Algebraic & Geometric Topology

Volume 24 (2024)

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Algebraic & Geometric Topology **24**:4 (2024) 2331–2350
DOI: 10.2140/agt.2024.24.2331
Published: 16 July 2024

An algebraic C_2–equivariant Bézout theorem

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One interpretation of Bézout's theorem, nonequivariantly, is as a calculation of the Euler class of a sum of line bundles over complex projective space, expressing it in terms of the rank of the bundle and its degree. We generalize this calculation to the C_2–equivariant context, using the calculation of the cohomology of C_2–complex projective spaces from an earlier paper, which used ordinary C_2–cohomology with Burnside ring coefficients and an extended grading necessary to define the Euler class. We express the Euler class in terms of the equivariant rank of the bundle and the degrees of the bundle and its fixed subbundles. We do similar calculations using constant $\mathbb{Z}$ coefficients and Borel cohomology and compare the results.

55N91; 14N10, 14N15, 55R40, 55R91

Introduction

Suppose that we have n nonzero homogeneous polynomials f_i for $1 \le i \le n$ in N variables where $n < N$, let d_i be the degree of f_i, and let $\Delta = d_1 d_2 \cdots d_n$. If $\mathbb{P}^{N-1}$ is the complex projective space, we can consider each f_i as giving a section of the complex line bundle $O(d_i)$, the d_i–fold tensor power of the dual of the tautological line bundle over $\mathbb{P}^{N-1}$. Each f_i determines a hypersurface $H_i \subset \mathbb{P}^{N-1}$, its zero locus. In this context, the (nonequivariant) Bézout theorem, as given by Fulton [5], for example, can be stated in several ways. Geometrically, it says that the intersection of the hypersurfaces H_i, counted with multiplicities, is generically rationally equivalent to Δ copies of $\mathbb{P}^{N-n-1}$. In the classical case, when $n = N-1$, the hypersurfaces intersect in Δ points.

We can restate Bézout's theorem as a purely algebraic statement: in the cohomology ring

$$H^*(\mathbb{P}^{N-1};\mathbb{Z}) \cong \mathbb{Z}[\hat{c}]/\langle \hat{c}^N \rangle,$$

the Euler class of $F = O(d_1) \oplus O(d_2) \oplus \cdots \oplus O(d_n)$ is

$$e(F) = \Delta \hat{c}^n,$$

where $\hat{c} = e(O(1))$. As a consequence, $e(F)$ determines and is completely determined by the rank n of F (that is, the complex dimension of each of its fibers) and its degree Δ. (The connection to the geometric statement is via the Chow ring, isomorphic to cohomology in this case, in which $\hat{c}^n$ is represented by $\mathbb{P}^{N-n-1}$.) Here we want to generalize the algebraic calculation, including giving a generalization of the notions of rank and degree, and discussing how they determine and are determined by the Euler class; in a followup paper we will pursue a geometric interpretation.

In [3] we began to examine how this generalizes in the presence of an action of the two-element group C_2. Let $\mathbb{C}$ denote the trivial complex representation of C_2 and let $\mathbb{C}^\sigma$ denote the nontrivial representation. If $p \ge 0$ and $q \ge 0$ are integers, let $\mathbb{C}^{p+q\sigma}$ be the sum of p copies of $\mathbb{C}$ and q copies of $\mathbb{C}^\sigma$, and let $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ be its (complex) projective space, a C_2–space. Using the equivariant ordinary cohomology with extended grading defined in [4], we computed the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ in [3] with Burnside ring coefficients. We also gave the zero-dimensional version of an equivariant Bézout theorem, showing that the equivariant Euler class in equivariant ordinary cohomology with Burnside ring coefficients allows us to determine the finite C_2–set in $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ given by the intersection of $p+q-1$ equivariant hypersurfaces.

Let us set up the context for a generalization to higher dimensions. As mentioned above, if F is a nonequivariant vector bundle over $\mathbb{P}^{N-1}$, its Euler class has the form $e(F) = \Delta\hat{c}^n$, where n is the rank of F, Δ is its degree, and we set $\Delta = 0$ if $n \ge N$.

Now suppose that we have $(n<p+q)$–many C_2–line bundles over $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ with direct sum F. We let Δ be the nonequivariant degree of F. We can also consider the fixed-set bundle F^{C_2} over $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2} = \mathbb{P}(\mathbb{C}^p) \sqcup \mathbb{P}(\mathbb{C}^{q\sigma})$. Let n_0 denote the rank of the restriction of F^{C_2} to $\mathbb{P}(\mathbb{C}^p)$ and let Δ_0 be its degree. We know that $n_0 \le n$, and, to keep the situation geometrically meaningful, we would like the generic intersection of the corresponding hypersurfaces in $\mathbb{P}(\mathbb{C}^p)$ to have dimension no more than the dimension of the intersection of all the hypersurfaces in $\mathbb{P}(\mathbb{C}^{p+q\sigma})$. For that, we require that $p-n_0-1 \le p+q-n-1$, that is, $n_0 \ge n-q$. Similarly, let n_1 denote the rank of F^{C_2} over $\mathbb{P}(\mathbb{C}^{q\sigma})$ and let Δ_1 be its degree; we require that $n_1 \ge n-p$. We record these notations and conditions for later reference.

Bézout context 0.1 F is the sum of n–many C_2–line bundles over $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ and Δ is its nonequivariant degree. The restriction of F^{C_2} to $\mathbb{P}(\mathbb{C}^p)$ has rank n_0 and degree Δ_0, while its restriction to $\mathbb{P}(\mathbb{C}^{q\sigma})$ has rank n_1 and degree Δ_1. We assume that

$$n < p+q, \quad n-q \le n_0 \le n \quad \text{and} \quad n-p \le n_1 \le n.$$

We call the triple (n, n_0, n_1) the *C_2–ranks* of F and the triple $(\Delta, \Delta_0, \Delta_1)$ the *C_2–degrees* of F.

In this context we will calculate the Euler class $e(F)$ as an element of the equivariant cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$, as computed in [3].

Bézout theorem, part I *In the context above, the Euler class $e(F)$ is completely determined by the ranks (n, n_0, n_1) and the degrees $(\Delta, \Delta_0, \Delta_1)$. Moreover, these ranks and degrees can be recovered from $e(F)$. The ranks are additive and the degrees are multiplicative.*

This will be proved as Theorem 2.11. When we say that the degrees are multiplicative, we really mean the following: Suppose that we have two such bundles F and F' with ranks (n, n_0, n_1) and (n', n_0', n_1'), respectively, and corresponding degrees. We assume that $F \oplus F'$ still satisfies the conditions of the Bézout context above. This allows the possibility that $n_0 + n_0' \ge p$, in which case the corresponding degree is not $\Delta_0\Delta_0'$ but 0, and similarly if $n_1 + n_1' \ge q$.

Nonequivariantly, the cohomology of $\mathbb{P}^{N-1}$ is a free $\mathbb{Z}$–module with a basis given by the powers of $\hat{c}$. Explicitly,

$$H^*(\mathbb{P}^{N-1}) \cong \bigoplus_{i=0}^{N-1} \hat{c}^i \mathbb{Z}. \tag{0.2}$$

The nonequivariant Bézout theorem can be viewed as expressing $e(F)$ in terms of this basis. In any given grading, there is at most one basis element, so there is only one coefficient to specify, which turns out to be the degree Δ. Equivariantly, the result is more complicated. In [3] we showed that the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ is free over the $\mathrm{RO}(C_2)$–graded equivariant cohomology of a point and gave an explicit basis that maps to the nonequivariant one. That is, we have a decomposition similar to (0.2), with $\mathbb{Z}$ replaced by the $\mathrm{RO}(C_2)$–graded cohomology of a point and the powers of $\hat{c}$ replaced by our preferred basis. Because the cohomology of a point is not concentrated in grading 0 outside of the $\mathbb{Z}$–graded part, in any given grading of the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ there are up to $p+q$ basis elements that can contribute, so an element potentially requires a $(p+q)$–tuple of coefficients (from the cohomology of a point) to specify. Our second main result is summarized as follows:

Bézout theorem, part II *In the context above, the Euler class $e(F)$ is the linear combination of at most three basis elements.*

This is proved as Theorem 2.12, which also gives the details as to which three basis elements are involved and what their coefficients are. The three basis elements are determined by (p and q and) the ranks (n, n_0, n_1). The coefficients are determined by the degrees $(\Delta, \Delta_0, \Delta_1)$, but are not simply equal to them.

This paper is structured as follows. In Section 1 we review the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ as computed in [3], including our preferred basis. In Section 2 we give the main results, proving the two theorems above. There are two other equivariant ordinary cohomology theories in common use: cohomology with constant $\mathbb{Z}$ coefficients and Borel cohomology. In Section 3 we discuss how the computation changes if we use constant $\mathbb{Z}$ coefficients rather than Burnside ring coefficients, and in Section 4 we discuss the similar computation in Borel cohomology. There are maps from cohomology with Burnside ring coefficients to cohomology with constant $\mathbb{Z}$ coefficients, and from that theory to Borel cohomology, both respecting Euler classes, and we will see that the Euler classes in the last two theories carry less information than the Euler class in cohomology with Burnside ring coefficients. In particular, we cannot recover the degrees Δ_0 and Δ_1 from the Euler class in cohomology with constant $\mathbb{Z}$ coefficients or the class in Borel cohomology.

Acknowledgements Hudson and Tilson were partially supported by the DFG through the SPP 1786: *Homotopy theory and algebraic geometry*, project number 405468058: *C_2–equivariant Schubert calculus of homogeneous spaces.* While Hudson and Tilson were affiliated to the Bergische Universität Wuppertal, the research was conducted in the framework of the research training group GRK 2240: *Algebro-geometric methods in algebra, arithmetic and topology*, which is funded by the DFG,

1 The cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$

1.1 Ordinary cohomology

We will use C_2–equivariant ordinary cohomology with the extended grading developed in [4]. This is an extension of Bredon's ordinary cohomology to be graded on representations of the fundamental groupoids of C_2–spaces. We review here some of the notation and computations we will be using. A more detailed description of this theory can be found in [3].

For an ex-C_2–space Y over X, we write $H_{C_2}^{RO(\Pi X)}(Y;\underline{T})$ for the ordinary cohomology of Y with coefficients in a Mackey functor $\underline{T}$, graded on $RO(\Pi X)$, the representation ring of the fundamental groupoid of X. Through most of this paper we will use the Burnside ring Mackey functor $\underline{A}$ as the coefficients, and write simply $H_{C_2}^{RO(\Pi X)}(Y)$.

In [4; 3] we considered cohomology to be Mackey functor-valued, which is useful for many computations, and wrote $\underline{H}_{C_2}^{RO(\Pi X)}(Y)$ for the resulting theory. Here we concentrate on the values at level C_2/C_2, and write

$$H_{C_2}^{RO(\Pi X)}(Y) = \underline{H}_{C_2}^{RO(\Pi X)}(Y)(C_2/C_2).$$

However, we will still refer to the restriction functor ρ from equivariant cohomology to nonequivariant cohomology, and the transfer map τ going in the other direction.

For all X and Y, $H_{C_2}^{RO(\Pi X)}(Y)$ is a graded module over

$$\mathbb{H} = \mathbb{H}^{RO(C_2)} = H_{C_2}^{RO(C_2)}(S^0),$$

the cohomology of a point. The grading on the latter is just $RO(C_2)$, the real representation ring of C_2, which is free abelian on 1, the class of the trivial representation $\mathbb{R}$, and σ, the class of the sign representation $\mathbb{R}^\sigma$. The cohomology of a point was calculated by Stong in an unpublished manuscript, and first published by Lewis in [6]. We can picture the calculation as in Figure 1, in which a group in grading $a+b\sigma$ is plotted at the point (a,b), and the spacing of the grid lines is 2 (which is more convenient for other graphs we will give). The box at the origin is a copy of $A(C_2)$, the Burnside ring of C_2, closed circles are copies of $\mathbb{Z}$, and open circles are copies of $\mathbb{Z}/2$.

Recall that $A(C_2)$ is the Grothendieck group of finite C_2–sets, with multiplication given by products of sets. Additively, it is free abelian on the classes of the orbits of C_2, for which we will write $1 = [C_2/C_2]$ and $g = [C_2/e]$. The multiplication is given by $g^2 = 2g$. We will also write $\kappa = 2 - g$. Other important elements are shown in the figure: The group in degree σ is generated by an element e, while the group in degree $-2+2\sigma$ is generated by an element ξ. The groups in the second quadrant are generated by the products $e^m\xi^n$, with $2e\xi = 0$. We have $g\xi = 2\xi$ and $ge = 0$. The groups in gradings $-m\sigma$, for $m \geq 1$, are generated by elements $e^{-m}\kappa$, so named because $e^m \cdot e^{-m}\kappa = \kappa$. We also have $ge^{-m}\kappa = 0$.

To explain $\tau(\iota^{-2})$, we think for a moment about the nonequivariant cohomology of a point. If we grade it on $RO(C_2)$, we get $H^{RO(C_2)}(S^0;\mathbb{Z}) \cong \mathbb{Z}[\iota^{\pm 1}]$, where $\deg \iota = -1+\sigma$. (Nonequivariantly, we cannot tell

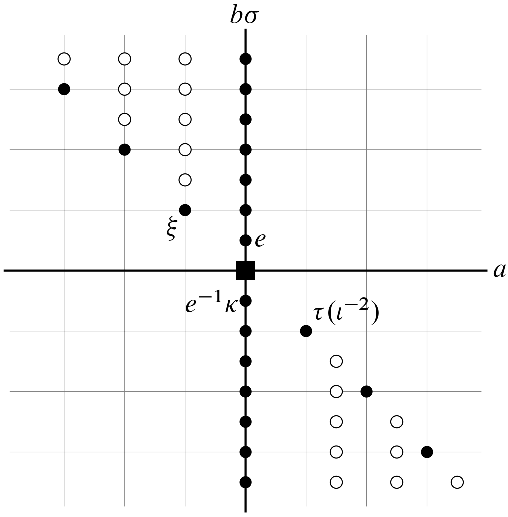

Figure 1: $\mathbb{H}^{RO(C_2)}$.

the difference between $\mathbb{R}$ and $\mathbb{R}^\sigma$.) We have $\rho(\xi) = \iota^2$ and $\tau(\iota^2) = g\xi = 2\xi$. Note also that $\tau(1) = g$. In the fourth quadrant the group in grading $n(1-\sigma)$ for $n \geq 2$ is generated by $\tau(\iota^{-n})$. The remaining groups in the fourth quadrant will not concern us here. For more details, see [2; 3].

1.2 The cohomology of projective space

As described in the introduction, the form of Bézout's theorem we shall give expresses the Euler class of a bundle over $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ in terms of a basis of its cohomology. We now review the structure of that cohomology as calculated in [3].

Write $B = \mathbb{P}(\mathbb{C}^{\infty+\infty\sigma})$. Its fixed set is

$$B^{C_2} = \mathbb{P}(\mathbb{C}^\infty) \sqcup \mathbb{P}(\mathbb{C}^{\infty\sigma}) = B^0 \sqcup B^1,$$

where we use the indices 0 and 1 to evoke the trivial and nontrivial representations of C_2, respectively. (We will use this convention, that a subscript 0 refers to something related to B^0 and subscript 1 refers to something related to B^1, throughout.) Representations of ΠB are determined by their restrictions to B^0 and B^1, which are elements of $RO(C_2)$ that must have the same nonequivariant rank and the same parity for the ranks of their fixed-point representations. As shown in [3, Section 2.2, page 13] this leads to the calculation

$$RO(\Pi B) = \mathbb{Z}\{1, \sigma, \Omega_0, \Omega_1\}/\langle \Omega_0 + \Omega_1 = 2\sigma - 2\rangle,$$

where Ω_0 is the representation whose value on B^0 is $2\sigma - 2$ and on B^1 is 0, while Ω_1 is the representation whose value on B^0 is 0 and on B^1 is $2\sigma - 2$. For any $\alpha \in RO(\Pi B)$, write $|\alpha| \in \mathbb{Z}$ for its underlying nonequivariant rank, and α_0 and $\alpha_1 \in RO(C_2)$ for its restrictions to B^0 and B^1, respectively. What we said above can be phrased as: α is completely determined by the triple of ranks $(|\alpha|, |\alpha_0^{C_2}|, |\alpha_1^{C_2}|)$, where the last two ranks have the same parity.

We think of the finite projective spaces as spaces over B by the evident inclusions $\mathbb{P}(\mathbb{C}^{p+q\sigma}) \to \mathbb{P}(\mathbb{C}^{\infty+\infty\sigma})$, and so will grade their cohomologies on $RO(\Pi B)$. Let ω denote the tautological line bundle over B, let $\omega^\vee$ be its dual bundle, let $\chi\omega = \omega \otimes_{\mathbb{C}} \mathbb{C}^\sigma$, and let $\chi\omega^\vee$ be the dual of $\chi\omega$. We will also use the notation from algebraic geometry in which $\omega = O(-1)$ and $\omega^\vee = O(1)$; we write $\chi O(-1) = \chi\omega$ and $\chi O(1) = \chi\omega^\vee$.

Associated to any bundle over B is a representation in $RO(\Pi B)$ that we think of as the equivariant rank of the bundle; this representation is given by the fiber representations over B^0 and B^1. In the case of ω and $\chi\omega$, we have

$$\omega = 2 + \Omega_1 \quad \text{and} \quad \chi\omega = 2 + \Omega_0,$$

where we write ω and $\chi\omega$ again for the associated elements of $RO(\Pi B)$.

Let $\hat{c}_\omega$ and $\hat{c}_{\chi\omega}$ denote the Euler classes of $\omega^\vee$ and $\chi\omega^\vee$, respectively. The cohomology of $\mathbb{P}(\mathbb{C}^{\infty+\infty\sigma})$ was calculated in [2] as follows:

Theorem 1.1 *$H_{C_2}^{RO(\Pi B)}(B_+)$ is an algebra over $\mathbb{H}$ generated by the Euler classes $\hat{c}_\omega$ and $\hat{c}_{\chi\omega}$ together with classes ζ_0 and ζ_1. These elements live in gradings*

$$\operatorname{grad} \hat{c}_\omega = \omega, \quad \operatorname{grad} \hat{c}_{\chi\omega} = \chi\omega, \quad \operatorname{grad} \zeta_1 = \omega - 2 \quad \textit{and} \quad \operatorname{grad} \zeta_0 = \chi\omega - 2.$$

They satisfy the relations

$$\zeta_0\zeta_1 = \xi \quad \textit{and} \quad \zeta_1\hat{c}_{\chi\omega} = (1-\kappa)\zeta_0\hat{c}_\omega + e^2,$$

which completely determine the algebra. Moreover, $H_{C_2}^{RO(\Pi B)}(B_+)$ is free as a module over $\mathbb{H}$. □

There are two restriction maps we will use,

$$\rho\colon H_{C_2}^\alpha(B_+) \to H^{|\alpha|}(B_+),$$

restriction to nonequivariant cohomology, and

$$(-)^{C_2}\colon H_{C_2}^\alpha(B_+) \to H^{\alpha_0^{C_2}}(B_+^0) \oplus H^{\alpha_1^{C_2}}(B_+^1),$$

the fixed-point map. These are ring maps and their values on the multiplicative generators are given by the following:

$$\begin{gathered} \rho(\zeta_0) = 1, \qquad \rho(\zeta_1) = 1, \qquad \rho(\hat{c}_\omega) = \hat{c}, \qquad \rho(\hat{c}_{\chi\omega}) = \hat{c}, \\ \zeta_0^{C_2} = (0,1), \qquad \zeta_1^{C_2} = (1,0), \qquad \hat{c}_\omega^{C_2} = (\hat{c},1), \qquad \hat{c}_{\chi\omega}^{C_2} = (1,\hat{c}). \end{gathered}$$

Here $\hat{c}$ denotes the first nonequivariant Chern class of $O(1)$. We also need the values of the similar restriction maps

$$\rho\colon \mathbb{H}^\alpha \to H^{|\alpha|}(S^0) \quad \text{and} \quad (-)^{C_2}\colon \mathbb{H}^\alpha \to H^{\alpha^{C_2}}(S^0).$$

The particular values we will need are

$$\rho(\tau(\iota^{2k})) = 1, \quad \rho(e^{-k}\kappa) = 0, \quad \rho(e^k) = 0, \quad \tau(\iota^{2k})^{C_2} = 0, \quad (e^{-k}\kappa)^{C_2} = 2 \quad \text{and} \quad (e^k)^{C_2} = 1.$$

Moving now to finite projective spaces, on pulling back along the inclusion $\mathbb{P}(\mathbb{C}^{p+q\sigma}) \hookrightarrow \mathbb{P}(\mathbb{C}^{\infty+\infty\sigma})$, the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ contains elements we will also call $\hat{c}_\omega$, $\hat{c}_{\chi\omega}$, ζ_0, and ζ_1.

Theorem 1.2 [3, Theorem A] *Let $0 \le p, q < \infty$ with $p+q > 0$. Then $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ is a free module over $\mathbb{H}$, and as a (graded) commutative algebra over $\mathbb{H}$ the ring $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ is generated by $\hat{c}_\omega$, $\hat{c}_{\chi\omega}$, ζ_0, and ζ_1, together with the following classes: $\hat{c}_\omega^p$ is infinitely divisible by ζ_0, meaning that, for $k \ge 1$, there are unique elements $\zeta_0^{-k}\hat{c}_\omega^p$ such that*

$$\zeta_0^k \cdot \zeta_0^{-k}\hat{c}_\omega^p = \hat{c}_\omega^p.$$

Similarly, $\hat{c}_{\chi\omega}^q$ is infinitely divisible by ζ_1, so for $k \ge 1$ there are unique elements $\zeta_1^{-k}\hat{c}_{\chi\omega}^q$ such that

$$\zeta_1^k \cdot \zeta_1^{-k}\hat{c}_{\chi\omega}^q = \hat{c}_{\chi\omega}^q.$$

The generators satisfy the following further relations:

$$\zeta_0\zeta_1 = \xi, \quad \zeta_1\hat{c}_{\chi\omega} = (1-\kappa)\zeta_0\hat{c}_\omega + e^2 \quad \text{and} \quad \hat{c}_\omega^p\hat{c}_{\chi\omega}^q = 0. \qquad \square$$

We also gave an explicit basis for $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ over $\mathbb{H}$, which we can describe as follows. We define sets $F_{p,q}(m)$, recursively on p and q, that give bases for $H_{C_2}^{m\omega+\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$. For $m \in \mathbb{Z}$, let

$$F_{p,0}(m) := \{\zeta_1^m, \zeta_1^{m-1}\hat{c}_\omega, \zeta_1^{m-2}\hat{c}_\omega^2, \ldots, \zeta_1^{m-p+1}\hat{c}_\omega^{p-1}\}$$

and

$$F_{0,q}(m) := \{\zeta_0^m, \zeta_0^{m-1}\hat{c}_{\chi\omega}, \zeta_0^{m-2}\hat{c}_{\chi\omega}^2, \ldots, \zeta_0^{m-q+1}\hat{c}_{\chi\omega}^{q-1}\}.$$

(Note that ζ_1 is invertible in the first case and ζ_0 is invertible in the second.) For $p, q > 0$ we then define

$$F_{p,q}(m) := \begin{cases} \{\zeta_1^m\} \cup i_! F_{p-1,q}(m-1) & \text{if } m \ge 0, \\ \{\zeta_0^{|m|}\} \cup j_! F_{p,q-1}(m+1) & \text{if } m < 0, \end{cases}$$

where $i\colon \mathbb{P}(\mathbb{C}^{p-1+q\sigma}) \to \mathbb{P}(\mathbb{C}^{p+q\sigma})$ and $j\colon \mathbb{P}(\mathbb{C}^{p+(q-1)\sigma}) \to \mathbb{P}(\mathbb{C}^{p+q\sigma})$ are the inclusions. The pushforward $i_!$ is given algebraically by multiplication by $\hat{c}_\omega$, and $j_!$ is multiplication by $\hat{c}_{\chi\omega}$.

It is possible from this description to write down the bases explicitly, but the results are messy, having to be broken down by cases depending on where m falls in relation to p and q; this is done in [3, Proposition 4.7]. However, we can make the following general statements.

(1) For fixed p, q, and m, there are exactly $p+q$ basis elements lying in $H_{C_2}^{m\omega+\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$.

(2) Those basis elements have gradings of the form $m(\omega-2) + 2a_i + 2b_i\sigma$ for $0 \le i \le p+q-1$, where $a_i + b_i = i$.

(3) The basis element with grading $m(\omega-2) + 2a + 2b\sigma$ restricts to the nonequivariant class $\hat{c}^{a+b}$, where again $\hat{c}$ is the first nonequivariant Chern class of $O(1)$.

(4) For a given integer k, there are at most two indices i such that $a_i = k$.

Figure 2 illustrates, in the case of $\mathbb{P}(\mathbb{C}^{4+5\sigma})$, how the basis elements can be arranged for various values of m. In each case, the basis element with grading $m(\omega-2) + 2a + 2b\sigma$ is marked by a dot at (a, b).

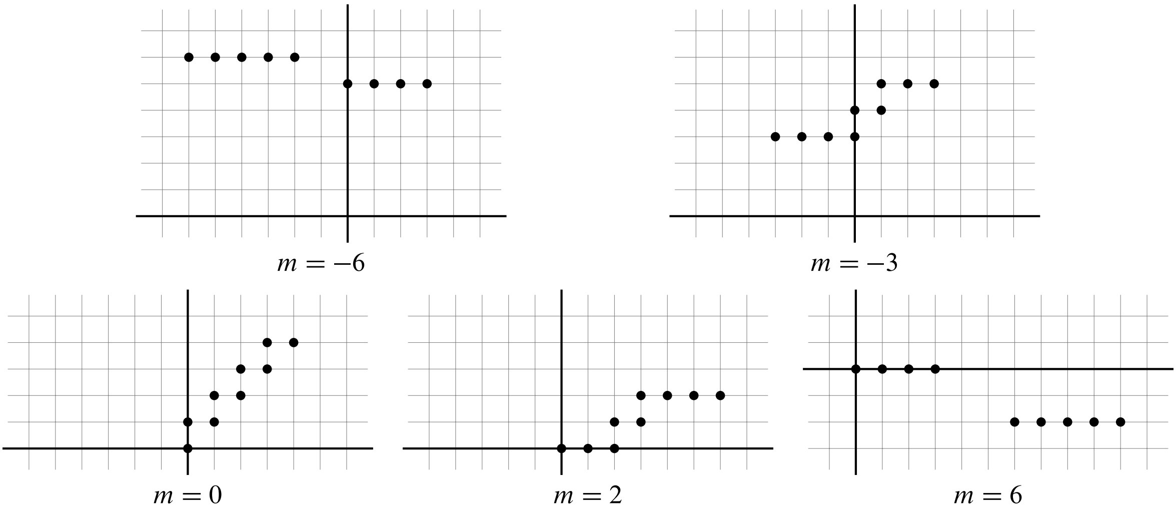

Figure 2: Bases for $H_{C_2}^{m\omega+\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{4+5\sigma})_+)$.

For ease of reference, we will write the bases as

$$F_{p,q}(m)=\{P_0^{(m)},P_1^{(m)},\dots,P_{p+q-1}^{(m)}\},$$

where $P_i^{(m)}$ is the basis element in $H_{C_2}^{m\omega+\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ restricting to the element $\hat{c}^i$ nonequivariantly. When m is understood, we will simply write P_i for $P_i^{(m)}$. We can also say that P_i is the basis element in grading $m(\omega-2)+2a+2b\sigma$ with $a+b=i$, as illustrated for $m=0$ in Figure 3.

Definition 1.3 Given any element $x\in H_{C_2}^{m\omega+\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$, we can write x uniquely as

$$x=\sum_{i=0}^{p+q-1}\alpha_i P_i^{(m)}$$

with each coefficient $\alpha_i\in\mathbb{H}$. We call the $(p+q)$–tuple (α_i) the *coefficient vector* of x.

Because elements of $\mathbb{H}$ lie in a restricted set of gradings, the number of nonzero coefficients possible for a given x may be limited, depending on the grading of x, though there are elements x for which all coefficients are nonzero.

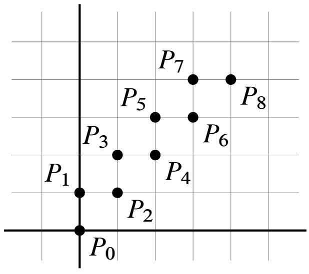

Figure 3: Basis for $H_{C_2}^{\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{4+5\sigma})_+)$.

2 The algebraic equivariant Bézout theorem

It is possible to take the calculation of Euler classes in [3] and, by brute force, work out their expression in terms of the basis for the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ discussed in the preceding section. Instead, we will take advantage of some features of the cohomology of a point to give a more conceptual approach that shows better why the calculation works the way it does.

Definition 2.1
- Let $T \subset \mathbb{H}$ consist of the elements $a\tau(\iota^{2\ell})$ for $a \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, the elements $ae^{-m}\kappa$ for $a \in \mathbb{Z}$ and $m \geq 1$, the elements ae^m for $a \in \mathbb{Z}$ and $m \geq 1$, and all of $A(C_2) = \mathbb{H}^0$.
- Let $I_e \subset T$ consist of the elements $a\tau(\iota^{2\ell})$ for $a \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, $ae^m\kappa$ for $a \in \mathbb{Z}$ and $m \in \mathbb{Z}$, and $a + bg \in A(C_2)$ such that a is even.

Note that $e^m\kappa = 2e^m$ if $m > 0$.

Proposition 2.2 *I_e is an ideal of $\mathbb{H}$.*

Proof This is a straightforward check from the known structure of $\mathbb{H}$, as given in [3]. □

On the other hand, T is not an ideal, because $e\xi \notin T$ while $e \in T$. But T is an additive subgroup.

An important fact about T is that, as shown in Figure 4, all of its elements lie in gradings of the form $n\sigma$ or $2n(1-\sigma)$, that is, on the vertical line through the origin or the diagonal through the origin with slope -1. Closed circles indicate copies of $\mathbb{Z}$, while the box at the origin is $A(C_2)$. T is a free $\mathbb{Z}$–module.

Another fact that follows from the known structure of $\mathbb{H}$ is that the quotient ring $\mathbb{H}/I_e$ is all 2–torsion.

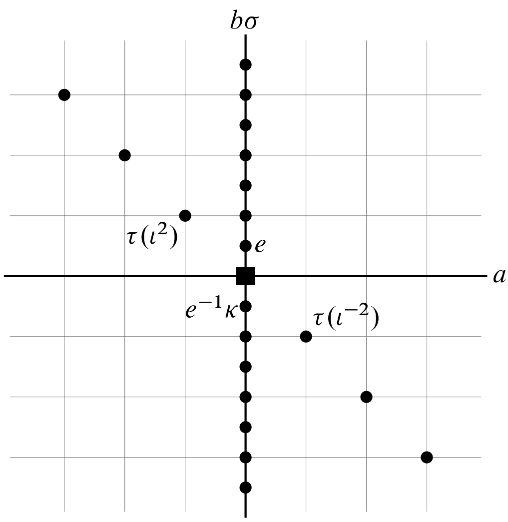

Figure 4: The subset T of $\mathbb{H}$.

Remark 2.3 The ideal I_e is almost, but not quite, the kernel of the restriction map

$$\mathbb{H} = H_{C_2}^{\mathrm{RO}(C_2)}(S^0; \underline{A}) \to H_{C_2}^{\mathrm{RO}(C_2)}(S^0; \underline{\mathbb{Z}/2}).$$

That kernel would not contain all the elements $a\tau(\iota^{-2m})$ for $m \geq 1$, but only those of the form $2a\tau(\iota^{-2m})$. Either ideal would serve our purpose here, but we chose to use the one that is slightly simpler to describe.

Definition 2.4
- Denote the set of linear combinations of elements of our preferred basis of $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ with coefficients in T by $\widetilde{T} \subset H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$.
- Let J_e be the ideal defined by

$$J_e = I_e H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+) \subset H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+).$$

Every element of J_e is a linear combination of elements from our preferred basis with coefficients in I_e (and this would be true for any basis we used). Because $J_e \subset \widetilde{T}$, the following facts about $\widetilde{T}$ apply to J_e as well.

Lemma 2.5 *Every element $x \in \widetilde{T}$ is a linear combination of at most three basis elements: if x lies in grading $m(\omega - 2) + a + b\sigma$, the only basis elements that can contribute to x are the one (if any) lying on the same diagonal as x, that is, in a grading $m(\omega - 2) + a' + b'\sigma$ with $a' + b' = a + b$, and the two (at most) lying in the same vertical line as x, that is, in gradings $m(\omega - 2) + a + b'\sigma$.*

Proof This follows from the description of the locations of the basis elements given in the preceding section together with the locations of the elements of T. □

See the example in Remark 2.15 below for an illustration of this lemma.

Proposition 2.6 *If $x \in \widetilde{T}$, then x is determined by its restrictions $\rho(x)$ and x^{C_2}.*

Proof By the preceding lemma, x can be written as a linear combination of at most three elements from our standard basis. There are various cases that should be considered. Suppose, for example, that x lies on the same diagonal as a basis element P_n and lies above two basis elements P_k and P_{k-1}. Then we can write

$$x = \alpha\tau(\iota^{2\ell})P_n + \beta e^m P_k + \gamma e^{m+2} P_{k-1}$$

for some integers α, β, γ, ℓ, and m. We now appeal to [3, Proposition 4.6], where we showed that our standard basis restricts to a nonequivariant basis for $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ and a nonequivariant basis for $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}$. We have $\rho(x) = 2\alpha\rho(P_n)$, so α is determined by $\rho(x)$. On the other hand, $x^{C_2} = \beta P_k^{C_2} + \gamma P_{k-1}^{C_2}$, so β and γ are determined by x^{C_2}.

There are other cases, for example, where x lies below two basis elements rather than above, or where it lies in the same grading as a basis element. Each of these can be handled in the same way as the case above. □

Note that this is not true for general elements of $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ because there are elements of $\mathbb{H}$ that vanish under both ρ and $(-)^{C_2}$.

For any $x \in H^{RO(\Pi B)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ we have

$$\rho(x) \in H^{\mathbb{Z}}(\mathbb{P}(\mathbb{C}^{p+q})_+),$$

so $\rho(x) = \Delta \hat{c}^k$ for some integers Δ and k, or it is 0, in which case we set $\Delta = 0$. We also have

$$x^{C_2} \in H^{\mathbb{Z}}(\mathbb{P}(\mathbb{C}^p)_+) \oplus H^{\mathbb{Z}}(\mathbb{P}(\mathbb{C}^q)_+),$$

so $x^{C_2} = (\Delta_0 \hat{c}^i, \Delta_1 \hat{c}^j)$ for some integers Δ_0, Δ_1, i, and j. (Again, we set $\Delta_0 = 0$ if $\Delta_0 \hat{c}^i = 0$ and $\Delta_1 = 0$ if $\Delta_1 \hat{c}^j = 0$.)

Definition 2.7 We call the triple of integers $(\Delta, \Delta_0, \Delta_1)$ determined as above the *C_2–degrees* of x.

Corollary 2.8 *If $x \in \widetilde{T}$, then x is determined by its grading and its C_2–degrees.*

Proof Suppose that x lies in grading $m(\omega - 2) + a + b\sigma$ and that the degrees of x are $(\Delta, \Delta_0, \Delta_1)$. By the structure of $\widetilde{T}$ and the locations of the basis elements, we can assume that a is even. Then

$$\rho(x) = \begin{cases} \Delta \hat{c}^{(a+b)/2} & \text{if } b \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad x^{C_2} = (\Delta_0 \hat{c}^{a/2}, \Delta_1 \hat{c}^{a/2-m}).$$

Thus the grading of x and its degrees determine $\rho(x)$ and x^{C_2}, so the result follows from the preceding proposition. □

In order to apply these results to derive the two parts of our Bézout theorem, we need to know a little more about the line bundles that are the summands of F as in Bézout context 0.1. In [3] we showed that the line bundles over $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ all have the form $O(d)$ or $\chi O(d)$. It is useful to further break these down into four types:

type I	bundles of the form $O(2d+1)$,
type II	bundles of the form $O(2d)$,
type III	bundles of the form $\chi O(2d+1)$,
type IV	bundles of the form $\chi O(2d)$.

The fixed points $O(2d+1)^{C_2}$ of a bundle of type I have fiber $\mathbb{C}$ over $\mathbb{P}(\mathbb{C}^p)$ and 0 over $\mathbb{P}(\mathbb{C}^{q\sigma})$, while the reverse is true for a bundle of type III. The fixed points $O(2d)^{C_2}$ of a bundle of type II have fiber $\mathbb{C}$ over both components of $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}$, while the fixed points of a bundle of type IV have fiber 0 over both components.

In [3], for $\dagger \in \{\text{I,II,III,IV}\}$ we wrote $n_\dagger$ for the number of summands of type $\dagger$ and $d_\dagger$ for the products of their degrees. These are related to the ranks and C_2–degrees of F by

$$n = n_{\mathrm{I}} + n_{\mathrm{II}} + n_{\mathrm{III}} + n_{\mathrm{IV}}, \quad n_0 = n_{\mathrm{I}} + n_{\mathrm{II}}, \quad n_1 = n_{\mathrm{II}} + n_{\mathrm{III}}, \quad \Delta = d_{\mathrm{I}} d_{\mathrm{II}} d_{\mathrm{III}} d_{\mathrm{IV}},$$

(2.9)
$$\Delta_0 = \begin{cases} d_{\mathrm{I}} d_{\mathrm{II}} & \text{if } n_0 < p, \\ 0 & \text{if } n_0 \geq p, \end{cases}$$

(2.10)
$$\Delta_1 = \begin{cases} d_{\mathrm{II}} d_{\mathrm{III}} & \text{if } n_1 < q, \\ 0 & \text{if } n_1 \geq q. \end{cases}$$

Now, d_{I} and d_{III} are always odd, and d_{II} and d_{IV} are even if and only if there is a summand of type II or IV, respectively. Notice that, when $n_{\mathrm{II}} > 0$, the quantities Δ, Δ_0, and Δ_1 will all be even. If $n_{\mathrm{II}} = 0$, then $n_0 + n_1 \le n$, which implies that

$$n_0 \le n - n_1 \le n - (n-p) = p$$

and $n_1 \le q$, similarly, with equality possible only if $n_{\mathrm{IV}} = 0$. So, if $n_{\mathrm{II}} = 0$ but $n_{\mathrm{IV}} > 0$, we will have Δ even and both Δ_0 and Δ_1 odd. When $n_{\mathrm{II}} = 0$ and $n_{\mathrm{IV}} = 0$, we will have Δ odd while Δ_0 and Δ_1 will be odd if nonzero.

Theorem 2.11 (Bézout theorem, part I) *Let F be as in Bézout context 0.1. Then $e(F)$ lies in $\widetilde{T}$, and hence is determined by its grading, which is*

$$(n_0 - n_1)(\omega - 2) + 2n_0 + 2(n - n_0)\sigma,$$

and its C_2–degrees, which are $(\Delta, \Delta_0, \Delta_1)$. Moreover, the grading and degrees can be recovered from $e(F)$. The ranks (n, n_0, n_1) are additive while the degrees are multiplicative.

Proof The additivity of the grading and the multiplicativity of the degrees are clear (but see the caveat about multiplicativity given in the introduction).

Given that n is the nonequivariant (complex) rank of F and n_0 and n_1 are the ranks of the restriction of F^{C_2} to $\mathbb{P}(\mathbb{C}^p)$ and $\mathbb{P}(\mathbb{C}^{q\sigma})$, respectively, $e(F)$ must lie in the grading given, which is the grading α with $|\alpha| = n$, $\alpha_0 = 2n_0 + 2(n-n_0)\sigma$, and $\alpha_1 = n_1 + 2(n-n_1)\sigma$.

Conversely, if $e(F)$ lies in grading $m(\omega-2) + 2a + 2b\sigma$, then we can recover $n = a+b$, $n_0 = a$, and $n_1 = a - m$.

The degrees $(\Delta, \Delta_0, \Delta_1)$ are, by the nonequivariant Bézout theorem, given by

$$\rho(e(F)) = \Delta \hat{c}^n \quad \text{and} \quad e(F)^{C_2} = (\Delta_0 \hat{c}^{n_0}, \Delta_1 \hat{c}^{n_1}),$$

using the fact that ρ and $(-)^{C_2}$ preserve Euler classes. Thus, we can recover the degrees from $e(F)$.

It remains to show that $e(F)$ is determined by its grading and C_2–degrees.

Recall the discussion above of the four types of line bundles over $\mathbb{P}(\mathbb{C}^{p+q\sigma})$. In [3, Proposition 6.5] we computed their Euler classes, which are

$$\begin{aligned} e(O(2d+1)) &= \hat{c}_\omega + d(\tau(1)\hat{c}_\omega + e^{-2}\kappa\zeta_1\hat{c}_\omega\hat{c}_{\chi\omega}) \equiv \hat{c}_\omega \pmod{J_e}, \\ e(O(2d)) &= d(\tau(\iota^{-2})\zeta_0\hat{c}_\omega + e^{-2}\kappa\hat{c}_\omega\hat{c}_{\chi\omega}) \equiv 0 \pmod{J_e}, \\ e(\chi O(2d+1)) &= \hat{c}_{\chi\omega} + d(\tau(1)\hat{c}_{\chi\omega} + e^{-2}\kappa\zeta_0\hat{c}_\omega\hat{c}_{\chi\omega}) \equiv \hat{c}_{\chi\omega} \pmod{J_e}, \\ e(\chi O(2d)) &= e^2 + d\tau(1)\zeta_0\hat{c}_\omega \equiv e^2 \pmod{J_e}. \end{aligned}$$

From (2.9) and (2.10), we see that Δ_0 and Δ_1 are both even if and only if F contains at least one summand of the form $O(2d)$ (type II). If F does not contain such a summand, then n_0 is the number of summands of the form $O(2d+1)$ and n_1 is the number of summands of the form $\chi O(2d+1)$, and we will have $n_0+n_1 \le n$. From the congruences above, we have, modulo J_e, that

$$e(F) \equiv \begin{cases} 0 & \text{if } \Delta_0 \text{ and } \Delta_1 \text{ are even,} \\ e^{2(n-n_0-n_1)}\hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{n_1} & \text{if } \Delta_0 \text{ or } \Delta_1 \text{ is odd.} \end{cases}$$

When Δ_0 or Δ_1 is odd, $n_0 \le p$ and $n_1 \le q$, with at least one of the inequalities being strict, so $\hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{n_1}$ is a basis element and $e^{2(n-n_0-n_1)}\hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{n_1} \in \widetilde{T}$. It follows that $e(F) \in \widetilde{T}$, and then the fact that $e(F)$ is determined by its grading and C_2–degrees follows from Corollary 2.8. □

By Lemma 2.5, the Euler class $e(F)$ can be written as a linear combination of just three basis elements. We next work out the explicit expression, which, by Theorem 2.11, is determined by the grading of $e(F)$ and its C_2–degrees.

Theorem 2.12 (Bézout theorem, part II) *Let F be as in Bézout context 0.1. Then we can write*

$$e(F) = \alpha P_n^{(m)} + \beta P_k^{(m)} + \gamma P_{k-1}^{(m)}$$

for some $1 \le k < p+q$ and some coefficients α, β, and γ in $\mathbb{H}$, so the coefficient vector of $e(F)$ has at most three nonzero components. Allowing for the possibility that $n=k$ or $n=k-1$, we can arrange that the coefficient α is always an integer multiple of $\tau(\iota^{2i})$ for some $i \in \mathbb{Z}$, and the coefficients β and γ are always integer multiples of e^{2i} or $e^{-2i}\kappa$ for some $i \ge 0$.

Use the briefer notation P_n and write $\epsilon = 0$ or 1 for the remainder on dividing $n+n_0+n_1$ by 2. We have

$$P_n = \begin{cases} \zeta_0^{-(n+n_0-n_1-2p)}\hat{c}_\omega^{p}\hat{c}_{\chi\omega}^{n-p} & \text{if } n+n_0-n_1 > 2p, \\ \zeta_1^{-(n-n_0+n_1-2q)}\hat{c}_\omega^{n-q}\hat{c}_{\chi\omega}^{q} & \text{if } n-n_0+n_1 > 2q, \\ \zeta_0^{\epsilon}\hat{c}_\omega^{(n+n_0-n_1+\epsilon)/2}\hat{c}_{\chi\omega}^{(n-n_0+n_1-\epsilon)/2} & \text{otherwise,} \end{cases}$$

$$P_k = \begin{cases} \zeta_0\hat{c}_\omega^{n_0+1}\hat{c}_{\chi\omega}^{n_1} & \text{if } n_0 < p, \\ \zeta_0^{-(n_0-p)}\hat{c}_\omega^{p}\hat{c}_{\chi\omega}^{n_1} & \text{if } n_0 \ge p, \end{cases} \quad \text{and} \quad P_{k-1} = \begin{cases} \hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{n_1} & \text{if } n_1 < q, \\ \zeta_1^{-(n_1-q)}\hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{q} & \text{if } n_1 \ge q. \end{cases}$$

The coefficient α will be an integer multiple of

$$\tau_n = \begin{cases} \tau(\iota^{2(n-n_1-p)}) & \text{if } n+n_0-n_1 > 2p, \\ \tau(\iota^{2(n-n_0-q)}) & \text{if } n-n_0+n_1 > 2q, \\ \tau(\iota^{n-n_0-n_1-\epsilon}) & \text{otherwise.} \end{cases}$$

Finally, write $\bar{n}_0 = \min\{n_0, p-1\}$ and $\bar{n}_1 = \min\{n_1, q\}$. Then we break the result into the following cases:

(1) *If Δ is even, then*

$$\alpha = \tfrac{1}{2}\Delta\tau_n, \quad \beta = \tfrac{1}{2}(\Delta_1-\Delta_0)e^{-2(\bar{n}_0+\bar{n}_1-n+1)}\kappa, \quad \gamma = \tfrac{1}{2}(\Delta_0)e^{-2(\bar{n}_0+\bar{n}_1-n)}\kappa, \quad k = \bar{n}_0+\bar{n}_1+1.$$

(2) *If Δ is odd and $\Delta_0 \ne 0$, then*

$$\alpha = \tfrac{1}{2}(\Delta-\Delta_0)\tau(1), \quad \beta = \tfrac{1}{2}(\Delta_1-\Delta_0)e^{-2}\kappa, \quad \gamma = \Delta_0, \quad k = n+1.$$

(3) *If Δ is odd and $\Delta_0 = 0$, then*

$$\alpha = \tfrac{1}{2}(\Delta-\Delta_1)\tau(1), \quad \beta = 0, \quad \gamma = \Delta_1, \quad k = n+1.$$

Remark 2.13 We should point out some abuses of notation we are indulging in. The formulas for P_k and P_{k-1} evaluate to 0, not basis elements, when both $n_0 \ge p$ and $n_1 \ge q$. In the case $n_0 < p-1$ and $n_1 \ge q$, the formula for P_k is not a basis element, but we know that its coefficient will be a multiple of $e^m\kappa$ for some integer m, and the product $e^m\kappa P_k = 0$ in that case because of the relations in the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$. A similar vanishing happens in the case of P_{k-1} when $n_0 > p$ and $n_1 < q$. Finally, the formulas for P_k and P_{k-1} coincide when $n_0 = p$ and $n_1 < q$, but in that case $\Delta_0 = 0$ so only one copy of this basis element appears in the formula for $e(F)$.

Proof Theorem 2.11 and Lemma 2.5 imply the first claim, that we can write $e(F)$ in terms of just three basis elements.

To determine P_n, P_k, and P_{k-1}, we recall from [3, Proposition 4.7] that the basis elements take one of the six possible forms

$$\zeta_1^m \hat{c}_\omega^a \quad \text{for } m>1, a<p, \qquad \zeta_0^m \hat{c}_{\chi\omega}^b \quad \text{for } m>1, b<q, \qquad \hat{c}_\omega^a \hat{c}_{\chi\omega}^b \quad \text{for } a\le p, b\le q,$$

$$\zeta_0 \hat{c}_\omega^a \hat{c}_{\chi\omega}^b \quad \text{for } a\le p, b<q, \qquad \zeta_0^{-m} \hat{c}_\omega^p \hat{c}_{\chi\omega}^b \quad \text{for } m>0, b<q, \qquad \zeta_1^{-m} \hat{c}_\omega^a \hat{c}_{\chi\omega}^q \quad \text{for } m>0, a<p,$$

where we recall that $\hat{c}_\omega^p \hat{c}_{\chi\omega}^q = 0$, so we do not have $a = p$ and $b = q$ above.

We noted earlier that $e(F)$ lies in grading

$$\operatorname{grad} e(F) = (n_0 - n_1)(\omega - 2) + 2n_0 + 2(n - n_0)\sigma.$$

P_n is the unique basis element having grading in $(n_0 - n_1)(\omega - 2) + \mathrm{RO}(C_2)$ restricting to $\hat{c}^n$, and we can check that the formula given in the statement of the theorem has those properties. Similarly, P_k and P_{k-1} are the (at most) two basis elements having gradings of the form $(n_0 - n_1)(\omega - 2) + 2n_0 + 2b\sigma$, and we can check that the formulas given have that property. The coefficient τ_n is the element of the form $\tau(\iota^{2i})$ such that $\tau_n P_n$ lies in the same grading as $e(F)$. The terms of the form $e^m\kappa$ multiplying P_k and P_{k-1} in the formulas for $e(F)$ are determined similarly.

To verify the coefficients of P_n, P_k, and P_{k-1}, we use the fact that $e(F)$ is determined by the nonequivariant elements

$$\rho(e(F)) = \Delta \hat{c}^n \quad \text{and} \quad e(F)^{C_2} = (\Delta_0 \hat{c}^{n_0}, \Delta_1 \hat{c}^{n_1}),$$

so we simply need to check that the formulas of the theorem have the correct values on applying these restriction maps.

First note that, regardless of which case we fall in, we will always have

$$\rho(\tau_n P_n) = 2\hat{c}^n \quad \text{and} \quad (\tau_n P_n)^{C_2} = (0, 0).$$

For P_k and P_{k-1} we have

$$\rho(P_k) = \hat{c}^k, \quad \rho(P_{k-1}) = \hat{c}^{k-1}, \quad P_k^{C_2} = (0, \hat{c}^{n_1}) \quad \text{and} \quad P_{k-1}^{C_2} = (\hat{c}^{n_0}, \hat{c}^{n_1}),$$

which includes the possibility that $P_{k-1}^{C_2} = (\hat{c}^{n_0}, 0)$ if $n_1 \ge q$.

Now, when Δ is even, in the formulas given, β and γ each have a factor of the form $e^m\kappa$, and $\rho(e^m\kappa)=0$ and $(e^m\kappa)^{C_2}=2$. Combined with the formulas above, this verifies case (1) of the theorem, except that we should say something about the parities of Δ_0 and Δ_1. From the discussion before Theorem 2.11, because Δ is even, Δ_0 and Δ_1 have the same parity. There is a possibility that Δ_0 is odd, but this can happen only when $n_{\mathrm{II}}=0$ and $n_{\mathrm{IV}}>0$, in which case $n_0<p$, $n_1<q$, and $n_0+n_1<n$. The coefficient γ in that case is

$$\gamma = \tfrac{1}{2}\Delta_0 e^{-2(n_0+n_1-n)}\kappa = \tfrac{1}{2}(\Delta_0)2e^{2(n-n_0-n_1)},$$

which we interpret as $\Delta_0 e^{2(n-n_0-n_1)}$ by another abuse of notation. (The abuse is that division by 2 is not well defined in $\mathbb{H}$.) We then use that $\rho(e^m)=0$ and $(e^m)^{C_2}=1$ for $m>0$.

If Δ is odd, then $n=n_0+n_1$, $n_0\le p$, and $n_1\le q$. If Δ_0 and Δ_1 are both nonzero, then $n_0<p$ and $n_1<q$, $P_n=P_{k-1}$, and the formula in case (2) of the theorem is easily verified.

If $\Delta_0\ne 0$ but $\Delta_1=0$, then $n_0<p$ and $n_1=q$. In this case,

$$e^{-2}\kappa P_k = e^{-2}\kappa\zeta_0\hat{c}_\omega^{n_0+1}\hat{c}_{\chi\omega}^q = 0,$$

so we allow the abuse of notation that $\Delta_1-\Delta_0$ is odd in the formula for β. With that caveat, the verification of case (2) can be completed.

In case (3), since $\Delta_0=0$ we must have $\Delta_1\ne 0$ and odd. The verification is then just as for the previous cases.

The asymmetry in these formulas comes from an asymmetry in our preferred basis regarding $\hat{c}_\omega$ vs $\hat{c}_{\chi\omega}$. □

Remark 2.14 Theorems 2.11 and 2.12 give us two related ways of determining $e(F)$: by the ranks (n,n_0,n_1) and the C_2–degrees $(\Delta,\Delta_0,\Delta_1)$, and also by its triple of nonzero coefficients. The advantage of using the degrees is that they are multiplicative. This is simpler to calculate with, and also parallels the result of the nonequivariant Bézout theorem that degrees are multiplicative under intersection of projective varieties.

Remark 2.15 The summary of Theorem 2.12 is that $e(F)$ can be expressed in terms of at most three basis elements. This is not a restriction imposed by the locations of the basis elements. As an example, consider $\mathbb{P}(\mathbb{C}^{5+5\sigma})$ and the bundle $F=4\chi O(2)$, the sum of four copies of $\chi O(2)$, so $n=4$ and $n_0=n_1=0$. This Euler class lives in grading

$$(n_0-n_1)(\omega-2)+2n_0+2(n-n_0)\sigma = 8\sigma.$$

Figure 5 shows the location of $e(F)$, the "×" at 8σ, and the locations of the basis elements in the $\mathrm{RO}(C_2)$–grading. The five basis elements within the shaded area have nonzero multiples in degree 8σ, so could conceivably contribute to $e(F)$, but the theorem says that it can be written in terms of just three of them: P_4, the one on the same diagonal as $e(F)$, and the two below it, P_0 and P_1.

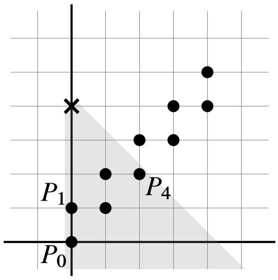

Figure 5: Location of $e(4\chi O(2))$.

In fact, we are in Theorem 2.12(1), with $\Delta = 8$ and $\Delta_0 = \Delta_1 = 1$, so

$$e(4\chi O(2)) = 8\tau(\iota^4)P_4 + 0P_1 + e^8 P_0 = 8\tau(\iota^4)\hat{c}_\omega^2\hat{c}_{\chi\omega}^2 + e^8.$$

As it happens, P_1 does not actually contribute in this example.

Remark 2.16 In [3], we looked in detail at the case $n = p + q - 1$, where the hypersurfaces associated with the line bundle summands of F intersect generically in a C_2–set of points in $\mathbb{P}(\mathbb{C}^{p+q\sigma})$. In that case, we showed that the explicit formula for $e(F)$ can be read as telling us how that collection of points breaks down as free orbits versus fixed points in each of the components of $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}$. In a followup to this paper, we will show how the Euler class more generally gives us geometric information about the intersection of hypersurfaces.

3 Comparison with constant $\mathbb{Z}$ coefficients

Another equivariant cohomology theory commonly used is ordinary cohomology with coefficients in $\underline{\mathbb{Z}}$, the constant-$\mathbb{Z}$ Mackey functor. We calculate the Euler class $e(F)$ with $\underline{\mathbb{Z}}$ coefficients and compare it to the class obtained with Burnside ring coefficients.

As shown in [2], $H_{C_2}^{\mathrm{RO}(C_2)}(S^0;\underline{\mathbb{Z}})$ is obtained from $\mathbb{H}$ by setting $\kappa = 0$. This has the effect of removing the elements $e^{-n}\kappa$ and making $2e = 0$. Since $\kappa = 2 - g$, it also has the effect of setting $g = 2$. Put another way, this theory cannot distinguish between a free orbit and two fixed points.

Because the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ with $\underline{A}$ coefficients is free over the cohomology of a point, we obtain the cohomology with $\underline{\mathbb{Z}}$ coefficients by setting $\kappa = 0$. The result is the following:

Theorem 3.1 [3, Corollary 5.4] *Let $0 \le p, q < \infty$ with $p + q > 0$. Then $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+;\underline{\mathbb{Z}})$ is a free module over $H_{C_2}^{\mathrm{RO}(C_2)}(S^0;\underline{\mathbb{Z}})$. Its structure as a graded commutative algebra over $H_{C_2}^{\mathrm{RO}(C_2)}(S^0;\underline{\mathbb{Z}})$ is described as in Theorem 1.2, except that the relation $\zeta_1\hat{c}_{\chi\omega} = (1-\kappa)\zeta_0\hat{c}_\omega + e^2$ is replaced by the relation*

$$\zeta_1\hat{c}_{\chi\omega} = \zeta_0\hat{c}_\omega + e^2. \qquad \square$$

Setting $\kappa = 0$ in Theorem 2.12, remembering that e^m is 2–torsion, and paying attention to the abuses of notation mentioned in the proof of that theorem, we get the following:

Theorem 3.2 (Bézout's theorem for constant $\mathbb{Z}$ coefficients) *Let F be as in Bézout context 0.1. Then the Euler class $e_{\mathbb{Z}}(F) \in H^{\mathrm{RO}(\Pi B)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+;\underline{\mathbb{Z}})$ is given by*

$$e_{\mathbb{Z}}(F) = \begin{cases} \frac{1}{2}\Delta\tau_n P_n^{(m)} & \text{if } \Delta,\ \Delta_0 \text{ and } \Delta_1 \text{ are even,} \\ \frac{1}{2}\Delta\tau_n P_n^{(m)} + e^{2(n-n_0-n_1)} P_{k-1}^{(m)} & \text{if } \Delta \text{ is even and } \Delta_0 \text{ or } \Delta_1 \text{ is odd,} \\ \Delta P_n^{(m)} & \text{if } \Delta \text{ is odd,} \end{cases}$$

where, writing $\epsilon = 0$ or 1 for the remainder on dividing $n+n_0+n_1$ by 2, we set

$$P_n^{(m)} = \begin{cases} \zeta_0^{-(n+n_0-n_1-2p)} \hat{c}_\omega^p \hat{c}_{\chi\omega}^{n-p} & \text{if } n+n_0-n_1 > 2p, \\ \zeta_1^{-(n-n_0+n_1-2q)} \hat{c}_\omega^{n-q} \hat{c}_{\chi\omega}^{q} & \text{if } n-n_0+n_1 > 2q, \\ \zeta_0^{\epsilon} \hat{c}_\omega^{(n+n_0-n_1+\epsilon)/2} \hat{c}_{\chi\omega}^{(n-n_0+n_1-\epsilon)/2} & \text{otherwise,} \end{cases}$$

$$\tau_n = \begin{cases} \tau(\iota^{2(n-n_1-p)}) & \text{if } n+n_0-n_1 > 2p, \\ \tau(\iota^{2(n-n_0-q)}) & \text{if } n-n_0+n_1 > 2q, \\ \tau(\iota^{n-n_0-n_1-\epsilon}) & \text{otherwise,} \end{cases}$$

and, when Δ is even and Δ_0 or Δ_1 is odd,

$$P_{k-1}^{(m)} = \hat{c}_\omega^{n_0} \hat{c}_{\chi\omega}^{n_1}. \qquad \square$$

While this result has the benefit of relative simplicity, it carries significantly less information than Theorem 2.12. In particular, we cannot reconstruct Δ_0 and Δ_1 from $e_{\mathbb{Z}}(F)$. This follows from the formula in the theorem, but we can also look again at the fixed-point map $(-)^{C_2}$ to see why this must happen. As defined in [4], the fixed-point map takes G–equivariant cohomology with coefficients in a Mackey functor $\underline{T}$ to nonequivariant cohomology with coefficients in $\underline{T}^G$. In the case of the group C_2, we have

$$\underline{T}^{C_2} = \underline{T}(C_2/C_2)/\tau(\underline{T}(C_2/e)).$$

This gives $\underline{A}^{C_2} = \mathbb{Z}$, but $\underline{\mathbb{Z}}^{C_2} = \mathbb{Z}/2$. We then get the following:

Corollary 3.3 *With F as in Bézout context 0.1, we have*

$$e_{\mathbb{Z}}(F)^{C_2} = (\Delta_0 \hat{c}^{n_0}, \Delta_1 \hat{c}^{n_1}) \in H^{2a}(\mathbb{P}(\mathbb{C}^p)_+;\mathbb{Z}/2) \oplus H^{2(a-m)}(\mathbb{P}(\mathbb{C}^{q\sigma})_+;\mathbb{Z}/2),$$

so

$$e_{\mathbb{Z}}(F)^{C_2} = \begin{cases} (0,0) & \text{if } \Delta_0 \text{ and } \Delta_1 \text{ are even,} \\ (\hat{c}^{n_0}, \hat{c}^{n_1}) & \text{if } \Delta_0 \text{ or } \Delta_1 \text{ is odd.} \end{cases}$$

Proof From the commutativity of the diagram

$$\begin{array}{ccc} H^{\mathrm{RO}(\Pi B)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+;\underline{A}) & \longrightarrow & H^{\mathrm{RO}(\Pi B)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+;\underline{\mathbb{Z}}) \\ \downarrow{\scriptstyle (-)^{C_2}} & & \downarrow{\scriptstyle (-)^{C_2}} \\ H^{\mathbb{Z}}(\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}_+;\mathbb{Z}) & \longrightarrow & H^{\mathbb{Z}}(\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}_+;\mathbb{Z}/2) \end{array}$$

where the horizontal arrows are given by change of coefficients, $e_{\mathbb{Z}}(F)^{C_2}$ is just the reduction of $e(F)^{C_2}$ modulo 2. $\square$

Thus, from this Euler class we cannot recover Δ_0 and Δ_1, only their parities. This goes back to the fact that, because $g = 2$, cohomology with $\underline{\mathbb{Z}}$ coefficients cannot distinguish between a free orbit and two fixed points, and hence retains only parity information about fixed points.

For example, in the case $n = p+q-1$ discussed in detail in [3], we can think of the Euler class in terms of the finite C_2–set given by the zero locus of a section of F, or the intersection of the hypersurfaces given by the zero loci of sections of the line bundles making up F. The Euler class with Burnside ring coefficients completely determines this C_2–set, including how many fixed points lie in each component of $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}$. The Euler class with constant $\mathbb{Z}$ coefficients can tell us only the parity of the number of fixed points in each component.

4 Comparison with Borel cohomology

Borel cohomology was the first theory thought of as equivariant ordinary cohomology, but is a considerably weaker theory than Bredon cohomology. (See, for example, May's discussion in [7].) There is a map from ordinary cohomology with $\underline{\mathbb{Z}}$ coefficients to Borel cohomology, so the latter is also weaker than cohomology with $\underline{\mathbb{Z}}$ coefficients. To see how much weaker, let us look at the calculation of $e(F)$ in Borel cohomology.

We take Borel cohomology to be the $\mathrm{RO}(C_2)$–graded theory defined on based C_2–spaces by

$$BH_{C_2}^{\mathrm{RO}(C_2)}(X) = H_{C_2}^{\mathrm{RO}(C_2)}((EC_2)_+ \wedge X),$$

where, as usual, we use Burnside ring coefficients on the right, but suppress them from the notation. (Because EC_2 is free, and $\underline{A} \to \underline{\mathbb{Z}}$ is an isomorphism at the C_2/e level, we could instead use $\underline{\mathbb{Z}}$ coefficients and get naturally isomorphic results.) This is the usual Borel cohomology with $\mathbb{Z}$ coefficients, but we have expanded the grading from the common $\mathbb{Z}$ to $\mathrm{RO}(C_2)$. As shown in [2], the Borel cohomology of a point is $\mathbb{H}$ with ξ inverted:

$$BH_{C_2}^{\mathrm{RO}(C_2)}(S^0) \cong \mathbb{Z}[e, \xi, \xi^{-1}]/\langle 2e \rangle.$$

Here $\deg e = \sigma$ and $\deg \xi = 2\sigma - 2$, as before. In the map $\mathbb{H} \to BH_{C_2}^{\mathrm{RO}(C_2)}(S^0)$, κ goes to 0. As with cohomology with $\underline{\mathbb{Z}}$ coefficients, Borel cohomology cannot tell the difference between g and 2.

Note that, if we restrict to the $\mathbb{Z}$ grading, as is usually done, we get a polynomial algebra in $e^2\xi^{-1}$ modulo $2e^2\xi^{-1} = 0$, a copy of the group cohomology of C_2 with $\mathbb{Z}$ coefficients. If we restrict the grading to $\sigma + \mathbb{Z}$, we see the group cohomology of C_2 with twisted $\mathbb{Z}$ coefficients. That the twisted and untwisted cohomologies can be combined in a single algebra like this seems to have been first observed by Čadek [1].

Because the ordinary C_2–cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ is free over the cohomology of a point, we obtain its Borel cohomology also by inverting ξ. On doing so, the elements ζ_0 and ζ_1 become invertible, with the result that, if we continued to grade on $\mathrm{RO}(\Pi B)$, the groups outside the $\mathrm{RO}(C_2)$ grading would all be isomorphic to groups in the $\mathrm{RO}(C_2)$ grading via multiplication by an appropriate power of, say, ζ_0.

So we lose nothing by considering the $RO(C_2)$–graded part only. To give the explicit result, let $\hat{c}$ be the image of $\zeta_0\hat{c}_\omega$ in $BH^{2\sigma}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$. The following is then a corollary of Theorem 1.2:

Corollary 4.1 *Let $0 \le p, q < \infty$ with $p+q>0$. Then $BH^{RO(C_2)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ is a free module over $BH^{RO(C_2)}_{C_2}(S^0)$, and as a (graded) commutative algebra over $BH^{RO(C_2)}_{C_2}(S^0)$, $BH^{RO(\Pi B)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ is generated by $\hat{c}$ in degree 2σ, which satisfies the single relation*

$$\hat{c}^p(\hat{c}+e^2)^q = 0. \qquad \square$$

Of course, we could also use as a generator the element $c' = \xi^{-1}\hat{c}$ in degree 2, but the relation is then

$$(c')^p(c'+e^2\xi^{-1})^q = 0.$$

For the simplicity of the relation, and to keep the generator more closely related to an element from ordinary cohomology, we prefer to use $\hat{c}$.

We view $\hat{c}$ as the Euler class of $\omega^\vee$. The Euler class of $\chi\omega^\vee$ is then $\hat{c}+e^2$, the image of $\zeta_1\hat{c}_{\chi\omega}$. In doing this, we are choosing to say that ω is a rank-2σ bundle over $EC_2 \times \mathbb{P}(\mathbb{C}^{p+q\sigma})$. Because EC_2 is free, we are as free to say ω has rank 2σ as to say it has rank 2.

Another way of seeing that $e(\chi\omega) = \hat{c}+e^2$ is to recall that $\chi\omega = \omega \otimes_{\mathbb{C}} \mathbb{C}^\sigma$, then use the additive formal group law of nonequivariant ordinary cohomology and the fact that $e(\mathbb{C}^\sigma) = e^2$.

Now consider the Euler classes of the bundles $O(d)$ and $\chi O(d)$, all of which we will think of as having rank 2σ. As a corollary of [3, Proposition 6.5], or as a consequence of the formal group law for nonequivariant cohomology, we have the following:

Proposition 4.2 *In the Borel cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ we have*

$$e(O(d)) = d\hat{c} \quad \text{and} \quad e(\chi O(d)) = d\hat{c}+e^2$$

for every $d \in \mathbb{Z}$. $\square$

Theorem 4.3 (Bézout's theorem for Borel cohomology) *Let F be as in Bézout context 0.1. The Euler class of F in the Borel cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ is*

$$e_{BH}(F) = \begin{cases} \Delta\hat{c}^n & \text{if } \Delta, \Delta_0 \text{ and } \Delta_1 \text{ are even,} \\ \Delta\hat{c}^n + e^{2(n-n_0-n_1)}\hat{c}^{n_0}(\hat{c}+e^2)^{n_1} & \text{if } \Delta \text{ is even and } \Delta_0 \text{ or } \Delta_1 \text{ is odd,} \\ \Delta\hat{c}^{n_0}(\hat{c}+e^2)^{n-n_0} & \text{if } \Delta \text{ is odd.} \end{cases}$$

Proof These formulas can be derived from the preceding proposition or from Theorem 3.2, using the fact that $\tau(1) = 2$ in Borel cohomology. $\square$

As we saw with ordinary cohomology with $\underline{\mathbb{Z}}$ coefficients, the Euler class in Borel cohomology contains significantly less information than the one in ordinary cohomology with Burnside ring coefficients. The fixed-point map would be

$$(-)^{C_2}\colon H^{RO(C_2)}_{C_2}((EC_2)_+ \wedge \mathbb{P}(\mathbb{C}^{p+q\sigma})_+) \to H^{\mathbb{Z}}(((EC_2)_+ \wedge \mathbb{P}(\mathbb{C}^{p+q\sigma})_+)^{C_2}; \mathbb{Z}) = H^{\mathbb{Z}}(*; \mathbb{Z}) = 0.$$

Thus, Borel cohomology contains no information at all about fixed points.

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Received: 9 November 2022 Revised: 21 March 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 4 (pages 1809–2387) 2024