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Integral generalized equivariant cohomologies of weighted Grassmann orbifolds

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We introduce a new definition of weighted Grassmann orbifolds. We study their several invariant q–CW complex structures and the orbifold singularities on the q–cells of these q–CW complexes. We discuss when the integral cohomology of a weighted Grassmann orbifold has no p–torsion. We compute the equivariant K–theory ring of weighted Grassmann orbifolds with rational coefficients. We introduce divisive weighted Grassmann orbifolds and show that they have invariant CW complex structures. We calculate the equivariant cohomology ring, equivariant K–theory ring and equivariant cobordism ring of a divisive weighted Grassmann orbifold with integer coefficients. We discuss how to compute the weighted structure constants for the integral equivariant cohomology ring of a divisive weighted Grassmann orbifold.

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1. Introduction 2209
2. Weighted Grassmann orbifolds and their invariant q–CW complexes 2213
3. Integral cohomology of certain weighted Grassmann orbifolds 2221
4. Equivariant cohomology, cobordism and K–theory of weighted Grassmann orbifolds 2232
5. Equivariant Schubert calculus for divisive weighted Grassmann orbifolds 2238
References 2242

1 Introduction

We consider the n–dimensional complex vector space $\mathbb{C}^n$ and a positive integer d satisfying $1 \leq d < n$. Then the set of all d–dimensional vector subspaces of $\mathbb{C}^n$ is called a (complex) Grassmann manifold and denoted by $\mathrm{Gr}(d,n)$. In particular, the space $\mathrm{Gr}(1,n)$ is called the $(n-1)$–dimensional complex projective space. The space $\mathrm{Gr}(d,n)$ has a manifold structure of dimension $d(n-d)$; see Mukherjee [24, Chapter 1]. This is a projective variety via the Plücker embedding. The natural $(\mathbb{C}^*)^n$–action on $\mathbb{C}^n$ induces a $(\mathbb{C}^*)^n$–action on $\mathrm{Gr}(d,n)$. Grassmann manifolds are central objects of study in algebraic geometry, algebraic topology and differential geometry. Several interesting topological and geometrical properties of Grassmann manifolds can be found in Laksov [21], Knutson and Tao [20] and Jiao and Peng [18].

The orbifold version of a complex projective space was introduced in Kawasaki [19] and was called a twisted projective space. Orbifolds, a generalization of manifolds, were introduced by Satake [27; 28] with the name V–manifolds. Later, Thurston [31] used the terminology orbifolds instead. In the past two decades, several developments have appeared to study orbifolds arising in algebraic geometry, differential geometry and string topology. Some cohomology theories, such as de Rham cohomology (see Adem, Leida and Ruan [2, Chapter 2]), singular cohomology (see Hatcher [16]), Dolbeault cohomology (see Baily [5]), the Chen–Ruan cohomology ring [6] and orbifold K–theory [2, Chapter 3] for a class of orbifolds were studied either with rational, real or complex coefficients. One can construct a CW complex structure on an effective orbifold following Goresky [11]. However, in general, the computation of the singular integral cohomology of an orbifold is considerably difficult.

Let G be a topological group and X a G–space. Then the equivariant map $X \to \{\mathrm{pt}\}$ induces a graded $\mathscr{E}^*_G(\{\mathrm{pt}\})$–algebra structure on $\mathscr{E}^*_G(X)$. The readers are referred to May [22] for the definitions and several results on the G–equivariant generalized cohomology theory $\mathscr{E}^*_G$. If $\mathscr{E}^*_G = H^*_G$, then it is known as the equivariant cohomology theory defined by

$$H^*_G(X) := H^*(EG \times_G X).$$

The ring $H^*_G(X)$ is called the Borel equivariant cohomology of X. If $\mathscr{E}^*_G = K^*_G$, then it is known as the equivariant K–theory. If X is compact, then $K^0_G(X)$ is the equivalence classes of G–equivariant complex vector bundles on X; see Segal [29]. If X is a point with trivial action, then $K^*_G(\{\mathrm{pt}\})$ is isomorphic to $R(G)[z, z^{-1}]$, where $R(G)$ is complex representation ring of G and z is the Bott element of cohomological dimension -2. The G–equivariant ring $\mathrm{MU}^*_G(X)$ is known as the equivariant complex cobordism ring; see tom Dieck [9]. Sinha [30] and Hanke [13] have shown several developments on MU^*_G. However, many interesting questions on $\mathrm{MU}^*_G(X)$ remain undetermined. For example, $\mathrm{MU}^*_G(\{\mathrm{pt}\})$ is not completely known for nontrivial groups G.

Corti and Reid [7] introduced the weighted projective analogs of a class of Grassmann manifolds and called them weighted Grassmannians. Then Abe and Matsumura [1] defined weighted Grassmannians explicitly and studied their equivariant cohomology ring of weighted Grassmannians with rational coefficients. The weighted Grassmannians are projective varieties with orbifold singularities. The simplest weighted Grassmannians are the weighted projective spaces. Kawasaki [19] proved that the integral cohomology of weighted projective spaces has no torsion and is concentrated in even degrees. The equivariant cohomology ring of a weighted projective space has been studied in Bahri, Franz and Ray [3] in terms of piecewise polynomials. The equivariant K–theory and equivariant cobordism rings of divisive weighted projective spaces have been discussed in Harada, Holm, Ray and Williams [15] in terms of piecewise Laurent polynomials and piecewise cobordism forms, respectively.

Inspired by the above works, we introduce a different definition of weighted Grassmann orbifolds and study their several topological properties such as torsion in the integral cohomology, equivariant cohomology ring, equivariant K–theory ring and equivariant cobordism ring with integer coefficients. We note that

Abe and Matsumura [1] and Corti and Reid [7] used the name "weighted Grassmannians". However, keeping other names in mind like Milnor manifolds and Seifert manifolds, we prefer to use Grassmann manifolds and weighted Grassmann orbifolds.

The paper is organized as follows. In Section 2, analogously to the definition of Grassmann manifold discussed in Mukherjee [24], we introduce another definition of a weighted Grassmann orbifold $\mathrm{WGr}(d,n)$ for $d<n$, $a\in\mathbb{Z}_{\geq 1}$ and a "weight vector" $W:=(w_1,\dots,w_n)\in(\mathbb{Z}_{\geq 0})^n$. Interestingly, this definition is equivalent to the previous one that appeared in Abe and Matsumura [1]. We recall the definition of Schubert symbols for $d<n$ and discuss how to get a total ordering on the Schubert symbols. Using this total order we show that there is an equivariant embedding from a weighted Grassmann orbifold to a weighted projective space; see Lemma 2.5. We describe a q–CW complex structure of $\mathrm{WGr}(d,n)$ in Proposition 2.7. Then we discuss a $(\mathbb{C}^*)^n$–invariant filtration

$$\{\mathrm{pt}\}=X_0\subset X_1\subset X_2\subset\cdots\subset X_m=\mathrm{WGr}(d,n)$$

of $\mathrm{WGr}(d,n)$ using the q–CW complex structure, where $m:=\binom{n}{d}-1$. Here, we consider q–CW complex structure in the sense of Poddar and Sarkar [25, Section 4]. We note that one may get different q–CW complex structures depending on the choice of the total orderings on the set of all Schubert symbols for $d<n$. Accordingly, one may obtain different $(\mathbb{C}^*)^n$–invariant filtrations of $\mathrm{WGr}(d,n)$.

In Section 3, first we recall that there is an equivariant homeomorphism from $\mathbb{W}P(rc_0,rc_1,\dots,rc_m)$ to $\mathbb{W}P(c_0,c_1,\dots,c_m)$ for any $1\leq r\in\mathbb{N}$. Using this technique, we show how the orbifold singularity on a q–cell of some subcomplexes of $\mathrm{WGr}(d,n)$ can be reduced; see Lemma 3.3. Consequently, we get a new q–CW complex structure of these subcomplexes, including $\mathrm{WGr}(d,n)$, possibly with less singularity on each q–cell; see Theorem 3.4. We show in Theorem 3.5 that two weighted Grassmann orbifolds are weakly equivariantly homeomorphic if their weight vectors differ by a permutation $\sigma\in S_n$. We define "admissible permutation" $\sigma\in S_n$ for a prime p and $\mathrm{WGr}(d,n)$; see Definition 3.8. The following result says when $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ has no p–torsion.

Theorem A (Theorem 3.10) *If there exists an admissible permutation $\sigma\in S_n$ for a prime p and $\mathrm{WGr}(d,n)$, then $H^{\mathrm{odd}}(\mathrm{WGr}(d,n);\mathbb{Z}_p)$ is trivial and $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ has no p–torsion.*

We introduce "divisive" weighted Grassmann orbifolds. We note that this definition coincides with the concept of divisive weighted projective space of Harada, Holm, Ray and Williams [15] when $1=d<n$. We prove the following.

Theorem B (Theorem 3.19) *If $\mathrm{WGr}(d,n)$ is a divisive weighted Grassmann orbifold, then it has a $(\mathbb{C}^*)^n$–invariant CW complex structure. Moreover, the $(\mathbb{C}^*)^n$–action on each cell of this CW complex can be described explicitly.*

This result implies that the integral cohomology of a divisive weighted Grassmann orbifold has no torsion and is concentrated in even degrees. We discuss a class of nontrivial examples of divisive weighted

Grassmann orbifolds. We remark that the weighted Grassmann orbifold in Example 3.12 is not divisive. However, its integral cohomology has no torsion.

In Section 4, we show that the $(\mathbb{C}^*)^n$–invariant stratification

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset \cdots \subset X_m = \mathrm{WGr}(d,n)$$

has the following property. The quotient X_i/X_{i-1} is homeomorphic to the Thom space of an orbifold $(\mathbb{C}^*)^n$–bundle

$$\xi^i : \mathbb{C}^{\ell(\lambda^i)}/G_i \to \{\mathrm{pt}\}$$

for some $\ell(\lambda^i) \in \mathbb{Z}_{\geq 1}$ and finite groups G_i for $i = 1, \ldots, m$; see Proposition 4.1. Then considering $T^n := (S^1)^n \subset (\mathbb{C}^*)^n$, we compute the equivariant K–theory ring of any weighted Grassmann orbifolds with rational coefficients; see Theorem 4.4. If $\mathrm{WGr}(d,n)$ is divisive then G_i is trivial for $i = 1, \ldots, m$. The following result describes the integral equivariant cohomology of certain weighted Grassmann orbifolds.

Theorem C (Theorem 4.7) *Let* $\mathrm{WGr}(d,n)$ *be a divisive weighted Grassmann orbifold for* $d < n$*. Then the generalized* T^n*–equivariant cohomology with integer coefficients* $\mathscr{E}^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ *can be given by*

$$\left\{ (f_i) \in \bigoplus_{i=0}^{m} \mathscr{E}^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z}) \;\middle|\; e_{T^n}(\xi^{ij}) \text{ divides } f_i - f_j \text{ for } j < i \text{ and } |\lambda^j \cap \lambda^i| = d-1 \right\}$$

for $\mathscr{E}^*_{T^n} = H^*_{T^n}, K^*_{T^n}$ *and* $\mathrm{MU}^*_{T^n}$*.*

The computation of $e_{T^n}(\xi^{ij})$ is discussed in (4-4). We compute the equivariant cohomology ring of some weighted Grassmann orbifold with integer coefficients which are not divisive; see Theorem 4.10. For $m \geq 2$, corresponding to each pair of positive integers (n,d) such that $d < n$ and $m+1 = \binom{n}{d}$, we have a T^n–action on $\mathbb{W}P(c_0, c_1, \ldots, c_m)$. For each pair (n,d), we discuss the generalized T^n–equivariant cohomology of a divisive $\mathbb{W}P(c_0, c_1, \ldots, c_m)$ with integer coefficients; see Theorem 4.11.

In Section 5, we show that there exist equivariant Schubert classes $\{w\widetilde{S}_{\lambda^i}\}_{i=0}^m$ which form a basis for the integral T^n–equivariant cohomology of a divisive weighted Grassmann orbifold; see Proposition 5.3. We study some properties of weighted structure constants; see Lemma 5.5. Then we show the following multiplication rule.

Proposition D (weighted Pieri rule, Proposition 5.7)

$$w\widetilde{S}_{\lambda^1} w\widetilde{S}_{\lambda^j} = (w\widetilde{S}_{\lambda^1}|_{\lambda^j}) w\widetilde{S}_{\lambda^j} + \sum_{\lambda^i \to \lambda^j} \frac{c_0}{c_j} w\widetilde{S}_{\lambda^i}.$$

Moreover, we deduce a recurrence relation which helps to compute the weighted structure constants $\{wc_{ij}^k\}$ corresponding to this Schubert basis $\{w\widetilde{S}_{\lambda^i}\}_{i=0}^m$ with integral coefficients.

Proposition E (Proposition 5.8) *For any three Schubert symbols λ^i, λ^j and λ^k, we have the recurrence relation*

$$(w\widetilde{S}_{\lambda^1}|_{\lambda^k} - w\widetilde{S}_{\lambda^1}|_{\lambda^i})wc_{ij}^k = \left(\sum_{\lambda^s \to \lambda^i} \frac{c_0}{c_i} wc_{sj}^k - \sum_{\lambda^k \to \lambda^t} \frac{c_0}{c_t} wc_{ij}^t \right).$$

2 Weighted Grassmann orbifolds and their invariant q–CW complexes

In this section, we introduce another definition of weighted Grassmann orbifold $\mathrm{WGr}(d,n)$, where $d < n$. We recall the definition of a Schubert symbol for $d < n$ and discuss some (total) ordering on the set of Schubert symbols. We show that there is an equivariant embedding from a weighted Grassmann orbifold to a weighted projective space. We show that our definition of weighted Grassmann orbifold is equivalent to the previous one, which appeared in [1]. We study the orbifold and q–CW complex structures of weighted Grassmann orbifolds generalizing the Grassmann manifolds counterpart discussed in [23].

Let $M_d(n,d)$ be the set of all complex $n \times d$ matrices of rank d, and $\mathrm{GL}(d,\mathbb{C})$ the set of all nonsingular complex matrices of order d. We denote a matrix $A \in M_d(n,d)$ by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nd} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix}, \quad \text{where } \boldsymbol{a}_i \in \mathbb{C}^d \text{ for } i = 1, \dots, n.$$

Definition 2.1 Let $W := (w_1, w_2, \dots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \in \mathbb{Z}_{\geq 1}$. Define an equivalence relation $\sim_w$ on $M_d(n,d)$ by

$$\begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} \sim_w \begin{pmatrix} t^{w_1}\boldsymbol{a}_1 \\ t^{w_2}\boldsymbol{a}_2 \\ \vdots \\ t^{w_n}\boldsymbol{a}_n \end{pmatrix} T$$

for $T \in \mathrm{GL}(d,\mathbb{C})$ and $t \in \mathbb{C}^*$ such that $t^a = \det(T) \in \mathbb{C}^*$. We denote the identification space by

$$\mathrm{WGr}(d,n) := M_d(n,d)/\sim_w.$$

The quotient map

$$\pi_w : M_d(n,d) \to \mathrm{WGr}(d,n) \tag{2-1}$$

is defined by $\pi_w(A) = [A]_{\sim_w}$. The topology on $\mathrm{WGr}(d,n)$ is given by the quotient topology via the map π_w.

Remark 2.2 If $W = (0,0,\dots,0)$ and $a = 1$, then $\mathrm{WGr}(d,n)$ is the Grassmann manifold $\mathrm{Gr}(d,n)$. We denote the corresponding quotient map by

$$\pi : M_d(n,d) \to \mathrm{Gr}(d,n). \tag{2-2}$$

The space $\mathrm{Gr}(d,n)$ is a $d(n-d)$–dimensional smooth manifold and represents the set of all d–dimensional vector subspaces in $\mathbb{C}^n$. Several basic properties, such as the manifold and CW complex structure of $\mathrm{Gr}(d,n)$, can be found in [23]. In this paper, by dimension we mean complex dimension unless specified explicitly.

Remark 2.3 If $d=1$, then $M_d(n,d)=M_1(n,1)=\mathbb{C}^n\setminus\{0\}$ and $\mathrm{GL}(1,\mathbb{C})=\mathbb{C}^*$. The corresponding $\sim_w$ is given by

$$(z_1,z_2,\ldots,z_n)\sim_w (t^{a+w_1}z_1,t^{a+w_2}z_2,\ldots,t^{a+w_n}z_n).$$

The quotient space $M_1(n,1)/{\sim_w}$ is called the weighted projective space with weights

$$(a+w_1,a+w_2,\ldots,a+w_n),$$

and is denoted by $\mathbb{W}P(c_0,c_1,\ldots,c_{n-1})$, where $c_i=a+w_{i+1}$ for $i\in\{0,1,\ldots,n-1\}$. For the weighted projective space, we denote $\sim_w$ by $\sim_c$ when $c=(c_0,c_1,\ldots,c_{n-1})$. This identification $\sim_c$ is called a weighted $\mathbb{C}^*$–action on $\mathbb{C}^n\setminus\{0\}$ with weights $(c_0,c_1,\ldots,c_{n-1})$. In addition, if $W=(0,0,\ldots,0)$ and $a=1$, then $c_0=1=c_1=\cdots=c_{n-1}$ and $\mathbb{W}P(c_0,c_1,\ldots,c_{n-1})$ is $\mathbb{C}P^{n-1}=\mathrm{Gr}(1,n)$.

A Schubert symbol λ for $d<n$ is a sequence of d integers $(\lambda_1,\lambda_2,\ldots,\lambda_d)$ such that $1\le\lambda_1<\lambda_2<\cdots<\lambda_d\le n$. The length $\ell(\lambda)$ of a Schubert symbol $\lambda:=(\lambda_1,\lambda_2,\ldots,\lambda_d)$ is defined by

$$\ell(\lambda):=(\lambda_1-1)+(\lambda_2-2)+\cdots+(\lambda_d-d).$$

There are $\binom{n}{d}$ many Schubert symbols for $d<n$. One can define a partial order $\preceq$ on the Schubert symbols for $d<n$ by

$$\lambda\preceq\mu \quad \text{if} \quad \lambda_i\le\mu_i \quad \text{for all } i=1,2,\ldots,d. \tag{2-3}$$

Then the set of all Schubert symbols for $d<n$ form a poset with respect to this partial order $\preceq$.

Definition 2.4 Let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_d)$ and $\mu=(\mu_1,\mu_2,\ldots,\mu_d)$ be two Schubert symbols for $d<n$. We say that $\lambda<\mu$ if $\ell(\lambda)<\ell(\mu)$, otherwise we use the dictionary order if $\ell(\lambda)=\ell(\mu)$.

This gives a total order on the set of all Schubert symbols. Note that the total order $<$ in Definition 2.4 preserves the partial order $\preceq$ in (2-3). That is, for two Schubert symbols λ and μ, if $\lambda\preceq\mu$ then $\lambda\le\mu$, but the converse may not be true in general. Observe that there may exist several other total orders on the set of all Schubert symbols which preserve the partial order $\preceq$. For example, the dictionary order also gives a total order on the Schubert symbols. By a total order on the set of all Schubert symbols for $d<n$, we mean one of these total orders on it. For $m=\binom{n}{d}-1$, let

$$\lambda^0<\lambda^1<\lambda^2<\cdots<\lambda^m \tag{2-4}$$

be a total order on the Schubert symbols for $d<n$.

For $W=(w_1,w_2,\dots,w_n)\in(\mathbb{Z}_{\geq 0})^n$, $a\in\mathbb{Z}_{\geq 1}$ and $i\in\{0,1,\dots,m\}$, let

$$c_i := a+\sum_{j=1}^{d} w_{\lambda^i_j}, \tag{2-5}$$

where $\lambda^i=(\lambda^i_1,\lambda^i_2,\dots,\lambda^i_d)$ is the i^{th} Schubert symbol given in (2-4). Then $c_i\geq 1$ for any $i\in\{0,\dots,m\}$. Therefore, one can define the weighted projective space $\mathbb{W}P(c_0,c_1,\dots,c_m)$ from Remark 2.3. We denote the associated orbit map $\mathbb{C}^{m+1}\setminus\{0\}\to\mathbb{W}P(c_0,c_1,\dots,c_m)$ by π'_c, which can be written as

$$\pi'_c(z_0,z_1,\dots,z_m)=[z_0:z_1:\cdots:z_m]_{\sim_c}. \tag{2-6}$$

Note that when $c_0=c_1=\cdots=c_m=1$, the corresponding orbit map is denoted by

$$\pi'\colon\mathbb{C}^{m+1}\setminus\{0\}\to\mathbb{C}P^m.$$

Let $(t_1,t_2,\dots,t_n)\in(\mathbb{C}^*)^n$ and $A=(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n)^{\text{tr}}\in M_d(n,d)$. Then $(\mathbb{C}^*)^n$ acts on $M_d(n,d)$ by

$$(t_1,\dots,t_n)(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n)^{\text{tr}} := (t_1\boldsymbol{a}_1,t_2\boldsymbol{a}_2,\dots,t_n\boldsymbol{a}_n)^{\text{tr}}. \tag{2-7}$$

This induces a natural $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}(d,n)$ such that the orbit map π_w of (2-1) is $(\mathbb{C}^*)^n$–equivariant.

The standard ordered basis $\{e_1,e_2,\dots,e_n\}$ of $\mathbb{C}^n$ induces an ordered basis $\{e_{\lambda^0},e_{\lambda^1},\dots,e_{\lambda^m}\}$ of $\Lambda^d(\mathbb{C}^n)$, where $e_\lambda=e_{\lambda_1}\wedge\cdots\wedge e_{\lambda_d}$ for the Schubert symbol $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_d)$ for $d<n$. Therefore, we can identify $\Lambda^d(\mathbb{C}^n)$ with $\mathbb{C}^{m+1}(=\mathbb{C}\{e_{\lambda^0},e_{\lambda^1},\dots,e_{\lambda^m}\})$. The standard action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ induces an action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^{m+1}\setminus\{0\}$, which is defined by

$$(t_1,t_2,\dots,t_n)\Big(\sum_{i=0}^{m} a_i e_{\lambda^i}\Big)=\sum_{i=0}^{m} a_i t_{\lambda^i} e_{\lambda^i}, \tag{2-8}$$

where $t_\lambda=t_{\lambda_1}\cdots t_{\lambda_d}$ for the Schubert symbol $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_d)$. This induces a $(\mathbb{C}^*)^n$–action on the weighted projective space $\mathbb{W}P(c_0,c_1,\dots,c_m)$ such that the orbit map π'_c in (2-6) is $(\mathbb{C}^*)^n$–equivariant.

For each Schubert symbol $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_d)$, let A_λ be the matrix with row vectors $\boldsymbol{a}_{\lambda_1},\boldsymbol{a}_{\lambda_2},\dots,\boldsymbol{a}_{\lambda_d}$. Define a map $P\colon M_d(n,d)\to\mathbb{C}^{m+1}\setminus\{0\}$ by

$$P(A)=\boldsymbol{v}_1\wedge\boldsymbol{v}_2\wedge\cdots\wedge\boldsymbol{v}_d=\sum_{i=0}^{m}\det(A_{\lambda^i})e_{\lambda^i}, \tag{2-9}$$

where $\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_d\in\mathbb{C}^n$ are the columns of A. Observe that $P(A)\neq 0$ as $A\in M_d(n,d)$ has rank d.

From (2-9) we have

$$P(DAT)=\sum_{i=0}^{m}\det((DAT)_{\lambda^i})e_{\lambda^i}=\sum_{i=0}^{m} t^{c_i}\det(A_{\lambda^i})e_{\lambda^i},$$

where $T \in \mathrm{GL}(d, \mathbb{C})$, $D = \mathrm{diag}(t^{w_1}, t^{w_2}, \dots, t^{w_n})$ is the diagonal matrix for $t \in \mathbb{C}^*$ such that $t^a = \det(T)$, and c_i is defined in (2-5) for $i = 0, 1, 2, \dots, m$. Therefore, the map P in (2-9) induces a map

$$\mathrm{Pl}_w : \mathrm{WGr}(d, n) \to \mathbb{W}P(c_0, c_1, c_2, \dots, c_m) \tag{2-10}$$

defined by $\mathrm{Pl}_w([A]_{\sim_w}) = [\det(A_{\lambda^0}) : \det(A_{\lambda^1}) : \dots : \det(A_{\lambda^m})]_{\sim_c}$.

The map Pl_w satisfies the following commutative diagram:

$$\begin{array}{ccc} M_d(n,d) & \xrightarrow{\quad P \quad} & \mathbb{C}^{m+1} \setminus \{0\} \\ \big\downarrow \pi_w & & \big\downarrow \pi'_c \\ \mathrm{WGr}(d,n) & \xrightarrow{\mathrm{Pl}_w} & \mathbb{W}P(c_0, c_1, \dots, c_m) \end{array}$$

Thus the map Pl_w is continuous, since π_w and π'_c are quotient maps.

Lemma 2.5 *The map* $\mathrm{Pl}_w : \mathrm{WGr}(d, n) \to \mathbb{W}P(c_0, c_1, c_2, \dots, c_m)$ *is an embedding.*

Proof Consider $[A]_{\sim_w} \in \mathrm{WGr}(d, n)$ for some $A \in M_d(n, d)$. There exists a Schubert symbol λ^i such that $\det(A_{\lambda^i}) \neq 0$. Without loss of generality, we can assume that $A_{\lambda^i} = \mathrm{I}_d$, where I_d is the identity matrix of order d. If $A_{\lambda^i} \neq \mathrm{I}_d$ then one can calculate $s \in \mathbb{C}^*$ such that $s^{c_i} = 1/\det(A_{\lambda^i})$. Now we consider the matrices $D = \mathrm{diag}(s^{w_1}, s^{w_2}, \dots, s^{w_n})$ and $T = (D_{\lambda^i} A_{\lambda^i})^{-1}$. Then $\det(T) = s^a$ and $(DAT)_{\lambda^i} = \mathrm{I}_d$. Note that $[DAT]_{\sim_w} = [A]_{\sim_w} \in \mathrm{WGr}(d, n)$.

We prove that Pl_w is injective. Let $[A]_{\sim_w}, [B]_{\sim_w} \in \mathrm{WGr}(d, n)$ be such that $\mathrm{Pl}_w([A]_{\sim_w}) = \mathrm{Pl}_w([B]_{\sim_w})$ for some $A, B \in M_d(n, d)$. Now

$$\mathrm{Pl}_w([A]_{\sim_w}) = \mathrm{Pl}_w([B]_{\sim_w}) \implies \det(A_{\lambda^j}) = t^{c_j} \det(B_{\lambda^j}) \tag{2-11}$$

for some $t \in \mathbb{C}^*$ and for all $j \in \{0, 1, \dots, m\}$. Since $A \in M_d(n, d)$ there exists a Schubert symbol $\lambda^i = (\lambda^i_1, \dots, \lambda^i_d)$ such that $\det(A_{\lambda^i}) \neq 0$. Then using (2-11), $\det(B_{\lambda^i}) \neq 0$. So we can assume $A_{\lambda^i} = B_{\lambda^i} = \mathrm{I}_d$. Then $t^{c_i} = 1$. Consider the matrices $D = \mathrm{diag}(t^{w_1}, t^{w_2}, \dots, t^{w_n})$ and $T = \mathrm{diag}(t^{-w_{\lambda^i_1}}, \dots, t^{-w_{\lambda^i_d}})$. Thus, we have $B_{\lambda^i} = (DAT)_{\lambda^i}$.

For $k \notin (\lambda^i_1, \dots, \lambda^i_d)$ and $1 \le l \le d$, let a_{kl} and b_{kl} be the (k, l) entries of the matrices A and B, respectively. For a fixed l, let λ^j be the Schubert symbol obtained by replacing λ^i_l by k in λ^i and then ordering the latter set. Then $\det(A_{\lambda^j}) = a_{kl}$ and $\det(B_{\lambda^j}) = b_{kl}$. Thus using (2-11), we get

$$b_{kl} = t^{c_j} a_{kl} \implies b_{kl} = t^{c_j - c_i} a_{kl} \implies b_{kl} = t^{w_k - w_{\lambda^i_l}} a_{kl}.$$

The above condition holds for all $1 \le k \le n$ and $1 \le l \le d$. This gives $B = DAT$. Then we have $[A]_{\sim_w} = [B]_{\sim_w}$. Hence, Pl_w is an injective map.

Observe that, if $W = (0, 0, \dots, 0)$ and $a = 1$, then the map Pl_w is the usual Plücker map

$$\mathrm{Pl} : \mathrm{Gr}(d, n) \to \mathbb{C}P^m.$$

It is well known that Pl is an embedding. Moreover, we have the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{WGr}(d,n) & \xrightarrow{\mathrm{Pl}_w} & \mathbb{W}P(c_0,c_1,\dots,c_m) \\ \uparrow{\scriptstyle \pi_w} & & \uparrow{\scriptstyle \pi'_c} \\ M_d(n,d) & \xrightarrow{P} & \mathbb{C}^{m+1}\setminus\{0\} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathrm{Gr}(d,n) & \xrightarrow{\mathrm{Pl}} & \mathbb{C}P^m \end{array} \tag{2-12}$$

Let U be an open subset of $\mathrm{WGr}(d,n)$. Then $\pi_w^{-1}(U)$ is an open subset of $M_d(n,d)$. Since the map π in (2-2) is an orbit map, $\pi(\pi_w^{-1}(U))$ is an open subset of $\mathrm{Gr}(d,n)$. Thus $\mathrm{Pl}(\pi(\pi_w^{-1}(U))) = \pi'(P(\pi_w^{-1}(U)))$ is an open subset of $\mathrm{Pl}(\mathrm{Gr}(d,n))$. Then $P(\pi_w^{-1}(U))$ is an open subset of $P(M_d(n,d))$. Therefore, $\mathrm{Pl}_w(U) = \pi'_c(P(\pi_w^{-1}(U)))$ is an open subset of $\mathrm{Pl}_w(\mathrm{WGr}(d,n))$. Thus Pl_w is an embedding. □

We call the embedding Pl_w the *weighted Plücker embedding*. Note that the actions of $(\mathbb{C}^*)^n$ on $\mathrm{WGr}(d,n)$ and $\mathbb{W}P(c_0,c_1,\dots,c_m)$ imply that the weighted Plücker embedding Pl_w is $(\mathbb{C}^*)^n$–equivariant, and $\mathrm{Pl}_w(\mathrm{WGr}(d,n))$ is a $(\mathbb{C}^*)^n$–invariant subset of $\mathbb{W}P(c_0,c_1,\dots,c_m)$. Thus all the maps in the diagram (2-12) are $(\mathbb{C}^*)^n$–equivariant.

Now we show that Definition 2.1 is equivalent to the definition of a weighted Grassmannian studied in [1]. The algebraic torus $(\mathbb{C}^*)^{n+1}$ acts on $\Lambda^d(\mathbb{C}^n)$ by

$$(t_1,t_2,\dots,t_n,t)\sum_{i=0}^m a_{\lambda^i}e_{\lambda^i} = \sum_{i=0}^m t\cdot t_{\lambda^i}a_{\lambda^i}e_{\lambda^i},$$

where $t_\lambda = t_{\lambda_1}\cdots t_{\lambda_d}$ for $\lambda = (\lambda_1,\dots,\lambda_d)$. Consider the subgroup WD of $(\mathbb{C}^*)^{n+1}$ defined by

$$\mathrm{WD} := \{(t^{w_1},t^{w_2},\dots,t^{w_n},t^a) \mid t \in \mathbb{C}^*\}.$$

Then the restricted action of WD on $\Lambda^d(\mathbb{C}^n)\setminus\{0\}$ is given by

$$(t^{w_1},t^{w_2},\dots,t^{w_n},t^a)\sum_{i=0}^m a_{\lambda^i}e_{\lambda^i} = \sum_{i=0}^m t^{c_i}a_{\lambda^i}e_{\lambda^i}.$$

Observe that this action of WD is same as the weighted $\mathbb{C}^*$–action in Remark 2.3. Then we have $\Lambda^d(\mathbb{C}^n)\setminus\{0\}/\mathrm{WD} = \mathbb{W}P(c_0,\dots,c_m)$ and by the commutativity of the diagram (2-12) we have

$$\mathrm{Pl}_w(\mathrm{WGr}(d,n)) = \frac{P(M_d(n,d))}{\mathrm{WD}}.$$

Therefore the topologies on $\mathrm{WGr}(d,n)$ and $P(M_d(n,d))/\mathrm{WD}$ are equivalent. Abe and Matsumura [1] called the quotient $P(M_d(n,d))/\mathrm{WD}$ a weighted Grassmannian and showed that it has an orbifold structure. We call $\mathrm{WGr}(d,n)$ a weighted Grassmann orbifold associated to the pair (W,a).

Next, we recall the Schubert cell decomposition of $\mathrm{Gr}(d,n)$ following [23]. For $k \le n$, we identify

$$\mathbb{C}^k = \{(z_1, z_2, \dots, z_k, 0, \dots, 0) \in \mathbb{C}^n\}.$$

For the Schubert symbol $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$, the Schubert cell $E(\lambda)$ is defined by

$$E(\lambda) := \{X \in \mathrm{Gr}(d,n) \mid \dim(X \cap \mathbb{C}^{\lambda_j}) = j, \dim(X \cap \mathbb{C}^{\lambda_j - 1}) = j-1 \text{ for all } j \in [d]\},$$

where $[d] := \{1, 2, \dots, d\}$. We have the following homeomorphism from [23, Chapter 6]:

$$E(\lambda) \cong \left\{ \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \\ 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & * \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \;\middle|\; * \in \mathbb{C} \text{ and } e_j \text{ is the } \lambda_j^{\mathrm{th}} \text{ row for } j \in [d] \right\}. \tag{2-13}$$

Note that the j^{th} column in the matrices in (2-13) has λ_j^{th} entry 1 and all subsequent entries of this column are zero for $j \in [d]$. Then $E(\lambda)$ is an open cell of dimension $\ell(\lambda) = (\lambda_1 - 1) + (\lambda_2 - 2) + \cdots + (\lambda_d - d)$.

We recall some basic properties of q–cell and finite q–CW complex from [25; 4]. Let D^n be the open unit disc in $\mathbb{R}^n$ and G a finite group acting on $\bar{D}^n$ such that $\partial \bar{D}^n$ is invariant. Then D^n/G is called a q–cell of real dimension n. Let Y be a space and $\phi\colon \partial \bar{D}^n/G \to Y$ a continuous map. Then the mapping cone

$$X := \left(Y \sqcup \frac{\bar{D}^n}{G}\right) \Big/ \left\{x \sim \phi(x) \text{ for } x \in \frac{\partial \bar{D}^n}{G}\right\}$$

is obtained from Y by attaching the q–cell D^n/G. As a set, we can write $X = Y \sqcup (D^n/G)$ whenever the attaching map is clear. If a space X is obtained from a finite set by attaching finitely many q–cells, then X is called a finite q–CW complex.

Let k be a positive integer and $G(k)$ the group of k^{th} roots of unity defined by

$$G(k) := \{t \in \mathbb{C}^* \mid t^k = 1\}.$$

Then we have the following.

Lemma 2.6 *Let S be a $\mathbb{C}^*$–space, and suppose that $\mathbb{C}^*$ acts on $S \times \mathbb{C}^*$ by $t \cdot (x, \alpha) = (t \cdot x, t^k \alpha)$. Then*

$$\frac{S \times \mathbb{C}^*}{\mathbb{C}^*} \cong \frac{S}{G(k)},$$

where $G(k)$ acts on S by restriction of the $\mathbb{C}^*$–action.

Proof The inclusion map $S \to S \times \mathbb{C}^*$ defined by $x \to (x, 1)$ induces a map

$$\bar{f} \colon S \to \frac{S \times \mathbb{C}^*}{\mathbb{C}^*}.$$

Note that every element in the codomain of $\bar{f}$ can be written as $[(u, 1)]$, where $u \in S$. To verify this, consider an element $[(x, t)]$ in the codomain of $\bar{f}$, where $x \in S$ and $t \in \mathbb{C}^*$. Consider $s \in \mathbb{C}^*$ such that $s^k = 1/t$. Then $s \cdot (x, t) = (s \cdot x, 1)$. Hence $[(x, t)] = [(u, 1)]$, where $u = s \cdot x$. Thus $u \in S$ is the preimage of $[u, 1] \in \text{codomain}(f)$ and the map $\bar{f}$ becomes onto.

Now $G(k)$ is a finite subgroup of $\mathbb{C}^*$ acts on S as a restriction of the $\mathbb{C}^*$- action. For any $t \in G(k)$,

$$\bar{f}(t \cdot u) = [(t \cdot u, 1)] = [(t \cdot u, t^k)] = [(u, 1)] = \bar{f}(u).$$

Thus $\bar{f}$ induces an onto map $f : S/G(k) \to S \times \mathbb{C}^*/\mathbb{C}^*$ such that the following diagram commutes:

(2-14)
$$\begin{array}{ccc} S & \xrightarrow{\quad \bar{f} \quad} & \dfrac{S \times \mathbb{C}^*}{\mathbb{C}^*} \\ & {\scriptstyle \sim G(k)} \searrow \qquad \nearrow {\scriptstyle f} & \\ & \dfrac{S}{G(k)} & \end{array}$$

To check that f is one-to-one, if $[(x, 1)] = [(y, 1)]$ then $(y, 1) = t \cdot (x, 1) = (t \cdot x, t^k)$. This implies $y = t \cdot x$ for some $t \in G(k)$. Thus $[x] = [y]$ in $S/G(k)$. Therefore,

$$\frac{S \times \mathbb{C}^*}{\mathbb{C}^*} \cong \frac{S}{G(k)}. \qquad \square$$

The next result gives a q–CW complex structure on $\mathrm{WGr}(d, n)$.

Proposition 2.7 $\mathrm{WGr}(d, n)$ *is a finite q–CW complex for $0 < d < n$.*

Proof Consider a total order on the Schubert symbols for $d < n$ as in (2-4), which satisfies the partial order in (2-3). For each $i \in \{0, 1, \ldots, m\}$, define $\widetilde{E}(\lambda^i) := \pi^{-1}(E(\lambda^i))$, where the map π is defined in (2-2). The Schubert cell decomposition of $\mathrm{Gr}(d, n)$ gives that $\mathrm{Gr}(d, n) = \bigsqcup_{i=0}^m E(\lambda^i)$. This implies

(2-15)
$$M_d(n, d) = \bigsqcup_{i=0}^m \widetilde{E}(\lambda^i),$$

since the map π is surjective. Note that

$$\widetilde{E}(\lambda^i) = \{A \in M_d(n, d) \mid \det(A_{\lambda^i}) \neq 0, \det(A_{\lambda^j}) = 0 \text{ for } j > i\}.$$

Let $A \in \widetilde{E}(\lambda^i)$ and $A \sim_w B$ for a matrix $B \in M_d(n, d)$. Then $B \in \widetilde{E}(\lambda^i)$.

Therefore, we have the decomposition of $\mathrm{WGr}(d,n)$

$$\mathrm{WGr}(d,n) = \pi_w(\widetilde{E}(\lambda^0)) \sqcup \pi_w(\widetilde{E}(\lambda^1)) \sqcup \cdots \sqcup \pi_w(\widetilde{E}(\lambda^m)).$$

By the commutativity of the diagram (2-12), we get

$$\mathrm{Pl}_w(\pi_w(\widetilde{E}(\lambda^i))) = \pi'_c(P(\widetilde{E}(\lambda^i))) \quad \text{and} \quad P(\widetilde{E}(\lambda^i)) = (\pi')^{-1}(\mathrm{Pl}(E(\lambda^i))).$$

The map π' is a principal $\mathbb{C}^*$–bundle, and $E(\lambda^i)$ is contractible. So there is a bundle isomorphism

$$\phi_i : P(\widetilde{E}(\lambda^i)) \to E(\lambda^i) \times \mathbb{C}^*.$$

Indeed, this map can be defined by $\phi_i(P(A)) = (\pi(A), \det(A_{\lambda^i}))$. The inverse map is defined by $(\pi(A), s) \mapsto (s(\det(A_{\lambda^i}))^{-1} P(A))$.

Let $\pi(A) \in \mathrm{Gr}(d,n)$ for some $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)^{\mathrm{tr}} \in M_d(n,d)$ and $t \in \mathbb{C}^*$. There is an action of $\mathbb{C}^*$ on $\mathrm{Gr}(d,n)$ defined by

$$t \cdot \pi(A) = t \cdot \pi((\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)^{\mathrm{tr}}) := \pi((t^{w_1}\boldsymbol{a}_1, t^{w_2}\boldsymbol{a}_2, \ldots, t^{w_n}\boldsymbol{a}_n)^{\mathrm{tr}}). \tag{2-16}$$

If $\pi(A) = \pi(B)$, then $A = BT$ if and only if $DA = DBT$ for a diagonal matrix D and $T \in \mathrm{GL}(d,\mathbb{C})$. Thus $t \cdot \pi(A) = t \cdot \pi(B)$. Then ϕ_i becomes $\mathbb{C}^*$–equivariant with the following weighted $\mathbb{C}^*$–action on $E(\lambda^i) \times \mathbb{C}^*$ given by

$$t \cdot (\pi(A), s) = (t \cdot \pi(A), t^{c_i} s),$$

where $t \cdot \pi(A)$ is defined in (2-16) and c_i is defined in (2-5). Thus

$$\pi'_c(P(\widetilde{E}(\lambda^i))) = \frac{P(\widetilde{E}(\lambda^i))}{\text{weighted } \mathbb{C}^*\text{–action}} \cong \frac{E(\lambda^i) \times \mathbb{C}^*}{\text{weighted } \mathbb{C}^*\text{–action}} \cong \frac{E(\lambda^i)}{G(c_i)},$$

where the last identification follows from Lemma 2.6.

Now $E(\lambda^i)/G(c_i)$ is a q–cell of dimension $\ell(\lambda^i)$ as $E(\lambda^i)$ is an open cell of dimension $\ell(\lambda^i)$ and $|G(c_i)| < \infty$.

Let $C(i) := \{[z_0 : z_1 : \cdots : z_{i-1} : 1 : 0 : \cdots : 0] \in \mathbb{W}P(c_0, c_1, \ldots, c_m)\}$.

Consider

$$S^{2i-1} = \left\{ (z_0, z_1, \ldots, z_{i-1}, 0, \ldots, 0) \in \mathbb{C}^{m+1} \;\middle|\; \sum_{j=0}^{i-1} |z_j|^2 = 1 \right\}$$

and the $G(c_i)$–action on S^{2i-1} by $g(z_0, \ldots, z_{i-1}, 0, \ldots, 0) \mapsto (g^{c_0} z_0, \ldots, g^{c_{i-1}} z_{i-1}, 0, \ldots, 0)$. The orbit space is called an orbifold Lens space and denoted by $L(c_i; c')$, where $c' = (c_0, \ldots, c_{i-1})$. Then $C(i) = \mathbb{C}^i / G(c_i)$ is homeomorphic to the cone $C(L(c_i; c'))$ on $L(c_i; c')$. The space $\mathbb{W}P(c_0, \ldots, c_{i-1})$ can be obtained by the weighted S^1–action on S^{2i-1} with the weight vector c'. Thus there is a map

$$\phi_i : \frac{S^{2i-1}}{G(c_i)} = L(c_i; c') \to \frac{S^{2i-1}}{\text{weighted } S^1\text{–action}},$$

which plays the role of the attaching map for the q–cell $C(i)$; see [19].

Note that the set $E(\lambda^i) \cong \{[z_0 : \cdots : z_{i-1} : 1 : 0 : \cdots : 0] \in \mathrm{Pl}(\mathrm{Gr}(d,n))\} \subset \mathbb{C}P^m$. Then $E(\lambda^i) \cong \mathbb{C}^{\ell(\lambda^i)}$ can be considered as a $G(c_i)$–invariant subset of $\mathbb{C}^i$ as $\ell(\lambda^i) < i$. So $S^{2i-1} \cap E(\lambda^i)$ is a $G(c_i)$–invariant sphere of real dimension $2\ell(\lambda^i) - 1$. Thus, we have

$$S\left(\frac{E(\lambda^i)}{G(c_i)}\right) := \frac{S^{2i-1} \cap E(\lambda^i)}{G(c_i)} \hookrightarrow \frac{E(\lambda^i)}{G(c_i)} \hookrightarrow \frac{\mathbb{C}^i}{G(c_i)} = C(i).$$

Therefore, the attaching map for the q–cell $E(\lambda^i)/G(c_i)$ is the restriction on $S(E(\lambda^i)/G(c_i))$ and the following diagram commutes:

$$\begin{array}{ccc} S\left(\dfrac{E(\lambda^i)}{G(c_i)}\right) & \xrightarrow{\psi_i} & \{[(z_0 : \cdots : z_{i-1} : 0 : \cdots : 0)] \in \mathrm{Pl}_w(\mathrm{WGr}(d,n))\} \\ \big\downarrow{\scriptstyle \mathrm{Pl}_w} & & \big\downarrow \\ L(c_i, c') & \xrightarrow{\phi_i} & \mathbb{W}P(c_0, c_1, \ldots, c_{i-1}) \end{array}$$

Therefore, a q–CW complex structure on $\mathrm{WGr}(d,n)$ is given by

$$\mathrm{Pl}_w(\mathrm{WGr}(d,n)) = \frac{E(\lambda^0)}{G(c_0)} \sqcup \frac{E(\lambda^1)}{G(c_1)} \sqcup \frac{E(\lambda^2)}{G(c_2)} \sqcup \cdots \sqcup \frac{E(\lambda^m)}{G(c_m)}. \qquad \square$$

For each $k \in \{0, 1, 2, \ldots, m\}$, let

$$X_k := \bigsqcup_{i=0}^{k} \frac{E(\lambda^i)}{G(c_i)} \subset \mathrm{WGr}(d,n).$$

Here X_k is built inductively by attaching the q–cells $E(\lambda^0)/G(c_0), \ldots, E(\lambda^k)/G(c_k)$ so that X_k remains a subset of $\mathrm{WGr}(d,n)$. Then each X_k is a $(\mathbb{C}^*)^n$–invariant and we have the following filtration of $\mathrm{WGr}(d,n)$:

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = \mathrm{WGr}(d,n). \tag{2-17}$$

We note that the paper [1] discussed a q–CW complex structure of $\mathrm{WGr}(d,n)$. However, our approach is different and helps to study torsions in the integral cohomology of $\mathrm{WGr}(d,n)$.

3 Integral cohomology of certain weighted Grassmann orbifolds

In this section, we study several q–CW complex structures on a weighted Grassmann orbifold. We show how a permutation on the weight vector affects the weighted Grassmann orbifold. We define admissible permutation $\sigma \in S_n$ for a prime p and $\mathrm{WGr}(d,n)$. Then we discuss when $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ has no p–torsion. We introduce the concept of divisive weighted Grassmann orbifolds, which incorporates the divisive weighted projective spaces of [15]. We show that a divisive weighted Grassmann orbifold has a $(\mathbb{C}^*)^n$–invariant CW complex structure. We describe this action on each cell explicitly. As a consequence, we get that the integral cohomology of a divisive weighted Grassmann orbifold has no torsion and is concentrated in even degrees.

The following lemma is well known, but for our purpose we may need its proof.

Lemma 3.1 *The map $\pi'_c\colon \mathbb{C}^{m+1} - \{0\} \to \mathbb{W}P(c_0, c_1, \dots, c_m)$ induces an equivariant homeomorphism $\mathbb{W}P(rc_0, rc_1, \dots, rc_m) \to \mathbb{W}P(c_0, c_1, \dots, c_m)$ for any positive integer r.*

Proof The weighted $\mathbb{C}^*$–action on $\mathbb{C}^{m+1} \setminus \{0\}$ for $\mathbb{W}P(rc_0, rc_1, \dots, rc_m)$ is given by

$$t(z_0, z_1, \dots, z_m) = (t^{rc_0} z_0, t^{rc_1} z_1, \dots, t^{rc_m} z_m).$$

We denote the equivalence class by $[z_0 : z_1 : \dots : z_m]_{\sim_{rc}}$.

One can define a map $f\colon \mathbb{W}P(rc_0, rc_1, \dots, rc_m) \to \mathbb{W}P(c_0, \dots, c_m)$ by

$$f([z_0 : z_1 : \dots : z_m]_{\sim_{rc}}) = [z_0 : z_1 : \dots : z_m]_{\sim_c}$$

and a map $g\colon \mathbb{W}P(c_0, c_1, \dots, c_m) \to \mathbb{W}P(rc_0, rc_1, \dots, rc_m)$ by

$$g([z_0 : z_1 : \dots : z_m]_{\sim_c}) = [z_0 : z_1 : \dots : z_m]_{\sim_{rc}}.$$

Thus the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}^{m+1} \setminus \{0\} & \xrightarrow{\ \mathrm{Id}\ } & \mathbb{C}^{m+1} \setminus \{0\} \\ {\scriptstyle \pi'_{rc}}\downarrow & & \downarrow{\scriptstyle \pi'_c} \\ \mathbb{W}P(rc_0, \dots, rc_m) & \underset{f}{\overset{g}{\rightleftarrows}} & \mathbb{W}P(c_0, \dots, c_m) \end{array}$$

Observe that, we have $f \circ g = \mathrm{Id}_{\mathbb{W}P(c_0,\dots,c_m)}$ and $g \circ f = \mathrm{Id}_{\mathbb{W}P(rc_0,\dots,rc_m)}$. Thus f is a bijective map with the inverse map g.

Let U be an open subset of $\mathbb{W}P(c_0, \dots, c_m)$ Then $(\pi'_c)^{-1}(U) = (\pi'_{rc})^{-1} \circ f^{-1}(U)$. Since π'_c is a quotient map then $(\pi'_c)^{-1}(U)$ is an open subset of $\mathbb{C}^{m+1} \setminus \{0\}$. Thus $f^{-1}(U)$ is an open subset of $\mathbb{W}P(rc_0, \dots, rc_m)$ as π'_{rc} is a quotient map. Thus f is continuous. By similar arguments, we can show that g is continuous. Hence f is a homeomorphism and also it is equivariant with respect to the $(\mathbb{C}^*)^n$–action on $\mathbb{W}P(c_0, \dots, c_m)$ and $\mathbb{W}P(rc_0, \dots, rc_m)$ defined after (2-8). □

Lemma 3.2 *Let B be a subset of $\mathbb{C}^{m+1} \setminus \{0\}$. Let $B'_c := \pi'_c(B)$ and $B'_{rc} := \pi'_{rc}(B)$. Then the map $f|_{B'_{rc}}\colon B'_{rc} \to B'_c$ is a homeomorphism.*

Proof Consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\ \mathrm{Id}\ } & B \\ \downarrow{\scriptstyle \pi'_{rc}} & & \downarrow{\scriptstyle \pi'_c} \\ B'_{rc} & \xrightarrow{\ f|_{B'_{rc}}\ } & B'_c \end{array}$$

The map f is well defined and one-to-one. It follows that $f|_{B'_{rc}}$ is also well defined and one-to-one. Note that $f|_{B'_{rc}}$ is defined by $f|_{B'_{rc}}(\pi'_{rc}(b)) = \pi'_c(b)$. Therefore, $\pi'_{rc}(b) \in B'_{rc}$ is the inverse image of an element $\pi'_c(b) \in B'_c$. So $f|_{B'_{rc}}$ is bijective. Also $(f|_{B'_{rc}})^{-1} = g|_{B'_c}$. To conclude that $f|_{B'_{rc}}$ is a homomorphism, recall that the restriction of a continuous map is also continuous. □

We apply the previous result onto some subsets of $P(M_d(n,d)) \subseteq \mathbb{C}^{m+1} \setminus \{0\}$ for $m+1 = \binom{n}{d}$, where P is defined in (2-9). For all $k \in \{0, 1, \ldots, m\}$, consider $\widetilde{X}_k \subset M_d(n,d)$ defined by

$$\widetilde{X}_k := \{A \in M_d(n,d) \mid \det(A_{\lambda^j}) = 0 \text{ for } j > k\}.$$

Then $\widetilde{X}_k = \bigsqcup_{i=0}^k \widetilde{E}(\lambda^i) \subset M_d(n,d)$, where $\widetilde{E}(\lambda^i) = \pi^{-1}(E(\lambda^i)$ and

$$P(\widetilde{X}_k) = \bigsqcup_{i=0}^{k} P(\widetilde{E}(\lambda^i)) \subseteq P(M_d(n,d)).$$

Note that $P(\widetilde{X}_k) \subseteq \mathbb{C}^{k+1} \setminus \{0\} \subseteq \mathbb{C}^{m+1} \setminus \{0\}$ for $k \in \{0, 1, \ldots, m\}$.

One can calculate c_i for all $i \in \{0, 1, \ldots, m\}$ from (2-5) for a weighted Grassmann orbifold $\mathrm{WGr}(d,n)$. Let $r_k := \gcd\{c_0, c_1, \ldots, c_k\}$ for all $k \in \{1, 2, \ldots m\}$ and $G(r_k)$ be the group of r_k^{th} roots of unity. Since $G(c_i)$ is cyclic, let $G(c_i/r_k)$ be the unique cyclic subgroup of $G(c_i)$ of order c_i/r_k for $i \in \{0, 1, 2, \ldots, k\}$. Also $G(r_k)$ is a subgroup of $G(c_i)$ and $G(c_i)/G(r_k)$ is isomorphic to $G(c_i/r_k)$ for $i \in \{0, 1, 2, \ldots, k\}$. Now $G(c_k)$ acts on $E(\lambda^k)$ as a restriction of the weighted $\mathbb{C}^*$–action. Then we have a restricted $G(c_k/r_k)$–action on $E(\lambda^k)$.

Lemma 3.3 *The space $\pi'_c(P(\widetilde{X}_k))$ is homeomorphic to $\pi'_{c/r_k}(P(\widetilde{X}_k))$. Moreover, $E(\lambda^k)/G(c_k)$ is homeomorphic to $E(\lambda^k)/G(c_k/r_k)$.*

Proof The diagram

$$\begin{array}{ccc} P(\widetilde{X}_k) & \xrightarrow{\quad \mathrm{Id} \quad} & P(\widetilde{X}_k) \\ \big\downarrow \pi'_c & & \big\downarrow \pi'_{c/r_k} \\ \pi'_c(P(\widetilde{X}_k)) & \xrightarrow{f|_{\pi'_c(P(\widetilde{X}_k))}} & \pi'_{c/r_k}(P(\widetilde{X}_k)) \end{array}$$

is commutative. By Lemma 3.2, the lower horizontal map is a homeomorphism. The second statement of the lemma follows by similar arguments with $P(\widetilde{X}_k)$ is replaced by $P(\widetilde{E}(\lambda^k))$. □

Theorem 3.4 *The collection $\{E(\lambda^i)/G(c_i/r_k)\}_{i=0}^k$ gives a q–CW complex structure of $\pi'_{c/r_k}(P(\widetilde{X}_k))$ for $k = 1, 2, \ldots, m$. Moreover, $\{E(\lambda^i)/G(c_i/r_i)\}_{i=0}^m$ gives a q–CW complex structure of $\mathrm{WGr}(d,n)$, where $r_0 = c_0$.*

Proof Note that the sets $P(\widetilde{E}(\lambda^i))$ and $P(M_d(n,d)) = \bigsqcup_{i=0}^m P(\widetilde{E}(\lambda^i))$ are invariant under the weighted $\mathbb{C}^*$–action defined in Remark 2.3 for all $i = 0, 1, \ldots, m$. Then we have the commutative diagram

$$\begin{array}{ccc} P(\widetilde{X}_k) & \subset & \mathbb{C}^{k+1} \setminus \{0\} \\ \big\downarrow \pi'_{c/r_k} & & \big\downarrow \pi'_{c/r_k} \\ \pi'_{c/r_k}(P(\widetilde{X}_k)) & \subset & \mathbb{W}P\left(\dfrac{c_0}{r_k}, \dfrac{c_1}{r_k}, \ldots, \dfrac{c_k}{r_k}\right) \end{array}$$

Thus the first part follows from

$$\pi'_{c/r_k}(P(\widetilde{X}_k)) = \pi'_{c/r_k}\left(\bigsqcup_{i=0}^{k} P(\widetilde{E}(\lambda^i))\right) = \bigsqcup_{i=0}^{k} \pi'_{c/r_k}(P(\widetilde{E}(\lambda^i))) = \bigsqcup_{i=0}^{k} \frac{P(\widetilde{E}(\lambda^i))}{\sim_{c/r_k}} \cong \bigsqcup_{i=0}^{k} \frac{E(\lambda^i)}{G(c_i/r_k)}.$$

The second part follows from $\mathrm{WGr}(d,n) \cong \pi'_c(P(\widetilde{X}_m))$ and by applying Lemma 3.3 successively for every $k \in \{1, 2, \dots, m\}$. □

We show that two weighted Grassmann orbifolds are weakly equivariantly homeomorphic if the associated weight vectors differ by a permutation $\sigma \in S_n$. Let X and Y be two G–spaces. A map $f : X \to Y$ is called a weakly equivariant homeomorphism if f is a homeomorphism and $f(gx) = \eta(g) f(x)$ for some $\eta \in \mathrm{Aut}(G)$ and for all $(g, x) \in G \times X$. If η is the identity, then f is called an equivariant homeomorphism.

Let $W = (w_1, w_2, \dots, w_n) \in (\mathbb{Z}_{\geq 0})^n$, $0 < a \in \mathbb{Z}$ and $\sigma W := (w_{\sigma_1}, w_{\sigma_2}, \dots, w_{\sigma_n})$ for some $\sigma \in S_n$. Consider two weighted Grassmann orbifolds $\mathrm{WGr}(d,n)$ and $\mathrm{WGr}'(d,n)$ associated to (W, a) and $(\sigma W, a)$, respectively. The group $(\mathbb{C}^*)^n$ acts on $\mathrm{WGr}(d,n)$ described in (2-7). Also, there exists a $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}'(d,n)$ defined by

$$(t_1, \dots, t_n)[(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)^{\mathrm{tr}}] := [(t_{\sigma_1}\boldsymbol{a}_1, t_{\sigma_2}\boldsymbol{a}_2, \dots, t_{\sigma_n}\boldsymbol{a}_n)^{\mathrm{tr}}]. \tag{3-1}$$

Theorem 3.5 *There exists a weakly equivariantly homeomorphism between* $\mathrm{WGr}(d,n)$ *and* $\mathrm{WGr}'(d,n)$*. Moreover, this may induce different q–CW complex structures on* $\mathrm{WGr}(d,n)$ *for different* σ*.*

Proof The matrix $A = (a_{ij}) \in M_d(n,d)$ if and only if $\sigma A = (a_{\sigma_i j}) \in M_d(n,d)$. Thus the natural weakly equivariant homeomorphism $\bar{f}_\sigma : M_d(n,d) \to M_d(n,d)$ defined by $\bar{f}_\sigma(A) = \sigma A$ induces the commutative diagram

$$\begin{array}{ccc} M_d(n,d) & \xrightarrow{\bar{f}_\sigma} & M_d(n,d) \\ \downarrow{\scriptstyle \pi_w} & & \downarrow{\scriptstyle \pi_{\sigma w}} \\ \mathrm{WGr}(d,n) & \xrightarrow{f_\sigma} & \mathrm{WGr}'(d,n) \end{array} \tag{3-2}$$

Here π_w is the quotient map defined in Definition 2.1. Thus, (3-2) induces a weakly equivariant homeomorphism $f_\sigma : \mathrm{WGr}(d,n) \to \mathrm{WGr}'(d,n)$, where $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}(d,n)$ is defined in (2-7) and the $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}'(d,n)$ is defined in (3-1). Note that $f_\sigma([A]_{\sim w}) = [\sigma A]_{\sim \sigma w}$.

We discuss the effects of the permutation σ on the q–CW complex structure on $\mathrm{WGr}(d,n)$. Consider $\mathbb{C}^i = \{(x_1, x_2, \dots, x_n) \in \mathbb{C}^n \mid x_j = 0 \text{ for } j > i\}$. For $\sigma \in S_n$, define

$$\sigma\mathbb{C}^n := \{(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n})\} \quad \text{and} \quad \sigma\mathbb{C}^i := \{(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n}) \in \sigma\mathbb{C}^n \mid x_{\sigma_j} = 0 \text{ for } \sigma_j > i\}.$$

Let $\lambda = (\lambda_1, \dots, \lambda_d)$ be a Schubert symbol for $d < n$. Then

$$\begin{aligned} \sigma E(\lambda) &= \{\sigma Y \mid Y \in E(\lambda)\} \\ &= \{X \in \mathrm{Gr}(d,n) \mid \dim(X \cap \sigma\mathbb{C}^{\lambda_i}) = i \text{ and } \dim(X \cap \sigma\mathbb{C}^{\lambda_i - 1}) = i - 1 \text{ for } i \in [d]\}, \end{aligned}$$

where $[d] = \{1, 2, \dots, d\}$. Then $E(\lambda) \cong \sigma E(\lambda)$ and $\dim(\sigma E(\lambda)) = \ell(\lambda)$.

So the permutation of the coordinates in $\mathbb{C}^n$ determines another CW complex structure for $\mathrm{Gr}(d,n)$ given by $\mathrm{Gr}(d,n) = \sigma\,\mathrm{Gr}(d,n) = \bigsqcup_{i=0}^m \sigma E(\lambda^i)$. This induces the following decomposition of $M_d(n,d)$, similar to (2-15):

$$M_d(n,d) = \bigsqcup_{i=0}^{m} \sigma \widetilde{E}(\lambda^i) \quad \text{and} \quad P(M_d(n,d)) = \bigsqcup_{i=0}^{m} P(\sigma \widetilde{E}(\lambda^i)).$$

Recall that $\lambda^i = (\lambda^i_1, \dots, \lambda^i_d)$ is a Schubert symbol and c_i is defined in (2-5) for $i = 0, \dots, m$. Then $\sigma\lambda^i := (\sigma(\lambda^i_{i_1}), \dots, \sigma(\lambda^i_{i_d}))$, where $i_1, \dots, i_d \in \{1, \dots, d\}$ and $\sigma(\lambda^i_{i_1}) < \sigma(\lambda^i_{i_2}) < \cdots < \sigma(\lambda^i_{i_d})$. Let

$$\sigma c_i := a + \sum_{j=1}^{d} w_{\sigma(\lambda^i_{i_j})}. \tag{3-3}$$

Now from the commutativity of the diagram (2-12), we have

$$\pi_w(\sigma(\widetilde{E}(\lambda^i))) \cong \mathrm{Pl}_w(\pi_w(\sigma \widetilde{E}(\lambda^i))) = \frac{P(\sigma \widetilde{E}(\lambda^i))}{\text{weighted } \mathbb{C}^*\text{–action}}.$$

There exists a homeomorphism

$$P(\sigma \widetilde{E}(\lambda^i) \cong \sigma E(\lambda^i) \times \mathbb{C}^*$$

defined by $P(\sigma A) \to (\pi(\sigma A), \det(A_{\sigma\lambda^i}))$. This is a $\mathbb{C}^*$–equivariant homomorphism, where the weighted $\mathbb{C}^*$–action on the left side is same as the weighted $\mathbb{C}^*$–action on $\mathbb{C}^{m+1} \setminus \{0\}$, and the weighted $\mathbb{C}^*$–action on the right side is defined by

$$t \cdot (\pi(\sigma A), s) = (t \cdot \pi(\sigma A), t^{\sigma c_i} s),$$

where $t \cdot \pi(\sigma A)$ is defined in (2-16). Then using Lemma 2.6, we have

$$\frac{P(\sigma \widetilde{E}(\lambda^i))}{\text{weighted } \mathbb{C}^*\text{–action}} \cong \frac{\sigma E(\lambda^i)}{G(\sigma c_i)}.$$

Then we get a q–CW complex structure of the weighted Grassmann orbifold $\mathrm{WGr}(d,n)$ given by

$$\mathrm{WGr}(d,n) \cong \frac{\sigma E(\lambda^0)}{G(\sigma c_0)} \sqcup \frac{\sigma E(\lambda^1)}{G(\sigma c_1)} \sqcup \cdots \sqcup \frac{\sigma E(\lambda^m)}{G(\sigma c_m)}. \qquad \square$$

Remark 3.6 Applying the permutation σ on the rows of the matrices in $E(\lambda)$, we get the matrices of $\sigma E(\lambda)$. That is,

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in E(\lambda) \iff \begin{pmatrix} v_{\sigma_1} \\ v_{\sigma_2} \\ \vdots \\ v_{\sigma_n} \end{pmatrix} \in \sigma E(\lambda).$$

Proposition 3.7 [4, Theorem 1.1] *Let X be a q–CW complex with no odd-dimensional q–cells, and p a prime number. Let $\{\mathrm{pt}\} = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_s = X$ be a filtration of X such that X_i is obtained by attaching the q–cell $\mathbb{R}^{2k_i}/G_i$ to X_{i-1} for all $i \in \{1, 2, \dots, s\}$. If $\gcd\{p, |G_i|\} = 1$ for all $i \in \{1, 2, \dots, s\}$, then $H^*(X;\mathbb{Z})$ has no p–torsion and $H^{\mathrm{odd}}(X;\mathbb{Z}_p)$ is trivial.*

Recall σc_i as defined in (3-3) for $\mathrm{WGr}(d,n)$ associated to weight vector $W=(w_1,\dots,w_n)\in(\mathbb{Z}_{\geq 0})^n$ and $1\leq a\in\mathbb{Z}$.

Definition 3.8 A permutation $\sigma\in S_n$ is called admissible for a prime p and $\mathrm{WGr}(d,n)$ if

$$\gcd\left\{p,\frac{\sigma c_i}{d_i}\right\}=1,$$

where σc_i is defined in (3-3) and $d_i=\gcd\{\sigma c_0,\sigma c_1,\dots,\sigma c_i\}$ for $i\in\{1,2,\dots,m\}$.

Some examples of admissible permutations are discussed in Example 3.12.

Remark 3.9 There may not always exist an admissible permutation $\sigma\in S_n$ for a prime p and $\mathrm{WGr}(d,n)$. However if $d=1$, then $m=n-1$ and there always exists an admissible permutation $\sigma\in S_n$ for every prime p. The admissible permutation $\sigma\in S_n$ may not be unique.

The following result says when the integral cohomology of $\mathrm{WGr}(d,n)$ has no p–torsion.

Theorem 3.10 *If there exists an admissible permutation $\sigma\in S_n$ for a prime p and $\mathrm{WGr}(d,n)$, then $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ has no p–torsion and $H^{\mathrm{odd}}(\mathrm{WGr}(d,n);\mathbb{Z}_p)$ is trivial.*

Proof Suppose $\sigma\in S_n$ be an admissible permutation for p and $\mathrm{WGr}(d,n)$. Then

$$\gcd\left\{p,\frac{\sigma c_i}{d_i}\right\}=1$$

by Definition 3.8, where $d_i=\gcd\{\sigma c_0,\sigma c_1,\dots,\sigma c_i\}$ for all $i\in\{1,2,\dots,m\}$. By Theorem 3.5, we have the q–CW complex structure

$$\mathrm{WGr}(d,n)\cong\frac{\sigma E(\lambda^0)}{G(\sigma c_0)}\sqcup\frac{\sigma E(\lambda^1)}{G(\sigma c_1)}\sqcup\cdots\sqcup\frac{\sigma E(\lambda^m)}{G(\sigma c_m)},$$

where $\sigma E(\lambda^i)\cong E(\lambda^i)\cong\mathbb{C}^{\ell(\lambda^i)}$. Let

$$\sigma X_k=\bigsqcup_{i=0}^{k}\frac{\sigma E(\lambda^i)}{G(\sigma c_i)}\subseteq\mathrm{WGr}(d,n)\quad\text{for } k=0,1,\dots,m.$$

Then σX_k is a subcomplex of $\mathrm{WGr}(d,n)$ for $k=0,1,\dots,m$ and $\sigma X_m=\mathrm{WGr}(d,n)$. This gives a filtration

$$\{\mathrm{pt}\}=\sigma X_0\subset\sigma X_1\subset\cdots\subset\sigma X_m=\mathrm{WGr}(d,n)$$

such that $\sigma X_i\setminus\sigma X_{i-1}$ is homeomorphic to $\sigma E(\lambda^i)/G(\sigma c_i)$.

Using Lemma 3.3,

$$\frac{\sigma E(\lambda^i)}{G(\sigma c_i)}\cong\frac{\sigma E(\lambda^i)}{G(\sigma c_i/d_i)}.$$

That is, $\sigma X_i\setminus\sigma X_{i-1}$ is homeomorphic to $\mathbb{C}^{\ell(\lambda^i)}/G(\sigma c_i/d_i)$ for all $i=1,2,\dots,m$. Therefore, by Proposition 3.7, $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ has no p–torsion and the group $H^{\mathrm{odd}}(\mathrm{WGr}(d,n);\mathbb{Z}_p)$ is trivial. This completes the proof. □

Corollary 3.11 [19] *$H^*(\mathbb{W}P(c_0, c_1, \dots, c_m); \mathbb{Z})$ has no torsion.*

Proof This follows from Theorem 3.10 and Remarks 2.3 and 3.9. □

Example 3.12 Consider the weighted Grassmann orbifold $\mathrm{WGr}(2,4)$ for weight vector $W = (1,1,3,4)$ and $a = 2$. Here

$$n = 4, \quad d = 2, \quad \binom{n}{d} = 6, \quad m = \binom{n}{d} - 1 = 5.$$

So, in this case, we have six Schubert symbols, which are

$$\lambda^0 = (1,2) < \lambda^1 = (1,3) < \lambda^2 = (1,4) < \lambda^3 = (2,3) < \lambda^4 = (2,4) < \lambda^5 = (3,4),$$

ordered as in Definition 2.4. For the prime $p = 3$, consider the permutation $\sigma \in S_4$ defined by

$$\sigma_1 = 3, \quad \sigma_2 = 4, \quad \sigma_3 = 1, \quad \sigma_4 = 2.$$

Then

$$\sigma c_0 = 9, \quad \sigma c_1 = 6, \quad \sigma c_2 = 6, \quad \sigma c_3 = 7, \quad \sigma c_4 = 7, \quad \sigma c_5 = 4,$$

using (3-3). This σ is admissible for $p = 3$ and $\mathrm{WGr}(2,4)$. Thus $H^*(\mathrm{WGr}(2,4); \mathbb{Z})$ has no 3–torsion by Theorem 3.10.

For the prime $p = 7$, consider the permutation $\sigma \in S_4$ defined by

$$\sigma_1 = 4, \quad \sigma_2 = 2, \quad \sigma_3 = 1, \quad \sigma_4 = 3.$$

Then

$$\sigma c_0 = 7, \quad \sigma c_1 = 7, \quad \sigma c_2 = 9, \quad \sigma c_3 = 4, \quad \sigma c_4 = 6, \quad \sigma c_5 = 6,$$

using (3-3). This σ is admissible for $p = 7$ and $\mathrm{WGr}(2,4)$. Thus $H^*(\mathrm{WGr}(2,4); \mathbb{Z})$ has no 7–torsion by Theorem 3.10.

To compute that it has no 2–torsion, we need to consider a different total order on the Schubert symbols, given by

$$\lambda^0 = (1,2) < \lambda^1 = (1,3) < \lambda^2 = (2,3) < \lambda^3 = (1,4) < \lambda^4 = (2,4) < \lambda^5 = (3,4),$$

which preserves the partial order in (2-3). In this case, using (2-5),

$$c_0 = 4, \quad c_1 = 6, \quad c_2 = 6, \quad c_3 = 7, \quad c_4 = 7, \quad c_5 = 9.$$

The identity permutation in S_4 is admissible for $p = 2$ and this $\mathrm{WGr}(2,4)$. Then $H^*(\mathrm{WGr}(2,4); \mathbb{Z})$ has no 2–torsion by Theorem 3.10.

The only primes which divide the orders of the orbifold singularities of this $\mathrm{WGr}(2,4)$ are 2, 3 and 7. Hence the integral cohomology of $\mathrm{WGr}(2,4)$ of this example has no torsion. □

Remark 3.13 Considering the total order given in Definition 2.4 on the Schubert symbols, there may not exist an admissible permutation σ for a prime. However, one can take another total order on the Schubert symbols for which one can find σ satisfying the hypothesis in Theorem 3.10 for this prime.

The q–CW complex structure in Theorem 3.4 leads us to introduce the following definition, which generalizes the concept of divisive weighted projective spaces of [15].

Definition 3.14 A weighted Grassmann orbifold $\mathrm{WGr}(d,n)$ is called divisive if there exists $\sigma \in S_n$ such that σc_i divides σc_{i-1} for $i = 1, 2, \dots, m$, where σc_i is defined in (3-3).

Example 3.15 Consider the weighted Grassmann orbifold $\mathrm{WGr}(2,4)$ for weight vector $W = (1,6,1,1)$ and $a = 3$. We have the ordering on the six Schubert symbols given by

$$\lambda^0 = (1,2) < \lambda^1 = (1,3) < \lambda^2 = (1,4) < \lambda^3 = (2,3) < \lambda^4 = (2,4) < \lambda^5 = (3,4).$$

Consider the permutation $\sigma \in S_4$ defined by

$$\sigma_1 = 2, \quad \sigma_2 = 1, \quad \sigma_3 = 3, \quad \sigma_4 = 4.$$

Then

$$\sigma c_0 = 10, \quad \sigma c_1 = 10, \quad \sigma c_2 = 10, \quad \sigma c_3 = 5, \quad \sigma c_4 = 5, \quad \sigma c_5 = 5,$$

using (3-3). Thus σc_i divides σc_{i-1} for $i = 1, 2, \dots, 5$. So $\mathrm{WGr}(2,4)$ of this example is divisive. □

Example 3.16 Let α and γ be any two nonnegative integers and β be any positive integer such that $\beta > d\alpha$. Let $\mathrm{WGr}(d,n)$ be the corresponding weighted Grassmann orbifold for $W = (\alpha + \gamma\beta, \alpha, \dots, \alpha) \in (\mathbb{Z}_{\geq 0})^n$ and $a = \beta - d\alpha > 0$. Consider the total order $\{\lambda^0, \lambda^1, \dots, \lambda^m\}$ on the Schubert symbols induced by the dictionary order. Then

$$c_i = \begin{cases} (\gamma+1)\beta & \text{if } i = 0, 1, \dots, \binom{n-1}{d-1} - 1, \\ \beta & \text{if } i = \binom{n-1}{d-1}, \dots, m. \end{cases}$$

Then c_i divides c_{i-1} for all $i = 1, 2, \dots, m$. Therefore this $\mathrm{WGr}(d,n)$ is a divisive weighted Grassmann orbifold. □

Definition 3.17 Let λ be a Schubert symbol for $d < n$. Then a reversal of λ is a pair (k,k') such that $k \in \lambda$, $k' \notin \lambda$ and $k' < k$. We denote the set of all reversals of λ by $\mathrm{rev}(\lambda)$. If $(k,k') \in \mathrm{rev}(\lambda)$ then $(k,k')\lambda$ is the Schubert symbol obtained by replacing k by k' in λ and ordering the later set.

Remark 3.18 If $(k,k') \in \mathrm{rev}(\lambda)$ then $(k,k')\lambda \prec \lambda$ and $\ell(\lambda)$ is the cardinality of the set $\mathrm{rev}(\lambda)$ where $\ell(\lambda)$ is the length of λ. Knutson and Tao [20] and Abe and Matsumura [1] defined an inversion of a Schubert symbol λ as a pair (k,k') such that $k \in \lambda$, $k' \notin \lambda$ and $k < k'$. In some sense, our definition of reversal is dual to the definition of inversion. If $\mathrm{inv}(\lambda)$ is the set of all inversions of λ and $\ell'(\lambda)$ is the cardinality of the set $\mathrm{inv}(\lambda)$, then $\ell(\lambda) + \ell'(\lambda) = d(n-d)$. Also, if $(k,k') \in \mathrm{rev}(\lambda)$ and $(k,k')\lambda = \mu$, then $(k',k) \in \mathrm{inv}(\mu)$ and $(k',k)\mu = \lambda$.

Next, we discuss $(\mathbb{C}^*)^n$–action on some CW complex structure of a divisive weighted Grassmann orbifold. Recall the $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}(d,n)$ which is induced from (2-7). We retain the notation from Section 2.

Theorem 3.19 *If* $\mathrm{WGr}(d,n)$ *is a divisive weighted Grassmann orbifold, then it has a* $(\mathbb{C}^*)^n$*–invariant CW complex structure with cells* $\{\mathbb{C}^{\ell(\lambda^i)} \mid i = 0,1,\dots,m\}$.

Proof Let $\mathrm{WGr}(d,n)$ be a divisive weighted Grassmann orbifold corresponding to weight vector $W = (w_1,\dots,w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $1 \leq a \in \mathbb{Z}$. Then there exists $\sigma \in S_n$ such that σc_i divides σc_{i-1} for all $i = 1,2,\dots,m$. Let us assume that $\sigma = \mathrm{Id}$ (the identity permutation in S_n). Then c_i divides c_{i-1} for all $i = 1,2,\dots,m$. Then $\gcd\{c_0,c_1,\dots,c_i\} = c_i$ for all $i \in \{1,2,\dots,m\}$. Thus,

$$\pi_w(\widetilde{E}(\lambda^i)) \cong \frac{E(\lambda^i)}{G(c_i)} \cong \frac{E(\lambda^i)}{G(c_i/c_i)} \cong E(\lambda^i) \quad \text{for all } i = 1,2,\dots,m,$$

by Lemma 3.3. Thus, each element of $\pi_w(\widetilde{E}(\lambda^i))$ can be represented uniquely by the equivalence class of an $n \times d$ matrix defined in (2-13).

Let $\lambda^i = (\lambda_1,\dots,\lambda_d)$ be a Schubert symbol for $d < n$ and let $\boldsymbol{z} \in \mathbb{C}^{\ell(\lambda^i)}$. Since

$$\ell(\lambda^i) = (\lambda_1 - 1) + (\lambda_2 - 2) + \cdots + (\lambda_d - d),$$

we can write $\boldsymbol{z} = (\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_d)$, where

$$\boldsymbol{z}_l = (z_1^l, z_2^l, \dots, \widehat{z_{\lambda_1}^l}, \dots, \widehat{z_{\lambda_2}^l}, \dots, \widehat{z_{\lambda_{l-1}}^l}, \dots, z_{\lambda_l - 1}^l) \quad \text{for } l = 1,\dots,d.$$

For $(t_1,\dots,t_n) \in (\mathbb{C}^*)^n$, we define $s \in \mathbb{C}^*$ such that $s^{c_i} = t_{\lambda_1} \cdots t_{\lambda_d}$. Define $T \in \mathrm{GL}(d,\mathbb{C})$ by

$$T = \mathrm{diag}\biggl(\biggl(\frac{t_{\lambda_1}}{s^{w_{\lambda_1}}}\biggr), \biggl(\frac{t_{\lambda_2}}{s^{w_{\lambda_2}}}\biggr), \dots, \biggl(\frac{t_{\lambda_d}}{s^{w_{\lambda_d}}}\biggr)\biggr).$$

Then $\det(T) = s^a$.

Define $g_{\lambda^i} : \mathbb{C}^{\ell(\lambda^i)} \to \pi_w(\widetilde{E}(\lambda^i))$ by

$$g_{\lambda^i}(\boldsymbol{z}) := \begin{bmatrix} z_1^1 & z_1^2 & \cdots & z_1^d \\ \vdots & \vdots & & \vdots \\ z_{\lambda_1-1}^1 & z_{\lambda_1-1}^2 & \cdots & z_{\lambda_1-1}^d \\ 1 & 0 & \cdots & 0 \\ 0 & z_{\lambda_1+1}^2 & \cdots & z_{\lambda_1+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & z_{\lambda_2-1}^2 & \cdots & z_{\lambda_2-1}^d \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & z_{\lambda_2+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & z_{\lambda_d-1}^d \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then g_{λ^i} is a homeomorphism. Now we have

$$(t_1,t_2,\dots,t_n)g_{\lambda^i}(z)=\begin{bmatrix} t_1z_1^1 & t_1z_1^2 & \dots & t_1z_1^d \\ \vdots & \vdots & & \vdots \\ t_{\lambda_1-1}z_{\lambda_1-1}^1 & t_{\lambda_1-1}z_{\lambda_1-1}^2 & \dots & t_{\lambda_1-1}z_{\lambda_1-1}^d \\ t_{\lambda_1} & 0 & \dots & 0 \\ 0 & t_{\lambda_1+1}z_{\lambda_1+1}^2 & \dots & t_{\lambda_1+1}z_{\lambda_1+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & t_{\lambda_2-1}z_{\lambda_2-1}^2 & \dots & t_{\lambda_2-1}z_{\lambda_2-1}^d \\ 0 & t_{\lambda_2} & \dots & 0 \\ 0 & 0 & \dots & t_{\lambda_2+1}z_{\lambda_2+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & t_{\lambda_d-1}z_{\lambda_d-1}^d \\ 0 & 0 & \dots & t_{\lambda_d} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then

$$(t_1,t_2,\dots,t_n)g_{\lambda^i}(z)=\begin{bmatrix} \frac{s^{w_{\lambda_1}}}{t_{\lambda_1}}t_1z_1^1 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}}t_1z_1^2 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_1z_1^d \\ \vdots & \vdots & & \vdots \\ \frac{s^{w_{\lambda_1}}}{t_{\lambda_1}}t_{\lambda_1-1}z_{\lambda_1-1}^1 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}}t_{\lambda_1-1}z_{\lambda_1-1}^2 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_1-1}z_{\lambda_1-1}^d \\ \frac{s^{w_{\lambda_1}}}{t_{\lambda_1}}t_{\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}}t_{\lambda_1+1}z_{\lambda_1+1}^2 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_1+1}z_{\lambda_1+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}}t_{\lambda_2-1}z_{\lambda_2-1}^2 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_2-1}z_{\lambda_2-1}^d \\ 0 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}}t_{\lambda_2} & \dots & 0 \\ 0 & 0 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_2+1}z_{\lambda_2+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_d-1}z_{\lambda_d-1}^d \\ 0 & 0 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_d} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}\times T.$$

Thus, $(t_1, t_2, \dots, t_n)g_{\lambda^i}(z)$ is equal to

$$
D \times \begin{bmatrix}
\frac{s^{w_{\lambda_1}}}{t_{\lambda_1} s^{w_1}} t_1 z_1^1 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2} s^{w_1}} t_1 z_1^2 & \cdots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d} s^{w_1}} t_1 z_1^d \\
\vdots & \vdots & & \vdots \\
\frac{s^{w_{\lambda_1}}}{s^{w_{\lambda_1-1}} t_{\lambda_1}} t_{\lambda_1-1} z^1_{\lambda_1-1} & \frac{s^{w_{\lambda_2}}}{s^{w_{\lambda_1-1}} t_{\lambda_2}} t_{\lambda_1-1} z^2_{\lambda_1-1} & \cdots & \frac{s^{w_{\lambda_d}}}{s^{w_{\lambda_1-1}} t_{\lambda_d}} t_{\lambda_1-1} z^d_{\lambda_1-1} \\
1 & 0 & \cdots & 0 \\
0 & \frac{s^{w_{\lambda_2}}}{s^{w_{\lambda_1+1}} t_{\lambda_2}} t_{\lambda_1+1} z^2_{\lambda_1+1} & \cdots & \frac{s^{w_{\lambda_d}}}{s^{w_{\lambda_1+1}} t_{\lambda_d}} t_{\lambda_1+1} z^d_{\lambda_1+1} \\
\vdots & \vdots & & \vdots \\
0 & \frac{s^{w_{\lambda_2}}}{s^{w_{\lambda_2-1}} t_{\lambda_2}} t_{\lambda_2-1} z^2_{\lambda_2-1} & \cdots & \frac{s^{w_{\lambda_d}}}{s^{w_{\lambda_2-1}} t_{\lambda_d}} t_{\lambda_2-1} z^d_{\lambda_2-1} \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & \frac{s^{w_{\lambda_d}}}{s^{w_{\lambda_2+1}} t_{\lambda_d}} t_{\lambda_2+1} z^d_{\lambda_2+1} \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \frac{s^{w_{\lambda_d}}}{s^{w_{\lambda_d-1}} t_{\lambda_d}} t_{\lambda_d-1} z^d_{\lambda_d-1} \\
0 & 0 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix} \times T = DMT,
$$

where $D = \operatorname{diag}(s^{w_1}, \dots, s^{w_n})$ is a diagonal matrix. So by the equivalence relation $\sim_w$ as in Definition 2.1,

$$(t_1, t_2, \dots, t_n)g_{\lambda^i}(z) = M \in \pi_w(\widetilde{E}(\lambda^i)) \subset \mathrm{WGr}(d, n).$$

Let a_{kl} be the coefficient of z_k^l in the matrix M for $1 \le l \le d$, $1 \le k \le \lambda_l - 1$, $k \ne \lambda_1, \lambda_2, \dots, \lambda_{l-1}$. Then

$$a_{kl} = \frac{s^{w_{\lambda_l}} t_k}{s^{w_k} t_{\lambda_l}}.$$

Now for $1 \le k \le \lambda_l - 1$ with $k \ne \lambda_1, \lambda_2, \dots, \lambda_{l-1}$ we have $(\lambda_l, k) \in \operatorname{rev}(\lambda^i)$. Let $\lambda^j = (\lambda_l, k)\lambda^i$. Note that $\lambda^j < \lambda^i$. Recall c_i from (2-5). So

$$\frac{t_k s^{w_{\lambda_l}}}{s^{w_k} t_{\lambda_l}} = \frac{t_{\lambda^j}}{t_{\lambda^i}} s^{w_{\lambda_l} - w_k} = \frac{t_{\lambda^j}}{t_{\lambda^i}} s^{c_i - c_j} = \frac{t_{\lambda^j}}{t_{\lambda^i}} t_{\lambda^i}^{(c_i - c_j)/c_i} = t_{\lambda^j}(t_{\lambda^i})^{-c_j/c_i},$$

since $s^{c_i} = t_{\lambda_1} \cdots t_{\lambda_d} = t_{\lambda^i}$ and $t_{\lambda^j} = t_{\lambda_1} \cdots t_{\lambda_{l-1}} t_k t_{\lambda_{l+1}} \cdots t_{\lambda_d}$. Since $\mathrm{WGr}(d, n)$ is divisive and $\lambda^j < \lambda^i$, we have that c_i divides c_j.

Define a $(\mathbb{C}^*)^n$–action on $\mathbb{C}^{\ell(\lambda^i)}$ by

$$(t_1, t_2, \dots, t_n)(z_k^l) = (t_{\lambda^j}(t_{\lambda^i})^{-c_j/c_i} z_k^l)$$

for $1 \le l \le d$, $1 \le k \le \lambda_l - 1$, $k \ne \lambda_1, \lambda_2, \dots, \lambda_{l-1}$. With this action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^{\ell(\lambda^i)}$, the map g_{λ^i} becomes $(\mathbb{C}^*)^n$–equivariant.

If $\sigma \neq \mathrm{Id}$, consider the cell

$$\pi_w(\sigma \widetilde{E}(\lambda^i)) \cong \frac{\sigma E(\lambda^i)}{G(\sigma c_i)} \cong \frac{\sigma E(\lambda^i)}{G(\sigma c_i/\sigma c_i)} \cong \sigma E(\lambda^i) \quad \text{for all } i = 1, 2, \dots, m,$$

by Lemma 3.3. Hence, we get the map $\sigma g_{\lambda^i} : \mathbb{C}^{\ell(\lambda^i)} \to \pi_w(\sigma \widetilde{E}(\lambda^i))$ defined by $z \to \sigma g_{\lambda^i}(z)$. Then by similar arguments, we get the $(\mathbb{C}^*)^n$–action on $\mathbb{C}^{\ell(\lambda^i)}$ defined by

$$(t_1, t_2, \dots, t_n)(z_k^l) = (t_{\sigma\lambda^j}(t_{\sigma\lambda^i})^{-\sigma c_j/\sigma c_i} z_k^l). \tag{3-4}$$

□

Corollary 3.20 *If* $\mathrm{WGr}(d,n)$ *is divisive, then* $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ *has no torsion and is concentrated in even degrees.*

We remark that Corollary 3.20 also follows from the proof of Theorem 3.10 and Definition 3.14. However, Theorem 3.19 describes the representation of the $(\mathbb{C}^*)^n$–action on each invariant cell explicitly. We also get that a divisive weighted Grassmann orbifold is integrally equivariantly formal.

4 Equivariant cohomology, cobordism and K–theory of weighted Grassmann orbifolds

In this section, first we compute the equivariant K–theory ring of any weighted Grassmann orbifold with rational coefficients. Then we compute the equivariant cohomology ring, equivariant K–theory ring and equivariant cobordism ring of a divisive weighted Grassmann orbifold with integer coefficients. We discuss the computation of the equivariant Euler classes for some line bundles on a point. We also compute the integral equivariant cohomology ring of some nondivisive weighted Grassmann orbifolds. We retain the notation of previous sections.

We recall the $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}(d,n)$ which is induced by (2-7). Consider the standard torus $T^n = (S^1)^n \subset (\mathbb{C}^*)^n$. So we have the restricted T^n–action on $\mathrm{WGr}(d,n)$. For each Schubert symbol $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$, consider $C(\lambda) \in M_d(n,d)$ with column vectors given by $e_{\lambda_1}, e_{\lambda_2}, \dots, e_{\lambda_d}$, where $\{e_1, e_2, \dots, e_n\}$ is the standard basis for $\mathbb{C}^n$. Therefore $[C(\lambda)] \in \mathrm{WGr}(d,n)$, and it is a fixed point of the T^n–action on $\mathrm{WGr}(d,n)$.

Proposition 4.1 *Let* $\mathrm{WGr}(d,n)$ *be a weighted Grassmann orbifold corresponding to weight vector* $W = (w_1, w_2, \dots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ *and* $a \geq 1$. *Then there is a* $(\mathbb{C}^*)^n$*–invariant stratification*

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = \mathrm{WGr}(d,n)$$

such that for $i = 1, \dots, m$, *the quotient* X_i/X_{i-1} *is homeomorphic to the Thom space* $\mathrm{Th}(\xi^i)$ *of an orbifold* $(\mathbb{C}^*)^n$*–vector bundle*

$$\xi^i : \mathbb{C}^{\ell(\lambda^i)}/G(c_i) \to [C(\lambda^i)], \tag{4-1}$$

where $G(c_i)$ *is the cyclic group of the* c_i^{th} *roots of unity.*

Proof Recall the $(\mathbb{C}^*)^n$–invariant stratification

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = \mathrm{WGr}(d,n)$$

from (2-17), which is obtained from the q–CW complex structure of $\mathrm{WGr}(d,n)$ as in Proposition 2.7. Note that X_i/X_{i-1} is the one-point compactification of $E(\lambda^i)/G(c_i)$, which is the Thom space of the orbifold $(\mathbb{C}^*)^n$–vector bundle

$$\frac{E(\lambda^i)}{G(c_i)} \to [C(\lambda^i)],$$

where $[C(\lambda^i)]$ is the $(\mathbb{C}^*)^n$–fixed point corresponding to the Schubert symbol λ^i for $i = 1, \dots, m$. It remains to note that $E(\lambda^i)$ is $(\mathbb{C}^*)^n$–equivariantly homeomorphic to $\mathbb{C}^{\ell(\lambda^i)}$; see (2-13). □

Now corresponding to $\mathrm{rev}(\lambda^i)$, one can define a subset of Schubert symbols

$$R(\lambda^i) := \{\lambda^j \mid \lambda^j = (k,k')\lambda^i \text{ for } (k,k') \in \mathrm{rev}(\lambda^i)\}. \tag{4-2}$$

Then the cardinality of the set $R(\lambda^i)$ is $\ell(\lambda^i)$ for every $i \in \{0, 1, \dots, m\}$. Note that the bundle in (4-1) is also an orbifold T^n–bundle.

Proposition 4.2 *The orbifold T^n–bundle in (4-1) has a decomposition*

$$\xi^i : \frac{\mathbb{C}^{\ell(\lambda^i)}}{G(c_i)} \to [C(\lambda^i)] \cong \bigoplus_{j\,:\,\lambda^j \in R(\lambda^i)} \left(\xi^{ij} : \frac{\mathbb{C}_{ij}}{G(c_{ij})} \to [C(\lambda^i)]\right).$$

Proof Observe that

$$X_i \setminus X_{i-1} = \frac{E(\lambda^i)}{G(c_i)} \cong \frac{\mathbb{C}^{\ell(\lambda^i)}}{G(c_i)}.$$

Since T^n is abelian, the T^n action on $E(\lambda^i) \cong \mathbb{C}^{\ell(\lambda^i)}$ determines the decomposition

$$E(\lambda^i) \cong \bigoplus_{j:\lambda^j \in R(\lambda^i)} \mathbb{C}_{ij}$$

for some irreducible representation $\mathbb{C}_{ij}$ of T^n. By [10, Proposition 2.8] there exists a finite covering map $q \colon T^n \to T^n$ such that the projection map $\phi \colon E(\lambda^i) \to E(\lambda^i)/G(c_i)$ is equivariant via the map q, ie $\phi(tx) = q(t)\phi(x)$. Therefore,

$$\frac{E(\lambda^i)}{G(c_i)} \cong \bigoplus_{j:\lambda^j \in R(\lambda^i)} \frac{\mathbb{C}_{ij}}{G(c_{ij})}$$

for some positive integers c_{ij} which divide c_i. Hence the proof follows. □

Remark 4.3 (1) The attaching map $\eta_i : S(\xi^i) \to X_{i-1}$ for the q–CW complex structure in (2-17) satisfies $\eta_i|_{S(\xi^{ij})} = f_{ij} \circ \xi^{ij}$, where $f_{ij} : [C(\lambda^i)] \to [C(\lambda^j)]$ is the constant map.

(2) The equivariant Euler classes $\{e_{T^n}(\xi^{ij}) \mid j < i\}$ are nonzero divisors. They are pairwise prime by [14, Lemma 5.2] and the T^n–action on $E(\lambda^i)$ discussed in the proof of Theorem 3.19.

Theorem 4.4 *Let* $\mathrm{WGr}(d,n)$ *be a weighted Grassmann orbifold for* $d<n$, *corresponding to weight vector* $W=(w_1,w_2,\dots,w_n)\in(\mathbb{Z}_{\geq 0})^n$ *and* $a\geq 1$. *Then the generalized* T^n*–equivariant cohomology* $\mathscr{E}^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Q})$ *can be given by*

$$\left\{(f_i)\in\bigoplus_{i=0}^{m}\mathscr{E}^*_{T^n}(\{\mathrm{pt}\};\mathbb{Q})\;\middle|\; e_{T^n}(\xi^{ij}) \text{ divides } f_i-f_j \text{ for } j<i \text{ and } |\lambda^j\cap\lambda^i|=d-1\right\}$$

for $\mathscr{E}^*_{T^n}=K^*_{T^n},\,H^*_{T^n}$, *where* $e_{T^n}(\xi^{ij})$ *represents the equivariant Euler class of* ξ^{ij}.

Proof This follows from [26, Proposition 2.3] using Propositions 4.1 and 4.2, and Remark 4.3. □

We note that equivariant cohomology ring of $\mathrm{WGr}(d,n)$ with rational coefficients is discussed in [1]. In the rest, we give a description of the equivariant cohomology ring, equivariant K–theory ring and equivariant cobordism ring of a divisive weighted Grassmann orbifold with integer coefficients.

Proposition 4.5 *Let* $\mathrm{WGr}(d,n)$ *be a divisive weighted Grassmann orbifold for* $d<n$ *corresponding to* $W=(w_1,w_2,\dots,w_n)\in(\mathbb{Z}_{\geq 0})^n$ *and* $a\geq 1$. *Then there is a* T^n*–invariant stratification*

$$\{\mathrm{pt}\}=X_0\subset X_1\subset\cdots\subset X_m=\mathrm{WGr}(d,n)$$

such that for $i=1,\dots,m$, *the quotient* X_i/X_{i-1} *is homeomorphic to the Thom space* $\mathrm{Th}(\xi^i)$ *of the* T^n*–vector bundle*

$$\xi^i:\mathbb{C}^{\ell(\lambda^i)}\to[C(\lambda^i)].$$

Proof Since $\mathrm{WGr}(d,n)$ is divisive, there exists $\sigma\in S_n$ such that σc_i divides σc_{i-1} for $i=1,2,\dots,m$. Then $\gcd\{\sigma c_0,\sigma c_1,\dots,\sigma c_i\}=\sigma c_i$ for all i. By Theorem 3.5, one can write

$$\mathrm{WGr}(d,n)=\bigsqcup_{i=0}^{m}\frac{\sigma E(\lambda^i)}{G(\sigma c_i)}.$$

By Lemma 3.3, the q–cell $\sigma E(\lambda^i)/G(\sigma c_i)$ is homeomorphic to $\sigma E(\lambda^i)/G(\sigma c_i/\sigma c_i)\cong\mathbb{C}^{\ell(\lambda^i)}$ for $i=1,\dots,m$. Let $X_k=\bigsqcup_{i=0}^{k}\sigma E(\lambda^i)/G(\sigma c_i)$ for $i=0,1,\dots,m$. The rest follows from the proof of Proposition 4.1. □

Remark 4.6 For a divisive weighted Grassmann orbifold, Proposition 4.2 and Remark 4.3 hold with $c_{ij}=1$ for every $j<i$.

Theorem 4.7 *Let* $\mathrm{WGr}(d,n)$ *be a divisive weighted Grassmann orbifold for* $d<n$. *Then the generalized* T^n*–equivariant cohomology* $\mathscr{E}^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ *can be given by*

$$\left\{(f_i)\in\bigoplus_{i=0}^{m}\mathscr{E}^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})\;\middle|\; e_{T^n}(\xi^{ij}) \text{ divides } f_i-f_j \text{ for } j<i \text{ and } |\lambda^j\cap\lambda^i|=d-1\right\}$$

for $\mathscr{E}^*_{T^n}=H^*_{T^n},\,K^*_{T^n}$ *and* $\mathrm{MU}^*_{T^n}$.

Proof This follows from Proposition 4.5, Remark 4.6 and [14, Theorem 2.3]. □

Remark 4.8 Let λ^i and λ^j be two Schubert symbols with $j < i$. If $\mathrm{WGr}(d,n)$ is a divisive weighted Grassmann orbifold then there exists a permutation $\sigma \in S_n$ such that σc_i divides σc_j. We write

$$\sigma d_{ij} := \frac{\sigma c_j}{\sigma c_i} \in \mathbb{Z}.$$

Next we discuss how to compute $e_{T^n}(\xi^{ij})$. We recall that

$$H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z}) = H^*(BT^n;\mathbb{Z}) \cong \mathbb{Z}[y_1, y_2, \dots, y_n],$$

where $y_1, y_2, \dots, y_n$ be the standard basis of $H^2(BT^n;\mathbb{Z})$. Using (3-4) the character of the one-dimensional representation for the bundle ξ^{ij} is given by

$$(t_1, t_2, \dots, t_n) \to t_{\sigma\lambda^j}(t_{\sigma\lambda^i})^{-\sigma c_j/\sigma c_i}. \tag{4-3}$$

Also,

$$K^*_{T^n}(\{\mathrm{pt}\}) \cong R(T^n)[z, z^{-1}],$$

where $R(T^n)$ is the complex representation ring of T^n and z is the Bott element in $K^{-2}(\{\mathrm{pt}\})$. Note that the ring $R(T^n)$ is isomorphic to the ring of Laurent polynomials with n variables, ie $R(T^n) \cong \mathbb{Z}[\alpha_1, \dots, \alpha_n]_{(\alpha_1 \cdots \alpha_n)}$, where α_i is the irreducible representation corresponding to the projection on the i^{th} factor; see [17]. Therefore, using (4-3), one has, for $j < i$ and $|\lambda^j \cap \lambda^i| = d-1$,

$$e_{T^n}(\xi^{ij}) = \begin{cases} 1 - \alpha_{\sigma\lambda^j}\alpha_{\sigma\lambda^i}^{-\sigma d_{ij}} & \text{in } K^0_{T^n}(\{\mathrm{pt}\};\mathbb{Z}), \\ e_{T^n}(\alpha_{\sigma\lambda^j}\alpha_{\sigma\lambda^i}^{-\sigma d_{ij}}) & \text{in } \mathrm{MU}^2_{T^n}(\{\mathrm{pt}\};\mathbb{Z}), \\ Y_{\sigma\lambda^j} - \sigma d_{ij} Y_{\sigma\lambda^i} & \text{in } H^2_{T^n}(\{\mathrm{pt}\};\mathbb{Z}), \end{cases} \tag{4-4}$$

where $Y_\lambda := \sum_{i=1}^d y_{\lambda_i}$ and $\alpha_\lambda = \alpha_{\lambda_1} \cdots \alpha_{\lambda_d}$ for a Schubert symbol $\lambda = (\lambda_1, \dots, \lambda_d)$.

We remark that the structure of $\mathrm{MU}^*_{T^n}(\{\mathrm{pt}\})$ is unknown; however, it is referred to in [15] as the ring of *T^n–cobordism forms*.

Example 4.9 Consider the weighted Grassmann orbifold $\mathrm{WGr}(2,4)$ for $W = (12,2,2,2)$ and $a = 6$. We have the ordering on the six Schubert symbols given by

$$\lambda^0 = (1,2) < \lambda^1 = (1,3) < \lambda^2 = (1,4) < \lambda^3 = (2,3) < \lambda^4 = (2,4) < \lambda^5 = (3,4).$$

Then $c_0 = 20, c_1 = 20, c_2 = 20, c_3 = 10, c_4 = 10, c_5 = 10$ from (2-5). Here c_i divides c_{i-1} for all $i = 1,2,3,4,5$. Thus, $\mathrm{WGr}(2,4)$ is divisive for the identity permutation in S_4. Then $d_{ij} = c_j/c_i$ in Remark 4.8 gives

$$d_{ij} = \begin{cases} 1 & \text{if } j < i \text{ and both } i,j \in \{0,1,2\} \text{ or } \{3,4,5\}, \\ 2 & \text{if } j \in \{0,1,2\} \text{ and } i \in \{3,4,5\}. \end{cases}$$

Then one can calculate the equivariant Euler class $e_{T^n}(\xi^{ij})$ from (4-4). The generalized integral equivariant cohomology ring $\mathscr{E}^*_{T^n}(\mathrm{WGr}(2,4);\mathbb{Z})$ of this divisive weighted Grassmann orbifold $\mathrm{WGr}(2,4)$ can be described by Theorem 4.7. □

The fixed points of the T^n–action on $\mathrm{WGr}(d,n)$ are $V := \{[C(\lambda^i)]\}_{i=0}^m$. Two fixed points $[C(\lambda^i)]$ and $[C(\lambda^j)]$ are connected by a T^n–invariant $\mathbb{W}P(c_i, c_j) \subset \mathrm{WGr}(d,n)$ if and only if $\lambda^j = (k,k')\lambda^i$ for some (k,k'), where $(k,k')\lambda^i$ is described in Definition 3.17. In that case it is said that there is an edge e_{ij} between $[C(\lambda^i)]$ and $[C(\lambda^j)]$. Let $E := \{e_{ij} \mid \lambda^j = (k,k')\lambda^i \text{ for some } (k,k')\}$. Then $\Gamma = (V,E)$ is a $d(n-d)$ valent graph with $(m+1)$–vertices. Consider the connection θ on Γ defined similarly as the GKM–graph of the Grassmann manifold in [12, Theorem 1.11.4, equation (1.34)]. Note that the T^n–action on $\mathbb{W}P(c_i,c_j)$ is given by $(t_1,\dots,t_n)[z_i : z_j] = [t_{\lambda^i} z_i : t_{\lambda^j} z_j]$. This action induces a map

$$\alpha\colon E \to H^*(BT^n;\mathbb{Q}) = \mathbb{Q}[y_1,\dots,y_n]$$

defined by $\alpha(e) := (c_i Y_{\lambda^j} - c_j Y_{\lambda^i})/c_i$ if e is the oriented edge from $[C(\lambda^i)]$ to $[C(\lambda^j)]$ with $|\lambda^j \cap \lambda^i| = d-1$. Note that if $\overline{e}$ is the edge with the opposite orientation on e then $\alpha(\overline{e}) = (c_j Y_{\lambda^i} - c_i Y_{\lambda^j})/c_j$. Let $r_e = c_i$ and $r_{\overline{e}} = c_j$. Then

$$r_e\alpha(e) = -r_{\overline{e}}\alpha(\overline{e}) \in H^2(BT^n;\mathbb{Z}). \tag{4-5}$$

Let e and e' be two edges with the same initial vertex. Let e' be the oriented edge from $[C(\lambda^i)]$ to $[C(\lambda^l)]$. Then we have

$$c_j c_l\big(\alpha(\theta_e(e')) - \alpha(e')\big) = 0 \bmod r_e\alpha(e).$$

The map α is called the axial function on Γ. Therefore, (Γ,α,θ) satisfies the definition of orbifold GKM–graph [8, Definition 2.2]. Hence, (Γ,α,θ) is the orbifold GKM–graph for the weighted Grassmann orbifold.

The following result gives equivariant cohomology ring of some nondivisive weighted Grassmann orbifolds with integer coefficients.

Theorem 4.10 *Suppose that* $\mathrm{WGr}(d,n)$ *is a weighted Grassmann orbifold corresponding to the order* $\lambda^0 < \cdots < \lambda^m$ *such that* $r = \gcd\{c_0, c_1\}$ *and* $c_i | c_k$ *for* $k \leq i$ *with* $i \geq 2$*. Then the integral equivariant cohomology ring of* $\mathrm{WGr}(d,n)$ *is given by*

$$H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z}) = \left\{(f_i) \in \bigoplus_{i=0}^m \mathbb{Z}[y_1,y_2,\dots,y_n] \;\middle|\; \begin{array}{l}(Y_{\lambda^j} - d_{ij}Y_{\lambda^i}) \text{ divides } (f_i - f_j) \text{ if } j < i,\ |\lambda^j \cap \lambda^i| = d-1,\\ \qquad (i,j) \neq (0,1) \text{ and } c_1Y_{\lambda^0} - c_0Y_{\lambda^1} \text{ divides } r(f_1 - f_0)\end{array}\right\}.$$

Proof By the given condition $\gcd\{c_0, c_1, \dots, c_i\} = c_i$ for $i \geq 2$. So, by Lemma 3.3, $E(\lambda^i)/G(c_i)$ is homeomorphic to $E(\lambda^i)/G(c_i/c_i) \cong \mathbb{C}^{\ell(\lambda^i)}$ for $i = 1,\dots,m$. When $i = 1$, we have that X_1 is equivariantly homeomorphic to $\mathbb{W}P(c_0,c_1)$. Therefore, $\mathrm{WGr}(d,n)$ has a T^n–invariant CW complex structure. For the edge $e = e_{01}$, the minimum of r_e that satisfies (4-5) is r. Thus, by [8, Definition 2.3 and Theorem 2.9], we get the result. □

Next, we discuss the equivariant cohomology ring of the weighted projective space $\mathbb{W}P(b_0, b_1 \dots, b_m)$, where $(b_0, b_1 \dots, b_m) \in (\mathbb{Z}_{\geq 1})^{m+1}$, for several torus actions. By Remark 2.3, $\mathbb{W}P(b_0, b_1 \dots, b_m) = \mathrm{WGr}(1, m+1)$, where the latter is associated to the weight vector $W = (b_0 - 1, \dots, b_m - 1)$ and $a = 1$. The Schubert symbols for $1 < m+1$ are $\{1\}, \dots, \{m\}$ and $\{m+1\}$. Assume that $\mathrm{WGr}(1, m+1)$ is divisive corresponding to this order, ie b_i divides b_{i-1} for $i = 1, 2, \dots, m$. Then

$$E(i+1) \cong \{[(u_0, u_1, \dots, u_{i-1}, 1, 0, \dots, 0)] \in \mathbb{W}P(b_0, b_1 \dots, b_m)\} \cong \mathbb{C}^i \quad \text{for } i = 0, 1, \dots, m.$$

Let (n, d) be such that $d < n$ and $\binom{n}{d} = m+1$. Then (2-8) gives a T^n–action on $\mathbb{W}P(b_0, b_1 \dots, b_m)$. Recall t_{λ^i} from (2-8) for the Schubert symbols $\lambda^0, \lambda^1, \dots, \lambda^m$ corresponding to $d < n$. We have

$$\begin{aligned}(t_1, t_2, \dots, t_n)&[(u_0, u_1, \dots, u_{i-1}, 1, 0, \dots, 0)] \\ &= [(t_{\lambda^0} u_0, t_{\lambda^1} u_1, \dots, t_{\lambda^{i-1}} u_{i-1}, t_{\lambda^i}, 0, \dots, 0)] \\ &= [((t_{\lambda^i})^{-b_0/b_i} t_{\lambda^0} u_0, (t_{\lambda^i})^{-b_1/b_i} t_{\lambda^1} u_1, \dots, (t_{\lambda^i})^{-b_{i-1}/b_i} t_{\lambda^{i-1}} u_{i-1}, 1, 0, \dots, 0)].\end{aligned}$$

Then $E(i+1)$ is T^n–invariant as well as T^{m+1}–invariant. Let

$$X_i := [(u_0, u_1, \dots, u_i, 0, \dots, 0)] \in \mathbb{W}P(b_0, b_1 \dots, b_m)\}.$$

Then X_i gives a filtration

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset \cdots \subset X_m = \mathbb{W}P(b_0, b_1, \dots, b_m). \tag{4-6}$$

Note that the filtration in (4-6) satisfies Proposition 4.5 and Remark 4.6. Thus in this case

$$\xi^i : E(i+1) \to [e_{i+1}] \cong \bigoplus_{j=0}^{i} (\xi^{ij} : \mathbb{C}_{ij} \to [e_{i+1}])$$

for some irreducible representation $\mathbb{C}_{ij}$. Using the proof of [15, Theorem 2.3] one can get the following result.

Theorem 4.11 *If $\mathbb{W}P(b_0, \dots, b_m)$ is divisive, then the generalized T^n–equivariant cohomology*

$$\mathscr{E}^*_{T^n}(\mathbb{W}P(b_0, \dots, b_m); \mathbb{Z})$$

for $\mathscr{E}^*_{T^n} = H^*_{T^n}, K^*_{T^n}$ and $\mathrm{MU}^*_{T^n}$ can be given by

$$\left\{ (f_i) \in \bigoplus_{i=0}^{m} \mathscr{E}^*_{T^n}(\{\mathrm{pt}\}; \mathbb{Z}) \;\middle|\; e_{T^n}(\xi^{ij}) \text{ divides } f_i - f_j \text{ for all } j < i \right\}.$$

We note that there are several pairs (n, d) such that $d < n$ and $\binom{n}{d} = m+1 > 2$. Now we discuss how to calculate the equivariant Euler class $e_{T^n}(\xi^{ij})$ in Theorem 4.11. The corresponding one-dimensional representation on the bundle ξ^{ij} for $j < i$ is determined by the character

$$(t_1, \dots, t_n) \to (t_{\lambda^i})^{-b_j/b_i} t_{\lambda^j}.$$

Thus, similar to (4-4), one can calculate the equivariant Euler class $e_{T^n}(\xi^{ij})$ of the bundle ξ^{ij} for $j < i$.

Example 4.12 For $m = 2$, we have $\binom{3}{1} = \binom{3}{2} = 3$. Thus, corresponding to two different pairs $(3,1)$ and $(3,2)$, we have two different T^3 actions on $\mathbb{W}P(b_0, b_1, b_2)$. The map $f : T^3 \to T^3$ defined by $(t_1, t_2, t_3) \to (t_1 t_2, t_1 t_3, t_2 t_3)$ is not an automorphism. So these actions are not equivalent. However, using Theorem 4.11, one can calculate the equivariant cohomology of $\mathbb{W}P(b_0, b_1, b_2)$ for both the actions if b_i divides b_{i-1} for $i = 1, 2$. □

5 Equivariant Schubert calculus for divisive weighted Grassmann orbifolds

In this section, we show that there exist equivariant Schubert classes which form a basis for the equivariant cohomology ring of a divisive weighted Grassmann orbifold with integer coefficients. We show some properties of the weighted structure constants. Moreover, we discuss some relations that help to compute the weighted structure constants corresponding to this equivariant Schubert basis with integer coefficients.

For $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n); \mathbb{Z})$, the support of x, denoted by supp(x), is the set of all Schubert symbols λ^i such that $\mathrm{x}|_{\lambda^i} \neq 0$. Recall the partial order $\preceq$ on the Schubert symbols defined in (2-3). We follow this partial order $\preceq$ and we say that an element $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n); \mathbb{Z})$ is supported above by λ^i if $\lambda^i \preceq \lambda^k$ for all $\lambda^k \in \mathrm{supp}(\mathrm{x})$.

Let $\mathrm{WGr}(d,n)$ be a divisive weighted Grassmann orbifold. Then there exists $\sigma \in S_n$ such that

$$\sigma c_i \text{ divides } \sigma c_{i-1} \quad \text{for } i = 1, 2, \dots, m. \tag{5-1}$$

Using Theorem 3.5, it is sufficient to consider $\sigma = \mathrm{Id}$, the identity permutation on S_n. For $\sigma = \mathrm{Id}$, (5-1) transforms to

$$c_i \text{ divides } c_{i-1} \quad \text{for } i = 1, 2, \dots, m.$$

Recall the definition of $R(\lambda^i)$ from (4-2). We introduce the following definition.

Definition 5.1 An element $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n); \mathbb{Z})$ is said to be an equivariant Schubert class corresponding to a Schubert symbol λ^i if the following conditions are satisfied:

(1) $\mathrm{x}|_{\lambda^k} \neq 0$ implies $\lambda^i \preceq \lambda^k$. (We say that x is supported above λ^i.)

(2) $\mathrm{x}|_{\lambda^i} = \prod_{\lambda^j \in R(\lambda^i)} (Y_{\lambda^j} - (c_j/c_i) Y_{\lambda^i})$.

(3) $\mathrm{x}|_{\lambda^k}$ is a homogeneous polynomial in $y_1, y_2, \dots, y_n$ of degree $\ell(\lambda^i)$.

Proposition 5.2 (uniqueness) *For each Schubert symbol λ^i, there is at most one equivariant Schubert class* x *corresponding to λ^i.*

Proof Suppose that there were two distinct equivariant Schubert classes x and x′ corresponding to λ^i. Let λ^j be the minimal Schubert symbol such that $(\mathrm{x} - \mathrm{x}')|_{\lambda^j} \neq 0$. By Definition 5.1(1)–(2), we get $\lambda^i \prec \lambda^j$. Then from the condition in the expression of the equivariant cohomology ring in Theorem 4.7, we get that $(\mathrm{x} - \mathrm{x}')|_{\lambda^j}$ is a multiple of $\prod_{\lambda^k \in R(\lambda^j)} (Y_{\lambda^k} - (c_k/c_j) Y_{\lambda^j})$, which is of degree $\ell(\lambda^j)$. This contradicts the fact that $\mathrm{x} - \mathrm{x}'$ is homogeneous of degree $\ell(\lambda^i) < \ell(\lambda^j)$. □

Let us denote the equivariant Schubert class corresponding to the Schubert symbol λ^i by $w\widetilde{S}_{\lambda^i}$ for $i=0,1,\dots,m$. We remark that the existence of $w\widetilde{S}_{\lambda^i}$ follows from [14, Proposition 4.3] and Theorem 4.7. Geometrically, $w\widetilde{S}_{\lambda^i}$ is the equivariant cohomology class corresponding to the closure of the cell $\sigma_r E(\sigma_r\lambda^i)$, where $\sigma_r \in S_n$ is the permutation defined by

$$\sigma_r := \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix}.$$

Using the arguments in the proof of [20, Proposition 1], one gets the following.

Proposition 5.3 *The equivariant Schubert classes $\{w\widetilde{S}_{\lambda^i}\}_{i=0}^m$ form a basis for $H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ as a module over $H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$. Moreover, any $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ can be written uniquely as an $H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$–linear combination of $w\widetilde{S}_{\lambda^i}$ using only those λ^i such that $\lambda^j \le \lambda^i$ for some $\lambda^j \in \mathrm{supp}(\mathrm{x})$.*

Proof To check that the set $\{w\widetilde{S}_{\lambda^i}\}_{i=0}^m$ is linearly independent, let $\sum_{i=0}^m a_i w\widetilde{S}_{\lambda^i} = 0$ for coefficients $a_i \in H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$ that are not identically zero. Let

$$k = \min\{i \in \{0,1,\dots,m\} \mid a_i \neq 0\}.$$

We also have that $w\widetilde{S}_{\lambda^i}|_{\lambda^k} = 0$ for $i > k$. Thus the restriction $\big(\sum_{i=0}^m a_i w\widetilde{S}_{\lambda^i}\big)|_{\lambda^k} = a_k w\widetilde{S}_{\lambda^k}|_{\lambda^k} \neq 0$, which is a contradiction.

Now, to prove that $\{w\widetilde{S}_{\lambda^i}\}$ spans, consider an element $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$. Let

$$j := \min\{i \in \{0,1,\dots,m\} \mid \lambda^i \in \mathrm{supp}(\mathrm{x})\}.$$

Then $\mathrm{x}|_{\lambda^j} = \beta_j w\widetilde{S}_{\lambda^j}|_{\lambda^j}$ using Theorem 4.7 and (4-4) for some $\beta_j \in \mathbb{Z}[y_1,\dots,y_n]$. Subtracting $\beta_j w\widetilde{S}_{\lambda^j}$, we can inductively reduce support of x upwards until it is empty. This uses only those λ^i such that $\lambda^j \le \lambda^i$ for some $\lambda^j \in \mathrm{supp}(\mathrm{x})$. □

Example 5.4 In Figure 1, we compute the equivariant Schubert class $w\widetilde{S}_{(2,3)} \in H^*_{T^4}(\mathrm{WGr}(2,4);\mathbb{Z})$, where $\mathrm{WGr}(2,4)$ is a divisive weighted Grassmann orbifold for some $W = (\alpha+\gamma\beta, \alpha, \alpha, \alpha) \in (\mathbb{Z}_{\ge 0})^4$ and $a = \beta - 2\alpha \in \mathbb{Z}_{>0}$. Figure 1, left, is the lattice of the Schubert symbols for $2<4$. Figure 1, right, gives the equivariant Schubert class corresponding to the Schubert symbol $(2,3)$. □

In the rest of this section, we compute the weighted structure constants for the equivariant cohomology of a divisive weighted Grassmann orbifold. Since the set $\{w\widetilde{S}_{\lambda^i}\}_{i=0}^m$ form a $H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$–basis for $H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$, for any two λ^i and λ^j, one has that

$$w\widetilde{S}_{\lambda^i}\, w\widetilde{S}_{\lambda^j} = \sum_{\lambda^k} wc_{ij}^k\, w\widetilde{S}_{\lambda^k}, \tag{5-2}$$

where $\lambda^k \in \{\lambda^0,\lambda^1,\dots,\lambda^m\}$. The constant $wc_{ij}^k \in H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$ in the formula is called a *weighted structure constant*.

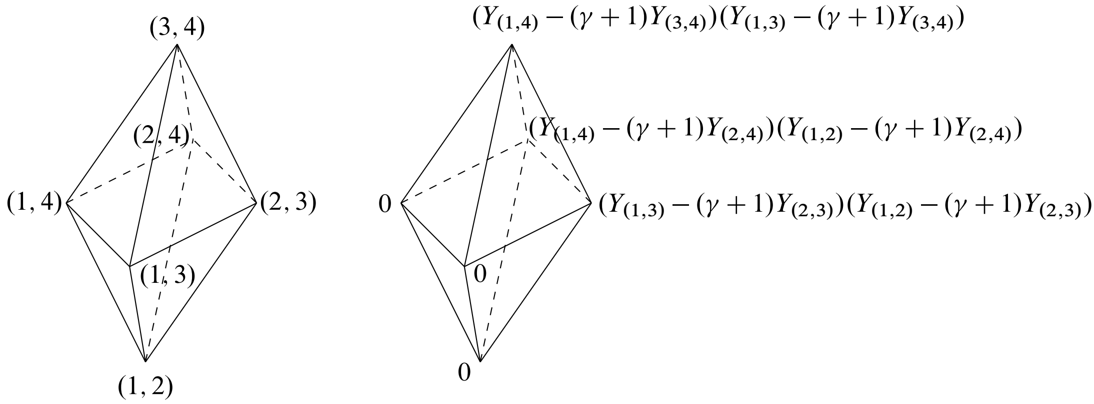

Figure 1

Lemma 5.5 *The weighted structure constants wc_{ij}^k have the following properties.*

(1) *The weighted structure constant wc_{ij}^k has degree $\ell(\lambda^i)+\ell(\lambda^j)-\ell(\lambda^k)$.*

(2) *The constant wc_{ij}^k is 0 unless $\ell(\lambda^k) \le \ell(\lambda^i)+\ell(\lambda^j)$ and $\lambda^k \succeq \lambda^i, \lambda^j$.*

(3) *When $i=k$, we have $wc_{ij}^i = w\widetilde{S}_{\lambda^j}|_{\lambda^i}$.*

Proof (1) The degree of $w\widetilde{S}_{\lambda^i}$ is $\ell(\lambda^i)$. So the degree of the weighted structure constant wc_{ij}^k is given by

$$\deg(wc_{ij}^k) = \deg(w\widetilde{S}_{\lambda^i}) + \deg(w\widetilde{S}_{\lambda^j}) - \deg(w\widetilde{S}_{\lambda^k}) = \ell(\lambda^i)+\ell(\lambda^j)-\ell(\lambda^k).$$

(2) The weighted structure constant $wc_{ij}^k = 0$ if $\ell(\lambda^i)+\ell(\lambda^j)-\ell(\lambda^k) < 0$. Also,

$$(w\widetilde{S}_{\lambda^i}\ w\widetilde{S}_{\lambda^j})|_{\lambda^s} \neq 0 \implies \lambda^s \succeq \lambda^i, \lambda^j.$$

Thus, by Proposition 5.3, $wc_{ij}^k \neq 0$ implies $\lambda^k \succeq \lambda^i, \lambda^j$.

(3) Comparing the $(\lambda^i)^{\text{th}}$ component of the both sides in (5-2), we get

$$w\widetilde{S}_{\lambda^i}|_{\lambda^i}\ w\widetilde{S}_{\lambda^j}|_{\lambda^i} = wc_{ij}^i\ w\widetilde{S}_{\lambda^i}|_{\lambda^i} + \sum_{k\neq i} wc_{ij}^k\ w\widetilde{S}_{\lambda^k}|_{\lambda^i}.$$

We have that $wc_{ij}^k = 0$ unless $\lambda^k \succeq \lambda^i$, but $w\widetilde{S}_{\lambda^k}|_{\lambda^i} = 0$ for $\lambda^k \succeq \lambda^i$, and $\lambda^k \neq \lambda^i$. Thus all the terms in the summation vanish. So the claim follows, since $w\widetilde{S}_{\lambda^i}|_{\lambda^i} \neq 0$. □

Now we introduce the equivariant Schubert divisor class. Note that $\ell(\lambda^i)=0$ if and only if $i=0$, and $\ell(\lambda^i)=1$ if and only if $i=1$. The equivariant Schubert class corresponding to the Schubert symbol λ^1 is called the *equivariant Schubert divisor class*.

Lemma 5.6 *The equivariant Schubert divisor class $w\widetilde{S}_{\lambda^1} \in H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ is given by*

$$w\widetilde{S}_{\lambda^1}|_{\lambda^i} = Y_{\lambda^0} - \frac{c_0}{c_i} Y_{\lambda^i}.$$

Proof Consider an element $\mathrm{x} \in \bigoplus_{i=0}^{m} H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$ defined by $\mathrm{x}|_{\lambda^i} = Y_{\lambda^0} - (c_0/c_i)Y_{\lambda^i}$. Let λ^i and λ^j be two Schubert symbols such that $\lambda^j \preceq \lambda^i$. Then

$$\mathrm{x}|_{\lambda^i} - \mathrm{x}|_{\lambda^j} = \frac{c_0}{c_j}\Big(Y_{\lambda^j} - \frac{c_j}{c_i}Y_{\lambda^i}\Big).$$

Thus $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ from Theorem 4.7 and (4-4). Note that $\mathrm{x}|_{\lambda^0} = 0$. If $\mathrm{x}|_{\lambda^k} \neq 0$ then $\lambda^1 \preceq \lambda^k$. Now $R(\lambda^1) = \{\lambda^0\}$ and

$$\mathrm{x}|_{\lambda^1} = Y_{\lambda^0} - \frac{c_0}{c_1}Y_{\lambda^1} = \prod_{\lambda^j \in R(\lambda^1)} \Big(Y_{\lambda^j} - \frac{c_j}{c_1}Y_{\lambda^1}\Big).$$

Also, $\mathrm{x}|_{\lambda^k}$ is a homogeneous polynomial of degree $1 = \ell(\lambda^1)$. Thus x satisfies all the conditions of Definition 5.1 for $i = 1$. Therefore, by the uniqueness of the equivariant Schubert classes, we have $\mathrm{x} = w\widetilde{S}_{\lambda^1}$. □

For any two Schubert symbols λ^i and λ^j, we write $\lambda^i \to \lambda^j$ if $\ell(\lambda^i) = \ell(\lambda^j) + 1$ and $\lambda^j \preceq \lambda^i$.

Proposition 5.7 (weighted Pieri rule) $$w\widetilde{S}_{\lambda^1}\, w\widetilde{S}_{\lambda^j} = (w\widetilde{S}_{\lambda^1}|_{\lambda^j})\, w\widetilde{S}_{\lambda^j} + \sum_{\lambda^i \to \lambda^j} \frac{c_0}{c_j} w\widetilde{S}_{\lambda^i}.$$

Proof Using the fact that $\deg(w\widetilde{S}_{\lambda^1}) = 1$, we have

$$w\widetilde{S}_{\lambda^1}\, w\widetilde{S}_{\lambda^j} = (wc^j_{1j})\, w\widetilde{S}_{\lambda^j} + \sum_{\lambda^i \to \lambda^j} (wc^i_{1j})\, w\widetilde{S}_{\lambda^i}.$$

From Lemma 5.5, we get $wc^j_{1j} = w\widetilde{S}_{\lambda^1}|_{\lambda^j}$. Fix λ^i such that $\lambda^i \to \lambda^j$ and compare the $(\lambda^i)^{\mathrm{th}}$ component of both sides; we get

$$\begin{aligned} w\widetilde{S}_{\lambda^1}|_{\lambda^i}\, w\widetilde{S}_{\lambda^j}|_{\lambda^i} &= (wc^j_{1j}) w\widetilde{S}_{\lambda^j}|_{\lambda^i} + (wc^i_{1j}) w\widetilde{S}_{\lambda^i}|_{\lambda^i} \\ \implies\ (wc^i_{1j}) w\widetilde{S}_{\lambda^i}|_{\lambda^i} &= (w\widetilde{S}_{\lambda^1}|_{\lambda^i} - w\widetilde{S}_{\lambda^1}|_{\lambda^j}) w\widetilde{S}_{\lambda^j}|_{\lambda^i} \\ \implies\ (wc^i_{1j}) w\widetilde{S}_{\lambda^i}|_{\lambda^i} &= \frac{c_0}{c_j}\Big(Y_{\lambda^j} - \frac{c_j}{c_i}Y_{\lambda^i}\Big) w\widetilde{S}_{\lambda^j}|_{\lambda^i}. \end{aligned}$$

Thus $wc^i_{1j} = c_0/c_j$ if $\lambda^i \to \lambda^j$. □

By applying Proposition 5.7 repeatedly, we can compute the following product, as well as the higher products:

$$\begin{aligned} (w\widetilde{S}_{\lambda^1})^2 w\widetilde{S}_{\lambda^j} &= w\widetilde{S}_{\lambda^1}\Big((w\widetilde{S}_{\lambda^1}|_{\lambda^j}) w\widetilde{S}_{\lambda^j} + \sum_{\lambda^i \to \lambda^j} \frac{c_0}{c_j} w\widetilde{S}_{\lambda^i}\Big) \\ &= (w\widetilde{S}_{\lambda^1}|_{\lambda^j})^2 w\widetilde{S}_{\lambda^j} + \sum_{\lambda^i \to \lambda^j} (w\widetilde{S}_{\lambda^1}|_{\lambda^j}) \frac{c_0}{c_j} w\widetilde{S}_{\lambda^i} + \sum_{\lambda^i \to \lambda^j} \frac{c_0}{c_j} (w\widetilde{S}_{\lambda^1}|_{\lambda^i}) w\widetilde{S}_{\lambda^i} \\ &\qquad + \sum_{\lambda^k \to \lambda^i \to \lambda^j} \frac{c_0}{c_j}\frac{c_0}{c_i} w\widetilde{S}_{\lambda^k}. \end{aligned}$$

Proposition 5.8 *For any three Schubert symbols λ^i, λ^j and λ^k, we have the recurrence relation*

$$(w\widetilde{S}_{\lambda^1}|_{\lambda^k} - w\widetilde{S}_{\lambda^1}|_{\lambda^i})wc_{ij}^k = \sum_{\lambda^s \to \lambda^i} \frac{c_0}{c_i} wc_{sj}^k - \sum_{\lambda^k \to \lambda^t} \frac{c_0}{c_t} wc_{ij}^t.$$

Proof We use the associativity of the multiplication in $H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ and weighted Pieri rule to expand $w\widetilde{S}_{\lambda^1} w\widetilde{S}_{\lambda^i} w\widetilde{S}_{\lambda^j}$ in two different ways:

$$\begin{aligned}(w\widetilde{S}_{\lambda^1} w\widetilde{S}_{\lambda^i})w\widetilde{S}_{\lambda^j} &= ((w\widetilde{S}_{\lambda^1}|_{\lambda^i})w\widetilde{S}_{\lambda^i} + \sum_{\lambda^s \to \lambda^i} \frac{c_0}{c_i} w\widetilde{S}_{\lambda^s})w\widetilde{S}_{\lambda^j}, \\ &= (w\widetilde{S}_{\lambda^1}|_{\lambda^i}) \sum_{\lambda^l} wc_{ij}^l w\widetilde{S}_{\lambda^l} + \sum_{\lambda^s \to \lambda^i} \frac{c_0}{c_i} \sum_{\lambda^l} wc_{sj}^l w\widetilde{S}_{\lambda^l},\end{aligned} \tag{5-3}$$

$$w\widetilde{S}_{\lambda^1}(w\widetilde{S}_{\lambda^i} w\widetilde{S}_{\lambda^j}) = w\widetilde{S}_{\lambda^1} \sum_{\lambda^l} wc_{ij}^l w\widetilde{S}_{\lambda^l} = \sum_{\lambda^l} wc_{ij}^l \Big((w\widetilde{S}_{\lambda^1}|_{\lambda^l})w\widetilde{S}_{\lambda^l} + \sum_{\lambda^r \to \lambda^l} \frac{c_0}{c_l} w\widetilde{S}_{\lambda^r}\Big). \tag{5-4}$$

Comparing the coefficient of $w\widetilde{S}_{\lambda^k}$ in (5-3) and (5-4) we get

$$(w\widetilde{S}_{\lambda^1}|_{\lambda^i})wc_{ij}^k + \sum_{\lambda^s \to \lambda^i} \frac{c_0}{c_i} wc_{sj}^k = wc_{ij}^k (w\widetilde{S}_{\lambda^1}|_{\lambda^k}) + \sum_{\lambda^k \to \lambda^t} \frac{c_0}{c_t} wc_{ij}^t. \qquad \square$$

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References

[1] **H Abe**, **T Matsumura**, *Equivariant cohomology of weighted Grassmannians and weighted Schubert classes*, Int. Math. Res. Not. 2015 (2015) 2499–2524 MR Zbl

[2] **A Adem**, **J Leida**, **Y Ruan**, *Orbifolds and stringy topology*, Cambridge Tracts in Math. 171, Cambridge Univ. Press (2007) MR Zbl

[3] **A Bahri**, **M Franz**, **N Ray**, *The equivariant cohomology ring of weighted projective space*, Math. Proc. Cambridge Philos. Soc. 146 (2009) 395–405 MR Zbl

[4] **A Bahri**, **D Notbohm**, **S Sarkar**, **J Song**, *On integral cohomology of certain orbifolds*, Int. Math. Res. Not. 2021 (2021) 4140–4168 MR Zbl

[5] **W L Baily, Jr**, *The decomposition theorem for V–manifolds*, Amer. J. Math. 78 (1956) 862–888 MR Zbl

[6] **W Chen**, **Y Ruan**, *A new cohomology theory of orbifold*, Comm. Math. Phys. 248 (2004) 1–31 MR Zbl

[7] **A Corti**, **M Reid**, *Weighted Grassmannians*, from "Algebraic geometry" (M C Beltrametti, F Catanese, C Ciliberto, A Lanteri, C Pedrini, editors), de Gruyter, Berlin (2002) 141–163 MR Zbl

[8] **A Darby**, **S Kuroki**, **J Song**, *Equivariant cohomology of torus orbifolds*, Canad. J. Math. 74 (2022) 299–328 MR Zbl

[9] **T tom Dieck**, *Bordism of G–manifolds and integrality theorems*, Topology 9 (1970) 345–358 MR Zbl

[10] **F Galaz-García**, **M Kerin**, **M Radeschi**, **M Wiemeler**, *Torus orbifolds, slice-maximal torus actions, and rational ellipticity*, Int. Math. Res. Not. 2018 (2018) 5786–5822 MR Zbl

[11] **R M Goresky**, *Triangulation of stratified objects*, Proc. Amer. Math. Soc. 72 (1978) 193–200 MR Zbl

[12] **V Guillemin**, **C Zara**, *1–skeleta, Betti numbers, and equivariant cohomology*, Duke Math. J. 107 (2001) 283–349 MR Zbl

[13] **B Hanke**, *Geometric versus homotopy theoretic equivariant bordism*, Math. Ann. 332 (2005) 677–696 MR Zbl

[14] **M Harada**, **A Henriques**, **T S Holm**, *Computation of generalized equivariant cohomologies of Kac–Moody flag varieties*, Adv. Math. 197 (2005) 198–221 MR Zbl

[15] **M Harada**, **T S Holm**, **N Ray**, **G Williams**, *The equivariant K–theory and cobordism rings of divisive weighted projective spaces*, Tohoku Math. J. 68 (2016) 487–513 MR Zbl

[16] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR Zbl

[17] **D Husemoller**, *Fibre bundles*, McGraw-Hill, New York (1966) MR Zbl

[18] **X Jiao**, **J Peng**, *Pseudo-holomorphic curves in complex Grassmann manifolds*, Trans. Amer. Math. Soc. 355 (2003) 3715–3726 MR Zbl

[19] **T Kawasaki**, *Cohomology of twisted projective spaces and lens complexes*, Math. Ann. 206 (1973) 243–248 MR Zbl

[20] **A Knutson**, **T Tao**, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. 119 (2003) 221–260 MR Zbl

[21] **D Laksov**, *Algebraic cycles on Grassmann varieties*, Adv. Math. 9 (1972) 267–295 MR Zbl

[22] **J P May**, *Equivariant homotopy and cohomology theory*, CBMS Reg. Conf. Ser. Math. 91, Amer. Math. Soc., Providence, RI (1996) MR Zbl

[23] **J W Milnor**, **J D Stasheff**, *Characteristic classes*, Ann. of Math. Stud. 76, Princeton Univ. Press (1974) MR Zbl

[24] **A Mukherjee**, *Differential topology*, 2nd edition, Hindustan, New Delhi (2015) MR Zbl

[25] **M Poddar**, **S Sarkar**, *On quasitoric orbifolds*, Osaka J. Math. 47 (2010) 1055–1076 MR Zbl

[26] **S Sarkar**, **J Song**, *GKM theory for orbifold stratified spaces and application to singular toric varieties*, Topology Appl. 288 (2021) art. id. 107472 MR Zbl

[27] **I Satake**, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. 42 (1956) 359–363 MR Zbl

[28] **I Satake**, *The Gauss–Bonnet theorem for V–manifolds*, J. Math. Soc. Japan 9 (1957) 464–492 MR Zbl

[29] **G Segal**, *Equivariant K–theory*, Inst. Hautes Études Sci. Publ. Math. 34 (1968) 129–151 MR Zbl

[30] **D P Sinha**, *Computations of complex equivariant bordism rings*, Amer. J. Math. 123 (2001) 577–605 MR Zbl

[31] **W P Thurston**, *The geometry and topology of three-manifolds*, lecture notes, Princeton Univ. (1979) Available at `https://url.msp.org/gt3m`

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MARA BELOTTI, ANTONIO LERARIO and ANDREW NEWMAN

Classical shadows of stated skein representations at roots of unity 2091

JULIEN KORINMAN and ALEXANDRE QUESNEY

Commensurators of thin normal subgroups and abelian quotients 2149

THOMAS KOBERDA and MAHAN MJ

Pushouts of Dwyer maps are $(\infty, 1)$–categorical 2171

PHILIP HACKNEY, VIKTORIYA OZORNOVA, EMILY RIEHL and MARTINA ROVELLI

A variant of a Dwyer–Kan theorem for model categories 2185

BORIS CHORNY and DAVID WHITE

Integral generalized equivariant cohomologies of weighted Grassmann orbifolds 2209

KOUSHIK BRAHMA and SOUMEN SARKAR

Projective modules and the homotopy classification of (G, n)–complexes 2245

JOHN NICHOLSON

Realization of Lie algebras of derivations and moduli spaces of some rational homotopy types 2285

YVES FÉLIX, MARIO FUENTES and ANICETO MURILLO

On the positivity of twisted L^2–torsion for 3–manifolds 2307

JIANRU DUAN

An algebraic C_2–equivariant Bézout theorem 2331

STEVEN R COSTENOBLE, THOMAS HUDSON and SEAN TILSON

Topologically isotopic and smoothly inequivalent 2–spheres in simply connected 4–manifolds whose complement has a prescribed fundamental group 2351

RAFAEL TORRES

Remarks on symplectic circle actions, torsion and loops 2367

MARCELO S ATALLAH

Correction to the article Hopf ring structure on the mod p cohomology of symmetric groups 2385

LORENZO GUERRA