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Hopf ring structure on the mod p cohomology of symmetric groups

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Correction to the article
Hopf ring structure on the mod p cohomology of symmetric groups

LORENZO GUERRA

We correct Propositions 5.4, 5.5 and 5.6 of the author's previous article (Algebr. Geom. Topol. 17 (2017) 957–982).

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In our previous paper [1], the formulas describing the action of the mod p Steenrod algebra on Hopf ring generators are incorrect. This is not caused by a flaw in the general argument, but by our misstatement of Hung and Minh's theorem on mod p modular invariants. In this note, we refer to our original article for notation and conventions.

We begin by providing the accurate statement of Hung and Minh's theorem. Although these two authors decided to compute coefficients explicitly, we believe that, for our purpose, their result is best restated using the total Steenrod class $\mathcal{P}^* = \sum_{r \geq 0} \mathcal{P}^r$.

Hung and Minh's theorem [2, Theorem 4.1, page 42] *In* $[\mathbb{Z}_p[y_1, \dots, y_n] \otimes \Lambda(x_1, \dots, x_n)]^{\mathrm{Gl}_n(\mathbb{Z}_p)}$, *the algebra of modular invariants for* $\mathrm{Gl}_n(\mathbb{Z}_p)$, *for all* $0 \leq s < n$ *and* $0 \leq s_1 < \dots < s_k < n$,

$$\mathcal{P}^*(d_{s,n-s}) = (d_{0,n} + \dots + d_{s,n-s})(d_{0,n} + \dots + d_{n-1,1} + 1)^{p-1} - (d_{0,n} + \dots + d_{s-1,n-s+1})^p,$$

$$\mathcal{P}^*(R_{n,s_1,\dots,s_k}) = \sum_{\substack{0 \leq t_1 < t_2 < \dots < t_{k+1} \leq n \\ s_{i-1} < t_i \leq s_i}} \sum_{i=1}^{k+1} [(-1)^{k+1-i} R_{n,t_1,\dots,\hat{t}_i,\dots,t_{k+1}} d_{t_i,n-t_i}](d_{0,n} + \dots + d_{n-1,1} + 1)^{p-2}.$$

In the expression above, we let, by convention, $s_{-1} = -1$, $s_{k+1} = n$, $R_{n,t_1,\dots,t_k,n} = 0$, *and* $d_{n,0} = 1$.

We now provide the corrected formulas for the Steenrod powers of generators.

Corrected version of Proposition 5.4 *Let* $0 \leq k < n$. *Let* $\mathrm{Outgrowth}(\gamma_{n-k,p^k})$ *be the set of full-width Hopf monomials* $x \in H^*(\Sigma_{p^n}; \mathbb{Z}_p)$ *of type* C *with* $\mathrm{effsc}(x) \geq n-k$ *and* $\mathrm{ht}(x) \leq p$. *Then, for all* $0 \leq r \leq p^n - p^k$,

$$\mathcal{P}^r(\gamma_{n-k,p^k}) = \sum_{\substack{x \in \mathrm{Outgrowth}(\gamma_{n-k,p^k}) \\ \deg(x) = 2(p^n - p^k + r(p-1))}} c_{n,k,x} x,$$

where $c_{n,k,x}$ is a scalar coefficient calculated as follows.

If $x = \prod_{i=1}^{n} \gamma_{i,p^{n-i}}^{e_i}$ $(e_i \geq 0)$ is a gathered block, then

$$c_{n,k,x} = (-1)^{n-k+\sum_{i=1}^{n} ie_i} \frac{1}{p} \frac{p!}{\prod_{i=0}^{n} e_i!} \Bigl(\sum_{i=n-k}^{n} e_i \Bigr),$$

where we let by convention $e_0 = p - \sum_{i=1}^{n} e_i$ and we reduce $c_{n,k,x}$ mod p after calculating its expression in $\mathbb{Z}$ first. If x is not a gathered block, we can write x as a nonzero scalar multiple of a transfer product $b_1 \odot \cdots \odot b_r$, where each b_j is a column of width equal to a power p^{l_j} of p. Then, we put $c_{n,k,x} = \prod_{j=1}^{r} c_{l_j, l_j - n + k, b_j}$.

Corrected version of Proposition 5.5 • *Let $1 \leq j \leq k$. Let $\mathrm{Outgrowth}(\alpha_{j,k})$ be the set of full-width Hopf monomials $x \in H^*(\Sigma_{p^k}; \mathbb{Z}_p)$ of type A with $\mathrm{effsc}(x) \geq k$ and $\mathrm{ht}(x) \leq p$. Then, for all $0 \leq r < p^k - p^{k-j}$,*

$$\mathcal{P}^r(\alpha_{j,k}) = \sum_{\substack{x \in \mathrm{Outgrowth}(\alpha_{j,k}) \\ \deg(x) = 2p^k - 2p^{k-j} - 1 + 2r(p-1)}} c'_{k,j,x} x,$$

where $c'_{k,j,x}$ is a scalar coefficient calculated as follows.

If $x = \alpha_{t,k} \prod_{i=1}^{k} \gamma_{i,p^{k-i}}^{e_i}$ with $e_i \geq 0$ and $1 \leq t \leq k$ is a gathered block, then

$$c'_{k,j,x} = \begin{cases} (-1)^{j+t+\sum_{i=1}^{k} ie_i} \bigl((p-1)!/\prod_{i=0}^{k} e_i!\bigr)\bigl(\sum_{l=j}^{k} e_l\bigr) & \text{if } 1 \leq t < j, \\ -(-1)^{j+t+\sum_{i=1}^{k} ie_i} \bigl((p-1)!/\prod_{i=0}^{k} e_i!\bigr)\bigl(\sum_{l=0}^{j-1} e_l\bigr) & \text{if } j \leq t \leq k, \end{cases}$$

where we let by convention $e_0 = p - 1 - \sum_{i=1}^{n} e_i$. All Hopf monomials $x \in \mathrm{Outgrowth}(\alpha_{j,k})$ have this form.

• *Let $1 \leq i < j \leq k$. Let $\mathrm{Outgrowth}(\beta_{i,j,p^{k-j}})$ be the set of full-width Hopf monomials $x \in H^*(\Sigma_{p^k}; \mathbb{Z}_p)$ of type B with $\mathrm{effsc}(x) \geq j$ and $\mathrm{ht}(x) \leq p$. Then, for all $0 \leq r \leq p^k - p^{k-j} - p^{k-i}$,*

$$\mathcal{P}^r(\beta_{i,j,p^{k-j}}) = \sum_{\substack{x \in \mathrm{Outgrowth}(\beta_{i,j,p^{k-j}}) \\ \deg(x) = 2(p^k - p^{k-j} - p^{k-i} + r(p-1))}} c''_{k,i,j,x} x,$$

where $c''_{k,i,j,x}$ is a scalar coefficient calculated as follows.

If $x = \beta_{t,u,p^{k-u}} \prod_{i=1}^{k} \gamma_{i,p^{k-i}}^{e_i}$ with $e_i \geq 0$ and $1 \leq t < u \leq k$ is a gathered block, then

$$c''_{k,i,j,x} = \begin{cases} -(-1)^{i+j+t+u+\sum_{m=1}^{k} me_m} \bigl((p-1)!/\prod_{m=0}^{k} e_m!\bigr)\bigl(\sum_{l=j}^{k} e_l\bigr) & \text{if } 1 \leq t < i \leq u < j \leq k, \\ (-1)^{i+j+t+u+\sum_{m=1}^{k} me_m} \bigl((p-1)!/\prod_{m=0}^{k} e_m!\bigr)\bigl(\sum_{l=i}^{j-1} e_l\bigr) & \text{if } 1 \leq t < i < j \leq u \leq k, \\ -(-1)^{i+j+t+u+\sum_{m=1}^{k} me_m} \bigl((p-1)!/\prod_{m=0}^{k} e_m!\bigr)\bigl(\sum_{l=0}^{i-1} e_l\bigr) & \text{if } i \leq t < j \leq u \leq k, \\ 0 & \text{otherwise,} \end{cases}$$

where we let by convention $e_0 = p - 1 - \sum_{m=1}^{k} e_m$. If x is not a gathered block, we can write x as a nonzero scalar multiple of a transfer product $b_1 \odot \cdots \odot b_r$, where each b_m is a column of width equal to a power p^{l_m} of p. Then, we put $c''_{k,i,j,x} = \prod_{m=1}^{r} c''_{l_m,i,j,b_m}$.

Corrected version of Proposition 5.6 *The following formulas hold*:

- $\beta(\alpha_{j,k}) = \gamma_{k,1}$ *if* $j = k$ *and is equal to* 0 *otherwise.*
- $\beta(\beta_{i,j,p^k}) = \alpha_{i,j} \odot \beta_{i,j,p^k - 1}$.
- $\beta(\gamma_{j,p^k}) = 0$.

The proof of these propositions is essentially unchanged from the author's original article, except that we use the correct statement of Hung and Mihn's theorem to compute coefficients.

References

[1] **L Guerra**, *Hopf ring structure on the* mod p *cohomology of symmetric groups*, Algebr. Geom. Topol. 17 (2017) 957–982 MR Zbl

[2] **N H V Hung**, **P A Minh**, *The action of the mod p Steenrod operations on the modular invariants of linear groups*, Vietnam J. Math. 23 (1995) 39–56 MR Zbl

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