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On the positivity of twisted L^2–torsion for 3–manifolds

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For any compact orientable irreducible 3–manifold N with empty or incompressible toral boundary, the twisted L^2–torsion is a nonnegative function defined on the representation variety $\mathrm{Hom}(\pi_1(N), \mathrm{SL}(n, \mathbb{C}))$. We show that if N has infinite fundamental group, then the L^2–torsion function is strictly positive. Moreover, this torsion function is continuous when restricted to the subvariety of upper triangular representations.

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1 Introduction

Let N be a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. The L^2–torsion of N is a numerical topological invariant of N that equals $\exp(\mathrm{Vol}(N)/6\pi)$, where $\mathrm{Vol}(N)$ is the simplicial volume of N; see [Lück 2002, Theorem 4.3]. The idea of twisting is to use a linear representation of $\pi_1(N)$ to define more L^2–torsion invariants. The first attempt was made by Li and Zhang [2006a; 2006b] in which they defined the L^2–Alexander invariants for knot complements, making use of the one-dimensional representations of the knot group. Later Dubois, Friedl and Lück [Dubois et al. 2015a] introduced the L^2–Alexander torsion for 3–manifolds which recovers the L^2–Alexander invariants. A recent breakthrough was made independently by Liu [2017] and Lück [2018] who proved that the L^2–Alexander torsion is always positive. More interesting properties of the L^2–Alexander torsion are revealed in [Liu 2017; Friedl and Lück 2019]; for example, we now know that the L^2–Alexander torsion is continuous and its limiting behavior recovers the Thurston norm of N.

Generally, let $\mathcal{R}_n(\pi_1(N)) := \mathrm{Hom}(\pi_1(N), \mathrm{SL}(n, \mathbb{C}))$ be the representation variety. One wishes to define L^2–torsion twisted by any representation $\rho \in \mathcal{R}_n(\pi_1(N))$, and we have this *twisted L^2–torsion function* abstractly defined on the representation variety of $\pi_1(N)$:

$$\rho \mapsto \tau^{(2)}(N, \rho) \in [0, +\infty), \quad \rho \in \mathcal{R}_n(\pi_1(N)).$$

A technical obstruction to defining a reasonable L^2–torsion is that the corresponding L^2–chain complex must be weakly L^2–acyclic and of determinant class (see Definition 2.3). If either condition is not satisfied, we define the L^2–torsion to be 0 by convention.

It is natural to question the positivity and continuity of this function. The first result of this paper is the following:

Theorem 1.1 *Let N be a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. Suppose N has infinite fundamental group; then the twisted L^2–torsion $\tau^{(2)}(N,\rho)$ is positive for any group homomorphism $\rho\colon \pi_1(N)\to \mathrm{SL}(n,\mathbb{C})$.*

When N is a graph manifold, the twisted L^2–torsion function is explicitly computed in Theorem 4.1. Other cases are dealt with in Theorem 4.5 where we only need to consider fibered 3–manifolds thanks to the virtual fibering arguments. We carefully construct a CW structure for N as in [Dubois et al. 2015a] and observe that the matrices in the corresponding twisted L^2–chain complex are in a special form so that we can apply Liu's result [2017, Theorem 5.1] to guarantee the positivity of the Fuglede–Kadison determinant.

We have the following partial result regarding continuity of the twisted L^2–torsion function, We say $\rho\in\mathcal{R}_n(\pi_1(N))$ is an upper triangular representation if $\rho(g)$ is an upper triangular matrix for every $g\in\pi_1(N)$.

Theorem 1.2 *Let N be a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. Suppose N has infinite fundamental group. Define $\mathcal{R}_n^{\mathrm{t}}(\pi_1(N))$ to be the subvariety of $\mathcal{R}_n(\pi_1(N))$ consisting of upper triangular representations. Then the twisted L^2–torsion function*

$$\rho\mapsto\tau^{(2)}(N,\rho)$$

is continuous with respect to $\rho\in\mathcal{R}_n^{\mathrm{t}}(\pi_1(N))$.

The continuity of the twisted L^2–torsion function in general is open. It is mainly because the Fuglede–Kadison determinant of an arbitrary matrix over $\mathbb{C}[\pi_1(N)]$ is very difficult to compute. However, the L^2–torsion twisted by upper triangular representations is simpler because we can reduce many problems to the one-dimensional case, which is well studied under the name of the L^2–Alexander torsion (see Section 5). We remark that the work of Bénard and Raimbault [2022] based on the strong acyclicity property by Bergeron and Venkatesh [2013] shows that the twisted L^2–torsion function is positive and real analytic near any holonomy representation $\rho_0\colon\pi_1(N)\to\mathrm{SL}(2,\mathbb{C})$ of a hyperbolic 3–manifold N.

The proof relies on the continuity of L^2–Alexander torsion with respect to the cohomology classes, which is conjectured by [Lück 2018, Chapter 10]. This is done by introducing the concept of Alexander multitwists (see Section 5). One can similarly define the "multivariable L^2–Alexander torsion" and our argument essentially shows that the multivariable function is multiplicatively convex (compare Theorem 5.7), generalizing [Liu 2017, Theorem 5.1]. This then applies to show the continuity as desired.

The organization of this paper is as follows. In Section 2, we introduce the terminology of this paper and some algebraic facts. In Section 3, we define the twisted L^2–torsion for CW complexes and state some basic properties. In Section 4, we prove Theorem 1.1 in two steps: first for graph manifolds, then for hyperbolic or mixed manifolds. In Section 5, we begin with the L^2–Alexander torsion and then prove Theorem 1.2.

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2 Notation and some algebraic facts

In this section we define the twisting functor and introduce L^2–torsion theory. The reader can refer to [Lück 2018] where discussions are taken on in a more general setting.

2.1 Twisting $\mathbb{C}G$–modules via $\mathrm{SL}(n, \mathbb{C})$ representations

Let G be a finitely generated group and let $\mathbb{C}G$ be its group ring. In this paper our main objects are finitely generated free left $\mathbb{C}G$–modules with a preferred ordered basis. We will abbreviate it as *based* $\mathbb{C}G$*–modules* unless otherwise stated. A natural example of a based $\mathbb{C}G$–module is $\mathbb{C}G^m$ as a free left $\mathbb{C}G$–module of rank m, with the natural ordered basis $\{\sigma_1, \dots, \sigma_m\}$ where σ_i is the unit element of the i^{th} direct summand. Any based $\mathbb{C}G$–module is canonically isomorphic to $\mathbb{C}G^m$ for some nonnegative integer m and this identification is used throughout.

We fix V to be an n–dimensional complex vector space with a fixed choice of basis $\{e_i\}_{i=1}^n$. Let $\rho\colon G \to \mathrm{SL}(n, \mathbb{C})$ be a group homomorphism. Then V can be viewed as a left $\mathbb{C}G$–module via ρ,

$$\gamma \cdot e_i = \sum_{j=1}^{n} \rho(\gamma^{-1})_{i,j} \cdot e_j, \quad \gamma \in G,$$

where $\rho(\gamma^{-1}) \in \mathrm{SL}(n, \mathbb{C})$ is a square matrix. We extend this action $\mathbb{C}$–linearly so that V is a left $\mathbb{C}G$–module. In other words, left action of γ corresponds to right multiplication to the row coordinate vector of the matrix $\rho(\gamma^{-1})$.

We are interested in twisting a based $\mathbb{C}G$–module via ρ. In literature, there are two different ways to twist a based $\mathbb{C}G$–module, namely the "diagonal twisting" and the "partial twisting" (compare [Lück 2018]). They are naturally isomorphic. We only consider the diagonal twisting.

Definition 2.1 Recall that $\mathbb{C}G^m$ is a based $\mathbb{C}G$–module with a natural basis $\{\sigma_i\}$ for $i = 1, \dots, m$. We define $(\mathbb{C}G^m \otimes_{\mathbb{C}} V)_{\mathrm{d}}$ to be the $\mathbb{C}G$–module with diagonal $\mathbb{C}G$–action, ie

$$(\mathbb{C}G^m \otimes_{\mathbb{C}} V)_{\mathrm{d}} := \mathbb{C}G^m \otimes_{\mathbb{C}} V, \quad g \cdot (u \otimes v) = gu \otimes gv$$

for any $g \in G$, $u \in \mathbb{C}G^m$ and $v \in V$, and then extend $\mathbb{C}$–linearly to define a $\mathbb{C}G$–module structure.

With the definition above, we can see that

$$(\mathbb{C}G^m \otimes_{\mathbb{C}} V)_{\mathrm{d}} = \bigoplus_{i=1}^{m} (\mathbb{C}G \otimes_{\mathbb{C}} V)_{\mathrm{d}}$$

is a based $\mathbb{C}G$–module with a basis

$$\{\sigma_1 \otimes e_1, \sigma_1 \otimes e_2, \dots, \sigma_1 \otimes e_n, \sigma_2 \otimes e_1, \dots, \sigma_m \otimes e_n\}.$$

Let $\mathscr{A}$ be the category whose objects are finitely generated free left $\mathbb{C}G$–modules with a preferred ordered basis and whose morphisms are $\mathbb{C}G$–linear homomorphisms. We consider the following "diagonal twisting" functor

$$\mathscr{D}(\rho)\colon \mathscr{A} \to \mathscr{A}$$

which sends any object M to the based $\mathbb{C}G$–module $(M \otimes_{\mathbb{C}} V)_{\mathrm{d}}$ and sends any morphism f to

$$\mathscr{D}(\rho) f := f \otimes_{\mathbb{C}} \mathrm{id}_V.$$

The following proposition describes how matrices behave under the twisting functor.

Proposition 2.2 *Let $\rho\colon G \to \mathrm{SL}(n, \mathbb{C})$ be any group homomorphism. Suppose that a homomorphism between based $\mathbb{C}G$–modules*

$$f\colon \mathbb{C}G^r \to \mathbb{C}G^s$$

is presented by a matrix $(\Lambda_{i,j})$ over $\mathbb{C}G$ of size $r \times s$; ie if

$$\{\sigma_1, \dots, \sigma_r\}, \quad \{\tau_1, \dots, \tau_s\}$$

are the natural bases of $\mathbb{C}G^r$ and $\mathbb{C}G^s$, respectively, then

$$f(\sigma_i) = \sum_{j=1}^{s} \Lambda_{i,j} \tau_j, \quad i = 1, \dots, r.$$

We form a new matrix Ω of size $nr \times ns$ by replacing each entry $\Lambda_{i,j}$ with an $n \times n$ square matrix $\Lambda_{i,j} \cdot \rho(\Lambda_{i,j})$. Then Ω is a matrix presenting the diagonal twisting morphism $\mathscr{D}(\rho) f$, under the natural bases

$$\{\sigma_1 \otimes e_1, \dots, \sigma_1 \otimes e_n, \sigma_2 \otimes e_1, \dots, \sigma_r \otimes e_n\},$$

$$\{\tau_1 \otimes e_1, \dots, \tau_1 \otimes e_n, \tau_2 \otimes e_1, \dots, \tau_s \otimes e_n\}$$

of the diagonal twisting based $\mathbb{C}G$–modules $\mathscr{D}(\rho)(\mathbb{C}G^r)$ and $\mathscr{D}(\rho)(\mathbb{C}G^s)$, respectively.

Proof Let $\Phi = (\Phi_{i,j})$ for $i = 1, \dots, r$ and $j = 1, \dots, s$ be a block matrix of size $nr \times ns$, with each entry $\Phi_{i,j}$ an $n \times n$ matrix, such that Φ is the matrix presenting $\mathscr{D}(\rho) f$ under the natural basis. We only need to verify that $\Phi_{i,j} = \Lambda_{i,j} \cdot \rho(\Lambda_{i,j})$. The submatrix $\Phi_{i,j}$ can be characterized as follows. Let $\pi_j\colon \mathscr{D}(\rho)(\mathbb{C}G^r) \to \mathscr{D}(\rho)(\mathbb{C}G)$ be the projection to the j^{th} direct component which is spanned by $\{(\sigma_j \otimes e_1)_{\mathrm{d}}, \dots, (\sigma_j \otimes e_n)_{\mathrm{d}}\}$. Then

$$\pi_j \circ \mathscr{D}(\rho) f \begin{pmatrix} (\sigma_i \otimes e_1)_{\mathrm{d}} \\ \vdots \\ (\sigma_i \otimes e_n)_{\mathrm{d}} \end{pmatrix} = \Phi_{i,j} \begin{pmatrix} (\tau_j \otimes e_1)_{\mathrm{d}} \\ \vdots \\ (\tau_j \otimes e_n)_{\mathrm{d}} \end{pmatrix}.$$

On the other hand, for any $k = 1, \dots, n$,

$$\begin{aligned} \pi_j \circ \mathscr{D}(\rho) f((\sigma_i \otimes e_k)_{\mathrm{d}}) &= \pi_j \biggl(\sum_{l=1}^{s} (\Lambda_{i,l} \tau_l \otimes e_k)_{\mathrm{d}} \biggr) = \pi_j \biggl(\sum_{l=1}^{s} \Lambda_{i,l} \cdot (\tau_l \otimes \Lambda_{i,l}^{-1} e_k)_{\mathrm{d}} \biggr) \\ &= \Lambda_{i,j} \cdot (\tau_j \otimes \Lambda_{i,j}^{-1} e_k)_{\mathrm{d}} = \Lambda_{i,j} \cdot \sum_{l=1}^{n} \rho(\Lambda_{i,j})_{k,l} (\tau_j \otimes e_l)_{\mathrm{d}}. \end{aligned}$$

This shows that $\Phi_{i,j} = \Lambda_{i,j} \cdot \rho(\Lambda_{i,j})$, and hence $\Phi = \Omega$. □

We now mention that the twisting functor can be naturally generalized to the category of *based $\mathbb{C}G$–chain complexes*. More explicitly, let C_* be a based $\mathbb{C}G$–chain complex, ie

$$C_* = (\cdots \to C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \to \cdots)$$

is a chain of based $\mathbb{C}G$–modules with $\mathbb{C}G$–linear connecting morphisms $\{\partial_p\}$ such that $\partial_{p-1} \circ \partial_p = 0$. We can apply the functor $\mathfrak{D}(\rho)$ to obtain a new $\mathbb{C}G$–chain complex

$$\mathfrak{D}(\rho)C_* = (\cdots \to \mathfrak{D}(\rho)C_{p+1} \xrightarrow{\mathfrak{D}(\rho)\partial_{p+1}} \mathfrak{D}(\rho)C_p \xrightarrow{\mathfrak{D}(\rho)\partial_p} \mathfrak{D}(\rho)C_{p-1} \to \cdots)$$

with connecting homomorphisms $\{\mathfrak{D}(\rho)\partial_p\}$. If f_* is a chain map between based $\mathbb{C}G$–chain complexes, the twisting chain map $\mathfrak{D}(\rho) f_*$ is a $\mathbb{C}G$–chain map between the corresponding twisted chain complexes. So $\mathfrak{D}(\rho)$ generalizes to be a functor of the category of based $\mathbb{C}G$–chain complexes.

2.2 L^2–torsion theory

Let

$$l^2(G) = \left\{ \sum_{g\in G} c_g \cdot g \;\middle|\; c_g \in \mathbb{C}, \sum_{g\in G} |c_g|^2 < \infty \right\}$$

be the Hilbert space orthonormally spanned by all elements in G. Since G is finitely generated, $l^2(G)$ is a separable Hilbert space with isometric left and right $\mathbb{C}G$–module structure. We denote by $\mathcal{N}(G)$ the *group von Neumann algebra* of G which consists of all bounded Hilbert operators of $l^2(G)$ that commute with the right $\mathbb{C}G$–action. We will treat $l^2(G)$ as a left $\mathcal{N}(G)$–module and a right $\mathbb{C}G$–module. The *l^2–completion* of a based $\mathbb{C}G$–chain complex C_* is then a *Hilbert $\mathcal{N}(G)$–chain complex* defined as

$$l^2(G) \otimes_{\mathbb{C}G} C_*,$$

and the l^2–completions of the connecting homomorphism ∂ and chain map f are $\mathrm{id} \otimes_{\mathbb{C}G} \partial$ and $\mathrm{id} \otimes_{\mathbb{C}G} f$, respectively. Note that each chain module of $l^2(G) \otimes_{\mathbb{C}G} C_*$ is simply a direct sum of $l^2(G)$,

$$l^2(G) \otimes_{\mathbb{C}G} C_p = l^2(G) \otimes_{\mathbb{C}G} \mathbb{C}G^{r_p} = l^2(G)^{r_p},$$

where r_p is the rank of C_p.

The l^2–completion process converts a based $\mathbb{C}G$–chain complex into a finitely generated, free Hilbert $\mathcal{N}(G)$–chain complex.

Definition 2.3 A finitely generated, free Hilbert $\mathcal{N}(G)$–chain complex is called *weakly acyclic* if the l^2–Betti numbers are all trivial. A finitely generated, free Hilbert $\mathcal{N}(G)$–chain complex is *of determinant class* if all the Fuglede–Kadison determinants of the connecting homomorphisms are positive real numbers.

Definition 2.4 Let C_* be a finitely generated, free Hilbert $\mathcal{N}(G)$–chain complex. Suppose C_* is of finite length, ie there exists an integer $N > 0$ such that $C_p = 0$ for $|p| > N$. Furthermore, if C_* is weakly acyclic and of determinant class, we define the L^2*–torsion* of C_* to be the alternating product of the Fuglede–Kadison determinants of the connecting homomorphisms:

$$\tau^{(2)}(C_*) = \prod_{p\in\mathbb{Z}} (\det_{\mathcal{N}(G)} \partial_p)^{(-1)^p}.$$

Otherwise, we artificially set $\tau^{(2)}(C_*) = 0$.

We recommend [Lück 2002] for the definition of the L^2–Betti number and the Fuglede–Kadison determinant. We remark that our notational convention follows [Dubois et al. 2015a; 2015b; Liu 2017], and the exponential of the torsion in [Lück 2002; 2018] is the multiplicative inverse of our torsion.

Let A be a $p \times p$ matrix over $\mathcal{N}(G)$. The *regular Fuglede–Kadison determinant* of A is defined to be

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(A) = \begin{cases} \det_{\mathcal{N}(G)}(A) & \text{if } A \text{ is full rank of determinant class,} \\ 0 & \text{otherwise.} \end{cases}$$

We will need the following two lemmas in order to do explicit calculations; the proof can be found in [Dubois et al. 2015b, Lemmas 2.6 and 3.2] combining with the basic properties of the Fuglede–Kadison determinant (see [Lück 2002, Theorem 3.14]).

Lemma 2.5 *Let $\mathbb{Z}^k$ be a free abelian subgroup of G generated by $z_1, \dots, z_k$. Let A be a $p \times p$ matrix over $\mathbb{C}\mathbb{Z}^k$. Identify $\mathbb{C}\mathbb{Z}^k$ with the k–variable Laurent polynomial ring $\mathbb{C}[z_1^{\pm}, \dots, z_k^{\pm}]$, and denote by $p(z_1, \dots, z_k)$ the ordinary determinant of A. Then*

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(A) = \mathrm{Mah}(p(z_1, \dots, z_k)),$$

where $\mathrm{Mah}(p(z_1, \dots, z_k))$ is the Mahler measure of the polynomial $p(z_1, \dots, z_k)$.

Lemma 2.6 *Let*

$$D_* = (0 \to \mathbb{C}G^j \xrightarrow{C} \mathbb{C}G^k \xrightarrow{B} \mathbb{C}G^{k+l-j} \xrightarrow{A} \mathbb{C}G^l \to 0)$$

be a complex, $L \subset \{1, \dots, k+l-j\}$ be a subset of size l and $J \subset \{1, \dots, k\}$ a subset of size j. Define

- *$A(J)$ to be the rows in A corresponding to J;*
- *$B(J, L)$ to be the result of deleting the columns of B corresponding to J and deleting the rows corresponding to L;*
- *$C(L)$ to be the columns of C corresponding to L.*

View A, B and C as matrices over $\mathcal{N}(G)$. If $\det^{\mathrm{r}}_{\mathcal{N}(G)}(A(J)) \neq 0$ and $\det^{\mathrm{r}}_{\mathcal{N}(G)}(C(L)) \neq 0$, then

$$\tau^{(2)}(l^2(G) \otimes_{\mathbb{C}G} D_*) = \det^{\mathrm{r}}_{\mathcal{N}(G)}(B(J, L)) \cdot \det^{\mathrm{r}}_{\mathcal{N}(G)}(A(J))^{-1} \cdot \det^{\mathrm{r}}_{\mathcal{N}(G)}(C(L))^{-1}.$$

3 Twisted L^2–torsion for CW complexes

Let X be a finite CW complex with fundamental group G. Denote by $\widehat{X}$ the universal cover of $|X|$ with the natural CW complex structure coming from X. Choose a lifting $\hat{\sigma}_i$ for each cell σ_i in the CW structure of X. The deck group G acts freely on the cellular chain complex of $\widehat{X}$ on the left, which makes the $\mathbb{C}$–coefficient cellular chain complex $C_*(\widehat{X})$ a based $\mathbb{C}G$–chain complex with basis $\{\hat{\sigma}_i\}$. Recall that $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ is any group homomorphism.

For future convenience, we introduce the concept of *admissible triple* for higher-dimensional linear representations, generalizing the admissibility condition in [Dubois et al. 2015b].

Definition 3.1 (admissible triple) Let $\gamma\colon G \to H$ be a homomorphism to a countable group H. We say that $(G,\rho;\gamma)$ forms an *admissible triple* if $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ factors through γ, ie for some homomorphism $\psi\colon H \to \mathrm{SL}(n,\mathbb{C})$, the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[dr, "\rho"'] & H \arrow[d, "\psi"] \\ & \mathrm{SL}(n,\mathbb{C}) \end{tikzcd}$$

Definition 3.2 Let $(G,\rho;\gamma)$ be an admissible triple. Consider $l^2(H)$ as a left Hilbert $\mathcal{N}(H)$–module, and a right $\mathbb{C}G$–module induced by γ. Define the L^2–chain complex of X twisted by $(G,\rho;\gamma)$ to be the Hilbert $\mathcal{N}(H)$–chain complex

$$C_*^{(2)}(X,\rho;\gamma) := l^2(H) \otimes_{\mathbb{C}G} \mathfrak{D}(\rho) C_*(\widehat{X}).$$

We define the L^2–torsion of X twisted by $(G,\rho;\gamma)$ as

$$\tau^{(2)}(X,\rho;\gamma) := \tau^{(2)}(C_*^{(2)}(X,\rho;\gamma)).$$

Proposition 3.3 *The definition of $\tau^{(2)}(X,\rho;\gamma)$ with respect to any admissible triple $(G,\rho;\gamma)$ does not depend on the order or orientation of the basis $\{\sigma_i\}$, nor the choice of lifting $\{\hat{\sigma}_i\}$. Moreover, let $\rho'\colon G \to \mathrm{SL}(n,\mathbb{C})$ be conjugate to ρ, ie there exists a matrix $T \in \mathrm{SL}(n,\mathbb{C})$, such that $\rho' = T \cdot \rho \cdot T^{-1}$. Then $(G,\rho';\gamma)$ is also an admissible triple and $\tau^{(2)}(X,\rho;\gamma) = \tau^{(2)}(X,\rho';\gamma)$.*

Proof The property of being weakly L^2–acyclic does not depend on the choices in the statement. We only need to analyze how these choices change the Fuglede–Kadison determinant of the connecting morphisms.

Abbreviate by $C_*(\widehat{X},\rho) := \mathfrak{D}(\rho)C_*(\widehat{X};\mathbb{C})$ the diagonal twisting chain complex. Suppose the based cellular chain complex of $\widehat{X}$ has the form

$$C_*(\widehat{X}) = (\cdots \to \mathbb{C}G^{r_{i+1}} \xrightarrow{\partial_{i+1}} \mathbb{C}G^{r_i} \xrightarrow{\partial_i} \mathbb{C}G^{r_{i-1}} \to \cdots),$$

where ∂_i is an $r_i \times r_{i-1}$ matrix over $\mathbb{C}G$ for all i. Then the diagonal twisting chain complex $C_*(\widehat{X},\rho)$ has the form

$$C_*(\widehat{X},\rho) = (\cdots \to \mathbb{C}G^{nr_{i+1}} \xrightarrow{\partial^{\rho}_{i+1}} \mathbb{C}G^{nr_i} \xrightarrow{\partial^{\rho}_i} \mathbb{C}G^{nr_{i-1}} \to \cdots),$$

where $\partial_i^\rho = \mathfrak{D}(\rho)\partial_i$ is an $nr_i \times nr_{i-1}$ matrix over $\mathbb{C}G$ for all i. An explicit formula for ∂_i^ρ is presented in Proposition 2.2. Then the L^2–chain complex of X twisted by $(G,\rho;\gamma)$ has the form

$$C_*^{(2)}(X,\rho;\gamma) = \left(\cdots \to l^2(H)^{nr_{i+1}} \xrightarrow{\gamma(\partial_{i+1}^\rho)} l^2(H)^{nr_i} \xrightarrow{\gamma(\partial_i^\rho)} l^2(H)^{nr_{i-1}} \to \cdots\right),$$

where $\gamma(\partial_i^\rho)$ means applying the group homomorphism γ to each monomial of any entry of the matrix ∂_i^ρ, resulting in a matrix over $\mathbb{C}H \subset \mathcal{N}(H)$.

We now analyze how the choices affect the value of $\tau^{(2)}(X,\rho;\gamma)$. If the basis of $C_i(X)$ is permuted, and the orientations are changed, then $\gamma(\partial_i^\rho)$ and $\gamma(\partial_{i+1}^\rho)$ change by multiplying a permutation matrix, with entries ± 1.

If one choose another lifting $g\hat{\sigma}$ instead of $\hat{\sigma}$ for some $g \in G$, then $\gamma(\partial_i^\rho)$ and $\gamma(\partial_{i+1}^\rho)$ change by multiplying a block matrix of the form

$$\begin{pmatrix} I^{n\times n} & & & & \\ & \ddots & & & \\ & & \rho(g)^{\pm 1}\cdot I^{n\times n} & & \\ & & & \ddots & \\ & & & & I^{n\times n} \end{pmatrix}.$$

If one replaces ρ by $\rho' = T\cdot\rho\cdot T^{-1}$ for a matrix $T \in \mathrm{SL}(n,\mathbb{C})$, the corresponding connecting homomorphism is of the form

$$\gamma(\partial_i^{\rho'}) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \gamma(\partial_i^\rho) \begin{pmatrix} T^{-1} & & \\ & \ddots & \\ & & T^{-1} \end{pmatrix}.$$

In all cases, the regular Fuglede–Kadison determinant of $\gamma(\partial_i^\rho)$ and $\gamma(\partial_{i+1}^\rho)$ are unchanged by basic properties of Fuglede–Kadison determinant; see [Lück 2002, Theorem 3.14]. □

Note that the "moreover" part of the previous lemma tells us that we don't need to worry about the different choices of the base point when identifying the fundamental group $\pi_1(X)$ with G.

Lemma 3.4 *Let T be a two-dimensional torus. For any admissible triple*

$$(T, \rho\colon \pi_1(T) \to \mathrm{SL}(n,\mathbb{C}); \gamma\colon \pi_1(T) \to H),$$

if $\operatorname{im}\gamma$ *is infinite, then*

$$\tau^{(2)}(T,\rho;\gamma) = 1.$$

Proof We consider the standard CW structure for T constructed by identifying pairs of sides of a square. Let P be the 0–cell, let E_1 and E_2 be the 1–cells, and let

$$e_1 = [E_1] \in \pi_1(T), \quad e_2 = [E_2] \in \pi_1(T).$$

Then $\pi_1(T)$ is the free abelian group generated by e_1 and e_2. There is a 2–cell σ whose boundary is the loop $E_1E_2E_1^{-1}E_2^{-1}$. Let $\widehat{T}$ be the universal covering of T with the induced CW structure. It is easy to see that the L^2–chain complex of T twisted by $(\pi_1(T),\rho;\gamma)$ is

$$C_*^{(2)}(T,\rho;\gamma) = (0 \to l^2(H)\langle\sigma\rangle \otimes_{\mathbb{C}} V \xrightarrow{\gamma(\partial_2^\rho)} l^2(H)\langle E_1,E_2\rangle \otimes_{\mathbb{C}} V \xrightarrow{\gamma(\partial_1^\rho)} l^2(H)\langle P\rangle \otimes_{\mathbb{C}} V \to 0)$$

in which

$$\gamma(\partial_2^\rho) = \begin{pmatrix} I^{n\times n} - \gamma(e_2)\rho(e_2) & -I^{n\times n} + \gamma(e_1)\rho(e_1) \end{pmatrix}, \quad \gamma(\partial_1^\rho) = \begin{pmatrix} \gamma(e_1)\rho(e_1) - I^{n\times n} \\ \gamma(e_2)\rho(e_2) - I^{n\times n} \end{pmatrix}.$$

We assume without loss of generality that $\gamma(e_1)$ has infinite order. Set $p(z) := \det(z\rho(e_1) - I^{n\times n})$ as a polynomial of indeterminant z. Then by Lemma 2.5,

$$\det{}^{\mathrm{r}}_{\mathcal{N}(H)}(\gamma(e_1)\rho(e_1) - I^{n\times n}) = \mathrm{Mah}(p(z)) \neq 0.$$

The conclusion follows from [Dubois et al. 2015b, Lemma 3.1] which is a formula analogous to Lemma 2.6 but applies to shorter chain complexes. □

There is another way to define twisted L^2–torsion, following Lück [2018]. Let H be a finitely generated group. Recall that $\widetilde{X}$ is called a *finite free H–CW complex* if $\widetilde{X}$ is a regular covering space of a finite CW complex X, with deck transformation group H acting on $\widetilde{X}$ on the left. Choose an H–equivariant CW structure for $\widetilde{X}$, and choose one representative cell for each H–orbit. Then the cellular chain complex $C_*(\widetilde{X})$ becomes a based $\mathbb{C}H$–chain complex. For any group homomorphism $\phi\colon H \to \mathrm{SL}(n,\mathbb{C})$, we form the diagonal twisting chain complex $\mathfrak{D}(\phi)C_*(\widetilde{X})$ (recall the definition of the twisting functor $\mathfrak{D}$ in Section 2). The *ϕ–twisted L^2–torsion* of the H–CW complex $\widetilde{X}$ is defined to be

$$\rho_H^{(2)}(\widetilde{X},\phi) := \log \tau^{(2)}(l^2(H) \otimes_{\mathbb{C}H} \mathfrak{D}(\phi)C_*(\widetilde{X})).$$

Note that ϕ is a unimodular representation in our setting; this torsion does not depend on a specific $\mathbb{C}H$–basis for $C_*(\widetilde{X})$ (compare Proposition 3.3). We point out in the following proposition that both definitions of twisted L^2–torsion are essentially the same.

Proposition 3.5 *Following the notation above, let G be the fundamental group of $X = H\backslash\widetilde{X}$. There is a natural quotient map $\gamma\colon G \to H$ by covering space theory, and it is obvious that $(G,\phi\circ\gamma;\gamma)$ is an admissible triple. Then*

$$\tau^{(2)}(X,\phi\circ\gamma;\gamma) = \exp \rho_H^{(2)}(\widetilde{X},\phi).$$

Proof Let $\widehat{X}$ be the universal covering space of X, with the natural CW structure coming from X. Choose a lifting for each cell in X and then $C_*(\widehat{X})$ becomes a based $\mathbb{C}G$–chain complex. It is a pure algebraic fact that the two based $\mathbb{C}H$–chain complexes are $\mathbb{C}H$–isomorphic:

$$(*) \qquad \mathfrak{D}(\phi)C_*(\widetilde{X}) \cong \mathbb{C}H \otimes_{\mathbb{C}G} \mathfrak{D}(\phi\circ\gamma)C_*(\widehat{X}).$$

Indeed, the $\mathbb{C}H$–chain complex $\mathbb{C}H \otimes_{\mathbb{C}G} \mathfrak{D}(\phi\circ\gamma)C_*(\widehat{X})$ is obtained from

$$C_*(\widehat{X}) = (\cdots \to \mathbb{C}G^{r_{i+1}} \xrightarrow{\partial_{i+1}} \mathbb{C}G^{r_i} \xrightarrow{\partial_i} \mathbb{C}G^{r_{i-1}} \to \cdots)$$

by the following two operations:

(1) **The diagonal twist** First, replace every direct summand $\mathbb{C}G$ by its n^{th} power $\mathbb{C}G^n$, and replace any entry $\Lambda_{i,j}$ of the matrix ∂_* by a block matrix $\Lambda_{i,j}\phi\circ\gamma(\Lambda_{i,j})$, as in Proposition 2.2, resulting in a new matrix $\partial_*^{\phi\circ\gamma}$.

(2) **Tensoring with $\mathbb{C}H$** Then replace every direct summand $\mathbb{C}G$ of the chain module by $\mathbb{C}H$, and apply γ to every entry of $\partial_*^{\phi\circ\gamma}$, resulting in a block matrix whose i,j–submatrix is $\gamma(\Lambda_{i,j})\phi\circ\gamma(\Lambda_{i,j})$.

The resulting chain complex is exactly the chain complex $\mathfrak{D}(\phi)(\mathbb{C}H \otimes_{\mathbb{C}G} C_*(\widehat{X}))$ (this can be seen by doing the above operations in the reversed order, thanks to the admissible condition). Combined with the well-known $\mathbb{C}H$–isomorphism $C_*(\widetilde{X}) \cong \mathbb{C}H \otimes_{\mathbb{C}G} C_*(\widehat{X})$, the isomorphism $(*)$ follows.

Finally, we tensor $l^2(H)$ on the left of both $\mathbb{C}H$–chain complexes and the conclusion follows from taking L^2–torsion of each. □

The following useful properties are obtained by translating the statements of [Lück 2018, Theorem 6.7] into our terminology.

Lemma 3.6 *Some basic properties of twisted L^2–torsions*:

(1) ***G–homotopy equivalence*** *Let X and Y be two finite CW complexes with fundamental group G. For any admissible triple $(G,\rho;\gamma)$, suppose there is a simple homotopy equivalence $f\colon X\to Y$ such that the induced homomorphism $f_*\colon G\to G$ preserves $\ker\gamma$. Then*

$$\tau^{(2)}(X,\rho;\gamma) = \tau^{(2)}(Y,\rho;\gamma).$$

(2) ***Restriction*** *Let X be a finite CW complex with fundamental group G. Let $\widetilde{X}$ be a finite regular cover of X with the induced CW structure. Suppose $\pi_1(\widetilde{X}) = \widetilde{G} \lhd G$ is a normal subgroup of index d. Let $\tilde{\rho}\colon \widetilde{G}\to \mathrm{SL}(n,\mathbb{C})$ be the restriction of $\rho\colon G\to\mathrm{SL}(n,\mathbb{C})$. Then*

$$\tau^{(2)}(\widetilde{X},\tilde{\rho}) = \tau^{(2)}(X,\rho)^d.$$

(3) ***Sum formula*** *Let X be a finite CW complex with fundamental group G and $\rho\colon G\to\mathrm{SL}(n,\mathbb{C})$ be a homomorphism. Let*

$$i_1\colon X_1 \hookrightarrow X, \quad i_2\colon X_2\hookrightarrow X, \quad i_0\colon X_1\cap X_2 \hookrightarrow X$$

be subcomplex of X with $X_1\cup X_2 = X$. Let

$$\rho_1 = \rho|_{\pi_1(X_1)}, \quad \rho_2 = \rho|_{\pi_1(X_2)}, \quad \rho_0 = \rho|_{\pi_1(X_1\cap X_2)}$$

be the restriction of ρ. If $\tau^{(2)}(X_1\cap X_2,\rho_0;i_{0*})\neq 0$, then

$$\tau^{(2)}(X,\rho) = \tau^{(2)}(X_1,\rho_1;i_{1*})\cdot\tau^{(2)}(X_2,\rho_2;i_{2*})/\tau^{(2)}(X_1\cap X_2,\rho_0;i_{0*}).$$

4 Twisted L^2–torsion for 3–manifolds

For the remainder of this paper, we will assume that N is a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. We denote by G the fundamental group of N and assume G is infinite. It is well known that G is finitely generated and residually finite [Hempel 1987]. For any group homomorphism $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ and $\gamma\colon G \to H$, we say $(N,\rho;\gamma)$ is an admissible triple if $(G,\rho;\gamma)$ is. In this case, we define the *twisted L^2–torsion of* $(N,\rho;\gamma)$ by

$$\tau^{(2)}(N,\rho;\gamma) := \tau^{(2)}(X,\rho;\gamma),$$

where X is any CW structure for N. This definition does not depend on the choice of X, thanks to Lemma 3.6. Indeed, if X and Y are two CW structures for N, and $f\colon X \to Y$ is the corresponding homeomorphism, then f is a simple homotopy equivalence by Chapman [1974, Theorem 1] and certainly preserves $\ker\gamma$. So $\tau^{(2)}(X,\rho;\gamma) = \tau^{(2)}(Y,\rho;\gamma)$.

The remaining part of this section is devoted to the proof of Theorem 1.1.

4.1 Twisted L^2–torsion for graph manifolds

We prove Theorem 1.1 for a graph manifold N with infinite fundamental group G.

Theorem 4.1 *Suppose M is a Seifert-fibered piece of the graph manifold N. Let $h \in \pi_1(M)$ be represented by the regular fiber of M. Consider the product of all eigenvalues (with multiplicity) of $\rho(h)$ whose modulus is not greater than 1, and denote by Λ the modulus of this product. Suppose the orbit space M/S^1 has orbifold Euler characteristic χ_{orb}. Then*

$$\tau^{(2)}(N,\rho) = \prod_{M\subset N \text{ is a Seifert piece}} \Lambda^{\chi_{\mathrm{orb}}}.$$

The proof is a direct generalization of [Bénard and Raimbault 2022, Proposition 4.3], though the technique in both proofs essentially goes back to T Kitano [1994], where he computed the $\mathrm{SL}(2,\mathbb{C})$–twisted Reidemeister torsion of graph manifolds.

Proof Fix any Seifert-fibered piece M of the JSJ decomposition of N. Then $\pi_1(M)$ is infinite as well. Suppose that M is isomorphic to a model

$$M(g,b;q_1/p_1,\dots,q_k/p_k), \quad k \geqslant 1,\ p_1,\dots,p_k > 0,$$

following Hatcher [2007]. More explicitly, take a surface of genus g with b boundary components, namely $E_1,\dots,E_b$, then drill out k–disjoint disks from it to form a new surface Σ with k additional boundary circles $F_1,\dots,F_k$. These k boundary circles correspond to k boundary tori of $\Sigma\times S^1$, namely $T_1,\dots,T_k$. Then M is obtained by a Dehn filling of slope $(q_1/p_1,\dots,q_k/p_k)$ along $(T_1,\dots,T_k)$, respectively. So

$$M = (\Sigma\times S^1)\cup_{T_1} D_1 \cup_{T_2} \cdots \cup_{T_k} D_k,$$

in which D_i is a solid torus whose meridian $(0,1)$–curve is attached to the (q_i, p_i)–curve of T_i. The orbit space can be viewed as a 2–dimensional orbifold, whose underlying topological space is a surface $\Sigma_{g,b}$ with k singularities of indices $p_1, \dots, p_k$, respectively. The orbifold Euler characteristic is

$$\chi_{\mathrm{orb}} = 2 - 2g - b - \sum_{i=1}^{k}\Bigl(1 - \frac{1}{p_i}\Bigr).$$

More details can be found in [Scott 1983].

Retract Σ along the boundary circle F_k to an 1–dimensional complex X; it is a bunch of circles with one common vertex P, and edges

$$A_1, B_1, \dots, A_g, B_g, E_1, \dots, E_b, F_1, \dots, F_{k-1}$$

where $A_1, B_1, \dots, A_g, B_g$ come from the standard polygon representation of a closed surface Σ_g. Suppose that A_i, B_i, E_i and F_i represent a_i, b_i, e_i and f_i, respectively, in $\pi_1(M)$. Let H be the 1–cell of S^1 representing $h \in \pi_1(M)$. Then $\Sigma \times S^1$ is given the product CW structure, with the cells in each dimension being

$$\{A_1 \times H, B_1 \times H, \dots, A_g \times H, B_g \times H, E_1 \times H, \dots, E_b \times H, F_1 \times H, \dots, F_{k-1} \times H\},$$
$$\{A_1, B_1, \dots, A_g, B_g, E_1, \dots, E_b, F_1, \dots, F_{k-1}, H\}, \quad \{P\}.$$

We have $f_i^{p_i} h^{q_i} = 1$ for $i = 1, \dots, k-1$ by the Dehn filling.

Denote by

$$\kappa\colon \Sigma \times S^1 \hookrightarrow N, \quad \iota_i\colon T_i \hookrightarrow N, \quad \zeta_i\colon D_i \hookrightarrow N, \quad i = 1, \dots, k,$$

the inclusion maps to the ambient manifold N. Our strategy is as follows: cut N along all JSJ tori and all tori $\{T_1, \dots, T_k\}$ that appear in each Seifert piece of the JSJ decomposition of N as above. By Lemma 3.4, the JSJ tori do not contribute to the L^2–torsion. Then, by the sum formula of Lemma 3.6,

$$\tau^{(2)}(N,\rho) = \prod_{M \subset N \text{ is a Seifert piece}} \frac{\tau^{(2)}(\Sigma \times S^1, \rho \circ \kappa_*; \kappa_*) \prod_{i=1}^{k} \tau^{(2)}(D_i, \rho \circ \zeta_{i*}; \zeta_{i*})}{\prod_{i=1}^{k} \tau^{(2)}(T_i, \rho \circ \iota_{i*}; \iota_{i*})}. \tag{1}$$

It remains to calculate the terms appearing in (1).

First, the easiest part. Since $\iota_{i*}(\pi_1(T_i))$ has infinite order in G, the twisted L^2–torsion of the admissible triple $(T_i, \rho \circ \iota_{i*}; \iota_{i*})$ is trivially 1 by Lemma 3.4.

We now compute $\tau^{(2)}(\Sigma \times S^1, \rho \circ \kappa_*; \kappa_*)$. Set $\pi := \pi_1(\Sigma \times S^1)$. The CW chain complex of the universal cover $\widehat{\Sigma \times S^1}$ is

$$C_*(\widehat{\Sigma \times S^1}) = (0 \to \mathbb{C}\pi^{2g+b+k-1} \xrightarrow{\partial_2} \mathbb{C}\pi^{2g+b+k} \xrightarrow{\partial_1} \mathbb{C}\pi \xrightarrow{\partial_0} 0)$$

in which

$$\partial_2 = \begin{pmatrix} 1-h & 0 & \cdots & 0 & * \\ 0 & 1-h & & \vdots & \vdots \\ \vdots & & \ddots & 0 & * \\ 0 & \cdots & 0 & 1-h & * \end{pmatrix}, \quad \partial_1 = \begin{pmatrix} * \\ \vdots \\ * \\ 1-h \end{pmatrix}.$$

Then the L^2–chain complex of $\Sigma\times S^1$ twisted by $(\pi,\rho\circ\kappa_*;\kappa_*)$ is

$$C_*^{(2)}(\Sigma\times S^1,\rho\circ\kappa_*;\kappa_*)=(0\to l^2(G)^{2g+b+k-1}\xrightarrow{\partial_2^\rho} l^2(G)^{2g+b+k}\xrightarrow{\partial_1^\rho} l^2(G)\to 0)$$

in which

$$\partial_2^\rho=\begin{pmatrix} I^{n\times n}-h\rho(h) & 0 & \cdots & 0 & * \\ 0 & I^{n\times n}-h\rho(h) & & \vdots & \vdots \\ \vdots & & \ddots & 0 & * \\ 0 & \cdots & 0 & I^{n\times n}-h\rho(h) & * \end{pmatrix},\quad \partial_1^\rho=\begin{pmatrix} * \\ \vdots \\ * \\ I^{n\times n}-h\rho(h)\end{pmatrix}.$$

We have identified h with its image under κ_* in $\pi_1(N)=G$ for notational convenience. If the modulus of all eigenvalues of $\rho(h)$ are $\lambda_1,\dots,\lambda_n$, by properties of the regular Fuglede–Kadison determinant and Lemmas 2.5 and 2.6, we know that

$$\begin{aligned}\tau^{(2)}(\Sigma\times S^1,\rho\circ\kappa_*;\kappa_*)&=\det{}^{\mathrm{r}}_{\mathcal{N}(G)}(I^{n\times n}-h\rho(h))^{2g+b+k-2}\\ &=\mathrm{Mah}\Big(\prod_{r=1}^{n}(1-z\lambda_r)\Big)^{2g+b+k-2}=\Lambda^{-(2g+b+k-2)}.\end{aligned}$$

Then we compute $\tau^{(2)}(D_i,\rho\circ\zeta_{i*};\zeta_{i*})$. It is easy to see that the generator of $\pi_1(D_i)$ is represented by $h^{m_i}f_i^{n_i}$, where (m_i,n_i) is a pair of integers such that $m_ip_i-n_iq_i=1$. Then

$$\tau^{(2)}(D_i,\rho\circ\zeta_{i*};\zeta_{i*})=\det{}^{\mathrm{r}}_{\mathcal{N}(G)}(I^{n\times n}-h^{m_i}f_i^{n_i}\cdot\rho(h^{m_i}f_i^{n_i}))^{-1},$$

where h and f_i are again viewed as elements in G. Since h and f_i commute and are simultaneously upper triangularizable, the modulus of all eigenvalues of $\rho(h^{m_i}f_i^{n_i})$ are $\lambda_1^{1/p_i},\dots,\lambda_n^{1/p_i}$. Note that $h^{m_i}f_i^{n_i}$ is an infinite order element. By Lemma 2.5,

$$\det{}^{\mathrm{r}}_{\mathcal{N}(G)}(I^{n\times n}-h^{m_i}f_i^{n_i}\cdot\rho(h^{m_i}f_i^{n_i}))=\mathrm{Mah}\Big(\prod_{r=1}^{n}(1-z\lambda_r^{1/p_i})\Big)=\Lambda^{-1/p_i},$$

and then $\tau^{(2)}(D_i,\rho\circ\zeta_{i*};\zeta_{i*})=\Lambda^{1/p_i}$.

Finally, combining the calculations above,

$$\begin{aligned}\frac{\tau^{(2)}(\Sigma\times S^1,\rho\circ\kappa_*;\kappa_*)\prod_{i=1}^{k}\tau^{(2)}(D_i,\rho\circ\zeta_{i*};\zeta_{i*})}{\prod_{i=1}^{k}\tau^{(2)}(T_i,\rho\circ\iota_{i*};\iota_{i*})}&=\Lambda^{-(2g+b+k-2)+\sum_{i=1}^{k}1/p_i}\\ &=\Lambda^{2-2g-b-\sum_{i=1}^{k}(1-1/p_i)}=\Lambda^{\chi_{\mathrm{orb}}},\end{aligned}$$

and the conclusion follows from (1). □

4.2 Twisted L^2–torsion for hyperbolic or mixed manifolds

In this part, we assume that N is not a graph manifold, or equivalently, N contains at least one hyperbolic piece in its geometrization decomposition. Then N is either hyperbolic or so-called mixed. By Agol's

RFRS criterion [2008] for virtual fibering and the virtual specialness of 3–manifolds having at least one hyperbolic piece [Agol 2013; Przytycki and Wise 2018], we can assume that N has a regular finite cover that fibers over the circle.

For future convenience, we introduce the following notions.

Definition 4.2 Let G be a finitely generated, residually finite group. For any cohomology class $\psi \in H^1(G;\mathbb{R})$, and any real number $t > 0$, there is an 1–dimensional representation

$$\psi_t : G \to \mathbb{C}^\times, \quad g \mapsto t^{\psi(g)}.$$

This representation can be used to twist $\mathbb{C}G$, determining a $\mathbb{C}G$–homomorphism

$$\kappa(\psi,t) : \mathbb{C}G \to \mathbb{C}G, \qquad g \mapsto t^{\psi(g)} g, \quad g \in G,$$

and extend $\mathbb{C}$–linearly. The $\mathbb{C}G$–homomorphism $\kappa(\psi,t)$ is called the *Alexander twist of* $\mathbb{C}G$ *associated to* (ψ,t).

Definition 4.3 A positive function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is multiplicatively convex if the function

$$F : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \log f(e^t),$$

is a convex function. In particular, a multiplicatively convex function is continuous and everywhere positive.

Our main technical tool is the following theorem due to Liu [2017, Theorem 5.1].

Theorem 4.4 *Let G be a finitely generated, residually finite group. For any square matrix A over $\mathbb{C}G$ and any 1–cohomology class $\psi \in H^1(G;\mathbb{R})$, the function*

$$t \mapsto \det^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\psi,t)A), \quad t > 0,$$

is either constantly zero or multiplicatively convex (and in particular everywhere positive).

With the above preparations, we are now ready to prove Theorem 1.1 for hyperbolic or mixed 3–manifolds.

Theorem 4.5 *Suppose N is a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. Assume that N is hyperbolic or mixed. Then $\tau^{(2)}(N,\rho) > 0$.*

Proof Since twisted L^2–torsion behaves multiplicatively with respect to finite covers by Lemma 3.6, we may assume without loss of generality that N itself fibers over the circle.

The following procedure is analogous to [Dubois et al. 2015b, Theorem 8.5]. Denote by Σ a fiber of N, and $f : \Sigma \to \Sigma$ the monodromy such that N is homeomorphic to the mapping torus

$$T_f(N) = \Sigma \times [-1,1]/(x,-1) \sim (f(x),1).$$

We can assume by isotopy that f has a fixed point P. Construct a CW structure X modeled on Σ with a single 0–cell P, k 1–cells $E_1,\dots,E_n$, and a 2–cell σ. By CW approximation, there is a cellular map $g\colon \Sigma\to\Sigma$ homotopic to f. Then the mapping torus $T_g(\Sigma)$ is homotopy equivalent to N, which is a simple homotopy equivalence since the Whitehead group of a fibered 3–manifold is trivial; see [Waldhausen 1978, Theorems 19.4 and 19.5]. Hence, by Lemma 3.6,

$$\tau^{(2)}(N,\rho)=\tau^{(2)}(T_g(\Sigma),\rho).$$

We proceed to describe a CW complex for the mapping torus $T_g(\Sigma)$. Suppose $\pi_1(N)=\pi_1(T_g(\Sigma))=:G$. The cells in each dimensions are

$$\{\sigma\times I\},\quad \{\sigma, E_1\times I,\dots,E_k\times I\},\quad \{E_1,\dots,E_k,P\times I\},\quad \{P\},$$

where $I=[-1,1]$. Let $e_i:=[E_i]\in G$ and $h:=[P\times I]\in G$ be the fundamental group elements represented by the corresponding loops. Denote by $\psi\in H^1(G;\mathbb{R})$ the 1–cohomology class dual to the fiber Σ. Then

$$\psi(h)=1,\quad \psi(e_1)=\cdots=\psi(e_k)=0.$$

The CW chain complex of $\widehat{T_g(\Sigma)}$ has the form

$$C_*(\widehat{T_g(\Sigma)})=(0\to\mathbb{C}G\xrightarrow{\partial_3}\mathbb{C}G^{k+1}\xrightarrow{\partial_2}\mathbb{C}G^{k+1}\xrightarrow{\partial_1}\mathbb{C}G\xrightarrow{\partial_0}0)$$

in which

$$\partial_3=(1-h,*,\dots,*),\quad \partial_2=\begin{pmatrix} * & * \\ I^{k\times k}-h\cdot A & * \end{pmatrix},\quad \partial_1=\begin{pmatrix} * \\ 1-h \end{pmatrix}$$

where "$*$" stands for matrices of appropriate size, and A is a matrix over $\mathbb{C}[\ker\psi]$ of size $k\times k$. Denote by A_ρ the matrix A twisted by ρ, as in Proposition 2.2. Then the L^2–chain complex of $T_g(\Sigma)$ twisted by $(G,\rho;\mathrm{id}_G)$ is

$$C_*^{(2)}(T_g(\Sigma),\rho)=(0\to l^2(G)^n\xrightarrow{\partial_3^\rho}l^2(G)^{n(k+1)}\xrightarrow{\partial_2^\rho}l^2(G)^{n(k+1)}\xrightarrow{\partial_1^\rho}l^2(G)^n\to 0)$$

in which

$$\partial_3^\rho=(I^{n\times n}-h\rho(h),*,\dots,*),\quad \partial_2^\rho=\begin{pmatrix} * & * \\ I^{nk\times nk}-h\cdot\rho(h)A_\rho & * \end{pmatrix},\quad \partial_1^\rho=\begin{pmatrix} * \\ I^{n\times n}-h\rho(h) \end{pmatrix}.$$

Consider the matrices

$$S:=I^{n\times n}-h\rho(h),\quad T:=I^{nk\times nk}-h\rho(h)A_\rho,$$

and the matrices under the Alexander twist associated to (ψ,t),

$$S(t):=\kappa(\psi,t)S=I^{n\times n}-t\cdot h\rho(h),\quad T(t):=\kappa(\psi,t)T=I^{nk\times nk}-t\cdot h\rho(h)A_\rho.$$

For any real number $t>0$ sufficiently small, the two matrices $S(t)$ and $T(t)$ are both invertible with regular Fuglede–Kadison determinant equal to 1; see [Dubois et al. 2015b, Proposition 8.8]. Then Liu's Theorem 4.4 applies to show that these two Fuglede–Kadison determinants are positive when $t=1$. It follows from Lemma 2.6 that $\tau^{(2)}(N,\rho)=\det^{\mathrm{r}}_{\mathcal{N}(G)}T(1)\cdot\det^{\mathrm{r}}_{\mathcal{N}(G)}S(1)^{-2}$ is positive. ◻

Theorem 1.1 then follows from Theorems 4.1 and 4.5.

5 Continuity of twisted L^2–torsion on representation varieties

Let N be any compact orientable irreducible 3–manifold with empty or incompressible toral boundary, and set $G := \pi_1(N)$. Suppose that G is infinite, and denote by $\mathcal{R}_n(G) := \mathrm{Hom}(G, \mathrm{SL}(n, \mathbb{C}))$ the representation variety. Then Theorem 1.1 implies that the twisted L^2–torsion can be viewed as a positive function

$$\rho \mapsto \tau^{(2)}(N, \rho), \quad \rho \in \mathcal{R}_n(G).$$

The continuity of this torsion function is an interesting but rather hard question. Work of Liu [2017, Theorem 1.2] has shown that the torsion function is continuous in $\mathrm{Hom}(G, \mathbb{R})$ along the Alexander twists. We remark that in his article the twist is not unimodular, and an equivalence class for torsion functions is introduced to guarantee well-definedness. If N is hyperbolic, $\rho_0 \colon G \to \mathrm{PSL}(2, \mathbb{C})$ is a holonomy representation associated to the hyperbolic structure, and $\rho \in \mathcal{R}_2(G)$ is a lifting of ρ_0 (such lifting always exists, see [Culler 1986, Corollary 2.2]), then Bénard and Raimbault [2022] proved that the torsion function is analytic near ρ. The continuity of the torsion function in general is wide open. In this section we present a partial result on the continuity of the twisted L^2–torsion function, namely Theorem 1.2. We start with a brief discussion of the L^2–Alexander torsions since it is closely related to the proof of Theorem 1.2.

5.1 L^2–Alexander torsion

The L^2–torsion twisted by 1–dimensional representations is called L^2*–Alexander torsion*. To be precise, for any 1–cohomology class $\psi \in H^1(G; \mathbb{R})$ and any real number $t > 0$, the L^2*–Alexander torsion* of N associated to (ψ, t) is defined to be

$$A^{(2)}(N, \psi, t) := \tau^{(2)}(C_*^{(2)}(N, \psi_t)).$$

Recall $\psi_t \colon G \to \mathbb{C}^\times$ that maps $g \in G$ to $t^{\psi(g)}$ is the representation associated to (ψ, t). Since ψ_t is not a unimodular representation, the L^2–Alexander torsion depends on the based $\mathbb{C}G$–chain complex $C_*(\widehat{N})$. Indeed, altering the $\mathbb{C}G$–basis of $C_*(\widehat{N})$, the base change matrix for $C_*^{(2)}(N, \psi_t)$ will be a permutation matrix with entries $\pm t^{\pm\psi(g_i)} g_i$ (compare Proposition 3.3), whose regular Fuglede–Kadison determinant is $t^{\sum_i \pm\psi(g_i)}$. Since $g_i \in G$ are independent of ψ and t, the continuity of $A^{(2)}(N, \psi, t)$ as a function of $(\psi, t) \in H^1(G; \mathbb{R}) \times \mathbb{R}_+$ is independent of the choice of cellular basis; here $H^1(N; \mathbb{R})$ is given the usual real vector space topology.

In [Dubois et al. 2015a; 2015b], one considers $A^{(2)}(N, \psi, t)$ as a function of t, and introduces an equivalence relation between functions. Namely, two functions $f_1, f_2 \colon \mathbb{R}_+ \to [0, +\infty)$ are equivalent if and only if there exists a real number r such that

$$f_1(t) = t^r \cdot f_2(t)$$

holds for all $t > 0$. In this case we denote by $f_1 \doteq f_2$. So the equivalence class of $A^{(2)}(N, \psi, t)$ as a function of t does not depend on the choice of cellular basis.

Another way to cure the ambiguity is to modify ψ_t to be a unimodular 2–dimensional representation. Set

$$\psi_t \oplus \psi_{t^{-1}} \colon G \to \mathrm{SL}(2,\mathbb{C}), \quad g \mapsto \begin{pmatrix} t^{\psi(g)} & 0 \\ 0 & t^{-\psi(g)} \end{pmatrix}.$$

Then it is easy to observe that $C_*^{(2)}(N,\psi_t \oplus \psi_{t^{-1}}) = C_*^{(2)}(N,\psi_t) \oplus C_*^{(2)}(N,\psi_{t^{-1}})$, and hence by Lück [2002, Theorem 3.35],

$$A^{(2)}(N,\psi,t) \cdot A^{(2)}(N,\psi,t^{-1}) = \tau^{(2)}(N,\psi_t \oplus \psi_{t^{-1}}),$$

which does not depend on the choice of cellular basis. This fact motivates the following definition.

Definition 5.1 For any $\psi \in H^1(G;\mathbb{R})$ and $t > 0$, we define the *symmetric L^2–Alexander torsion of N associated to* (ψ,t) to be

$$A^{(2)}_{\mathrm{sym}}(N,\psi,t) := \tau^{(2)}(N,\psi_t \oplus \psi_{t^{-1}})^{1/2}.$$

It is shown in [Dubois et al. 2015a, Chapter 6] that the L^2–Alexander torsion satisfies

$$A^{(2)}(N,\psi,t) = t^{-\psi(c_1(e))} \cdot A^{(2)}(N,\psi,t^{-1})$$

where $c_1(e) \in H_1(N;\mathbb{Z})$ is independent of (ψ,t). This shows that

$$A^{(2)}_{\mathrm{sym}}(N,\psi,t) = t^r \cdot A^{(2)}(N,\psi,t)$$

for some real number r. We remark that, as a function of (ψ,t), the continuity of $A^{(2)}(N,\psi,t)$ defined by any CW structure is equivalent to the continuity of $A^{(2)}_{\mathrm{sym}}(N,\psi,t)$.

As an illustration of the various definitions, we rediscover the L^2–Alexander torsion $A^{(2)}(N,\psi,t)$ for a graph manifold N using Theorem 4.1. The calculation is first carried out by Herrmann [2017] for Seifert fibering space and by [Dubois et al. 2015a] for graph manifolds.

Theorem 5.2 *Let N be a graph manifold with infinite fundamental group. Suppose that $N \neq S^1 \times D^2$ and $N \neq S^1 \times S^2$. Then a representative of the L^2–torsion twisted by (ψ,t) is*

$$A^{(2)}(N,\psi,t) = \max\{1, t^{x_N(\psi)}\},$$

where x_N is the Thurston norm for $H^1(N;\mathbb{R})$.

Proof For $t \geqslant 1$, set $\rho := \psi_t \oplus \psi_{t^{-1}}$. Then, by Theorem 4.1,

$$A^{(2)}_{\mathrm{sym}}(N,\psi,t)^2 = \tau^{(2)}(N,\psi_t \oplus \psi_{t^{-1}}) = \prod_{M \subset N \text{ is a Seifert piece}} t^{-|\psi(h)|\cdot\chi_{\mathrm{orb}}},$$

where $h \in H^1(M;\mathbb{R})$ is represented by the regular fiber of M and χ_{orb} is the orbifold Euler characteristic of M/S^1. By our assumption on N, we know that $\chi_{\mathrm{orb}} \leqslant 0$, so $-|\psi(h)|\cdot\chi_{\mathrm{orb}} = x_M(\psi)$ by [Herrmann

2017, Lemma A], where x_M is the Thurston norm for $H^1(M;\mathbb{R})$. Then by [Eisenbud and Neumann 1985, Proposition 3.5],

$$\sum_{M\subset N \text{ is a Seifert piece}} x_M(\psi)=x_N(\psi)$$

and so

$$A^{(2)}_{\text{sym}}(N,\psi,t)^2=t^{x_N(\psi)},\quad t\geqslant 1.$$

Since the symmetric L^2–Alexander torsion is by definition symmetric,

$$A^{(2)}_{\text{sym}}(N,\psi,t)=\max\{t^{\frac{1}{2}x_N(\psi)},t^{-\frac{1}{2}x_N(\psi)}\}\doteq\max\{1,t^{x_N(\psi)}\}.\qquad\square$$

It follows that the L^2–Alexander torsion of graph manifolds is continuous in $(\psi,t)\in H^1(G;\mathbb{R})\times\mathbb{R}^+$. For a general 3–manifold N, the continuity of the L^2–Alexander torsion is a hard question. Liu [2017] and Lück [2018] independently proved that the L^2–Alexander torsion function is always positive. Moreover Liu proved in the same article that $A^{(2)}(N,\psi,t)$ is continuous with respect to t. Lück [2018, Chapter 10] conjectured that this function is continuous with respect to $(\psi,t)\in H^1(N;\mathbb{R})\times\mathbb{R}^+$. We will see that this statement is true.

Theorem 5.3 *Let N be a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. Suppose $\pi_1(N)=G$ is infinite. Then any representative of the L^2–Alexander torsion function $A^{(2)}(N,\psi,t)$ is continuous with respect to $(\psi,t)\in H^1(N;\mathbb{R})\times\mathbb{R}^+$.*

Theorem 1.2 is now a corollary of Theorem 5.3, as we restate here.

Theorem 5.4 *Let N be a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. Suppose $\pi_1(N)=G$ is infinite. Define $\mathcal{R}^{\mathrm{t}}_n(G)$ to be the subvariety of $\mathcal{R}_n(G)$ consisting of upper triangular representations. Then the twisted L^2–torsion function*

$$\rho\mapsto\tau^{(2)}(N,\rho)$$

is continuous with respect to $\rho\in\mathcal{R}^{\mathrm{t}}_n(G)$.

Proof Fix a CW structure for N and fix a choice of cell-lifting to $\widehat{N}$, so we can talk about the L^2–Alexander torsion unambiguously. For any $\rho\in\mathcal{R}^{\mathrm{t}}_n(G)$, we can assume that

$$\rho(g)=\begin{pmatrix}\chi_1(g) & \cdots & * \\ & \ddots & \vdots \\ & & \chi_n(g)\end{pmatrix},$$

where $\chi_k\colon G\to\mathbb{C}^\times$ are characters. The modulus of those characters can be written as

$$|\chi_k|=e^{\phi_k},\quad g\mapsto e^{\phi_k(g)},$$

for some real 1–cohomology classes $\phi_k\in H^1(G;\mathbb{R})$. The classes $\phi_1,\dots,\phi_n$ are continuous with respect to $\rho\in\mathcal{R}^{\mathrm{t}}_n(G)$.

Let V_n be the G–invariant subspace of V corresponding to χ_n, and let $V' := V/V_n$, then there is an exact sequence of G–representations

$$0 \to V_n \to V \to V' \to 0,$$

where the G–actions are given by

$$\rho_n(g) = \chi_n(g), \quad \rho(g) = \begin{pmatrix} \chi_1(g) & \cdots & * \\ & \ddots & \vdots \\ & & \chi_n(g) \end{pmatrix}, \quad \rho'(g) = \begin{pmatrix} \chi_1(g) & \cdots & * \\ & \ddots & \vdots \\ & & \chi_{n-1}(g) \end{pmatrix},$$

respectively. Then, by Lück [2018, Lemma 3.3],

$$\tau^{(2)}(N,\rho) = \tau^{(2)}(N,\rho_n)\tau^{(2)}(N,\rho').$$

Since unitary twists have no effect on L^2–torsions by Lück [2018, Theorem 4.1], we have

$$\tau^{(2)}(N,\rho_n) = \tau^{(2)}(N,e^{\phi_n}) = A^{(2)}(N,\phi_n,e).$$

The above process can then be applied to ρ' and finally we have the formula

$$\tau^{(2)}(N,\rho) = A^{(2)}(N,\phi_1,e)\cdots A^{(2)}(N,\phi_n,e).$$

Since the cohomology classes $\phi_1,\dots,\phi_n$ vary continuously with respect to $\rho \in \mathscr{R}_n^{\mathrm{t}}(G)$, the conclusion follows from Theorem 5.3. □

The remaining part of this section is devoted to the proof of Theorem 5.3. We will need the notion of Alexander multitwists.

5.2 Alexander multitwists of matrices

Recall that G is any finitely generated, residually finite group. For any collection of 1–cohomology classes $\Phi = (\phi_1,\dots,\phi_n) \in \prod_{i=1}^n H^1(G;\mathbb{R})$ and any collection of positive real numbers $T = (t_1,\dots,t_n) \in \mathbb{R}_+^n$, we define a $\mathbb{C}G$–homomorphism

$$\kappa(\Phi,T)\colon \mathbb{C}G \to \mathbb{C}G, \quad g \to t_1^{\phi_1(g)}\cdots t_n^{\phi_n(g)}\cdot g, \quad g \in G.$$

This is called the *Alexander multitwist of* $\mathbb{C}G$ *associated to* (Φ,T).

Proposition 5.5 *Basic properties of the Alexander multitwist*:

(1) ***Associativity*** *Suppose* $\Phi = (\phi_1,\dots,\phi_n)$ *and* $T = (t_1,\dots,t_n)$. *Then*

$$\kappa(\Phi,T) = \kappa(\phi_1,t_1)\circ\cdots\circ\kappa(\phi_n,t_n).$$

(2) ***Commutativity*** $\kappa(\phi_1,t_1)\circ\kappa(\phi_2,t_2) = \kappa(\phi_2,t_2)\circ\kappa(\phi_1,t_1)$.

(3) ***Change of coordinate*** *Let* $r_1,r_2 \in \mathbb{R}$; *then*

$$\kappa(r_1\phi_1 + r_2\phi_2,t) = \kappa(\phi_1,t^{r_1})\circ\kappa(\phi_2,t^{r_2}),$$
$$\kappa(\phi,t_1^{r_1}t_2^{r_2}) = \kappa(r_1\phi,t_1)\circ\kappa(r_2\phi,t_2).$$

The Alexander multitwist extends to an endomorphism of the matrix algebra with entries in $\mathbb{C}G$.

In the following part of this section, we shall fix a square matrix Ω over $\mathbb{C}G$, and suppose that $\det^{\mathrm{r}}_{\mathcal{N}(G)}(\Omega)$ is not zero. For any collection of 1–cohomology classes $\Phi = (\phi_1, \dots, \phi_n)$ and positive real numbers $T = (t_1, \dots, t_n)$, we introduce the notation

$$V_\Phi(T) := \det^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\Phi, T)\Omega).$$

Proposition 5.6 *For any fixed choice of Φ, the multivariable function $V_\Phi(T)$ is everywhere positive and is multiplicatively convex in each coordinate with respect to $T = (t_1, \dots, t_n) \in \mathbb{R}^n_+$.*

Proof By associativity and commutativity of the Alexander multitwist,

$$\kappa(\Phi, T)\Omega = \kappa(\phi_i, t_i) \circ \kappa(\Phi', T')\Omega$$

where (Φ', T') are variables other than (ϕ_i, t_i). The conclusion then follows from applying Theorem 4.4 to each i. $\square$

Theorem 5.7 *For any fixed choice of Φ, the multivariable real function $V_\Phi(T)$ is multiplicatively convex with respect to $T = (t_1, \dots, t_n) \in \mathbb{R}^n_+$.*

Proof We will prove that for any fixed choice of Φ and every positive integer $k \leqslant n$, the function $V_\Phi(T)$ is multiplicatively convex with respect to the first k coordinates.

The case $k = 1$ is proved by Proposition 5.6. Assume the claim holds for $(k-1)$ and consider

$$V_{\phi_1,\dots,\phi_k}(t_1, \dots, t_k) = V_\Phi(T)$$

as a function of the first k variables of Φ and T. It suffices to prove that for any $\theta \in (0,1)$ and any collection of positive numbers $r_1, \dots, r_k > 0$ and $s_1, \dots, s_k > 0$,

$$\big(V_{\phi_1,\dots,\phi_k}(r_1, \dots, r_k)\big)^\theta \cdot \big(V_{\phi_1,\dots,\phi_k}(s_1, \dots, s_k)\big)^{1-\theta} \geqslant V_{\phi_1,\dots,\phi_k}(r_1^\theta s_1^{1-\theta}, \dots, r_k^\theta s_k^{1-\theta}).$$

We can assume that $r_1 \neq s_1$, otherwise this inequality degenerates to the $(k-1)$ case after permuting the coordinates. Consider $\psi_1 = \phi_1 + \lambda\phi_k$ for a real number λ which will be determined later. We have the identity that for all $t_1, \dots, t_k > 0$,

$$V_{\psi_1,\phi_2,\dots,\phi_k}(t_1, \dots, t_{k-1}, t_k) = V_{\phi_1,\phi_2,\dots,\phi_k}(t_1, \dots, t_{k-1}, t_1^\lambda t_k).$$

By the induction hypothesis, for all $r > 0$,

$$\big(V_{\psi_1,\phi_2,\dots,\phi_k}(r_1, \dots, r_{k-1}, r)\big)^\theta \cdot \big(V_{\psi_1,\phi_2,\dots,\phi_k}(s_1, \dots, s_{k-1}, r)\big)^{1-\theta} \geqslant V_{\psi_1,\phi_2,\dots,\phi_k}(r_1^\theta s_1^{1-\theta}, \dots, r_{k-1}^\theta s_{k-1}^{1-\theta}, r),$$

which is equivalent to

$$\big(V_{\phi_1,\dots,\phi_k}(r_1, \dots, r_{k-1}, r_1^\lambda r)\big)^\theta \cdot \big(V_{\phi_1,\dots,\phi_k}(s_1, \dots, s_{k-1}, s_1^\lambda r)\big)^{1-\theta} \geqslant V_{\phi_1,\dots,\phi_k}\big(r_1^\theta s_1^{1-\theta}, \dots, r_{k-1}^\theta s_{k-1}^{1-\theta}, (r_1^\lambda r)^\theta \cdot (s_1^\lambda r)^{1-\theta}\big).$$

Since $r_1 \neq s_1$, we can prescribe $\lambda \in \mathbb{R}$ and $r > 0$ by solving the equations

$$r_1^\lambda r = r_k, \quad s_2^\lambda r = s_k.$$

This finishes the induction. □

Corollary 5.8 *For any fixed $(\Phi, T) \in \prod_{i=1}^n H^1(G;\mathbb{R}) \times \mathbb{R}_+^n$, the function $W_{\Phi,T} : \mathbb{R}^n \to \mathbb{R}$,*

$$W_{\Phi,T}(s_1, \dots, s_n) := \log(V_{s_1\phi_1,\dots,s_n\phi_s}(T)),$$

is convex. In particular, it is continuous.

Proof This follows from the identity

$$W_{\Phi,T}(s_1, \dots, s_n) := \log(V_{s_1\phi_1,\dots,s_n\phi_s}(T)) = \log(V_\Phi(t_1^{s_1}, \dots, t_n^{s_n}))$$

and the multiplicative convexity of $V_\Phi(T)$. □

Theorem 5.9 *The regular Fuglede–Kadison determinant map $\det^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\phi,t)\Omega)$ is continuous with respect to $(\phi,t) \in H^1(G;\mathbb{R}) \times \mathbb{R}_+$.*

Proof Let $\Psi = (\psi_1, \dots, \psi_k)$ be a basis for the real vector space $H^1(G;\mathbb{R})$. Suppose

$$\phi = \sum_{i=1}^k c_j \psi_j, \quad 1 \leqslant i \leqslant n,$$

where the coefficients c_j are continuous with respect to $\phi \in H^1(G;\mathbb{R})$. Then

$$\begin{aligned}\kappa(\phi,t)\Omega &= \kappa(c_1\psi_1, t) \circ \cdots \circ \kappa(c_k\psi_k, t)\Omega \\ &= \kappa(c_1 \log t \cdot \psi_1, e) \circ \cdots \circ \kappa(c_k \log t \cdot \psi_k, e)\Omega \\ &= \kappa\big((c_1 \log t \cdot \psi_1, \dots, c_k \log t \cdot \psi_k), (e, \dots, e)\big)\Omega.\end{aligned}$$

By definition,

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\phi,t)\Omega) = \exp W_{\Psi,(e,\dots,e)}(c_1 \log t, \dots, c_k \log t).$$

The continuity follows from Corollary 5.8. □

5.3 Applications to 3–manifolds

Proof of Theorem 5.3 If N is a graph manifold, then Theorem 5.2 offers an explicit formula for the L^2–Alexander torsion; the theorem holds since the Thurston norm is continuous in $H^1(N;\mathbb{R})$.

If N is a compact connected orientable irreducible 3–manifold which is hyperbolic or mixed, then as in the proof of Theorem 4.5, we can find a regular finite covering $p: \widetilde{N} \to N$ of degree d such that $\widetilde{N}$ fibers over the circle. Since by Lemma 3.6 we have

$$\tau^{(2)}(N, \psi_t \oplus \psi_{t^{-1}})^d = \tau^{(2)}(\widetilde{N}, p^*\psi_t \oplus p^*\psi_{t^{-1}}),$$

it follows that $A^{(2)}_{\mathrm{sym}}(N,\psi,t)^d = A^{(2)}_{\mathrm{sym}}(\widetilde{N}, p^*\psi, t)$. Note that the pullback map $p^*: H^1(N;\mathbb{R}) \to H^1(\widetilde{N};\mathbb{R})$ is a continuous embedding, so we only need to prove the theorem for $\widetilde{N}$. We can assume without loss of generality that our manifold N fibers over circle. From the proof of Theorem 4.5, we see that

$$A^{(2)}(N,\psi,t) = \det{}^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\psi,t)T)\cdot \det{}^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\psi,t)S)^{-2},$$

where $T = I^{k\times k} - hA_\rho$ and $S = 1-h$ are square matrices over $\mathbb{C}G$ with positive regular Fuglede–Kadison determinant. The conclusion follows immediately from Theorem 5.9. □

The continuity result can be used to improve previous calculations of the L^2–Alexander torsion associated to fibered classes. In [Dubois et al. 2015b, Theorem 8.2], the calculation is carried out for rational homology classes only. Liu's result [2017, Theorem 1.2] shows that the asymptotic degree of the L^2–Alexander torsion associated to any class equals its Thurston norm, but does not offer an explicit formula.

Theorem 5.10 *Let N be any compact, connected, irreducible, orientable 3–manifold with empty or incompressible toral boundary. Suppose $\pi_1(N)$ is infinite, $N \neq S^1\times D^2$ and $N \neq S^1 \times S^2$. Let $\phi \in H^1(N;\mathbb{R})$ be in the interior of a fibered cone. Then there exists a representative of the L^2–Alexander torsion associated to (ϕ, t) such that*

$$A^{(2)}(N,\phi,t) = \begin{cases} 1 & \text{if } t < 1/h(\phi),\\ t^{x_N(\phi)} & \text{if } t > h(\phi),\end{cases}$$

where $h(\phi)$ is the entropy function defined on the fibered cone of $H^1(N;\mathbb{R})$ (compare [Dubois et al. 2015b, Section 8]).

Proof Let $\phi_n \in H^1(N;\mathbb{Q})$ be a sequence in the fibered cone that converge to ϕ. By [Dubois et al. 2015b, Theorem 8.5], for any n,

$$A^{(2)}(N,\phi_n,t) = \begin{cases} 1 & \text{if } t < 1/h(\phi_n),\\ t^{x_N(\phi_n)} & \text{if } t > h(\phi_n).\end{cases}$$

By Theorem 5.3,

$$A^{(2)}(N,\phi_n,t) \to A^{(2)}(N,\phi,t), \quad n\to\infty,$$

for any $t \in \mathbb{R}$. Since the entropy and the Thurston norm are continuous functions of $H^1(N;\mathbb{R})$,

$$h(\phi_n)\to h(\phi), \quad x_N(\phi_n)\to x_N(\phi), \quad n\to\infty.$$ □

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