^m \equiv b^m \pmod{K^g}.$$

Thus, the commutators

$$x_m := [\gamma^{mN}, z] = a^m b^{-m}$$

have the property that $x_m \in K^g$ for all $m \in \mathbb{Z}$. It is also clear that $x_m \in \Gamma$ for all $m \in \mathbb{Z}$.

Since K and K^g are commensurable, the collection of elements

$$\{x_m = a^m b^{-m}\}_{m \in \mathbb{Z}}$$

has the property that for some $s \neq t$, the elements $x_s = a^s b^{-s}$ and $x_t = a^t b^{-t}$ lie in the same right coset of $K \cap K^g$ in K^g , as follows immediately from the pigeonhole principle.

It follows that there exists an element $k \in K$ such that

$$k a^s b^{-s} = a^t b^{-t}.$$

Therefore, we see that

$$a^{-t} k a^s = b^{s-t},$$

which furnishes an element $k' \in K$ such that $k' a^{s-t} = b^{s-t}$.

Thus, there exists $M = s - t \neq 0$ such that $a^M \equiv b^M \pmod{K}$. In particular, z preserves Q -lines with holes, the desired conclusion. \square

In the sequel, we will be interested in specific cases in which Q -lines with holes are preserved, and especially the case where Q is the integral homology of Γ/K .

We now discuss a mild generalization of the notion of preserving homological lines with holes in **Definition 3.2**. Let $Q = \Gamma/K$ be a quotient group. Clearly, $H_1(Q, \mathbb{Z})$ is a quotient of $H_1(\Gamma, \mathbb{Z})$.

Let $\gamma \in \Gamma$ and let $g \in \text{Comm}_G(\Gamma)$. There is an integer $N > 0$ such that $\{\gamma^n, (\gamma^g)^n\} \subset \Gamma$ for all $n \in N\mathbb{Z}$. We can then compare the homology classes of γ^n and $(\gamma^g)^n$ in $H_1(\Gamma, \mathbb{Z})$, and hence in $H_1(Q, \mathbb{Z})$. As before, we say that g preserves homological lines with holes in Q if for all $\gamma \in \Gamma$, there exists an integer $N > 0$ such that for all $n \in N\mathbb{Z}$, the homology classes of γ^n and $(\gamma^g)^n$ in $H_1(Q, \mathbb{Z})$ are equal.

Let Q^{ab} denote the abelianization of Q . Then the condition that g preserves homological lines with holes in Q is equivalent to saying that g preserves Q^{ab} -lines with holes in the sense of **Definition 3.1**.

When $b_1(Q) > 0$ then [Theorem 3.4](#) above furnishes the following commensurability criterion, whose proof is straightforward now.

Theorem 3.5 *Let $Q = \Gamma/K$, let*

$$g \in \text{Comm}_G \Gamma \cap \text{Comm}_G K.$$

Then K^g preserves homological lines with holes in Q .

Proof Let Q_0 be a quotient of Q , and let $h \in K^g$. Since h preserves Q -lines with holes by [Theorem 3.4](#), it also preserves Q_0 -lines with holes. Specializing to $Q_0 = Q^{\text{ab}}$ proves the result. \square

In particular, when the commensurator of Γ in G contains the commensurator of K , we have that

$$K^G = \langle K^g \mid g \in \text{Comm}_G(K) \rangle$$

preserves homological lines with holes in Q . We remark that in our applications, $\text{Comm}_G K < \text{Comm}_G \Gamma$ by [Proposition 2.4](#).

4 Homological lines with holes and Hodge theory

The goal of this section is to translate between preservation of lines with holes and the existence of commensuration-invariant harmonic 1-forms. We shall first deduce cohomological consequences of preserving homological lines with holes.

4.1 Preserving homological lines with holes and cohomological consequences

For the purposes of this subsection, let G denote a semisimple Lie group with no compact factors, with associated symmetric space of nonpositive curvature X . Let Γ be a lattice in G and let $g \in \text{Comm}_G(\Gamma)$. We write $S = X/\Gamma$ and $S^g = X/\Gamma^g$. Since $g \in \text{Comm}_G(\Gamma)$, the group $\Gamma \cap \Gamma^g$ is of finite index in both Γ and Γ^g . Let $W = X/(\Gamma \cap \Gamma^g)$ denote the corresponding common cover of S and S^g . We shall refer to S and S^g as *conjugate manifolds* and W as their *minimal common cover*. Here, W depends on g . However, since g will be fixed throughout, we will suppress it from the notation. We will also fix a differential 1-form ω on S . Let $p: X \rightarrow S$ denote the universal covering map. Note that the 1-form $p^*\omega$ is a 1-form on X . In applications in the sequel, ω will be a harmonic form.

The element $g \in G$ is an isometry of X and hence acts on differential forms on X via pullback. The form $g^*p^*\omega$ is a 1-form on X which is invariant under Γ^g and hence descends to S^g . The resulting 1-form on the quotient manifold S^g is denoted by ω^g . Let $q: W \rightarrow S$ and $q^g: W \rightarrow S^g$ denote the natural covering maps. Denote $q^*\omega$ by ω_W and $(q^g)^*\omega^g$ by ω_W^g .

We shall also need to set up notation for g -conjugates of cycles and loops, as basepoints will play an important role in what follows. Let $o \in W$ be a basepoint. By choosing a lift $\tilde{o} \in X$ and by joining \tilde{o} to $g.\tilde{o}$

by a geodesic segment in X and projecting back to W , we obtain a natural geodesic segment $[o, g.o]$ in W , where $g.o$ denotes the image of $g.\tilde{o}$ under the covering projection. Thus, $g.o$ may be regarded as a new basepoint for integrating chains against a pulled back form.

Now suppose that α is a loop in W representing an element $h \in \pi_1(W)$ such that h^g also belongs to $\pi_1(W)$, where here we have identified $\pi_1(W)$ with $\Gamma \cap \Gamma^g$. Lifting α to a path $\tilde{\alpha}$ in X , translating by g and quotienting X by $\Gamma \cap \Gamma^g$ we obtain a new loop denoted $g.\alpha$ on W based at $g.o$. Here, we use notation that is similar to the case of a genuine g -action on W , though the action is well-defined only on the universal cover X .

The concatenation $[o, g.o] * g.\alpha * \overline{[o, g.o]}$ gives a loop based at o , where $\overline{[o, g.o]}$ denotes $[o, g.o]$ parametrized in the opposite direction from $g.o$ to o . We denote this loop as α^g :

$$\alpha^g = [o, g.o] * g.\alpha * \overline{[o, g.o]}.$$

Finally, for σ any closed, oriented loop on W , based at o say, the n^{th} power of the loop σ will be the loop which traverses the loop σ a total of n times. The result will be denoted by σ^n .

Remark 4.1 A subtlety in the following lemma needs to be noted. On the one hand, the hypothesis is about preserving homological lines with holes in Γ . The conclusion, on the other hand, is about cohomology classes in the common minimal cover W . The reason for this is that the pullback of ω to X and its pullback by g are both invariant under $\Gamma \cap \Gamma^g$, though not necessarily by Γ nor Γ^g . Thus, ω_W^g is well-defined as a form on W , but does not necessarily live in S .

Lemma 4.2 *Let*

$$\{\Gamma, S, g, S^g, W, \omega_W, \omega_W^g\}$$

be as above, Suppose that g preserves homological lines with holes in Γ . Then we have $[\omega_W] = [\omega_W^g]$ as elements of $H^1(W, \mathbb{R})$.

The importance of [Lemma 4.2](#) will become apparent in [Section 4.2](#), particularly [Corollary 4.7](#). It follows from the Hodge theorem that if ω_W is a harmonic form representing $[\omega_W] \in H^1(W, \mathbb{R})$, then $\omega_W = \omega_W^g$ as forms, and not just as cohomology classes.

Proof We continue with the notation from the discussion before the statement of the lemma. Let σ be any closed loop on W based at o . Since g commensurates Γ , we may choose $n > 0$ such that σ^n and $(\sigma^n)^g$ are both cycles, and so are viewed as loops based at o . Observe that if h denotes the element of $\pi_1(W, o)$ represented by σ^n then the loop $(\sigma^n)^g$ represents the group element $h^g \in \pi_1(W, o)$.

Since g is assumed to preserve homological lines with holes in Γ , there exists an integer $N > 0$ such that σ^N and $(\sigma^N)^g$ represent the same element of $H_1(S, \mathbb{Z})$. Indeed, for any differential 1-form ω on S , we have

$$(1) \quad \int_{q(\sigma^N)} \omega = \int_{q((\sigma^N)^g)} \omega = \int_{q(g.\sigma^N)} \omega,$$

where $q: W \rightarrow S$ is the covering projection, and where the second inequality holds because the integrals of ω along $[o, g \cdot o]$ and $[\overline{o, g \cdot o}]$ cancel each other. Note that the integrals in equation (1) are over S .

Next, by the definition of the pullback form $\omega_W = q^*\omega$, we have that

$$\int_{\sigma^N} \omega_W = \int_{q(\sigma^N)} \omega \quad \text{and} \quad \int_{(\sigma^N)^g} \omega_W = \int_{q((\sigma^N)^g)} \omega.$$

Combining the equations above, we obtain

$$(2) \quad \int_{\sigma^N} \omega_W = \int_{(\sigma^N)^g} \omega_W = \int_{g \cdot \sigma^N} \omega_W,$$

where all the integrals in equation (2) are over W .

Finally, we observe that by the definition of the pullback ω_W^g , we have

$$(3) \quad \int_{g \cdot (\sigma^N)} \omega_W = \int_{\sigma^N} \omega_W^g,$$

again using the fact that the integrals of ω_W along $[o, g \cdot o]$ and $[\overline{o, g \cdot o}]$ cancel each other.

Putting all these equalities together, we obtain

$$(4) \quad \int_{\sigma^N} \omega_W = \int_{\sigma^N} \omega_W^g.$$

Since

$$\int_{\sigma^N} \omega_W = N \int_{\sigma} \omega_W,$$

we conclude that

$$(5) \quad \int_{\sigma} \omega_W = \int_{\sigma} \omega_W^g$$

for any closed loop σ in W based at o . The forms ω_W and ω_W^g represent well-defined elements of $H^1(W, \mathbb{R})$, by their very definition. By equation (5) above they have the same periods, and since they are both closed differential forms, they are cohomologous. □

The cohomological consequence of preserving homological lines with holes in quotients is the following (cf Remark 4.1):

Lemma 4.3 *Let $Q = \Gamma/K$, let $g \in \text{Comm}_G(\Gamma)$ preserve homological lines with holes in Q , and let $\omega \in H^1(Q, \mathbb{R})$. Then the periods of $[\omega_W]$ and $[\omega_W^g]$ agree, where W is the common minimal cover of $S = X/\Gamma$ and its conjugate manifold $S^g = X/\Gamma^g$, and where ω_W is the pullback of ω to $H^1(W, \mathbb{R})$.*

Proof Let $\omega \in H^1(Q, \mathbb{R})$ be a nontrivial cohomology class. Then the quotient map $q: \Gamma \rightarrow Q$ induces a pullback form $q^*\omega \in H^1(\Gamma, \mathbb{R})$, which can be viewed as a differential form on $S = X/\Gamma$. The map q also induces a map $q_*: H_1(\Gamma, \mathbb{Z}) \rightarrow H_1(Q, \mathbb{Z})$. If σ is any 1-cycle on X/Γ then by definition

$$\int_{\sigma} q^*\omega = \omega(q_*\sigma),$$

where the right-hand side denotes the evaluation of ω on $q_*(\sigma)$ (recall ω is a cohomology class of Q).

Writing ω_W for the form on W given by pullback of $q^*\omega$ along the covering map $p: W \rightarrow S$, we have that ω_W^g and ω_W have the same periods, provided that g preserves homological lines with holes in Q . A justification of this claim is identical to that in the proof of [Lemma 4.2](#). \square

We note the following easy observation (cf [Observation 3.3](#) above).

Observation 4.4 *Consider the set $C \subset \text{Comm}_G(\Gamma)$ consisting of elements which preserve homological lines with holes in Q . Then C is closed under multiplication of group elements and contains the identity, and is therefore a monoid. In particular, if $K_1, K_2 \subset C$ are subgroups, then the group $\langle K_1, K_2 \rangle$ is contained in C .*

4.2 Hodge theory

Hodge theory will allow us to leverage preservation of homological lines with holes in order to promote equality of cohomology classes to equality of forms. We recall the necessary tools from Hodge theory and L^2 -cohomology that we shall need. Let M be a (not necessarily compact) Riemannian manifold. We fix notation: Ω^k will denote the space of smooth k -forms, d will denote the differential on forms, $*$ will denote the Hodge star operator, d^* will denote the adjoint of d , and $\Delta = dd^* + d^*d$ will denote the Laplacian on forms. A form $\omega \in \Omega^k$ is a *harmonic k -form* for the given metric on M if $\Delta\omega = 0$. Harmonic forms are closed and coclosed.

Theorem 4.5 [[35](#), Chapter 6] *Let M be a compact Riemannian manifold. Then for all k and every real cohomology class $[\omega] \in H^k(M, \mathbb{R})$, there exists a unique harmonic form ω_{harm} representing $[\omega]$.*

We shall need a version of [Theorem 4.5](#) for noncompact complete manifolds M . The appropriate cohomology theory used is L^2 -cohomology. Let Ω_2^k denote the space of smooth square-integrable k -forms. The reduced L^2 -cohomology groups are given by

$$H_{(2)}^k(M) = \ker(d) / \overline{\text{Im}(d)},$$

where $\overline{\text{Im}(d)}$ denotes the closure of the image of d . We refer the reader to [[10](#)] for more details. We shall need only the following special case (see [[10](#), Lemma 1.5] due to Gaffney, or [[11](#)] for instance):

Theorem 4.6 *Let M be a complete negatively curved manifold of finite volume modeled on \mathbb{H}^n or $\mathbb{C}\mathbb{H}^n$. Then for every real cohomology class $[\omega] \in H_{(2)}^1(M, \mathbb{R})$, there exists a unique L^2 harmonic form ω_{harm} representing $[\omega]$.*

Note that a compactly supported cohomology class is an L^2 class. Thus in our context, if X/Γ has nontrivial real cohomology with compact supports, then we can find nontrivial L^2 harmonic forms representing such cohomology classes. In our analysis of the case $b_1(Q) > 0$ for groups arising as quotients of nonuniform lattices Γ , the absence of a nonzero L^2 harmonic 1-form will (roughly) allow us to assume that $H_c^1(X/\Gamma, \mathbb{R}) = 0$. See the proof of [Theorem 5.1](#) below.

We recall the setup of [Lemma 4.2](#) in a slightly restricted setting: we are given a lattice Γ in a group $G \in \{\mathrm{SO}(n, 1), \mathrm{SU}(n, 1)\}_{n \geq 2}$ with associated symmetric space of noncompact type X , and an element $g \in G$ commensurating Γ . We have an orbifold $S = X/\Gamma$, the conjugate manifold $S^g = X/\Gamma^g$, the common refinement $W = X/(\Gamma \cap \Gamma^g)$ and a cohomology class $\omega \in H^1(S, \mathbb{R})$. We assume the existence of a (possibly L^2) harmonic representative ω_{harm} of ω , whose uniqueness is then guaranteed by [Theorems 4.5](#) and [4.6](#). Note that such a harmonic representative may not exist only in the case where S is noncompact.

We will also call the resulting harmonic form ω as it will not cause confusion. Recall the notation

$$p: X \rightarrow S, \quad W, \quad \omega_W, \quad \omega_W^g,$$

from [Section 4.1](#). For convenience, we will denote $p^*\omega$ by ω_X and $g^*\omega_X$ by ω_X^g , where g^* is the action on 1-forms induced by the isometry g of X .

Corollary 4.7 *Assume the above setup, and suppose that g preserves homological lines with holes in Γ . Then the harmonic representatives of ω_W and ω_W^g are equal as differential 1-forms on W . In particular, the harmonic representatives of ω_X and ω_X^g are equal.*

Proof Since g acts on X by an isometry, the pullback of a harmonic form under g is also harmonic; see [Section 4](#) of [\[14\]](#), for example. Thus, ω_W^g is a form on W which is cohomologous to the form ω_W , by [Lemma 4.2](#). Since $\Gamma \cap \Gamma^g$ has finite index in Γ , we have that W still has finite volume and hence the suitable Hodge theorem ([Theorem 4.5](#) or [4.6](#)) applies, whence the harmonic representatives of ω_W and ω_W^g are equal. The equality of forms on X is immediate. \square

A part of the remainder of the paper will deal with the case where there is no harmonic form representing a nontrivial homology class, which is to say a complement to [Corollary 4.7](#) adapted to cusped orbifolds.

4.3 The commensurator of a form

The notion of the commensurator of a form will now be introduced. It will be shown that under suitable hypotheses, K^G lies in the commensurator of a harmonic form, as is forced by preservation of homological lines with forms. The rigid nature of the harmonic form will force it to be zero whenever K^G fails to be discrete, which only occurs if $\mathrm{Comm}_G(K)$ is dense. As before, cohomology with compact supports will be denoted by $H_c^*(\cdot)$.

Definition 4.8 Let $\Gamma < G$ be a lattice in a semisimple Lie group G with associated symmetric space X , and let $S = X/\Gamma$. Let ω be a closed form such that $[\omega] \in H^p(S, \mathbb{Q})$ or $[\omega] \in H_c^p(S, \mathbb{Q})$ is a nonzero cohomology class. Let $p: X \rightarrow S$ denote the universal cover. The *commensurator* $\mathrm{Comm}(\omega)$ of the form ω is defined as

$$\mathrm{Comm}(\omega) = \{h \in G \mid h^*p^*\omega = p^*\omega\}.$$

A subgroup H of G is said to commensurate ω if $H < \mathrm{Comm}(\omega)$. It is immediate the $\mathrm{Comm}(\omega)$ is itself a group.

We have the following general discreteness result that applies to the isometries of real and complex hyperbolic spaces. We will not consider isometries of quaternionic hyperbolic spaces or the Cayley plane; see the remarks following [Theorem 1.3](#). We direct the reader to [\[34; 1\]](#), from which the main idea used in the following proposition is taken.

Proposition 4.9 *Let X be \mathbb{H}^n or $\mathbb{C}\mathbb{H}^n$. For Γ a torsion-free lattice, let $S = X/\Gamma$. Let ω be a nonzero harmonic or L^2 -harmonic 1-form according to whether S is compact or noncompact. Then $\text{Comm}(\omega)$ is discrete.*

Proof Let $p: X \rightarrow S$ denote the universal cover. We now argue by contradiction. Suppose that $\text{Comm}(\omega)$ is not discrete. Since the associated Lie group G (ie $\text{SO}(n, 1)$ or $\text{SU}(n, 1)$) is simple, it follows that $\text{Comm}(\omega)$ is dense in G , as $\text{Comm}(\omega)$ contains the Zariski-dense subgroup Γ . Also, since $\text{Comm}(\omega)$ preserves $p^*(\omega)$, we have that G must preserve $p^*(\omega)$, since G is identified with the group of isometries of X . That is, $p^*(\omega)$ is a G -invariant nonzero harmonic 1-form on X . (Note that here, compactness or noncompactness of S is not relevant, as $p^*(\omega)$ being defined on X is all that we are concerned with at this stage.) Hence $p^*(\omega)$ gives a nonzero harmonic differential 1-form ω^* on the compact dual of \mathbb{H}^n or $\mathbb{C}\mathbb{H}^n$; see Venkataramana [\[34\]](#) and Agol [\[1\]](#), cf Sections 2 and 3 of Chapter II in [\[9\]](#). Since the compact duals S^n and $\mathbb{C}\mathbb{P}^n$ of \mathbb{H}^n and $\mathbb{C}\mathbb{H}^n$ respectively have trivial first cohomology (at least when $n \geq 2$), this is a contradiction. \square

From [Lemma 4.3](#), we obtain the following consequence:

Corollary 4.10 *Suppose Γ is torsion-free. Let $Q = \Gamma/K$, and let $C \subset \text{Comm}_G(\Gamma)$ denote the set of elements which preserve homological lines with holes in Q . If there exists a (possibly L^2) harmonic form on $S = X/\Gamma$ representing a pullback of a nonzero cohomology class of Q , then C is discrete.*

Proof Let ω be the harmonic representative of a form on S arising by pullback from Q , and let $g \in C$. Then by [Lemma 4.3](#) and [Corollary 4.7](#), we have that $\omega_W = \omega_W^g$ as forms, by either classical or L^2 -Hodge theory, and where here W is the common refinement of S and its conjugate S^g . Pulling back these forms to the universal cover X , we have that $g \in \text{Comm}(\omega)$. By [Proposition 4.9](#), we conclude that C is discrete. \square

5 Abelian quotients and harmonic 1-forms

We are now in a position to assemble the pieces to prove [Theorem 1.3](#). The ideas to establish the result naturally bifurcate:

- (i) The vanishing cuspidal case, amenable to L^2 -cohomology techniques. For $\text{PSL}_2(\mathbb{R})$, this is the case where the underlying hyperbolic surface has genus greater than zero. This part of the argument uses Hodge theory.

- (ii) The nonvanishing cuspidal case, where discrete patterns of horoballs are used to obtain discreteness of the commensurator; see the [appendix](#). For $\mathrm{PSL}_2(\mathbb{R})$, this is the case where the underlying hyperbolic surface has genus equal to zero, and compactly supported cohomology vanishes. This part of the argument borrows heavily from the ideas in [21].

5.1 Proof of Theorem 1.3

We now establish part of the main result of this paper:

Theorem 5.1 *Let $\Gamma < G$ be a lattice in a rank-one simple Lie group. Let $K < \Gamma$ be an infinite normal subgroup, and let $Q = \Gamma/K$. If the first Betti number of Q satisfies $b_1(Q) > 0$ then $\mathrm{Comm}_G(K)$ is discrete.*

Here, the lattice may or may not be torsion-free, and may or may not be uniform. As remarked in the introduction, we only consider lattices in $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$.

Proof We begin by passing to a torsion-free finite-index subgroup Γ' of Γ , and by replacing K with the corresponding finite-index subgroup of K given by the corresponding intersection $K \cap \Gamma'$. The resulting subgroup of K is commensurable with K and hence has the same commensurator in G as K . Moreover, by restricting the quotient map $\Gamma \rightarrow Q$ to Γ' , we get a finite-index subgroup $Q' < Q$ which also has positive first Betti number. Thus without loss of generality, we will assume that $\Gamma = \Gamma'$.

Recall that we write

$$K^G = \langle K^g \mid g \in \mathrm{Comm}_G(K) \rangle$$

for the subgroup generated by the collection $\{K^g\}$, as g ranges over $\mathrm{Comm}_G(K)$. By [Proposition 2.4](#), we have that $\mathrm{Comm}_G(K) < \mathrm{Comm}_G(\Gamma)$. By [Theorem 3.5](#) and [Observation 4.4](#), we have that if $y \in K^G$, then y preserves homological lines with holes in Q .

By hypothesis, we have $H^1(Q, \mathbb{R}) \neq 0$. Writing $S = X/\Gamma$ as usual, we have that $H^1(S, \mathbb{R}) \neq 0$ since Q is a quotient of Γ and since $\Gamma = \pi_1(S)$. We have that S is metrically complete and is either compact or noncompact, which yields two possible cases concerning cohomology:

(i) **S is compact** By [Theorem 4.5](#), there is a harmonic form ω on S which represents the pullback of a nontrivial cohomology class of Q .

(ii) **S is not compact** This case bifurcates into further possibilities:

(a) The composition

$$H^1(Q, \mathbb{R}) \rightarrow H^1(S, \mathbb{R}) \rightarrow H^1(\partial S, \mathbb{R})$$

has a nontrivial kernel, where the first map is the pullback along the quotient map $\Gamma \rightarrow Q$ and the second map is the pullback along the inclusion map $\partial S \rightarrow S$. Note that the first arrow is an injection. Furthermore,

$$H^1((S, \partial S), \mathbb{R}) = H_c^1(S, \mathbb{R}) = H_{(2)}^1(S, \mathbb{R}).$$

See [23, Lemma 1.93]. Hence, by Theorem 4.6, there is a nonzero cohomology class of S that is represented by a nonzero L^2 harmonic form ω such that $[\omega] \in H^1_{(2)}(S, \mathbb{R})$ is the pullback of a cohomology class of Q .

(b) The composition

$$H^1(Q, \mathbb{R}) \rightarrow H^1(S, \mathbb{R}) \rightarrow H^1(\partial S, \mathbb{R})$$

is injective.

In case (ii), we interpret ∂S in the usual way, ie by removing a small horoball around the cusps of S , whereby the boundary of S becomes the image of the horosphere bounding the horoball.

Suppose first that there exists a nontrivial (possibly L^2) harmonic form on $S = X/\Gamma$ representing a pullback of a nontrivial class in $H^1(Q, \mathbb{Q})$, as in case (i) or (ii)(a) above. Then K^G is discrete by Corollary 4.10. That $\text{Comm}_G(K)$ is discrete now follows from Lemma 2.6.

If no such form exists, then we are in case (ii)(b). Writing $q: \Gamma \rightarrow Q$ for the quotient map, we have that

$$q_* \circ i_*: H_1(\partial S, \mathbb{Q}) \rightarrow H_1(Q, \mathbb{Q})$$

is surjective, where $i: \partial S \rightarrow S$ denotes inclusion. Because $H_1(Q, \mathbb{Q}) \neq 0$ by hypothesis, there exists a finite collection of cusps $\{T_1, \dots, T_k\}$ of S which contain homology classes $z_j \in H_1(T_j, \mathbb{Q})$ for which

$$q_* \circ i_*(z_j) \neq 0.$$

For $1 \leq j \leq k$, let $t_j \in \partial X$ denote the basepoint (at infinity) of a horoball lift of T_j to X . Let \mathcal{T}_j denote the set of the Γ -translates of t_j in ∂X . Also, let \mathcal{H}_j (resp. $\partial\mathcal{H}_j$) denote the collection of horoballs (resp. horospheres) in X that are lifts of T_j (resp. ∂T_j). These are an instance of a *discrete pattern* in the sense of Schwartz [29]; see Definition A.3 below, for instance. Let $\Gamma_j < G$ denote the subgroup preserving the collection $\partial\mathcal{H}_j$. By [27, Propositions 5.3 and 5.4] (see Lemma A.6 for instance), the group Γ_j is a lattice containing Γ as a subgroup of finite index.

We complete the proof assuming Claim 5.2 below. It follows from Claim 5.2 that each element of K^G has a uniformly bounded power contained in the discrete group $\bigcap_{s=1}^k \Gamma_s$. Hence K^G is discrete by Lemma 2.3. Lemma 2.6 now implies that $\text{Comm}_G(K)$ itself is discrete. □

Claim 5.2 *There is an $N > 0$ such that for all $y \in K^G$, we have*

$$y^N \in \bigcap_{s=1}^k \Gamma_s.$$

Proof By Theorem 3.5, we know that K^G preserves homological lines with holes in Q . Choose parabolic subgroups $\{G_1, \dots, G_k\}$ of G , which we use to identify $\pi_1(T_j)$ as a subgroup of $\pi_1(S)$ for $1 \leq j \leq k$, and let $\{x_1, \dots, x_k\} \subset \partial X$ be their respective fixed points. Let $\gamma \in \Gamma$ be a parabolic isometry representing $z_j \in H_1(T_j, \mathbb{Z})$, and such that $q_* \circ i_*(z_j)$ is nonzero. Replacing γ by a conjugate in Γ if necessary,

γ fixes x_j and hence lies in G_j . Let $y \in K^G$. Since y preserves homological lines with holes in Q , there exists a positive integer m such that

$$[(\gamma^m)^y] = [\gamma^m] = m \cdot q_* \circ i_*(z_j),$$

where $[\cdot]$ denotes the corresponding homology class in $H_1(Q, \mathbb{Z})$, and where elements of Γ acquire homology classes in $H_1(Q, \mathbb{Z})$ via q_* . Since $y \in G$, we have that $(\gamma^m)^y$ is also parabolic. Since y commensurates Γ (by Proposition 2.4) and preserves homological lines with holes in Q , we have that there exists $r \in \Gamma$ such that $(\gamma^m)^{y^r} \in G_\ell$ for some $1 \leq \ell \leq k$. Thus, y preserves homological lines with holes in Q but may “change the cusp” which supports a given cuspidal homology class. Since there are only k many cusps of S which contribute to the homology of Q via $q_* \circ i_*$, for $N = k!$ we may assume that $(\gamma^m)^{y^N}$ is conjugate into G_j by an element $r \in \Gamma$. We thus have that $y^N r \in G_j$.

Now, any element of the parabolic subgroup G_j can be decomposed as $A_\lambda N_\lambda$, where A_λ acts on $\partial X \setminus \{x\}$ by a conformal homothety and N_λ acts by an isometry. Here, the metric on $\partial X \setminus \{x\}$ is obtained by identifying it with a reference horosphere in X based at x via projection along geodesics from x .

For $X = \mathbb{H}^n$, these are all Euclidean similarities and for $X = \mathbb{C}\mathbb{H}^n$, these are all Heisenberg similarities (see [29, Section 8.1]). In particular, for any j , and for any $g \in G_j$, g scales all distances on the reference horosphere by a fixed $r_g > 0$. We call r_g the *scale factor* of g . Let

$$\hat{g}: H_1(T_j) \rightarrow H_1(T_j)$$

denote the induced map on $H_1(T_j)$ thought of as a subset of $\partial X \setminus \{x\}$. Here, we use the notation \hat{g} in place of g_* to avoid confusing with the action on homology of the cusp per se. Since g scales the length of all elements by r_g , it follows that $\hat{g}(u) = r_g \cdot u$ for all $u \in H_1(T_j)$. Let $A_\lambda(y^N r) > 0$ denote the scale factor of the homothety component of $y^N r$. Write $\mathcal{H}_{x_j} \in \mathcal{H}_j$ for the horoball in X based at x_j .

Since

$$[(\gamma^m)^{y^N r}] = A_\lambda(y^N r)[\gamma^m] \in H_1(Q, \mathbb{Q}),$$

the scale factor $A_\lambda(y^N r)$ must equal one. But $A_\lambda(y^N r) = 1$ if and only if $y^N r$ preserves the horosphere $\partial\mathcal{H}_{x_j}$. Since $r \in \Gamma$ necessarily preserves $\partial\mathcal{H}_j$, it follows that y^N stabilizes $\partial\mathcal{H}_j$, ie $y^N \in \Gamma_j$. Since $y \in K^G$ and $1 \leq j \leq k$ were arbitrary, and since Γ_j contains $\bigcap_{s=1}^k \Gamma_s$ with finite index (as follows easily from Lemma A.6) this completes the proof of the claim. □

5.2 Applications

We conclude this section by giving three sets of examples to which Theorem 5.1 applies.

Irrational pencils in complex hyperbolic manifolds Many cocompact arithmetic lattices in $SU(2, 1)$ admit irrational pencils, ie $S = X/\Gamma$ admits a holomorphic fibration (with singular fibers) onto a Riemann surface of genus greater than zero. Let F denote the general fiber and $i: F \rightarrow S$ denote inclusion. Then $K = i_*(\pi_1(F))$ is normal in Γ and $Q = \Gamma/K$ is a surface group. Theorem 5.1 applies

to show that $\text{Comm}_G(K)$ is discrete. We note that M Kapovich in unpublished work [19] (see Biswas, Mj and Pancholi [6] for a small generalization) established that K is never finitely presented.

Real hyperbolic manifolds that algebraically fiber Agol [2] shows that hyperbolic 3-manifolds virtually fiber over the circle with surface group fibers. The resulting normal surface subgroups were dealt with in [22] without the arithmeticity hypothesis. However, a new family of examples of finitely generated (but not necessarily finitely presented) normal subgroups of arithmetic hyperbolic n -manifolds has recently been discovered. A classical result of Dodziuk [13] (see also Anghel [3]) shows that the first L^2 -Betti number of a hyperbolic manifold of dimension greater than 2 vanishes. Kielak [20] shows that a cubulated hyperbolic group Γ is virtually algebraically fibered (ie Γ admits a virtual surjection to \mathbb{Z} with a finitely generated kernel) if and only if $\beta_{(2)}^1(\Gamma) = 0$. On the other hand, Bergeron, Haglund and Wise [5] show that standard cocompact arithmetic congruence subgroups Γ of $\text{SO}(n, 1)$ are cubulated. Thus standard cocompact arithmetic congruence subgroups Γ of $\text{SO}(n, 1)$ admit finitely generated normal subgroups K with quotient \mathbb{Z} . This furnishes a family of examples K to which [Theorem 5.1](#) applies to show that $\text{Comm}_G(K)$ is discrete (since $b_1(Q) = b_1(\mathbb{Z}) = 1$ in this case).

Uncountably many pairwise nonisomorphic 2-generated groups P Hall produced uncountably many pairwise nonisomorphic quotients of a free group F_2 on two generators; see [17, III.C.40], for instance. Evidently, the free group on two generators can be realized as a lattice in a rank-one simple Lie group. Hall's construction produces uncountable families of 2-generated torsion-free solvable groups, and each of his groups surjects to \mathbb{Z} . This furnishes a continuum's worth of thin normal subgroups of lattices to which [Theorem 5.1](#) applies.

Appendix Discrete patterns of horoballs

In the course of the proof of [Theorem 5.1](#), case (ii)(b), we have used the fact that a certain *discrete pattern of horoballs* is preserved by K^g . Since the notion of a discrete pattern also makes its appearance in earlier approaches to [Question 1.2](#), we give a quick account here.

Let G be a rank-one semisimple Lie group and let X be the associated symmetric space. The space X is, in a natural way, a Riemannian manifold endowed with a left-invariant metric [18]. Following [29; 30; 27; 7] we define the following (see [27, Definition 1.6] in particular):

Definition A.3 Let $\Gamma < G$ be a lattice and $S = X/\Gamma$. A Γ -*discrete pattern of points* on X is a nonempty Γ -invariant set $\mathcal{S} \subset X$ such that \mathcal{S}/Γ is finite.

Let $\Gamma < G$ be a nonuniform lattice, and let $S = X/\Gamma$. A Γ -*discrete pattern of horoballs* in X is a nonempty Γ -invariant collection $\mathcal{S} \subset X$ of closed horoballs such that \mathcal{S}/Γ is a disjoint union of neighborhoods of cusps.

Definition A.4 Let $\Gamma < G$ be a lattice. A subgroup H of G is said to *preserve* a Γ -discrete pattern \mathcal{S} of points if $h(\mathcal{S}) \subset \mathcal{S}$ for all $h \in H$.

Propositions 3.5 and 3.7 of [27] show that a subgroup H of G preserving a Γ -discrete pattern \mathcal{S} is closed and totally disconnected. Since any such subgroup of G is necessarily discrete, we have the following:

Lemma A.5 [27, Propositions 3.5 and 3.7] *Let $\Gamma < G$ be a lattice and \mathcal{S} a Γ -discrete pattern (of points or geodesics). Then the subgroup H of G preserving \mathcal{S} is discrete, and $[H : \Gamma] < \infty$.*

Propositions 5.3 and 5.4 of [27] (see also [25, Theorem 3.11]) prove that the subgroup H of G preserving a Γ -discrete pattern of horoballs is closed and totally disconnected. It follows that:

Lemma A.6 [27, Propositions 5.3 and 5.4] *Let $\Gamma < G$ be a nonuniform lattice in a rank-one Lie group and $S = X/\Gamma$, where X is the associated symmetric space. Let \mathcal{S} be a Γ -discrete pattern of horoballs. Then the subgroup H of G preserving \mathcal{S} is discrete, and $[H : \Gamma] < \infty$.*

As an aside, we mention that for lattices in $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SO}(2, 1)$, there are more direct ways of understanding discrete patterns, and in particular Proposition 4.9 above, that are inspired by ideas from Teichmüller theory. In this context, one can view the commensurator of a nontrivial harmonic form as explicitly producing a Γ -discrete pattern. Specifically, one can use the fact that a harmonic form is the real part of an abelian differential on the Riemann surface \mathbb{H}^2/Γ . In the case of a cocompact lattice, one can use the fact that the set of zeros of the form is nonempty and discrete, and preserved by the commensurator of the form. Then Lemma A.5 gives discreteness of the commensurator itself. In the case of a nonuniform lattice, one uses saddle connections in the Baily–Borel–Satake compactification of \mathbb{H}^2/Γ , and the fact that these are invariant under the commensurator. Again, Lemma A.5 gives discreteness of the commensurator.

Acknowledgements

The authors are grateful to D Witte Morris, T N Venkataramana, S Varma and C S Rajan for helpful discussions and correspondence, in particular for telling us about Proposition 2.4. Special thanks are due to D Fisher and W van Limbeek for many discussions and for catching an error in an earlier version of this paper. We thank D Wise for his lectures on coherence at IMPAN, Warsaw, and for references on L^2 -Betti numbers [13]. We thank Y Shalom for his interest and perceptive comments.

Koberda was partially supported by an Alfred P Sloan Foundation Research Fellowship, by NSF grant DMS-1711488, and by NSF grant DMS-2002596 while this research was carried out. Mj is supported by the Department of Atomic Energy, Government of India, under project 12-R&D-TFR-5.01-0500DST, and also in part by a Department of Science and Technology J C Bose Fellowship, and an endowment from the Infosys Foundation. Both authors were partially supported by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund (code BCSim-2019-s11).

References

- [1] **I Agol**, *Virtual Betti numbers of symmetric spaces*, preprint (2006) [arXiv math/0611828](#)
- [2] **I Agol**, *The virtual Haken conjecture*, Doc. Math. 18 (2013) 1045–1087 [MR](#) [Zbl](#)
- [3] **N Anghel**, *The L^2 -harmonic forms on rotationally symmetric Riemannian manifolds revisited*, Proc. Amer. Math. Soc. 133 (2005) 2461–2467 [MR](#) [Zbl](#)
- [4] **Y Benoist**, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. 7 (1997) 1–47 [MR](#) [Zbl](#)
- [5] **N Bergeron**, **F Haglund**, **D T Wise**, *Hyperplane sections in arithmetic hyperbolic manifolds*, J. Lond. Math. Soc. 83 (2011) 431–448 [MR](#) [Zbl](#)
- [6] **I Biswas**, **M Mj**, **D Pancholi**, *Homotopical height*, Int. J. Math. 25 (2014) art. id. 1450123 [MR](#) [Zbl](#)
- [7] **K Biswas**, **M Mj**, *Pattern rigidity in hyperbolic spaces: duality and PD subgroups*, Groups Geom. Dyn. 6 (2012) 97–123 [MR](#) [Zbl](#)
- [8] **A Borel**, *Density and maximality of arithmetic subgroups*, J. Reine Angew. Math. 224 (1966) 78–89 [MR](#) [Zbl](#)
- [9] **A Borel**, **N Wallach**, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, 2nd edition, Math. Surv. Monogr. 67, Amer. Math. Soc., Providence, RI (2000) [MR](#) [Zbl](#)
- [10] **G Carron**, *L^2 harmonic forms on non compact manifolds*, course notes (2006) [arXiv 0704.3194](#)
- [11] **J Cheeger**, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. 76 (1979) 2103–2106 [MR](#) [Zbl](#)
- [12] **C Chevalley**, **A Weil**, **E Hecke**, *Über das Verhalten der integrale, I: Gattung bei automorphismen des funktionenkörpers*, Abh. Math. Sem. Univ. Hamburg 10 (1934) 358–361 [MR](#) [Zbl](#)
- [13] **J Dodziuk**, *L^2 harmonic forms on rotationally symmetric Riemannian manifolds*, Proc. Amer. Math. Soc. 77 (1979) 395–400 [MR](#) [Zbl](#)
- [14] **J Eells**, **L Lemaire**, *A report on harmonic maps*, Bull. Lond. Math. Soc. 10 (1978) 1–68 [MR](#) [Zbl](#)
- [15] **W Gaschütz**, *Über modulare Darstellungen endlicher Gruppen, die von freien Gruppen induziert werden*, Math. Z. 60 (1954) 274–286 [MR](#) [Zbl](#)
- [16] **L Greenberg**, *Commensurable groups of Moebius transformations*, from “Discontinuous groups and Riemann surfaces” (L Greenberg, editor), Ann. of Math. Stud. 79, Princeton Univ. Press (1974) 227–237 [MR](#) [Zbl](#)
- [17] **P de la Harpe**, *Topics in geometric group theory*, Univ. Chicago Press (2000) [MR](#) [Zbl](#)
- [18] **S Helgason**, *Differential geometry, Lie groups, and symmetric spaces*, Pure Appl. Math. 80, Academic, New York (1978) [MR](#) [Zbl](#)
- [19] **M Kapovich**, *On the normal subgroups in the fundamental groups of complex surfaces*, preprint (1998) Available at <http://www.math.ucdavis.edu/~Ekapovich/EPR/koda.pdf>
- [20] **D Kielak**, *Residually finite rationally solvable groups and virtual fibring*, J. Amer. Math. Soc. 33 (2020) 451–486 [MR](#) [Zbl](#)
- [21] **T Koberda**, **M Mj**, *Commutators, commensurators, and $\mathrm{PSL}_2(\mathbb{Z})$* , J. Topol. 14 (2021) 861–876 [MR](#) [Zbl](#)
- [22] **C Leininger**, **DD Long**, **A W Reid**, *Commensurators of finitely generated nonfree Kleinian groups*, Algebr. Geom. Topol. 11 (2011) 605–624 [MR](#) [Zbl](#)

- [23] **W Lück**, *L^2 -invariants: theory and applications to geometry and K -theory*, Ergebnisse der Math. 44, Springer (2002) [MR](#) [Zbl](#)
- [24] **G A Margulis**, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Math. 17, Springer (1991) [MR](#) [Zbl](#)
- [25] **M Mj**, *Relative rigidity, quasiconvexity and C -complexes*, Algebr. Geom. Topol. 8 (2008) 1691–1716 [MR](#) [Zbl](#)
- [26] **M Mj**, *On discreteness of commensurators*, Geom. Topol. 15 (2011) 331–350 [MR](#) [Zbl](#)
- [27] **M Mj**, *Pattern rigidity and the Hilbert–Smith conjecture*, Geom. Topol. 16 (2012) 1205–1246 [MR](#) [Zbl](#)
- [28] **P Sarnak**, *Notes on thin matrix groups*, from “Thin groups and superstrong approximation” (E Breuillard, H Oh, editors), Math. Sci. Res. Inst. Publ. 61, Cambridge Univ. Press (2014) 343–362 [MR](#) [Zbl](#)
- [29] **R E Schwartz**, *The quasi-isometry classification of rank one lattices*, Inst. Hautes Études Sci. Publ. Math. 82 (1995) 133–168 [MR](#) [Zbl](#)
- [30] **R E Schwartz**, *Symmetric patterns of geodesics and automorphisms of surface groups*, Invent. Math. 128 (1997) 177–199 [MR](#) [Zbl](#)
- [31] **Y Shalom**, *Rigidity of commensurators and irreducible lattices*, Invent. Math. 141 (2000) 1–54 [MR](#) [Zbl](#)
- [32] **Y Shalom**, *Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group*, Ann. of Math. 152 (2000) 113–182 [MR](#) [Zbl](#)
- [33] **Y Shalom, G A Willis**, *Commensurated subgroups of arithmetic groups, totally disconnected groups and adelic rigidity*, Geom. Funct. Anal. 23 (2013) 1631–1683 [MR](#) [Zbl](#)
- [34] **T N Venkataramana**, *Virtual Betti numbers of compact locally symmetric spaces*, Israel J. Math. 166 (2008) 235–238 [MR](#) [Zbl](#)
- [35] **F W Warner**, *Foundations of differentiable manifolds and Lie groups*, Scott, Foresman, Glenview, IL (1971) [MR](#) [Zbl](#)
- [36] **D Witte Morris**, *Introduction to arithmetic groups*, Deductive Press (2015) [MR](#) [Zbl](#)
- [37] **R J Zimmer**, *Ergodic theory and semisimple groups*, Monogr. Math. 81, Birkhäuser, Basel (1984) [MR](#) [Zbl](#)

Department of Mathematics, University of Virginia
Charlottesville, VA, United States

School of Mathematics, Tata Institute of Fundamental Research
Mumbai, India

thomas.koberda@gmail.com, mahan@math.tifr.res.in

<https://sites.google.com/view/koberdat>, <http://www.math.tifr.res.in/~mahan>

Received: 9 May 2022 Revised: 16 April 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Kathryn Hess
kathryn.hess@epfl.ch
École Polytechnique Fédérale de Lausanne

BOARD OF EDITORS

Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Robert Lipshitz	University of Oregon lipshitz@uoregon.edu
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu	Norihiko Minami	Yamato University minami.norihiko@yamato-u.ac.jp
Tara E Brendle	University of Glasgow tara.brendle@glasgow.ac.uk	Andrés Navas	Universidad de Santiago de Chile andres.navas@usach.cl
Indira Chatterji	CNRS & Univ. Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr	Thomas Nikolaus	University of Münster nikolaus@uni-muenster.de
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Robert Oliver	Université Paris 13 bobol@math.univ-paris13.fr
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Jessica S Purcell	Monash University jessica.purcell@monash.edu
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Birgit Richter	Universität Hamburg birgit.richter@uni-hamburg.de
David Futer	Temple University dfuter@temple.edu	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk	Vesna Stojanoska	Univ. of Illinois at Urbana-Champaign vesna@illinois.edu
Ian Hambleton	McMaster University ian@math.mcmaster.ca	Zoltán Szabó	Princeton University szabo@math.princeton.edu
Matthew Hedden	Michigan State University mhedden@math.msu.edu	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Nathalie Wahl	University of Copenhagen wahl@math.ku.dk
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu	Chris Wendl	Humboldt-Universität zu Berlin wendl@math.hu-berlin.de
Thomas Koberda	University of Virginia thomas.koberda@virginia.edu	Daniel T Wise	McGill University, Canada daniel.wise@mcgill.ca
Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr		


See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US \$705/year for the electronic version, and \$1040/year (+\$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**

nonprofit scientific publishing

<https://msp.org/>

© 2024 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 4 (pages 1809–2387) 2024

Möbius structures, quasimetrics and completeness	1809
MERLIN INCERTI-MEDICI	
$\mathbb{Z}/p \times \mathbb{Z}/p$ actions on $S^n \times S^n$	1841
JIM FOWLER and COURTNEY THATCHER	
\mathbb{Z}_k -stratifolds	1863
ANDRÉS ÁNGEL, CARLOS SEGOVIA and ARLEY FERNANDO TORRES	
Relative systoles in hyperelliptic translation surfaces	1903
CORENTIN BOISSY and SLAVYANA GENINSKA	
Smooth singular complexes and diffeological principal bundles	1913
HIROSHI KIHARA	
Natural symmetries of secondary Hochschild homology	1953
DAVID AYALA, JOHN FRANCIS and ADAM HOWARD	
The shape of the filling-systole subspace in surface moduli space and critical points of the systole function	2011
YUE GAO	
Moduli spaces of geometric graphs	2039
MARA BELOTTI, ANTONIO LERARIO and ANDREW NEWMAN	
Classical shadows of stated skein representations at roots of unity	2091
JULIEN KORINMAN and ALEXANDRE QUESNEY	
Commensurators of thin normal subgroups and abelian quotients	2149
THOMAS KOBERDA and MAHAN MJ	
Pushouts of Dwyer maps are $(\infty, 1)$ -categorical	2171
PHILIP HACKNEY, VIKTORIYA OZORNOVA, EMILY RIEHL and MARTINA ROVELLI	
A variant of a Dwyer–Kan theorem for model categories	2185
BORIS CHORNY and DAVID WHITE	
Integral generalized equivariant cohomologies of weighted Grassmann orbifolds	2209
KOUSHIK BRAHMA and SOUMEN SARKAR	
Projective modules and the homotopy classification of (G, n) -complexes	2245
JOHN NICHOLSON	
Realization of Lie algebras of derivations and moduli spaces of some rational homotopy types	2285
YVES FÉLIX, MARIO FUENTES and ANICETO MURILLO	
On the positivity of twisted L^2 -torsion for 3-manifolds	2307
JIANRU DUAN	
An algebraic C_2 -equivariant Bézout theorem	2331
STEVEN R COSTENOBLE, THOMAS HUDSON and SEAN TILSON	
Topologically isotopic and smoothly inequivalent 2-spheres in simply connected 4-manifolds whose complement has a prescribed fundamental group	2351
RAFAEL TORRES	
Remarks on symplectic circle actions, torsion and loops	2367
MARCELO S ATALLAH	
Correction to the article Hopf ring structure on the mod p cohomology of symmetric groups	2385
LORENZO GUERRA	