Algebraic & Geometric Topology

Volume 24 (2024)

Issue 4 (pages 1809–2387)

msp

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US $705/year for the electronic version, and $1040/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

Algebraic & Geometric Topology **24**:4 (2024) 1809–1840
DOI: 10.2140/agt.2024.24.1809
Published: 16 July 2024

Möbius structures, quasimetrics and completeness

MERLIN INCERTI-MEDICI

We study cross ratios from an axiomatic viewpoint, also known as the study of Möbius spaces. We characterise cross ratios induced by quasimetrics in terms of topological properties of their image. Furthermore, we generalise the notions of Cauchy sequences and completeness to Möbius spaces and prove the existence of a unique completion under an extra assumption that, again, can be expressed in terms of the image of the cross ratio.

20F65; 53C23

1 Introduction

Let Z be a set, ρ a metric on Z, possibly with a point at infinity; see Section 2 for definitions. We can define the cross ratio induced by ρ with the formula

$$\mathrm{cr}(z_1, z_2, z_3, z_4) := \frac{\rho(z_1, z_2)\rho(z_3, z_4)}{\rho(z_1, z_3)\rho(z_2, z_4)}, \tag{1-1}$$

where the quotient of any two infinite distances equals 1, ie infinite distances cancel each other. Provided that no three points in the quadruple $(z_1, \dots, z_4)$ coincide, this yields a well-defined number in $[0, \infty]$.

Cross ratios arise naturally in the study of negatively curved spaces: If X is a CAT(-1) space, we can define its boundary at infinity, which can be endowed with a family of metrics $\{\rho_x\}_{x \in X}$, called visual metrics. It is a classical result by Bourdon that, for a CAT(-1) space, all visual metrics induce the same cross ratio on the boundary. Therefore, the cross ratio provides us with an intrinsic geometric structure on the boundary at infinity. This allows us to think about the pair $(\partial X, \mathrm{cr})$ as a topological space with a geometric structure of its own, which leads to the study of cross ratios from an axiomatic viewpoint; see for example [Hamenstädt 1997; Buyalo 2016]. In this context, cross ratios are also referred to as Möbius structures and a set equipped with a Möbius structure will be called a Möbius space.

Buyalo [2016] showed how Möbius structures give rise to a topology, called Möbius topology. Furthermore, he showed that every Möbius structure is induced by a semimetric, ie every Möbius structure arises from formula (1-1) if ρ is a semimetric, that is, it satisfies the same properties as a metric, except for the triangle inequality. Between semimetrics and metrics there is the notion of a K–quasimetric, which satisfies a weak triangle inequality; see Section 2 for precise definitions. Quasimetrics are of particular interest in the study of cross ratios because of involutions. Given a metric ρ, its involution at a point $o \in Z$ is defined by

$$\rho_o(z, z') = \frac{\rho(z, z')}{\rho(z, o)\rho(o, z')}.$$

A direct computation shows that ρ_o induces the same cross ratio as ρ. However, if ρ is a metric, the map ρ_o may no longer be a metric which leads to technical complications when studying cross ratios purely from a metric point of view. Quasimetrics have the advantage that, given a quasimetric ρ, the involution ρ_o is again a quasimetric; cf Proposition 5.3.6 in [Buyalo and Schroeder 2007]. Quasimetrics are weaker than metrics in many ways. For example, they do not enjoy the same continuity properties as metrics, as we will see in Example 4.5. However, Möbius structures induced by quasimetrics have several nice topological features, which, together with the observation on involutions above, motivates their study.

When studying Möbius structures that appear on boundaries at infinity, there are many results that only require for one to 'roughly' know the Möbius structure. More specifically, a map $f : Z \to Z'$ between metric spaces — which induce cross ratios cr and cr$'$ — is called a *quasi-Möbius map* if there exists a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ such that for all quadruples Q of distinct points in Z, we have $\mathrm{cr}'(f(Q)) \leq \eta(\mathrm{cr}(Q))$. It is called a *quasi-Möbius equivalence* if it is invertible and the inverse is quasi-Möbius as well. There are instances where it is much easier to define a Möbius structure only up to quasi-Möbius equivalence (eg on boundaries of δ–hyperbolic spaces) and, in fact, sometimes we only know how to define the cross ratio up to quasi-Möbius equivalences (at the time of writing, this is the case for Morse boundaries [Charney et al. 2019]). Studying the quasi-Möbius class of a Möbius structure is of interest as the quasi-Möbius class of the Möbius structure on a boundary often characterises the interior space up to quasi-isometry [Paulin 1996; Charney et al. 2019]. If one wishes to determine a (sufficiently) negatively curved space from its boundary more precisely, one needs to utilise a finer structure on the boundary than the quasi-Möbius class. When the Möbius structure can be defined (eg on boundaries of CAT(-1) spaces [Bourdon 1995], Roller boundaries of CAT(0) cube complexes [Beyrer et al. 2021], or boundaries of rank-one Hadamard manifolds [Incerti-Medici 2020]), one can obtain stronger rigidity results, where the Möbius structure on the boundary determines the interior space up to a $(1, C)$–quasi-isometry or even up to isometry; see [Biswas 2015; Beyrer et al. 2021; Incerti-Medici 2020]. For this reason both the Möbius structure and its quasi-Möbius class have become separate objects of interest and study. Some of their properties are shared or analogous, but there are also some notable differences. For example, we will see in Example 4.12 that Theorem A does not hold for the quasi-Möbius class.

In this paper, we put our attention to Möbius structures. We provide a characterisation of those Möbius structures that are induced by quasimetrics in terms of the image of the cross ratio. We then study the Möbius topology introduced by Buyalo and show that, if the cross ratio is induced by a metric, the metric topology and the Möbius topology coincide. Finally, if a Möbius structure is induced by a quasimetric that satisfies an additional symmetry condition, we can define the notion of Cauchy sequences for such a Möbius structure. The main results of this paper are the following:

Theorem A *Let (Z,ρ) be a metric space, M the Möbius structure induced by ρ. Denote the metric topology induced by ρ by $\mathcal{T}_\rho$ and the Möbius topology induced by M by $\mathcal{T}_M$. Then $\mathcal{T}_\rho = \mathcal{T}_M$.*

Theorem B *Let (Z,ρ) be a (possibly extended) metric space and denote the induced Möbius structure by M. The following are equivalent:*

(1) *(Z,M) is complete as a Möbius space.*
(2) *(Z,ρ) is complete as a metric space and is either bounded or has a point at infinity.*

Theorem C *Let (Z,M) be a Möbius space that satisfies the symmetry condition. Then there exists a complete Möbius space $(\overline{Z},\overline{M})$ with a Möbius embedding $i_Z\colon Z \hookrightarrow \overline{Z}$ such that $i_Z(Z)$ is dense in $\overline{Z}$.*

Furthermore, if (Z',M') is a complete Möbius space with a Möbius embedding $i\colon Z \hookrightarrow Z'$ such that $i(Z)$ is dense in Z', then there exists a unique Möbius equivalence $f\colon \overline{Z} \to Z'$ such that $i = f \circ i_Z$.

The rest of the paper is organised as follows. In Section 2, we give precise definitions for the terminology we will require. In Section 3, we show the characterisation of Möbius structures induced by quasimetrics. In Section 4, we review Buyalo's definition of the Möbius topology and prove Theorem A. In Section 5, we introduce Cauchy sequences and prove Theorem B. In Section 6, we construct the completion and prove Theorem C.

Acknowledgements The author is grateful to Viktor Schroeder for many discussions and helpful advice.

2 Preliminaries

Let Z be a set, $\rho\colon Z \times Z \to \mathbb{R}$ a map. We say that ρ is a *semimetric* if it is symmetric, nonnegative and $\rho(z,z') = 0$ if and only if $z = z'$. We say that ρ is a *K–quasimetric*, where $K \geq 1$, if it is a semimetric and for all $x,y,z \in Z$, we have $\rho(x,z) \leq K \max(\rho(x,y),\rho(y,z))$. Finally, we say ρ is a *metric* if it is a semimetric and for all $x,y,z \in Z$, we have $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. Generalising the definition of a metric, we say that $\rho\colon Z \times Z \to [0,\infty]$ is an *extended metric* if there exists exactly one point $\omega \in Z$, such that for all $x \in Z \setminus \{\omega\}$, $\rho(x,\omega) = \infty$, $\rho(\omega,\omega) = 0$ and the restriction of ρ to $(Z \setminus \{\omega\}) \times (Z \setminus \{\omega\})$ is a metric. We call ω the *point at infinity* with respect to ρ. A motivating example for this notion is the Riemannian sphere, seen as the union $\mathbb{C} \cup \{\infty\}$. We define the notions of extended semimetrics and extended K–quasimetrics analogously.

We call an n–tuple $(z_1,\dots,z_n) \in Z^n$ *nondegenerate* if and only if for all $i \neq j$, we have $z_i \neq z_j$.

Given a semimetric ρ, we can define a cross ratio. The cross ratio will be defined on admissible quadruples.

Definition 2.1 A quadruple $(z_1,z_2,z_3,z_4) \in Z^4$ is admissible if there exists no triple $i \neq j \neq k \neq i$ such that $z_i = z_j = z_k$. We denote the set of admissible quadruples by $\mathscr{A}$.

We define the cross ratio induced by ρ as follows: for $(z_1,z_2,z_3,z_4) \in \mathscr{A}$,

$$\mathrm{cr}(z_1,z_2,z_3,z_4) := \frac{\rho(z_1,z_2)\rho(z_3,z_4)}{\rho(z_1,z_3)\rho(z_2,z_4)} \in [0,\infty].$$

Admissible quadruples are exactly those quadruples, for which the expression above does not yields division of zero by zero for any permutation of the points z_i.

We also define the cross ratio triple. Write

$$\Delta := \{(a:b:c) \in \mathbb{R}P^2 \mid a,b,c > 0\}, \quad \overline{\Delta} := \Delta \cup \{(0:1:1),(1:0:1),(1:1:0)\}.$$

The cross ratio triple induced by ρ is a map $\mathrm{crt}\colon \mathscr{A} \to \overline{\Delta}$ defined by

$$\mathrm{crt}(z_1,z_2,z_3,z_4) := (\rho(z_1,z_2)\rho(z_3,z_4) : \rho(z_1,z_3)\rho(z_2,z_4) : \rho(z_1,z_4)\rho(z_2,z_3)).$$

Admissible quadruples are exactly those quadruples, for which at most one entry of the cross ratio triple is zero.

We can generalise these definitions to extended semimetrics by using the following convention. Let $\omega \in Z$ be the point at infinity with respect to ρ. Fractions of the form $\rho(\omega,z)/\rho(\omega,z')$ for $z,z' \in Z \setminus \{\omega\}$ can be replaced by 1, based on the principle that "infinite distances cancel each other". In other words, if $z_1,z_2,z_3 \in Z \setminus \{\omega\}$, then

$$\mathrm{cr}(z_1,z_2,z_3,\omega) = \frac{\rho(z_1,z_2)}{\rho(z_1,z_3)}, \quad \mathrm{cr}(z_1,z_2,\omega,\omega) = 0, \quad \mathrm{cr}(z_1,\omega,\omega,z_2) = 1,$$
$$\mathrm{crt}(z_1,z_2,z_3,\omega) = (\rho(z_1,z_2) : \rho(z_1,z_3) : \rho(z_2,z_3)), \quad \mathrm{crt}(z_1,z_2,\omega,\omega) = (0:1:1).$$

It turns out that the maps cr and crt determine each other. If $\mathrm{crt}(z_1,z_2,z_3,z_4) = (a:b:c)$, then $\mathrm{cr}(z_1,z_2,z_3,z_4) = a/b$. On the other hand, if we write

$$\mathrm{cr}(z_1,z_3,z_4,z_2) := \alpha, \quad \mathrm{cr}(z_1,z_4,z_2,z_3) := \beta, \quad \mathrm{cr}(z_1,z_2,z_3,z_4) := \gamma,$$

then

$$\mathrm{crt}(z_1,z_2,z_3,z_4) = (\gamma^{1/3}\beta^{-1/3} : \alpha^{1/3}\gamma^{-1/3} : \beta^{1/3}\alpha^{-1/3}).$$

In order to study the properties of the cross ratio, it is sometimes useful to reformulate the cross ratio in an additive manner. Write

$$\overline{L}_4 := \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\} \cup \{(0,\infty,-\infty), (-\infty,0,\infty), (\infty,-\infty,0)\}.$$

We define the cross difference $M\colon \mathcal{A} \to \overline{L}_4$ induced by ρ to be

$$M(z_1,z_2,z_3,z_4) := \bigl(\ln(\mathrm{cr}(z_1,z_3,z_4,z_2)), \ln(\mathrm{cr}(z_1,z_4,z_2,z_3)), \ln(\mathrm{cr}(z_1,z_2,z_3,z_4))\bigr).$$

The maps M and cr determine each other.

We end this section with a construction that allows us to construct different semimetrics that induce the same cross ratio. Let ρ be an extended semimetric and let $o \in Z$ be a point such that for all $z \neq o$, $\rho(z,o) > 0$. We define the *involution of* ρ *at* o by

$$\rho_o(x,y) := \frac{\rho(x,y)}{\rho(x,o)\rho(o,y)}.$$

Note that o lies at infinity with respect to ρ_o and, if ω is a point at infinity with respect to ρ, then

$$\rho_o(x,\omega) = \frac{1}{\rho(x,o)}.$$

Note that, if ρ was an extended semimetric, then ρ_o is again an extended semimetric. Buyalo and Schroeder [2007, Proposition 5.3.6] prove that for any extended K–quasimetric ρ, its involution ρ_o is a K'^2–quasimetric for some $K' \geq K$. A direct computation shows that ρ and ρ_o induce the same cross ratio.

3 Möbius structures and quasimetrics

Consider the ordered triple $\bigl((12)(34), (13)(42), (14)(23)\bigr)$. The symmetric group of four elements $\mathcal{S}_4$ acts on this triple by permuting the numbers 1–4. Whenever $\sigma \in \mathcal{S}_4$ acts on the numbers, it induces a permutation on the triple. Define $\varphi(\sigma) \in \mathcal{S}_3$ to be the permutation on the triple induced by the action of σ. It is easy to check that $\varphi\colon \mathcal{S}_4 \to \mathcal{S}_3$ is a group homomorphism. One can interpret the expression $(12)(34)$ as denoting two opposite edges of a tetrahedron whose corners are labelled by the numbers 1–4. In this interpretation, φ is the group homomorphism that sends a permutation of the corners to the induced permutation of pairs of opposite edges.

Let Z be a set with at least three points. For any semimetric, denote its set of admissible quadruples by $\mathcal{A}$ (recall that all semimetrics have the same admissible quadruples). We can now define a cross ratio axiomatically.

Definition 3.1 Let Z be a set with at least three points. A map $M\colon \mathcal{A} \to \overline{L}_4$ is called a *Möbius structure* if and only if it satisfies the following conditions:

(1) For all $P \in \mathcal{A}$ and all $\pi \in \mathcal{S}_4$, we have

$$M(\pi P) = \mathrm{sgn}(\pi)\varphi(\pi)M(P).$$

(2) For $P \in \mathcal{A}$, $M(P) \in L_4$ if and only if P is nondegenerate.

(3) For $P = (x, x, y, z)$, we have $M(P) = (0, \infty, -\infty)$.

(4) Let $(x, y, \omega, \alpha, \beta)$ be an admissible 5–tuple $(x, y, \omega, \alpha, \beta)$ such that (ω, α, β) is a nondegenerate triple, $\alpha \neq x \neq \beta$ and $\alpha \neq y \neq \beta$. Then there exists some $\lambda = \lambda(x, y, \omega, \alpha, \beta) \in \mathbb{R} \cup \{\pm\infty\}$ such that

$$M(\alpha x \omega \beta) + M(\alpha \omega y \beta) - M(\alpha x y \beta) = (\lambda, -\lambda, 0).$$

Moreover, when (ω, α, β) is nondegenerate, $x \neq \beta$ and $y \neq \alpha$, the first component of the left-hand side expression is well defined. Analogously, the second component of the left-hand side expression is well defined when (ω, α, β) is nondegenerate, $x \neq \alpha$ and $y \neq \beta$.

The pair (Z, M) is called a *Möbius space*.

Given M, we obtain a map $\mathrm{cr}\colon \mathcal{A} \to [0, \infty]$ and a map $\mathrm{crt}\colon \mathcal{A} \to \overline{\Delta}$ using the formulas from Section 2. Abusing notation, we will also refer to (Z, cr) and (Z, crt) as Möbius spaces.

It is a straightforward computation to show that for any semimetric ρ, the induced cross difference M is a Möbius structure. Buyalo [2016] proved that the converse is true as well: Every Möbius structure is the cross difference of a semimetric. We also have a characterisation of Möbius structures that are induced by quasimetrics.

Definition 3.2 Let Z be a set with at least three points. A map $M\colon \mathcal{A} \to \overline{L}_4$ is called a *strong Möbius structure* if it is a Möbius structure and the induced map crt satisfies the following condition:

Corner There exist open neighbourhoods of $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$, such that the image of crt doesn't intersect these neighbourhoods.

The remainder of this section is devoted to proving the following result.

Proposition 3.3 *Let (Z, M) be a Möbius structure. There exists an extended quasimetric ρ inducing M if and only if M is a strong Möbius structure.*

Furthermore, whenever there exists an extended K–quasimetric inducing M, there exists a bounded K^2–quasimetric inducing M.

We begin by proving that quasimetrics induce strong Möbius structures.

Lemma 3.4 *Let Z be a set, ρ a quasimetric on Z and* crt *the cross ratio induced by ρ. Then* crt *satisfies the corner condition and, therefore, the induced cross difference M is a strong Möbius structure.*

Proof Let ρ be a K–quasimetric on Z, M the induced Möbius structure and crt the induced cross ratio triple. Let (w, x, y, z) be an admissible quadruple. We want to show that $\mathrm{crt}(w, x, y, z)$ cannot be close to any of the three corner points. We will show this for the corner point $(0:0:1)$. The others work analogously.

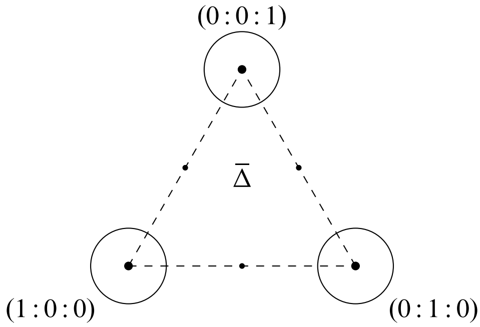


Figure 1: A Möbius structure crt satisfies the corner condition if and only if we can find open neighbourhoods, as depicted above, such that the image of crt in $\overline{\Delta}$ doesn't intersect these neighbourhoods.

In order for the point $\mathrm{crt}(w,x,y,z)$ to be close to $(0:0:1)$, the ratio between the first and third component has to be small, as does the ratio between the second and the third component. We will show that this cannot happen. To prove this, we need to make several case distinctions. We leave it to the reader to check that all cases can be handled analogously by simply permuting the roles and properties of w,x,y,z.

Let $\epsilon > 0$. Consider $\mathrm{crt}(w,x,y,z) = (\rho(w,x)\rho(y,z) : \rho(w,y)\rho(x,z) : \rho(w,z)\rho(x,y))$ and suppose

$$\max(\rho(w,x)\rho(y,z), \rho(w,y)\rho(x,z)) = \epsilon.$$

We want to bound $\rho(w,z)\rho(x,y)$ in terms of ϵ, proving that the ratios

$$\frac{\rho(w,x)\rho(y,z)}{\rho(w,z)\rho(x,y)} \quad \text{and} \quad \frac{\rho(w,y)\rho(x,z)}{\rho(w,z)\rho(x,y)}$$

cannot become too small. Assume without loss of generality that

$$\rho(w,x) \le \rho(y,z) \quad \text{and} \quad \rho(w,y) \le \rho(x,z),$$

and thus

$$\rho(w,x) \le \sqrt{\epsilon} \quad \text{and} \quad \rho(w,y) \le \sqrt{\epsilon}.$$

Since ρ is a K–quasimetric, we have

$$\rho(x,y) \le K \max(\rho(w,x), \rho(w,y)).$$

Swapping x and y if necessary (which does not change any of the inequalities obtained above), we may assume without loss of generality that $\rho(w,x) \ge \rho(w,y)$, and hence

$$\rho(x,y) \le K\rho(w,x).$$

Further, we have

$$\rho(z,w) \le K \max(\rho(z,y), \rho(y,w)).$$

We now combine the inequalities above, distinguishing between two cases. If $\rho(z,y) \ge \rho(y,w)$, then

$$\rho(x,y)\rho(z,w) \le K^2\rho(w,x)\rho(z,y) \le K^2\epsilon,$$

as $\epsilon = \max(\rho(w,x)\rho(y,z), \rho(w,y)\rho(x,z))$. If $\rho(z,y) < \rho(y,w)$, then we use the previously obtained inequalities $\rho(w,x), \rho(w,y) \leq \sqrt{\epsilon}$ to estimate

$$\rho(x,y)\rho(z,w) \leq K^2\rho(w,x)\rho(y,w) \leq K^2\epsilon.$$

We see that, in either case, $\rho(x,y)\rho(z,w) \leq K^2\epsilon$. We use this to show that crt stays away from the corner points. Consider the triple

$$(a,b,c) := (\rho(w,x)\rho(y,z), \rho(w,y)\rho(x,z), \rho(w,z)\rho(x,y)) \in \mathbb{R}^3.$$

The argument above shows that

$$c \leq K^2 \max(a,b).$$

Projecting (a,b,c) to projective space, this implies that

$$(a:b:c) \notin \left\{ (a':b':1) \in \mathbb{R}P^2 \;\middle|\; a' < \frac{1}{K^2}, b' < \frac{1}{K^2} \right\},$$

which is an open neighbourhood of $(0:0:1)$ in $\mathbb{R}P^2$. Since $(a:b:c) = \mathrm{crt}(w,x,y,z)$, we found an open neighbourhood of $(0:0:1)$ that doesn't intersect with Im(crt). Using analogous arguments, we find neighbourhoods of $(1:0:0)$ and $(0:1:0)$ that don't intersect with Im(crt). This completes the proof. □

The other direction of the characterisation is based on the following result.

Lemma 3.5 *Let ρ be a semimetric on the set Z such that ρ has a point at infinity. Then ρ is a quasimetric if and only if its induced Möbius structure is a strong Möbius structure.*

Proof Lemma 3.4 immediately implies one direction of the proof. Suppose now crt satisfies the corner condition. We want to show that ρ is a quasimetric.

Denote the point at infinity with respect to ρ by ω. Let $x,y,z \in Z$. If two of the points are the same, or if one of the three points equals ω, then the inequality for quasimetrics is immediately satisfied. So assume x,y,z are mutually different and different from ω. Then (x,y,z,ω) is a nondegenerate quadruple and we can look at the cross ratio triple

$$\mathrm{crt}(x,y,z,\omega) = (\rho(x,y) : \rho(x,z) : \rho(y,z)).$$

Since crt satisfies the corner condition, we know that there is an open neighbourhood of $(1:0:0)$, independent of x, y and z, such that $\mathrm{crt}(x,y,z,\omega)$ doesn't lie within that neighbourhood. We find $\epsilon > 0$ such that $\mathrm{crt}(x,y,z,\omega) \notin N_\epsilon$, where

$$N_\epsilon := \{(1:b:c) \mid b,c \in (-\epsilon,\epsilon)\}.$$

This implies that

$$\max\left(\frac{\rho(x,z)}{\rho(x,y)}, \frac{\rho(y,z)}{\rho(x,y)}\right) \geq \epsilon,$$

or, equivalently,

$$\frac{1}{\epsilon}\max(\rho(x,z), \rho(z,y)) \geq \rho(x,y).$$

Thus, ρ is a $(1/\epsilon)$–quasimetric. □

Lemmas 3.4 and 3.5 together with Buyalo's result that every Möbius structure is induced by a semimetric prove the first part of Proposition 3.3. We are left to prove the second part.

Proof of Proposition 3.3 Let ρ be a K–quasimetric on Z with a point at infinity. Denote the point at infinity by ω. Choose a base point $o \in Z$. Now define, for all $x, y, z \in Z$,

$$\lambda(z) := \max(1, \rho(z, o)) \quad \text{and} \quad \widetilde{\rho}(x, y) := \frac{\rho(x, y)}{\lambda(x)\lambda(y)}.$$

By Proposition 5.3.6 from [Buyalo and Schroeder 2007], $\widetilde{\rho}$ is a K'^2–quasimetric for some $K' \geq K$. Furthermore,

$$\widetilde{\rho}(x, y) = \frac{\rho(x, y)}{\lambda(x)\lambda(y)} \leq K \frac{\max(\rho(x, o), \rho(o, y))}{\lambda(x)\lambda(y)} \leq K,$$

and thus, $\widetilde{\rho}$ is a bounded quasimetric on Z. A straightforward computation shows that ρ and $\widetilde{\rho}$ induce the same cross ratio and therefore, the same M. □

4 The Möbius topology

Let (Z, M) be a Möbius space. In order to construct a topology on Z, we will recall Buyalo's construction of a family of extended semimetrics, each of which induces M. We will then use those semimetrics to define a topology.

Since $M(w, x, y, z) \in \overline{L}_4$ is a triple, we write $M = (a, b, c)$, where $a, b, c \colon \mathscr{A} \to [-\infty, \infty]$ are the components of M. Condition (4) in the definition of Möbius structures now implies that for all nondegenerate triples (ω, α, β) and $x, y \in Z \setminus \{\omega\}$, we have

$$a(\alpha, x, \omega, \beta) + a(\alpha, \omega, y, \beta) - a(\alpha, x, y, \beta) = b(\alpha, x, y, \beta) - b(\alpha, x, \omega, \beta) - b(\alpha, \omega, y, \beta).$$

Therefore, writing $A := (\omega, \alpha, \beta)$, we can define

$$\rho_A(x, y) := \begin{cases} 0 & \text{if } x = y, \\ e^{a(\alpha,x,\omega,\beta)+a(\alpha,\omega,y,\beta)-a(\alpha,x,y,\beta)} & \text{if } x \neq \beta \text{ and } y \neq \alpha, \\ e^{b(\alpha,x,y,\beta)-b(\alpha,x,\omega,\beta)-b(\alpha,\omega,y,\beta)} & \text{if } x \neq \alpha \text{ and } y \neq \beta. \end{cases}$$

Buyalo [2016] proved the following properties of ρ_A.

Theorem 4.1 [Buyalo 2016] *Let (Z, M) be a Möbius space, and ρ_A the map induced by A for any nondegenerate triple A in Z. Let M_A be the cross difference induced by ρ_A. Then the following hold:*

(1) *Every ρ_A is an extended semimetric on Z, ie ρ_A is symmetric, nonnegative and nondegenerate.*

(2) *For all $x \neq \omega$, $\rho_{(\omega,\alpha,\beta)}(x, \omega) = \infty$. Moreover, $\rho_{(\omega,\alpha,\beta)}(\alpha, \beta) = 1$.*

(3) *Let $A = (\omega, \alpha, \beta)$, $A' = (\omega, \beta, \alpha)$, $A'' = (\beta, \alpha, \omega)$. Then*

$$\rho_A = \rho_{A'} \quad \text{and} \quad \rho_{A''}(x, y) = \frac{\rho_A(x, y)}{\rho_A(x, \beta)\rho_A(\beta, y)}.$$

(4) *Let* $(\omega, \alpha, \beta, b)$ *be a nondegenerate quadruple in* Z*. Then* $\rho_{(\omega,\alpha,\beta)} = \lambda \rho_{(\omega,\alpha,b)}$ *for some constant* $\lambda > 0$.

(5) *For each nondegenerate triple* A, $M_A = M$*.*

The following result is a straightforward computation.

Lemma 4.2 *If* M *is induced by a semimetric* ρ*, then for every nondegenerate triple* A *and for all* $x \neq y$*,*

$$\rho_A(x, y) = \frac{\rho(x, y)}{\rho(x, \omega)\rho(\omega, y)} \frac{\rho(\alpha, \omega)\rho(\omega, \beta)}{\rho(\alpha, \beta)}.$$

Proof Let A be a nondegenerate triple and let $x, y \in Z$. Suppose, $x \neq \beta$ and $y \neq \alpha$. Then

$$\begin{aligned}\rho_A(x, y) &= e^{a(\alpha,x,\omega,\beta)+a(\alpha,\omega,y,\beta)-a(\alpha,x,y,\beta)} \\ &= \mathrm{cr}(\alpha, \omega, \beta, x) \cdot \mathrm{cr}(\alpha, y, \beta, \omega) \cdot \mathrm{cr}(\alpha, y, \beta, x)^{-1} \\ &= \frac{\rho(\alpha, \omega)\rho(\beta, x)\rho(\alpha, y)\rho(\beta, \omega)\rho(\alpha, \beta)\rho(x, y)}{\rho(\alpha, \beta)\rho(\omega, x)\rho(\alpha, \beta)\rho(y, \omega)\rho(\alpha, y)\rho(\beta, x)} \\ &= \frac{\rho(x, y)}{\rho(x, \omega)\rho(\omega, y)} \frac{\rho(\alpha, \omega)\rho(\omega, \beta)}{\rho(\alpha, \beta)}.\end{aligned}$$

The case when $x \neq \alpha$ and $y \neq \beta$ is analogous. □

We see that $\{\rho_A\}_A$ is a family of extended semimetrics that can be constructed from a Möbius structure M. In [Buyalo 2016], these semimetrics are used to define the following topology.

Let $A = (\omega, \alpha, \beta)$ be a nondegenerate triple. For $y \in Z \setminus \{\omega\}$ and $r > 0$, define

$$B_{A,r}(y) := \{x \in Z \mid \rho_A(x, y) < r\}$$

to be the *open ball* around y of radius r with respect to ρ_A. We take the family of all open balls for all nondegenerate triples A, all positive radii r and all points $y \in Z \setminus \{\omega\}$ as a subbasis to define a topology $\mathcal{T}_M$ on Z. This is the *topology on* Z *induced by* M. From now on, whenever we speak of a Möbius space (Z, M), we assume it to be endowed with the topology induced by M, unless stated otherwise.

Lemma 4.3 *Consider* $[0, \infty]$ *with the topology where open neighbourhoods of* ∞ *are complements of compact sets in* $[0, \infty)$ *and open neighbourhoods of other points are just the standard euclidean open neighbourhoods. Let* (Z, M) *be a Möbius space,* A *a nondegenerate triple in* Z *and* $y \in Z$*. Then the maps* $\rho_A(\cdot\,, y), \rho_A(y, \cdot)\colon Z \to [0, \infty]$ *are continuous with respect to* $\mathcal{T}_M$*.*

Proof First note that if $y = \omega$, then $\rho_A(\cdot\,, y) \equiv \infty$ is constant and hence continuous. If $y \neq \omega$, we start by defining the set

$$C_{A,r}(y) := \{x \in Z \mid \rho_A(x, y) > r\},$$

which can be thought of as the complement of a "closed" ball (again $y \neq \omega$). Let (a,b) be an open interval in $\mathbb{R}$ (possibly unbounded) and consider the map $f := \rho_A(\,\cdot\,, y)$ for some fixed $y \neq \omega$. Then $f^{-1}((a,b)) = B_{A,b}(y) \cap C_{A,a}(y)$ and continuity of f follows, if $C_{A,a}(y)$ is open for all $a \geq 0$.

By Theorem 4.1, we know that for any nondegenerate triple (ω,α,y) and every $x \in Z \setminus \{y,\omega\}$,

$$\rho_{(\omega,\alpha,y)}(x,y)\rho_{(y,\alpha,\omega)}(x,\omega) = 1 \quad \text{and} \quad \rho_{(\omega,\alpha,\beta)}(x,y) = \lambda\rho_{(\omega,\alpha,o)}(x,y).$$

Therefore, we see that

$$\rho_{(y,\alpha,o)}(x,\omega) = \lambda\rho_{(y,\alpha,\omega)}(x,\omega) = \frac{\lambda}{\rho_{(\omega,\alpha,y)}(x,y)} = \frac{\lambda}{\mu\rho_{(\omega,\alpha,\beta)}(x,y)}$$

for $y, \omega, \alpha, \beta, o$ mutually different and $\lambda, \mu > 0$ depending only on α, ω, y, o and α, β, ω, y, respectively. This immediately implies that $B_{(\omega,\alpha,\beta),r}(y) = C_{(y,\alpha,o),\lambda/(\mu r)}(\omega)$ for some $\lambda, \mu > 0$ (notice that the points ω and y behave nicely). Since this is true for all ω, α, β, y, o and r as above, we see that $C_{A,r}(y)$ is open for all nondegenerate triples A, all $r > 0$ and all $y \in Z$. This implies the lemma. □

Remark 4.4 The proof of the continuity of ρ_A relies on the fact that we take the open balls of *all* semimetrics ρ_A. It is not sufficient to take just one — or some — of the nondegenerate triples. Only collectively can they define a topology such that $\rho_A(\,\cdot\,, y)$ is continuous. In particular, the involution plays a critical role. The following example illustrates how the topology induced by a single quasimetric does not have this.

Example 4.5 Let $X = [0,1]$ and define

$$\rho(x,y) := \begin{cases} |x-y| & \text{if } |x-y| < 1, \\ 2|x-y| & \text{if } |x-y| \geq 1. \end{cases}$$

Since for all $x, y, z \in X$ we have

$$\rho(x,y) \leq 2|x-y| \leq 2(|x-z| + |z-y|) \leq 4\max(\rho(x,z), \rho(z,y)),$$

we see that ρ is a 4–quasimetric. Consider the sequence $x_n := 1 - 1/n$ and the topology generated by the "open balls" $B_r(x) := \{y \in Z \mid \rho(x,y) < r\}$. The sequence x_n converges to 1 in the topology induced by ρ. However,

$$\rho(0,x_n) = 1 - \frac{1}{n} \xrightarrow{n\to\infty} 1 \neq 2 = \rho(0,1),$$

and therefore, ρ is not continuous with respect to the topology it induces. This is in significant contrast to metric spaces or the maps ρ_A with the Möbius topology.

Lemma 4.6 *The topological space $(Z, \mathcal{T}_M)$ is Hausdorff.*

Proof Let $x, y \in Z$ be two different points. Choose a point $\alpha \in Z \setminus \{x,y\}$. We know that for every $z \in Z$,

$$\rho_{(y,\alpha,x)}(x,z) = \frac{1}{\rho_{(x,\alpha,y)}(y,z)}.$$

Therefore, the intersection of the two open balls $B_{(y,\alpha,x),1}(x)$, $B_{(x,\alpha,y),1}(y)$ is empty. □

Consider two Möbius spaces (Z, M) and (Z', M'). We want to have a notion of maps that are compatible with the Möbius structures.

Definition 4.7 Let (Z, M) and (Z', M') be Möbius spaces. A map $f : Z \to Z'$ is called a *Möbius map* if and only if for every admissible quadruple $(w, x, y, z) \in \mathscr{A}$, we have

$$M(w, x, y, z) = M'(f(w), f(x), f(y), f(z)).$$

If a Möbius map f is bijective, we call it a *Möbius equivalence*.

Lemma 4.8 *Let (Z, M) and (Z', M') be two Möbius spaces and $f : Z \to Z'$ a Möbius equivalence. Then f is a homeomorphism when we equip Z and Z' with their respective Möbius topologies.*

Proof Let $A = (\omega, \alpha, \beta)$ be a nondegenerate triple in Z. Since f is a bijection, it sends A to a nondegenerate triple, denoted by $f(A)$, in Z'. Looking at the definition of the semimetric ρ_A, we immediately see that, since f preserves the Möbius structure, we have for all $x, y \in Z$ that

$$\rho_A(x, y) = \rho_{f(A)}(f(x), f(y)).$$

Thus, the map f sends an open ball $B_{A,r}(x)$ in Z to the open ball $B_{f(A),r}(f(x))$ in Z' and a subbasis of $\mathcal{T}_M$ to a subbasis of $\mathcal{T}_{M'}$. The same is true for f^{-1}, which proves the lemma. □

Classically, Möbius structures arise in the study of metric spaces. When a Möbius structure arises from a metric, the topology constructed above coincides with the topology induced by the metric.

Theorem 4.9 *Let (Z, ρ) be a metric space. Let $\mathcal{T}_\rho$ denote the topology on Z induced by ρ, and denote the induced Möbius structure by M. Let $\mathcal{T}_M$ be the topology induced by M and let $\{\rho_A\}_A$ be the family of semimetrics induced by M. Then $\mathcal{T}_\rho = \mathcal{T}_M$.*

Proof Since Z is a metric space, Lemma 4.2 tells us that for all nondegenerate triples A and for all $x \neq y$, we have

$$\rho_A(x, y) = \frac{\rho(x, y)}{\rho(x, \omega)\rho(\omega, y)} \frac{\rho(\alpha, \omega)\rho(\omega, \beta)}{\rho(\alpha, \beta)}.$$

In particular, $\rho_A(x, y)$ is continuous in x with respect to $\mathcal{T}_\rho$ as long as $x \in Z \setminus \{\omega\}$.

We need to show that the open balls in ρ are open with respect to $\mathcal{T}_M$, and that the open balls with respect to the ρ_A are open with respect to $\mathcal{T}_\rho$. We denote by

$$B_s(y) := \{x \in Z \mid \rho(x, y) < s\}$$

the open ball of radius s with respect to ρ, and by

$$B_{A,s}(y) := \{x \in Z \mid \rho_A(x, y) < s\}$$

the open ball of radius s with respect to ρ_A. These sets generate $\mathcal{T}_\rho$ and $\mathcal{T}_M$, respectively.

We first show that $B_{A,r}(y)$ is open with respect to $\mathcal{T}_\rho$ for all nondegenerate triples A, and all $r > 0$ and $y \in Z \setminus \{\omega\}$. Let $x \in B_{A,r}(y)$, ie $\rho_A(x, y) < r$. Since ρ_A is continuous with respect to $\mathcal{T}_\rho$, there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subset B_{A,r}(y)$. We conclude that $B_{A,r}(y)$ is open in $\mathcal{T}_\rho$ and that $\mathcal{T}_\rho$ is finer than $\mathcal{T}_M$.

In order to show that $\mathcal{T}_M$ is finer than $\mathcal{T}_\rho$, we consider the open ball $B_r(y)$ for $r > 0$ and $y \in Z$. Let $x \in B_r(y)$. Since ρ is a metric, there exists a smaller ball around x contained in $B_r(y)$, ie there exists $r' < r$ such that $B_{r'}(x) \subset B_r(y)$. Replacing r' by ϵ, it is now enough to show that for every $\epsilon > 0$, we can find $\delta > 0$ and a nondegenerate triple A such that $B_{A,\delta}(x) \subset B_\epsilon(x)$.

Choose $\omega, \alpha \in Z$ such that $A := (\omega, \alpha, x)$ is nondegenerate. We claim that there exist $\delta > 0$ and $C > 0$ such that $\rho(z, \omega) < C$ for all $z \in B_{A,\delta}(x)$. Suppose not. Then we find a sequence z_n such that $\rho_A(z_n, x) \to 0$ and $\rho(z_n, \omega) \to \infty$. However,

$$0 \leftarrow \rho_A(z_n, x) = \frac{\rho(z_n, x)\rho(\alpha, \omega)}{\rho(z_n, \omega)\rho(\alpha, x)} \geq \frac{\rho(z_n, \omega) - \rho(\omega, x)}{\rho(z_n, \omega)} \frac{\rho(\alpha, \omega)}{\rho(\alpha, x)} \to \frac{\rho(\alpha, \omega)}{\rho(\alpha, x)} \neq 0.$$

Let

$$0 < \delta' < \min\left(\epsilon \frac{\rho(\alpha, \omega)}{C\rho(\alpha, x)}, \delta\right) \quad \text{and} \quad z \in B_{A,\delta'}(x).$$

Then

$$\rho(z, x) = \frac{\rho(z, x)\rho(\alpha, \omega)}{\rho(z, \omega)\rho(\alpha, x)} \frac{\rho(z, \omega)\rho(\alpha, x)}{\rho(\alpha, \omega)} \leq \rho_A(z, x) \frac{C\rho(\alpha, x)}{\rho(\alpha, \omega)} \leq \epsilon.$$

In other words, $B_{A,\delta'}(x) \subseteq B_\epsilon(x)$. This implies that balls with respect to ρ are open with respect to $\mathcal{T}_M$. Hence $\mathcal{T}_M$ is finer than $\mathcal{T}_\rho$, which completes the proof of Theorem 4.9. □

Remark 4.10 This proof easily extends to extended metric spaces which have a point at infinity: Let ∞ denote the point at infinity in the metric space (Z, ρ). Then, for any nondegenerate triple $A = (\infty, \alpha, \beta)$, we have $\rho_A = \lambda\rho$ for some positive number λ. This immediately implies that $\mathcal{T}_\rho \subseteq \mathcal{T}_M$. To prove equality, one modifies the proof provided above.

Applying Lemma 4.8 in the context of Theorem 4.9 immediately yields the following corollary.

Corollary 4.11 *Let (Z, ρ) and (Z', ρ') be — possibly extended — metric spaces, let M_ρ and $M_{\rho'}$ be the induced Möbius structures, and $f : (Z, M_\rho) \to Z'$ a Möbius equivalence. Then f is a homeomorphism with respect to the metric topologies $\mathcal{T}_\rho$ and $\mathcal{T}_{\rho'}$.*

Proof We know from Lemma 4.8 that f is a homeomorphism with respect to the topologies $\mathcal{T}_M, \mathcal{T}_{M'}$. By Theorem 4.9, the Möbius topologies and the metric topologies coincide, ie $\mathcal{T}_M = \mathcal{T}_\rho$ and $\mathcal{T}_{M'} = \mathcal{T}_{\rho'}$. The statement follows. □

It is worth noting that the Möbius topology is not preserved under quasi-Möbius equivalences; see Section 1 for the definition. This is illustrated by the following example.

Example 4.12 Let $X = [0,1]$ and define

$$\rho(x,y) := \begin{cases} 3|x-y| & \text{if } |x-y| < 1, \\ |x-y| & \text{if } |x-y| = 1. \end{cases}$$

One easily checks that ρ is a quasimetric and bi-Lipschitz equivalent to the standard metric on X, which we shall denote by d. Since d and ρ are bi-Lipschitz equivalent, their induced Möbius structures are quasi-Möbius equivalent. Let $\mathcal{T}_M$ denote the Möbius topology coming from the Möbius structure induced by ρ and $\mathcal{T}_{\mathrm{std}}$ denote the standard topology, which is the one induced by d. By Theorem 4.9, $\mathcal{T}_{\mathrm{std}}$ is the Möbius topology of the Möbius structure induced by d. We will now show that $\mathcal{T}_{\mathrm{std}} \neq \mathcal{T}_M$, providing an example where two quasi-Möbius equivalent Möbius structures do not induce the same topology.

We will show our claim by proving that $1 \in X$ is an isolated point with respect to $\mathcal{T}_M$. Let $A = \left(\frac{1}{2}, 0, 1\right)$ and compute

$$\rho_A(x,y) := \begin{cases} \dfrac{3|x-y|}{9\left|x-\frac{1}{2}\right|\left|y-\frac{1}{2}\right|} C_A & \text{if } |x-y| < 1, \\ \dfrac{1}{9\frac{1}{4}} \cdot C_A & \text{if } |x-y| = 1, \end{cases}$$

where $C_A = \rho\left(0, \frac{1}{2}\right)\rho\left(1, \frac{1}{2}\right)/\rho(0,1)$ depends on A but not on x, y. Theorem 4.1(2) implies that $C_A = \frac{9}{4}$. If we fix $x = 0$, we obtain

$$\rho_A(0,y) := \begin{cases} \dfrac{3}{2} \dfrac{y}{\left|y-\frac{1}{2}\right|} & \text{if } |x-y| < 1, \\ 1 & \text{if } |x-y| = 1. \end{cases}$$

Since $\frac{3}{2}y/\left|y-\frac{1}{2}\right| \geq \frac{3}{2}$ for all $y > \frac{1}{2}$, we see that $B_{1+\epsilon,A}(0) = [0,t) \cup \{1\}$ for some $\epsilon > 0$ sufficiently small and $t < \frac{1}{2}$ depending on ϵ. On the other hand, we have

$$\rho_A(1,y) := \begin{cases} \dfrac{3}{2} \dfrac{|1-y|}{\left|y-\frac{1}{2}\right|} & \text{if } |x-y| < 1, \\ 1 & \text{if } |x-y| = 1, \end{cases}$$

which only approaches zero for $y \to 1$. We see that for ϵ sufficiently small, $B_{1+\epsilon,A}(0) \cap B_{\epsilon,A}(1) = \{1\}$, implying that $\{1\}$ is an open set in the Möbius topology of the Möbius structure induced by ρ.

We conclude that $\mathcal{T}_M \neq \mathcal{T}_{\mathrm{std}}$, showing that the Möbius topology is not preserved under quasi-Möbius maps in general, even if the inducing quasimetrics ρ, d are bi-Lipschitz equivalent.

5 Cauchy sequences and completeness

The next two sections are devoted to the notion of Cauchy sequences. We show how to define Cauchy sequences on strong Möbius spaces in a way that is compatible with the situation when the strong Möbius structure is induced by a metric space. In the next section, we show how to construct a completion under an additional symmetry assumption.

Let (Z, ρ) be a metric space, M its induced strong Möbius structure. We recall that a Cauchy sequence — in its usual sense on a metric space — is a sequence $(x_n)_n$ in Z such that for all $\epsilon > 0$ there exists a natural number N_ϵ such that for all $m, n \geq N_\epsilon$, we have $\rho(x_m, x_n) < \epsilon$. Our goal is to generalise this notion to strong Möbius spaces. It may be tempting to simply generalise the statement above to quasi- and semimetrics and use that as a definition, but since a Möbius structure can be induced by many different semimetrics, a definition relying only on the Möbius structure itself is more desirable.

Before we formulate the key insight, we need some notation. Let ρ be a (possibly extended) semimetric that induces M. If ρ has a point at infinity, we denote that point by ω. We write $(y \mid z) := -\ln(\rho(y, z))$ for all $y, z \in Z$. Further, consider a sequence $(x_{n,m})_{n,m\in\mathbb{N}}$ in Z. We say that $\lim_{n,m\to\infty} x_{n,m} = y$, if and only if for all $\epsilon > 0$ there exists an N_ϵ such that for all $n, m \geq N_\epsilon$, we have $\rho(x_{n,m}, y) < \epsilon$.

In what follows, we will often consider a sequence $(x_n)_n$ and a pair of points $y, z \in Z \setminus \{\omega\}$ such that $y \neq z$ and neither $\rho(x_n, y)$ nor $\rho(x_n, z)$ converges to zero. Given a sequence $(x_n)_n$, we will refer such a pair y, z as a *good pair*.

Recall that we write $M = (a, b, c)$, where a, b, c denote the components of M. We can now characterise Cauchy sequences in terms of the Möbius structure.

Lemma 5.1 *Let (Z, ρ) be a metric space, and $(x_n)_{n\in\mathbb{N}}$ a sequence in Z. The following are equivalent*:

(1) *The sequence $(x_n)_n$ is either a Cauchy sequence, or $\rho(x_n, y) \xrightarrow{n\to\infty} \infty$ for all $y \in Z$.*

(2) *There exists a good pair $y, z \in Z$ such that $\lim_{n,m\to\infty} \mathrm{crt}(x_n, x_m, y, z) = (0:1:1)$.*

(3) *There exists a good pair $y, z \in Z$ such that $\lim_{n,m\to\infty} c(x_n, x_m, y, z) = -\infty$.*

Further, if (1) *holds, then* (2) *and* (3) *hold for all good pairs $y, z \in Z$. In addition,* (2) *holds for a good pair y, z if and only if* (3) *holds for the same good pair y, z.*

The equivalence of (1) and (2) is stated in Lemma 2.2 of [Beyrer and Schroeder 2017]. Furthermore, it is easy to see from the proof that (1) implies (2) for every good pair. We are left to prove (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1). For this, we require an auxiliary result. Since it is our goal to generalise Cauchy sequences beyond the realm of metric spaces, we will formulate this result in a more general context.

Lemma 5.2 *Let (Z, M) be a strong Möbius structure and ρ a quasimetric that induces M. Let $(x_n)_n$ be a sequence in Z and suppose there exists a good pair $y, z \in Z$ such that $c(x_n, x_m, y, z) \xrightarrow{n,m\to\infty} -\infty$. Then one of the following two statements holds*:

(a) *For every $x \in Z \setminus \{\omega\}$, there exists some $B_x > 0$ such that $\rho(x_n, x) < B_x$ for all n. Furthermore, $\rho(x_n, x_m) \xrightarrow{n,m\to\infty} 0$. We say that x_n is bounded.*

(b) *For every $x \in Z \setminus \{\omega\}$, we have $\rho(x_n, x) \xrightarrow{n\to\infty} \infty$. We say that x_n diverges to infinity and write $x_n \to \infty$.*

Lemma 5.2 is a generalisation of the statement (3) $\Longrightarrow$ (1) in Lemma 5.1.

Remark 5.3 Lemmas 5.1 and 5.2 also hold for extended metric spaces. One can prove (1) $\Longrightarrow$ (2) for the case $y = \omega$ separately (and, by symmetry, the same proof works for $z = \omega$). The proof of (2) $\Longrightarrow$ (3) that we see below immediately generalises to extended metric spaces. For (3) $\Longrightarrow$ (1), we can use the fact that by Lemma 5.2, this statement also holds for quasimetrics. If $y = \omega$ for a given quasimetric, we can perform involution of ρ at any point $x \in Z \setminus \{y, z\}$. This provides us with a quasimetric that induces the same strong Möbius structure, but neither y nor z lies at infinity.

Proof of Lemma 5.2 Let $(x_n)_n$ be a sequence in the strong Möbius space (Z, M), let ρ be a quasimetric that induces M and let y, z be a good pair such that $c(x_n, x_m, y, z) \xrightarrow{n,m\to\infty} -\infty$. By definition of the Möbius structure induced by ρ, we can write

$$c(x_n, x_m, y, z) = (x_n|y) + (x_m|z) - (x_n|x_m) - (y|z) = \ln\left(\frac{\rho(x_n, x_m)\rho(y, z)}{\rho(x_n, y)\rho(x_m, z)}\right).$$

Using this equality, the statement $c(x_n, x_m, y, z) \xrightarrow{n,m\to\infty} -\infty$ becomes equivalent to

$$\frac{\rho(x_n, x_m)\rho(y, z)}{\rho(x_n, y)\rho(x_m, z)} \xrightarrow{n,m\to\infty} 0. \tag{5-1}$$

We will distinguish between two cases, which will turn out to be exactly the distinction between case (a) and case (b). Suppose there exists some $x \in Z \setminus \{\omega\}$ and some constant $B > 0$ such that $\rho(x_n, x) < B$ for all n. We want to show that we are in case (a).

Since ρ is a quasimetric, we have that for all $x' \in Z \setminus \{\omega\}$,

$$\rho(x_n, x') \le K \max(\rho(x_n, x), \rho(x, x')) \le K \max(B, \rho(x, x')).$$

Therefore, we see that $\rho(x_n, x')$ is bounded for all $x' \in Z \setminus \{\omega\}$. In particular, $\rho(x_n, y)$ and $\rho(x_n, z)$ are both bounded by some constant $B > 0$. We obtain

$$\frac{\rho(x_n, x_m)\rho(y, z)}{\rho(x_n, y)\rho(x_m, z)} \ge \rho(x_n, x_m)\frac{\rho(y, z)}{B^2}.$$

Since the left-hand side of this equation goes to zero by assumption, the right-hand side has to go to zero as well. Hence we see that $\rho(x_n, x_m) \xrightarrow{n,m\to\infty} 0$.

We are left to show that we end up in case (b) whenever there is no $x \in Z \setminus \{\omega\}$ such that $\rho(x_n, x)$ is bounded. Suppose $\rho(x_n, x)$ is unbounded for all $x \in Z \setminus \{\omega\}$. Then there exists a subsequence $(x_{n_i})_i$ of $(x_n)_n$ such that $\rho(x_{n_i}, x) \to \infty$ for one (and hence all, since ρ is a quasimetric) $x \in Z \setminus \{\omega\}$.

Suppose by contradiction that $\rho(x_n, x)$ does not converge to infinity for one and hence all $x \in Z \setminus \{\omega\}$. Then we find another subsequence $(x_{m_j})_j$ of $(x_n)_n$, which is bounded. In particular, we find a constant $B > 0$ such that

$$\rho(x_{m_j}, y) \le B \quad \text{and} \quad \rho(x_{m_j}, z) \le B$$

for all j. From our treatment of case (a), we know that for this subsequence, $\rho(x_{m_j}, x_{m_{j'}}) \xrightarrow{j,j'\to\infty} 0$. In particular, we find a number J such that for all $j, j' \geq J$, we have

$$\rho(x_{m_j}, x_{m_{j'}}) < 1.$$

Now we estimate the distance between the two subsequences $(x_{m_j})_j$ and $(x_{n_i})_i$. For this, we need to take x_{m_J} as an auxiliary point. Since x_{n_i} diverges to infinity, there is a number I such that

$$\rho(x_{m_J}, x_{n_i}) > \max(K, K \cdot B) \quad \text{for all } i \geq I.$$

Now we use the fact that ρ is a quasimetric to get that for all $i \geq I$ and $j \geq J$ we have

$$\max(K, K\cdot B) \leq \rho(x_{m_J}, x_{n_i}) \leq K \max(\rho(x_{m_J}, x_{m_j}), \rho(x_{m_j}, x_{n_i})) = K\rho(x_{m_j}, x_{n_i}),$$

where the last equality follows from the fact that $\rho(x_{m_J}, x_{m_j}) < 1$ for all $j \geq J$. Now consider, for $i \geq I$ and $j \geq J$,

$$\begin{aligned}
\frac{\rho(x_{m_j}, x_{n_i})\rho(y,z)}{\rho(x_{m_j}, y)\rho(x_{n_i}, z)} \geq \frac{\rho(x_{m_j}, x_{n_i})\rho(y,z)}{B\rho(x_{n_i}, z)} &\geq \frac{\rho(x_{m_j}, x_{n_i})\rho(y,z)}{BK\max(\rho(x_{n_i}, x_{m_j}), \rho(x_{m_j}, z))} \\
&= \frac{\rho(x_{m_j}, x_{n_i})\rho(y,z)}{BK\rho(x_{n_i}, x_{m_j})} \\
&= \frac{\rho(y,z)}{BK},
\end{aligned}$$

where in the second-to-last step we use the fact that $\rho(x_{n_i}, x_{m_j}) \geq \max(1, B) \geq B \geq \rho(x_{m_j}, z)$ for all $i \geq I$ and $j \geq J$. This inequality shows that $\rho(x_{m_j}, x_{n_i})\rho(y,z)/(\rho(x_{m_j}, y)\rho(x_{n_i}, z))$ is bounded from below by a positive constant. But by assumption, $\rho(x_{m_j}, x_{n_i})\rho(y,z)/(\rho(x_{m_j}, y)\rho(x_{n_i}, z))$ converges to zero, a contradiction. We see that, if a subsequence $(x_{n_i})_i$ diverges to infinity, the sequence $(x_n)_n$ has to diverge to infinity as well. Thus, we are in case (b), which completes the proof. □

Proof of Lemma 5.1 Let (Z, ρ) be a nonextended metric space, $(x_n)_n$ a sequence in Z and $y, z \in Z$ such that $\lim_{n\to\infty} x_n \neq y, z$.

(1) $\Longrightarrow$ (2) Instead of proving just (1) $\Longrightarrow$ (2), which follows directly from [Beyrer and Schroeder 2017], we will also prove the second part of the lemma, ie that $\lim_{n,m\to\infty} \operatorname{crt}(x_n, x_m, y, z) = (0:1:1)$ for all good pairs y, z.

Step 1 We start by proving that for every Cauchy sequence, we have

$$\lim_{n,m\to\infty} \operatorname{crt}(x_n, x_m, y, z) = (0:1:1).$$

Suppose (x_n) is a Cauchy sequence. Note that this implies that $\rho(x_n, x)$ converges for all $x \in Z$. Let $\epsilon > 0$. We find some $N_\epsilon \in \mathbb{N}$ such that for all $n, m \geq N_\epsilon$, we have $\rho(x_n, x_m) < \epsilon$. Since y, z is a good pair, we can choose ϵ sufficiently small such that there is an N_ϵ such that, additionally, $\rho(x_n, y), \rho(x_n, z) > \epsilon^{1/4}$ for all $n \geq N_\epsilon$. Therefore, we get

$$\frac{\rho(x_n, x_m)\rho(y,z)}{\rho(x_n, y)\rho(x_m, z)} < \frac{\epsilon\rho(y,z)}{\rho(x_n, y)\rho(x_m, z)} < \frac{\epsilon}{\sqrt{\epsilon}}\rho(y,z) = \sqrt{\epsilon}\rho(y,z).$$

Thus we see that
$$\frac{\rho(x_n,x_m)\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)} \xrightarrow{n,m\to\infty} 0.$$
For symmetry reasons, we immediately see that also
$$\frac{\rho(x_n,x_m)\rho(y,z)}{\rho(x_n,z)\rho(x_m,y)} \xrightarrow{n,m\to\infty} 0.$$
We are left to show that
$$\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} \xrightarrow{n,m\to\infty} 1$$
in order to prove that $\operatorname{crt}(x_n,x_m,y,z) \xrightarrow{n,m\to\infty} (0:1:1)$. Since y,z is a good pair, we have
$$\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} \le \frac{\rho(x_n,y)(\rho(x_n,z)+\rho(x_n,x_m))}{\rho(x_n,z)(\rho(x_n,y)-\rho(x_n,x_m))} = \frac{1+\dfrac{\rho(x_n,x_m)}{\rho(x_n,z)}}{1-\dfrac{\rho(x_n,x_m)}{\rho(x_n,y)}} \xrightarrow{n,m\to\infty} 1,$$
$$\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} \ge \frac{\rho(x_n,y)(\rho(x_n,z)-\rho(x_n,x_m))}{\rho(x_n,z)(\rho(x_n,y)+\rho(x_n,x_m))} = \frac{1-\dfrac{\rho(x_n,x_m)}{\rho(x_n,z)}}{1+\dfrac{\rho(x_n,x_m)}{\rho(x_n,y)}} \xrightarrow{n,m\to\infty} 1.$$
It follows that
$$\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} \xrightarrow{n,m\to\infty} 1$$
and hence $\operatorname{crt}(x_n,x_m,y,z) \xrightarrow{n,m\to\infty} (0:1:1)$. Note that we relied on the triangle inequality for this part of the proof.

Step 2 We show that if (x_n) diverges to infinity, we get
$$\lim_{n,m\to\infty} \operatorname{crt}(x_n,x_m,y,z) = (0:1:1).$$
Suppose that $\rho(x_n,x)\to\infty$ for all $x\in Z$ as n goes to infinity (except for the point $x\in Z$ that may lie at infinity). Then, for any $y,z\in Z$ that do not lie at infinity, we have
$$\begin{aligned}\frac{\rho(x_n,x_m)\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)} &\le \frac{(\rho(x_n,y)+\rho(y,x_m))\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)}\\ &= \frac{\rho(y,z)}{\rho(x_m,z)} + \frac{\rho(x_m,y)\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)}\\ &\le \frac{\rho(y,z)}{\rho(x_m,z)} + \frac{(\rho(x_m,z)+\rho(z,y))\rho(y,z)}{\rho(x_n,y)\rho(x_m,z)}\\ &= \frac{\rho(y,z)}{\rho(x_m,z)} + \frac{\rho(y,z)}{\rho(x_n,y)} + \frac{\rho(y,z)^2}{\rho(x_n,y)\rho(x_m,z)} \xrightarrow{n,m\to\infty} 0.\end{aligned}$$

We are left to show that $\rho(x_n,y)\rho(x_m,z)/(\rho(x_n,z)\rho(x_m,y)) \xrightarrow{n,m\to\infty} 1$. For this, we do the estimate

$$\begin{aligned}\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} &\le \frac{(\rho(x_n,z)+\rho(y,z))(\rho(x_m,y)+\rho(y,z))}{\rho(x_n,z)\rho(x_m,y)} \\ &= 1+\frac{\rho(y,z)}{\rho(x_n,z)}+\frac{\rho(y,z)}{\rho(x_m,y)}+\frac{\rho(y,z)^2}{\rho(x_n,z)\rho(x_m,y)} \xrightarrow{n,m\to\infty} 1.\end{aligned}$$

In the same way, we have

$$\begin{aligned}\frac{\rho(x_n,y)\rho(x_m,z)}{\rho(x_n,z)\rho(x_m,y)} &\ge \frac{(\rho(x_n,z)-\rho(y,z))(\rho(x_m,y)-\rho(y,z))}{\rho(x_n,z)\rho(x_m,y)} \\ &= 1-\frac{\rho(y,z)}{\rho(x_n,z)}-\frac{\rho(y,z)}{\rho(x_m,y)}+\frac{\rho(y,z)^2}{\rho(x_n,z)\rho(x_m,y)} \xrightarrow{n,m\to\infty} 1.\end{aligned}$$

From these two estimates, we conclude that $\rho(x_n,y)\rho(x_m,z)/(\rho(x_m,y)\rho(x_n,z)) \xrightarrow{n,m\to\infty} 1$. This concludes the proof of Step 2 and the proof that (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3) Recall that, by definition,

$$c(w,x,y,z)=\ln\left(\frac{\rho(w,x)\rho(y,z)}{\rho(w,y)\rho(x,z)}\right),$$

which is a continuous map with respect to the metric topology. In particular, if $\mathrm{crt}(w,x,y,z)\to(0:1:1)$, then

$$\ln\left(\frac{\rho(w,x)\rho(y,z)}{\rho(w,y)\rho(x,z)}\right)\to-\infty.$$

We see that (2) $\Longrightarrow$ (3). In particular, if (2) holds for a given pair y,z then (3) holds for the same pair y,z.

(3) $\Longrightarrow$ (1) This is a special case of Lemma 5.2. Since we have seen that (1) $\Longrightarrow$ (2) for all good pairs y,z, we also see that, if (3) holds for a good pair y,z, then (2) holds for the same good pair y,z. This concludes the proof of Lemma 5.1 □

Among other things, Lemma 5.1 tells us that for metric spaces, we only need to find one good pair y,z that satisfies condition (2) or (3) to get the same condition for all good pairs y,z that aren't the limit of $(x_n)_n$. It would be good to have the same condition in any strong Möbius space that isn't necessarily induced by a metric. Then we could define a sequence in a strong Möbius space to be a Cauchy sequence if for one good pair y,z and hence all good pairs, we have $\mathrm{crt}(x_n,x_m,y,z)\to(0:1:1)$, which would be much easier to check in practice than if we had to check all good pairs. The next lemma tells us that this is actually true in the case of condition (3).

Lemma 5.4 *Let (Z,M) be a strong Möbius space. Let $(x_n)_n$ be a sequence in Z. Suppose there is a good pair y,z such that*

$$c(x_n,x_m,y,z)\xrightarrow{n,m\to\infty}-\infty.$$

Then the same holds for all good pairs $y',z'\in Z$.

Proof Let ρ be a quasimetric that induces M. By Lemma 5.2, we know that (x_n) is either bounded or diverges to infinity. Let y', z' be a good pair. As we have seen in the proofs of Lemma 5.1 and 5.2, we get the right convergence of $c(x_n, x_m, y', z')$ if $\rho(x_n, x_m)\rho(y', z')/(\rho(x_n, y')\rho(x_m, z'))$ converges to zero.

Case 1 Suppose $(x_n)_n$ is bounded. Since y', z' is a good pair, we find some $\epsilon > 0$ and a subsequence $(x_{n_i})_i$ such that $\rho(x_{n_i}, y') \geq \epsilon$ for all i. From Lemma 5.2, we know that $\rho(x_n, x_m) \xrightarrow{n,m\to\infty} 0$ and we find a number N such that for all $n, m \geq N$, $\rho(x_n, x_m) < \epsilon/(2K)$. Thus, we have for all $n \geq N$,

$$\epsilon \leq \rho(x_{n_i}, y') \leq K \max(\rho(x_{n_i}, x_n), \rho(x_n, y')).$$

Since $K\rho(x_{n_i}, x_n) \leq \frac{1}{2}\epsilon < \epsilon$, we see that

$$\frac{\epsilon}{K} \leq \frac{1}{K}\rho(x_{n_i}, y') \leq \rho(x_n, y')$$

for $n \geq N$. This implies the sequence $(x_n)_n$ stays away from y' for large n; specifically, $\rho(x_n, y') \geq \epsilon/K$ for $n \geq N$. The same is true for $(x_n)_n$ and z' and some other $\tilde{\epsilon} > 0$. Hence, we have

$$\frac{\rho(x_n, x_m)\rho(y', z')}{\rho(x_n, y')\rho(x_m, z')} \leq K^2 \frac{\rho(x_n, x_m)\rho(y', z')}{\epsilon\tilde{\epsilon}} \xrightarrow{n,m\to\infty} 0.$$

We see that $\rho(x_n, x_m)\rho(y', z')/(\rho(x_n, y')\rho(x_m, z'))$ converges to zero; hence $c(x_n, x_m, y', z') \to -\infty$.

Case 2 Suppose x_n diverges to infinity. We can find a number N such that $\rho(x_n, y') \geq \rho(y', z')$ and $\rho(x_n, z') \geq \rho(y', z')$ for all $n \geq N$. Then we have

$$\begin{aligned}\frac{\rho(x_n, x_m)\rho(y', z')}{\rho(x_n, y')\rho(x_m, z')} &\leq \frac{K \max(\rho(x_n, y'), \rho(y', x_m))\rho(y', z')}{\rho(x_n, y')\rho(x_m, z')} \\ &\leq \frac{K^2 \max(\rho(x_n, y'), \rho(y', z'), \rho(z', x_m))\rho(y', z')}{\rho(x_n, y')\rho(x_m, z')} \\ &= K^2 \frac{\rho(y', z')}{\min(\rho(x_n, y'), \rho(x_m, z'))} \to 0.\end{aligned}$$

Hence, we see that also in this case, $\rho(x_n, x_m)\rho(y', z')/(\rho(x_n, y')\rho(x_m, z'))$ converges to zero and, therefore, $c(x_n, x_m, y', z') \to -\infty$. This completes the proof. □

One might hope that an analogous statement for condition (2) holds. However, the following example illustrates that Lemmas 5.2 and 5.4 are the best that we can hope for.

Example 5.5 Consider the circle, represented as $S^1 = \mathbb{R}/4\mathbb{Z}$. We will mostly use representatives in $[-2, 4]$ to represent points on the circle. Consider the space $Z := S^1 \setminus \{[0]\}$ and define a map $\rho\colon Z \times Z \to [0, \infty)$ by

$$\rho([x], [y]) := \begin{cases} |x-y| & \text{if } (x, y) \in (0, 2]^2 \cup [1, 3]^2 \cup [2, 4)^2 \cup ([-1, 1]\setminus\{0\})^2 \\ 2|x-y| & \text{if } (x, y) \in ((0, 1)\times(2, 3)) \cup ((2, 3)\times(0, 1)) \cup ((1, 2)\times(3, 4)) \cup ((3, 4)\times(1, 2)). \end{cases}$$

Notice the use of different representatives depending on the case. Geometrically, (Z, ρ) can be thought of as follows. Think of Z as a subset of the circle of circumference 4 with the shortest path metric. This circle can be embedded into $\mathbb{R}^2$ such that it is centred at the origin, ie it is the boundary of a disk centred at the origin.

We can now consider the intersection of the circle with each quarter of $\mathbb{R}^2$. We call them the upper-right, upper-left, lower-left and lower-right segments of S^1, based on their position in the standard coordinate system of $\mathbb{R}^2$.

The distance $\rho(x,y)$ between two points x and y is now defined to be the same as on S^1 if x and y lie on the same segment of S^1 or if they lie on segments that are neighbours of each other. If x and y lie on segments of S^1 that lie opposite to each other, then $\rho(x,y)$ is exactly twice the length of the path from x to y that passes through the point $(0,-1)$.

A straightforward computation with several case-distinctions shows that ρ is a 12–quasimetric. Thus, we get a strong Möbius space (Z, M_ρ). Consider now the following sequence in Z:

$$x_n = \left[\frac{1}{n}(-1)^n\right].$$

One can show that there is a good pair for $(x_n)_n$ that satisfies condition (3), but not condition (2). Furthermore, one can even find another good pair for $(x_n)_n$ that satisfies both conditions (2) and (3). Specifically, choose $y = 1.5$, $z = -1.5$ for the first case, and $y = 1.5$, $z = 1.6$ for the second case.

The issue at hand is that even if we understand the convergence behaviour of

$$\frac{\rho(x_n, x_m)\rho(y,z)}{\rho(x_n, y)\rho(x_m, z)},$$

we cannot control the convergence behaviour of

$$\frac{\rho(x_n, y)\rho(x_m, z)}{\rho(x_n, z)\rho(x_m, y)}$$

if ρ is not a metric. So we have found a quasimetric — and thus a strong Möbius structure M_ρ — for which the statement "(3) $\Longrightarrow$ (2)", that we have proven for metrics in Lemma 5.1, does not hold.

This example illustrates the relationship between the different possible conditions one could use to define Cauchy sequences in a strong Möbius space. If condition (2) holds for one good pair y, z, this does not imply that condition (2) holds for all good pairs, unless we work with a metric space. In the same way, if condition (3) holds for all good pairs, this doesn't imply the same for condition (2). However, from Lemma 5.4 we know that, if condition (3) holds for one good pair, it holds for all of them.

Example 5.5 leads us to the following definition of Cauchy sequences in a strong Möbius space.

Definition 5.6 Let (Z, M) be a strong Möbius space. A sequence $(x_n)_n$ in Z is called a *Cauchy sequence* if and only if for one (and hence all) good pairs y, z in Z, we have

$$c(x_n, x_m, y, z) \xrightarrow{n,m\to\infty} -\infty.$$

Definition 5.7 A strong Möbius space (Z, M) is called *complete* if and only if all Cauchy sequences in (Z, M) converge.

Using the previous lemma, the following results are easy to see.

Proposition 5.8 *Let (Z, M) and (Z', M') be two strong Möbius spaces, and let $f : Z \to Z'$ be a Möbius equivalence between them.*

(1) *Let $(x_n)_n$ be a sequence in Z. Then $(x_n)_n$ is a Cauchy sequence in (Z, M) if and only if $(f(x_n))_n$ is a Cauchy sequence in (Z', M').*

(2) *The strong Möbius space (Z, M) is complete if and only if (Z', M') is.*

Proof (1) The sequence $(x_n)_n$ is a Cauchy sequence if and only if for some good pair y, z in Z,

$$c(x_n, x_m, y, z) \to -\infty.$$

Since f is a Möbius equivalence, this implies

$$c'(f(x_n), f(x_m), f(y), f(z)) = c(x_n, x_m, y, z) \to -\infty.$$

Since f is a homeomorphism by Lemma 4.8 and y, z is a good pair, so is $f(y), f(z)$ for $(f(x_n))_n$. Thus, $(f(x_n))_n$ is a Cauchy sequence in (Z', M').

(2) Suppose (Z, M) is complete and let $(x'_n)_n$ be a Cauchy sequence in (Z', M'). By part (1), $(f^{-1}(x'_n))_n$ is a Cauchy sequence in (Z, M) which converges to some $x \in Z$ by completeness. Since f is a homeomorphism, $(x'_n)_n$ has to converge to $f(x)$. This implies completeness. □

The notion of completeness defined above compares to the notion of completeness defined in metric spaces as follows:

Theorem 5.9 *Let (Z, ρ) be a (possibly extended) metric space, and denote the induced Möbius structure by M. The following are equivalent:*

(1) *(Z, M) is complete as a strong Möbius space.*

(2) *(Z, ρ) is complete as a metric space and is either bounded or has a point at infinity.*

Proof (1) $\Longrightarrow$ (2) Suppose (Z, M) is complete as a strong Möbius space and let $(x_n)_n$ be a Cauchy sequence in the metric sense. By Lemma 5.1, $(x_n)_n$ is also a Cauchy sequence in the Möbius sense. Hence, (x_n) has to converge in the Möbius topology. Since the Möbius topology is the same as the metric topology on a metric space by Theorem 4.9, $(x_n)_n$ converges in the metric topology and (Z, ρ) is complete in the metric sense.

(2) $\Longrightarrow$ (1) Suppose (Z, ρ) is complete as a metric space and let $(x_n)_n$ be a Cauchy sequence in the Möbius sense. By Lemma 5.1, $(x_n)_n$ is either a Cauchy sequence in the metric sense, or it diverges to infinity. If it is a Cauchy sequence in the metric sense, it converges in the metric topology (and thus in the Möbius topology) by metric completeness. If x_n diverges to infinity, the metric space cannot be bounded. Hence, it has a point at infinity by assumption, and x_n converges to the point at infinity in the metric and Möbius topologies. □

6 Constructing the completion

Now that we have a notion of Cauchy sequences and a notion of completeness for strong Möbius spaces, an obvious question is whether every strong Möbius space has a naturally unique completion, as metric spaces do.

Certainly, if we take a metric space (Z,ρ) and consider the induced Möbius structure M, the metric completion $(\bar{Z},\bar{\rho})$ is either complete with respect to the induced Möbius structure $\overline{M}$, which is just an extension of M, or one has to add one point at infinity to make it complete in the Möbius sense. Adding a point at infinity doesn't change that Z is dense in its completion and it is easy to see that uniqueness up to isometry for the metric case implies uniqueness up to Möbius equivalence (even up to isometry) in the Möbius sense.

We want to see whether we can create a completion beyond the metric case. It turns out that this requires an extra condition. We start by doing the same construction that is used to obtain the metric completion. We will point out where the construction fails, and distil the extra condition needed. Let (Z,crt) be a strong Möbius space. Define the set

$$\bar{Z} := \{(x_n)_n \mid (x_n) \text{ a Cauchy sequence in } (Z,\mathrm{crt})\}/\sim,$$

where $(x_n)\sim(x'_n)$ if and only if, for every pair $y\neq z$ in Z that is a good pair for both (x_n) and (x'_n), we have

$$c(x_n,x'_n,y,z)\to-\infty.$$

There is a canonical embedding of Z into $\bar{Z}$ defined by sending x to the constant sequence $x_n=x$. This is clearly a Cauchy sequence and the map $x\mapsto[(x)_n]$ is injective, since two different constant sequences are not equivalent in the sense defined above.

The next step is to extend the Möbius structure crt to $\bar{Z}$. We would like to define

$$\overline{\mathrm{crt}}\big([(w_n)],[(x_n)],[(y_n)],[(z_n)]\big) := \lim_{n\to\infty}\mathrm{crt}(w_n,x_n,y_n,z_n).$$

There are two questions that arise immediately when stating this definition. Does the limit on the right-hand side exist and is it independent of the choice of representative of a point $[(w_n)]\in\bar{Z}$? In general, the answer to these two questions is no. The reason for that has already appeared in Example 5.5, namely that, if $\rho(x_n,x_m)\to 0$, we cannot make sure that $\rho(x_n,y)$ converges for all $y\in Z$. Specifically, the sequence x_n discussed in Example 5.5 satisfies $\mathrm{crt}(x_{2n},x_{2n+1},y,z)\to(0:1:4)$ and $\mathrm{crt}(x_{2n},x_{2n+2},y,z)\to(0:1:1)$. Therefore $\lim_{n,m\to\infty}\mathrm{crt}(x_n,x_m,y,z)$ does not exist. This example is a special case that will appear in the definition of $\overline{\mathrm{crt}}$ given above and makes this construction not well defined in general.

As mentioned in Example 5.5, the problem at hand is that we cannot control the behaviour of the ratio $\rho(x_n,y)\rho(x_m,z)/(\rho(x_m,y)\rho(x_n,z))$ for a Cauchy sequence (x_n). If we knew that $\mathrm{crt}(w_n,x_n,y_n,z_n)$ could only converge to points in $\mathbb{R}P^2$ that are allowed to be obtained by a Möbius structure, then we could resolve this problem (as we will see below). The following property makes sure that these issues cannot arise.

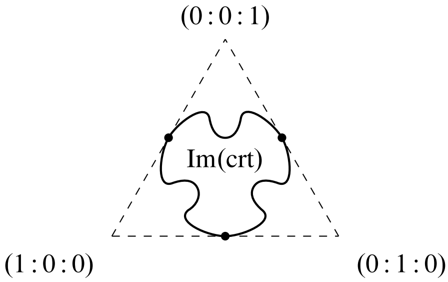


Figure 2: A Möbius structure crt satisfies the symmetry condition if and only if no point in the boundary of $\overline{\Delta}$ can be approximated by a sequence of points in Im(Δ) except for $\left(\frac{1}{2} : \frac{1}{2} : 0\right)$, $\left(\frac{1}{2} : 0 : \frac{1}{2}\right)$ and $\left(0 : \frac{1}{2} : \frac{1}{2}\right)$. In other words, the image doesn't touch the boundary at any other than those three points.

Definition 6.1 A Möbius structure crt or a Möbius space (Z, crt) satisfies the *symmetry* condition if and only if

$$\overline{\mathrm{Im(crt)}} \subseteq \overline{\Delta} = \{(a : b : c) \mid a, b, c > 0\} \cup \{(0 : 1 : 1), (1 : 0 : 1), (1 : 1 : 0)\},$$

where $\overline{\mathrm{Im(crt)}}$ denotes the closure of the image of crt in $\mathbb{R}P^2$.

To interpret this definition, it is useful to think of $\Delta \subset \mathbb{R}P^2$ as a triangle. Specifically, consider the triangle $\{(x, y, z) \in \mathbb{R} \mid x + y + z = 1,\ x, y, z \geq 0\}$. The projection of this triangle onto $\mathbb{R}P^2$ is exactly the topological closure of Δ. The symmetry condition tells us that any sequence of cross ratio triples $\mathrm{crt}(w_n, x_n, y_n, z_n)$ can only accumulate at points in the interior of this triangle or at one of the three distinct points on the boundary of the triangle that are assumed by degenerate quadruples. It turns out that this is the property needed to construct a completion.

Theorem 6.2 *Let (Z, M) be a Möbius space that satisfies the symmetry condition. Then there exists a complete strong Möbius space $(\overline{Z}, \overline{\mathrm{crt}})$ with a Möbius embedding $i_Z \colon Z \hookrightarrow \overline{Z}$ — that is, satisfying $\overline{\mathrm{crt}}(i_Z(w), i_Z(x), i_Z(y), i_Z(z)) = \mathrm{crt}(w, x, y, z)$ for all admissible quadruples (w, x, y, z) — such that $i_Z(Z)$ is dense in $\overline{Z}$.*

Furthermore, if (Z', crt') is a complete strong Möbius space such that there exists a Möbius embedding $i : Z \hookrightarrow Z'$ such that $i(Z)$ is dense in Z', then there exists a unique Möbius equivalence $f : \overline{Z} \to Z'$ such that $i = f \circ i_Z$.

The space $(\overline{Z}, \overline{\mathrm{crt}})$ is going to be the one constructed above. Suppose (Z, crt) satisfies the symmetry condition. Let ρ be a quasimetric inducing crt, (x_n) a Cauchy sequence in the Möbius sense and y, z a good pair for (x_n). By symmetry of x_n, x_m we see that $\rho(x_n, x_m)\rho(y, z)/(\rho(x_n, y)\rho(x_m, z))$ and $\rho(x_n, x_m)\rho(y, z)/(\rho(x_m, y)\rho(x_n, z))$ both converge to zero as n and m tend to infinity. Therefore, the sequence $\mathrm{crt}(x_n, x_m, y, z)$ can be written in the form $(a_n : b_n : c_n)$ with all three entries being nonnegative, where we scale a_n, b_n, c_n so that $a_n + b_n + c_n = 2$. By the convergence statements above, a_n has to converge to zero. Since crt satisfies the symmetry condition, the only point $(0:b:c)$ that can be approximated arbitrarily well in Im(crt) is $(0:1:1)$. Therefore, $\operatorname{crt}(x_n, x_m, y, z) \xrightarrow{n,m\to\infty} (0:1:1)$.

Remark 6.3 Theorem 6.2 has an analogue for the quasi-Möbius class. Given a strong Möbius space (Z, M), one can choose a bounded quasimetric ρ that induces the given Möbius structure by Proposition 3.3. Using the fact that 2–quasimetrics can be deformed into metrics (see for example [Heinonen 2005]), we find that there exists some $\epsilon > 0$ and a metric d such that d is bi-Lipschitz-equivalent to ρ^ϵ and the Möbius structures induced by ρ and d respectively are quasi-Möbius equivalent. Since d is a metric, it has a completion, which is still bounded, and by Theorem 5.9 the Möbius space induced by the completion of (Z, d) is complete as a Möbius space. In other words, every strong Möbius space is quasi-Möbius equivalent to a Möbius space that is induced by a metric and admits a completion. This is in contrast to the situation where we stay within the same Möbius class, where not every strong Möbius structure admits a completion, as Example 5.5 shows.

The symmetry condition allows us to prove a result about convergence that will be useful in proving Theorem 6.2.

Proposition 6.4 *Let (Z, crt) be a strong Möbius structure satisfying the symmetry condition. Let (x_n) and (y_n) be Cauchy sequences in Z, let $y \in Z$ and let ρ be a quasimetric that induces* crt *and has a point at infinity (eg $\rho = \rho_A$). Then $\rho(x_n, y)$ and $\rho(x_n, y_n)$ converge, possibly to infinity.*

Recall that every sequence $(x_{n,m})$ in $\mathbb{R}$ parametrised by $\mathbb{N}^2$ with the property that $\lim_{n\to\infty} x_{n,m}$ exists for every m, $\lim_{m\to\infty} x_{n,m}$ exists for every n and $\lim_{n,m\to\infty} x_{n,m}$ exists, satisfies

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{n,m} = \lim_{m\to\infty}\lim_{n\to\infty} x_{n,m} = \lim_{n,m\to\infty} x_{n,m}.$$

Proof Denote the point at infinity with respect to ρ by ∞. By Lemma 5.2, x_n is either bounded or diverges to infinity. If x_n diverges to infinity with respect to ρ, then $\rho(x_n, y) \to \infty$. Now assume the Cauchy sequence x_n is bounded with respect to ρ. By Lemma 5.2, we know that $\rho(x_n, x_m) \xrightarrow{n,m\to\infty} 0$. In particular, since ρ is a quasimetric, either $\rho(x_n, y) \xrightarrow{n\to\infty} 0$, or there exists $\epsilon > 0$, such that $\rho(x_n, y) \geq \epsilon$ for all n sufficiently large. Suppose $\rho(x_n, y)$ does not converge to zero. Then y, ∞ are a good pair for (x_n), $c(x_n, x_m, y, \infty) \xrightarrow{n,m\to\infty} -\infty$ and, by the symmetry condition,

$$\operatorname{crt}(x_n, x_m, y, \infty) \xrightarrow{n,m\to\infty} (0:1:1).$$

This implies that

$$\frac{\rho(x_n, y)}{\rho(x_m, y)} \xrightarrow{n,m\to\infty} 1. \tag{6-1}$$

We can now use this to prove that $\rho(x_n, y)$ converges for every Cauchy sequence (x_n) and any $y \in Z$. If (x_n) converges to y, then $\rho(x_n, y) \to 0$ by definition. If (x_n) diverges to infinity with respect to ρ, then

$\rho(x_n, y) \to \infty$. If (x_n) is bounded with respect to ρ, then $0 \le \rho(x_n, y) \le B$ and hence — by compactness — has a convergent subsequence $\rho(x_{n_i}, y)$. Applying equation (6-1) in the case $m = n_i$ yields

$$\frac{\rho(x_n, y)}{\rho(x_{n_i}, y)} \xrightarrow{n,i\to\infty} 1.$$

Since $\rho(x_{n_i}, y)$ converges, this implies that the limit of $\rho(x_n, y)$ exists and

$$\lim_{n\to\infty} \rho(x_n, y) = \lim_{i\to\infty} \rho(x_{n_i}, y).$$

Now consider the two Cauchy sequences (x_n) and (y_n). If one of the sequences is bounded and the other diverges to infinity, then $\rho(x_n, y_n) \to \infty$. If both sequences diverge to infinity, replace ρ with an involution ρ_o at any point $o \in Z$. Both (x_n) and (y_n) are bounded with respect to ρ_o. Convergence of $\rho_o(x_n, y_n)$ and the fact that ρ is the involution of ρ_o at the point $\infty \in Z$ will imply convergence of $\rho(x_n, y_n)$.

We are left to prove convergence of $\rho(x_n, y_n)$ when both sequences are bounded. In this situation, we know that $\rho(x_n, x_m), \rho(y_n, y_m) \xrightarrow{n,m\to\infty} 0$. Suppose $\rho(x_n, y_n)$ does not converge to zero. Then the limits above and the fact that ρ is a quasimetric imply that there exists some $\epsilon > 0$ such that for all n sufficiently large, $\rho(x_n, y_n) > \epsilon$. We conclude that

$$\mathrm{cr}(x_n, x_m, y_n, \infty) = \frac{\rho(x_n, x_m)}{\rho(x_n, y_n)} \xrightarrow{n,m\to\infty} 0.$$

Since crt satisfies the symmetry condition, this implies that

$$\frac{\rho(y_n, x_n)}{\rho(y_n, x_m)} = \mathrm{cr}(y_n, x_n, x_m, \infty) \xrightarrow{n,m\to\infty} 1. \tag{6-2}$$

Furthermore, replacing either n or m by a subsequence does not change this convergence behaviour. The same argument with the roles of (x_n) and (y_n) swapped implies

$$\frac{\rho(x_n, y_n)}{\rho(x_n, y_m)} \xrightarrow{n,m\to\infty} 1.$$

Since (x_n) and (y_n) are bounded, there exist subsequences (x_{n_i}) and (y_{n_i}) such that $\rho(x_{n_i}, y_{n_i})$ converges. Equation (6-2) now implies that

$$\lim_{i,m\to\infty} \frac{\rho(y_{n_i}, x_{n_i})}{\rho(y_{n_i}, x_m)} = 1$$

and, therefore,

$$\lim_{i\to\infty} \rho(y_{n_i}, x_{n_i}) = \lim_{i,n\to\infty} \rho(y_{n_i}, x_n).$$

Using equation (6-2) with the roles of (x_n), (y_n) swapped, we obtain

$$\lim_{i,n\to\infty} \frac{\rho(x_n, y_n)}{\rho(x_n, y_{n_i})} = 1$$

and, therefore,

$$\lim_{n\to\infty} \rho(x_n, y_n) = \lim_{i,n\to\infty} \rho(x_n, y_{n_i}).$$

This implies that $\rho(x_n, y_n)$ converges whenever both sequences are Cauchy sequences (provided that ρ has a point at infinity). □

Proof of Theorem 6.2 Let $\bar{Z}$ and $\overline{\mathrm{crt}}$ be as defined before. We start by proving that $\overline{\mathrm{crt}}$ is well defined. Let (w_n), (x_n), (y_n) and (z_n) be Cauchy sequences in Z. By Proposition 6.4, $\rho(\cdot_n, \cdot_n)$ converges for any two of the sequences. Therefore, $\mathrm{crt}(w_n, x_n, y_n, z_n)$ converges as well and, by the symmetry condition, it converges to a point in $\overline{\mathrm{Im(crt)}} \subseteq \bar{\Delta}$.

We are left to show that $\lim_{n\to\infty} \mathrm{crt}(w_n, x_n, y_n, z_n) = \lim_{n\to\infty} \mathrm{crt}(w'_n, x'_n, y'_n, z'_n)$ for $(w_n) \sim (w'_n)$, $(x_n) \sim (x'_n)$, $(y_n) \sim (y'_n)$ and $(z_n) \sim (z'_n)$. Again, we will prove the statement for $\rho(x_n, y)$ and a quasimetric ρ that induces crt and has a point at infinity. Repeating this argument then implies, as above, that the statement for $\mathrm{crt}(w_n, x_n, y_n, z_n)$ holds.

So let ρ be a quasimetric that induces crt and has a point at infinity, denoted by ∞. Let $(x_n) \sim (x'_n)$. Since $c(x_n, x'_n, y, z) \to -\infty$ for all good pairs, it is easy to see that either $\rho(x_n, x'_n) \to 0$ or x_n and x'_n both diverge to infinity.

If (x_n) diverges to ∞, then x'_n has to diverge to infinity too; hence $\lim_{n\to\infty} \rho(x_n, y) = \lim_{n\to\infty} \rho(x'_n, y)$ for all $y \in Z$.

Now suppose (x_n) does not diverge to ∞. It has to be bounded by Lemma 5.2, and $\rho(x_n, x'_n) \xrightarrow{n\to\infty} 0$. By Proposition 6.4, $\rho(x_n, y)$ and $\rho(x'_n, y)$ both converge. Suppose $\rho(x_n, y) \xrightarrow{n\to\infty} 0$. Then

$$\rho(x'_n, y) \le K \max(\rho(x'_n, x_n), \rho(x_n, y)) \xrightarrow{n\to\infty} 0.$$

Thus, $\lim_{n\to\infty} \rho(x'_n, y) = 0 = \lim_{n\to\infty} \rho(x_n, y)$.

Finally, suppose $\rho(x_n, y) \to r$ for some positive real number. Then, by swapping x_n and x'_n in the argument above, $\rho(x'_n, y)$ doesn't converge to zero. Therefore and because (x_n) and (x'_n) are both bounded, y, ∞ is a good pair for both sequences. Since the two sequences are equivalent by assumption,

$$c(x_n, x'_n, y, \infty) \to -\infty.$$

The symmetry condition implies

$$\mathrm{crt}(x_n, x'_n, y, \infty) \to (0:1:1).$$

In other words,

$$\mathrm{crt}(x_n, x'_n, y, \infty) = \frac{\rho(x_n, y)}{\rho(x'_n, y)} \to 1$$

and, therefore,

$$\lim_{n\to\infty} \rho(x_n, y) = \lim_{n\to\infty} \rho(x'_n, y).$$

Analogously to the second half of the proof of Proposition 6.4, we show that $\lim_{n\to\infty} \rho(x_n, y_n) = \lim_{n\to\infty} \rho(x'_n, y_n)$ for all Cauchy sequences $(x_n) \sim (x'_n), (y_n)$. Thus, $\lim_{n\to\infty} \mathrm{crt}(w_n, x_n, y_n, z_n) = \lim_{n\to\infty} \mathrm{crt}(w'_n, x'_n, y'_n, z'_n)$ and therefore, $\overline{\mathrm{crt}}$ is well defined.

Given a Möbius space (Z, crt) that satisfies the symmetry condition, we have constructed a new strong Möbius space $(\bar{Z}, \overline{\mathrm{crt}})$. We also have a canonical map of Z into $\bar{Z}$ that preserves the Möbius structure (hence it is also a topological embedding).

We are left to show that $\overline{Z}$ is complete and that $\overline{Z}$ is unique. We prove completeness first. Suppose that $\xi^m = [(x_n^{(m)})_n] \in \overline{Z}$ is such that $(\xi^m)_m$ is a Cauchy sequence in $\overline{Z}$. We will often identify ξ^m with the representative $(x_n^{(m)})$. Choose a quasimetric ρ on Z that induces crt and let $\overline{\rho}$ be the extension to $\overline{Z}$. Clearly, $\overline{\rho}$ induces $\overline{\mathrm{crt}}$. By Lemma 5.2, $(\xi^m)_m$ either diverges to infinity, or it is bounded with respect to $\overline{\rho}$.

We analyse the point at infinity in $\overline{Z}$ with respect to $\overline{\rho}$. Let it be represented by a Cauchy sequence (z_n) in Z. Then $\overline{\rho}((z_n),(y_n)) = \infty$ for all Cauchy sequences (y_n) in Z that are not equivalent to (z_n). This means that

$$\infty = \overline{\rho}((z_n),(y_n)) = \lim_{n\to\infty} \rho(z_n, y_n),$$

which is the same as saying that (z_n) diverges to infinity. So the point at infinity with respect to $\overline{\rho}$ is the equivalence class of all sequences in Z that diverge to infinity with respect to ρ.

Before we study the convergence of our sequence $(\xi^m)_m$, we need to take a look at convergence in the Möbius topology. Given a strong Möbius space (Z', M'), a sequence x_n in Z' converges to x if and only if, for all nondegenerate triples $A = (\omega, \alpha, \beta)$ in Z' and all $y \in Z'$ such that y does not lie at infinity with respect to ρ_A, we have $\rho_A(x_n, y) \to \rho_A(x, y)$. By Lemmas 3.5 and 4.2, if a Möbius structure crt is induced by a quasimetric ρ, then the induced semimetrics ρ_A are quasimetrics and have the form

$$\rho_A(x,y) = \frac{\rho(x,y)}{\rho(x,\omega)\rho(\omega,y)} \frac{\rho(\alpha,\omega)\rho(\omega,\beta)}{\rho(\alpha,\beta)}.$$

We see that, as long as x_n does not diverge to infinity with respect to ρ, it is sufficient to prove that $\rho(x_n, y) \to \rho(x, y)$ for all y. In particular, since every strong Möbius structure is induced by a bounded quasimetric ρ by Proposition 3.3, we can simply use such a quasimetric to study convergence.

Returning to the space $(\overline{Z}, \overline{\mathrm{crt}})$ constructed above, if we pick a bounded quasimetric ρ that induces crt, then $\overline{\rho}$ will be a bounded quasimetric as well. The discussion above implies that a sequence $(\xi^m)_m$ converges to a point ξ if and only if $\overline{\rho}(\xi^m, \eta) \to \overline{\rho}(\xi, \eta)$ for all $\eta \in \overline{Z}$.

Back to the sequence $(\xi^m)_m$. Since we assume ρ to be bounded, any Cauchy sequence in (Z, M) is bounded with respect to ρ. We need to find a Cauchy sequence $(x_l)_l$ in Z such that $(\xi^m)_m$ converges to that sequence in the Möbius topology as m tends to infinity. Since ρ is bounded, $\xi^m = [(x_n^{(m)})_n]$ can be represented by a bounded Cauchy sequence for every m. By Lemma 5.2,

$$\rho(x_n^{(m)}, x_{n'}^{(m)}) \xrightarrow{n,n'\to\infty} 0.$$

Thus, for every fixed m and every $\epsilon > 0$, we find a natural number N_m such that for all $n, n' \geq N_m$, we have

$$\rho(x_n^{(m)}, x_{n'}^{(m)}) < \epsilon.$$

Let (y_n) be a Cauchy sequence in Z. Since $\overline{\rho}$ is bounded, the sequence $(\xi^m)_m$ is bounded and we find some constant $B > 0$ such that $\overline{\rho}(\xi^m, (y_n)) < B$ for all $m \in \mathbb{N}$. Therefore, for every m we find some natural number $\overline{N}_m$ such that for all $n \geq \overline{N}_m$, we have

$$\rho(x_n^{(m)}, y_n) \leq 2B.$$

Since $(\xi^m)_m$ is a bounded Cauchy sequence by assumption, we also find for every $\epsilon > 0$ a natural number M such that for all $m, m' \geq M$,

$$\overline{\rho}(\xi^m, \xi^{m'}) < \epsilon.$$

We now use the following technical lemma.

Lemma 6.5 *There exists a sequence $(x_l)_l$ in Z satisfying the following properties*:

(1) $x_l = x_{n_l}^{(m_l)}$.

(2) *The sequences m_l and n_l are increasing.*

(3) *For every $l \in \mathbb{N}$ and all $n \geq n_l$, we have $\rho(x_{n_l}^{(m_l)}, x_n^{(m_l)}) < 1/(lK)$.*

(4) *For every $l \in \mathbb{N}$ and all $m, m' \geq m_l$, we have $\overline{\rho}(\xi^m, \xi^{m'}) \leq 1/(2lK)$.*

(5) *For all $l \leq l' \in \mathbb{N}$ and all $n \geq n_{l'}$, we have $\rho(x_n^{(m_l)}, x_n^{(m_{l'})}) < 1/(lK)$.*

We first show how the lemma completes the proof of Theorem 6.2. Given such a sequence $(x_l)_l$, one immediately sees that for all l and all $l' \geq l$, we have

$$\rho(x_l, x_{l'}) = \rho(x_{n_l}^{(m_l)}, x_{n_{l'}}^{(m_{l'})}) \leq K \max\bigl(\rho(x_{n_l}^{(m_l)}, x_{n_{l'}}^{(m_l)}), \rho(x_{n_{l'}}^{(m_l)}, x_{n_{l'}}^{(m_{l'})})\bigr) \leq K \frac{1}{lK} = \frac{1}{l}.$$

This implies that x_l is bounded and a Cauchy sequence. Furthermore, for any $l_0 \in \mathbb{N}^+$ and $m \geq m_{l_0}$,

$$\begin{aligned}\overline{\rho}(\xi^m, (x_l)_l) &= \lim_{l\to\infty} \rho(x_l^{(m)}, x_{n_l}^{(m_l)}) \\ &\leq \lim_{l\to\infty} K^3 \max\bigl(\rho(x_l^{(m)}, x_l^{(m_{l_0})}), \rho(x_l^{(m_{l_0})}, x_{n_{l_0}}^{(m_{l_0})}), \rho(x_{n_{l_0}}^{(m_{l_0})}, x_{n_l}^{(m_{l_0})}), \rho(x_{n_l}^{(m_{l_0})}, x_{n_l}^{(m_l)})\bigr).\end{aligned}$$

For sufficiently large l, we can estimate each of the four expressions in the maximum. By property (4) above, the limit of the first expression is at most $1/(2l_0K)$. The second and third expression are both bounded by $1/(l_0K)$ due to property (3) for $l \geq \max(n_{l_0}, l_0)$. The fourth expression is bounded by $1/(l_0K)$ due to property (5) for $l \geq l_0$. We conclude

$$\begin{aligned}\overline{\rho}(\xi^m, (x_l)_l) &\leq \lim_{l\to\infty} K^3 \max\bigl(\rho(x_l^{(m)}, x_l^{(m_{l_0})}), \rho(x_l^{(m_{l_0})}, x_{n_{l_0}}^{(m_{l_0})}), \rho(x_{n_{l_0}}^{(m_{l_0})}, x_{n_l}^{(m_{l_0})}), \rho(x_{n_l}^{(m_{l_0})}, x_{n_l}^{(m_l)})\bigr) \\ &\leq \lim_{l\to\infty} \frac{K^2}{l_0} = \frac{K^2}{l_0}.\end{aligned}$$

Thus $\overline{\rho}(\xi^m, (x_l)_l) \xrightarrow{m\to\infty} 0$ and for any other point $(y_l)_l \in \overline{Z}$, we find ϵ_y such that $\overline{\rho}(\xi^m, (y_l)_l) > \epsilon_y$ for m sufficiently large. This implies that

$$\overline{\mathrm{crt}}(\xi^m, (x_l)_l, (y_l)_l, (z_l)_l) \xrightarrow{m\to\infty} (0:1:1) \quad \text{for all } (y_l)_l, (z_l)_l \in \overline{Z} \setminus \{(x_l)_l\}.$$

Since we assume that $(\xi^m)_m$ does not diverge to infinity, we have that $(x_l)_l \neq \infty$ and we can choose $(z_l)_l = \infty$ (by having chosen the original ρ to have a point at infinity). Then, writing $y := (y_l)_l$ and $\infty = (\infty)_l$, this limit takes the form

$$\overline{\mathrm{crt}}(\xi^m, (x_l)_l, y, \infty) \xrightarrow{m\to\infty} (0:1:1).$$

By the definition of $\overline{\mathrm{crt}}$ this implies
$$\frac{\bar{\rho}(\xi^m, y)}{\bar{\rho}((x_l)_l, y)} \xrightarrow{m\to\infty} 1.$$
In other words, $\lim_{m\to\infty} \bar{\rho}(\xi^m, y) = \bar{\rho}((x_l)_l, y)$. This implies that ξ^m converges to (x_l).

We are left to prove the technical lemma and to show that the completion $(\bar{Z}, \overline{\mathrm{crt}})$ is unique up to unique Möbius equivalence. Let (Z', crt') be a complete strong Möbius space and $i : Z \hookrightarrow Z'$ a Möbius embedding, ie an injective map that is a Möbius equivalence onto its image. Further, assume $i(Z)$ is dense in Z' with its Möbius topology. Denote the canonical inclusion of Z into $\bar{Z}$ by i_Z. Since i and i_Z are both injective, we get a bijection $f : i(Z) \to i_Z(Z)$ which sends $i(x)$ to $i_Z(x)$. Since i and i_Z are Möbius equivalences onto their images, they are also homeomorphisms onto their images. Therefore, the map f is a homeomorphism with respect to the subspace topology on $i(Z)$ and $i_Z(Z)$. Since f preserves the Möbius structure and therefore Cauchy sequences and equivalent Cauchy sequences, it extends to a bijection $F : Z' \to \bar{Z}$.

We claim that F is a Möbius equivalence. Let (w, x, y, z) be a nondegenerate quadruple in Z' (clearly, F preserves the Möbius structure on degenerate, admissible quadruples). Then we can approximate these four points by sequences w_n, x_n, y_n, z_n in $i(Z)$. By definition of F,
$$F(w) = \lim_{n\to\infty} F(w_n), \quad F(x) = \lim_{n\to\infty} F(x_n), \quad F(y) = \lim_{n\to\infty} F(y_n), \quad F(z) = \lim_{n\to\infty} F(z_n),$$
and hence
$$\begin{aligned}\overline{\mathrm{crt}}(F(w)F(x)F(y)F(z)) &= \lim_{n\to\infty} \mathrm{crt}(F(w_n)F(x_n)F(y_n)F(z_n)) = \lim_{n\to\infty} \mathrm{crt}(f(w_n)f(x_n)f(y_n)f(z_n)) \\ &= \lim_{n\to\infty} \mathrm{crt}'(w_n, x_n, y_n, z_n) = \mathrm{crt}'(w, x, y, z).\end{aligned}$$
This shows that F preserves the Möbius structure on nondegenerate quadruples. Hence, F is a Möbius equivalence. Since all Möbius equivalences are homeomorphisms, uniqueness follows from the fact that $F|_{i(Z)} = f$ is given and the fact that $i(Z)$ is dense in Z'. This completes the proof of Theorem 6.2 up to the proof of Lemma 6.5. □

Proof of Lemma 6.5 We are left to construct the sequence x_l. We construct x_l inductively. The induction starts as follows: Since $(\xi^m)_m$ is a bounded Cauchy sequence, we find natural numbers $M_1 < M_2$ such that
$$\bar{\rho}(\xi^m, \xi^{m'}) < \begin{cases} \dfrac{1}{2K} & \text{for all } m, m' \geq M_1, \\ \dfrac{1}{4K} & \text{for all } m, m' \geq M_2. \end{cases}$$
Now we fix $m = M_1, m' = M_2$. We find a natural number N_1 such that
$$\rho(x_n^{(M_1)}, x_n^{(M_2)}) < \frac{1}{K} \quad \text{for all } n \geq N_1.$$
Since $(x_n^{(M_1)})_n$ is a bounded Cauchy sequence in Z, we can choose N_1 such that, additionally,
$$\rho(x_n^{(M_1)}, x_{n'}^{(M_1)}) < \frac{1}{K} \quad \text{for all } n, n' \geq N_1.$$

Set

$$x_1 := x_{N_1}^{(M_1)}.$$

We see that x_1 satisfies conditions (3) and (4) from above. Now we do the inductive construction.

Suppose we are given points $x_1, \dots, x_l$ in Z satisfying properties (1)–(5). Since $(\xi^m)_m$ is a Cauchy sequence in $\bar{Z}$, we find some $M_{l+1} > m_l$ such that

$$\bar{\rho}(\xi^m, \xi^{m'}) < \frac{1}{2(l+1)K} \quad \text{for all } m, m' \geq M_{l+1}.$$

Put $m_{l+1} := M_{l+1}$. Since we have chosen $M_{l+1} > m_l$, condition (2) stays satisfied for $(m_l)_l$. Furthermore, m_{l+1} satisfies condition (4). Since $\xi^{m_{l+1}}$ is a Cauchy sequence, we find some natural number N_0 such that

$$\rho(x_n^{(m_{l+1})}, x_{n'}^{(m_{l+1})}) < \frac{1}{(l+1)K} \quad \text{for all } n, n' \geq N_0.$$

Thus condition (3) is satisfied if we choose $n_{l+1} \geq N_0$. By condition (4), we know that

$$\bar{\rho}(\xi^{m_i}, \xi^{m_{l+1}}) < \frac{1}{2iK} \quad \text{for all } i < l+1.$$

Therefore, we find some natural numbers N_i such that

$$\rho(x_n^{(m_i)}, x_n^{(m_{l+1})}) < \frac{1}{iK} \quad \text{for all } n \geq N_i.$$

We put $N := \max(N_0, N_1, \dots, N_l, n_l)$ and get

$$\rho(x_n^{(m_i)}, x_n^{(m_{l+1})}) < \frac{1}{iK} \quad \text{for all } n \geq N \text{ and } i < l+1.$$

Put $n_{l+1} := N$ and put

$$x_{l+1} := x_{n_{l+1}}^{(m_{l+1})}.$$

By the definition of N, the sequence $(n_l)_l$ satisfies condition (2). Condition (3) is satisfied since $n_{l+1} \geq N_0$. Condition (4) is satisfied by choice of m_{l+1}. Finally, condition (5) is satisfied because $n_{l+1} \geq \max(N_1, \dots, N_l)$. Condition (1) is trivially satisfied and hence we have constructed a sequence with properties (1)–(5). We have seen before that such a sequence is a Cauchy sequence in (Z, crt) and $(\xi^m)_m$ converges to $(x_l)_l$ in $(\bar{Z}, \overline{\mathrm{crt}})$. Hence the Cauchy sequence $(\xi^m)_m$ converges. This implies that $(\bar{Z}, \overline{\mathrm{crt}})$ is complete. □

References

[Beyrer and Schroeder 2017] **J Beyrer**, **V Schroeder**, *Trees and ultrametric Möbius structures*, p–Adic Numbers Ultrametric Anal. Appl. 9 (2017) 247–256 MR Zbl

[Beyrer et al. 2021] **J Beyrer**, **E Fioravanti**, **M Incerti-Medici**, CAT(0) *cube complexes are determined by their boundary cross ratio*, Groups Geom. Dyn. 15 (2021) 313–333 MR Zbl

[Biswas 2015] **K Biswas**, *On Moebius and conformal maps between boundaries of* CAT(−1) *spaces*, Ann. Inst. Fourier (Grenoble) 65 (2015) 1387–1422 MR Zbl

[Bourdon 1995] **M Bourdon**, *Structure conforme au bord et flot géodésique d'un* CAT(-1)*–espace*, Enseign. Math. 41 (1995) 63–102 MR Zbl

[Buyalo 2016] **S V Buyalo**, *Möbius and sub-Möbius structures*, Algebra i Analiz 28 (2016) 1–20 MR Zbl In Russian; translated in St. Petersburg Math. J. 28 (2017) 555–568

[Buyalo and Schroeder 2007] **S Buyalo**, **V Schroeder**, *Elements of asymptotic geometry*, Eur. Math. Soc., Zürich (2007) MR Zbl

[Charney et al. 2019] **R Charney**, **M Cordes**, **D Murray**, *Quasi-Mobius homeomorphisms of Morse boundaries*, Bull. Lond. Math. Soc. 51 (2019) 501–515 MR Zbl

[Hamenstädt 1997] **U Hamenstädt**, *Cocycles, Hausdorff measures and cross ratios*, Ergodic Theory Dynam. Systems 17 (1997) 1061–1081 MR Zbl

[Heinonen 2005] **J Heinonen**, *Lectures on Lipschitz analysis*, Report. Univ. Jyväskylä Dept. Math. Stat. 100, University of Jyväskylä (2005) MR Zbl

[Incerti-Medici 2020] **M Incerti-Medici**, *Circumcenter extension maps for non-positively curved spaces*, preprint (2020) arXiv 2001.11472

[Paulin 1996] **F Paulin**, *Un groupe hyperbolique est déterminé par son bord*, J. Lond. Math. Soc. 54 (1996) 50–74 MR Zbl

Fakultät für Mathematik, Karlsruher Institut für Technologie
Karlsruhe, Germany

merlin.medici@gmail.com

Received: 5 May 2020 Revised: 20 November 2021

Algebraic & Geometric Topology **24**:4 (2024) 1841–1862
DOI: 10.2140/agt.2024.24.1841
Published: 16 July 2024

$\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions on $S^n \times S^n$

JIM FOWLER
COURTNEY THATCHER

We determine the homotopy type of quotients of $S^n \times S^n$ by free actions of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ where $2p > n+3$. Much like free $\mathbb{Z}_{/p}$ actions, they can be classified via the first p–localized k–invariant, but there are restrictions on the possibilities, and these restrictions are sufficient to determine every possibility in the $n = 3$ case. We use this to complete the classification of free $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions on $S^3 \times S^3$ for $p > 3$ by reducing the problem to the simultaneous classification of pairs of binary quadratic forms. Although the restrictions are not sufficient to determine which k–invariants are realizable in general, they can sometimes be used to rule out free actions by groups that contain $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ as a normal abelian subgroup.

57N65, 57S25

1 Introduction

The topological spherical space form problem asks: what groups can act freely on the sphere and how can these group actions be classified? Conditions for which groups can act were determined during the middle of the last century; see e.g. Smith [28], Milnor [22] and Madsen, Thomas and Wall [18]. The question of *how* free cyclic groups can act on spheres was addressed in the study of lens spaces, with the classification of all free cyclic group actions being completed recently; see Macko and Wegner [16; 17].

This question can easily be extended to actions on products of spheres. What groups can act has been addressed in a number of papers (see e.g. Conner [7], Heller [12], Oliver [26], Adem and Smith [1], Benson and Carlson [3], Hambleton and Ünlü [11] and Okay and Yalçin [25]), while the classification of *how* the simplest of groups do act on products of spheres and what invariants distinguish them has largely been skipped. Here we focus specifically on the *how* question.

To begin addressing how groups act, one might consider the simplest group actions. Free $\mathbb{Z}/p$ actions on $S^n \times S^n$ for $p > \frac{1}{2}(n+3)$ were addressed by Thatcher [30] — the homotopy type is determined completely by the homotopy groups and the first k–invariant. We consider quotients of free actions of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ on $S^n \times S^n$ with $n > 1$ odd and $p > \frac{1}{2}(n+3)$. It turns out that the homotopy classification is similar to the $\mathbb{Z}_{/p}$ case — the classes are determined by the first k–invariants, but the k–invariants are more complicated. A significant insight is the usefulness of localizing at a large prime — while the homotopy groups of spheres are replete with torsion, $\pi_i S^n$ has no p–torsion for $i \le 2n$ when p is reasonably large.

From this we see that only a couple of nontrivial stages in the localized Postnikov tower carry all the relevant data for our study.

We begin with a review of the cohomology of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ in Section 2 and then proceed with the classification. In Section 3 we determine the homotopy type in terms of a single k–invariant, or equivalently, in terms of the transgression in a certain spectral sequence, which the reader might also prefer to think of as an Euler class. The homotopy classification of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions on $S^n \times S^n$ then amounts to a choice of parameters in $\mathbb{Z}_{/p}$.

In Section 4 we find that there are strong restrictions on the possible k–invariants. In Section 5 we provide constructions of the possible homotopy classes based on these restrictions, and in Section 6 we show that this is the full homotopy classification of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions on $S^3 \times S^3$ by reducing the classification to that of pairs of binary quadratic forms. One of our main results is the following:

Theorem 6.6 *Let $p > 3$ be prime. If $p \equiv 1 \bmod 4$, then there are four homotopy classes of quotients of $S^3 \times S^3$ by free $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions. If $p \equiv 3 \bmod 4$, then there are two classes.*

Finally, in Section 8 we show that these restrictions can be used to rule out free actions by groups containing $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ as a normal abelian subgroup. This is consistent with the results about $\mathrm{Qd}(p)$ in a recent paper by Okay and Yalçin [25].

We note that a subsequent paper will provide the homeomorphism classification of these quotients in the case of linear actions.

Acknowledgements

The authors thank Ian Hambleton for helpful conversations and the reviewer for many useful suggestions, including repairing a mistake in an earlier version of this manuscript.

2 The cohomology of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$

To begin, we will need the integral cohomology of $X = (S^n \times S^n)/(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p})$. To determine this, we first need to consider the ring structure of the integral cohomology of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$. It is known that $H^*(\mathbb{Z}_{/p}; \mathbb{Z}_{/p}) = \mathbb{F}_p[a] \otimes \wedge(u)$, where $|u| = 1$, $|a| = 2$, and $\beta(u) = a$ with β the Bockstein homomorphism, and that $H^*(\mathbb{Z}_{/p}; \mathbb{Z}) = \mathbb{Z}[a]/(pa)$, where $|a| = 2$. It follows from the Künneth theorem that $H^*(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z}_{/p}) \cong \mathbb{F}_p[a, b] \otimes \wedge(u, v)$, where $|u| = |v| = 1$ and $|a| = |b| = 2$, but $H^*(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z})$ requires a bit more work.

The homology and cohomology groups themselves can be determined using the Künneth theorem and universal coefficients.

Proposition 2.1 *The integral homology groups of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ are*

$$H_k(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ (\mathbb{Z}_{/p})^{(k+3)/2} & \text{for } k > 0 \text{ odd}, \\ (\mathbb{Z}_{/p})^{k/2} & \text{for } k > 0 \text{ even}. \end{cases}$$

The integral cohomology groups are

$$H^k(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } k = 0, \\ 0 & \text{for } k = 1, \\ (\mathbb{Z}_{/p})^{(k-1)/2} & \text{for } k > 1 \text{ odd}, \\ (\mathbb{Z}_{/p})^{(k+2)/2} & \text{for } k > 1 \text{ even}. \end{cases}$$

The ring structure can then be determined by piecing together the exact sequences in cohomology associated to the short exact sequences $0 \to \mathbb{Z}_{/p} \to \mathbb{Z}^2_{/p} \to \mathbb{Z}_{/p} \to 0$ and $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_{/p} \to 0$. We take $G = \mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ for notational ease. Then the triangle in the diagram

$$\begin{tikzcd} H^n(G;\mathbb{Z}) \arrow[r, "\rho"] & H^n(G;\mathbb{Z}_{/p}) \arrow[r, "\tilde{\beta}"] \arrow[dr, "\beta"'] & H^{n+1}(G;\mathbb{Z}) \arrow[r, "p"] \arrow[d, "\rho"] & H^{n+1}(G;\mathbb{Z}) \\ & & H^{n+1}(G;\mathbb{Z}_{/p}) & \end{tikzcd}$$

commutes, where β is the Bockstein associated to the first short exact sequence above, $\tilde{\beta}$ is the Bockstein associated to the second one, ρ is the homomorphism induced by the map $\mathbb{Z} \to \mathbb{Z}_{/p}$, and p is the map induced by multiplication by p. This along with the ring structure of $H^*(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z}_{/p})$ allows one to find the ring structure of $H^*(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z})$. This ring structure is given, among other places, in [6; 27].

Theorem 2.2 *The integral cohomology ring of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ is*

$$H^*(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}; \mathbb{Z}) \cong \mathbb{Z}[a, b, c]/(pa, pb, pc, c^2),$$

where $|a| = |b| = 2$ and $|c| = 3$.

3 Homotopy equivalence and the k–invariants

Let $G = \mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ act freely on $S^n \times S^n$, and let X be the resulting quotient manifold, which may only be a TOP manifold. A simple example of such an action is given by the first $\mathbb{Z}_{/p}$ acting freely on the first S^n and the second $\mathbb{Z}_{/p}$ acting freely on the second S^n in such a way that the resulting quotient manifold is the product of two lens spaces. We wish to determine when two arbitrary free actions of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ result in homotopy equivalent quotients.

For $p > 3$, the fundamental group $\pi_1(X) = G$ acts trivially on the homology of the universal cover of X because $\mathrm{GL}_2(\mathbb{Z})$ has no p–torsion. So by [13, Remark 2.19], it follows that X is nilpotent, and hence X has a Postnikov tower that admits principal refinements and X can be p–localized.

Definition 3.1 A connected space X *n–simple* if $\pi_1(X)$ is abelian and acts trivially on $\pi_i(X)$ for $1 < i \leq n$.

An n–simple space has a Postnikov tower that consists of principal fibrations through the n^{th} stage. We briefly describe the construction, but more specific details can be found in [19]. The first stage is taken to be $X_1 = K(\pi_1(X), 1)$, with $f_1 \colon X \to X_1$ inducing an isomorphism on π_1. The map $p_i \colon X_i \to X_{i-1}$ is constructed iteratively as the fibration induced from the path space fibration over $K(\pi_i X, i+1)$ by the map $k^{i+1} \colon X_{i-1} \to K(\pi_i X, i+1)$. The k^{i+1} are called k–invariants, and are thought of as cohomology classes. There are maps $f_i \colon X \to X_i$ for $1 \le i \le n$ such that $p_i \circ f_i = f_{i-1}$, and each f_i induces an isomorphism on π_k for all $k \le i$. Additionally, $\pi_k(X_i) = 0$ for $k > i$.

The bottom of the Postnikov tower for an n–simple space generically looks like:

$$\begin{tikzcd}
& {} \arrow[d] & \\
& X_n \arrow[d] & \\
& \vdots \arrow[d] & \\
& X_3 \arrow[r, "k^5"] \arrow[d, "p_3"] & K(\pi_4(X), 5) \\
& X_2 \arrow[r, "k^4"] \arrow[d, "p_2"] & K(\pi_3(X), 4) \\
X \arrow[r] \arrow[uuuur, "f_n"] \arrow[uur, "f_3"] \arrow[ur, "f_2"] & X_1 = K(G, 1) \arrow[r, "k^3"] & K(\pi_2(X), 3)
\end{tikzcd}$$

Lemma 3.2 *Let $n \ge 3$. For $p > 3$, $X = (S^n \times S^n)/(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p})$ is n–simple.*

Proof Since $\pi_i(X) \cong \pi_i(S^n \times S^n) \cong \pi_i(S^n) \times \pi_i(S^n)$ we see that $\pi_2(X) \cong \pi_3(X) \cong \cdots \cong \pi_{n-1}(X) = 0$, and hence there is one nontrivial homotopy group $\pi_i X$ for $1 < i < n+1$: $\pi_n(X) \cong \pi_n(S^n \times S^n) = \mathbb{Z}^2$. Since $\mathrm{Aut}(\mathbb{Z}^2)$ only has 2– and 3–torsion and $p > 3$, π_1 acts trivially on $\pi_i(X)$ for $1 < i \le n$. $\square$

Since $\pi_i(X)$ is trivial for $1 < i < n$, $X_1 \simeq X_2 \simeq \cdots \simeq X_{n-1}$, and the bottom of the Postnikov tower becomes

$$\begin{tikzcd}
& {} \arrow[d] & \\
& X_n \arrow[d, "p_n"] & \\
X \arrow[r] \arrow[ur, "f_n"] & X_1 = K(G, 1) \arrow[r, "k^{n+1}"] & K((\mathbb{Z})^2, n+1)
\end{tikzcd}$$

As X is nilpotent, the Postnikov tower above the n^{th} step admits principal refinements. Specifically, using the notation in [20], there is a central $\pi_1(X)$–series $1 = G_{j,r_j} \subset \cdots \subset G_{j,0} = \pi_j(X)$ for each $j > n$ such that $A_{j,l} = G_{j,l}/G_{j,l+1}$ for $0 \le l < r_j$ is abelian and $\pi_1(X)$ acts trivially on $A_{j,l}$. The $(n+1)^{\text{st}}$ stage is then a finite collection of spaces $X_{n+1,l}$ constructed from maps $k^{n+2,l} \colon X_{n+1,l} \to K(A_{n+1,l}, n+2)$

and with $X_{n+1,0} = X_n$. Similarly, the $(n+i)^{\text{th}}$ stage is a finite collection of spaces $X_{n+i+1,l}$ constructed from maps $k^{n+i+1,l}\colon X_{n+i,l} \to K(A_{n+i,l}, n+i+1)$ and with $X_{n+i,0} = X_{n+i-1,r_{n+i-1}}$.

Additionally, since X is nilpotent, X can be p–localized. This is done by inductively p–localizing the Postnikov tower, i.e. the $(X_j)_{(p)}$ are inductively constructed using fibrations with the $K(\pi, j)$, where each π is a $\mathbb{Z}_{(p)}$–module; see for example [20, Theorem 5.3.2] or Sullivan's notes [29]. Specifically we localize the first stage $(X_1)_{(p)} = (K(\pi_1(X), 1))_{(p)} = K((\pi_1(X))_{(p)}, 1) = K((\mathbb{Z}/_p)^2, 1) = X_1$ and localize the n^{th} homotopy group $(\pi_n X)_{(p)} = \pi_n X \otimes \mathbb{Z}_{(p)} = (\mathbb{Z}_{(p)})^2$, and then consider the following diagram:

$$\begin{tikzcd}
K(\pi_n X, n) \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d, dashed, "\phi_{n+1}"] & X_1 \arrow[r, "k^{n+1}"] \arrow[d, "\phi_n"] & K(\pi_n X, n+1)) \arrow[d] \\
K((\pi_n X)_{(p)}, n) \arrow[r] & (X_n)_{(p)} \arrow[r] & (X_1)_{(p)} \arrow[r, "(k^{n+1})_{(p)}"] & K((\pi_n X)_{(p)}, n+1))
\end{tikzcd}$$

Here $(k^{n+1})_{(p)}$ is the p–localization of k^{n+1}. The right square commutes up to homotopy and there exists a map ϕ_{n+1}, that is, localization of X_n at p, such that the middle and left squares commute up to homotopy. Similar arguments can be made for the stages above n, and then we take $X_{(p)} = \lim (X_i)_{(p)}$ and $\phi = \lim \phi_i\colon X \to X_{(p)}$.

We note that the unique map (up to homotopy) ϕ localizes the homotopy and homology groups of X. In particular, $\phi_*(\pi_i X) = (\pi_i X)_{(p)}$, and further, $\phi_*\colon [X_{(p)}, Z] \to [X, Z]$ is an isomorphism for any p–local space Z [13; 20].

By [4] the unstable homotopy group $\pi_i(S^n)$ has no p–torsion for $i < 2p+n-3$. We restrict to $p > \frac{1}{2}(n+3)$, so that $\pi_i(X)$ has no p–torsion for $i \le 2n$. It follows that $A_{j,l}$ has no p–torsion for $n < j \le 2n$, and since $A_{j,l}$ is finite for all $j > n$, $(A_{j,l})_{(p)} = A_{j,l} \otimes Z_{(p)} = 0$, $K((A_{j,l})_{(p)}, j+1)$ is a point, and $(k^{m+1})_{(p)} = 0$, where $(k^{m+1})_{(p)}$ is the p–localized k–invariant associated with the m^{th} stage $(X_m = X_{j,l})$. Since the construction of the tower becomes formal after the dimension of X (after $2n$), the only nontrivial k–invariant in the localized Postnikov tower before it becomes formal is $(k^{n+1})_{(p)} \in H^{n+1}(X_1; (\mathbb{Z}_{(p)})^2) \cong (\mathbb{Z}/_p)^{n+3}$. Given an identification of π_1 and π_n, this p–localized first k–invariant then determines the homotopy type of the localization. In fact, the first nontrivial k–invariant characterizes X up to homotopy as well.

To state Theorem 3.3, we define G_n following [2]. For $n = 1, 3, 7$ define $G_n = \mathrm{GL}_2(\mathbb{Z})$, and for other positive odd n define G_n to be the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ generated by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Theorem 3.3 *Let X and Y be quotients of free $\mathbb{Z}/_p \times \mathbb{Z}/_p$ actions on $S^n \times S^n$ with odd $n \ge 3$, where $p > 3$ satisfies $2p+n-3 > 2n$, and let k_X^{n+1} and k_Y^{n+1} denote the first nontrivial k–invariant. The* *spaces X and Y are homotopy equivalent if and only if there are isomorphisms $g_1\colon \pi_1 X \to \pi_1 Y$ and $g_n\colon \pi_n X \to \pi_n Y$ with $g_n \in G_n$ and such that*

$$\begin{tikzcd} K(\pi_1(X),1) \arrow[r, "k_X^{n+1}"] \arrow[d, "g_{1\star}"] & K(\pi_n(X),n+1) \arrow[d, "g_{n\star}"] \\ K(\pi_1(Y),1) \arrow[r, "k_Y^{n+1}"] & K(\pi_n(Y),n+1) \end{tikzcd}$$

commutes up to homotopy, i.e. $k_X^{n+1} \in H^{n+1}(\pi_1 X; \pi_n X)$ and $k_Y^{n+1} \in H^{n+1}(\pi_1 Y; \pi_n Y)$ are identified through the maps induced by g_1 and g_n.

Lemmas 3.4 and 3.5 are used to prove this, and are related to [30, Lemmas 1 and 2], respectively.

Lemma 3.4 *Let X and Y be n–simple spaces with identifications $\pi_1(X) \cong \pi_1(Y) \cong G$ and $\pi_n(X) \cong \pi_n(Y) \cong H$. Further suppose $\pi_i(X) = \pi_i(Y) = 0$ for $1 < i < n$. If, as in Theorem 3.3, the identifications on π_1 and π_n provide an identification of the first nontrivial k–invariants of X and Y in $H^{n+1}(G;H)$, then the n^{th} stages of the Postnikov towers for X and Y are homotopy equivalent, i.e. $X_n \simeq Y_n$.*

Proof We have isomorphisms $g_1\colon \pi_1 X \to \pi_1 Y$ and $g_n\colon \pi_n X \to \pi_n Y$, and k_X^{n+1} and k_Y^{n+1} are the first nontrivial k–invariants of X and Y, respectively. The k–invariant is regarded as a map

$$k_X^{n+1}\colon K(\pi_1(X),1) \to K(\pi_n(X),n+1).$$

The isomorphism g_1 induces a homotopy equivalence $g_{1\star}\colon K(\pi_1(X),1) \to K(\pi_1(Y),1)$. Similarly, the isomorphism g_n induces a homotopy equivalence $g_{n\star}\colon K(\pi_n(X),n+1) \to K(\pi_n(Y),n+1)$. The identification of the first nontrivial k–invariant means that $g_{n\star} \circ k_X^{n+1}$ is homotopic to $k_Y^{n+1} \circ g_{1\star}$.

The n^{th} stage X_n of the Postnikov tower is constructed as the pullback of the path-space fibration over $K(\pi_n(X),n+1)$ and k_X^{n+1}:

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & (K(\pi_n(X),n+1))^I \arrow[d] \\ K(\pi_1(X),1) \arrow[r, "k_X^{n+1}"] & K(\pi_1(X),n+1) \end{tikzcd}$$

A similar construction is performed for Y_n. We have the following map of fibrations, and we want to define a map f on the fibers:

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, dashed, "f"] & K(\pi_1(X),1) \arrow[r, "k_X^{n+1}"] \arrow[d, "g_{1\star}"] & K(\pi_n(X),n+1) \arrow[d, "g_{n\star}"] \\ Y_n \arrow[r] & K(\pi_1(Y),1) \arrow[r, "k_Y^{n+1}"] & K(\pi_n(Y),n+1) \end{tikzcd}$$

The identification of the first nontrivial k–invariants means the square on the right commutes up to homotopy. Let $h\colon K(\pi_1(X),1) \times I \to K(\pi_n(Y),n+1)$ be a homotopy from $g_{n\star} \circ k_X^{n+1}$ to $k_Y^{n+1} \circ g_{1\star}$. With X_n defined as a pullback, a point in X_n consists of a pair (x,q) with $x \in K(\pi_1(X),1)$ and

$q\colon I \to K(\pi_n(X), n+1)$ satisfying $q(1) = k_X^{n+1}(x)$. Define $f\colon X_n \to Y_n$ by $f(x,q) = (y,r)$ with $y = g_{1\star}(x)$ and $r\colon I \to K(\pi_n(Y), n+1)$ given by

$$r(t) = \begin{cases} g_{n\star}q(t) & \text{if } t \le \frac{1}{2}, \\ h(x, 2t-1) & \text{if } t \ge \frac{1}{2}. \end{cases}$$

This provides a construction for $f\colon X_n \to Y_n$, and by a theorem of Milnor, the fibers X_n and Y_n have the homotopy types of CW complexes. Therefore we have a commuting diagram of homotopy groups

$$\begin{tikzcd} \pi_{j+1}K(\pi_1(X),1) \arrow[r] \arrow[d, "\cong"] & \pi_{j+1}K(\pi_n(X),n+1) \arrow[r] \arrow[d, "\cong"] & \pi_j X_n \arrow[r] \arrow[d] & \pi_j K(\pi_1(X),1) \arrow[r] \arrow[d, "\cong"] & \pi_j K(\pi_n(X),n+1) \arrow[d, "\cong"] \\ \pi_{j+1}K(\pi_1(Y),1) \arrow[r] & \pi_{j+1}K(\pi_n(Y),n+1) \arrow[r] & \pi_j Y_n \arrow[r] & \pi_j K(\pi_1(Y),1) \arrow[r] & \pi_j K(\pi_n(Y),n+1) \end{tikzcd}$$

The five lemma gives us that $\pi_j X_n \cong \pi_j Y_n$ for all j. Thus we have a weak equivalence between spaces having the homotopy type of CW complexes, so we have a homotopy equivalence. □

Lemma 3.5 *Let M and N be nilpotent spaces such that $H^n(M;\mathbb{Z}) = 0$ and $H^n(N;\mathbb{Z}) = 0$ for $n > m$, for some $m > 0$. If the m^{th} stage of the Postnikov tower for M is homotopy equivalent to the m^{th} stage of the Postnikov tower for N, then M is homotopy equivalent to N, i.e. if $M_m \simeq N_m$ then $M \simeq N$.*

We note that this lemma is essentially [30, Lemma 2] — the difference being the change of "m–dimensional" to the cohomology requirement above — and the obstruction argument proof works exactly as written.

Proof of Theorem 3.3 As has been our convention, let $G = \mathbb{Z}/_p \times \mathbb{Z}/_p$.

In one direction, we assume there is a homotopy equivalence from X to Y. On π_1, the homotopy equivalence provides an isomorphism which then yields a homotopy equivalence between the first stage of a Postnikov tower of X and the same of Y. The next nontrivial stage is stage n, and we have a commutative square

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\ X_1 \arrow[r] & Y_1 \end{tikzcd}$$

The commutativity of this square follows by using a functorial model for the Postnikov tower; see [10, Chapter VI.2]. The vertical maps are fibrations, and taking the cofibers of these vertical maps yields the commutative square displayed in the statement of Theorem 3.3. The condition that g_n, regarded as an element of $\mathrm{GL}_2(\mathbb{Z})$, lies in G_n is a consequence of [2, Theorem 6.3].

To prove the other direction, we assume $\pi_1 X$ and $\pi_1 Y$ are identified with G and that this gives an isomorphism $g_1\colon \pi_1 X \to \pi_1 Y$, and $\pi_n X$ and $\pi_n Y$ are identified with $(\mathbb{Z})^2$ and that this gives an isomorphism $g_n\colon \pi_n X \to \pi_n Y$. These maps induce identifications $\pi_1 X_{(p)} \cong \pi_1 Y_{(p)} \cong G \otimes \mathbb{Z}_{(p)} \cong G$ and $\pi_n X_{(p)} \cong \pi_n Y_{(p)} \cong (\mathbb{Z})^2 \otimes \mathbb{Z}_{(p)} \cong (\mathbb{Z}_{(p)})^2$ after localizing the Postnikov systems of both X and Y at p. We have that $(X_1)_{(p)} = K(\pi_1 X_{(p)}, 1) \simeq (Y_1)_{(p)} = K(\pi_1 Y_{(p)}, 1)$. Let k_X^{n+1} and k_Y^{n+1} be the first nontrivial k–invariants of X and Y, respectively, and take $(k_X^{n+1})_{(p)}$ and $(k_Y^{n+1})_{(p)}$ to be the p–localized first k–invariants, respectively. Since k_X^{n+1} and k_Y^{n+1} are in the same homotopy class of maps in $[K(G,1) : K((\mathbb{Z})^2, n+1)]$, $(k_X^{n+1})_{(p)}$ and $(k_Y^{n+1})_{(p)}$ will be in the same homotopy class of maps in $[K(G,1) : K((\mathbb{Z}_{(p)})^2, n+1)]$ by construction. Since $p > 3$, X and Y are n–simple by Lemma 3.2. Given that the localization of both spaces and their homotopy groups preserves this property, $X_{(p)}$ and $Y_{(p)}$ are both n–simple as well, and we can apply Lemma 3.4. It follows that $(X_n)_{(p)} \simeq (Y_n)_{(p)}$.

Since we are assuming $2p+n-3 > 2n$, we have that $(X_{2n+1,0})_{(p)} \simeq (X_n)_{(p)} \simeq (Y_n)_{(p)} \simeq (Y_{2n+1,0})_{(p)}$. It follows from Lemma 3.5 that $X_{(p)} \simeq Y_{(p)}$. The maps $l_1 \colon X_{(p)} \to X_{(0)}$ and $l_2 \colon Y_{(p)} \to Y_{(0)}$, given by inverting p, give via the naturality of localization a homotopy equivalence $\iota \colon X_{(0)} \xrightarrow{\simeq} Y_{(0)}$ and identifications of $\pi_n X_{(0)}$ and $\pi_n Y_{(0)}$ with $(\mathbb{Q})^2$. The following commutes up to homotopy:

$$\begin{tikzcd} X_{(p)} \arrow[r, "\simeq"] \arrow[d, "l_1"] & Y_{(p)} \arrow[d, "l_2"] \\ X_{(0)} \arrow[r, "\iota"] & Y_{(0)} \end{tikzcd}$$

On the other hand, we can consider localization away from p. For X we have the commutative diagram

$$\begin{tikzcd} S^n \times S^n \arrow[r] \arrow[d, "q"'] & (S^n \times S^n)[1/p] \arrow[d, "{q[1/p]}"] \\ X \arrow[r] & X[1/p] \end{tikzcd}$$

Since $\pi_1(X[1/p]) = G \otimes \mathbb{Z}[1/p] = 0$, we see that $\pi_j(X[1/p]) \cong \pi_j((S^n \times S^n)[1/p])$ for all j. Thus $q[1/p]$ induces an isomorphism on every homotopy group, and is a homotopy equivalence since $(S^n \times S^n)[1/p]$ and $X[1/p]$ both have the homotopy types of CW complexes. Similarly we have a homotopy equivalence $(S^n \times S^n)[1/p] \simeq Y[1/p]$. Invoking [2, Theorem 6.3], we can realize any element of $\mathrm{GL}_2(\mathbb{Z})$ via a homotopy equivalence $S^n \times S^n \simeq S^n \times S^n$. Composing these equivalences yields

$$X[1/p] \simeq (S^n \times S^n)[1/p] \simeq (S^n \times S^n)[1/p] \simeq Y[1/p],$$

so the map $X[1/p] \to Y[1/p]$ on π_n is $g_n \otimes \mathbb{Z}[1/p]$.

Since we have maps $X[1/p] \to X_{(0)}$ and $Y[1/p] \to Y_{(0)}$ given by inverting everything else, the naturality of localization gives us a map $\iota' \colon X_{(0)} \xrightarrow{\simeq} Y_{(0)}$. It is a homotopy equivalence because it induces an isomorphism on all of the homotopy groups. We have a diagram that commutes up to homotopy:

$$\begin{tikzcd} X[1/p] \arrow[r, "\simeq"] \arrow[d, "L_1"] & Y[1/p] \arrow[d, "L_2"] \\ X_{(0)} \arrow[r, "\iota'"] & Y_{(0)} \end{tikzcd}$$

Since $X_{(0)}$ and $Y_{(0)}$ are $K((\mathbb{Q})^2, n)$, homotopy classes of maps from $X_{(0)}$ to $Y_{(0)}$ are identified with elements of $\mathrm{Hom}(\pi_n X \otimes \mathbb{Q}, \pi_n Y \otimes \mathbb{Q})$, but by construction, ι and ι' are identified by their action on π_n.

The space X is the homotopy pullback of $X_{(p)}$ and $X[1/p]$ along l_1 and L_1. For $x \in X$, write x_1 and x_2 for the images of x in $X_{(p)}$ and $X[1/p]$, respectively, so $l_1(x_1)$ and $L_1(x_2)$ are connected by a path in $X_{(0)}$. To map into Y, a homotopy pullback, it is enough to provide maps $X \to Y_{(p)}$ and $X \to Y[1/p]$ which agree up to a path in $Y_{(0)}$. Combining the localization squares for X and Y and all of the maps we have constructed between the squares, we have the following cube that commutes up to homotopy, thereby providing maps $X \to Y_{(p)}$ and $X \to Y[1/p]$ which agree up to homotopy:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & X_{(p)} \arrow[dr, "\simeq"] \arrow[dd, "l_1" near start] & \\
& Y \arrow[rr, crossing over] \arrow[dd] & & Y_{(p)} \arrow[dd, "l_2"] \\
X[1/p] \arrow[rr, "L_1" near end] \arrow[dr, "\simeq"'] & & X_{(0)} \arrow[dr, dashed] & \\
& Y[1/p] \arrow[rr, "L_2"] & & Y_{0)}
\end{tikzcd}$$

From this we obtain maps of short exact sequences on homotopy for all j:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \pi_j X \arrow[r] \arrow[d] & \pi_j X_{(p)} \oplus \pi_j X[1/p] \arrow[r] \arrow[d, "\cong"] & \pi_j X_{(0)} \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, equal] \\
0 \arrow[r] & \pi_j Y \arrow[r] & \pi_j Y_{(p)} \oplus \pi_j Y[1/p] \arrow[r] & \pi_j Y_{(0)} \arrow[r] & 0
\end{tikzcd}$$

The five lemma gives isomorphisms on the homotopy groups of X and Y. This then gives a homotopy equivalence from X to Y as they are both CW complexes. $\square$

4 Restrictions on the first k–invariant

Throughout this section we will continue to let $G = \mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ and $X := (S^n \times S^n)/G$, where $p > 3$ is an odd prime and $n \geq 3$ is odd. The first stage of the Postnikov system provides a fibration: $K(\pi_n(X), n) \xrightarrow{j} X_n \to X_1 = K(\pi_1 X, 1)$. The space X_n is induced from the path-space fibration over $K(\pi_n(X), n+1)$, so the fundamental group $\pi_1(X_1) = G$ acts trivially on the homology of $K(\pi_n(X), n)$. This results in an exact sequence

$$\cdots \to H^n(X_n; \pi_n(X)) \xrightarrow{j^*} H^n(K(\pi_n(X), n); \pi_n(X)) \xrightarrow{\tau} H^{n+1}(X_1; \pi_n(X)),$$

where τ is the transgression. By [21, Section 6.2], τ is also the differential $\tau = d_{n+1} \colon E^{0,n}_{n+1} \to E^{n+1,0}_{n+1}$ in the Serre spectral sequence of the fibration. As described in [8, Section 3.7], the fundamental classes of the fiber $K(\pi_n(X), n)$ and the base X_1 correspond under the transgression. If $\iota \in H^n(K(\pi_n(X), n); \pi_n(X))$ is the fundamental class of the fiber, the k–invariant $k^{n+1} \in H^{n+1}(X_1; \pi_n(X))$ is the pullback of the fundamental class of the base space, and $\tau(\iota) = k^{n+1}$.

On the other hand, consider the Borel fibration

$$S^n \times S^n \xrightarrow{i} (S^n \times S^n)_{hG} \to BG,$$

where $(S^n \times S^n)_{hG} := (EG \times S^n \times S^n)/G \simeq (S^n \times S^n)/G = X$.

There is a map of fibrations

$$\begin{tikzcd} S^n \times S^n \arrow[r, "i"] \arrow[d, "\phi_n"] & X \arrow[r, "f_1"] \arrow[d, "f_n"] & BG \arrow[d, "="] \\ K(\pi_n(X), n) \arrow[r, "j"] & X_n \arrow[r, "p_n"] & BG \end{tikzcd}$$

where the map $\phi_n \colon S^n \times S^n \to K(\pi_n(X), n)$ classifies the fundamental class in $H^n(S^n \times S^n; \mathbb{Z}^2)$, and $f_n \colon X \to X_n$ is the n–equivalence in the Postnikov tower. Since $\pi_1(BG) = G$ is a finite group generated by odd-order elements, it acts trivially on the cohomology of the fiber (see [25]), and we obtain maps between the induced exact sequences in cohomology:

$$\begin{tikzcd} \cdots \arrow[r] & H^n(X; \pi_n(X)) \arrow[r, "i^*"] & H^n(S^n \times S^n; \pi_n(X)) \arrow[r, "\bar{\tau}"] & H^{n+1}(X_1; \pi_n(X)) \\ \cdots \arrow[r] & H^n(X_n; \pi_n(X)) \arrow[r, "j^*"] \arrow[u, "f_n^*"] & H^n(K(\pi_n(X), n); \pi_n(X)) \arrow[r, "\tau"] \arrow[u, "\phi_n^*"] & H^{n+1}(X_1; \pi_n(X)) \arrow[u, "="] \end{tikzcd}$$

It follows that $\bar{\tau}(\phi_n^*(\iota)) = \tau(\iota) = k^{n+1}$ for the fundamental class $\iota \in H^n(K(\pi_n(X), n), \pi_n(X))$, which corresponds to the identity map under the equivalence $H^n(K(\pi_n(X), n), \pi_n(X)) \cong \mathrm{Hom}(\pi_n(X), \pi_n(X))$ by the universal coefficient theorem. Further, since $H_{n-1}(S^n \times S^n) = 0$, the universal coefficient theorem also gives $H^n(S^n \times S^n; \mathbb{Z}^2) \cong H^n(S^n \times S^n; \mathbb{Z}) \oplus H^n(S^n \times S^n; \mathbb{Z})$.

We write $(0, 1)$ for the element of $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$ sending (x, y) to y, write $(1, 0)$ for the element of $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$ sending (x, y) to x, and set $\iota = (1,0) \oplus (0,1) \in H^n(K(\pi_n(X), n); \pi_n(X)) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}^2) \cong \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$. Then $\phi_n^*(\iota) = \phi_n^*((1, 0) \oplus (0, 1)) = (\alpha, 0) \oplus (0, \gamma) \in H^n(S^n \times S^n; \mathbb{Z}^2) \cong H^n(S^n \times S^n; \mathbb{Z}) \oplus H^n(S^n \times S^n; \mathbb{Z})$. Here α and γ are preferred generators for $H^n(S^n \times S^n; \mathbb{Z}) \cong \mathbb{Z}^2$. It can now be seen that $k^{n+1} = \bar{\tau}((\alpha, 0) \oplus (0, \gamma))$.

It suffices to examine the transgression from the Serre spectral sequence with integral coefficients for the Borel fibration in order to find out information about the first nontrivial k–invariant, k^{n+1}. In particular, for $S^n \times S^n \to X \to BG$,

$$E_2^{p,q} = H^p(BG; H^q(S^n \times S^n; \mathbb{Z})) \Rightarrow H^{p+q}(X; \mathbb{Z}).$$

The first nontrivial differential is d_{n+1}, and the transgression

$$d_{n+1} \colon H^0(BG; H^n(S^n \times S^n; \mathbb{Z})) \to H^{n+1}(BG; H^0(S^n \times S^n; \mathbb{Z}))$$

here satisfies $d_{n+1}(\alpha) = \bar{\tau}(\alpha, 0)$ and $d_{n+1}(\gamma) = \bar{\tau}(0, \gamma)$. It follows that $k^{n+1} = d_{n+1}(\alpha) \oplus d_{n+1}(\gamma)$. The cohomology ring of $H^*(BG; \mathbb{Z})$ is given in Theorem 2.2 and we use the same notation by taking the generators to be a, b and c, with $|a| = |b| = 2$, $|c| = 3$ and $pa = pb = pc = c^2 = 0$. Additionally, we take α and γ to be the generators in degree n of $H^*(S^n \times S^n; \mathbb{Z})$ with $\alpha^2 = \gamma^2 = 0$, as described above. We see that the $E_2 \cong E_{n+1}$ page reads

$$
\begin{tikzcd}[row sep=large]
2n & \alpha\gamma \arrow[drrrrrrrr, "d_{n+1}"] & 0 & \begin{matrix}\alpha\gamma b\\ \alpha\gamma a\end{matrix} & \alpha\gamma c & \begin{matrix}3\\ \text{gens}\end{matrix} & \begin{matrix}\alpha\gamma ac\\ \alpha\gamma bc\end{matrix} & & & \begin{matrix}\frac{1}{2}(n+3)\\ \text{gens}\end{matrix} \\
n & \alpha, \gamma \arrow[drrrrrrrr, "d_{n+1}"] & 0 & \begin{matrix}\alpha a, \gamma a\\ \alpha b, \gamma b\end{matrix} & \begin{matrix}\alpha c\\ \gamma c\end{matrix} & \begin{matrix}6\\ \text{gens}\end{matrix} & \begin{matrix}\alpha ac, \alpha bc\\ \gamma ac, \gamma bc\end{matrix} & & & \begin{matrix}n+3\\ \text{gens}\end{matrix} \\
0 & 1 & 0 & a, b & c & \begin{matrix}a^2, ab,\\ b^2\end{matrix} & \begin{matrix}ac\\ bc\end{matrix} & & & \begin{matrix}a^{(n+1)/2}, \dots\\ \dots, b^{(n+1)/2}\end{matrix} \\
 & 0 & 1 & 2 & 3 & 4 & 5 & \cdots & \cdots & n+1
\end{tikzcd}
$$

where $\alpha\gamma b$ is $\alpha\gamma \otimes b$, etc, by abuse of notation. Note that the blank entries are not necessarily 0.

By virtue of its codomain being generated by suitable powers of a and b, the transgression

$$d_{n+1} \colon E_{n+1}^{0,n} = H^0(BG; H^n(S^n \times S^n; \mathbb{Z})) \to E_{n+1}^{n+1,0} = H^{n+1}(BG; H^0(S^n \times S^n; \mathbb{Z}))$$

satisfies

$$d_{n+1}(\alpha) = \sum_{i=0}^{(n+1)/2} q_{\alpha,i} a^{(n+1)/2-i} b^i \quad \text{and} \quad d_{n+1}(\gamma) = \sum_{j=0}^{(n+1)/2} q_{\gamma,j} a^{(n+1)/2-j} b^j,$$

where the $q_{\alpha,i}$ and $q_{\gamma,j}$ are elements of $\mathbb{Z}_{/p}$.

This spectral sequence converges to the integral cohomology of X, and since X is a finite manifold of dimension $2n$, there are restrictions on what the coefficients $q_{\alpha,i}$ and $q_{\gamma,j}$ can be.

Proposition 4.1 *The coefficients $q_{\alpha,0}$ and $q_{\gamma,0}$ (which are coefficients for $a^{(n+1)/2}$) cannot both be zero. Similarly, the coefficients $q_{\alpha,(n+1)/2}$ and $q_{\gamma,(n+1)/2}$ (which are coefficients for $b^{(n+1)/2}$) cannot both be zero.*

Proof Since G acts freely and $H^{2n}((S^n \times S^n)/G; \mathbb{Z}) \cong \mathbb{Z}$, only quotients of the groups generated by the $E_2^{p,q} \cong E_{n+1}^{p,q}$ terms with $p+q < 2n$ or $p = 0$ and $q = 2n$ can survive. Assume the transgression $d_{n+1} \colon E_{n+1}^{0,n} \to E_{n+1}^{n+1,0}$ satisfies $d_{n+1}(\alpha) = q_{\alpha,1} a^{(n-1)/2} b + \cdots + q_{\alpha,(n+1)/2} b^{(n+1)/2}$ and $d_{n+1}(\gamma) = q_{\gamma,1} a^{(n-1)/2} b + \cdots + q_{\gamma,(n+1)/2} b^{(n+1)/2}$ for some $q_{\alpha,i}, q_{\gamma,j} \in \mathbb{Z}_{/p}$ with $1 \le i, j \le \frac{1}{2}(n+1)$. In other words, both $q_{\alpha,0}$ and $q_{\gamma,0}$ vanish.

The $(n+1)^{\text{st}}$ differential takes the generators in $E_{n+1}^{n-1,n}$ to combinations of the generators in $E_{n+1}^{2n,0}$. By Leibniz, d_{n+1} sends $\alpha \otimes a^{(n-1)/2}$ to $q_{\alpha,1} a^{n-1} b + \cdots + q_{\alpha,(n+1)/2} a^{(n-1)/2} b^{(n+1)/2}$, and similarly for the other generators. It is not hard to see that the only other nontrivial differential, d_{n+1}, does not hit the subgroup generated by a^{n+1}, and there are no other differentials that map to this group. Therefore the generated $\mathbb{Z}_{/p}$ is present in $H^{2n}((S^n \times S^n)/G;\mathbb{Z})$ and other cohomology groups in higher degrees. Since $H^{2n}((S^n \times S^n)/G;\mathbb{Z})$ is torsion free and the highest nontrivial degree, we get a contradiction.

The argument for $q_{\alpha,(n+1)/2}$ and $q_{\gamma,(n+1)/2}$ both being nontrivial is similar. □

Observe that Proposition 4.1 also implies that neither $d_{n+1}(\alpha)$ nor $d_{n+1}(\gamma)$ can map to 0. We also see that it holds after replacing the specified generators with their images under an automorphism of G.

Corollary 4.2 *For nonzero $\lambda \in H^2(G;\mathbb{Z})$, either $d_{n+1}(\alpha)$ or $d_{n+1}(\gamma)$ is nonzero in*

$$H^{n+1}(G;\mathbb{Z})/\lambda^{(n+1)/2}.$$

Proof Suppose φ is an automorphism of G chosen so that $\varphi_\star\lambda = a \in H^2(G;\mathbb{Z})$. After twisting by φ the action of G on $S^n \times S^n$, the resulting quotient is homeomorphic (albeit not equivariantly homeomorphic) to the original quotient space. In particular, in that quotient the coefficients $q_{\alpha,0}$ and $q_{\gamma,0}$, namely the coefficients for $a^{(n+1)/2}$, cannot both be zero, which corresponds in the original space to the condition in the corollary. □

5 Constructions

Now we construct examples which are more complicated than lens spaces cross lens spaces. In this section, we take the dimension of the spheres we are acting on to be $n = 2m-1$ to avoid fractions appearing in subscripts. Let $R = (r_1,\dots,r_m,r'_1,\dots,r'_m)$ and $Q = (q_1,\dots,q_m,q'_1,\dots,q'_m)$ be elements of $(\mathbb{Z}_{/p})^{2m}$ so that R and Q together generate a copy of $(\mathbb{Z}_{/p})^2$ inside $(\mathbb{Z}_{/p})^{2m}$. We refer to these $4m$ parameters as "rotation numbers" in analogy with the case of a lens space.

Let S^{2m-1} be the unit sphere in $\mathbb{C}^m$, so $S^{2m-1} \times S^{2m-1}$ is a submanifold of $\mathbb{C}^m \times \mathbb{C}^m$. Then R acts on $S^{2m-1} \times S^{2m-1}$ by

$$\begin{aligned} R\cdot(z,z') &= (r,r')\cdot(z,z') = (r,r')\cdot(z_1,\dots,z_m,z'_1,\dots,z'_m) \\ &= (e^{2\pi i r_1/p}z_1,\dots,e^{2\pi i r_1/p}z_m,e^{2\pi i r'_1/p}z'_1,\dots,e^{2\pi i r'_m/p}z'_m), \end{aligned}$$

and similarly Q acts on $S^{2m-1} \times S^{2m-1}$. This provides an action of the group $(\mathbb{Z}_{/p})^2 \cong \langle R,Q\rangle$ on $S^{2m-1} \times S^{2m-1}$. In analogy with the lens space case, we call such actions "linear" and we write the quotient as $L(p,p;R,Q)$. In the case of lens spaces, the k–invariant is the product of rotation numbers. We now compute the first nontrivial k–invariant in the case of $L(p,p;R,Q)$. We will denote this first nontrivial k–invariant by k in what follows.

Lemma 5.1 *Let $L = L(p,p;R,Q)$ and suppose $p > m$. Then $k(L) \in H^{2m}((\mathbb{Z}_{/p})^2;\mathbb{Z}^2)$ is*

$$\left(\prod_{i=1}^{m}(r_i a + q_i b), \prod_{i=1}^{m}(r'_i a + q'_i b)\right),$$

where a and b are generators of $H^2((\mathbb{Z}_{/p})^2;\mathbb{Z})$ as described in Section 2.

In keeping with the analogy to the lens space, Lemma 5.1 states that the k–invariant is the product of rotation *classes* in $H^2((\mathbb{Z}_{/p})^2;\mathbb{Z})$.

Proof The k–invariant $k(L) \in H^{2m}(K(G,1);\mathbb{Z}^2)$ is a homotopy class of maps

$$K(\pi_1 L, 1) \to K(\pi_{2m-1} L, 2m).$$

The proof makes use of the naturality of the k–invariant. Suppose a $\mathbb{Z}_{/p}$ subgroup of $(\mathbb{Z}_{/p})^2$ is generated by (α, β). Then we have a cover

$$\bar{L} = (S^{2m-1} \times S^{2m-1})/\mathbb{Z}_{/p} \to L(p, p; R, Q).$$

By [30, page 396], the k–invariant $k(\bar{L}) \in H^{2m}(\mathbb{Z}_{/p};\mathbb{Z}^2)$ associated to the quotient of $S^{2m-1} \times S^{2m-1}$ by the subgroup $\langle(\alpha,\beta)\rangle \cong \mathbb{Z}_{/p}$ is

$$k(\bar{L}) = \left(\prod_{i=1}^{m} (r_i\alpha + q_i\beta)\omega, \prod_{i=1}^{m} (r_i'\alpha + q_i'\beta)\omega \right),$$

where ω is the generator in $H^2(\mathbb{Z}_{/p};\mathbb{Z})$, which is identified with the generator of $\mathbb{Z}_{/p}$ via $H^2(\mathbb{Z}_{/p};\mathbb{Z}) \cong \mathrm{Ext}(H_1(\mathbb{Z}_{/p};\mathbb{Z}),\mathbb{Z}) \cong \mathbb{Z}_{/p}$.

By universal coefficients and the fact that the cohomology (except in degree zero) of $\mathbb{Z}_{/p}$ and $(\mathbb{Z}_{/p})^2$ is torsion, we have $\mathrm{Ext}(H_1((\mathbb{Z}_{/p})^2;\mathbb{Z}),\mathbb{Z}) \cong H^2((\mathbb{Z}_{/p})^2;\mathbb{Z}) \cong (\mathbb{Z}_{/p})^2$ and $\mathrm{Ext}(H_1(\mathbb{Z}_{/p};\mathbb{Z}),\mathbb{Z}) \cong H^2(\mathbb{Z}_{/p};\mathbb{Z}) \cong \mathbb{Z}_{/p}$, and $H^2((\mathbb{Z}_{/p})^2;\mathbb{Z}) \to H^2(\mathbb{Z}_{/p};\mathbb{Z})$ is dual to the inclusion map $\mathbb{Z}_{/p} \hookrightarrow (\mathbb{Z}_{/p})^2$; the inclusion map sends the generator of $\mathbb{Z}_{/p}$ to (α,β), so the dual map sends $xa + yb \in H^2((\mathbb{Z}_{/p})^2;\mathbb{Z})$ to $(\alpha a + \beta b)\omega$.

By naturality of the k–invariant, the map $H^{2m}((\mathbb{Z}_{/p})^2;\mathbb{Z}^2) \to H^{2m}(\mathbb{Z}_{/p};\mathbb{Z}^2)$ sends $k(L)$ to $k(\bar{L})$. We consider only the left-hand factor of $k(L)$; this is some homogeneous polynomial of degree n in the classes $a, b \in H^2((\mathbb{Z}_{/p})^2;\mathbb{Z})$. Write this polynomial as $f(a,b)$.

Then the map $H^{2m}((\mathbb{Z}_{/p})^2;\mathbb{Z}) \to H^{2m}(\mathbb{Z}_{/p};\mathbb{Z})$ sends $f(a,b)$ to

$$f(\alpha,\beta)\omega \in H^{2m}(\mathbb{Z}_{/p};\mathbb{Z}),$$

and therefore, for $\alpha, \beta \in \mathbb{Z}_{/p}$,

$$f(\alpha,\beta) = \prod_{i=1}^{m} (r_i\alpha + q_i\beta).$$

Now assuming $m < p$, this equality of polynomials as functions gives rise to the desired equality

$$f(a,b) = \prod_{i=1}^{m} (r_i a + q_i b).$$

The right-hand factor of $k(L)$ is computed the same way. □

6 The $S^3 \times S^3$ classification

Suppose $p > 3$. We now classify $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions on $S^3 \times S^3$ up to homotopy. By Theorem 3.3, this boils down to the k–invariants encoded by the transgression

$$d_4(\alpha) = q_{\alpha,0}a^2 + q_{\alpha,1}ab + q_{\alpha,2}b^2 \quad \text{and} \quad d_4(\gamma) = q_{\gamma,0}a^2 + q_{\gamma,1}ab + q_{\gamma,2}b^2.$$

We therefore package $(d_4(\alpha), d_4(\gamma))$ as a pair (Q_1, Q_2) of binary quadratic forms over $\mathbb{Z}_{/p}$. The homotopy classification of $(S^3 \times S^3)/(\mathbb{Z}_{/p} \times \mathbb{Z}_{/p})$ amounts, algebraically, to classifying pairs of binary quadratic forms over $\mathbb{Z}_{/p}$, up to the action of automorphisms of $\mathbb{Z}^2$ on the pair (Q_1, Q_2). For example, the pair (Q_1, Q_2) determines the same equivariant oriented homotopy type as $(Q_1 + Q_2, Q_1)$. Note that $\mathrm{Aut}(\mathbb{Z}^2)$ amounts to the action of

$$\mathrm{SL}_2^{\pm}(\mathbb{Z}_{/p}) := \{M \in \mathrm{GL}_2(\mathbb{Z}_{/p}) \mid \det M = \pm 1\}$$

on pairs (Q_1, Q_2). In what follows regard this as a *left* action of $\mathrm{SL}_2^{\pm}(\mathbb{Z}_{/p})$ so that $M = (m_{ij}) \in \mathrm{SL}_2^{\pm}(\mathbb{Z}_{/p})$ acts via

$$(*) \qquad M \cdot (Q_1, Q_2) = (m_{11}Q_1 + m_{12}Q_2, m_{21}Q_1 + m_{22}Q_2).$$

Now we determine the classification disregarding the identification of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ with π_1. On the levels of quadratic forms, we may replace the pair (Q_1, Q_2) by (Q_1', Q_2') where Q_1 and Q_1' (as well as Q_2 and Q_2') are related by a common change of coordinates, i.e. an automorphism of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$, which amounts to $\mathrm{GL}_2(\mathbb{Z}_{/p})$. In what follows, regard this as a *right* action of $\mathrm{GL}_2(\mathbb{Z}_{/p})$ on pairs (Q_1, Q_2).

Lemma 6.1 *Let z be a quadratic nonresidue in $\mathbb{Z}_{/p}$. A pair of binary quadratic forms (Q_1, Q_2) satisfying the condition in Proposition 4.1 is equivalent to (xa^2, yb^2) or equivalent to $(a^2 + xb^2, 2ab)$ for $x, y \in \mathbb{Z}_{/p}$.*

Proof There are five [24, Theorem IV.10] equivalence classes of binary quadratic forms modulo p, namely the trivial form $Q(a,b) = 0$, two degenerate forms a^2 and za^2, and two nondegenerate quadratic forms $a^2 + b^2$ and $a^2 + zb^2$.

Suppose Q_1 is degenerate, so $Q_1(a,b) = a^2$ or $Q_1(a,b) = za^2$. Through an automorphism of $\mathbb{Z}^2$ subtracting a multiple of Q_1, the form Q_2 becomes $xab + yb^2$ for some $x, y \in \mathbb{Z}_{/p}$. By Proposition 4.1, it cannot be that $y = 0$. Since $y \neq 0$, the automorphism of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ sending a to a and b to $-ax/(2y) + b$ preserves Q_1 but transforms Q_2 into $yb^2 - (x^2/(4y))a^2$. Subtracting off a multiple of Q_1 via an automorphism of $\mathbb{Z}^2$ finally transforms Q_2 into yb^2. Therefore $(Q_1, Q_2) \simeq (xa^2, yb^2)$ for some $x, y \in \mathbb{Z}_{/p}$.

On the other hand, suppose Q_1 is nondegenerate, meaning $Q_1(a,b) = a^2 + b^2$ or $Q_1(a,b) = a^2 + zb^2$. As before, by subtracting off a multiple of Q_1, the form Q_2 becomes $xab + yb^2$ for some $x, y \in \mathbb{Z}_{/p}$. Either $y \neq 0$ or $y = 0$. If $y \neq 0$, then as before Q_2 is equivalent to $yb^2 - (x^2/(4y))a^2$, which via an automorphism of $\mathbb{Z}^2$ is transformed into a multiple of b^2, and this case is then handled by the above case in which Q_1 is degenerate. If $y = 0$, then we are in the situation $(a^2 + zb^2, xab)$ for nonzero z

and x. We apply the automorphism given by $a \mapsto a$ and $b \mapsto 2b/x$ to reduce to a situation of the form $(a^2 + wb^2, 2ab)$ for some nonzero w. □

Proposition 6.2 *A pair of the form (xa^2, yb^2) for nonzero $x, y \in \mathbb{Z}_{/p}$ is equivalent to $(a^2 + wb^2, 2ab)$ for $w \in \mathbb{Z}_{/p}$.*

Proof By independently scaling a and b, and depending on whether or not x and y are quadratic residues, the pair (xa^2, yb^2) is equivalent to

$$(a^2, b^2), \quad (a^2, zb^2), \quad (za^2, b^2) \quad \text{or} \quad (za^2, zb^2)$$

for a quadratic nonresidue $z \in \mathbb{Z}^\star_{/p}$. By exchanging the roles of a and b and swapping the components of the tuple, the pair (a^2, zb^2) is equivalent to (za^2, b^2). It is also the case that $(za^2, zb^2) \simeq (a^2, b^2)$ because

$$(za^2, zb^2) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1/z \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \cdot (a^2, b^2).$$

To conclude the proof, we show $(a^2, wb^2) \simeq (a^2 + 4w^2b^2, 2ab)$ for $w \in \mathbb{Z}_{/p}$. To see this, applying the equivalence given by $a \mapsto a/(2w) - b$ and $b \mapsto a + 2wb$ shows

$$(a^2, wb^2) \simeq \Big(\frac{1}{4w^2}a^2 - \frac{1}{w}ab + b^2, wa^2 + 4w^2ab + 4w^3b^2\Big),$$

and then applying the automorphism of $\mathbb{Z}^2$ corresponding to

$$\begin{pmatrix} 1/(4w^2) & -1/(2w) \\ w & 2w^2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_{/p})$$

implies that

$$(a^2 + 4w^2b^2, 2ab) \simeq \Big(\frac{1}{4w^2}a^2 - \frac{1}{w}ab + b^2, wa^2 + 4w^2ab + 4w^3b^2\Big),$$

so $(a^2, wb^2) \simeq (a^2 + 4w^2b^2, 2ab)$. □

It remains to check that $(a^2 + b^2, 2ab)$ is *not* equivalent to $(a^2 + zb^2, 2ab)$.

Lemma 6.3 *If $(a^2 + \delta b^2, 2ab)$ is equivalent to $(a^2 + \delta' b^2, 2ab)$ for nonzero δ and δ', then $\delta'/\delta \in \mathbb{Z}^{\star 4}_{/p}$.*

Proof We follow the argument in [9]. Suppose $(a^2 + \delta b^2, 2ab)$ is equivalent to $(a^2 + \delta' b^2, 2ab)$ for nonzero δ and δ'. Then there is an $R \in \mathrm{GL}_2(\mathbb{Z}_{/p})$ and $S \in \mathrm{SL}_2^{\pm}(\mathbb{Z}_{/p})$ such that

$$(a^2 + \delta b^2, 2ab) \cdot R = S \cdot (a^2 + \delta' b^2, 2ab). \tag{1}$$

Equality of the first component in each tuple yields

$$(r_{11}^2 + \delta r_{21}^2)a^2 + 2(r_{11}r_{12} + \delta r_{21}r_{22})ab + (r_{12}^2 + \delta r_{22}^2)b^2 = s_{11}a^2 + 2s_{12}ab + \delta' s_{11}b^2. \tag{2}$$

Equality of the coefficients of a^2 and b^2 in (2) yields

$$s_{11} = \delta r_{21}^2 + r_{11}^2 \quad \text{and} \quad \delta' s_{11} = \delta r_{22}^2 + r_{12}^2,$$

respectively, and therefore

$$\delta r_{22}^2 + r_{12}^2 = \delta'\delta r_{21}^2 + \delta' r_{11}^2. \tag{3}$$

Equality of the second component in (1) yields

$$2r_{11}r_{21}a^2 + (2r_{12}r_{21} + 2r_{11}r_{22})ab + 2r_{12}r_{22}b^2 = s_{21}a^2 + 2s_{22}ab + \delta' s_{21}b^2. \tag{4}$$

Equality of the coefficients of a^2 and b^2 in (4) yields

$$s_{21} = 2r_{11}r_{21} \quad \text{and} \quad \delta' s_{21} = 2r_{12}r_{22},$$

respectively. We conclude

$$r_{12}r_{22} = \delta' r_{11}r_{21}. \tag{5}$$

Squaring both sides of (3) and subtracting 4δ times (5) squared yields

$$(\delta r_{22}^2 - r_{12}^2)^2 = (\delta'\delta r_{21}^2 - \delta' r_{11}^2)^2,$$

and so

$$\delta r_{22}^2 - r_{12}^2 = \pm(\delta'\delta r_{21}^2 - \delta' r_{11}^2). \tag{6}$$

The sign in (6) cannot be positive; if it were, then adding (6) to (3) yields

$$2\delta r_{22}^2 = 2\delta'\delta r_{21}^2,$$

so $r_{22}^2 = \delta' r_{21}^2$. But multiply both sides of (5) by r_{21}^2 and we deduce

$$r_{12}r_{22}r_{21}^2 = \delta' r_{11}r_{21}^2 = r_{22}^2 r_{11}.$$

So either $r_{22} = 0$, in which case $r_{21} = 0$ and the second row of R is zero, or $r_{12}r_{21} = r_{22}r_{11}$ and so $\det R = 0$. In either case we contradict the assumption $R \in \mathrm{GL}_2(\mathbb{Z}_{/p})$, and so the sign in (6) must be negative, meaning

$$\delta r_{22}^2 - r_{12}^2 = -\delta'\delta r_{21}^2 + \delta' r_{11}^2. \tag{7}$$

The difference of (3) and (7) yields

$$2r_{12}^2 = 2\delta\delta' r_{21}^2,$$

so $\delta\delta'$ is a square in $\mathbb{Z}_{/p}$. And if our only requirement is that $R \in \mathrm{GL}_2(\mathbb{Z}_{/p})$, then the necessary condition that $\delta'\delta \in \mathbb{Z}_{/p}^{\star 2}$ would suffice, but we also required $S \in \mathrm{SL}_2^{\pm}(\mathbb{Z}_{/p})$, or equivalently that $(\det S)^2 = 1$. From (2) and (4),

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} \delta r_{21}^2 + r_{11}^2 & \delta r_{21}r_{22} + r_{11}r_{12} \\ 2r_{11}r_{21} & r_{12}r_{21} + r_{11}r_{22} \end{pmatrix},$$

which means

$$\det S = (\delta r_{21}^2 - r_{11}^2)(r_{12}r_{21} - r_{11}r_{22}).$$

Squaring $\det R$ results in

$$\begin{aligned}(\det R)^2 = (r_{11}r_{22} - r_{12}r_{21})^2 = r_{11}^2 r_{22}^2 - 2r_{11}r_{12}r_{21}r_{22} + r_{12}^2 r_{21}^2 = r_{11}^4 \frac{\delta'}{\delta} - 2\delta' r_{11}^2 r_{21}^2 + \delta\delta' r_{21}^4 \\ = \frac{\delta'}{\delta}(\delta r_{21}^2 - r_{11}^2)^2\end{aligned}$$

by invoking (5) and applying the identities $r_{12}^2 = \delta\delta' r_{21}^2$ and $\delta r_{22}^2 = \delta' r_{11}^2$, which follow from taking the sum and difference of (3) and (7). Consequently,

$$(\det S)^2 = (\delta r_{21}^2 - r_{11}^2)^2 (\det R)^2 = \frac{\delta'}{\delta}(\delta r_{21}^2 - r_{11}^2)^4,$$

so δ'/δ must be a fourth power. □

In particular, $(a^2+b^2, 2ab)$ is not equivalent to $(a^2+zb^2, 2ab)$ because z was chosen specifically to be a quadratic nonresidue.

Lemma 6.4 *For nonzero* $\delta, w \in \mathbb{Z}_{/p}$, *the pair* $(a^2+\delta b^2, 2ab)$ *is equivalent to* $(a^2+\delta w^4 b^2, 2ab)$.

Proof Choose $r_1, r_2 \in \mathbb{Z}_{/p}$ so that

$$\delta {r_1}^2 - r_2^2 \equiv 1/w^3 \pmod p. \tag{8}$$

This is possible; in fact, there are $p-(\delta/p)$ solutions to (8). Then set

$$R := \begin{pmatrix} w^2 r_2 & \delta w^2 r_1 \\ r_1 & r_2 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} \delta w^4 {r_1}^2 + w^4 r_2^2 & 2\delta w^4 r_1 r_2 \\ 2w^2 r_1 r_2 & \delta w^2 r_1^2 + w^2 r_2^2 \end{pmatrix}.$$

Because of (8), we have

$$\det R = -w^2(\delta {r_1}^2 - {r_2}^2) = -1/w \neq 0 \quad \text{and} \quad \det S = w^6(\delta {r_1}^2 - {r_2}^2)^2 = 1,$$

so $R \in \mathrm{GL}_2(\mathbb{Z}_{/p})$ and $S \in \mathrm{SL}_2(\mathbb{Z}_{/p})$.

We finish the proof by verifying

$$(a^2+\delta w^4 b^2, 2ab) \cdot R = S \cdot (a^2+\delta b^2, 2ab). \tag{9}$$

Comparing the first coordinates each side of (9) shows

$$(r_2 w^2 a + \delta r_1 w^2 b)^2 + \delta w^4 (r_1 a + r_2 b)^2 = (\delta w^4 {r_1}^2 + w^4 {r_2}^2) \cdot (a^2+\delta b^2) + 2\delta w^4 r_1 r_2 \cdot 2ab.$$

Similarly, the second coordinates are equal because

$$2(w^2 r_2 a + \delta w^2 r_1 b)(r_1 a + r_2 b) = 2w^2 r_1 r_2 (a^2 + b^2\delta) + (\delta w^2 {r_1}^2 + w^2 {r_2}^2) 2ab. \qquad \square$$

It is easier to see that $(a^2+\delta b^2, 2ab)$ is equivalent to $(a^2+\delta w^8 b^2, 2ab)$. Simply replace a by aw and b by b/w^3 to show $(a^2+\delta w^8 b^2, 2ab) \simeq (w^2 a^2 + \delta w^2 b^2, (2/w^2)ab)$, and then scale the first by $1/w^2$ and the second by w^2 to see that this is equivalent to $(a^2+\delta b^2, 2ab)$. The challenge of Lemma 6.4 lies in replacing w^8 with w^4.

Combining Lemmas 6.3 and 6.4 yields the following:

Proposition 6.5 *Equivalence classes of pairs of the form* $(a^2+wb^2, 2ab)$ *are in one-to-one correspondence with elements of* $\mathbb{Z}_{/p}^\times/(\mathbb{Z}_{/p}^\times)^4$, *where* $\mathbb{Z}_{/p}^\times$ *denotes units modulo* p.

Observe that the size of $\mathbb{Z}_{/p}^\star/(\mathbb{Z}_{/p}^\times)^4$ depends on $p \bmod 4$. Specifically, for $p \equiv 1 \bmod 4$, there are *four* equivalence classes. These are given by $(a^2+zb^2, 2ab)$ for z representatives of classes $\mathbb{Z}_{/p}^\star/\mathbb{Z}_{/p}^{\star 4}$.

For $p \equiv 3 \bmod 4$, there are *two* equivalence classes. For nonzero $x, x', y, y' \in \mathbb{Z}_{/p}$, the pair (xa^2, yb^2) is equivalent to $(x'a^2, y'b^2)$, and every pair is equivalent to either $(a^2+b^2, 2ab)$ or $(a^2+zb^2, 2ab)$ for a quadratic nonresidue z. So the only possibilities are $(a^2+b^2, 2ab) \simeq (a^2, b^2)$ and $(a^2+zb^2, 2ab)$.

All of this algebra encodes the homotopy type of the quotients, as summarized in the following:

Theorem 6.6 *Let $p > 3$ be prime. If $p \equiv 1 \bmod 4$, then there are four homotopy classes of quotients of $S^3 \times S^3$ by free $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions. If $p \equiv 3 \bmod 4$, then there are two classes.*

Proof We must construct quotients of $S^3 \times S^3$ by free $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions which exhibit these possible k–invariants. For this, we rely on Lemma 5.1. We note that $(a^2+wb^2, 2ab)$ is equivalent to

$$(a^2+wb^2+(1+w)ab, 2ab) = ((a+b)(a+wb), 2ab),$$

so let $R = (1,1,2,0)$ and $Q = (1,w,0,1)$, and then $L(p,p;R,Q)$ has k–invariant equivalent to $(a^2+wb^2, 2ab)$. We must impose the additional condition $w \neq 0$ in order to ensure that this is a *free* action. With this construction in hand, the classification of quotients then follows from Proposition 6.5. □

Remark 6.7 There are precedents for considering the simultaneous equivalence of forms. The case of simultaneous equivalence of forms over $\mathbb{Z}$ is discussed in [23], but our situation over $\mathbb{Z}_{/p}$ is easier. To make the situation even more concrete, instead of forms, consider matrices; equivalence of forms amounts to congruence of matrices. That setup fits into the work of Corbas and Williams [9] which considers the action of $\mathrm{GL}_2(\mathbb{Z}_{/p}) \times \mathrm{GL}_2(\mathbb{Z}_{/p})$ on pairs (A, B) of matrices, where GL_2 acts on the right by congruence and on the left as in $(*)$.

7 Lens cross lens

Section 6 completed the classification of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ actions on $S^3 \times S^3$, but now we narrow in on a special case. Consider $L_3(p;1,x) \times L_3(p;1,y)$, i.e. the product of two lens spaces with rotation numbers x and y, respectively. Viewed as a quotient of $S^3 \times S^3$ by $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$, this product has k–invariant (xa^2, yb^2).

We can classify $L_3(p;1,x) \times L_3(p;1,y)$ up to (simple) homotopy equivalence. When $p \equiv 3 \bmod 4$, any product of 3–dimensional lens spaces is (simple) homotopy equivalent to any other such product.

Proposition 7.1 *Suppose $p \equiv 3 \bmod 4$. Then for nonzero $x, x', y, y' \in \mathbb{Z}_{/p}$, the pair (xa^2, yb^2) is equivalent to $(x'a^2, y'b^2)$.*

Proof As in the proof of Proposition 6.2, the pair (xa^2, yb^2) is equivalent to

$$(a^2, b^2) \simeq (za^2, zb^2) \quad \text{or} \quad (a^2, zb^2) \simeq (za^2, b^2)$$

for a quadratic nonresidue $z \in \mathbb{Z}_{/p}^\star$. But when $p \equiv 3 \bmod 4$, the quantity $-z$ is a square, and so

$$(a^2, zb^2) \simeq (a^2, -zb^2) \simeq (a^2, b^2),$$

meaning *all* pairs of the form (xa^2, yb^2) are equivalent. □

When $p \equiv 1 \mod 4$, since

$$(xa^2, yb^2) \simeq (a^2, (y/x)b^2) \simeq (a^2 + 4(y/x)^2 b^2, 2ab),$$

the classification boils down to whether or not $2(y/x)$ is a square modulo p.

This is related to work of Kwasik and Schultz; they proved squares of lens spaces are diffeomorphic.

Theorem 7.2 [14] *For p odd and rotation numbers r and q, there is a diffeomorphism*

$$L_3(p;1,r) \times L_3(p;1,r) \cong L_3(p;1,q) \times L_3(p;1,q).$$

A future paper completes the homeomorphism classification of spaces resulting from "linear" actions such as these products of lens spaces.

8 Some comments on groups containing $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$

While we know that $\mathbb{Z}_{/p}$ and $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ can act freely on $S^n \times S^n$, the exact conditions for a group to be able to act freely on $S^n \times S^n$ remains open. Conner [7] and Heller [12] showed that for a group to act freely on $S^n \times S^n$ the group must have rank at most 2, but Oliver [26] showed that A_4 cannot act on $S^n \times S^n$, and so every rank-2 simple group is also ruled out [1]. Explicit examples of free actions by subgroups of a nonabelian extension of S^1 by $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ have been constructed [11], but Okay and Yalçin [25] have shown that $\mathrm{Qd}(p) = (\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}) \rtimes SL_2(\mathbb{F}_p)$ cannot act freely on $S^n \times S^n$. In this section we show how the restrictions on the k–invariant as described in Section 4 can be useful in determining whether or not a group G containing $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ as a normal abelian subgroup can act freely on $X = S^n \times S^n$. We continue to take $p > 3$ to be an odd prime and $n \geq 3$ to be odd. We align some of our notation with that in [25] to better show the parallel calculations.

Similar to the approach in Section 4, we can consider the Borel fibration

$$X \xrightarrow{i} X_{hG} \to BG,$$

and the associated Serre spectral sequence

$$E_2^{p,q} = H^p(BG; H^q(X;\mathbb{Z})) \Rightarrow H^{p+q}(X_{hG};\mathbb{Z})$$

with the first nontrivial differential d_{n+1}. If α and γ are the generators in degree n of $H^*(X;\mathbb{Z})$ with $\alpha^2 = \gamma^2 = 0$, then $d_{n+1}(\alpha) = \bar{\tau}(\alpha, 0)$, $d_{n+1}(\gamma) = \bar{\tau}(0, \gamma)$ and $k^{n+1} = d_{n+1}(\alpha) \oplus d_{n+1}(\gamma)$.

Set K to be the normal abelian subgroup of $\mathbb{Z}_{/p} \times \mathbb{Z}_{/p}$ in G, and consider the restriction of the spectral sequence associated to the Borel fibration to the K action. Then Proposition 4.1 and Corollary 4.2 can sometimes be used to determine if G can act freely on X.

The transgression for the first nontrivial differential of the restriction of the spectral sequence associated to the Borel fibration to K is

$$(d_{n+1})_K \colon H^0(BK; H^n(X;\mathbb{Z})) \to H^{n+1}(BK; H^0(X;\mathbb{Z})).$$

Let $\mathrm{Res}^G_K\colon H^*(G)\to H^*(K)$ be induced by the inclusion of K into G. Since the Borel construction is natural, it follows that the k–invariant in the restricted case is $k_K^{n+1}=\mathrm{Res}^G_K(d_{n+1}(\alpha))\oplus\mathrm{Res}^G_K(d_{n+1}(\gamma))$.

Suppose G acts freely on X, so $H^*(X_{hG};\mathbb{Z})\cong H^*(X/G;\mathbb{Z})$ is finite-dimensional in each degree and vanishes above $2n$. It follows that the restriction to K gives that $H^*(X_{hK};\mathbb{Z})\cong H^*(X/K;\mathbb{Z})$ is also finite-dimensional in each degree and vanishes above $2n$ as K acts freely. If both $(d_{n+1})_K(\alpha)$ and $(d_{n+1})_K(\gamma)$ are zero in $H^{n+1}(K;\mathbb{Z})/\lambda^{(n+1)/2}$, for some nonzero $\lambda\in H^2(K;\mathbb{Z})$, then X/K will fail to be finite-dimensional by Corollary 4.2, and we get a contradiction. Hence G cannot act freely.

As an example, consider $G=\mathrm{Qd}(p)=(\mathbb{Z}_{/p})^2\rtimes SL_2(\mathbb{Z}_{/p})$. We show that one can use the restrictions on the k–invariants and some of the arguments in [25] to see that $\mathrm{Qd}(p)$ cannot act freely on $S^n\times S^n$ for p an odd prime and n odd. This result is consistent with [25, Theorem 5.1].

Since cohomology is taken with $\mathbb{Z}_{/p}$ coefficients in [25], we first set up a relationship between generators from the different coefficient groups. Suppose the first nontrivial differential takes α and γ, also the generators of $H^n(S^n\times S^n;\mathbb{Z}_{/p})$ by slight abuse of notation, to μ_1 and μ_2 in $H^{n+1}(G;\mathbb{Z}_{/p})$. Taking K to be the normal elementary abelian subgroup $\mathbb{Z}_{/p}\times\mathbb{Z}_{/p}$ in $G=\mathrm{Qd}(p)$ and restricting the action to K, we have that $\theta_1,\theta_2\in H^{n+1}(K;\mathbb{Z}_{/p})$ are such that $\theta_1=\mathrm{Res}^G_K(\mu_1)$ and $\theta_2=\mathrm{Res}^G_K(\mu_2)$.

Recall the commuting triangle, from Section 2,

$$\begin{tikzcd}
H^n(K;\mathbb{Z}) \arrow[r, "\rho"] & H^n(K;\mathbb{Z}_{/p}) \arrow[r, "\tilde\beta"] \arrow[dr, "\beta"'] & H^{n+1}(K;\mathbb{Z}) \arrow[r, "p"] \arrow[d, "\rho"] & H^{n+1}(K;\mathbb{Z}) \\
 & & H^{n+1}(K;\mathbb{Z}_{/p}) &
\end{tikzcd}$$

Since p is the 0 map, the vertical ρ is injective and $\tilde\beta$ is surjective. We can write $H^*(K;\mathbb{Z}/p)=\mathbb{F}_p[x,y]\otimes\wedge(u,v)$, where $|x|=|y|=2$, $|u|=|v|=1$, $\beta(u)=x$, $\beta(v)=y$ and $H^*(K;\mathbb{Z})=\mathbb{F}_p[a,b]\otimes\wedge(c)$, with $|a|=|b|=2$ and $|c|=3$. It is not hard to see that $\tilde\beta(x)=a$, $\tilde\beta(y)=b$ and $\tilde\beta(uv)=c$.

Now the Bockstein generally satisfies $\beta(\delta\varepsilon)=\beta(\delta)\varepsilon+(-1)^{|\delta|}\delta\beta(\varepsilon)=\delta\beta(\varepsilon)$ for δ being x^iy^j and ε being u, v or uv. We see that

$$\beta(H^n(K;\mathbb{Z}_{/p}))\subseteq\langle x^{(n+1)/2},x^{(n-1)/2}y,\dots,y^{(n+1)/2}\rangle\subseteq\mathbb{F}_p[x,y],$$

since $n+1$ is even. Similarly, $\tilde\beta$ satisfies $\tilde\beta(\delta\varepsilon)=\delta\tilde\beta(\varepsilon)$ for δ being x^iy^j and ε being u, v or uv. Again we see that $\tilde\beta(H^n(K;\mathbb{Z}_{/p}))\subseteq\langle a^{(n+1)/2},a^{(n-1)/2}b,\dots,b^{(n+1)/2}\rangle\subseteq\mathbb{F}_p[a,b]$. As $\tilde\beta$ is surjective, ρ is injective, and $\beta=\rho(\tilde\beta)$. It follows that the k–invariant $\theta_1\oplus\theta_2$ comes from elements in $H^{n+1}(K;\mathbb{Z})$ for some K action on $S^n\times S^n$: $k^{n+1}=\rho^{-1}(\theta_1)\oplus\rho^{-1}(\theta_2)$.

In [25] it is shown that the ideal generated by θ_1 and θ_2 is in fact generated by $\zeta^{(n+1)/2(p+1)}$, where $\zeta=xy^p-yx^p$ (which is in part based on calculations in [15]). Since no power of ζ will contain $x^{(n+1)/2}$ or $y^{(n+1)/2}$, we see that $d_{n+1}(\alpha)$ and $d_{n+1}(\gamma)$, where α and γ generate $H^n(S^n\times S^n;\mathbb{Z})$, have both $q_{\alpha,0}$ and $q_{\gamma,0}$ zero (where $q_{\alpha,0}$ and $q_{\gamma,0}$ are the coefficients in Proposition 4.1). We derive a contradiction.

It is worth noting that, in [25], the calculations show that the free actions of $\mathrm{Qd}(p)$ must have p smaller than n, and $n+1$ divisible by $2(p+1)$. The argument also finds a contradiction to finiteness, but relies on [5]. We also note that while we take p to be large in our homotopy type calculations, the only restrictions that were required in Section 4 (and hence in this section) were that $p > 3$ be an odd prime and $n \geq 3$ be odd. Further, there may be a way to show a contradiction to finiteness using Proposition 4.1 more directly (without needing to make arguments with $\mathbb{Z}_{/p}$ coefficients).

A similar argument could hold for any group containing $(\mathbb{Z}_{/p})^2$ that has a restriction that forces the transgression to behave in such a way.

References

[1] **A Adem**, **J H Smith**, *On spaces with periodic cohomology*, Electron. Res. Announc. Amer. Math. Soc. 6 (2000) 1–6 MR Zbl

[2] **H J Baues**, *On the group of homotopy equivalences of a manifold*, Trans. Amer. Math. Soc. 348 (1996) 4737–4773 MR Zbl

[3] **D J Benson**, **J F Carlson**, *Complexity and multiple complexes*, Math. Z. 195 (1987) 221–238 MR Zbl

[4] **C-F Bödigheimer**, **H-W Henn**, *A remark on the size of $\pi_q(S^n)$*, Manuscripta Math. 42 (1983) 79–83 MR Zbl

[5] **G Carlsson**, *On the rank of abelian groups acting freely on $(S^n)^k$*, Invent. Math. 69 (1982) 393–400 MR Zbl

[6] **G R Chapman**, *The cohomology ring of a finite abelian group*, Proc. Lond. Math. Soc. 45 (1982) 564–576 MR Zbl

[7] **P E Conner**, *On the action of a finite group on $S^n \times S^n$*, Ann. of Math. 66 (1957) 586–588 MR Zbl

[8] **G E Cooke**, *The Hauptvermutung according to Casson and Sullivan*, from "The Hauptvermutung book", K–Monogr. Math. 1, Kluwer, Dordrecht (1996) 165–187 MR Zbl

[9] **B Corbas**, **G D Williams**, *Congruence of two-dimensional subspaces in $M_2(K)$ (characteristic $\neq 2$)*, Pacific J. Math. 188 (1999) 225–235 MR Zbl

[10] **P G Goerss**, **J F Jardine**, *Simplicial homotopy theory*, Progr. Math. 174, Birkhäuser, Basel (1999) MR Zbl

[11] **I Hambleton**, **Ö Ünlü**, *Free actions of finite groups on $S^n \times S^n$*, Trans. Amer. Math. Soc. 362 (2010) 3289–3317 MR Zbl

[12] **A Heller**, *A note on spaces with operators*, Illinois J. Math. 3 (1959) 98–100 MR Zbl

[13] **P Hilton**, **G Mislin**, **J Roitberg**, *Localization of nilpotent groups and spaces*, North-Holland Math. Stud. 15, North-Holland, Amsterdam (1975) MR Zbl

[14] **S Kwasik**, **R Schultz**, *All $\mathbb{Z}_q$ lens spaces have diffeomorphic squares*, Topology 41 (2002) 321–340 MR Zbl

[15] **J H Long**, *The cohomology rings of the special affine group of* $\mathbb{F}_p^2$ *and of* $\mathrm{PSL}(3, p)$, PhD thesis, University of Maryland, College Park (2008) Available at `https://www.proquest.com/docview/304560975`

[16] **T Macko**, **C Wegner**, *On fake lens spaces with fundamental group of order a power of 2*, Algebr. Geom. Topol. 9 (2009) 1837–1883 MR Zbl

[17] **T Macko**, **C Wegner**, *On the classification of fake lens spaces*, Forum Math. 23 (2011) 1053–1091 MR Zbl

[18] **I Madsen**, **C B Thomas**, **C T C Wall**, *The topological spherical space form problem, II: Existence of free actions*, Topology 15 (1976) 375–382 MR Zbl

[19] **J P May**, *A concise course in algebraic topology*, Univ. of Chicago Press (1999) MR Zbl

[20] **J P May**, **K Ponto**, *More concise algebraic topology: localization, completion, and model categories*, Univ. Chicago Press (2012) MR Zbl

[21] **J McCleary**, *A user's guide to spectral sequences*, 2nd edition, Cambridge Stud. Adv. Math. 58, Cambridge Univ. Press (2001) MR Zbl

[22] **J Milnor**, *Groups which act on S^n without fixed points*, Amer. J. Math. 79 (1957) 623–630 MR Zbl

[23] **J Morales**, *The classification of pairs of binary quadratic forms*, Acta Arith. 59 (1991) 105–121 MR Zbl

[24] **M Newman**, *Integral matrices*, Pure Appl. Math. 45, Academic, New York (1972) MR Zbl

[25] **C Okay**, **E Yalçin**, *Dimension functions for spherical fibrations*, Algebr. Geom. Topol. 18 (2018) 3907–3941 MR Zbl

[26] **R Oliver**, *Free compact group actions on products of spheres*, from "Algebraic topology" (J L Dupont, I H Madsen, editors), Lecture Notes in Math. 763, Springer (1979) 539–548 MR Zbl

[27] **C D Roberts**, *The cohomology ring of a finite abelian group*, PhD thesis, University of Waterloo (2013) Available at `http://hdl.handle.net/10012/7276`

[28] **P A Smith**, *Permutable periodic transformations*, Proc. Nat. Acad. Sci. U.S.A. 30 (1944) 105–108 MR Zbl

[29] **D P Sullivan**, *Geometric topology: localization, periodicity and Galois symmetry*, K–Monogr. Math. 8, Springer (2005) MR Zbl

[30] **C M Thatcher**, *On free $\mathbb{Z}_p$ actions on products of spheres*, Geom. Dedicata 148 (2010) 391–415 MR Zbl

Department of Mathematics, The Ohio State University
Columbus, OH, United States

Department of Mathematics and Computer Science, University of Puget Sound
Tacoma, WA, United States

fowler@math.osu.edu, courtneythatcher@gmail.com

https://math.osu.edu/~fowler.291/

Received: 28 August 2020 Revised: 16 December 2022 msp

Algebraic & Geometric Topology **24**:4 (2024) 1863–1901
DOI: 10.2140/agt.2024.24.1863
Published: 16 July 2024

$\mathbb{Z}_k$–stratifolds

ANDRÉS ÁNGEL
CARLOS SEGOVIA
ARLEY FERNANDO TORRES

Generalizing the ideas of $\mathbb{Z}_k$–manifolds from Sullivan and stratifolds from Kreck, we define $\mathbb{Z}_k$–stratifolds. We show that the bordism theory of $\mathbb{Z}_k$–stratifolds is sufficient to represent all homology classes of a CW–complex with coefficients in $\mathbb{Z}_k$. We present a geometric interpretation of the Bockstein long exact sequences and the Atiyah–Hirzebruch spectral sequence for $\mathbb{Z}_k$–bordism for k an odd number. Finally, for p an odd prime, we give geometric representatives of all classes in $H_*(B\mathbb{Z}_p;\mathbb{Z}_p)$ using $\mathbb{Z}_p$–stratifolds.

57R90, 58A35, 58A40; 55N20

1 Introduction

Various geometric models of homology classes use the notion of bordism. For instance, Baas [3] constructs a generalized homology theory using the bordism of manifolds with singularities. Buoncristiano, Rourke and Sanderson [5] give a geometric treatment of generalized homology. Certain singular spaces called $\mathbb{Z}_k$–manifolds were introduced initially by Sullivan [18; 19; 20], although Morgan and Sullivan [15] gave the first formal study of this subject. The theory of $\mathbb{Z}_k$–manifolds gives a geometric model for $\mathbb{Z}_k$–homology classes, but Sullivan pointed out that $\mathbb{Z}_k$–manifolds are not general enough to represent $\mathbb{Z}_k$–homology. For example, the generator of $H_8(K(\mathbb{Z},3);\mathbb{Z}_3)$ is not represented by a $\mathbb{Z}_3$–manifold; see Sullivan [21]. Moreover, Brumfiel [4] shows that the nonzero classes in $H_{2p}(K(\mathbb{Z}_p,1);\mathbb{Z}_p)$ cannot be represented by $\mathbb{Z}_p$–manifolds whenever p is prime. In this work, we show that for an odd prime number p, there exists a class $\alpha_{2i} \in H_{2i}(B\mathbb{Z}_p;\mathbb{Z}_p)$, with $i \geq p$, that cannot be represented by $\mathbb{Z}_p$–manifolds. Thus a geometric model is needed to represent every homology class with $\mathbb{Z}_k$–coefficients. For this purpose, we focus on the theory of stratifolds developed by Kreck [12], where the homology groups with $\mathbb{Z}$–coefficients and $\mathbb{Z}_2$–coefficients are represented by the bordism theories of stratifold homology $SH_*(X)$ and stratifold homology with $\mathbb{Z}_2$–coefficients (this only works for $\mathbb{Z}_2$–coefficients).

We consider the generalized homology theory of bordism of $\mathbb{Z}_k$–manifolds with continuous maps to X, denoted by $\Omega_*(X;\mathbb{Z}_k)$. There is a long exact sequence satisfying the commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & \Omega_n(X) \arrow[r, "\times k"] \arrow[d, "h"] & \Omega_n(X) \arrow[r, "r"] \arrow[d, "h"] & \Omega_n(X;\mathbb{Z}_k) \arrow[r, "\delta"] \arrow[d, "h_{\mathbb{Z}_k}"] & \Omega_{n-1}(X) \arrow[r] \arrow[d, "h"] & \cdots \\
\cdots \arrow[r] & H_n(X) \arrow[r, "\times k"] & H_n(X) \arrow[r, "r"] & H_n(X;\mathbb{Z}_k) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}
\tag{1}
$$

where $\delta\colon \Omega_*(X;\mathbb{Z}_k) \to \Omega_{n-1}(X)$ is the Bockstein homomorphism, $r\colon \Omega_n(X) \to \Omega_n(X;\mathbb{Z}_k)$ is the reduction homomorphism obtained by considering a closed manifold as a $\mathbb{Z}_k$–manifold with empty Bockstein, and $h_{\mathbb{Z}_k}: \Omega_*(X;\mathbb{Z}_k) \to H_*(X;\mathbb{Z}_k)$ is the Hurewicz homomorphism provided by the existence of fundamental $\mathbb{Z}_k$–homology classes.

Generalizing the ideas of Sullivan and Kreck, we define the bordism theory of $\mathbb{Z}_k$–stratifolds, and we can consider the generalized homology theory of bordism of $\mathbb{Z}_k$–stratifolds with continuous maps to X, denoted by $SH_*(X;\mathbb{Z}_k)$. We call this theory *$\mathbb{Z}_k$–stratifold homology*. Again, we have a long exact sequence satisfying the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & SH_n(X) \arrow[r, "\times k"] \arrow[d, "h"] & SH_n(X) \arrow[r, "r"] \arrow[d, "h"] & SH_n(X;\mathbb{Z}_k) \arrow[r, "\delta"] \arrow[d, "h_{\mathbb{Z}_k}"] & SH_{n-1}(X) \arrow[r] \arrow[d, "h"] & \cdots \\
\cdots \arrow[r] & H_n(X) \arrow[r, "\times k"] & H_n(X) \arrow[r, "r"] & H_n(X;\mathbb{Z}_k) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd} \tag{2}$$

In this case, the Hurewicz homomorphism $h_{\mathbb{Z}_k}: SH_*(X;\mathbb{Z}_k) \to H_*(X;\mathbb{Z}_k)$ is constructed in the same vein as in the theory of $\mathbb{Z}_k$–manifolds. We show that $\mathbb{Z}_k$–stratifold homology satisfies the Eilenberg–Steenrod axioms on CW–complexes, in particular, we show that the Mayer–Vietoris sequence axiom holds by using a regularity argument for $\mathbb{Z}_k$–stratifolds; see Kreck [12]. The main result of this paper is the following.

Theorem 1.1 *An isomorphism exists between $\mathbb{Z}_k$–stratifold homology theory and singular homology with $\mathbb{Z}_k$–coefficients. This isomorphism is valid for all CW–complexes and is compatible with the Bockstein homomorphisms.*

Führing [9] develops a smooth version of the Baas–Sullivan theory of manifolds with singularities that is applied to the positive scalar curvature problem. In a way, stratifolds and $\mathbb{Z}_k$–stratifolds are another kind of smooth version of the Baas–Sullivan theory of manifolds with singularities. One of the advantages of stratifolds and $\mathbb{Z}_k$–stratifolds is a very concrete description of the filtration of the Atiyah–Hirzebruch spectral sequence (AHSS) for oriented bordism and $\mathbb{Z}_k$–bordism. This geometric description of the AHSS for $\mathbb{Z}$–coefficients was given by Tene [23], and for $\mathbb{Z}_k$–coefficients has the following form.

Theorem 1.2 *For k an odd number, the filtration for the AHSS of $\mathbb{Z}_k$–bordism*

$$E^{\infty}_{n,0} \subseteq \cdots \subseteq E^{r+2}_{n,0} \subseteq \cdots \subseteq E^{2}_{n,0} \cong H_n(X;\mathbb{Z}_k) = SH_n(X;\mathbb{Z}_k) \tag{3}$$

coincides with the set of classes generated by singular $\mathbb{Z}_k$–stratifolds in X, where the singular part is of dimension at most $n-r-2$.

A fascinating application is the existence of homology classes $\alpha_{2i} \in H_{2p}(B\mathbb{Z}_p;\mathbb{Z}_p)$, for an odd prime number p and $i \geq p$, that cannot be represented by a $\mathbb{Z}_p$–manifold. This is similar to the counterexample of Thom for the Steenrod problem [24, Chapter III], which we explain geometrically in [2].

We organize the article as follows: Section 2 outlines some basic facts about $\mathbb{Z}_k$–manifolds studied by Morgan and Sullivan [15]. In Section 3, we briefly introduce the language of stratifolds from Kreck [12; 13]. Section 4 introduces the main theorems of this work, where we combine the theory of $\mathbb{Z}_k$–manifolds from Sullivan and the theory of stratifolds from Kreck. Then we define $\mathbb{Z}_k$–stratifolds and develop the basic theory of these objects. We show that the usual properties of stratifolds still remain valid. We show that $\mathbb{Z}_k$–stratifold homology satisfies the Eilenberg–Steenrod axioms on CW–complexes. Section 6 develops the existence of the fundamental class, and we postpone the proof of the existence of the Mayer–Vietoris sequence until the appendix. In Section 7, we apply the results of Tene [23] to give a geometric description of the Atiyah–Hirzebruch spectral sequence for $\mathbb{Z}_k$–bordism, for k an odd number. In Section 8, we use this description to find homology classes with $\mathbb{Z}_k$–coefficients that cannot be represented by $\mathbb{Z}_k$–manifolds. Finally, in Section 9, the two possible ways to represent homology with $\mathbb{Z}_2$–coefficients using stratifolds are related, providing an explicit isomorphism between the two theories.

Acknowledgements We thank the Math Institute UNAM-Oaxaca and Universidad de los Andes for the hospitality and financial support that made this collaboration possible. Ángel acknowledges and thanks the hospitality and financial support provided by the Max Planck Institute for Mathematics in Bonn. This work was partially supported by the grant #INV-2019-84-1860 from the Fondo de Investigaciones de la Facultad de Ciencias de la Universidad de los Andes. Segovia is supported by cátedras CONACYT and Proyecto CONACYT Ciencias básicas 2016, #284621. Torres' PhD thesis [25] contains part of these results under the supervision of Ángel. The maturity of the present paper is due to the guidance of Segovia during two visits by Torres to the Math Institute UNAM-Oaxaca; without this invaluable contribution, this work would not have been possible. Torres would like to thank the Universidad Pontificia Javeriana for the help provided after his PhD, and especially the Universidad Externado de Colombia, where he has been a professor in the mathematics department since 2020. Finally, we thank the reviewer for the careful reading of our manuscript. We sincerely appreciate all the valuable comments and suggestions which helped us improve its quality.

2 $\mathbb{Z}_k$–manifolds

Suppose that $k \geq 2$ is a positive integer. In what follows, we outline some basic facts about $\mathbb{Z}_k$–manifolds introduced by Morgan and Sullivan [15].

Note 2.1 Unless otherwise indicated, let us set the convention that the manifolds are oriented and compact. Also, all the diffeomorphisms and embeddings are orientation-preserving.

Definition 2.2 A closed n–dimensional $\mathbb{Z}_k$*–manifold* is given by the triple $\mathcal{M} = (M, \delta M, \theta_i)$, where

(1) M is a compact n–manifold, with boundary ∂M,

(2) δM is a compact $(n-1)$–manifold without boundary, called the Bockstein, and

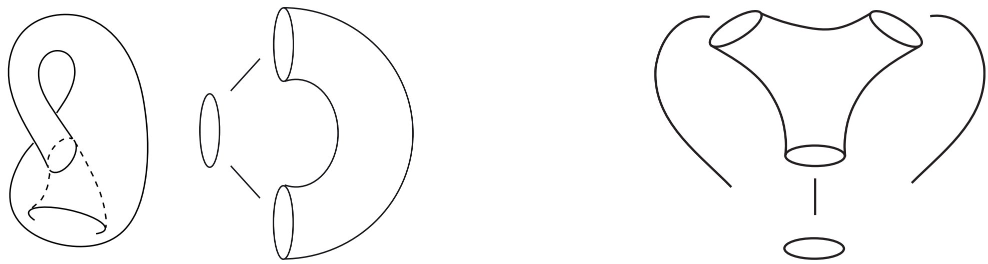


Figure 1: Left: a representation of the Klein bottle as the quotient space of a $\mathbb{Z}_2$–manifold. Right: a closed $\mathbb{Z}_3$–manifold.

(3) $\theta_i\colon \delta M \hookrightarrow \partial M$, with $i \in \mathbb{Z}_k$, are k disjoint embeddings such that we have a diffeomorphism $\partial M = \bigsqcup_{i \in \mathbb{Z}_k} \theta_i(\delta M)$.

Definition 2.3 There is an associated *quotient* space $\widetilde{M}$ given by the identification on M of the k copies of δM together using the embeddings θ_i.

Example 2.4 A closed oriented manifold is a $\mathbb{Z}_0$–manifold (or equivalently a $\mathbb{Z}$–manifold) where the Bockstein δM is empty.

Example 2.5 The typical example of a $\mathbb{Z}_2$–manifold is the cylinder $M = S^1 \times [0,1]$, $\delta M = S^1$ and embeddings $\theta_1, \theta_2\colon S^1 \hookrightarrow S^1 \times \{0\} \sqcup S^1 \times \{1\}$, with $\theta_1(S^1) = S^1 \times \{0\}$ and $\theta_2(S^1) = S^1 \times \{1\}$ (with the reverse orientation on $S^1 \times \{1\}$). The quotient space $K := \widetilde{M}$ is the well-known Klein bottle; see Figure 1, left.

Here we observe that even though the second integral homology group is zero for the Klein bottle, we can obtain a fundamental class after we change to $\mathbb{Z}_2$ coefficients, ie $H_2(K;\mathbb{Z}_2) \cong \mathbb{Z}_2$. In Section 6, we show this fundamental class always exists for a $\mathbb{Z}_k$–stratifold.

Example 2.6 Consider the pair of pants P with boundary $\partial P = S^1 \sqcup S^1 \sqcup S^1$ and Bockstein $\delta P = S^1$; see Figure 1, right.

Definition 2.7 An $(n+1)$–dimensional $\mathbb{Z}_k$*–manifold with boundary* is given by the triple $\mathcal{B} = (B, \delta B, \psi_i)$, where

(1) B is a compact $(n+1)$–dimensional manifold, with boundary ∂B,

(2) δB is a compact n–dimensional manifold, called the Bockstein, with boundary $\partial\delta B$, and

(3) $\psi_i\colon \delta B \hookrightarrow \partial B$, with $i \in \mathbb{Z}_k$, are k disjoint embeddings such that the triple
$$\left(\partial B - \operatorname{int}\left(\bigsqcup_{i \in \mathbb{Z}_k} \psi_i(\delta B)\right), \partial\delta B, \psi_i|_{\partial\delta B}\right)$$
defines a closed n–dimensional $\mathbb{Z}_k$–manifold $(M, \delta M, \theta_i)$.

This closed n–dimensional $\mathbb{Z}_k$–manifold is called the $\mathbb{Z}_k$*–boundary* of the $\mathbb{Z}_k$–manifold with boundary $\mathcal{B}$ and is denoted by $\partial\mathcal{B} = (M, \delta M, \theta_i)$.

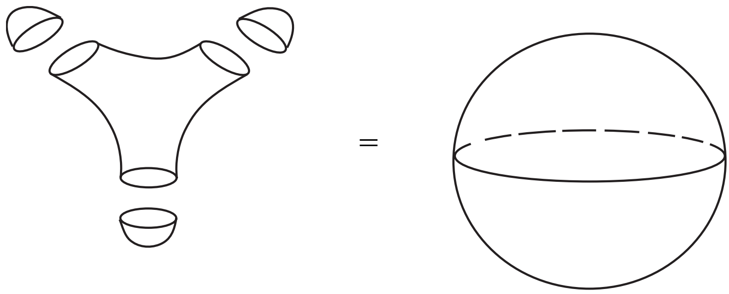


Figure 2: A $\mathbb{Z}_3$–manifold with boundary.

Definition 2.8 As before, there is the *quotient* space $\widetilde{B}$ which results from the identification on B of the k embedded copies of δB together using the embeddings ψ_i.

Example 2.9 Consider the three-dimensional $\mathbb{Z}_3$–manifold with boundary $\mathscr{B} = (B, \delta B, \psi_i)$, where $B = D^3$ is the three-dimensional closed ball (hence $\partial B = S^2$), $\delta B = D^2$ is the two-dimensional closed disc and the $\psi_i \colon D^2 \to S^2$ for $i \in \mathbb{Z}_3$ are given by three disjoint embedded discs inside the sphere. The $\mathbb{Z}_3$–boundary $\partial\mathscr{B} = (M, \delta M, \theta_i)$ is the two-dimensional $\mathbb{Z}_3$–manifold from Example 2.6, where M is the pair of pants and δM is the circle. See Figure 2 for an illustration.

Example 2.10 Consider the two-dimensional $\mathbb{Z}_3$–manifold with boundary $\mathscr{B} = (B, \delta B, \psi_i)$, where B is a connected surface of genus one with only one boundary circle, the Bockstein δB is the interval $[0, 1]$, and the $\psi_i \colon [0, 1] \to \partial B = S^1$ for $i \in \mathbb{Z}_3$ are given by three disjoint embedded intervals inside the circle. The $\mathbb{Z}_3$–boundary of the $\mathbb{Z}_3$–manifold $\mathscr{B}$ is a one-dimensional $\mathbb{Z}_3$–manifold $\partial\mathscr{B} = (M, \delta M, \theta_i)$, where M is the disjoint union of three copies of the interval, δM is the disjoint union of two points and the embeddings θ_i are given by the restrictions $\psi_i|_{\delta M}$. In Figure 3, we illustrate the $\mathbb{Z}_3$–stratifold $(B, \delta B, \psi_i)$, where on the right-hand side we depict the boundary ∂B after the quotient.

Definition 2.11 Let X be a topological space and n a natural number. An n–dimensional *singular $\mathbb{Z}_k$–manifold* in X is a closed n–dimensional $\mathbb{Z}_k$–manifold $\mathscr{M} = (M, \delta M, \theta_i)$ together with a continuous map $f \colon M \to X$ such that $f \circ \theta_i = f \circ \theta_j$ for $i, j \in \mathbb{Z}_k$. A *singular $\mathbb{Z}_k$–bordism* between two n–dimensional

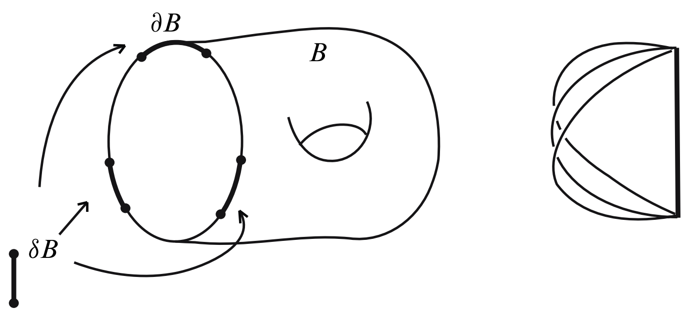


Figure 3: A $\mathbb{Z}_3$–manifold with boundary, left, and the boundary ∂B after quotient, right.

singular $\mathbb{Z}_k$–manifolds $(\mathcal{M}, f)$ and $(\mathcal{M}', f')$ is a $\mathbb{Z}_k$–manifold with boundary $\mathcal{B} = (B, \delta B, \psi_i)$, with $\mathbb{Z}_k$–boundary $\partial\mathcal{B} = (M + M', \delta M + \delta M', f + f')$ together with a continuous map $F \colon B \to X$ such that $F \circ \psi_i = F \circ \psi_j$ for $i, j \in \mathbb{Z}_k$, extending f and f'. Recall that the $\mathbb{Z}_k$–manifolds are oriented. In this definition, the sum of $\mathbb{Z}_k$–manifolds is given by

$$(M + M', \delta M + \delta M', f + f') = (M \sqcup -M', \delta M \sqcup -\delta M', f \sqcup f').$$

The $\mathbb{Z}_k$*–bordism group* group $\Omega_n(X; \mathbb{Z}_k)$ is given by the equivalence classes of n–dimensional singular $\mathbb{Z}_k$–manifolds $(\mathcal{M}, f)$ under this $\mathbb{Z}_k$–bordism relation. The elements of this group are denoted by $[\mathcal{M}, f]$.

The $\mathbb{Z}_k$–bordism groups $\Omega_n(X; \mathbb{Z}_k)$ are a generalized homology theory (this follows by Section 4 or see [5, Chapter III]). The existence of the fundamental class $[\mathcal{M}]_{\mathbb{Z}_k} \in H_n(\widetilde{M}; \mathbb{Z}_k)$, see Section 6, induces the Hurewicz homomorphism $h_{\mathbb{Z}_k} \colon \Omega_n(X; \mathbb{Z}_k) \to H_n(X; \mathbb{Z}_k)$. In addition, we have the reduction map $r \colon \Omega_n(X) \to \Omega_n(X; \mathbb{Z}_k)$. This map considers an n–dimensional closed manifold as a $\mathbb{Z}_k$–manifold with $\delta M = \varnothing$. Moreover, we have the Bockstein sequence, which fits into the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \Omega_n(X) \arrow[r, "\times k"] \arrow[d, "h"] & \Omega_n(X) \arrow[r, "r"] \arrow[d, "h"] & \Omega_n(X; \mathbb{Z}_k) \arrow[r, "\delta"] \arrow[d, "h_{\mathbb{Z}_k}"] & \Omega_{n-1}(X) \arrow[r] \arrow[d, "h"] & \cdots \\ \cdots \arrow[r] & H_n(X) \arrow[r, "\times k"] & H_n(X) \arrow[r, "r"] & H_n(X; \mathbb{Z}_k) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots \end{tikzcd} \tag{4}$$

for $n \geq 1$.

3 Stratifolds

We briefly introduce the language of stratifolds from Kreck [12; 13]. For this purpose, we need the notion of differential space [17; 12; 13].

Definition 3.1 A *differential space* is a pair $(X, \mathcal{C})$ where X is a topological Hausdorff space with a countable basis and $\mathcal{C} \subset C^0(X)$ is a sheaf of real-valued continuous functions such that for $f_1, \ldots, f_k$ in $\mathcal{C}$ and f a smooth function on $\mathbb{R}^k$, the composition $f(f_1, \ldots, f_k)$ is in $\mathcal{C}$.

For a differential space, each point $x \in X$ has associated a tangent space, denoted by $T_x X$, which is the space of all derivations of the germ $\Gamma_x(\mathcal{C})$ of smooth functions at x. A smooth manifold is a natural example of a differential space, which is locally diffeomorphic to $\mathbb{R}^n$ equipped with the sheaf of all smooth functions.

Definition 3.2 [13, Definition 1] An n–dimensional *stratifold* is a differential space $(S, \mathcal{C})$ where the sheaf $\mathcal{C}$ induces a suitable stratification $S^k := \{x \in S : \dim T_x S = k\}$. The union of all strata of dimension $\leq k$ is called the k–skeleton S_k. In addition, we assume:

(i) For each k, the stratum S^k, together with the restriction sheaf $\mathcal{C}|_{S^k}$, is a smooth k–dimensional manifold as a differential space.

(ii) All skeleta are closed subsets of S.

(iii) All strata of dimension $> n$ are empty.

(iv) For each $x \in S$ and open neighborhood U with $x \in U$, there is a so-called bump function $\rho: S \to \mathbb{R}_{\geq 0}$ in $\mathscr{C}$ such that $\operatorname{supp} \rho \subset U$ and $\rho(x) > 0$.

(v) For each $x \in S^k$, the restriction gives an isomorphism $\Gamma_x(\mathscr{C}) \to \Gamma_x(\mathscr{C}|_{S^k})$.

Definition 3.3 A continuous map $f: (S, \mathscr{C}) \to (S', \mathscr{C}')$ is *smooth* if the precomposition by f sends every element of $\mathscr{C}'$ to an element of $\mathscr{C}$. If f and the inverse f^{-1} are smooth, then f is called a *diffeomorphism of stratifolds*. Similarly, we can define the notion of a (smooth) *embedding of stratifolds* by requiring that the restriction to the image is a diffeomorphism of stratifolds.

Example 3.4 [12, Example 1, page 19] The open cone of an n–dimensional manifold,

$$\mathring{CM} := M \times [0,1)/_{M \times \{0\}},$$

is an example of an $(n+1)$–dimensional stratifold, where $\mathscr{C}$ consists of all continuous functions on $\mathring{CM}$ which are constant on some open neighborhood of the point produced by collapsing $M \times \{0\}$, and whose restriction to $M \times (0,1)$ is smooth.

Definition 3.5 Let W be a smooth manifold. A *collar* is a homeomorphism $c: \partial W \times [0,\epsilon) \to U$ with $\epsilon > 0$, where U is an open neighborhood of ∂W in W such that $c|_{\partial W \times \{0\}} = \mathrm{id}_{\partial W}$ and $c|_{\partial W \times (0,\epsilon)}$ is a diffeomorphism onto $U - \partial W$.

Definition 3.6 Let $(T, \partial T)$ be a pair of topological spaces. Assume $\mathring{T} = T - \partial T$ and ∂T are stratifolds of dimensions n and $n-1$, with $\partial T \subset T$ a closed subspace. A *collar* of ∂T into T is a homeomorphism $c: \partial T \times [0,\epsilon) \to U$ with $\epsilon > 0$, where U is an open neighborhood of ∂T in T such that $c|_{\partial T \times \{0\}} = \mathrm{id}_{\partial T}$ and $c|_{\partial T \times (0,\epsilon)}$ is a diffeomorphism of stratifolds onto $U - \partial T$.

Definition 3.7 An $(n+1)$–dimensional *stratifold with boundary* is a pair of topological spaces $(T, \partial T)$, together with a collar c of ∂T into T, where $T - \partial T$ is an $(n+1)$–dimensional stratifold and ∂T is an n–dimensional stratifold, which is a closed subspace of T. We call ∂T the *boundary* of T.

The following example is crucial in the theory of stratifolds.

Example 3.8 [12, page 36] The closed cone $C(S)$ of a stratifold S has underlying topological space $T = S \times [0,1]/_{S \times \{0\}}$, whose interior is $S \times [0,1)/_{S \times \{0\}}$ and whose boundary is $S \times \{1\}$. The collar is given by the map $S \times \left[0, \frac{1}{2}\right) \to C(S)$ mapping (x,t) to $(x, 1-t)$.

Now, we define some important classes of stratifolds [12].

Definition 3.9 [12, page 79] An n–dimensional stratifold S is *oriented* if the top stratum S^n is an oriented manifold and the stratum S^{n-1} is empty.

Definition 3.10 [12, page 43] An n–dimensional stratifold S is *regular* if for each $x \in S^i$, where $0 \le i \le n$, there is an open neighborhood U of x in S, a stratifold F with F^0 a single point, an open subset V of S^i, and a diffeomorphism of stratifolds $\phi\colon V \times F \to U$, whose restriction to $V \times F^0$ is the identity.

Remark 3.11 [12, page 24] In this paper, we restrict to a special class of stratifolds called *p–stratifolds*. The construction of a p–stratifold is as follows: we start with a zero-dimensional p–stratifold, which is a zero-dimensional manifold. Assume we construct by induction a $(k-1)$–dimensional p–stratifold $(S, \mathscr{C})$ and let W be a k–dimensional manifold with a smooth and proper map $f\colon \partial W \to S$. Then we define the k–dimensional p–stratifold $(W \sqcup_f S, \mathscr{C}')$, where $\mathscr{C}'$ is constructed using a collar $c\colon \partial W \times [0,\epsilon) \to U$. More precisely, the function g belongs to $\mathscr{C}'$ if and only if $g|_S$ and $g|_{W-\partial W}$ are smooth and for some $\delta < \epsilon$ we have $gc(x,t) = gf(x)$ for all $x \in \partial W$ and $t < \delta$.

Note 3.12 A stratifold with boundary T is an oriented/regular stratifold if both $T - \partial T$ and ∂T are oriented/regular stratifolds (the collar preserves the product orientation for oriented stratifolds). Similarly, T is a p–stratifold if both $T - \partial T$ and ∂T are p–stratifolds.

From Section 4, until the end of this paper, all statements about stratifolds are meant as statements about p–stratifolds; see Note 4.1.

As Kreck mentions in [13, page 303]: “The following observation is central for our construction of the zoo of bordism groups.” For two stratifolds T and T' with the same boundary $\partial T = \partial T'$, there is a stratifold structure for the gluing of stratifolds $T \cup_{\partial T} T'$, where the two collars are combined to produce a *bicollar*; see the details in [12, pages 36–37].

Definition 3.13 Let X be a topological space and n a natural number. An n–dimensional *singular stratifold* in X is a closed (compact without boundary) n–dimensional stratifold S together with a continuous map $f\colon S \to X$. A *singular bordism* between two n–dimensional singular stratifolds (S, f) and (S', f') is a compact stratifold with boundary T, with boundary $(S + S', f + f')$ together with a continuous map $F\colon T \to X$ extending f and f'. The sum of oriented stratifolds is given by

$$(S + S', f + f') = (S \sqcup -S', f \sqcup f).$$

Since one can glue n–dimensional singular stratifolds over a common boundary component, singular bordism is an equivalence relation. The *oriented stratifold homology* group $SH_n(X)$ consists of the equivalence classes of n–dimensional oriented singular stratifolds (S, f) under this bordism relation. The elements of these groups are denoted by $[S, f]$.

The significance of the previous bordism groups lies in the positive solution for the Steenrod problem [7] of showing that a geometric object represents integral homology classes. The precise statement is:

Theorem 3.14 (Kreck [12, Theorem 20.1, page 186]) *The functor SH_* defines a homology theory. Moreover, there exists a natural transformation h from $SH_*(\cdot)$ to singular homology $H_*(\cdot\,;\mathbb{Z})$ such that h is an isomorphism for all CW–complexes.*

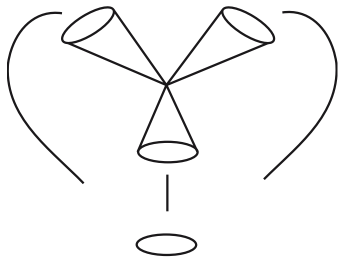


Figure 4: A closed $\mathbb{Z}_3$–stratifold.

4 $\mathbb{Z}_k$–stratifolds

Now we combine the theory of $\mathbb{Z}_k$–manifolds from Sullivan and the theory of stratifolds from Kreck.

Note 4.1 Unless otherwise indicated, let us set the convention that the stratifolds are oriented, regular p–stratifolds. Also, all the diffeomorphisms and embeddings of stratifolds are orientation-preserving.

Definition 4.2 A closed n–dimensional $\mathbb{Z}_k$–*stratifold* is given by the triple $\mathcal{S} = (S, \delta S, \theta_i)$, where

(1) S is a compact, n–dimensional stratifold, with boundary ∂S,

(2) δS is a compact $(n-1)$–dimensional stratifold without boundary, called the Bockstein, and

(3) the $\theta_i : \delta S \to \partial S$ for $i \in \mathbb{Z}_k$ are k disjoint embeddings of stratifolds such that we have a diffeomorphism of stratifolds $\partial S = \bigsqcup_{i \in \mathbb{Z}_k} \theta_i(\delta S)$.

Definition 4.3 There is an associated *quotient* space $\widetilde{S}$ given by the identification on S of the k copies of δS together using the embeddings θ_i.

Example 4.4 The class of closed stratifolds and the class of $\mathbb{Z}_k$–manifolds are the first examples of $\mathbb{Z}_k$–stratifolds.

Example 4.5 Consider the two-dimensional $\mathbb{Z}_3$–stratifold given by the closed cone of the disjoint union of three circles $S = C(S^1 \sqcup S^1 \sqcup S^1)$, where the boundary is $\partial S = S^1 \sqcup S^1 \sqcup S^1$, and the Bockstein is $\delta S = S^1$; see Figure 4.

Definition 4.6 An $(n+1)$–dimensional $\mathbb{Z}_k$–*stratifold with boundary* is given by the triple $\mathcal{T} = (T, \delta T, \psi_i)$, where

(1) T is a compact $(n+1)$–dimensional stratifold, with boundary ∂T,

(2) δT is a compact n–dimensional stratifold with boundary, called the Bockstein, with boundary $\partial \delta T$, and

(3) the $\psi_i : \delta T \hookrightarrow \partial T$ for $i \in \mathbb{Z}_k$ are k disjoint embeddings of stratifolds such that the triple
$$\left(\partial T - \operatorname{int}\left(\bigsqcup_{i\in\mathbb{Z}_k} \psi_i(\delta T)\right), \partial\delta T, \psi_i|_{\partial\delta T}\right)$$
defines a closed n–dimensional $\mathbb{Z}_k$–stratifold $(S, \delta S, \theta_i)$.

This closed n–dimensional $\mathbb{Z}_k$–stratifold is called the $\mathbb{Z}_k$*–boundary* of the $\mathbb{Z}_k$–stratifold $\mathcal{T}$ and is denoted by $\partial\mathcal{T} = (S, \delta S, \theta_i)$.

Definition 4.7 There is a *quotient* space $\widetilde{T}$ resulting from the identification on T of the k copies of δT together using the embeddings ψ_i.

Example 4.8 A $\mathbb{Z}_k$–manifold with boundary is an example of a $\mathbb{Z}_k$–stratifold with boundary.

Example 4.9 Consider the three-dimensional $\mathbb{Z}_3$–stratifold with boundary $\mathcal{T} = (T, \delta T, \psi_i)$, where T is the wedge of three closed balls $D^3 \vee D^3 \vee D^3$ by the north pole of the boundary spheres, hence the boundary is $\partial T = S^2 \vee S^2 \vee S^2$. The stratifold structure over the wedge point is given by the open cone of the disjoint union of three discs. The Bockstein is the two-dimensional closed disc $\delta T = D^2$, and the $\psi_i : D^2 \to S^2 \vee S^2 \vee S^2$ for $i \in \mathbb{Z}_3$ are given by the embeddings of D^2 on each of the three southern hemispheres. The $\mathbb{Z}_3$–boundary $\partial\mathcal{T} = (S, \delta S, \theta_i)$ is the two-dimensional $\mathbb{Z}_3$–stratifold from Example 4.5, where $S = C(S^1 \sqcup S^1 \sqcup S^1)$ and the Bockstein is $\delta S = S^1$. See Figure 5 for an illustration.

Definition 4.10 The *cone* of a $\mathbb{Z}_k$–stratifold $(S, \delta S, \theta_i)$ is defined as follows: take the closed cone $C(\delta S)$ (see [12, page 36] or Example 3.8) and use k copies $kC(\delta S) := \bigsqcup_{i\in\mathbb{Z}_k}(C(\delta S) \times \{i\})$ to get the closed stratifold $S' := kC(\delta S) \sqcup_{\partial S} S$. Now take the cone $C(S')$, which is an $(n+1)$–dimensional stratifold. The cone of the $\mathbb{Z}_k$–stratifold $(S, \delta S, \theta_i)$ is given by the $(n+1)$–dimensional $\mathbb{Z}_k$–stratifold with boundary $\mathcal{T} := (C(S'), C(\delta S), \psi_i)$, where ψ_i is the canonical inclusion in the i–component. The $\mathbb{Z}_k$–boundary of $\mathcal{T}$ is the original $\mathbb{Z}_k$–stratifold $(S, \delta S, \theta_i)$.

Note 4.11 For an n–dimensional $\mathbb{Z}_k$–stratifold $(S, \delta S, \theta_i)$, we need $n \geq 2$ in order to for $C(S')$ and $C(\delta S)$ to be oriented stratifolds.

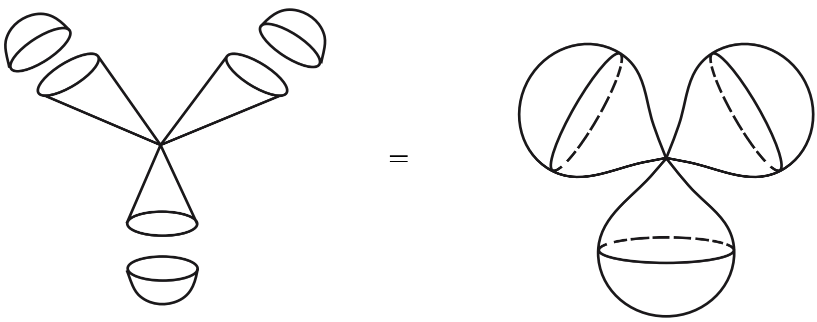


Figure 5: A $\mathbb{Z}_3$–stratifold with boundary.

The technique to show that the cartesian product of two differentiable manifolds has a differentiable structure is called *straightening the angle*. We follow the exposition given by Conner and Floyd in [6, Section I.3]. Let $\mathbb{R}_+ \subset \mathbb{R}$ consist of all nonnegative real numbers. We have the homeomorphism $\tau\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \times \mathbb{R}_+$, defined using polar coordinates by $\tau(\rho,\theta) = (\rho, 2\theta)$ with $0 \le \theta \le \pi/2$, such that the restriction τ is a diffeomorphism of $\mathbb{R}_+ \times \mathbb{R}_+ \setminus (0,0)$ onto $\mathbb{R} \times \mathbb{R}_+ \setminus (0,0)$. Consider the product of two differentiable manifolds B_1 and B_2 with collars U_1 and U_2 of the boundaries ∂B_1 and ∂B_2, respectively. There are diffeomorphisms $\Phi_1\colon U_1 \to \partial B_1 \times \mathbb{R}_+$ and $\Phi_2\colon U_2 \to \partial B_2 \times \mathbb{R}_+$. Let $U = U_1 \times U_2$. Then $\Phi = \Phi_1 \times \Phi_2$ is a homeomorphism of U onto $\partial B_1 \times \partial B_2 \times \mathbb{R}_+ \times \mathbb{R}_+$ and the composition with $\tau' = \mathrm{id} \times \tau$ produces a homeomorphism $\tau' \circ \Phi\colon U \to \partial B_1 \times \partial B_2 \times \mathbb{R} \times \mathbb{R}_+$. The differentiable structure of $\partial B_1 \times \partial B_2 \times \mathbb{R} \times \mathbb{R}_+$ induces a differentiable structure on U such that $\tau' \circ \Phi$ is a diffeomorphism. Then U and $B_1 \times B_2 \setminus \partial B_1 \times \partial B_2$ have differentiable structures, and they induce the same differentiable structure on their intersection. This structure is referred to as obtained by straightening the angle.

Proposition 4.12 *If $\mathcal{S} = (S, \delta S, \theta_i)$ is a closed n–dimensional $\mathbb{Z}_k$–stratifold, then after straightening the angle we obtain an $(n+1)$–dimensional $\mathbb{Z}_k$–stratifold with boundary $\mathcal{S} \times [0,1] := (S \times [0,1], \delta S \times [0,1], \psi_i)$, where the $\mathbb{Z}_k$–boundary $(S', \delta S', \theta'_i)$ is given by*

- $S' = S \times \{0\} \sqcup -S \times \{1\}$,
- $\delta S' = \delta S \times \{0\} \sqcup -\delta S \times \{1\}$,
- $\theta'_i = \theta_i \times \{0\} \sqcup \theta_i \times \{1\}$.

Proof The technique of straightening the angle works similarly for the product of two stratifolds with boundary. In fact, from Kreck [12, Sections A.1–A.2], we can use local retractions to show that the product of stratifolds has a stratifold structure.

Consequently, the product space $S \times [0,1]$ has the structure of compact $(n+1)$–dimensional stratifold with boundary, where $\partial(S \times [0,1]) = (\partial S \times [0,1]) \cup (S \times \{0,1\})$ is also a stratifold with a collar into $S \times [0,1]$. Similarly, the product $\delta S \times [0,1]$ is a compact n–dimensional stratifold with boundary, and we have embeddings $\theta_i \times \mathrm{id}_{[0,1]}\colon \delta S \times [0,1] \hookrightarrow \partial S \times [0,1]$ for $i \in \mathbb{Z}_k$. Denote by ψ_i the embedding obtained as the composition of $\theta_i \times \mathrm{id}_{[0,1]}$ with the inclusion $\partial S \times [0,1] \hookrightarrow \partial(S \times [0,1])$. We associate the $\mathbb{Z}_k$–stratifold with boundary $(T, \delta T, \psi_i)$, where $T := S \times [0,1]$ and the Bockstein $\delta T := \delta S \times [0,1]$.

From Definition 4.6, it remains to show that the triple $(S', \delta S', \theta'_i) := (\partial T - \mathrm{int}(\partial S \times [0,1]), \partial \delta T, \psi_i|_{\partial \delta T})$ is a closed n–dimensional $\mathbb{Z}_k$–stratifold. We have $S' = S \times \{0,1\}$, $\delta S' = \delta S \times \{0,1\}$ and the embeddings are $\theta'_i = \psi_i|_{\delta S'} = \theta_i \times \{0,1\}$. The orientation of $S \times [0,1]$ induces opposite orientations for the two copies of S associated to $\{0,1\}$, and similarly for δS. The embedding $\theta_i \times \{0\}$ preserves the orientation, while the embedding $\theta_i \times \{1\}$ reverses the orientation. This shows that $(S', \delta S', \theta'_i)$ is a $\mathbb{Z}_k$–stratifold which is the $\mathbb{Z}_k$–boundary of $\mathcal{S} \times [0,1]$. □

Now we state a gluing lemma for $\mathbb{Z}_k$–stratifolds. This result is a direct application of Proposition A.1 in Kreck's book [12, page 194].

Lemma 4.13 *Let $\mathcal{T} := (T, \delta T, \psi_i)$ and $\mathcal{T}' := (T', \delta T', \psi_i')$ be $\mathbb{Z}_k$–stratifolds with $\mathbb{Z}_k$–boundaries $\partial\mathcal{T} = \mathcal{S} \sqcup \mathcal{S}'$ and $\partial\mathcal{T}' = \mathcal{S} \sqcup \mathcal{S}''$, where $\mathcal{S} = (S, \delta S, \theta_i)$, $\mathcal{S}' = (S', \delta S', \theta_i')$ and $\mathcal{S}'' = (S'', \delta S'', \theta_i'')$ are closed $\mathbb{Z}_k$–stratifolds. Then there is a $\mathbb{Z}_k$–stratifold with boundary*

$$\mathcal{T} \sqcup_{\mathcal{S}} \mathcal{T}' := (T \sqcup_S T', \delta T \sqcup_{\delta S} \delta T', \psi_i \sqcup_{\delta S} \psi_i'),$$

where the $\mathbb{Z}_k$–boundary is $\mathcal{S}' \sqcup \mathcal{S}''$.

Proof We consider the stratifolds $Y_1 := S' \sqcup_{\partial S'} \bigsqcup_{i\in\mathbb{Z}_k} \psi_i(\delta T)$ and $Y_2 := S'' \sqcup_{\partial S''} \bigsqcup_{i\in\mathbb{Z}_k} \psi_i'(\delta T')$. Thus the boundary of the stratifold T and T' are $\partial T = S \sqcup_{\partial S} Y_1$ and $\partial T' = S \sqcup_{\partial S} Y_2$, respectively. The work of Kreck [12, Proposotion A.1, page 194] implies that the gluing $T \sqcup_S T'$ is a stratifold with boundary, where $\partial(T \sqcup_S T') = Y_1 \sqcup_{\partial S} Y_2$. Similarly, the gluing $\delta T \sqcup_{\delta S} \delta T'$ is a stratifold with boundary, which is the Bockstein. Thus the $\mathbb{Z}_k$–boundary is precisely $(S' \sqcup S'', \delta S' \sqcup \delta S'', \theta_i' \sqcup \theta_i'')$, and the lemma follows. □

Definition 4.14 Let X be a topological space and n a natural number. An n–dimensional *singular $\mathbb{Z}_k$–stratifold* in X is a closed n–dimensional $\mathbb{Z}_k$–stratifold $\mathcal{S} = (S, \delta S, \theta_i)$ together with a continuous map $f : S \to X$ such that $f \circ \theta_i = f \circ \theta_j$ for $i, j \in \mathbb{Z}_k$. A *singular $\mathbb{Z}_k$–bordism* between two n–dimensional singular $\mathbb{Z}_k$–stratifolds $(\mathcal{S}, f)$ and $(\mathcal{S}', f')$ is a $\mathbb{Z}_k$–stratifold with boundary $\mathcal{T} = (T, \delta T, \psi_i)$, with $\mathbb{Z}_k$–boundary $\partial\mathcal{T} = (S + S', \delta S + \delta S', f + f')$ together with a continuous map $F : T \to X$ such that $F \circ \psi_i = F \circ \psi_j$ for $i, j \in \mathbb{Z}_k$, extending f and f'. Recall that the $\mathbb{Z}_k$–stratifolds consist of oriented, regular p–stratifolds. In this definition, the sum of $\mathbb{Z}_k$–stratifolds is given by

$$(S + S', \delta S + \delta S', f + f') = (S \sqcup -S', \delta S \sqcup -\delta S', f \sqcup f').$$

Again, one can glue n–dimensional singular $\mathbb{Z}_k$–stratifolds over a common boundary component. We state in Proposition 4.15 that singular $\mathbb{Z}_k$–bordism is an equivalence relation. The *$\mathbb{Z}_k$–stratifold homology* group $SH_n(X; \mathbb{Z}_k)$ is given by the equivalence classes of n–dimensional singular $\mathbb{Z}_k$–stratifolds $(\mathcal{S}, f)$ under the $\mathbb{Z}_k$–stratifold bordism relation. We denote by $[\mathcal{S}, f]$ the elements of this group.

As a consequence of Proposition 4.12 and the gluing result of Lemma 4.13, we obtain the following.

Proposition 4.15 *The $\mathbb{Z}_k$–stratifold bordism relation is an equivalence relation.*

To any closed n–dimensional stratifold S, there is an associated closed n–dimensional stratifold given by the disjoint union $kS := \bigsqcup_{i\in\mathbb{Z}_k} S \times \{i\}$. This assignment produces the homomorphism

$$\times k : SH_n(X) \to SH_n(X). \tag{5}$$

To any closed n–dimensional $\mathbb{Z}_k$–stratifold $\mathcal{S} = (S, \delta S, \theta_i)$, there is an associated closed n–dimensional $\mathbb{Z}_k$–stratifold given by the disjoint union $kS := \bigsqcup_{i\in\mathbb{Z}_k} S \times \{i\}$, where the Bockstein is the whole boundary ∂S and the embeddings $\psi_i\colon \partial S \to \bigsqcup_{i\in\mathbb{Z}_k} \partial S\times\{i\}$ are the canonical inclusions. Moreover, the boundary $\partial S = \bigsqcup_{i\in\mathbb{Z}_k} \theta_i(\delta S)$ can be considered as a k–disjoint union and we can denote $(kS, k\delta S, \psi_i) := (kS, \partial S, \psi_i)$. This assignment produces the homomorphism

$$\times k^k\colon SH_n(X;\mathbb{Z}_k)\to SH_n(X;\mathbb{Z}_k), \tag{6}$$

which we show below is trivial.

Proposition 4.16 *For every integer $n\geq 0$, the homomorphism $\times k^k\colon SH_n(X;\mathbb{Z}_k)\to SH_n(X;\mathbb{Z}_k)$ is zero.*

Proof Take $(\mathcal{S}, f) = ((S,\delta S), f)$ a closed singular $\mathbb{Z}_k$–stratifold. Consider the stratifold with boundary given by the cylinder $T := kS\times[0,1]$ and the Bockstein $\delta T := (\partial S\times[0,1])\sqcup_{\partial S\times\{1\}}(-S\times\{1\})$ with embeddings

$$\psi_i\colon \delta T\hookrightarrow \partial T = \big[(S\times\{0\})\sqcup_{\partial S\times\{0\}}(\partial S\times[0,1])\sqcup_{\partial S\times\{1\}}(-S\times\{1\})\big]\times\{i\},$$

which are the canonical inclusions. The $\mathbb{Z}_k$–boundary of the $\mathbb{Z}_k$–stratifold $(T,\delta T,\psi_i)$ is the k–disjoint union of $(S,\delta S)$. $\square$

Similar to the work of Morgan and Sullivan [15], we have the Bockstein sequence, which fits into the commutative diagram

$$\begin{tikzcd}
\arrow[r] & SH_n(X) \arrow[r, "\times k"] \arrow[d, "h"] & SH_n(X) \arrow[r, "r"] \arrow[d, "h"] & SH_n(X;\mathbb{Z}_k) \arrow[r, "\delta"] \arrow[d, "h_{\mathbb{Z}_k}"] & SH_{n-1}(X) \arrow[r] \arrow[d, "h"] & \cdots SH_0(X;\mathbb{Z}_k) \arrow[d] \\
\arrow[r] & H_n(X) \arrow[r, "\times k"] & H_n(X) \arrow[r, "r"] & H_n(X;\mathbb{Z}_k) \arrow[r] & H_{n-1}(X) \arrow[r] & \cdots H_0(X;\mathbb{Z}_k)
\end{tikzcd} \tag{7}$$

The description of the maps is as follows:

- The reduction $r\colon SH_n(X)\to SH_n(X;\mathbb{Z}_k)$ is obtained by considering an n–dimensional closed stratifold as a $\mathbb{Z}_k$–stratifold, ie $(S,\delta S,\theta_i)$ with $\delta S=\varnothing$.
- Multiplication $\times k\colon SH_n(X)\to SH_n(X)$ takes a singular stratifold (S, f) in X and assigns the class of the k–disjoint union of S, denoted by $[kS, kf]$.
- The Bockstein $\delta\colon SH_n(X;\mathbb{Z}_k)\to SH_{n-1}(X)$ assigns to a singular $\mathbb{Z}_k$–stratifold $(\mathcal{S}, f)$, where $\mathcal{S}=(S,\delta S,\theta_i)$, the class $[\delta S, f|_{\delta S}]$.
- The Hurewicz homomorphism for stratifolds, $h\colon SH_n(X)\to H_n(X)$ for $n\geq 0$, was constructed by Kreck [12, pages 186–187].
- The Hurewicz homomorphism for $\mathbb{Z}_k$–stratifolds, $h_{\mathbb{Z}_k}\colon SH_n(X;\mathbb{Z}_k)\to H_n(X;\mathbb{Z}_k)$ for $n\geq 0$, is constructed in Section 6, where we show the existence of the fundamental class for $\mathbb{Z}_k$–stratifolds.

We leave the proof of the exactness of (7) for Section 5, where the commutativity follows after we construct the fundamental class in Section 6.

Finally, we spend the rest of the section discussing the properties of $SH_*(\,\cdot\,;\mathbb{Z}_k)$ as a functor. Kreck [12] proves the Eilenberg–Steenrod axioms for the bordism groups $SH_*(\,\cdot\,)$ in the category of CW–complexes. We have a functor, ie $\mathrm{id}_* = \mathrm{id}$ and $(g\circ f)_* = g_*\circ f_*$, which is homotopy invariant, has the Mayer–Vietoris sequence, $SH_n(*) = 0$ for $n\neq 0$ and $SH_0(*) = \mathbb{Z}$. Similarly, the $\mathbb{Z}_k$–stratifold homology satisfies the Eilenberg–Steenrod axioms, that we show in detail below. The proof of the Mayer–Vietoris sequence is in Section A.2.

Definition 4.17 A continuous map $g\colon X\to Y$ defines a morphism between the $\mathbb{Z}_k$–stratifold bordism groups by

$$g_*\colon SH_n(X;\mathbb{Z}_k)\to SH_n(Y;\mathbb{Z}_k),\quad [\mathcal{S}, f]\mapsto [\mathcal{S}, g\circ f],$$

for $\mathcal{S} = (S,\delta S,\theta_i)$ a closed n–dimensional $\mathbb{Z}_k$–stratifold.

This defines a functor which is homotopy invariant, as in the following proposition.

Proposition 4.18 *If g and g' are homotopic maps from X to Y, then*

$$g_* = {g'}_*\colon SH_n(X;\mathbb{Z}_k)\to SH_n(Y;\mathbb{Z}_k).$$

Proof There is a homotopy $G\colon X\times[0,1]\to Y$ between g and g'. Take $[\mathcal{S}, f]\in SH_n(X;\mathbb{Z}_k)$, and hence $[\mathcal{S}\times[0,1], G\circ(f\times\mathrm{id})]$ is a singular $\mathbb{Z}_k$–stratifold bordism (see Proposition 4.12) between $g_*([\mathcal{S}, f])$ and ${g'}_*([\mathcal{S}, f])$. □

Proposition 4.19 *For the $\mathbb{Z}_k$–stratifold bordism group, we have*

$$SH_n(*;\mathbb{Z}_k) = \begin{cases}\mathbb{Z}_k & \text{for } n = 0,\\ 0 & \text{for } n\neq 0.\end{cases}$$

Proof An important assumption here is that every n–dimensional $\mathbb{Z}_k$–stratifold $(S,\delta S)$ is formed by oriented stratifolds S and δS. For $n\geq 2$, we use the first horizontal long exact sequence of (7), with $SH_n(*) = 0$ and $SH_{n-1}(*) = 0$, and we conclude $SH_n(*;\mathbb{Z}_k) = 0$. For $n = 1$, the sequence (7) becomes

$$0\to SH_1(*;\mathbb{Z}_k)\to\mathbb{Z}\xrightarrow{\times k}\mathbb{Z}\xrightarrow{r} SH_0(*;\mathbb{Z}_k)\to 0,$$

then $SH_1(*;\mathbb{Z}_k) = 0$ and $SH_0(*;\mathbb{Z}_k) = \mathbb{Z}_k$. □

A geometric approach for the previous proposition is as follows: for any closed n–dimensional $\mathbb{Z}_k$–stratifold $\mathcal{S} = (S,\delta S,\theta_i)$, with $n>1$, we take the cone as in Definition 4.10. Thus we consider the usual cone $C(\delta S)$ and use k copies $kC(\delta S)$ to get the closed stratifold $S' := kC(\delta S)\sqcup_{\partial S} S$. Then we form the $(n{+}1)$–dimensional $\mathbb{Z}_k$–stratifold with boundary $\mathcal{T} := (C(S'), C(\delta S),\psi_i)$ where ψ_i is the canonical inclusion on the i^{th} component. The $\mathbb{Z}_k$–boundary of $\mathcal{T}$ is the original $\mathbb{Z}_k$–stratifold $(S,\delta S,\theta_i)$. For $n = 1$, we have a disjoint union of circles and intervals with orientation. Since each interval has the boundary $\{+,-\}$, then the number of intervals must be divided by k. Thus, after capping the circles with discs by Proposition 4.16, this element is trivial in $SH_1(*;\mathbb{Z}_k)$. Finally, for $n = 0$, the generator of $SH_0(*;\mathbb{Z}_k)$ is the closed zero-dimensional $\mathbb{Z}_k$–stratifold $(*,\varnothing,\mathrm{id}_\varnothing)$, where we use Proposition 4.16.

5 The Bockstein sequence

Previously, we have defined the k–disjoint union homomorphisms for stratifolds and $\mathbb{Z}_k$–stratifolds. These homomorphisms are as follows $\times k\colon SH_n(X)\to SH_n(X)$ and $\times k^k\colon SH_n(X;\mathbb{Z}_k)\to SH_n(X;\mathbb{Z}_k)$, defined in (5) and (6), respectively. The second is the trivial homomorphism by Proposition 4.16. There is a third k–disjoint union homomorphism of the form

$$\times k^{k^2}\colon SH_n(X;\mathbb{Z}_k)\to SH_n(X;\mathbb{Z}_{k^2}), \tag{8}$$

which assigns to an n–dimensional $\mathbb{Z}_k$–stratifold $(S,\delta S)$ the n–dimensional $\mathbb{Z}_{k^2}$–stratifold $(kS,\delta S)$. There is a projection homomorphism

$$p\colon SH_n(X;\mathbb{Z}_{k^2})\to SH_n(X;\mathbb{Z}_k)$$

which assigns to an n–dimensional $\mathbb{Z}_{k^2}$–stratifold $(S,\delta S)$ the n–dimensional $\mathbb{Z}_k$–stratifold $(S,k\delta S)$.

We skip the embeddings and singular maps in defining these homomorphisms to simplify the notation.

These homomorphisms satisfy a compatibility condition with the reduction and the Bockstein homomorphisms from the last section.

Proposition 5.1 *Let $r\colon SH_n(X)\to SH_n(X;\mathbb{Z}_k)$ and $r\colon SH_n(X)\to SH_n(X;\mathbb{Z}_{k^2})$ be the reduction homomorphisms and let $\delta\colon SH_n(X;\mathbb{Z}_{k^2})\to SH_{n-1}(X)$ be the Bockstein homomorphism for $\mathbb{Z}_{k^2}$–stratifolds. We have the following commutative diagrams*:

$$\begin{tikzcd}
SH_n(X;\mathbb{Z}_{k^2}) \arrow[r, "p"] \arrow[d, "\delta"] & SH_n(X;\mathbb{Z}_k) \arrow[d, "\delta"] \\
SH_{n-1}(X) \arrow[r, "\times k"] & SH_{n-1}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
SH_n(X) \arrow[r, "\times k"] \arrow[d, "r"] & SH_n(X) \arrow[d, "r"] \\
SH_n(X;\mathbb{Z}_k) \arrow[r, "\times k^k"] & SH_n(X;\mathbb{Z}_k)
\end{tikzcd}$$

$$\begin{tikzcd}
SH_n(X) \arrow[r, "\times k"] \arrow[d, "r"] & SH_n(X) \arrow[d, "r"] \\
SH_n(X;\mathbb{Z}_k) \arrow[r, "\times k^{k^2}"] & SH_n(X;\mathbb{Z}_{k^2})
\end{tikzcd}$$

$$\begin{tikzcd}
SH_n(X) \arrow[d, "r"] \arrow[dr, "r"] & \\
SH_n(X;\mathbb{Z}_{k^2}) \arrow[r, "p"] & SH_n(X;\mathbb{Z}_k)
\end{tikzcd}
\qquad
\begin{tikzcd}
SH_n(X;\mathbb{Z}_{k^2}) \arrow[dr, "p"] & \\
SH_n(X;\mathbb{Z}_k) \arrow[u, "\times k^{k^2}"'] \arrow[r, "\times k^k"] & SH_n(X;\mathbb{Z}_k)
\end{tikzcd}$$

Proof We show the commutativity of the first three squares. Take $(S,\delta S)$ an n–dimensional $\mathbb{Z}_{k^2}$–stratifold. We have $k\delta S := \times k(\delta S) = \times k\circ\delta(S,\delta S)$ and $k\delta S=\delta(S,k\delta S)=\delta\circ p(S,\delta S)$. Now, for S a closed n–dimensional stratifold, we obtain $r\circ\times k(S)=(kS,\varnothing)$ and $\times k^k\circ r(S)=\times k^k(S,\varnothing)=(kS,\varnothing)$ in $SH_n(X;\mathbb{Z}_k)$. Similarly, we can show the commutativity of the third diagram with $(kS,\varnothing)$

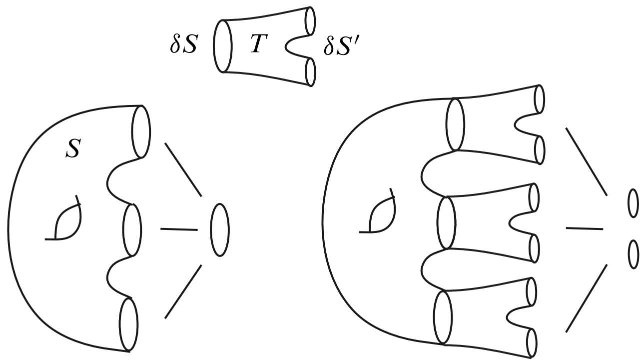


Figure 6: The bordism T from δS and $\delta S'$ and the two $\mathbb{Z}_k$–bordant $\mathbb{Z}_k$–stratifolds.

in $SH_n(X;\mathbb{Z}_{k^2})$. Finally, we show the commutativity of the last two diagrams. We have $r(S)=(S,\varnothing)=p(S,\varnothing)=p(r(S))$ and $p\circ\times k^{k^2}(S,\delta S)=p(kS,\delta S)=(kS,k\delta S)=\times k^k(S,\delta S)$. The commutativity of the second and fifth diagrams means that the composition is trivial by Proposition 4.16. □

The following result shows how a stratifold bordism gives rise to a $\mathbb{Z}_k$–stratifold bordism.

Proposition 5.2 *Assume that δS and $\delta S'$ are two n–dimensional closed stratifolds such that there is a bordism of stratifolds T with boundary $\partial T=\delta S\sqcup -\delta S'$. In addition, suppose the pair $(S,\delta S)$ is an n–dimensional $\mathbb{Z}_k$–stratifold. Then $(S,\delta S)$ is $\mathbb{Z}_k$–bordant to $(S\sqcup_{\partial S}-kT,\delta S')$.*

Proof This is similar to Proposition 4.12. Consider the product space $T':=(S\sqcup_{\partial S}-kT)\times[0,1]$ and the Bockstein $\delta T':=(\delta S'\times[0,1])\sqcup_{\delta S'\times\{1\}}-T$ with embeddings $\psi_i\colon\delta T'\hookrightarrow\partial T'$, where

$$\partial T'=\big((S\sqcup_{\partial S}-kT)\times\{0\}\big)\sqcup_{k\delta S'\times\{0\}}k(\delta S'\times[0,1])\sqcup_{k\delta S'\times\{1\}}\big((S\sqcup_{\partial S}-kT)\times\{1\}\big).$$

The $\mathbb{Z}_k$–stratifold $(T',\delta T',\psi_i)$ is a $\mathbb{Z}_k$–bordism between $(S,\delta S)$ and $(S\sqcup_{\partial S}-kT,\delta S')$. □

Remark 5.3 Because of the relevance of the previous result for our work, in Figure 6 we illustrate two $\mathbb{Z}_k$–stratifolds that are $\mathbb{Z}_k$–bordant by the previous proposition. Notice that, whenever it is possible to connect δS to the empty set by a bordism T, then the $\mathbb{Z}_k$–stratifold $(S,\delta S)$ is $\mathbb{Z}_k$–bordant to $(S\sqcup_{\partial S}-kT,\varnothing)$.

Similar to the work of Morgan and Sullivan [15], the $\mathbb{Z}_k$–stratifolds bordisms groups have a Bockstein exact sequence associated with $0\to\mathbb{Z}\xrightarrow{\times k}\mathbb{Z}\to\mathbb{Z}_k\to 0$. There is also the other Bockstein exact sequence associated with $0\to\mathbb{Z}_k\xrightarrow{\times k}\mathbb{Z}_{k^2}\to\mathbb{Z}_k\to 0$. These two sequences are part of the commutative diagram

$$\begin{tikzcd}
{}\arrow[r] & SH_n(X)\arrow[r,"\times k"]\arrow[d,"r"] & SH_n(X)\arrow[r,"r"]\arrow[d,"r"] & SH_n(X;\mathbb{Z}_k)\arrow[r,"\delta"]\arrow[d,"="] & SH_{n-1}(X)\arrow[r]\arrow[d,"r"] & {}\\
{}\arrow[r] & SH_n(X;\mathbb{Z}_k)\arrow[r,"\times k"] & SH_n(X;\mathbb{Z}_{k^2})\arrow[r,"p"] & SH_n(X;\mathbb{Z}_k)\arrow[r,"\widetilde{\delta}"] & SH_{n-1}(X;\mathbb{Z}_k)\arrow[r] & {}
\end{tikzcd}\tag{9}$$

The primary purpose of the present section is to show the exactness of the two Bockstein exact sequences.

Proposition 5.4 *The sequence*

$$\cdots \to SH_n(X) \xrightarrow{\times k} SH_n(X) \xrightarrow{r} SH_n(X;\mathbb{Z}_k) \xrightarrow{\delta} SH_{n-1}(X) \xrightarrow{\times k} \cdots$$

is exact.

Proof We have $r\circ(\times k) = (\times k^k)\circ r = 0$ by Proposition 5.1. In addition, we obtain $\delta\circ r = 0$ since the Bockstein of a (closed) stratifold is empty. Moreover, $\times k\circ\delta = 0$ since we start with a $\mathbb{Z}_k$–stratifold $(S,\delta S,\theta_i)$, where the boundary ∂S is diffeomorphic to $\bigsqcup_{i\in\mathbb{Z}_k}\theta_i(\delta S)$.

Now, we show exactness.

- **$\ker r \subset \operatorname{im}(\times k)$** Consider an n–dimensional singular stratifold (S,f) with $r([S,f]) = 0$. Then there is an $(n+1)$–dimensional $\mathbb{Z}_k$–bordism $(\mathcal{T},F) = ((T,\delta T),F)$ such that the $\mathbb{Z}_k$–boundary $\partial(T,\delta T) = (S,\varnothing)$ and F extends f. Consequently, we obtain $\partial\delta T = \delta S = \varnothing$ and hence $\partial T = S\sqcup k\delta T$, and we can take the singular stratifolds given by $(\delta T, F|_{\partial T})$ with the reverse orientation. We have $k[-\delta T, -F|_{\partial T}] = [S,f]$.
- **$\ker\delta \subset \operatorname{im} r$** Consider an n–dimensional singular $\mathbb{Z}_k$–stratifold $(\mathcal{S},f) = ((S,\delta S),f)$ such that $\delta([\mathcal{S},f]) = 0$. Then $(\delta S, f|_{\delta S})$ is the boundary of an n–dimensional singular stratifold (T,F), ie $\partial T = \delta S$ and F extends $f|_{\partial S}$. Proposition 5.2 and Remark 5.3 imply that the $\mathbb{Z}_k$–stratifold $(S\sqcup_{\partial S} -kT,\varnothing)$ is $\mathbb{Z}_k$–bordant to $\mathbb{Z}_k$–stratifold $(S,\delta S)$. There is a map $f'\colon S\sqcup_{\partial S} -kT \to X$ which extends the singular map f. Therefore, the singular $\mathbb{Z}_k$–stratifold $((S\sqcup_{\partial S} -kT,\varnothing),f')$ is $\mathbb{Z}_k$–bordant to the original singular $\mathbb{Z}_k$–stratifold $((S,\delta S),f)$.
- **$\ker(\times k) \subset \operatorname{im}\delta$** Consider an $(n-1)$–dimensional singular stratifold (S,f) with $\times k([S,f]) = 0$. Then there exists an n–dimensional singular stratifold (T,F) with $\partial T = kS$ and F extends kf. Thus we can take the n–dimensional singular $\mathbb{Z}_k$–stratifold $((T,S),F)$ and we obtain $\delta([(T,S),F]) = [S,f]$. □

Denote by $\tilde{\delta}$ the composition $SH_n(X;\mathbb{Z}_k) \xrightarrow{\delta} SH_{n-1}(X) \xrightarrow{r} SH_{n-1}(X;\mathbb{Z}_k)$.

Proposition 5.5 *The sequence*

$$\cdots \to SH_n(X;\mathbb{Z}_k) \xrightarrow{\times k^{k^2}} SH_n(X;\mathbb{Z}_{k^2}) \xrightarrow{p} SH_n(X;\mathbb{Z}_k) \xrightarrow{\tilde{\delta}} SH_{n-1}(X;\mathbb{Z}_k) \xrightarrow{\times k^{k^2}} \cdots$$

is exact.

Proof We have $p\circ(\times k^{k^2}) = \times k^k = 0$ by Proposition 5.1. Again we use Proposition 5.1, and we get

$$\tilde{\delta}\circ p = r\circ\delta\circ p = (r\circ(\times k))\circ\delta = 0.$$

Similarly, we obtain

$$(\times k^{k^2})\circ\tilde{\delta} = (\times k^{k^2})\circ r\circ\delta = (r\circ(\times k))\circ\delta = 0.$$

Now we show exactness.

- **$\ker p \subset \operatorname{im}(\times k^{k^2})$** Consider an n–dimensional singular $\mathbb{Z}_{k^2}$–stratifold $(\mathcal{S},f) = ((S,\delta S),f)$ with $p([\mathcal{S},f]) = 0$. Then there exists an $(n+1)$–dimensional singular $\mathbb{Z}_k$–stratifold with boundary $(\mathcal{T},F) = ((T,\delta T),F)$ such that the $\mathbb{Z}_k$–boundary is $\partial\mathcal{T} = (S,k\delta S)$. Thus we can consider k copies of δT with the reverse orientation, which are glued with S to form a closed stratifold $S \sqcup_{\partial S} -k\delta T$, which is the boundary of T. There are k disjoint embeddings $c_i : \delta S \times [0,\epsilon) \hookrightarrow \delta T$ induced by the collar of ∂S into the k copies of δT. Write $\overline{\delta T} := \delta T - \bigsqcup_{i \in \mathbb{Z}_k} c_i(\delta S \times [0,\epsilon/2])$. We consider the $\mathbb{Z}_{k^2}$–stratifold with boundary $(T, \delta S \times [0,\epsilon/2], \psi_i)$, where $\psi_i = c_i|_{\delta S \times [0,\epsilon/2]}$. This is a $\mathbb{Z}_{k^2}$–bordism between $(S, \delta S)$ and $(k\overline{\delta T}, \delta S)$. This means that $(\times k^{k^2})([\overline{\delta T}, \delta S]) = [k\overline{\delta T}, \delta S] = [S, \delta S]$.

- $\mathbf{ker}\, \widetilde{\boldsymbol{\delta}} \subset \mathbf{im}\, \boldsymbol{p}$ Consider an n–dimensional singular $\mathbb{Z}_k$–stratifold $(\mathcal{S}, f) = ((S, \delta S), f)$ such that $\widetilde{\delta}([\mathcal{S}, f]) = 0$. Since $\widetilde{\delta} = r \circ \delta$, this means that there exists an n–dimensional singular $\mathbb{Z}_k$–bordism $(\mathcal{T}, F) = ((T, \delta T), F)$ such that the $\mathbb{Z}_k$–boundary is $((\delta S, \varnothing), f|_{\delta S})$. Therefore, $\partial T = \delta S \sqcup k\delta T$, F extends $f|_{\delta S}$ and $\partial\delta T = \varnothing$. Consequently, we consider k copies of T with the reverse orientation, glued with S to form the n–dimensional stratifold with boundary $S' = -kT \sqcup_{\partial S} S$. There is a map $f' : S' \to X$ also constructed by the gluing. Thus we have an n–dimensional singular $\mathbb{Z}_{k^2}$–stratifold $((S', \delta T), f')$. We have $p([(S', \delta T), f']) = [(S', k\delta T), f']$, which is equal to $(\mathcal{S}, f)$ by Proposition 5.2.
- $\mathbf{ker}(\times \boldsymbol{k}^{\boldsymbol{k}^2}) \subset \mathbf{im}(\widetilde{\boldsymbol{\delta}})$ Consider an $(n-1)$–dimensional singular $\mathbb{Z}_k$–stratifold $(\mathcal{S}, f) = ((S, \delta S, \theta_i), f)$ with $\times k^{k^2}([\mathcal{S}, f]) = 0$. Then there is an n–dimensional singular $\mathbb{Z}_{k^2}$–stratifold $(\mathcal{T}, F) = ((T, \delta T, \psi_i), F)$ with $\mathbb{Z}_{k^2}$–boundary $((kS, \delta S), kf)$. Therefore, $\partial T = kS \sqcup_{\partial kS} -k^2\delta T$ is a closed n–dimensional stratifold. By the definition of the $\mathbb{Z}_{k^2}$–boundary of a $\mathbb{Z}_{k^2}$–stratifold with boundary (Definition 4.6), hence $\delta S = \partial\delta T$ and the embeddings are $\theta_i = \psi_i|_{\partial\delta T}$. Therefore, the gluing $S \sqcup_{\partial S} k\delta T$ is a closed n–dimensional stratifold and, in addition, we obtain ∂T is the disjoint union of k copies of $S \sqcup_{\partial S} k\delta T$. Consequently, we take the $(n+1)$–dimensional singular $\mathbb{Z}_k$–stratifold $((T, S \sqcup_{\partial S} k\delta T), F)$ and $\widetilde{\delta}([(T, S \sqcup_{\partial S} k\delta T), F]) = [(S \sqcup_{\partial S} k\delta T, \varnothing), F|_{S \sqcup_{\partial S} k\delta T}]$, which is $\mathbb{Z}_k$–bordant to $((S, \delta S, \theta_i), f)$ by Proposition 5.2. □

6 Fundamental classes

Recall from Section 2 that a closed $\mathbb{Z}_k$–manifold $(M, \delta M, \theta_i)$ has an associated quotient space $\widetilde{M}$. Similarly, we write $\widetilde{\partial M}$ to mean the quotient space given by the identification on ∂M of the k copies of δM. Notice that in this case, we have $\widetilde{\partial M} \cong \delta M$. Similarly, for a $\mathbb{Z}_k$–manifold with boundary $(B, \delta B, \psi_i)$, we denote by $\widetilde{B}$ and $\widetilde{\partial B}$ the quotient spaces obtained by the identification of the k copies of δB on B and ∂B, respectively.

In this section, we will construct a natural transformation from $\mathbb{Z}_k$–bordism stratifold homology to homology with $\mathbb{Z}_k$–coefficients

$$\Phi : SH_*(X; \mathbb{Z}_k) \to H_*(X; \mathbb{Z}_k). \tag{10}$$

We can define this natural transformation for $\mathbb{Z}_k$–manifolds [15]. There is no formal proof of this fact in the literature, so we provide a detailed argument below. The case of $\mathbb{Z}_k$–stratifolds uses some results of Tene [22]. We give the details of these statements at the end of this section.

Assume that $\mathcal{M} = (M, \partial M, \theta_i)$ is a closed n–dimensional $\mathbb{Z}_k$–manifold and that there is a continuous map $f : M \to X$ to the topological space X. There exists the *fundamental class* $[\mathcal{M}]_{\mathbb{Z}_k} \in H_n(\widetilde{M}; \mathbb{Z}_k)$, and for an element $[\mathcal{M}, f] \in \Omega_n(X; \mathbb{Z}_k)$, there is a natural transformation defined by

$$\Phi([\mathcal{M}, f]) = \tilde{f}_*([\mathcal{M}]_{\mathbb{Z}_k}), \tag{11}$$

where $\tilde{f}\colon \widetilde{M} \to X$ is the induced map from the quotient space $\widetilde{M}$.

We can find the fundamental class $[\mathcal{M}]_{\mathbb{Z}_k}$ using the commutative diagram

$$\begin{tikzcd}
\arrow[r] & H_n(\partial M; \mathbb{Z}_k) \arrow[r] \arrow[d, "q_*"] & H_n(M; \mathbb{Z}_k) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_n(M, \partial M; \mathbb{Z}_k) \arrow[r, "\partial"] \arrow[d, "q_*"] & H_{n-1}(\partial M; \mathbb{Z}_k) \arrow[r] \arrow[d, "q_*"] & {} \\
\arrow[r] & H_n(\widetilde{\partial M}; \mathbb{Z}_k) \arrow[r] & H_n(\widetilde{M}; \mathbb{Z}_k) \arrow[r, "i_*"] & H_n(\widetilde{M}, \widetilde{\partial M}; \mathbb{Z}_k) \arrow[r, "\partial"] & H_{n-1}(\widetilde{\partial M}; \mathbb{Z}_k) \arrow[r] & {}
\end{tikzcd} \tag{12}$$

In the previous diagram, the rows are the long exact sequences associated with the pairs $(M, \partial M)$ and $(\widetilde{M}, \widetilde{\partial M})$. The quotient map induces the vertical morphisms. We start with the well-known fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{Z}_k)$ which satisfies $\partial([M, \partial M]) = [\partial M]$ and

$$H_{n-1}(\widetilde{\partial M}; \mathbb{Z}_k) \xrightarrow{\cong} H_{n-1}(\delta M; \mathbb{Z}_k), \quad q_*([\partial M]) \mapsto k[\delta M]. \tag{13}$$

Thus $q_*([\partial M]) = 0$ by the coefficients. We have the isomorphism $q_*\colon H_n(M, \partial M; \mathbb{Z}_k) \to H_n(\widetilde{M}, \widetilde{\partial M}; \mathbb{Z}_k)$ and $H_n(\widetilde{\partial M}; \mathbb{Z}_k) \cong H_n(\delta M; \mathbb{Z}_k) = 0$. Therefore, there exists a unique class $[\mathcal{M}]_{\mathbb{Z}_k} \in H_n(\widetilde{M}; \mathbb{Z}_k)$ with the property

$$i_*([\mathcal{M}]_{\mathbb{Z}_k}) = q_*([M, \partial M]). \tag{14}$$

The following lemma is needed to show the existence of relative fundamental classes for $\mathbb{Z}_k$–manifolds.

Lemma 6.1 *Let M be a closed compact oriented manifold of dimension n. Assume M is the gluing of two compact oriented manifolds with boundary of dimension n, ie*

$$M = M_1 \sqcup_{\partial M_1 = \partial M_2} M_2. \tag{15}$$

Then the composition

$$H_n(M) \xrightarrow{i_*} H_n(M, M_1) \xrightarrow{\cong} H_n(M_2, \partial M_2)$$

sends the fundamental class $[M] \in H_n(M)$ to the relative fundamental class $[M_2, \partial M_2] \in H_n(M_2, \partial M_2)$, where the isomorphism $H_n(M, M_1) \xrightarrow{\cong} H_n(M_2, \partial M_2)$ is provided by excision.

Proof We have the commutative diagram

$$\begin{tikzcd}
 & H_n(M_2, \partial M_2) \arrow[r] \arrow[d, "\mathrm{exc}"] & H_n(M_2, M_2 - \{x\}) \arrow[d, "\cong"] \\
H_n(M) \arrow[r] & H_n(M, M_1) \arrow[r] & H_n(M, M - \{x\})
\end{tikzcd} \tag{16}$$

where $x \in \mathring{M}_2 = M_2 - \partial M_2$. By classic algebraic topology [11, Lemma 3.27], the two rows send the fundamental classes to the generators associated with the point x, which shows the lemma. □

Now we show the existence of a *relative fundamental class* of an $(n+1)$–dimensional $\mathbb{Z}_k$–manifold with boundary $\mathcal{B} = (B, \partial B, \psi_i)$, where the $\mathbb{Z}_k$–boundary is $\partial\mathcal{B} = (M, \delta M, \theta_i)$. We find the fundamental class $[\mathcal{B}, \partial\mathcal{B}]_{\mathbb{Z}_k}$ using the commutative diagram

$$\begin{tikzcd}
H_{n+1}(\partial B, M;\mathbb{Z}_k) \arrow[r] \arrow[d, "q_*"] & H_{n+1}(B, M;\mathbb{Z}_k) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_{n+1}(B, \partial B;\mathbb{Z}_k) \arrow[r, "\partial"] \arrow[d, "q_*"] & H_n(\partial B, M;\mathbb{Z}_k) \arrow[d, "q_*"] \\
H_{n+1}(\widetilde{\partial B}, \widetilde{M};\mathbb{Z}_k) \arrow[r] & H_{n+1}(\widetilde{B}, \widetilde{M};\mathbb{Z}_k) \arrow[r, "i_*"] & H_{n+1}(\widetilde{B}, \widetilde{\partial B};\mathbb{Z}_k) \arrow[r, "\partial"] & H_n(\widetilde{\partial B}, \widetilde{M};\mathbb{Z}_k)
\end{tikzcd} \tag{17}$$

In the previous diagram, the rows are the long exact sequences associated with the triples $(B, \partial B, M)$ and $(\widetilde{B}, \widetilde{\partial B}, \widetilde{M})$, respectively, and the quotient map induces the vertical morphisms. We start with the relative fundamental class $[B, \partial B] \in H_n(B, \partial B;\mathbb{Z}_k)$ and using Lemma 6.1 we have $\partial[B, \partial B] = [k\delta B, \partial M]$, where $k\delta B := \bigsqcup_{i\in\mathbb{Z}_k} \psi_i(\delta B)$, and

$$H_n(\widetilde{\partial B}, \widetilde{M};\mathbb{Z}_k) \xrightarrow{\cong} H_n(\delta B, \delta M;\mathbb{Z}_k), \qquad q_*[k\delta B, \partial M] \mapsto k[\delta B, \delta M]. \tag{18}$$

Thus $q_*[k\delta B, \partial M] = 0$ by the coefficients. We have isomorphisms of the form

$$q_*\colon H_{n+1}(B, \partial B;\mathbb{Z}_k) \xrightarrow{\cong} H_{n+1}(\widetilde{B}, \widetilde{\partial B};\mathbb{Z}_k) \quad \text{and} \quad H_{n+1}(\widetilde{\partial B}, \widetilde{M};\mathbb{Z}_k) \cong H_{n+1}(\delta B, \delta M;\mathbb{Z}_k) = 0.$$

Therefore, there exists a unique class $[\mathcal{B}, \partial\mathcal{B}]_{\mathbb{Z}_k} \in H_{n+1}(\widetilde{B}, \widetilde{M};\mathbb{Z}_k)$ with the property

$$i_*([\mathcal{B}, \partial\mathcal{B}]_{\mathbb{Z}_k}) = q_*([B, \partial B]). \tag{19}$$

Proposition 6.2 *Let $\mathcal{B} = (B, \partial B, \psi_i)$ be an $(n+1)$–dimensional $\mathbb{Z}_k$–manifold with boundary, where the $\mathbb{Z}_k$–boundary is $\partial\mathcal{B} = (M, \delta M, \theta_i)$. Then the class $[\partial\mathcal{B}]_{\mathbb{Z}_k}$ is the image of $[\mathcal{B}, \partial\mathcal{B}]_{\mathbb{Z}_k}$ under the map $\partial\colon H_{n+1}(\widetilde{B}, \widetilde{M};\mathbb{Z}_k) \to H_n(\widetilde{M};\mathbb{Z}_k)$.*

Proof We apply the differential maps to the middle square in (17), and we obtain the commutative cube

$$\begin{tikzcd}
H_{n+1}(B, M;\mathbb{Z}_k) \arrow[rr, "i_*"] \arrow[dr, "\partial"] \arrow[dd, "q_*"] & & H_{n+1}(B, \partial B;\mathbb{Z}_k) \arrow[dr, "\partial"] \arrow[dd, "q_*"] & \\
& H_n(M;\mathbb{Z}_k) \arrow[rr, "i_*"] \arrow[dd, "q_*"] & & H_n(\partial B;\mathbb{Z}_k) \arrow[dd, "q_*"] \\
H_{n+1}(\widetilde{B}, \widetilde{M};\mathbb{Z}_k) \arrow[rr, "i_*"] \arrow[dr, "\partial"] & & H_{n+1}(\widetilde{B}, \widetilde{\partial B};\mathbb{Z}_k) \arrow[dr, "\partial"] & \\
& H_n(\widetilde{M};\mathbb{Z}_k) \arrow[rr, "i_*"] & & H_n(\widetilde{\partial B};\mathbb{Z}_k)
\end{tikzcd} \tag{20}$$

We continue with the long exact sequence of the pairs $(\partial B, k\delta B)$ and $(\widetilde{\partial B}, k\widetilde{\partial B})$ for the front square of (20), and we obtain the middle square in the commutative diagram

$$\begin{tikzcd}
H_n(M;\mathbb{Z}_k) \arrow[r, "i_*"] \arrow[d, "q_*"] & H_n(\partial B;\mathbb{Z}_k) \arrow[r, "j_*"] \arrow[d, "q_*"] & H_n(\partial B, k\delta B;\mathbb{Z}_k) \arrow[r, "\cong", "\text{exc}"'] \arrow[d, "q_*"] & H_n(M, \partial M;\mathbb{Z}_k) \arrow[d, "q*", "\cong"'] \\
H_n(\widetilde{M};\mathbb{Z}_k) \arrow[r, "i_*"] & H_n(\widetilde{\partial B};\mathbb{Z}_k) \arrow[r, "j*"] & H_n(\widetilde{\partial B}, k\widetilde{\delta}B;\mathbb{Z}_k) \arrow[r, "\cong", "\text{exc}"'] & H_n(\widetilde{M}, \widetilde{\partial M};\mathbb{Z}_k)
\end{tikzcd} \tag{21}$$

In the previous commutative diagram, we use excision for the third square on the right. Notice that the composition of the horizontal maps in (21) are the maps $i_*\colon H_n(M;\mathbb{Z}_k) \to H_n(M,\partial M;\mathbb{Z}_k)$ and $i_*\colon H_n(\widetilde{M};\mathbb{Z}_k) \to H_n(\widetilde{M},\partial\widetilde{M};\mathbb{Z}_k)$.

We chase the class $[\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k} \in H_{n+1}(\widetilde{B},\widetilde{M};\mathbb{Z}_k)$ in the diagrams (20) and (21), where we obtain, as consequences,

$$i_*\partial([\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k}) = \partial i_*([\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k}) = \partial q_*([B,\partial B]) = q_*\partial([B,\partial B]) = q_*([\partial B]).$$

By Lemma 6.1, we have the equation $j_*([\partial B]) = [M,\partial M]$. Thus, we obtain the property (14) and the result follows. □

Proposition 6.3 *The natural transformation $\Phi\colon \Omega_*(X;\mathbb{Z}_k) \to H_*(X;\mathbb{Z}_k)$ is well defined.*

Proof For an n–dimensional singular $\mathbb{Z}_k$–manifold $(\mathcal{M}, f)$ which is null $\mathbb{Z}_k$–bordant, there exists an $(n+1)$–dimensional $\mathbb{Z}_k$–bordism $(\mathcal{B}, F)$ with $\partial\mathcal{B} = \mathcal{M}$, where F extends f. We have the commutative diagram

$$\begin{array}{ccc} [\mathcal{B},\partial\mathcal{B}]_{\mathbb{Z}_k} \in H_n(\widetilde{B},\widetilde{M};\mathbb{Z}_k) & \longrightarrow & H_n(X,X;\mathbb{Z}_k) = 0 \\ \big\downarrow \partial & & \big\downarrow \partial \\ [\mathcal{M}]_{\mathbb{Z}_k} \in H_n(\widetilde{M};\mathbb{Z}_k) & \longrightarrow & H_n(X;\mathbb{Z}_k) \end{array} \tag{22}$$

This ends the proposition. □

In the case of stratifolds, the fundamental classes are defined by Tene [22]. More precisely, let S be a compact oriented regular p–stratifold of dimension n and denote by $(M,\partial M)$ the smooth manifold we attach as top stratum. We have isomorphisms

$$H_n(M,\partial M) \xrightarrow[\text{exc}]{\cong} H_n(S,S_{n-2}) \xleftarrow{\cong} H_n(S), \tag{23}$$

where S_{n-2} is the $(n-2)$–skeleton of S. The *fundamental class* $[S] \in H_n(S)$ is defined as the image of $[M,\partial M] \in H_n(M,\partial M)$.

Let $(T,\partial T)$ be a compact oriented regular p–stratifold of dimension $n+1$ with boundary and denote by $(B,\partial B)$ the smooth manifold with boundary and collar attached as the top stratum. Then

$$H_{n+1}(B,\partial B) \xrightarrow[\text{exc}]{\cong} H_{n+1}(T,T_{n-1}\cup\partial T) \xleftarrow{\cong} H_{n+1}(T,\partial T), \tag{24}$$

where T_{n-1} is the $(n-1)$–skeleton of T. The *relative fundamental class* $[T,\partial T] \in H_{n+1}(T,\partial T)$ is defined as the image of $[B,\partial B] \in H_{n+1}(B,\partial B)$.

Proposition 6.4 [22, Lemma 3.9] *Let T be a compact oriented regular stratifold of dimension $n+1$, where the boundary is ∂T. Then the image of $[T,\partial T]$ under the map $\partial\colon H_{n+1}(T,\partial T) \to H_n(\partial T)$ is the class $[\partial T]$.*

Assume $\mathcal{S} = (S, \delta S, \theta_i)$ is a closed n–dimensional $\mathbb{Z}_k$–stratifold, where both S and δS are compact oriented regular p–stratifolds. Similarly as in diagram (12), we can find *the fundamental class* $[\mathcal{S}]_{\mathbb{Z}_k}$ in $H_n(\widetilde{S}; \mathbb{Z}_k)$ using the commutative diagram

$$\begin{array}{ccccccc} H_n(\partial S;\mathbb{Z}_k) & \longrightarrow & H_n(S;\mathbb{Z}_k) & \xrightarrow{i_*} & H_n(S,\partial S;\mathbb{Z}_k) & \xrightarrow{\partial} & H_{n-1}(\partial S;\mathbb{Z}_k) \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} \\ H_n(\widetilde{\partial S};\mathbb{Z}_k) & \longrightarrow & H_n(\widetilde{S};\mathbb{Z}_k) & \xrightarrow{i_*} & H_n(\widetilde{S},\widetilde{\partial S};\mathbb{Z}_k) & \xrightarrow{\partial} & H_{n-1}(\widetilde{\partial S};\mathbb{Z}_k) \end{array} \tag{25}$$

In the previous diagram, the rows are the long exact sequences associated with the pairs $(S, \partial S)$ and $(\widetilde{S}, \widetilde{\partial S})$. The quotient map induces the vertical morphisms. Again, we have the isomorphism $q_* \colon H_n(S, \partial S; \mathbb{Z}_k) \to H_n(\widetilde{S}, \widetilde{\partial S}; \mathbb{Z}_k)$ and $H_n(\widetilde{\partial S}; \mathbb{Z}_k) \cong H_n(\delta S; \mathbb{Z}_k) = 0$. The same arguments as those for $\mathbb{Z}_k$–manifolds, show that there exists a unique fundamental class $[\mathcal{S}]_{\mathbb{Z}_k} \in H_n(\widetilde{S}; \mathbb{Z}_k)$ with the property

$$i_*([\mathcal{S}]_{\mathbb{Z}_k}) = q_*([S, \partial S]). \tag{26}$$

The local orientations at each point define the fundamental class of a manifold. This property also follows for stratifolds considering points inside the interior of the top stratum. Therefore, we use this fact to generalize Lemma 6.1 for stratifolds. More precisely, let S be a compact oriented regular p–stratifold of dimension n, which is the gluing $S = S' \sqcup_{\partial S' = \partial S''} S''$, then in the next diagram, we have that the fundamental classes are mapped to the generators associated with the point x:

$$\begin{array}{ccccc} [S'', \partial S''] \in H_n(S'', \partial S'') & \xrightarrow{\cong} & H_n(S'', (S'')_{n-2} \cup \partial S'') & \longrightarrow & H_n(S'', S'' - \{x\}) \\ \cong \uparrow {\scriptstyle \mathrm{exc}} & & & & \\ H_n(S, S') & & & & \downarrow \cong \\ \uparrow & & & & \\ [S] \in H_n(S) & \xrightarrow{\cong} & H_n(S, S_{n-2}) & \longrightarrow & H_n(S, S - \{x\}) \end{array} \tag{27}$$

Here $(S'')_{n-2}$ and S_{n-2} are the $(n-2)$–skeletons of S'' and S.

Similarly, we show the existence of a *relative fundamental class* of an $(n+1)$–dimensional $\mathbb{Z}_k$–stratifold with boundary $\mathcal{T} = (T, \partial T, \psi_i)$. The $\mathbb{Z}_k$–boundary is $\partial\mathcal{T} = (S, \delta S, \theta_i)$ and all stratifolds are compact oriented regular p–stratifolds. We can find the fundamental class $[\mathcal{T}, \partial\mathcal{T}]_{\mathbb{Z}_k}$ using the commutative diagram

$$\begin{array}{ccccccc} H_{n+1}(\partial T, S;\mathbb{Z}_k) & \longrightarrow & H_{n+1}(T, S;\mathbb{Z}_k) & \xrightarrow{i_*} & H_{n+1}(T,\partial T;\mathbb{Z}_k) & \xrightarrow{\partial} & H_n(\partial T, S;\mathbb{Z}_k) \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} \\ H_{n+1}(\widetilde{\partial T}, \widetilde{S};\mathbb{Z}_k) & \longrightarrow & H_{n+1}(\widetilde{T}, \widetilde{S};\mathbb{Z}_k) & \xrightarrow{i_*} & H_{n+1}(\widetilde{T},\widetilde{\partial T};\mathbb{Z}_k) & \xrightarrow{\partial} & H_n(\widetilde{\partial T}, \widetilde{S};\mathbb{Z}_k) \end{array} \tag{28}$$

where the rows are the long exact sequences associated with the triples $(T, \partial T, S)$ and $(\widetilde{T}, \widetilde{\partial T}, \widetilde{S})$, respectively, and the vertical morphisms are induced by considering the quotient spaces. The same arguments show the existence of the fundamental class $[\mathcal{T}, \partial\mathcal{T}]_{\mathbb{Z}_k} \in H_{n+1}(\widetilde{T}, \widetilde{S}; \mathbb{Z}_k)$ with the property

(29) $$i_*([\mathcal{T}, \partial\mathcal{T}]_{\mathbb{Z}_k}) = q_*([T, \partial T]).$$

The same arguments as those for $\mathbb{Z}_k$–manifolds, show that the image of $[\mathcal{T}, \partial\mathcal{T}]_{\mathbb{Z}_k}$ under the map $\partial\colon H_{n+1}(\widetilde{T}, \widetilde{S}; \mathbb{Z}_k) \to H_n(\widetilde{S}; \mathbb{Z}_k)$ is the class $[\partial\mathcal{T}]_{\mathbb{Z}_k}$.

As a consequence, the following result is straightforward.

Proposition 6.5 *There is a well-defined natural transformation $\Phi'\colon SH_*(X; \mathbb{Z}_k) \to H_*(X; \mathbb{Z}_k)$, which fits into the commutative diagram*

(30) $$\begin{tikzcd} \Omega_*(X; \mathbb{Z}_k) \arrow[r, "\Phi"] \arrow[d] & H_*(X; \mathbb{Z}_k) \\ SH_*(X; \mathbb{Z}_k) \arrow[ur, "\Phi'"'] & \end{tikzcd}$$

In addition, Φ' is an isomorphism for all CW–complexes.

7 A geometric description of the Atiyah–Hirzebruch spectral sequence for $\mathbb{Z}_k$–coefficients

We assume all spaces are CW–complexes, and for a CW–complex X we denote by X^k its k^{th} skeleton. For a generalized homology theory h, a Postnikov tower is a sequence of homology theories $h^{(r)}$ and natural transformations

(31) $$\begin{tikzcd} & h \arrow[d] \arrow[drrr] \arrow[drrrr] \arrow[drrrrr] & & & & & \\ \cdots \arrow[r] & h^{(r)} \arrow[r] & \cdots \arrow[r] & h^{(2)} \arrow[r] & h^{(1)} \arrow[r] & h^{(0)} \end{tikzcd}$$

such that

- $h_n(*) \to h_n^{(r)}(*)$ is an isomorphism for $n \le r$, and
- $h_n^{(r)}(*)$ is trivial for $n > r$.

These properties determine $h^{(r)}$ completely, see [16, Chapter II, 4.13-4.18].

Every generalized homology theory h, has an associated Atiyah–Hirzebruch spectral sequence $(E^r_{s,t}, d^r_{s,t})$. For $r \ge 2$, Tene [23] constructs a natural isomorphism of spectral sequences $E^r_{s,t} \to \widehat{E}^r_{s,t}$, where

$$E^r_{s,t} = \frac{\operatorname{Im}\bigl(h_{s+t}(X^s, X^{s-r}) \to h_{s+t}(X^s, X^{s-1})\bigr)}{\operatorname{Im}\bigl(h_{s+t+1}(X^{s+r-1}, X^s) \to h_{s+t}(X^s, X^{s-1})\bigr)}, \quad \widehat{E}^r_{s,t} = \operatorname{Im}\bigl(h^{(t+r-2)}_{s+t}(X^s) \to h^{(t)}_{s+t}(X^{s+r-1})\bigr).$$

The argument of Tene [23, Section 4] that gives the isomorphisms

$$E^r_{s,t} = \frac{\operatorname{Im}(f')}{\operatorname{Im}(f)} \cong \operatorname{Im}(f_1) \cong \operatorname{Im}(f_2) \cong \operatorname{Im}(f_3) = \widehat{E}^r_{s,t}$$

we now explain with diagram (32):

(32)
$$\begin{tikzcd}
& & h^{(t+r-2)}_{s+t}(X^s) \arrow[r, "f_3"] \arrow[d, two heads] & h^{(t)}_{s+t}(X^{s+r-1}) \arrow[dd, "\cong"] \\
& & h^{(t+r-2)}_{s+t}(X^s, X^{s-r}) \arrow[dr] & \\
& h_{s+t}(X^s, X^{s-r}) \arrow[r] \arrow[ur, two heads] \arrow[rr, bend left, "f_2"] \arrow[d, "f'"] \arrow[dr, "f_1"] & h_{s+t}(X^{s+r-1}, X^{s-r}) \arrow[r] \arrow[d] & h^{(t)}_{s+t}(X^{s+r-1}, X^{s-r}) \arrow[d, tail] \\
h_{s+t+1}(X^{s+r-1}, X^s) \arrow[r, "f"] \arrow[ur] & h_{s+t}(X^s, X^{s-1}) \arrow[r] & h_{s+t}(X^{s+r-1}, X^{s-1}) \arrow[r, "\cong"] & h^{(t)}_{s+t}(X^{s+r-1}, X^{s-1})
\end{tikzcd}$$

The differential $\widehat{d}^r_{s,t}\colon \widehat{E}^r_{s,t} \to \widehat{E}^r_{s-r,t+r-1}$ is the homomorphism induced by the diagram

(33)
$$\begin{tikzcd}
& h^{(t+r-2)}_{s+t}(X^s) \arrow[r] \arrow[d, "\Phi"] & h^{(t)}_{s+t}(X^{s+r-1}) \arrow[d, "\Phi"] \\
& h_{s+t-1}(X^{s-r+1}) \arrow[r] \arrow[d, "\Psi"] & h_{s+t-1}(X^{s-1}) \arrow[d, "\Psi"] \\
h^{(t+2r-3)}_{s+t-1}(X^{s-r}) \arrow[r] & h^{(t+2r-3)}_{s+t-1}(X^{s-r+1}) \arrow[r] & h^{(t+r-1)}_{s+t-1}(X^{s-1})
\end{tikzcd}$$

where the natural transformation Φ is defined by the composition

$$h^{(r)}_n(X) \to h^{(r)}_n(X, X^{n-r-1}) \xrightarrow{\cong} h_n(X, X^{n-r-1}) \to h_{n-1}(X^{n-r-1}),$$

and Ψ is the natural transformation given by the composition of the natural transformations in the Postnikov tower.

For oriented bordism Ω_*, Tene [23] has a geometric description of the Atiyah–Hirzebruch spectral sequence, coming from a geometric description of Postnikov tower $SH^{(r)}$. This description of the spectral sequence is similar in spirit to the Conner–Floyd spectral sequence appearing in equivariant bordism [6] and the spectral sequence for orbifold cobordism of [1]. The bordism theory $SH^{(r)}$ is defined using oriented p–stratifolds, with all strata of codimension $0 < k < r+2$ empty. Thus, a singular stratifold S in X, of the form $f\colon S \to X$, gives an element of $SH^{(r)}_n(X)$ if S is an n–dimensional stratifold with singular part of dimension at most $n-r-2$. We put a similar restriction to the stratifold bordisms, which are $(n+1)$–dimensional stratifolds with boundary, and the singular part is of dimension at most $n-r-1$.

Therefore, we have natural transformations $\Omega_n \to SH^{(r)}_n$ such that $\Omega_n(*) \to SH^{(r)}_n(*)$ are isomorphisms for $n \le r$, and $SH^{(r)}_n(*)$ is trivial for $n > r$. Among other properties, we obtain that $SH^{(r)}_n(X^k)$ is trivial for $k + r < n$.

For $r \geq 2$, write

$$\widehat{E}^r_{s,t} = \mathrm{Im}(SH^{(t+r-2)}_{s+t}(X^s) \to SH^{(t)}_{s+t}(X^{s+r-1})), \tag{34}$$

and the differential $\widehat{d}^r_{s,t} \colon \widehat{E}^r_{s,t} \to \widehat{E}^r_{s-r,t+r-1}$ is the homomorphism induced by the diagram

$$\begin{tikzcd} & SH^{(t+r-2)}_{s+t}(X^s) \arrow[r] \arrow[d, "\Phi"'] & SH^{(t)}_{s+t}(X^{s+r-1}) \arrow[d, "\Phi"'] \\ & \Omega_{s+t-1}(X^{s-r+1}) \arrow[r] \arrow[d, "\Psi"'] & \Omega_{s+t-1}(X^{s-1}) \arrow[d, "\Psi"'] \\ SH^{(t+2r-3)}_{s+t-1}(X^{s-r}) \arrow[r] & SH^{(t+2r-3)}_{s+t-1}(X^{s-r+1}) \arrow[r] & SH^{(t+r-1)}_{s+t-1}(X^{s-1}) \end{tikzcd} \tag{35}$$

where Φ is a natural transformation defined by

$$SH^{(r)}_n(X) \to SH^{(r)}_n(X, X^{n-r-1}) \xrightarrow{\cong} \Omega_n(X, X^{n-r-1}) \to \Omega_{n-1}(X^{n-r-1}). \tag{36}$$

The isomorphism $SH^{(r)}_n(X, X^{n-r-1}) \xrightarrow{\cong} \Omega_n(X, X^{n-r-1})$ is the restriction to the top stratum and the map $\Omega_n(X, X^{n-r-1}) \to \Omega_{n-1}(X^{n-r-1})$ is the boundary homomorphism. The natural transformation Ψ is the composition of the natural transformations in the Postnikov tower. Therefore, for a stratifold S of dimension $s+t$, with a map $f \colon S \to X^s$, the image of the differential $d^r_{s,t}$ is induced by

$$[f \colon S \to X^s] \mapsto [f|_{\mathrm{sing}(S)} \circ g \colon \partial W \to X^{s-1}], \tag{37}$$

where W is the top stratum of S and $g \colon \partial W \to \mathrm{sing}(S)$ is the attaching map used to glue W to the singular part $\mathrm{sing}(S)$.

The $\mathbb{Z}_k$–bordism groups $\Omega_n(X; \mathbb{Z}_k)$ form a generalized homology theory (this follows by Section 6 or see [5, Chapter III]). The authors define bordism theory for resolutions with abelian groups in that book. The standard resolution for $\mathbb{Z}_k$ and the theory of this section coincide with that given by the definition of $\mathbb{Z}_k$–manifolds. We construct a Postnikov tower $SH^{(r)}(\,\cdot\,; \mathbb{Z}_k)$ defined with oriented $\mathbb{Z}_k$–stratifolds, with all strata of codimension $0 < k < r+2$ empty. Thus a singular $\mathbb{Z}_k$–stratifold in X, of the form $f \colon (S, \delta S) \to X$, represents an element of $SH^{(r)}_n(X; \mathbb{Z}_k)$ if

- S is an n–dimensional $\mathbb{Z}_k$–stratifold with singular part of dimension at most $n-r-2$, and
- δS is an $(n-1)$–dimensional $\mathbb{Z}_k$–stratifold with singular part of dimension at most $n-r-3$.

Similarly, the stratifold bordism $(T, \delta T)$ should be such that

- T is an $(n+1)$–dimensional $\mathbb{Z}_k$–stratifold with boundary, the singular part is of dimension at most $n-r-1$, and
- δT is an n–dimensional $\mathbb{Z}_k$–stratifold with boundary, and the singular part is of dimension at most $n-r-2$.

Notice that we obtain $SH^{(0)}(\,\cdot\,; \mathbb{Z}_k) = SH(\,\cdot\,; \mathbb{Z}_k)$. In what follows, we use the important property that $\Omega_*(*)$ has no odd torsion and just 2–torsion; see [14].

Theorem 7.1 *For k an odd number, the homology theories $SH_*^{(r)}(\,\cdot\,;\mathbb{Z}_k)$ give the Postnikov tower of the generalized homology theory $\Omega_*(\,\cdot\,;\mathbb{Z}_k)$.*

Proof We have natural transformations

$$
\begin{tikzcd}
 & \Omega_*(\,\cdot\,;\mathbb{Z}_k) \arrow[d] \arrow[rrr] \arrow[rrrr] \arrow[rrrrr] & & & & \\
{} \arrow[r] & SH_*^{(r)}(\,\cdot\,;\mathbb{Z}_k) \arrow[r] & \cdots \arrow[r] & SH_*^{(2)}(\,\cdot\,;\mathbb{Z}_k) \arrow[r] & SH_*^{(1)}(\,\cdot\,;\mathbb{Z}_k) \arrow[r] & SH_*^{(0)}(\,\cdot\,;\mathbb{Z}_k)
\end{tikzcd}
\tag{38}
$$

The conditions of the Postnikov tower are proven as follows:

- Assume $n \le r$, hence $n-r-2 \le -2$ and $n-r-1 \le -1$. Thus the $\mathbb{Z}_k$–stratifolds are $\mathbb{Z}_k$–manifolds and the $\mathbb{Z}_k$–stratifolds bordism are $\mathbb{Z}_k$–manifolds with boundary. Therefore, the maps $\Omega_n(*,\mathbb{Z}_k) \to SH_n^{(r)}(*,\mathbb{Z}_k)$ are isomorphisms for $n \le r$.
- Assume $n > r+1$, hence $n-r-1 \ge 1$ and $n-r-2 \ge 0$. Thus for an n–dimensional $\mathbb{Z}_k$–stratifold $(S,\delta S)$ in $SH_n^{(r)}(*;\mathbb{Z}_k)$, we construct the cone as in Definition 4.10. As a consequence, $SH_n^{(r)}(*;\mathbb{Z}_k)=0$ for $n > r+1$.
- Assume $n=r+1$, hence $n-r-2=-1$ and $n-r-3=-2$. Thus an n–dimensional $\mathbb{Z}_k$–stratifold in $SH_n^{(r)}(*;\mathbb{Z}_k)$ is a $\mathbb{Z}_k$–manifold $(M,\delta M)$. Because $n-r-1=0$ and $n-r-2=-1$, we allow $\mathbb{Z}_k$–stratifold bordisms with singular points of dimension at most 0 and the Bockstein has to be an n–dimensional manifold with boundary. In $\Omega_{n-1}(*)$ we have $k[\delta M]=0$, but since Ω_* *has no odd torsion*, then there exists an n–dimensional manifold with boundary N where $\partial N = \delta M$. Consider the $\mathbb{Z}_k$–stratifold bordism $(C(kN \sqcup_{\partial M} M), N)$ where $C(kN \sqcup_{\partial M} M)$ is the closed cone. The $\mathbb{Z}_k$–boundary is precisely the $\mathbb{Z}_k$–manifold $(M,\delta M)$ which shows that $SH_n^{(r)}(*;\mathbb{Z}_k)=0$ for $n=r+1$.

For $k=2$, this argument fails, and we cannot work around it using the cone of δM because we obtain singular points of dimension ≥ 1. □

The same arguments of Tene [23] give a geometric description of the Atiyah–Hirzebruch spectral sequence for $\mathbb{Z}_k$–bordism. For $r \ge 2$ and X a CW–complex, define

$$\widehat{E}^r_{s,t} = \operatorname{Im}(SH^{(t+r-2)}_{s+t}(X^s;\mathbb{Z}_k) \to SH^{(t)}_{s+t}(X^{s+r-1};\mathbb{Z}_k)), \tag{39}$$

and the differential $\widehat{d}^r_{s,t}\colon \widehat{E}^r_{s,t} \to \widehat{E}^r_{s-r,t+r-1}$ is the homomorphism induced by the diagram

$$
\begin{tikzcd}
 & SH^{(t+r-2)}_{s+t}(X^s;\mathbb{Z}_k) \arrow[r] \arrow[d, "\Phi"'] & SH^{(t)}_{s+t}(X^{s+r-1};\mathbb{Z}_k) \arrow[d, "\Phi"'] \\
 & \Omega_{s+t-1}(X^{s-r+1};\mathbb{Z}_k) \arrow[r] \arrow[d, "\Psi"'] & \Omega_{s+t-1}(X^{s-1};\mathbb{Z}_k) \arrow[d, "\Psi"'] \\
SH^{(t+2r-3)}_{s+t-1}(X^{s-r};\mathbb{Z}_k) \arrow[r] & SH^{(t+2r-3)}_{s+t-1}(X^{s-r+1};\mathbb{Z}_k) \arrow[r] & SH^{(t+r-1)}_{s+t-1}(X^{s-1};\mathbb{Z}_k)
\end{tikzcd}
\tag{40}
$$

Therefore, for a singular $\mathbb{Z}_k$–stratifold $((S,\delta S), f\colon S \to X^s)$, we consider the top stratum, which is a $\mathbb{Z}_k$–manifold with boundary $(W,\delta W)$. Denote the $\mathbb{Z}_k$–boundary by $(M,\delta M) := \partial(W,\delta W)$ and $g\colon M \to \mathrm{sing}(S)$ the attaching map used to glue W to the singular part which is of dimension at most $s-r$. The image of the differential $d^r_{s,t}$ is induced by

$$[(S,\delta S), f\colon S \to X^s] \longmapsto [(M,\delta M), f|_{\mathrm{sing}(S)} \circ g\colon M \to X^{s-r}]. \tag{41}$$

We have finally proved:

Theorem 7.2 *For k an odd number, the filtration of the Atiyah–Hirzebruch spectral sequence of $\mathbb{Z}_k$–bordism*

$$E^\infty_{n,0} \subseteq \cdots \subseteq E^{r+2}_{n,0} \subseteq \cdots \subseteq E^2_{n,0} \cong H_n(X;\mathbb{Z}_k), \tag{42}$$

coincides with

$$E^r_{n,0} = \mathrm{Im}\big(SH_n^{(r-2)}(X;\mathbb{Z}_k) \to SH_n^{(0)}(X;\mathbb{Z}_k) \cong H_n(X;\mathbb{Z}_k)\big), \tag{43}$$

ie the set of classes generated by singular $\mathbb{Z}_k$–stratifolds in X with singular part of dimension at most $n-r-2$.

Notice that the Atiyah–Hirzebruch spectral sequence is trivial for $k=2$; hence, the last theorem does not apply.

8 Geometric representatives of nonrepresentable classes

The present section is motivated by the authors' counterexamples of the Steenrod problem in [2].

The Steenrod problem [7] states the following: if $z \in H_n(X)$ is an integral homology class, does there exist an oriented manifold M and a map $f\colon M \to X$ such that z is the image of the generator of $H_n(M)$?

Conner and Floyd [6] rephrased the Steenrod realization problem in terms of the Atiyah–Hirzebruch spectral sequence $(E^r_{s,t}, d^r_{s,t})$. More precisely, the homomorphism from oriented bordism to integral homology $\Omega_*(X) \to H_*(X)$ is an epimorphism if and only if the differentials $d^r_{s,t}\colon E^r_{s,t} \to E^r_{s-r,t+r-1}$ are trivial for all $r \geq 2$.

Using the previous section, the Steenrod realization problem for $\mathbb{Z}_k$–coefficients has the following form.

Theorem 8.1 *If X is a CW–complex and k an odd number, then for the Atiyah–Hirzebruch spectral sequence $(E^r_{s,t}, d^r_{s,t})$, the differentials $d^r_{s,t}\colon E_{s,t} \to E_{s-r,t+r-1}$ are trivial for all $r \geq 2$ if and only if the map $\mu\colon \Omega_n(X;\mathbb{Z}_k) \to H_n(X;\mathbb{Z}_k)$ is an epimorphism for all $n \geq 0$.*

For the rest of this section, we assume that k is an odd prime number p. Following Conner and Floyd [6], we identify stratifolds with maps to $B\mathbb{Z}_p$ with stratifolds with free actions of $\mathbb{Z}_p$.

The Bockstein exact sequence of $B\mathbb{Z}_p$ implies the isomorphisms

$$H_{2n-1}(B\mathbb{Z}_p) \overset{\mathrm{mod}\ p}{\cong} H_{2n-1}(B\mathbb{Z}_p;\mathbb{Z}_p) \quad \text{and} \quad H_{2n}(B\mathbb{Z}_p;\mathbb{Z}_p) \overset{\beta}{\cong} H_{2n-1}(B\mathbb{Z}_p) \tag{44}$$

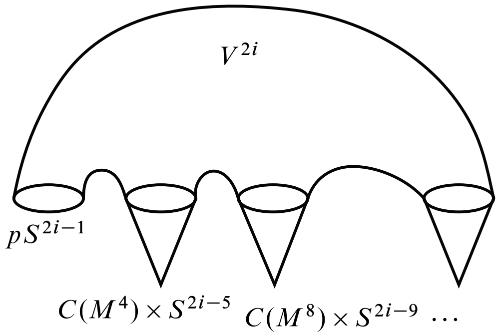


Figure 7: The class α_{2i}.

for $n>0$ (the map β was formerly denoted by δ). Take generators $\alpha_i \in H_i(B\mathbb{Z}_p;\mathbb{Z}_p)$ such that $\beta(\alpha_i)=\alpha_{i-1}$ for i even, and $\beta(\alpha_i)=0$ for i odd. The odd generators are represented by spheres. The generator α_{2i} is determined by the identity $\beta(\alpha_{2i})=\alpha_{2i-1}$. From Conner and Floyd [6, page 144], we know that the following equation holds in bordism of $B\mathbb{Z}_p$:

$$p\alpha_{2i-1}+[M^4]\alpha_{2i-5}+[M^8]\alpha_{2i-9}+\cdots=0 \quad \text{for } i\geq 1. \tag{45}$$

The manifolds M^{4k} for $k=1,2,\ldots$ are constructed inductively in [6]. Therefore, there is a compact oriented manifold V^{2i}, with a free action of $\mathbb{Z}_p$, such that

$$\partial V^{2i}=pS^{2i-1}\cup(M^4\times S^{2i-5})\cup(M^8\times S^{2i-9})\cup\cdots. \tag{46}$$

There are two representations of the generator α_{2i} by $\mathbb{Z}_p$–stratifolds, which we will show are $\mathbb{Z}_p$–bordant:

(i) Denote by $C(M^{4l})$ the cone of M^{4l} for $l=1,2,\ldots$, and take the gluing of V^{2i} with

$$(C(M^4)\times S^{2i-5})\cup(C(M^8)\times S^{2i-9})\cup\cdots.$$

The boundary of this construction is pS^{2i-1} and therefore the Bockstein is α_{2i-1}. We obtain a $2i$–dimensional $\mathbb{Z}_p$–stratifold $(S,\delta S)$, where $S=V\cup(C(M^4)\times S^{2i-5})\cup(C(M^8)\times S^{2i-9})\cup\cdots$ is a $2i$–dimensional $\mathbb{Z}_p$–stratifold with singular part $S^{2i-5}\cup S^{2i-9}\cup\cdots$, and the Bockstein $\delta S=S^{2i-1}$ is a $(2i-1)$–dimensional $\mathbb{Z}_p$–stratifold with empty singular part. We illustrate this construction in Figure 7.

(ii) The manifolds M^{4l}, with $4l<2p-2$, belong to $p\Omega_*$; see the paper by Floyd [8, page 336]. Therefore, there exist manifolds $M_l\in\Omega_{4l}$ such that $M^{4l}=pM_l$. For $p=2k+1$, consider the cone $C(M^{4m})$ for $m=k,k+1,\ldots$, and take the gluing of V^{2i} with

$$(C(M^{4k})\times S^{2i-4k-1})\cup(C(M^{4(k+1)})\times S^{2i-4(k+1)-1})\cup\cdots.$$

We obtain a $2i$–dimensional $\mathbb{Z}_p$–stratifold $(S,\delta S)$, where

$$S=V\cup(C(M^{4k})\times S^{2i-4k-1})\cup(C(M^{4(k+1)})\times S^{2i-4(k+1)-1})\cup\cdots$$

is a $2i$–dimensional $\mathbb{Z}_p$–stratifold with singular part $S^{2i-4k-1} \cup S^{2i-4(k+1)-1} \cup \cdots$, and the Bockstein $\delta S = S^{2i-1} \cup (M_1 \times S^{2i-5}) \cup \cdots \cup (M_{k-1} \times S^{2i-4(k-1)-1})$ is a $(2i-1)$–dimensional $\mathbb{Z}_p$–stratifold with empty singular part.

Notice that for a generic $\mathbb{Z}_p$–stratifold $(S, \delta S) \in H_{2i}(B\mathbb{Z}_p \times B\mathbb{Z}_p; \mathbb{Z}_p)$ the singular parts of S and δS are allowed up to dimensions $2i-2$ and $2i-3$, respectively. The two previous $\mathbb{Z}_p$–stratifolds satisfy these dimension conditions for the singular parts. The bordism of $\mathbb{Z}_p$–stratifolds is of the form $(T, \delta T)$, where the singular parts of T and δT are allowed up to dimensions $2i-1$ and $2i-2$, respectively. If we show the two $\mathbb{Z}_p$–stratifolds in (i) and (ii) are $\mathbb{Z}_p$–bordant, we will have two representations of the generator α_{2i}. Apply Proposition 5.2 using the bordism in stratifolds between $M_1 \times S^{2i-5} \cup \cdots \cup M_{k-1} \times S^{2i-4(k-1)-1}$ and the empty stratifold produced by the cone $C(M_1) \times S^{2i-5} \cup \cdots \cup C(M_{k-1}) \times S^{2i-4(k-1)-1}$, which has singular part $S^{2i-5} \cup \cdots S^{2i-4(k-1)-1}$. The proof of Proposition 5.2 uses a product with the interval producing a $\mathbb{Z}_p$–stratifold $(T', \delta T')$ with the singular parts of T' and $\delta T'$ of dimensions $2i-4$ and $2i-5$. This shows that the two $\mathbb{Z}_p$–stratifolds in (i) and (ii) are $\mathbb{Z}_p$–bordant, consequently both $\mathbb{Z}_p$–stratifolds represent the generator α_{2i}.

Theorem 8.2 *For $X = B\mathbb{Z}_p$, the differentials $d^r_{s,t}$ are trivial for $r \le 2p-2$. In particular, the differential $d^{2p-1}_{2i,0}$ is of the form*

$$d^{2p-1}_{2i,0} \colon H_{2i}(B\mathbb{Z}_p; \Omega_0(*; \mathbb{Z}_p)) \to H_{2p-2i+1}(B\mathbb{Z}_p; \Omega_{2p-2}(*; \mathbb{Z}_p)),$$

and the image of the class $\alpha_{2i} \in H_{2i}(B\mathbb{Z}_p; \mathbb{Z}_p)$ with $i \ge p$ under the differential d^{2p-1} is nontrivial.

Proof We can restrict to the differentials $d^r_{2i,0}$ since those starting on coordinates $(2i+1, 0)$ are always trivial since the classes α_{2i+1} are represented by spheres. From Section 7, the differential $d^r_{2i,0} \colon E^r_{2i,0} \to E^r_{2i-r,r-1}$ has the form

$$\begin{array}{c} \operatorname{Im}\big(SH^{(r-2)}_{2i}(X^{2i}; \mathbb{Z}_p) \to SH^{(0)}_{2i}(X^{2i+r-1}; \mathbb{Z}_p)\big) \\ \Big\downarrow{\scriptstyle d^r} \\ \operatorname{Im}\big(SH^{(2r-3)}_{2i-1}(X^{2i-r}; \mathbb{Z}_p) \to SH^{(r-1)}_{2i-1}(X^{2i-1}; \mathbb{Z}_p)\big) \end{array}$$

For $p = 2k+1$, recall the representation of the generator α_{2i} by a $2i$–dimensional $\mathbb{Z}_p$–stratifold $(S, \delta S)$, where

- $S = V \cup (C(M^{4k}) \times S^{2i-4k-1}) \cup (C(M^{4(k+1)}) \times S^{2i-4(k+1)-1}) \cup \cdots$ is a $2i$–dimensional $\mathbb{Z}_p$–stratifold with singular part $S^{2i-4k-1} \cup S^{2i-4(k+1)-1} \cup \cdots$, and
- $\delta S = S^{2i-1} \cup (M_1 \times S^{2i-5}) \cup \cdots \cup (M_{k-1} \times S^{2i-4(k-1)-1})$ is a $(2i-1)$–dimensional $\mathbb{Z}_p$–stratifold with empty singular part.

Since $r \le 2p-2 = 4k$, we obtain $2i-4k-1 \le 2i-(r-2)-2$; hence α_{2i} belongs to $SH^{(r-2)}_{2i}(X^{2i}; \mathbb{Z}_p)$. From Section 7, the representation of the differential $d^r(\alpha_{2i})$ is calculated with the top stratum, which is the

$\mathbb{Z}_p$–manifold $(V \cup M^{4k} \times [0,1] \times S^{2i-4k-1} \cup \cdots, S^{2i-1} \cup (M_1 \times S^{2i-5}) \cup \cdots (M_{k-1} \times S^{2i-4(k-1)-1}))$, which has $\mathbb{Z}_p$–boundary $((M^{4k} \times S^{2i-4k-1}) \cup (M^{4(k+1)} \times S^{2i-4(k+1)-1}) \cup \cdots, \varnothing)$. Therefore, $d^r(\alpha_{2i}) = (M^{4k} \times S^{2i-4k-1}) \cup (M^{4(k+1)} \times S^{2i-4(k+1)-1}) \cup \cdots$ and we can cone all the M^{4m} since the singular parts of the bordisms in $SH_{2i-1}^{(r-1)}(X^{2i-1}; \mathbb{Z}_k)$ are allowed up to dimension $2i-r-1$ and $2i-4k-1 \leq 2i-r-1$ precisely when $r \leq 4k$. Therefore, the differential $d^r(\alpha_{2i})$ is zero for $r \leq 4k = 2p-2$. In fact, we have $E^2 \cong \cdots \cong E^{2p-1}$ because we have a commutative diagram

$$\begin{tikzcd} E^r_{s,0} \otimes \Omega_t(*;\mathbb{Z}_p) \arrow[r] \arrow[d, "d^r \otimes \mathrm{id}"'] & E^r_{s,t} \arrow[d, "d^r"'] \\ E^r_{s-r,r-1} \otimes \Omega_t(*;\mathbb{Z}_p) \arrow[r] & E^r_{s-r,t+r-1} \end{tikzcd} \tag{47}$$

as in Conner and Floyd [6, pages 17 and 41], and we have by induction that the rows are isomorphisms for $r \leq 2p-2$. Finally, for $r = 2p-1$, the element $d^{2p-1}_{2i,0}(\alpha_{2i}) = M^{2p-2} \times S^{2i-2p+1}$ is not zero in $H_{2i-2p+1}(B\mathbb{Z}_p; \Omega_{2p-2}(*;\mathbb{Z}_p))$, since M^{2p-2} is a Milnor generator of $\Omega/p\Omega$. For $p=3$, M^4 can be taken to be $\mathbb{CP}^2$ and we find the obstruction to realizability with d^5. □

9 $\mathbb{Z}_2$–stratifold homology is stratifold homology with $\mathbb{Z}_2$–coefficients

Kreck [12, Chapter 4] introduces the theory of $\mathbb{Z}_2$–oriented stratifolds in order to represent homology with $\mathbb{Z}_2$–coefficients. He calls this theory *stratifold homology with $\mathbb{Z}_2$–coefficients*, denoted by $S\mathcal{H}_*(X;\mathbb{Z}_2)$. The elements are bordism classes of singular stratifolds where the stratum of codimension 1 is empty, but there is no requirement of an orientation of the top stratum. There is a natural isomorphism

$$S\mathcal{H}_*(X;\mathbb{Z}_2) \to H_*(X;\mathbb{Z}_2) \tag{48}$$

that, for a singular stratifold $(S, f\colon S \to X)$, takes the pushforward of the fundamental class $[S]$ in $H_*(S;\mathbb{Z}_2)$.

This article introduces the theory of $\mathbb{Z}_2$–stratifolds, which also represent homology with $\mathbb{Z}_2$–coefficients. This is called *$\mathbb{Z}_2$–stratifold homology*, denoted by $SH_*(X;\mathbb{Z}_2)$. The elements are $\mathbb{Z}_2$–bordism classes of singular $\mathbb{Z}_2$–stratifolds where the stratum of codimension 1 is empty, but we require an orientation of the top stratum. There is a natural isomorphism

$$SH_*(X;\mathbb{Z}_2) \to H_*(X;\mathbb{Z}_2) \tag{49}$$

that, for a singular $\mathbb{Z}_2$–stratifold $((S,\delta S), f\colon S \to X)$, takes the pushforward of the fundamental class $[S]_{\mathbb{Z}_2} \in H_n(\widetilde{S};\mathbb{Z}_2)$.

Therefore, we have the commutative diagram

$$\begin{tikzcd} SH_*(X;\mathbb{Z}_2) \arrow[rr, "q"] \arrow[dr, "\cong"'] & & S\mathcal{H}_*(X;\mathbb{Z}_2). \arrow[dl, "\cong"] \\ & H_*(X;\mathbb{Z}_2) & \end{tikzcd} \tag{50}$$

To define the map q, note that for an n–dimensional $\mathbb{Z}_2$–stratifold $(S, \delta S, \theta_i)$, the quotient space $\widetilde{S}$ is an n–dimensional $\mathbb{Z}_2$–oriented stratifold. This is true because the two disjoint collars associated with the two embedded copies of the Bockstein δS are combined to produce a bicollar on the quotient space $\widetilde{S}$. For $(\mathcal{S}, f)$ an n–dimensional singular $\mathbb{Z}_2$–stratifold with $\mathcal{S} = (S, \delta S, \theta_i)$, we have the map $q\colon SH_n(X;\mathbb{Z}_2) \to S\mathcal{H}_n(X;\mathbb{Z}_2)$ defined by $q([\mathcal{S}, f]) = [\widetilde{S}, \widetilde{f}]$, where $\widetilde{f}$ is the quotient map.

The description of the inverse for the isomorphism $q\colon SH_*(X;\mathbb{Z}_2) \to S\mathcal{H}_*(X;\mathbb{Z}_2)$ is an open question. Wall [26] shows a description for an n–dimensional manifold whose first Stiefel–Whitney class ω_1 in $H^1(M;\mathbb{Z}_2)$ is the restriction $\operatorname{mod} 2$ of a class with integer coefficients. Thus there is a map $f\colon M \to K(\mathbb{Z}, 1) = S^1$, which can be approximated by a smooth map. Take a regular value t and consider the cutting $f^{-1}(t)$. The manifold with boundary $M - f^{-1}(t)$ is orientable, and in that case $f^{-1}(t)$ is also orientable; this describes q^{-1} for this particular case.

Appendix

A.1 Regular values for $\mathbb{Z}_k$–stratifolds

In [12, page 27], Kreck defines a *regular value* for a smooth map $f\colon S \to N$ from a closed stratifold S to a boundaryless manifold N as a point $x \in N$ such that for all $y \in f^{-1}(x)$ the differential df_y is surjective, or, equivalently, x is a regular value of $f|_{S^i}$ for all i. Kreck [12, Propositions 2.6 and 2.7, pages 27–29] shows that the set of regular values of f is dense in N, and $f^{-1}(x)$ is a stratifold of dimension $\dim S - \dim N$.

In [12, page 35], Kreck defines a *smooth map* $f\colon T \to N$ from a stratifold with boundary T to a boundaryless manifold N as a continuous function whose restriction to $\mathring{T} = T - \partial T$ and to ∂T is smooth and which commutes with the collar $c\colon \partial T \times [0, \epsilon) \to U$, ie there is a $\delta > 0$ with $\delta \leq \epsilon$ such that $fc(x, t) = f(x)$ for all $x \in \partial T$ and $t < \delta$. Kreck [12, page 38] says $x \in N$ is a *regular value* if x is a regular value for $f|_{T-\partial T}$ and $f|_{\partial T}$. In this case, the preimage $f^{-1}(x)$ is a stratifold with boundary of dimension $\dim T - \dim N$. This fact is a generalization of a result of [12, Proposition 2.7] using local retractions for $T - \partial T$ and ∂T, together with Theorem A.1. Also, by Theorem A.1, the set of regular values is dense in N.

Theorem A.1 [10, pages 60–62] *Let $f\colon M \to N$ be a smooth map of a manifold M with boundary onto a boundaryless manifold N and let $x \in N$ a regular value of both f and ∂f. Then the preimage $f^{-1}(x)$ is a submanifold of M with boundary $f^{-1}(x) \cap \partial M$ of dimension* $\dim M - \dim N$. *Moreover, the set of critical values of both f and ∂f has measure zero.*

In what follows, we obtain the version for stratifolds with boundary of Propositions 4.2 and 4.3 of Kreck [12].

Proposition A.2 *Let T be an oriented, regular stratifold with boundary, $f: T \to \mathbb{R}$ a smooth function and t a regular value. Then $f^{-1}(t)$ is an oriented, regular stratifold with boundary.*

Proof We use the work of Kreck [12, Proposition 4.2, page 44] in order to show that $f|_{T-\partial T}{}^{-1}(t)$ and $f|_{\partial T}{}^{-1}(t)$ are regular stratifolds. We induce the collar by restriction. We notice $f^{-1}(t)$ is an oriented stratifold, since $T^{n-1} = \varnothing$ and the intersection with the top stratum is an oriented manifold. □

Remark A.3 In the case T is a p–stratifold with boundary, see Remark 3.11; hence the preimage $f^{-1}(t)$ is also a p–stratifold with boundary, for t a regular value. The construction of this p–stratifold is as follows: for t a regular value, on each stratum T_i the preimage $f|_{T_i}^{-1}(t)$ is a submanifold of T_i with boundary $f|_{T_i}^{-1} \cap \partial T_i$ by Theorem A.1. Similarly, the preimage $\partial f|_{\partial T_i}^{-1}(t)$ is a submanifold of ∂T_i. Moreover, these submanifolds come with collars and attaching maps that construct this p–stratifold with boundary inductively.

Proposition A.4 *Let T be a regular stratifold with boundary. Then the set of regular points of a smooth map $f: T \to \mathbb{R}$ is an open subset of T. If, in addition, T is compact, the regular values form an open set.*

Proof We know the regular points of $f|_{T-\partial T}$ and $f|_{\partial T}$ are open in $T - \partial T$ and ∂T, respectively. By definition $fc(x,t) = f(x)$ for some collar c in T. So, the regular points of $f|_{\partial T}$ extend to the collar by an open set. Thus, we obtain the first statement. Now, in the case T is compact, the singular points that are the complement of the regular points, form a closed set which is compact. Thus, the image under f is closed, implying that the regular values are an open set. □

A crucial fact for the Mayer–Vietoris sequence for stratifolds is the following:

Proposition A.5 [12, Proposition 2.8] *Let S be a closed n–dimensional, connected stratifold and A and B disjoint closed nonempty subsets of S. Then there is a nonempty $(n-1)$–dimensional stratifold P with $P \subset S - (A \cup B)$. That is, P separates A and B.*

Remark A.6 More precisely, Kreck [12, Proposition 2.4, page 26] constructs a smooth function $f: S \to \mathbb{R}$ which maps A to 1 and B to -1. The stratifold P is the preimage $f^{-1}(t)$ of a regular value $t \in (-1,1)$ such that $f^{-1}(t) \subset S - (A \cup B)$ and $A \subset f^{-1}(t,\infty)$ and $B \subset f^{-1}(-\infty,t)$. After composition with an appropriate translation, we can assume $t = 0$.

We extend Proposition A.5 to the theory of $\mathbb{Z}_k$–stratifolds. However, it is not enough to consider stratifolds with boundary. The reason is that the smooth function must be $\mathbb{Z}_k$–invariant on the boundary. One needs a smooth function that factors as

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}"'] & & \mathbb{R} \\ & \widetilde{S} \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

We need a $\mathbb{Z}_k$–stratifold version of the following result.

Proposition A.7 [12, Proposition 2.4] *Let $A \subset S$ be a closed subset of a stratifold S, let U be an open neighborhood of A, and $f: U \to \mathbb{R}$ a smooth function. Then there is a smooth function $g: S \to \mathbb{R}$ such that $g|_A = f|_A$.*

Proposition A.8 *Let $\mathcal{S} = (S, \delta S, \theta_i)$ be an n–dimensional compact closed $\mathbb{Z}_k$–stratifold, $A \subseteq \widetilde{S}$ a closed subset of the quotient space, U an open neighborhood of A and $f: U \to \mathbb{R}$ a smooth function. Then there exists a smooth function $G: S \to \mathbb{R}$ that factors through the quotient space $\widetilde{S}$ such that $G|_A = f|_A$ in the quotient space.*

Proof We construct a smooth function on S, which is the gluing of the following two functions:

- For the first function, consider δS inside the quotient space $\widetilde{S}$. By normality of S, there exists a closed subset $A_1 \subset \delta S$ such that $A \cap \delta S \subset \operatorname{int} A_1$ and $A_1 \subset \delta S \cap U$. By compactness and using the collar, $\mathrm{pr}: \delta S \times [0, \epsilon) \to \widetilde{S}$, we find $0 < t < \epsilon$ such that
$$\mathrm{pr}^{-1}(A) \cap (\partial S \times [0, 2t)) \subset \mathrm{pr}^{-1}(A_1) \times [0, 2t) \subset \mathrm{pr}^{-1}(U).$$
Proposition A.7 implies that it is possible to construct a smooth function $f_1: \delta S \to \mathbb{R}$ such that A_1 maps to 1 and $f_1(x) = 0$ for $x \in \delta S - U \cap \delta S$. Lift f_1 to a smooth function on the whole boundary ∂S and take the smooth function $g_1: \partial S \times [0, 2t) \to \mathbb{R}$ by writing $g_1(x, s) = f_1(x)$.
- For the second function, take the stratifold $S_1 := S - (\partial S \times [0, t])$ and again by Proposition A.7 we can construct a smooth function $g_2: S_1 \to \mathbb{R}$ such that $A \cap S_1$ maps to 1 and $g_2(x) = 0$ for $x \in S_1 - U \cap S_1$.

A partition of unity glues these two functions together into a smooth function $G: S \to \mathbb{R}$, which is $\mathbb{Z}_k$–invariant. Thus it descends to the quotient and sends A to 1 and $\widetilde{S} - U$ to 0. Using Proposition 2.4 of Kreck [12] (Proposition A.7), we apply the previous process to construct the function $G: S \to \mathbb{R}$, which is $\mathbb{Z}_k$–invariant and is such that $G|_A = f|_A$ in the quotient space. □

In conclusion, we obtain the $\mathbb{Z}_k$–stratifold version of Kreck [12, Proposition 2.8] (Proposition A.5).

Proposition A.9 *Let $(S, \delta S)$ be an n–dimensional, compact, connected $\mathbb{Z}_k$–stratifold and A and B disjoint closed nonempty subsets of the quotient space $\widetilde{S}$. Then there is a nonempty $(n-1)$–dimensional $\mathbb{Z}_k$–stratifold $(P, \delta P)$ with $\widetilde{P} \subset \widetilde{S} - (A \cup B)$ and $\delta P \subset \delta S - ((A \cup B) \cap \delta S)$.*

We construct a smooth function $G: S \to \mathbb{R}$ that factors through the quotient space $\widetilde{S}$, and maps A to 1 and B to -1. The $\mathbb{Z}_k$–stratifold $(P, \delta P)$ is provided by a regular value $t \in (-1, 1)$ of both S and ∂S, and we have $P = G^{-1}(t)$ and $\delta P = G|_{\delta S}^{-1}(t)$. The pair $(P, \delta P)$ is a $\mathbb{Z}_k$–stratifold because we choose a regular value by Proposition A.4 and the preimage $P = G^{-1}(t)$ is a stratifold with boundary, where $\partial P = G^{-1}(t) \cap \partial S = \bigsqcup_{i \in \mathbb{Z}_k} \theta_i(G^{-1}(t) \cap \delta S) = \bigsqcup_{i \in \mathbb{Z}_k} \theta_i(G|_{\delta S}^{-1}(t))$ and the Bockstein is $\delta P = G|_{\delta S}^{-1}(t)$.

A.2 Mayer–Vietoris sequence

Let U and V be open subsets of a space X. In this section we show that the long exact sequence

(51)
$$\begin{tikzcd} \cdots \arrow[r, "d"] & SH_n(U\cap V;\mathbb{Z}_k) \arrow[r, "i_*"] & SH_n(U;\mathbb{Z}_k)\oplus SH_n(V;\mathbb{Z}_k) \arrow[r, "j_*"] & SH_n(U\cup V;\mathbb{Z}_k) \arrow[dll, "d"] \\ & SH_{n-1}(U\cap V;\mathbb{Z}_k) \arrow[r, "i_*"] & \cdots & \end{tikzcd}$$

is exact for $n\geq 1$. Denote by i_U and i_V the inclusions $U\cap V\hookrightarrow U$ and $U\cap V\hookrightarrow V$. Denote by j_U and j_V the inclusions $U\hookrightarrow U\cup V$ and $V\hookrightarrow U\cup V$. We describe the homomorphisms as follows:

- $i_*\colon SH_n(U\cap V;\mathbb{Z}_k)\to SH_n(U;\mathbb{Z}_k)\oplus SH_n(V;\mathbb{Z}_k)$ is given by (i_{U*}, i_{V*}).
- $j_*\colon SH_n(U;\mathbb{Z}_k)\oplus SH_n(V;\mathbb{Z}_k)\to SH_n(U\cup V;\mathbb{Z}_k)$ is given by $j_{U*}-j_{V*}$.
- The connecting homomorphism $d\colon SH_n(U\cup V;\mathbb{Z}_k)\to SH_{n-1}(U\cap V;\mathbb{Z}_k)$ considers an element $[(S,\delta S),g]\in SH_n(U\cup V;\mathbb{Z}_k)$. For the projection pr: $S\to\widetilde{S}$, we obtain disjoint closed subsets of $\widetilde{S}$ given by $A:=\mathrm{pr}(g^{-1}(X-V))$ and $B:=\mathrm{pr}(g^{-1}(X-U))$. By Proposition A.9, we obtain an $(n-1)$–dimensional $\mathbb{Z}_k$–stratifold $(P,\delta P)$ such that $\widetilde{P}\subset\widetilde{S}-(A\cup B)$ and $\delta P\subset\delta S-((A\cup B)\cap\delta S)$. We define

$$d([(S,\delta S),g])=[(P,\delta P),g|_P]. \tag{52}$$

  In the case that A or B is empty, the $\mathbb{Z}_k$–stratifold $(P,\delta P)$ is empty, and the differential is zero.

Proof that d is well defined It was pointed out by Kreck [13, page 304] that in the case of bordism of smooth manifolds, the connecting homomorphism for the Mayer–Vietoris sequence is well defined because of the existence of a *bicollar* for $P:=G^{-1}(0)$, ie an isomorphism with $P\times(-\epsilon,\epsilon)$, where 0 is a regular value by the composition of a translation. For a stratifold S, this is only possible up to bordism where we naively change S by $S-P\cup(P\times(-\epsilon,\epsilon))$. The formal statement is [12, Lemma B.1, page 197], and the proof is as follows. Kreck's Proposition 4.3 in [12] (our Proposition A.4) allows us to choose $\delta>0$ such that $(-\delta,\delta)$ consists only of regular values of G. Consider a monotone smooth map $\mu\colon\mathbb{R}\to\mathbb{R}$ which is the identity for $|t|>\delta/2$ and 0 for $|t|<\delta/4$. Take $\eta\colon S\times\mathbb{R}\to\mathbb{R}$ mapping $(x,t)\mapsto G(x)-\mu(t)$, which has 0 as regular value. Kreck's Proposition 4.2 in [12] implies that $S'=\eta^{-1}(0)$ is a regular stratifold containing $P\times(-\delta/4,\delta/4)$, which is the required bicollar. It remains to construct a bordism between S and S'. Now take the function $\gamma\colon S\times\mathbb{R}\times[0,1]\to\mathbb{R}$ defined by

$$(x,t,s)\mapsto G(x)-(\zeta(s)\mu(t)+(1-\zeta(s))t),$$

where $\zeta\colon[0,1]\to\mathbb{R}$ is 0 near 0, and 1 near 1. This map has 0 as a regular value, and the preimage $Q:=\gamma^{-1}(0)$ is the bordism between S and S'.

For the case of $\mathbb{Z}_k$–stratifolds, we start with a closed $\mathbb{Z}_k$–stratifold $(S,\delta S,\theta_i)$ and we need to separate this $\mathbb{Z}_k$–stratifold by a bicollar over the regular $\mathbb{Z}_k$–stratifold

$$(P,\delta P,\theta_i|_{\delta P})=(G^{-1}(0),G|_{\delta S}^{-1}(0),\theta_i|_{G|_{\delta S}^{-1}(0)}).$$

In such a case, the bicollar consists of a pair of embedded cylinders $(G^{-1}(0)\times(-\epsilon,\epsilon), G|_{\delta S}^{-1}(0)\times(-\epsilon,\epsilon))$ which are consistent with respect to the embeddings θ_i. In order to reproduce Kreck's [12, Lemma B.1] in the context of $\mathbb{Z}_k$–stratifolds, we observe that the map $\eta\colon S\times\mathbb{R}\to\mathbb{R}$ is $\mathbb{Z}_k$–invariant for our case and we take $(S',\delta S') := (\eta^{-1}(0), \eta|_{\delta S\times\mathbb{R}}^{-1}(0))$, which is a regular $\mathbb{Z}_k$–stratifold by Proposition A.2. We construct the bicollar taking $\epsilon=\delta/4$ with δ as in the previous paragraph. The $\mathbb{Z}_k$–bordism between $((S,\delta S),\mathrm{id})$ and $((S',\delta S'),\pi_1)$, where π_1 is the projection on the first variable, is constructed similarly, as in the case of stratifolds.

The remaining steps to show d is well defined are analogous to the case of stratifolds [12, pages 199–200]. The idea is to assume that $[(S,\delta S), g]$ is trivial, then $[(S',\delta S'), g\circ\pi_1]$ is also trivial. For the modified $\mathbb{Z}_k$–stratifold $(S',\delta S')$, we can take the separating function given by the projection on the second variable. This means that there exists a $\mathbb{Z}_k$–bordism $(T,\delta T)$ that has as $\mathbb{Z}_k$–boundary $(S',\delta S')$. Moreover, the separating function extends to T. This function has a regular value t very close to 0, then $(P\times\{t\},\delta P\times\{t\})$ is null $\mathbb{Z}_k$–bordant taking the preimage of t. However, this last $\mathbb{Z}_k$–stratifold is $\mathbb{Z}_k$–bordant to $(P,\delta P)$. $\square$

The following results are required to show that (51) is exact.

Proposition A.10 *Suppose M is a manifold with boundary of dimension n and $g\colon M\to\mathbb{R}$ a smooth map with regular value 0. Then the preimage $g^{-1}(-\infty,0]$ is a manifold with boundary, and the boundary has the form*

$$g^{-1}(0)\sqcup_{(g^{-1}(0)\cap\partial M)}(g^{-1}(-\infty,0]\cap\partial M).$$

In addition, if M is oriented, then $g^{-1}(-\infty,0]$ is oriented.

Proof Here we will dismiss the orientation of the manifolds, which is understood depending on the case. From [10, page 62], we have that for a manifold N without boundary and $f\colon N\to\mathbb{R}$ a smooth map, the preimage $f^{-1}(-\infty,0]$ is a manifold with boundary given by $f^{-1}(0)$. Thus the restriction to the boundary $g|_{\partial M}$ is such that $g|_{\partial M}^{-1}(-\infty,0]=g^{-1}(-\infty,0]\cap\partial M$ is a manifold whose boundary is $g|_{\partial M}^{-1}(0)=g^{-1}(0)\cap\partial M$. Furthermore, we use Theorem A.1 (or [10, pages 60–62]) which shows that $g^{-1}(0)$ is also a manifold with boundary $g^{-1}(0)\cap\partial M$. Then we glue these two manifolds obtaining a boundaryless smooth manifold of dimension $n-1$. In Figure 8 we illustrate the boundary of $g^{-1}(-\infty,0]$.

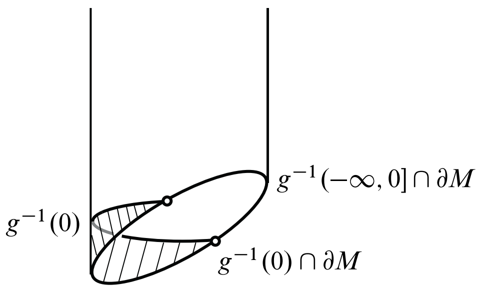


Figure 8: The boundary of $g^{-1}(-\infty,0]$.

Now we consider the restriction $g|_{M-\partial M}$ and we obtain a smooth structure for

$$g|_{M-\partial M}^{-1}(-\infty,0] = g^{-1}(-\infty,0] - (g^{-1}(-\infty,0] \cap \partial M)$$

with boundary $g^{-1}(0) - (g^{-1}(0) \cap \partial M)$. We can establish a collar around $g^{-1}(0)$. As g commutes with the collar of ∂M, there is a collar around $(g^{-1}(-\infty,0] \cap \partial M)$. Finally, similar to the proof of Proposition 4.15, we combine the two collars of $g^{-1}(0)$ and $(g^{-1}(-\infty,0] \cap \partial M)$, where we smooth the corners by straightening the angle [6, pages 9–10] (or see Section 4). □

Proposition A.10 follows for stratifolds with boundary (all p–stratifolds). Notice that

$$g^{-1}(-\infty,0] \cap \partial T = (g^{-1}(-\infty,0] \cap S) \cup (g^{-1}(-\infty,0] \cap k\delta T)$$

and hence $g^{-1}(-\infty,0]$ is a stratifold with boundary where

$$\partial g^{-1}(-\infty,0] = g^{-1}(0) \cup (g^{-1}(-\infty,0] \cap S) \cup (g^{-1}(-\infty,0] \cap k\delta T).$$

Thus we obtain the following application for $\mathbb{Z}_k$–stratifolds.

Corollary A.11 *Suppose $(T,\delta T)$ is a $\mathbb{Z}_k$–stratifold with boundary of dimension n, where the $\mathbb{Z}_k$–boundary is denoted by $(S,\delta S)$. Let $g\colon T \to \mathbb{R}$ be a smooth map which factors to the quotient space $\widetilde{T}$ with 0 as a regular value for g. Then the preimage*

$$\bigl(g^{-1}(-\infty,0],\, g^{-1}(-\infty,0] \cap \delta T\bigr)$$

is a $\mathbb{Z}_k$–stratifold whose $\mathbb{Z}_k$–boundary is the $\mathbb{Z}_k$–stratifold

$$\bigl(g^{-1}(0) \cup (g^{-1}(-\infty,0] \cap S),\, (g^{-1}(0) \cap \delta T) \cup (g^{-1}(-\infty,0]) \cap \delta S)\bigr).$$

Now we use these tools to show the exactness of the Mayer–Vietoris sequence.

Proof of exactness of (51) We follow the arguments used for the case of stratifolds [12, pages 200–208], where we will specify the additional details used for the case of $\mathbb{Z}_k$–stratifolds.

To show that we have a complex, we notice that both $j_U \circ i_U$ and $j_V \circ i_V$ are the canonical inclusion $U \cap V \hookrightarrow U \cup V$, therefore $j_* \circ i_* = 0$. We show the other cases $i_* \circ d = 0$ and $d \circ j_* = 0$ in what follows: for the first identity, we choose a representative for the homology class (with $\mathbb{Z}_k$–coefficients) in $U \cap V$ such that we can cut along the separating $\mathbb{Z}_k$–stratifold defining the boundary operator. The two pieces separated by this $\mathbb{Z}_k$–stratifold induce the null $\mathbb{Z}_k$–bordisms on the homology groups (with $\mathbb{Z}_k$–coefficients) associated with U and V. For the second identity, if $[(S,\delta S), g] \in SH(U;\mathbb{Z}_k)$, we can choose a smooth function and the regular value such that the separating regular $\mathbb{Z}_k$–stratifold is empty, therefore, $d(j_{U*}) = 0$. By the same argument $d(j_{V*}) = 0$.

Now we show exactness.

- **ker $\boldsymbol{j_*} \subset$ im $\boldsymbol{i_*}$** Consider $[(S,\delta S), f] \in SH_n(U;\mathbb{Z}_k)$ and $[(S',\delta S'), f'] \in SH_n(V;\mathbb{Z}_k)$ which are such that $j_{U*}([(S,\delta S), f]) = j_{V*}([(S',\delta S'), f'])$. There exists a $\mathbb{Z}_k$–bordism $((T,\delta T), F)$ between

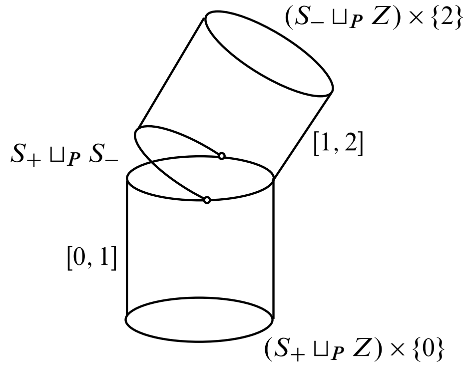


Figure 9: The $\mathbb{Z}_k$–bordism T.

$[(S, \delta S), j_U f]$ and $[(S', \delta S'), j_V f']$, where $F = \widetilde{F} \circ \mathrm{pr}$ for the quotient $\widetilde{F} \colon \widetilde{T} \to U \cup V$. For the closed disjoint subsets $A_T = \widetilde{S} \cup \widetilde{F}^{-1}(X - V)$ and $B_T = \widetilde{S}' \cup \widetilde{F}^{-1}(X - U)$, we construct a separating function $G \colon T \to \mathbb{R}$ which is $\mathbb{Z}_k$–invariant with $G(A_T) = 1$ and $G(B_T) = -1$ and with a regular value $-1 < s < 1$ (we can assume that $s = 0$) such that $(G^{-1}(0), G^{-1}(0) \cap \delta T)$ is a separating $\mathbb{Z}_k$–stratifold. We can find a bicollar around $G^{-1}(0)$ similarly to when we show that d is well defined. Therefore, Corollary A.11 implies that $((S, \delta S), f)$ and $((G^{-1}(0), G^{-1}(0) \cap \delta T), F|_{G^{-1}(0)})$ are $\mathbb{Z}_k$–bordant in U by the $\mathbb{Z}_k$–bordism $((G^{-1}[0, \infty), G^{-1}[0, \infty) \cap \delta T), F|_{G^{-1}[0,\infty)})$, and $((G^{-1}(0), G^{-1}(0) \cap \delta T), F|_{G^{-1}(0)})$ and $((S', \delta S'), f')$ are $\mathbb{Z}_k$–bordant in V by the $\mathbb{Z}_k$–bordism $((G^{-1}(-\infty, 0], G^{-1}(-\infty, 0] \cap \delta T), F|_{G^{-1}(-\infty,0]})$. Thus,

$$i_{U*}\big([(G^{-1}(0), G^{-1}(0) \cap \delta T), F|_{G^{-1}(0)}]\big) = [(S, \delta S), f],$$
$$i_{V*}\big([(G^{-1}(0), G^{-1}(0) \cap \delta T), F|_{G^{-1}(0)}]\big) = [(S', \delta S'), f'].$$

- **$\ker i_* \subset \operatorname{im} d$** Suppose we have $[(P, \delta P), r] \in SH_{n-1(U \cap V;\mathbb{Z}_k)}$ which satisfies $i_{U*}([(P, \delta P), r]) = 0$ and $i_{V*}([(P, \delta P), r]) = 0$. Then there exist null $\mathbb{Z}_k$–bordisms $((T_1, \delta T_1), R_1)$ and $((T_2, \delta T_2), R_2)$ of $i_{U*}([(P, \delta P), r])$ and $i_{V*}([(P, \delta P), r])$, respectively. We construct $((T_1 \sqcup_P T_2, \delta T_1 \sqcup_{\delta P} \delta T_2), R_1 \sqcup_r R_2)$ with image under d equal to $[(P, \delta P), r]$.

- **$\ker d \subset \operatorname{im} j_*$** Consider $[(S, \delta S), f] \in SH_n(U \cup V; \mathbb{Z}_k)$ with $d([(S, \delta S), f]) = 0$. For a separating function G with regular value s as in the definition of d, write $(P, \delta P) = (G^{-1}(s), G|_{\delta S}^{-1}(s))$, which has a bicollar. We put

$$(S_+, \delta S_+) = (G^{-1}[s, \infty), G|_{\delta S}^{-1}[s, \infty)) \quad \text{and} \quad (S_-, \delta S_-) = (G^{-1}(-\infty, 0], G|_{\delta S}^{-1}(-\infty, 0]).$$

Then $S = S_+ \sqcup_P S_-$ and $\delta S = \delta S_+ \sqcup_{\delta P} \delta S_-$. By the assumptions, there is $((Z, \delta Z), r)$ with $r \colon Z \to U \cap V$, which has the $\mathbb{Z}_k$–boundary $(P, \delta P)$ and $f|_P = r|_P$. Consider the continuous maps $f_+ \colon S_+ \sqcup_P Z \to U$ and $f_- \colon S_- \sqcup_P Z \to V$. The gluing $T := ((S_+ \sqcup_P Z) \times [0, 1]) \sqcup_Z ((S_- \sqcup_P Z) \times [1, 2])$ (similarly for the Bockstein δT) gives a $\mathbb{Z}_k$–bordism between

$$j_{U*}\big(((S_+ \sqcup_P Z, \delta S_+ \sqcup_{\delta P} \delta Z), f_+)\big) - j_{V*}\big(((S_- \sqcup_P Z, \delta S_- \sqcup_{\delta P} \delta Z), f_-)\big)$$

and $((S, \delta S), f)$. We show an illustrative picture of the $\mathbb{Z}_k$–bordism $(T, \partial T)$ in Figure 9. $\square$

References

[1] **A Ángel**, *A spectral sequence for orbifold cobordism*, from "Algebraic topology: old and new" (M Golasiński, Y Rudyak, P Salvatore, N Saveliev, N Wahl, editors), Banach Center Publ. 85, Polish Acad. Sci. Inst. Math., Warsaw (2009) 141–154 MR Zbl

[2] **A Angel**, **C Segovia**, **A F Torres**, *Thom's counterexamples for the Steenrod problem*, preprint (2021) arXiv 2105.01806 To appear in Homology Homotopy Appl.

[3] **N A Baas**, *On bordism theory of manifolds with singularities*, Math. Scand. 33 (1973) 279–302 MR Zbl

[4] **G W Brumfiel**, *Realizing homology classes by PL manifolds*, from "Algebraic and geometric topology" (K C Millett, editor), Lecture Notes in Math. 664, Springer (1978) 43–60 MR Zbl

[5] **S Buoncristiano**, **C P Rourke**, **B J Sanderson**, *A geometric approach to homology theory*, Lond. Math. Soc. Lect. Note Ser. 18, Cambridge Univ. Press (1976) MR Zbl

[6] **P E Conner**, **E E Floyd**, *Differentiable periodic maps*, Ergebnisse der Math. 33, Academic, New York (1964) MR Zbl

[7] **S Eilenberg**, *On the problems of topology*, Ann. of Math. 50 (1949) 247–260 MR Zbl

[8] **E E Floyd**, *Actions of $(Z_p)^k$ without stationary points*, Topology 10 (1971) 327–336 MR Zbl

[9] **S Führing**, *Positive scalar curvature and a smooth variation of Baas–Sullivan theory*, PhD thesis, Universität Augsburg (2013) Available at `https://opus.bibliothek.uni-augsburg.de/opus4/2446`

[10] **V Guillemin**, **A Pollack**, *Differential topology*, Prentice-Hall, Englewood Cliffs, NJ (1974) MR Zbl

[11] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR Zbl

[12] **M Kreck**, *Differential algebraic topology: from stratifolds to exotic spheres*, Grad. Stud. in Math. 110, Amer. Math. Soc., Providence, RI (2010) MR Zbl

[13] **M Kreck**, *A zoo of geometric homology theories*, J. Singul. 18 (2018) 300–306 MR Zbl

[14] **J W Milnor**, *Abstracts of contributed papers: On the cobordism ring Ω_**, Not. Amer. Math. Soc. 5 (1958) 457

[15] **J W Morgan**, **D P Sullivan**, *The transversality characteristic class and linking cycles in surgery theory*, Ann. of Math. 99 (1974) 463–544 MR Zbl

[16] **Y B Rudyak**, *On Thom spectra, orientability, and cobordism*, Springer (1998) MR Zbl

[17] **R Sikorski**, *Differential modules*, Colloq. Math. 24 (1971) 45–79 MR Zbl

[18] **D P Sullivan**, *Triangulating homotopy equivalences*, PhD thesis, Princeton University (1966) Available at `https://www.proquest.com/docview/302229406`

[19] **D Sullivan**, *On the Hauptvermutung for manifolds*, Bull. Amer. Math. Soc. 73 (1967) 598–600 MR Zbl

[20] **D Sullivan**, *Singularities in spaces*, from "Proceedings of Liverpool Singularities Symposium, II" (C T C Wall, editor), Lecture Notes in Math. 209, Springer (1971) 196–206 MR Zbl

[21] **D P Sullivan**, *Triangulating and smoothing homotopy equivalences and homeomorphisms*, from "The Hauptvermutung book" (A A Ranicki, editor), K–Monogr. Math. 1, Kluwer, Dordrecht (1996) 69–103 MR Zbl

[22] **H Tene**, *Stratifolds and equivariant cohomology theories*, PhD thesis, Universität Bonn (2010) Available at `https://hdl.handle.net/20.500.11811/4903`

[23] **H Tene**, *A geometric description of the Atiyah–Hirzebruch spectral sequence for B–bordism*, Münster J. Math. 10 (2017) 171–188 MR Zbl

[24] **R Thom**, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. 28 (1954) 17–86 MR Zbl

[25] **A F Torres Galindo**, $\mathbb{Z}_k$–*stratifolds*, PhD thesis, Universidad de los Andes (2018) Available at `https://tinyurl.com/TorresGalindo-PhD-thesis`

[26] **C T C Wall**, *Determination of the cobordism ring*, Ann. of Math. 72 (1960) 292–311 MR Zbl

Departamento de Matemáticas, Universidad de los Andes
Bogota, Colombia

Instituto de Matemáticas, UNAM Unidad Oaxaca
Oaxaca, Mexico

Departamento de Matemáticas, Universidad Externado de Colombia / Universidad Pontificia Javeriana
Bogotá, Colombia

`ja.angel908@uniandes.edu.co`, `csegovia@matem.unam.mx`, `arley.torres@uexternado.edu.co`

Received: 13 November 2020 Revised: 10 March 2023

Algebraic & Geometric Topology **24**:4 (2024) 1903–1912
DOI: 10.2140/agt.2024.24.1903
Published: 16 July 2024

Relative systoles in hyperelliptic translation surfaces

CORENTIN BOISSY
SLAVYANA GENINSKA

We prove that the systole function on a connected component of area-1 translation surfaces admits a local maximum that is not a global maximum if and only if the connected component is not hyperelliptic.

32G15; 30F30

1 Introduction

We deal with flat metrics defined by abelian differentials on compact Riemann surfaces (*translation surfaces*). Such flat metrics have conical singularities of angle $(k+1)2\pi$, where k is the order of the zero of the corresponding abelian differential. A stratum of the moduli space of abelian differentials corresponds to translation surfaces that share the same combinatorics of zeroes.

Connected components of the strata have been classified by Kontsevich and Zorich in [6]. In each genus $g \geq 2$, there are exactly two components that consist of hyperelliptic translation surfaces, the so-called *hyperelliptic connected components*.

A saddle connection on a translation surface S is a geodesic joining two singularities (possibly the same) and with no singularity in its interior. We define the *relative systole* $\mathrm{Sys}(S)$ to be the length of the shortest saddle connection of S. A sequence of area-1 translation surfaces $(S_n)_{n\in\mathbb{N}}$ in a stratum of the moduli space of translation surfaces leaves any compact set if and only if $\mathrm{Sys}(S_n) \to 0$; see Kerckhoff, Masur and Smillie [5, Proposition 1]. The set of translation surfaces with short relative systole and compactification issues of strata are related to dynamics and counting problems on translation surfaces, and have been widely studied in the last 30 years; see for instance Eskin, Kontsevich and Zorich [2], Eskin, Masur and Zorich [3] and Kerckhoff, Masur and Smillie [5].

Here we are interested in the opposite problem: we study surfaces that are "far" from the boundary. In [1], we have characterized global maxima for Sys, and we have shown that each stratum of genus greater than or equal to 3 contains local but nonglobal maxima for the function Sys.

We prove that there are no such local maxima in hyperelliptic connected components (Theorem 3.1), while they exist in every other connected component (Theorem 4.1). This gives us the following characterization:

Main Theorem *Let $\mathcal{C}$ be a connected component of a stratum of area-1 surfaces with no marked points. The relative systole function on $\mathcal{C}$ admits a local maximum that is not a global maximum if and only if $\mathcal{C}$ is not a hyperelliptic connected component.*

Note that our notion of relative systole is different from the "true systole" (ie shortest closed curve) that has been studied by Judge and Parlier in [4]. Henceforth, for simplicity, if not mentioned otherwise the term "systole" will mean "relative systole".

Acknowledgments The authors thank the referee for useful comments and D-M Nguyen for suggesting a reference.

2 Background

2.1 Translation surfaces

A *translation surface* is a (real compact connected) genus-g surface S with a translation atlas, ie a triple $(S, \mathcal{U}, \Sigma)$ such that Σ (whose elements are called *singularities*) is a finite subset of S and $\mathcal{U} = \{(U_i, z_i)\}$ is an atlas of $S \setminus \Sigma$ whose transition maps are translations of $\mathbb{C} \simeq \mathbb{R}^2$. We will require that for each $s \in \Sigma$ there is a neighborhood of s isometric to a Euclidean cone whose total angle is a multiple of 2π. One can show that the holomorphic structure on $S \setminus \Sigma$ extends to S and that the holomorphic 1–form $\omega = dz_i$ extends to a holomorphic 1–form on S where Σ corresponds to the zeroes of ω and maybe some marked points. We usually call ω an *abelian differential*. A zero of ω of order k corresponds to a singularity of angle $(k+1)2\pi$. By a slight abuse of notation, we allow the order of a zero to be 0, and in this case it corresponds to a (regular) marked point.

A *saddle connection* is a geodesic segment joining two singularities (possibly the same) and with no singularity in its interior. Integrating ω along the saddle connection we get a complex number. Considered as a planar vector, this complex number represents the affine holonomy vector of the saddle connection. In particular, its Euclidean length is the modulus of its holonomy vector.

For $g \geq 1$, we define the moduli space of abelian differentials $\mathcal{H}_g$ as the moduli space of pairs (X, ω) where X is a genus-g (compact connected) Riemann surface and ω a nonzero holomorphic 1–form defined on X. The term moduli space means that we identify the points (X, ω) and (X', ω') if there exists an analytic isomorphism $f : X \to X'$ such that $f^*\omega' = \omega$.

One can also see a translation surface obtained from a polygon (or a finite union of polygons) whose sides come by pairs, and for each pair, the corresponding segments are parallel and of the same length. These parallel sides are glued together by translation and we assume that this identification preserves the natural orientation of the polygons. In this context, two translation surfaces are identified in the moduli space of abelian differentials if and only if the corresponding polygons can be obtained from each other by cutting and gluing, and preserving the identifications.

The moduli space of abelian differentials is stratified by the combinatorics of the zeroes; we will denote by $\mathcal{H}(k_1,\dots,k_r)$ the stratum of $\mathcal{H}_g$ consisting of (classes of) pairs (X,ω) such that ω has exactly r zeroes, of order $k_1,\dots,k_r$. This space is (Hausdorff) complex analytic, and local coordinates for a stratum of abelian differentials are obtained by integrating the holomorphic 1–form along a basis of the relative homology $H_1(S,\Sigma;\mathbb{Z})$, where Σ denotes the set of conical singularities of S; see for instance [7; 8; 9]. We have the classical Gauss–Bonnet formula $\sum_i k_i = 2g-2$, where g is the genus of the underlying surfaces. We often restrict to the subset $\mathcal{H}_1(k_1,\dots,k_r)$ of *area*-1 surfaces.

2.2 Connected component of strata

Here we recall the Kontsevich–Zorich classification of the connected components of the strata of abelian differentials [6].

A translation surface (X,ω) is *hyperelliptic* if the underlying Riemann surface is hyperelliptic, ie there is an involution τ such that X/τ is the Riemann sphere. In this case ω satisfies $\tau^*\omega=-\omega$. A connected component of a stratum is said to be *hyperelliptic* if it consists only of hyperelliptic translation surfaces (note that a connected component which is not hyperelliptic may contain some hyperelliptic translation surfaces).

Let γ be a simple closed smooth curve parametrized by the arc length on a translation surface that avoids the singularities. Then $t\to\gamma'(t)$ defines a map from $\mathbb{S}^1$ to $\mathbb{S}^1$. We denote by $\mathrm{Ind}(\gamma)$ the index of this map. Assume that the translation surface S has only even-degree singularities $S\in\mathcal{H}(2k_1,\dots,2k_r)$. Let $(a_i,b_i)_{i\in\{1,\dots,g\}}$ be a collection of simple closed curves as above that represents a symplectic basis of the homology of S. Then

$$\sum_{i=1}^{g}(\mathrm{Ind}(a_i)+1)(\mathrm{Ind}(b_i)+1) \mod 2$$

is an invariant of connected components and is called the *parity of the spin structure* (see [6] for details).

Here is a reformulation of the classification of connected components of strata by Kontsevich and Zorich:

Theorem 2.1 [6, Theorems 1 and 2] *Let $\mathcal{H}=\mathcal{H}(k_1,\dots,k_r)$ be a stratum of genus $g\geq 2$ translation surfaces, without marked points.*

- *The stratum $\mathcal{H}$ contains a hyperelliptic connected component if and only if $\mathcal{H}=\mathcal{H}(2g-2)$ or $\mathcal{H}=\mathcal{H}(g-1,g-1)$. In this case there is only one hyperelliptic component. In genus 2, any stratum is connected (and hyperelliptic).*
- *If there exists i such that k_i is odd, or if $g=3$, then there exists a unique nonhyperelliptic connected component.*
- *If $g\geq 4$ and, for all i, k_i is even, then there are exactly two nonhyperelliptic connected components distinguished by the parity of the spin structure.*

The following lemma is classical and will be useful in the next section.

Lemma 2.2 *Let S be a translation surface in a hyperelliptic connected component and let γ be a saddle connection. Then $[\gamma] = -[\tau(\gamma)]$ in $H_1(S, \Sigma; \mathbb{Z})$.*

Proof If the images in S of γ and $\tau(\gamma)$ coincide, $[\gamma] = -[\tau(\gamma)]$ since they have opposite orientation.

Otherwise the images in S of γ and $\tau(\gamma)$ intersect at most at the ends of the curves. We note that in the case of the stratum $\mathcal{H}(g-1, g-1)$, the two singularities are interchanged by the involution; see [6, Section 2.1].

Hence, the image of γ (and $\tau(\gamma)$) in the quotient sphere S/τ is always a simple closed curve. Therefore it is the boundary of a subsurface that contains ramification points of the covering. Considering its preimage, we obtain that $\gamma \cup \tau(\gamma)$ is the boundary of a subsurface of S. □

3 Hyperelliptic connected component

In this section, we prove the first part of the Main Theorem.

Theorem 3.1 *Let $\mathcal{C}$ be a hyperelliptic connected component of the moduli space of abelian differentials. Let $S \in \mathcal{C}$ be a local maximum of the relative systole function* Sys*. Then S is a global maximum for* Sys *in $\mathcal{C}$.*

The proof uses the following technical lemma. We postpone its proof to the end of the section.

Lemma 3.2 *Let D be a translation surface that is topologically a disk and whose boundary consists of n–saddle connections (an "n–gon") with $n \geq 4$. We assume that all boundary saddle connections are of length greater than or equal to 1. Then we can continuously deform D so that its area decreases and the boundary saddle connections of length 1 remain of length 1.*

Proof of Theorem 3.1 Let $S \in \mathcal{C}$ be a translation surface that such that $\mathrm{Sys}(S)$ is not a global maximum. We use the same normalization as in [1]: after rescaling the surface we assume that $\mathrm{Sys}(S)$ equals 1, and then continuously deform S so that $\mathrm{Sys}(S)$ remains 1 and $\mathrm{Area}(S)$ decreases.

Let $\{\gamma_1, \dots, \gamma_r\}$ be the set of saddle connections realizing the systole. Recall that $\gamma_1, \dots, \gamma_r$ are sides of the Delaunay triangulation and that global maxima correspond to surfaces whose Delaunay cells are only equilateral triangles; see [1, Lemma 3.1 and Theorem 3.3]. Let $C_1, \dots, C_k$ be the connected components of $S \setminus \bigcup_i \gamma_i$. Up to renumbering we can assume that C_1 is not a triangle. We consider $\tau(C_1)$, where τ is the hyperelliptic involution. We study the two possible cases: whether $\tau(C_1)$ equals C_1 or not. Note that C_1 does not contain any singularity in its interior, since there are at most two singularities in S and if there are two singularities P_1 and P_2, we must have $\tau(P_1) = P_2$.

Case 1 We first assume that $\tau(C_1) \neq C_1$. Since the hyperelliptic involution preserves $\bigcup_i \gamma_i$, up to renumbering, $\tau(C_1) = C_2$.

We observe that C_1 has only one boundary component. Indeed, suppose that there are more than one such components and consider a saddle connection η in C_1 that joins a singularity of one boundary component to a singularity of another boundary component. Then $\tau(\eta)$ is a curve in C_2 and $[\tau(\eta)] = -[\eta]$ by Lemma 2.2. But $C_1 \setminus \eta$ is connected, and hence $S \setminus (\eta \cup \tau(\eta))$ is connected, which is a contradiction. Therefore C_1 is a disk because it embeds in S/τ, which is a sphere.

Since the boundary of C_1 consists of at least four saddle connections of length 1, by Lemma 3.2 we can continuously decrease its area while keeping the boundary saddle connections of length 1.

This continuous deformation of C_1 leads to the following area-decreasing continuous deformation of S:

The component C_2 is deformed in a symmetric way as C_1.

For each saddle connection γ in the boundary of C_1, the components of $S \setminus (\gamma \cup \tau(\gamma))$ correspond to components of the complement of $[\gamma]$ in the quotient sphere S/τ. Since $[C_1] = [C_2]$, we have that C_1 and C_2 are in the same connected component of $S \setminus (\gamma \cup \tau(\gamma))$. We denote by D_γ the other component. By construction, the boundary of D_γ consists of γ and $\tau(\gamma)$. Note that D_γ is empty if γ and $\tau(\gamma)$ have the same image in S. We observe that if γ and γ' are two distinct saddle connections in the boundary of C_1, then D_γ and $D_{\gamma'}$ are disjoint.

We denote by $\gamma_1, \dots, \gamma_k$ the boundary saddle connections of C_1. When continuously deforming C_1, each γ_i is rotated by an angle θ_i (with $\theta_1, \dots, \theta_k$ continuous functions) and $\tau(\gamma_i)$ is also rotated by θ_i since C_2 is deformed in a symmetric way. Since the components $D_{\gamma_1}, \dots, D_{\gamma_r}$ are disjoint, for each i we can glue by translation the component D_{γ_i} rotated by θ_i with the boundary saddle connections corresponding to γ_i and $\tau(\gamma_i)$.

Since the identifications are done by translation, we get a continuous family of translation surfaces and they are in the same stratum.

Case 2 Now we assume that $\tau(C_1) = C_1$.

We claim that we can cut C_1 along saddle connections and obtain two discs A and B such that $\tau(A) = B$ and for each saddle connection γ in the boundary of A either γ is of length 1 or $\tau(\gamma) = \gamma$ (equivalently, γ is also a boundary saddle connection of B).

To prove the claim, we first consider the Delaunay cells of S. Recall that the shortest geodesics (and hence the boundary saddle connections of C_1) are sides of the Delaunay cells; see [1, Lemma 3.1]. This induces a decomposition of C_1 into Delaunay cells, and this decomposition is preserved by the involution τ because of the uniqueness of the Delaunay cell decomposition. We define a Delaunay subdivision $\mathcal{D}$ in the following way: For each Delaunay cell d, if $\tau(d) \neq d$ then $d, \tau(d) \in \mathcal{D}$. If $\tau(d) = d$ (and since d is cyclic) it can be cut by a diagonal into two polygons d' and $d'' = \tau(d')$. Then $d', d'' \in \mathcal{D}$.

Now we use the following algorithm:

- We start from a pair $(d_0, \tau(d_0))$ in $\mathcal{D}^2$ and let $A_0 = d_0$ and $B_0 = \tau(d_0)$.
- Suppose we have constructed the disks A_k and B_k such that $\tau(A_k) = B_k$, and A_k and B_k are unions of elements in $\mathcal{D}$.

If $A_k \cup B_k \neq C_1$, there exists an element $d_{k+1} \in \mathcal{D}$ adjacent to A_k along a saddle connection γ_k (and $\tau(d_{k+1}) \in \mathcal{D}$ is adjacent to B_k along $\tau(\gamma_k)$). We define A_{k+1} by gluing A_k and d_{k+1} along γ_k. Note that γ_k is the only saddle connection in the common boundary of A_k and d_{k+1}, because otherwise $S \setminus (\gamma_k \cup \tau(\gamma_k))$ would be connected, which is impossible in the hyperelliptic connected component.

If $A_k \cup B_k = C_1$ we define $A = A_k$ and $B = B_k$.

The boundary of the disk A consists of $n \geq 3$ saddle connections of lengths at least 1.

If $n \geq 4$, then from Lemma 3.2 it can be continuously deformed so that the area decreases and the boundary saddle connections of length 1 remain of length 1.

If $n = 3$ then A is a triangle. Two of its sides are boundary saddle connections of C_1, and hence of length 1. The third side of A is a saddle connection inside C_1 and is, by construction of C_1, of length greater than 1 (recall that $C_1, \dots, C_r$ are obtained after removing all saddle connections of length 1). Such a triangle can be deformed so that the area decreases and the boundary saddle connections of length 1 remain of length 1.

We deform B in a symmetric way. Note that A and B are directly glued together in C_1 along the boundary saddle connections of lengths greater than 1. Therefore the possible changes of these saddle connections are not a problem. The deformation of $S \setminus C_1$ is treated as in the previous case. $\square$

Proof of Lemma 3.2 The sum of the boundary angles (coming from the intersection of two consecutive boundary saddle connections) of D equals $(n-2)\pi$. Therefore D has boundary angles smaller than π. If such a boundary angle has a corresponding boundary saddle connection which is of length greater than 1, then by slightly changing its length we can decrease the area of the corresponding triangle and hence of D.

So we can assume that for each boundary angle smaller than π the two adjacent saddle connections are of length 1. We claim that we can find two consecutive angles such that one is smaller than π and the other is smaller than 2π (note that since D is not necessarily embedded in the plane, it can have boundary angles greater than 2π). Indeed, consider the sequence of consecutive boundary angles of D. If each time an angle is smaller than π the following one is greater than or equal to 2π, then the global sum will be greater than $n\pi$, which is not possible.

Now we consider the three consecutive saddle connections corresponding to these two angles, and see them as a broken line on the plane. We close this line by adding a segment t to obtain a quadrilateral $\mathcal{Q}$ (that can be also crossed). Without loss of generality, we can assume that t is horizontal. We have

$$\mathrm{Area}(D) = \mathrm{Area}(D_0) + \mathrm{Area}_{\mathrm{alg}}(\mathcal{Q}),$$

where D_0 is the translation surface obtained by "replacing" the broken line by t (see Figure 1). Here $\mathrm{Area}_{\mathrm{alg}}(\mathcal{Q})$ means that the part of $\mathcal{Q}$ below the segment t is counted negatively.

Claim *We can continuously deform $\mathcal{Q}$ without changing the lengths of its sides so that* $\mathrm{Area}_{\mathrm{alg}}(\mathcal{Q})$ *decreases.*

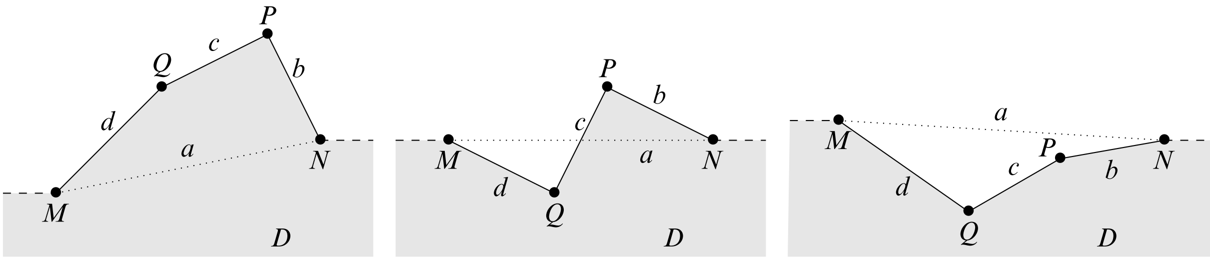


Figure 1: The disk D and the quadrilateral $\mathcal{Q}$ in three configurations.

Denote by $MNPQ$ the quadrilateral $\mathcal{Q}$, and by a, b, c and d the lengths of the sides of $\mathcal{Q}$ with a being the length of the segment $t = MN$. Denote by α the oriented angle from MN to MQ, and by γ its opposite angle in $\mathcal{Q}$ (ie the angle from PQ to PN). Without loss of generality we assume that $b = c = 1$, $d \geq 1$ and $0 < \gamma < \pi$ (in fact we must have $\gamma > \frac{1}{3}\pi$, otherwise there would be a smallest saddle connection). We also have $-\pi < \alpha < \pi$. Further, the sides NP and QM do not intersect since it would imply intersecting boundary saddle connections in D (see Figure 1).

Write $K = \mathrm{Area}_{\mathrm{alg}}(\mathcal{Q})$. We compute K by adding the (algebraic) area of the triangles MNQ and NPQ. We obtain

$$K = \tfrac{1}{2}(ad\sin(\alpha) + bc\sin(\gamma)). \tag{1}$$

The expression of the length of NQ gives the second equality:

$$a^2 + d^2 - 2ad\cos(\alpha) = b^2 + c^2 - 2bc\cos(\gamma). \tag{2}$$

These two equations imply Bretschneider's formula for $\mathcal{Q}$:

$$K^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos\bigl(\tfrac{1}{2}(\alpha+\gamma)\bigr). \tag{3}$$

Here $s = \frac{1}{2}(a+b+c+d)$.

From now on we fix a, b, c and d and study the variations of the area with respect to α and γ. Equation (2) implies that γ depends differentially on α. Hence we can write $K = K(\alpha)$. We need to prove that either $K'(\alpha) \neq 0$ or $K(\alpha)$ is a strict local maximum (note that α varies in an open set). We have

$$(K^2)'(\alpha) = abcd(1+\gamma'(\alpha))\sin\bigl(\tfrac{1}{2}(\alpha+\gamma)\bigr)\cos\bigl(\tfrac{1}{2}(\alpha+\gamma)\bigr).$$

We assume that $K'(\alpha) = 0$, and hence $(K^2)'(\alpha) = 0$, so we are in one of the following three cases:

(i) $\boldsymbol{\bigl(\sin\bigl(\tfrac{1}{2}(\alpha+\gamma)\bigr) = 0\bigr)}$ The conditions $-\pi < \alpha < \pi$ and $0 < \gamma < \pi$ imply $\alpha = -\gamma < 0$. Hence the quadrilateral $\mathcal{Q}$ has self-intersections. Since the sides NP and QM do not intersect, the sides MN and PQ intersect. The condition $\alpha = -\gamma$ implies that the points M, N, P and Q are cocyclic, and since $b = c = 1$ we must have $d < 1$, which is a contradiction.

(ii) $\boldsymbol{\bigl(\cos\bigl(\tfrac{1}{2}(\alpha+\gamma)\bigr) = 0\bigr)}$ Then $\alpha + \gamma = \pi$, and therefore $\alpha > 0$, and hence $K > 0$. From (2) and (3) we have a strict local maximum for K^2, and therefore for K.

(iii) $(\gamma'(\alpha) = -1)$ By differentiating (2) and using (1), we see that $K = 0$, and hence Q has a self-intersection $I = MN \cap PQ$. By differentiating (1) and using (2), we obtain $K'(\alpha) = 0 = \frac{1}{2}\big(\frac{1}{2}(a^2+d^2)-1\big)$, and hence $a^2 + d^2 = 2$. Since $d \geq 1$, we have $a \leq 1 \leq d$. However, triangle inequalities for INP and IMQ give $a + c > d + b$, and hence $a > d$, which is a contradiction. □

4 Nonhyperelliptic connected components

In this section, we prove the second part of the Main Theorem.

Theorem 4.1 *Each nonhyperelliptic connected component of each stratum of area-1 surfaces with no marked points contains local maxima of the function* Sys *that are not global.*

We will need the following lemma, which is a refinement of [1, Lemma 3.2(2)].

Lemma 4.2 *Let $\mathcal{C} \subset \mathcal{H}(2k_1, \dots, 2k_r)$ be a connected component of a stratum of abelian differentials with $2k_1, \dots, 2k_r \geq 0$. There exists a surface $S \in \mathcal{C}$ realizing the global maximum for the systole function, and such that there exists a shortest saddle connection γ joining a singularity of degree $2k_1$ to itself and* $\mathrm{Ind}([\gamma]) = 0$.

Proof We do as in the proof of [1, Lemma 3.2]. There exists a square-tiled surface in $\mathcal{C}$ with singularities on each corner of the squares as in Figure 2, and we can assume that the top left horizontal segment identifies with the bottom left horizontal segment (see Figure 2). After a suitable transformation as in the figure, we obtain the required surface. □

Proof of Theorem 4.1 In [1, Theorem 4.7] we have already constructed examples in each genus $g \geq 3$ stratum. By Theorem 3.1 each such example is in a nonhyperelliptic component. So it remains to construct new examples only in strata with more than one nonhyperelliptic connected component.

From the theorem of Kontsevich and Zorich (Theorem 2.1) there is more than one nonhyperelliptic connected component only for genus $g \geq 4$ strata with only even-degree singularities, and in this case there are two nonhyperelliptic components distinguished by the parity of the spin structure.

In Figure 3 we give surfaces $S_{2,0} \in \mathcal{H}(2,0)$ and $S_{2,0,0} \in \mathcal{H}(2,0,0)$ that are local but nonglobal maxima for the systole function.

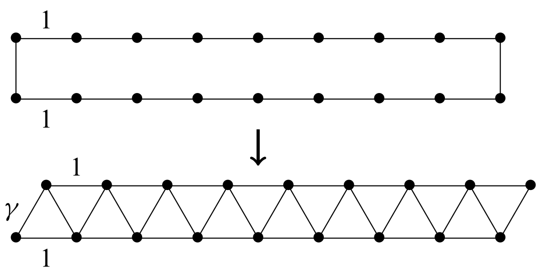


Figure 2: A global maximum with a closed shortest saddle connection γ satisfying $\mathrm{Ind}([\gamma]) = 0$.

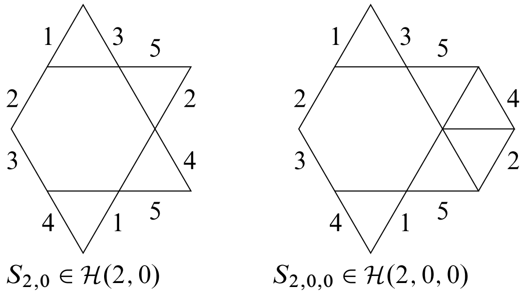


Figure 3: Local but nonglobal maxima in $\mathcal{H}(2,0)$ and $\mathcal{H}(2,0,0)$.

We consider the following construction: Start from the surface $S_{2,0}$ and a surface M that is a global maximum for Sys in $\mathcal{H}(2k_1,\dots,2k_r)$. There exists a shortest saddle connection γ_1 in $S_{2,0}$ joining the two singularities. By Lemma 4.2, we can assume that there exists a shortest saddle connection γ_2 in M joining the singularity of degree $2k_1$ to itself and such that $\mathrm{Ind}([\gamma_2]) = 0$. We can further assume that γ_1 and γ_2 are vertical and of the same length. Now we glue the two surfaces by the following classical surgery: cut the two surfaces along γ_1 and γ_2, and glue the left side of γ_1 with the right side of γ_2 and the right side of γ_1 with the left side of γ_2. We get a surface S in $\mathcal{H}(2k_1+4, 2k_2,\dots,2k_r)$ that satisfies the hypothesis of [1, Theorem 4.1], and hence is a local but nonglobal maximum. By Theorem 3.1, the surface S is necessarily in a nonhyperelliptic component.

We compute $\mathrm{Spin}(S)$: Choose a symplectic basis $(a_i,b_i)_i$ of $H_1(M,\mathbb{Z})$ such that $[\gamma_2] = a_1$. Then a simple computation gives

$$\mathrm{Spin}(S) = \mathrm{Spin}(S_{0,2}) + \mathrm{Spin}(M) + \mathrm{Ind}(a_1) + 1 \mod 2. \tag{4}$$

Since $\mathrm{Ind}(a_1) = 0$,

$$\mathrm{Spin}(S) = \mathrm{Spin}(S_{0,2}) + \mathrm{Spin}(M) + 1 \mod 2.$$

When $\sum_i 2k_i \geq 4$, we can prescribe any value of $\mathrm{Spin}(M)$ by choosing M in a suitable component, and in this way we can obtain any possible value for $\mathrm{Spin}(S)$. Note that this is also true for $M \in \mathcal{H}(4)$ or $M \in \mathcal{H}(2,2)$. Indeed, in these strata there are two components, hyperelliptic and nonhyperelliptic, and the spin structure distinguishes them; see [6, Theorem 2 and Corollary 5].

By this construction, we obtain a local but nonglobal maximum for Sys in any (nonhyperelliptic) connected component of any stratum $\mathcal{H}(2n_1,\dots,2n_r)$ for $r \geq 1$, as soon as $\sum_i 2n_i \geq 8$ and $2n_j \geq 4$ for at least one $j \in \{1,\dots,r\}$.

We do an analogous construction as above starting from $S_{2,0,0}$ (see Figure 3) and $M \in \mathcal{H}(0,2^r)$, with $\gamma_1 \in S_{2,0,0}$ joining the two marked points and $\gamma_2 \in M$ joining the marked point to itself. We obtain a local but nonglobal maximum in $\mathcal{H}(2^{r+2})$. For $r \geq 2$ we can choose the spin structure of M and thus get S in any nonhyperelliptic component of $\mathcal{H}(2^{r+2})$. Note that for $r = 1$ we get $S \in \mathcal{H}(2,2,2)$ with odd spin structure.

There remain the following cases:

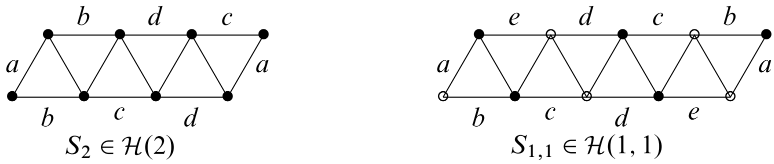


Figure 4: Global maxima in $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$.

- $\boldsymbol{\mathcal{H}(6)}$ We do the same construction as above, starting from $S_{2,0}$ and $M \in \mathcal{H}(2)$. We consider for $M \in \mathcal{H}(2)$ the surface S_2 in Figure 4. We see that $[a]$ and $[b]$ in this figure have different indices mod 2. Hence choosing $\gamma_2 = a$ or $\gamma_2 = b$ gives surfaces with different Spin structure; see (4).
- $\boldsymbol{\mathcal{H}(4,2)}$ We do the same as for $\mathcal{H}(6)$, starting from $S_{2,0,0}$ and $M = S_2$.
- **The even component of** $\boldsymbol{\mathcal{H}(2,2,2)}$ We do the same construction but starting from $S_{2,0,0}$ and $M \in \mathcal{H}(1,1)$, the surface $S_{1,1}$ in Figure 4. We consider $\gamma_2 = a$ (joining the two singularities of degree 1). By a direct computation, the above construction gives a surface $S \in \mathcal{H}(2,2,2)$ with $\mathrm{Spin}(S) = 0 \mod 2$. □

References

[1] **C Boissy**, **S Geninska**, *Systoles in translation surfaces*, Bull. Soc. Math. France 149 (2021) 417–438 MR Zbl

[2] **A Eskin**, **M Kontsevich**, **A Zorich**, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*, Publ. Math. Inst. Hautes Études Sci. 120 (2014) 207–333 MR Zbl

[3] **A Eskin**, **H Masur**, **A Zorich**, *Moduli spaces of abelian differentials: the principal boundary, counting problems, and the Siegel–Veech constants*, Publ. Math. Inst. Hautes Études Sci. 97 (2003) 61–179 MR Zbl

[4] **C Judge**, **H Parlier**, *The maximum number of systoles for genus two Riemann surfaces with abelian differentials*, Comment. Math. Helv. 94 (2019) 399–437 MR Zbl

[5] **S Kerckhoff**, **H Masur**, **J Smillie**, *Ergodicity of billiard flows and quadratic differentials*, Ann. of Math. 124 (1986) 293–311 MR Zbl

[6] **M Kontsevich**, **A Zorich**, *Connected components of the moduli spaces of abelian differentials with prescribed singularities*, Invent. Math. 153 (2003) 631–678 MR Zbl

[7] **H Masur**, *Interval exchange transformations and measured foliations*, Ann. of Math. 115 (1982) 169–200 MR Zbl

[8] **W A Veech**, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math. 115 (1982) 201–242 MR Zbl

[9] **W A Veech**, *Moduli spaces of quadratic differentials*, J. Anal. Math. 55 (1990) 117–171 MR Zbl

Institut Mathématique de Toulouse, Université Paul Sabatier
Toulouse, France
Institut Mathématique de Toulouse, Université Paul Sabatier
Toulouse, France

corentin.boissy@math.univ-toulouse.fr, geninska@math.univ-toulouse.fr

Received: 7 June 2021 Revised: 12 December 2022

Algebraic & Geometric Topology **24**:4 (2024) 1913–1951
DOI: 10.2140/agt.2024.24.1913
Published: 16 July 2024

Smooth singular complexes and diffeological principal bundles

HIROSHI KIHARA

In previous papers, we used the standard simplices Δ^p $(p \geq 0)$ endowed with diffeologies having several "good" properties to introduce the singular complex $S^{\mathcal{D}}(X)$ of a diffeological space X. (Here, $\mathcal{D}$ denotes the category of diffeological spaces.) On the other hand, Hector and Christensen–Wu used the standard simplices Δ^p_{sub} $(p \geq 0)$ endowed with the subdiffeology of $\mathbb{R}^{p+1}$ and the standard affine p–spaces $\mathbb{A}^p$ $(p \geq 0)$ to introduce the singular complexes $S^{\mathcal{D}}_{\text{sub}}(X)$ and $S^{\mathcal{D}}_{\text{aff}}(X)$, respectively, of a diffeological space X. We prove that $S^{\mathcal{D}}(X)$ is a fibrant approximation of both $S^{\mathcal{D}}_{\text{sub}}(X)$ and $S^{\mathcal{D}}_{\text{aff}}(X)$. This result immediately implies that the homotopy groups of $S^{\mathcal{D}}_{\text{sub}}(X)$ and $S^{\mathcal{D}}_{\text{aff}}(X)$ are isomorphic to the smooth homotopy groups of X, which enables us to give a positive answer to a conjecture of Christensen and Wu. Further, we characterize diffeological principal bundles (ie principal bundles in the sense of Iglesias-Zemmour) using the singular functor $S^{\mathcal{D}}_{\text{aff}}$. By using these results, we extend the characteristic classes for $\mathcal{D}$–numerable principal bundles to those for diffeological principal bundles.

58A40; 18F15, 55U10

1 Introduction

Let $\mathcal{D}$ denote the category of diffeological spaces. In [Kihara 2019], we constructed diffeologies on $\Delta^p = \{(x_0, \ldots, x_p) \in \mathbb{R}^{p+1} \mid \sum x_i = 1,\ x_i \geq 0 \text{ for any } i\}$ $(p \geq 0)$. We called them *good* because they allowed us to define the singular complex $S^{\mathcal{D}}(X)$ of a diffeological space X, which enables us to introduce a model structure on the category $\mathcal{D}$ (see Section 2.2). Further, in [Kihara 2023], we also used the singular functor $S^{\mathcal{D}}$ to introduce a simplicial category structure on $\mathcal{D}$, and developed a smooth homotopy theory based on the simplicial and model category structures on $\mathcal{D}$.

On the other hand, Hector [1995] used the sets Δ^p endowed with the subdiffeology of $\mathbb{R}^{p+1}$ $(p \geq 0)$ to define the singular complex $S^{\mathcal{D}}_{\text{sub}}(X)$ of a diffeological space X. His singular complex is also used in [Kuribayashi 2020]. Christensen and Wu [2014] also used the affine spaces

$$\mathbb{A}^p = \Big\{(x_0, \ldots, x_p) \in \mathbb{R}^{p+1} \;\Big|\; \sum x_i = 1\Big\}$$

endowed with the subdiffeology of $\mathbb{R}^{p+1}$ $(p \geq 0)$ to define the singular complex $S^{\mathcal{D}}_{\text{aff}}(X)$ in an attempt to construct a model structure on $\mathcal{D}$. Their singular complex is also used in [Bunk 2022; Kuribayashi 2020; 2021].

As is described in the references cited above, the singular complexes $S^{\mathcal{D}}(X)$, $S^{\mathcal{D}}_{\text{sub}}(X)$, and $S^{\mathcal{D}}_{\text{aff}}(X)$ are playing crucial roles in the smooth homotopical study of diffeological spaces. However, the natural weak equivalences between them have not yet been established.

In this paper, we show that the singular complexes $S^{\mathcal{D}}(X)$, $S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ are weakly equivalent (Theorem 1.1). As a corollary of this result, we identify the homotopy groups of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ and $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ with the smooth homotopy groups of X, proving a conjecture of Christensen and Wu (Corollary 1.2). Though we mainly use the singular functor $S^{\mathcal{D}}$, we also use the singular functor $S^{\mathcal{D}}_{\mathrm{aff}}$ to characterize diffeological principal bundles (ie principal bundles in the sense of Iglesias-Zemmour) (Theorem 1.3). This theorem, along with the weak equivalence between $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ and $S^{\mathcal{D}}(X)$, is used to extend the characteristic classes for $\mathcal{D}$–numerable principal G–bundles to those for diffeological principal G–bundles (Proposition 1.4).

Throughout this paper, $\mathcal{D}$ and $\mathcal{S}$ denote the category of diffeological spaces and the category of simplicial sets, respectively. (See [Goerss and Jardine 1999; May 1992; Kihara 2014] for the basics of simplicial homotopy theory.)

Weak equivalences between $S^{\mathcal{D}}(X)$, $S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $S^{\mathcal{D}}_{\mathrm{aff}}(X)$

The following theorem is the main result of this paper. Note that the canonical maps $\Delta^p \xrightarrow{\mathrm{id}} \Delta^p_{\mathrm{sub}} \hookrightarrow \mathbb{A}^p$ ($p \geq 0$) induce natural morphisms of simplicial sets $S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathcal{D}}(X)$ (see Lemma 3.1(3) and Proposition 3.4); note that the first and second canonical maps induce the second and first morphisms of singular complexes, respectively. Recall that $S^{\mathcal{D}}(X)$ is always Kan (ie fibrant in the category $\mathcal{S}$); see Corollary 2.6(1) (cf Remark 3.2(2)).

Theorem 1.1 *The natural morphisms of simplicial sets*

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathcal{D}}(X)$$

are weak equivalences. In particular, $S^{\mathcal{D}}(X)$ is a fibrant approximation of both $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ and $S^{\mathcal{D}}_{\mathrm{sub}}(X)$.

That $S^{\mathcal{D}}(X)$ is a fibrant approximation of $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ was announced in [Kihara 2019, Remark A.5].

Next we recall that $\pi_i(S^{\mathcal{D}}(X), x)$ is isomorphic to the smooth homotopy group $\pi_i^{\mathcal{D}}(X, x)$ (Theorem 2.7), and use Theorem 1.1 to identify the homotopy groups of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ and $S^{\mathcal{D}}_{\mathrm{sub}}(X)$; see Section 4.4 for the homotopy groups of a simplicial set which need not satisfy the Kan condition.

Corollary 1.2 *Let (X, x) be a pointed diffeological space. Then both $\pi_i(S^{\mathcal{D}}_{\mathrm{aff}}(X), x)$ and $\pi_i(S^{\mathcal{D}}_{\mathrm{sub}}(X), x)$ are naturally isomorphic to the smooth homotopy group $\pi_i^{\mathcal{D}}(X, x)$ for $i \geq 0$.*

Christensen and Wu [2014, Theorem 4.11] showed that if $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ is fibrant, then $\pi_i(S^{\mathcal{D}}_{\mathrm{aff}}(X), x)$ is isomorphic to the smooth homotopy group $\pi_i^{\mathcal{D}}(X, x)$ for $i \geq 0$, and conjectured that for every diffeological space X, $\pi_i(S^{\mathcal{D}}_{\mathrm{aff}}(X), x)$ is isomorphic to $\pi_i^{\mathcal{D}}(X, x)$ for $i \geq 0$ [Christensen and Wu 2014, page 1272]. Corollary 1.2 contains their conjecture.

(Co)homology of diffeological spaces Following [Kihara 2023, Section 3.1], we define the homology $H_*(X; A)$ and the cohomology $H^*(X; A)$ of a diffeological space X with coefficients in an abelian group A by

$$H_*(X; A) = H_*(\mathbb{Z}S^{\mathcal{D}}(X) \otimes A), \quad H^*(X; A) = H^*\big(\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}(X), A)\big),$$

where the simplicial abelian group $\mathbb{Z}K$ freely generated by a simplicial set K is regarded as a chain complex by setting $\partial = \sum(-1)^i d_i$. It follows from Theorem 1.1 that the (co)homology of X is naturally isomorphic to the (co)homologies defined using $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ and $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ instead of $S^{\mathcal{D}}(X)$. However, this fact is actually proved in Section 3.2 as a key to proving Theorem 1.1; the (co)homology of X is also naturally isomorphic to the cubic (co)homology introduced in [Iglesias-Zemmour 2013, pages 176–186] (Remark 3.6).

Application to diffeological principal bundles

Let G be a diffeological group. A $\mathcal{D}$–numerable principal G–bundle $\pi: P \to X$ is a principal G–bundle which admits a trivialization open cover $\{U_i\}$ of X and a smooth partition of unity subordinate to it. On the other hand, Iglesias-Zemmour introduced a weaker notion of a principal G–bundle; such a principal G–bundle, referred to as a diffeological principal G–bundle, is defined by local triviality of the pullback along any plot (Definition 5.1(2)).

Though we mainly use the singular complexes $S^{\mathcal{D}}(X)$ in smooth homotopy theory, the singular complexes $S^{\mathcal{D}}_{\mathrm{aff}}(X)$, along with Theorem 1.1 play an essential role in the study of diffeological principal bundles, as explained below.

Characterization of diffeological principal G–bundles Let $\mathscr{C}$ be a category with finite products, and G a group in $\mathscr{C}$. Then $\mathscr{C}G$ denotes the category of right G–objects of $\mathscr{C}$ (ie objects of $\mathscr{C}$ endowed with a right G–action). For $B \in \mathscr{C}$, $\mathscr{C}G/B$ denotes the category of objects of $\mathscr{C}G$ over B, where B is regarded as an object of $\mathscr{C}G$ with trivial G–action.

Since $S^{\mathcal{D}}_{\mathrm{aff}}: \mathcal{D} \to \mathcal{S}$ is a right adjoint (Remark 3.2(1)), $S^{\mathcal{D}}_{\mathrm{aff}}$ induces the functor $\mathcal{D}G/X$ to $\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)/S^{\mathcal{D}}_{\mathrm{aff}}(X)$. We then have the following characterization theorem for diffeological principal G–bundles (the notion of a simplicial principal bundle is introduced in Definition 5.3).

Theorem 1.3 (1) *Let $\pi: P \to X$ be an object of $\mathcal{D}G/X$. Then $\pi: P \to X$ is a diffeological principal G–bundle if and only if*

$$S^{\mathcal{D}}_{\mathrm{aff}}(\pi): S^{\mathcal{D}}_{\mathrm{aff}}(P) \to S^{\mathcal{D}}_{\mathrm{aff}}(X)$$

is a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle.

(2) *The functor $S^{\mathcal{D}}_{\mathrm{aff}}: \mathcal{D} \to \mathcal{S}$ induces a faithful functor from the category $\mathsf{P}\mathcal{D}G_{\mathrm{diff}}$ of diffeological principal G–bundles to the category $\mathsf{P}\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)$ of principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundles.*

The essential reason why $S^{\mathcal{D}}_{\mathrm{aff}}$ is useful in the study of diffeological principal G–bundles is because $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ can be regarded as the set of global plots of X. We can use Theorem 1.3 to calculate the (co)homology of exceptional diffeological spaces such as irrational tori and $\mathbb{R}/\mathbb{Q}$ (see Section 2.3 and Example 6.7); other cohomology theories of irrational tori were calculated by Iglesias-Zemmour and Kuribayashi (see Remark 6.8).

Characteristic classes of diffeological principal G–bundles We apply Theorem 1.3 to construct characteristic classes for diffeological principal G–bundles.

A *characteristic class* for a class $\mathcal{P}$ of smooth principal G–bundles is a rule assigning to a principal G–bundle $\pi : P \to X$ in $\mathcal{P}$ a cohomology class $\alpha(P)$ of X such that $\alpha(f^*P) = f^*\alpha(P)$. Christensen and Wu [2021, Theorem 5.10] constructed the universal $\mathcal{D}$–numerable principal G–bundle $\pi_G : EG \to BG$ and proved that the set of isomorphism classes of $\mathcal{D}$–numerable principal G–bundles over X bijectively corresponds to the smooth homotopy set $[X, BG]_{\mathcal{D}}$. Thus, a cohomology class $\alpha \in H^k(BG; A)$ defines the characteristic class $\alpha(\cdot)$ for the class of $\mathcal{D}$–numerable principal G–bundles. More precisely, the characteristic class $\alpha(P) \in H^k(X; A)$ of a $\mathcal{D}$–numerable principal G–bundle $\pi : P \to X$ is defined by

$$\alpha(P) = f_P^*\alpha,$$

where $f_P : X \to BG$ is a classifying map of P.

We would like to extend the characteristic class $\alpha(\cdot)$ to the class of diffeological principal G–bundles. Since pullbacks of EG are necessarily $\mathcal{D}$–numerable, the above definition of the characteristic class $\alpha(\cdot)$ does not apply to the class of diffeological principal G–bundles. Further, since the class of diffeological principal G–bundles does not have the homotopy invariance property with respect to pullback, it has no classifying space; see [Christensen and Wu 2021, Section 3].

Nevertheless, we can prove the following result.

Proposition 1.4 *Let G be a diffeological group and α an element of $H^k(BG; A)$. Then the characteristic class $\alpha(\cdot)$ for $\mathcal{D}$–numerable principal G–bundles extends to a characteristic class for diffeological principal G–bundles.*

This paper is organized as follows. In Section 2, we recall the basic notions and results on diffeological spaces and the singular functor $S^{\mathcal{D}}$. In Section 3, we briefly review the singular functors $S^{\mathcal{D}}_{\mathrm{sub}}$ and $S^{\mathcal{D}}_{\mathrm{aff}}$, and show that there exist natural morphisms between $S^{\mathcal{D}}_{\mathrm{aff}}(X)$, $S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $S^{\mathcal{D}}(X)$ which induce isomorphisms on (co)homology. We prove Theorem 1.1 and Corollary 1.2 in Section 4. In Section 5, we recall the notions of a diffeological principal bundle and a simplicial principal bundle, and prove Theorem 1.3. In Section 6, we prove Proposition 1.4 and discuss the sets of characteristic classes for the three classes $\mathsf{P}\mathcal{D}G$, $\mathsf{P}\mathcal{D}G_{\mathrm{num}}$, and $\mathsf{P}\mathcal{D}G_{\mathrm{diff}}$ of smooth principal G–bundles (see Definition 5.1(3) for these three classes).

2 Diffeological spaces

In this section, we first recall the convenient properties of the category $\mathcal{D}$ of diffeological spaces, along with the adjoint pair $\tilde{\cdot} : \mathcal{D} \rightleftarrows \mathcal{C}^0 : R$ of the underlying topological space functor and its right adjoint (Section 2.1). Then we recall the standard simplices Δ^p $(p \geq 0)$ and the adjoint pair $|\cdot|_{\mathcal{D}} : \mathcal{S} \rightleftarrows \mathcal{D} : S^{\mathcal{D}}$ of the realization and singular functors (see Section 2.2). Last, we make a brief review of some results of [Kihara 2023], in which the adjoint pairs $(\tilde{\cdot}, R)$ and $(|\cdot|_{\mathcal{D}}, S^{\mathcal{D}})$ play an essential role (Section 2.3).

2.1 Categories $\mathscr{D}$ and $\mathscr{C}^0$

In this subsection, we summarize the convenient properties of the category $\mathscr{D}$ of diffeological spaces, recalling the adjoint pair $\tilde{\cdot} : \mathscr{D} \rightleftarrows \mathscr{C}^0 : R$ of the underlying topological space functor and its right adjoint; see [Iglesias-Zemmour 2013; Kihara 2019] for full details.

Let us begin with the definition of a diffeological space. A *parametrization* of a set X is a (set-theoretic) map $p\colon U \to X$, where U is an open subset of $\mathbb{R}^n$ for some n.

Definition 2.1 (1) A *diffeological space* is a set X together with a specified set D_X of parametrizations of X satisfying the following conditions:

- (i) **Covering** Every constant parametrization $p\colon U \to X$ is in D_X.
- (ii) **Locality** Let $p\colon U \to X$ be a parametrization such that there exists an open cover $\{U_i\}$ of U satisfying $p|_{U_i} \in D_X$. Then p is in D_X.
- (iii) **Smooth compatibility** Let $p\colon U \to X$ be in D_X. Then for every $n \geq 0$, every open set V of $\mathbb{R}^n$, and every smooth map $F\colon V \to U$, $p \circ F$ is in D_X.

The set D_X is called the *diffeology* of X, and its elements are called *plots*.

(2) Let $X = (X, D_X)$ and $Y = (Y, D_Y)$ be diffeological spaces, and let $f : X \to Y$ be a (set-theoretic) map. We say that f is *smooth* if $f \circ p \in D_Y$ for every $p \in D_X$.

The convenient properties of $\mathscr{D}$ are summarized in the following proposition. Recall that a topological space X is called *arc-generated* if its topology is final for the continuous curves from $\mathbb{R}$ to X, and let $\mathscr{C}^0$ denote the category of arc-generated spaces and continuous maps. See [Frölicher and Kriegl 1988, pages 230–233] for initial and final structures with respect to the underlying set functor.

Proposition 2.2 (1) *The category $\mathscr{D}$ has initial and final structures with respect to the underlying set functor. In particular, $\mathscr{D}$ is complete and cocomplete.*

(2) *The category $\mathscr{D}$ is cartesian closed.*

(3) *The underlying set functor $\mathscr{D} \to$ Set is factored as the underlying topological space functor $\tilde{\cdot} : \mathscr{D} \to \mathscr{C}^0$ followed by the underlying set functor $\mathscr{C}^0 \to$ Set. Further, the functor $\tilde{\cdot} : \mathscr{D} \to \mathscr{C}^0$ has a right adjoint $R\colon \mathscr{C}^0 \to \mathscr{D}$.*

Proof See [Christensen et al. 2014, page 90; Iglesias-Zemmour 2013, pages 35–36; Kihara 2019, Propositions 2.1 and 2.10]. □

The following remark relates to Proposition 2.2.

Remark 2.3 (1) Let $\mathfrak{X}$ be a concrete category (ie a category equipped with a faithful functor to Set); the faithful functor $\mathfrak{X} \to$ Set is called the underlying set functor. See [Frölicher and Kriegl 1988, Section 8.8] for the notions of an $\mathfrak{X}$–embedding, an $\mathfrak{X}$–subspace, an $\mathfrak{X}$–quotient map, and an

$\mathfrak{X}$–quotient space. $\mathfrak{D}$–subspaces and $\mathfrak{D}$–quotient spaces are usually called diffeological subspaces and diffeological quotient spaces, respectively.

(2) For Proposition 2.2(3), recall that the underlying topological space $\widetilde{A}$ of a diffeological space $A = (A, D_A)$ is defined to be the set A endowed with the final topology for D_A [Iglesias-Zemmour 2013, 2.8] and that R assigns to an arc-generated space X the set X endowed with the diffeology

$$D_{RX} = \{\text{continuous parametrizations in } X\}.$$

Then we can easily see that $\tilde{\cdot} \circ R = \mathrm{Id}_{\mathscr{C}^0}$ and that the unit $A \to R\widetilde{A}$ of the adjoint pair $(\tilde{\cdot}, R)$ is set-theoretically the identity map.

(3) The notion of an arc-generated space is equivalent to that of a Δ–generated space (see [Christensen et al. 2014; Kihara 2019, Section 2.2]). The categories $\mathfrak{D}$ and $\mathscr{C}^0$ share convenient properties (1) and (2) in Proposition 2.2, which often enables us to deal with $\mathfrak{D}$ and $\mathscr{C}^0$ simultaneously (see [Kihara 2023]). See [Kihara 2023, Remark 2.4] for the reason why $\mathscr{C}^0$ is the most suitable category as a target category of the underlying topological space functor for diffeological spaces.

2.2 Standard simplices Δ^p

In this subsection, we recall the standard simplices Δ^p $(p \geq 0)$, along with the adjoint pair $|\cdot|_{\mathfrak{D}} : \mathcal{S} \rightleftarrows \mathfrak{D} : S^{\mathfrak{D}}$ of the realization and singular functors.

In [Kihara 2019], we introduced a model structure on the category $\mathfrak{D}$. The principal part of our construction of a model structure on $\mathfrak{D}$ is the construction of so-called good diffeologies on the sets

$$\Delta^p = \left\{(x_0, \dots, x_p) \in \mathbb{R}^{p+1} \;\middle|\; \sum x_i = 1,\, x_i \geq 0 \text{ for any } i\right\} \quad (p \geq 0)$$

which enable us to define weak equivalences, fibrations, and cofibrations and to verify the model axioms (see Remark 2.8). The required properties of the diffeologies on Δ^p $(p \geq 0)$ are expressed in the following four axioms:

Axiom 1 *The underlying topological space of Δ^p is the topological standard p–simplex Δ^p_{top} for $p \geq 0$.*

Recall that $f : \Delta^p \to \Delta^q$ is an *affine map* if f preserves convex combinations.

Axiom 2 *Any affine map $f : \Delta^p \to \Delta^q$ is smooth.*

For $K \in \mathcal{S}$, the *simplex category* $\Delta \downarrow K$ is defined to be the full subcategory of the overcategory $\mathcal{S} \downarrow K$ consisting of maps $\sigma : \Delta[n] \to K$. By Axiom 2, we can consider the diagram $\Delta \downarrow K \to \mathfrak{D}$ sending $\sigma : \Delta[n] \to K$ to Δ^n. Thus, we define the *realization functor*

$$|\cdot|_{\mathfrak{D}} : \mathcal{S} \to \mathfrak{D}$$

by $|K|_{\mathfrak{D}} = \mathrm{colim}_{\Delta \downarrow K}\, \Delta^n$.

Consider the smooth map $|\dot{\Delta}[p]|_{\mathfrak{D}} \hookrightarrow |\Delta[p]|_{\mathfrak{D}} = \Delta^p$ induced by the inclusion of the boundary $\dot{\Delta}[p]$ into $\Delta[p]$.

Axiom 3 *The canonical smooth injection*

$$|\dot{\Delta}[p]|_{\mathcal{D}} \hookrightarrow \Delta^p$$

is a $\mathcal{D}$*–embedding.*

The $\mathcal{D}$–homotopical notions, especially the notion of a $\mathcal{D}$–deformation retract, are defined in the same manner as in the category of topological spaces by using the unit interval $I = [0,1]$ endowed with a diffeology via the canonical bijection with Δ^1 [Kihara 2019, Section 2.4]. The k^{th} *horn* of Δ^p is a diffeological subspace of Δ^p defined by

$$\Lambda^p_k = \{(x_0, \dots, x_p) \in \Delta^p \mid x_i = 0 \text{ for some } i \neq k\}.$$

Axiom 4 *The* k^{th} *horn* Λ^p_k *is a* $\mathcal{D}$*–deformation retract of* Δ^p *for* $p \geq 1$ *and* $0 \leq k \leq p$*.*

For a subset A of the affine p–space $\mathbb{A}^p = \{(x_0, \dots, x_p) \in \mathbb{R}^{p+1} \mid \sum x_i = 1\}$, A_{sub} denotes the set A endowed with the subdiffeology of $\mathbb{A}^p$ (and hence of $\mathbb{R}^{p+1}$). The diffeological spaces Δ^p_{sub} $(p \geq 0)$ satisfy Axioms 1 and 2, but Δ^p_{sub} satisfies neither Axiom 3 nor 4 for $p \geq 2$ [Kihara 2019, Proposition A.2]. Thus, we must construct a new diffeology on Δ^p, at least for $p \geq 2$.

Let (i) denote the vertex $(0, \dots, 1_{(i)}, \dots, 0)$ of Δ^p, and let d^i denote the affine map from Δ^{p-1} to Δ^p, defined by

$$d^i((k)) = \begin{cases} (k) & \text{if } k < i, \\ (k+1) & \text{if } k \geq i. \end{cases}$$

Definition 2.4 We define the *standard p–simplices* Δ^p $(p \geq 0)$ inductively. Set $\Delta^p = \Delta^p_{\mathrm{sub}}$ for $p \leq 1$. Suppose that the diffeologies on Δ^k $(k < p)$ are defined. We define the map

$$\varphi_i : \Delta^{p-1} \times [0,1) \to \Delta^p$$

by $\varphi_i(x,t) = (1-t)(i) + t d^i(x)$, and endow Δ^p with the final structure for the maps $\varphi_0, \dots, \varphi_p$.

The following result is established in [Kihara 2019, Propositions 3.2, 5.1, 7.1, and 8.1].

Proposition 2.5 *The standard* p*–simplices* Δ^p $(p \geq 0)$ *in Definition 2.4 satisfy Axioms 1–4.*

Without explicit mention, the symbol Δ^p denotes the standard p–simplex defined in Definition 2.4 and a subset of Δ^p is endowed with the subdiffeology of Δ^p. Since the diffeology of Δ^p is the subdiffeology of $\mathbb{A}^p$ for $p \leq 1$, the $\mathcal{D}$–homotopical notions, especially the notion of a $\mathcal{D}$–deformation retract, coincide with the ordinary smooth homotopical notions in the theory of diffeological spaces [Iglesias-Zemmour 2013, page 108; Kihara 2019, Remark 2.14].

Since $\Delta^\bullet = \{\Delta^p\}$ is a cosimplicial diffeological space by Axiom 2, the singular complex $S^{\mathcal{D}}(X)$ is defined by

$$S^{\mathcal{D}}(X) = \mathcal{D}(\Delta^\bullet, X).$$

We can easily see that $|\cdot|_{\mathscr{D}} : \mathscr{S} \rightleftarrows \mathscr{D} : S^{\mathscr{D}}$ is an adjoint pair [Kihara 2019, Proposition 9.1]. Further, we can derive the following result from Proposition 2.5.

Corollary 2.6 (1) *The natural isomorphisms*

$$|\Delta[p]|_{\mathscr{D}} = \Delta^p, \quad |\dot{\Delta}[p]| = \dot{\Delta}^p \quad \text{and} \quad |\Lambda_k[p]|_{\mathscr{D}} = \Lambda^p_k$$

exist.

(2) *$S^{\mathscr{D}}X$ is a Kan complex for any diffeological space X.*

Proof (1) See [Kihara 2019, Proposition 9.2].

(2) See [Kihara 2019, Lemma 9.4(1)]. □

See [Christensen and Wu 2014, Section 3.1] or [Iglesias-Zemmour 2013, Chapter 5] for the smooth homotopy groups $\pi^{\mathscr{D}}_p(X,x)$ of a pointed diffeological space (X,x). Note that $S^{\mathscr{D}}X$ is always a Kan complex (Corollary 2.6(2)) and see [Goerss and Jardine 1999, page 25] for the homotopy groups $\pi_p(K,x)$ of a pointed Kan complex (K,x).

Theorem 2.7 *Let (X,x) be a pointed diffeological space. Then there exists a natural bijection*

$$\Theta_X : \pi^{\mathscr{D}}_p(X,x) \to \pi_p(S^{\mathscr{D}}X,x) \quad \text{for } p \geq 0,$$

that is an isomorphism of groups for $p > 0$.

Proof See [Kihara 2019, Theorem 1.4]. □

Remark 2.8 (1) Define a map $f : X \to Y$ in $\mathscr{D}$ to be

(i) a *weak equivalence* if $S^{\mathscr{D}}f : S^{\mathscr{D}}X \to S^{\mathscr{D}}Y$ is a weak equivalence in the category of simplicial sets,

(ii) a *fibration* if the map f has the right lifting property with respect to the inclusions $\Lambda^p_k \hookrightarrow \Delta^p$ for all $p > 0$ and $0 \leq k \leq p$, and

(iii) a *cofibration* if the map f has the left lifting property with respect to all maps that are both fibrations and weak equivalences.

Then $\mathscr{D}$ is a compactly generated model category whose object is always fibrant. In fact, the sets of morphisms of $\mathscr{D}$,

$$\mathscr{I} = \{\dot{\Delta}^p \hookrightarrow \Delta^p \mid p \geq 0\},$$
$$\mathscr{J} = \{\Lambda^p_k \hookrightarrow \Delta^p \mid p > 0,\ 0 \leq k \leq p\},$$

are the sets of generating cofibrations and generating trivial cofibrations, respectively [Kihara 2019, Theorem 1.3]. See [May and Ponto 2012, Definition 15.2.1] for a compactly generated model category.

By Theorem 2.7, weak equivalences in $\mathscr{D}$ are just smooth maps inducing isomorphisms on smooth homotopy groups.

(2) The adjoint pairs

$$|\cdot|_{\mathscr{D}} : \mathscr{S} \rightleftarrows \mathscr{D} : S^{\mathscr{D}} \quad \text{and} \quad \tilde{\cdot} : \mathscr{D} \rightleftarrows \mathscr{C}^0 : R$$

are pairs of Quillen equivalences [Kihara 2023, Theorem 1.5]. Note that the composite of these adjoint pairs is just the adjoint pair

$$|\cdot| : \mathscr{S} \rightleftarrows \mathscr{C}^0 : S$$

of the topological realization and singular functors.

2.3 Homotopy type of $S^{\mathscr{D}}(X)$

In this subsection, we recall from [Kihara 2023] the basic results on the homotopy type of $S^{\mathscr{D}}(X)$; they are not essential in the later sections, but they are related to a few results in Section 6.

For a diffeological space X, consider the unit id: $X \to R\tilde{X}$ of the adjoint pair $\tilde{\cdot} : \mathscr{D} \rightleftarrows \mathscr{C}^0 : R$. By applying $S^{\mathscr{D}} (= \mathscr{D}(\Delta^{\bullet}, \cdot))$, we have the natural inclusion

$$S^{\mathscr{D}}(X) \hookrightarrow S(\tilde{X})$$

(see Proposition 2.5, in particular Axiom 1).

If X is a nice diffeological space such as a cofibrant object or a C^{∞}–manifold in the sense of [Kriegl and Michor 1997, Section 27], then $S^{\mathscr{D}}(X) \hookrightarrow S(\tilde{X})$ is a weak equivalence [Kihara 2023, Corollary 1.6, Proposition 2.6, and Theorem 11.2]. Hence, we can calculate the homotopy groups and the (co)homology groups of such nice diffeological spaces as those of the underlying topological spaces.

Conversely, if X is an exceptional diffeological space such as an irrational torus, then $S^{\mathscr{D}}(X) \hookrightarrow S(\tilde{X})$ is not a weak equivalence; see [Kihara 2023, Appendix A]. See Section 6.2 for an approach to the homotopy type of $S^{\mathscr{D}}(X)$ of exceptional diffeological spaces X such as irrational tori and $\mathbb{R}/\mathbb{Q}$.

Remark 2.9 The (co)homology and homotopy groups of diffeological spaces have the same desirable properties as those of topological spaces. Further, the (co)homology and homotopy groups of a diffeological space are just those of its singular complex. Thus, we can apply various algebraic topological and simplicial homotopical tools to the calculation of the (co)homology and homotopy groups of a diffeological space X whether or not X is a nice diffeological space; see [Kihara 2023, Section 3.1], Theorem 2.7, and Remark 5.8.

3 Smooth singular complexes

In this section, we summarize the basic notions and results on the smooth singular complexes $S^{\mathscr{D}}_{\mathrm{sub}}(X)$ and $S^{\mathscr{D}}_{\mathrm{aff}}(X)$ (Section 3.1), and then show that there exist natural morphisms between $S^{\mathscr{D}}_{\mathrm{aff}}(X)$, $S^{\mathscr{D}}_{\mathrm{sub}}(X)$, and $S^{\mathcal{D}}(X)$ which induce chain homotopy equivalences, and hence isomorphisms on (co)homology (Section 3.2). We also show that the singular functors $S^{\mathcal{D}}_{\mathrm{aff}}$, $S^{\mathcal{D}}_{\mathrm{sub}}$, and $S^{\mathcal{D}}$ transform diffeological coverings to simplicial coverings (Section 3.3); this result is used to reduce the proof of Theorem 1.1.

3.1 Smooth singular complexes $S^{\mathcal{D}}(X)$, $S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $S^{\mathcal{D}}_{\mathrm{aff}}(X)$

By using the cosimplicial diffeological space $\Delta^{\bullet} = \{\Delta^p\}$, the singular complex $S^{\mathcal{D}}(X)$ is defined by

$$S^{\mathcal{D}}(X) = \mathcal{D}(\Delta^{\bullet}, X),$$

which is intensively studied in [Kihara 2019; 2023] (see Section 2.2).

Let $\mathbb{A}^p$ denote the affine p–space $\{(x_0, \dots, x_p) \in \mathbb{R}^{p+1} \mid \sum x_i = 1\}$ endowed with the subdiffeology of $\mathbb{R}^{p+1}$. Since $\mathbb{A}^{\bullet} = \{\mathbb{A}^p\}$ is a cosimplicial diffeological space, the singular complex $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ is defined by

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) = \mathcal{D}(\mathbb{A}^{\bullet}, X).$$

The singular complex $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ was introduced by Christensen and Wu [2014]; they used the singular functor $S^{\mathcal{D}}_{\mathrm{aff}}$ to define the classes of weak equivalences, fibrations, and cofibrations in $\mathcal{D}$, but the model axioms are not yet verified.

Let Δ^p_{sub} denote the set Δ^p endowed with the subdiffeology of $\mathbb{A}^p$. Since $\Delta^{\bullet}_{\mathrm{sub}} = \{\Delta^p_{\mathrm{sub}}\}$ is a cosimplicial diffeological space, the singular complex $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ is defined by

$$S^{\mathcal{D}}_{\mathrm{sub}}(X) = \mathcal{D}(\Delta^{\bullet}_{\mathrm{sub}}, X).$$

The singular complex $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ was used by Hector [1995] to study diffeological spaces by homotopical means such as singular (co)homology.

Now, we summarize the basic properties of Δ^p, Δ^p_{sub}, and $\mathbb{A}^p$, and the relations among them, which are needed later. A subset A of $\mathbb{A}^p$ endowed with the subdiffeology of $\mathbb{A}^p$ is denoted by A_{sub}. The notion of $\mathcal{D}$–contractibility (or smooth contractibility) is defined in the obvious manner (a $\mathcal{D}$–contractible diffeological space is often called simply a contractible diffeological space if there is no confusion in context).

Lemma 3.1 (1) *The diffeological spaces Δ^p, Δ^p_{sub}, and $\mathbb{A}^p$ are smoothly contractible.*

(2) *The underlying topological space of Δ^p and Δ^p_{sub} is just the standard topological p–simplex. The underlying topological space of $\mathbb{A}^p$ is just the set $\mathbb{A}^p$ endowed with the usual topology.*

(3) *The map $\mathrm{id}\colon \Delta^p \to \Delta^p_{\mathrm{sub}}$ is smooth, which restricts to the diffeomorphism*

$$\mathrm{id}\colon \Delta^p - \mathrm{sk}_{p-2}\,\Delta^p \xrightarrow[\cong]{} (\Delta^p - \mathrm{sk}_{p-2}\,\Delta^p)_{\mathrm{sub}},$$

where $\mathrm{sk}_{p-2}\,\Delta^p$ denotes the $(p-2)$–skeleton of Δ^p.

Proof (1) The smooth contractibility of Δ^p_{sub} and $\mathbb{A}^p$ are obvious. See [Kihara 2019, Remark 9.3] for the smooth contractibility of Δ^p.

(2) The result for Δ^p follows from Proposition 2.5. The results for Δ^p_{sub} and $\mathbb{A}^p$ follow from [Kihara 2019, Lemma 2.12].

(3) See [Kihara 2019, Lemmas 3.1 and 4.2]. □

Remark 3.2 In this remark, we recall the left adjoints of $S^{\mathcal{D}}_{\mathrm{sub}}$ and $S^{\mathcal{D}}_{\mathrm{aff}}$, and see that $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ and $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ need not be Kan.

(1) As mentioned above, the realization functor $|\cdot|_{\mathcal{D}} : \mathcal{S} \to \mathcal{D}$ is a left adjoint of the singular functor $S^{\mathcal{D}} : \mathcal{D} \to \mathcal{S}$, and the composite of the adjoint pairs $|\cdot|_{\mathcal{D}} : \mathcal{S} \rightleftarrows \mathcal{D} : S^{\mathcal{D}}$ and $\tilde{\cdot} : \mathcal{D} \rightleftarrows \mathcal{C}^0 : R$ is just the adjoint pair $|\cdot| : \mathcal{S} \rightleftarrows \mathcal{C}^0 : S$ (see Remark 2.8(2)).

Similarly, we can define the realization functor $|\cdot|'_{\mathcal{D}} : \mathcal{S} \to \mathcal{D}$ by

$$|K|'_{\mathcal{D}} = \operatorname*{colim}_{\Delta \downarrow K} \Delta^n_{\mathrm{sub}},$$

which is a left adjoint of the singular functor $S^{\mathcal{D}}_{\mathrm{sub}} : \mathcal{D} \to \mathcal{S}$. The composite of the adjoint pairs $|\cdot|'_{\mathcal{D}} : \mathcal{S} \rightleftarrows \mathcal{D} : S^{\mathcal{D}}_{\mathrm{sub}}$ and $\tilde{\cdot} : \mathcal{D} \rightleftarrows \mathcal{C}^0 : R$ is also just the adjoint pair $|\cdot| : \mathcal{S} \rightleftarrows \mathcal{C}^0 : S$ (see Lemma 3.1(2)).

The realizations $|K|_{\mathcal{D}}$ and $|K|'_{\mathcal{D}}$ of a simplicial complex K viewed as a simplicial set [May 1992, Example 1.4] are just the diffeological polyhedra $|K|_{\mathcal{D}}$ and $|K|'_{\mathcal{D}}$ respectively [Kihara 2023, Section 8.1]; they played an essential role in the proof of the homotopy cofibrancy theorem [Kihara 2023, Theorem 1.10].

Christensen and Wu [2014] defined the realization functor $|\cdot|''_{\mathcal{D}} : \mathcal{S} \to \mathcal{D}$ by

$$|K|''_{\mathcal{D}} = \operatorname*{colim}_{\Delta \downarrow K} \mathbb{A}^n,$$

which is a left adjoint of the singular functor $S^{\mathcal{D}}_{\mathrm{aff}} : \mathcal{D} \to \mathcal{S}$.

(2) Let us see that $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ need not be Kan. For this, we consider the extension problem in $\mathcal{S}$

$$\begin{tikzcd} \Lambda_0[2] \arrow[r, "d^1+d^2"] \arrow[d, hook] & S^{\mathcal{D}}_{\mathrm{sub}}(\Lambda^2_{0\,\mathrm{sub}}) \\ \Delta[2] \arrow[ur, dashed] & \end{tikzcd}$$

where $\Lambda_0[2] \xrightarrow{d^1+d^2} S^{\mathcal{D}}_{\mathrm{sub}}(\Lambda^2_{0\,\mathrm{sub}})$ is the simplicial map whose restriction to the i^{th} face corresponds to (the corestriction of) $d^i : \Delta^1 \to \Delta^2$ for $i = 1, 2$. Suppose that this extension problem has a solution r. Then we have the commutative diagram in $\mathcal{D}$

$$\begin{tikzcd} {|\Lambda_0[2]|'_{\mathcal{D}}} \arrow[r, "d^1+d^2"] \arrow[d, hook] & \Lambda^2_{0\,\mathrm{sub}} \\ \Delta^2_{\mathrm{sub}} \arrow[ur, "r"'] & \end{tikzcd}$$

(see part (1)). Noticing that $|\Lambda_0[2]|'_{\mathcal{D}}$ can be set-theoretically identified with Λ^2_0, we see that r is a $\mathcal{D}$–retraction of Δ^2_{sub} onto $\Lambda^2_{0\,\mathrm{sub}}$, which is a contradiction [Kihara 2019, Proposition A.2(2)]; see also [Kihara 2023, Remark 8.2].

Similarly, we can use [Bröcker and Jänich 1982, Theorem 5.13] to see that $S^{\mathcal{D}}_{\mathrm{aff}}((d^1\mathbb{A}^1 \cup d^2\mathbb{A}^1)_{\mathrm{sub}})$ is not Kan; however, it has already been shown that $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ need not be Kan [Christensen and Wu 2014, Section 4.3].

3.2 Natural transformations between $S^{\mathcal{D}}_{\mathrm{aff}}$, $S^{\mathcal{D}}_{\mathrm{sub}}$, and $S^{\mathcal{D}}$

In this subsection, we construct natural morphisms between $S^{\mathcal{D}}_{\mathrm{aff}}(X)$, $S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $S^{\mathcal{D}}(X)$, and show that they induce chain homotopy equivalences between $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $\mathbb{Z}S^{\mathcal{D}}(X)$, and hence isomorphisms on the (co)homology with arbitrary coefficients.

First, we show that the singular functors $S^{\mathcal{D}}$, $S^{\mathcal{D}}_{\mathrm{sub}}$, and $S^{\mathcal{D}}_{\mathrm{aff}}$ preserve homotopy. Recall the $\mathcal{D}$–homotopical notions from Section 2.2 and let $\simeq_{\mathcal{D}}$ denote the $\mathcal{D}$–homotopy relation.

Lemma 3.3 *For smooth maps $f, g\colon X \to Y$, consider the conditions*

(i) $f \simeq_{\mathcal{D}} g\colon X \to Y$,

(ii) $S^{\mathcal{D}} f \simeq S^{\mathcal{D}} g\colon S^{\mathcal{D}}(X) \to S^{\mathcal{D}}(Y)$,

(iii) $H_*(f;\mathbb{Z}) = H_*(g;\mathbb{Z})\colon H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$.

The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) *hold. The same conclusion applies to the functors $S^{\mathcal{D}}_{\mathrm{sub}}$ and $S^{\mathcal{D}}_{\mathrm{aff}}$, and their homologies.*

Proof For $S^{\mathcal{D}}$: see [Kihara 2019, Lemma 9.4(2)] for (i) $\Longrightarrow$ (ii), and [May 1992, pages 12–13] for (ii) $\Longrightarrow$ (iii).

For $S^{\mathcal{D}}_{\mathrm{sub}}$: recall that $\Delta^1 = \Delta^1_{\mathrm{sub}}$; then a similar argument applies.

For $S^{\mathcal{D}}_{\mathrm{aff}}$: observe that $f \simeq_{\mathcal{D}} g$ if and only if there exists a smooth map $H\colon X \times \mathbb{A}^1 \to Y$ such that $H(\,\cdot\,,(0)) = f$ and $H(\,\cdot\,,(1)) = g$; then a similar argument applies. □

Using Lemmas 3.1 and 3.3, we can prove the following result.

Proposition 3.4 *There exist natural morphisms of simplicial sets*

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathcal{D}}(X)$$

which induce chain homotopy equivalences

$$\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X) \to \mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X) \to \mathbb{Z}S^{\mathcal{D}}(X).$$

Proof We prove the result in three steps.

Step 1: construction of natural morphisms By Lemma 3.1(3), we have the canonical morphisms of cosimplicial diffeological spaces

$$\Delta^\bullet \xrightarrow{\mathrm{id}} \Delta^\bullet_{\mathrm{sub}} \hookrightarrow \mathbb{A}^\bullet,$$

which induce natural morphisms

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) \xrightarrow{\kappa} S^{\mathcal{D}}_{\mathrm{sub}}(X) \xhookrightarrow{\iota} S^{\mathcal{D}}(X).$$

(Note that the first and second morphisms of cosimplicial diffeological spaces induce the second and first morphisms of singular complexes, respectively.)

Step 2 We show that for $p \geq 0$,

$$H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(\Delta^p)) \cong H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(\Delta^p)) \cong H_*(\mathbb{Z}S^{\mathcal{D}}(\Delta^p)) \cong \mathbb{Z}[0],$$

where $\mathbb{Z}[0]$ denotes the graded module with $\mathbb{Z}[0]_0 = \mathbb{Z}$ and $\mathbb{Z}[0]_i = 0$ $(i \neq 0)$. It is easily seen that these isomorphisms hold for $p = 0$. Thus, they hold for any $p \geq 0$ by Lemmas 3.3 and 3.1(1).

Step 3 To prove the rest of the statement, we "augment" the singular chain complexes $\mathbb{Z}S^{\mathcal{D}}(X)$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)$ in a canonical manner (see [Eilenberg and Mac Lane 1953, page 194]); the augmented singular chain complexes are denoted by $\mathbb{Z}S^{\mathcal{D}}(X)\tilde{}$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)\tilde{}$, and $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)\tilde{}$. Then

$$H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(\Delta^p)\tilde{}) = H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(\Delta^p)\tilde{}) = H_*(\mathbb{Z}S^{\mathcal{D}}(\Delta^p)\tilde{}) = 0$$

(by Step 2). Since each component of degree ≥ 0 of $\mathbb{Z}S^{\mathcal{D}}(X)\tilde{}$ (resp. $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)\tilde{}$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)\tilde{}$) is representable for the set of model objects $\{\Delta^p\}_{p\geq 0}$ (resp. $\{\Delta^p_{\mathrm{sub}}\}_{p\geq 0}$, $\{\mathbb{A}^p\}_{p\geq 0}$) in the sense of [Eilenberg and Mac Lane 1953, page 189], we can use [Eilenberg and Mac Lane 1953, Theorem II] to construct chain homotopy inverses of the augmented natural chain maps

$$\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)\tilde{} \xrightarrow{\mathbb{Z}\kappa\tilde{}} \mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)\tilde{} \xrightarrow{\mathbb{Z}\iota\tilde{}} \mathbb{Z}S^{\mathcal{D}}(X)\tilde{}$$

such that they restrict to chain homotopy inverses of the natural chain maps

$$\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X) \xrightarrow{\mathbb{Z}\kappa} \mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X) \xrightarrow{\mathbb{Z}\iota} \mathbb{Z}S^{\mathcal{D}}(X)$$

(see Step 1). □

Recall the definitions of $H_*(X;A)$ and $H^*(X;A)$ from Section 1.

Corollary 3.5 *Let A be an abelian group.*

(1) *The natural morphisms of simplicial sets*

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) \xrightarrow{\kappa} S^{\mathcal{D}}_{\mathrm{sub}}(X) \xhookrightarrow{\iota} S^{\mathcal{D}}(X)$$

induce isomorphisms of graded modules

$$H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)\otimes A) \xrightarrow[\cong]{\kappa_*} H_*(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)\otimes A) \xrightarrow[\cong]{\iota_*} H_*(X;A),$$
$$H^*\big(\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X),A)\big) \xleftarrow[\cong]{\kappa^*} H^*\big(\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X),A)\big) \xleftarrow[\cong]{\iota^*} H^*(X;A).$$

(2) *If A is a commutative associative ring with unit, then $H^*(X;A)$, $H^*\big(\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X), A)\big)$, and $H^*\big(\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X), A)\big)$ have natural commutative graded A–algebra structures and the isomorphisms between them are isomorphisms of graded A–algebras.*

Proof (1) The result is immediate from Proposition 3.4.

(2) See [Kihara 2023, Remark 3.8(2)] for $H^*(X;A)$. The argument there can also be applied to $H^*\,\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X), A)$ and $H^*\,\mathrm{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X), A)$. Since the cohomology isomorphisms in part (1) are induced by the natural simplicial maps

$$S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathcal{D}}(X),$$

they are isomorphisms of graded A–algebras. □

Remark 3.6 In the study of differential forms and de Rham cohomology of diffeological spaces, Iglesias-Zemmour [2013, pages 182–183] introduced the complex $\mathsf{C}_\star(X)$ of reduced groups of cubic chains for a diffeological space X, and called its homology $\mathsf{H}_*(X)$ the cubic homology of X.

We can easily see that $\mathsf{H}_*(X)$ is a smooth homotopy invariant. In fact, given a smooth homotopy $H\colon \mathbb{R}\times X \to Y$ between f and g, a chain homotopy $H_\sharp\colon \mathsf{C}_*(X) \to \mathsf{C}_{*+1}(Y)$ between $\mathsf{C}_*(f)$ and $\mathsf{C}_*(g)$ is defined by

$$\mathbb{R}^p \xrightarrow{\sigma} X \mapsto \mathbb{R}^{p+1} = \mathbb{R}\times\mathbb{R}^p \xrightarrow{1\times\sigma} \mathbb{R}\times X \xrightarrow{H} Y.$$

Thus, by an argument similar to that in the proof of Proposition 3.4, we can use [Eilenberg and Mac Lane 1953, Theorem II] to construct a natural chain homotopy equivalence between $\mathsf{C}_*(X)$ and $\mathbb{Z}S^{\mathcal{D}}(X)$, showing that $\mathsf{H}_*(X)$ is naturally isomorphic to $H_*(X)$.

The basic idea of the proof that $\mathbb{Z}S^{\mathcal{D}}(X)$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $\mathsf{C}_*(X)$ are chain homotopy equivalent was briefly discussed in [Kihara 2023, Remark 3.9]. It is also shown in [Kuribayashi 2020, Section 4.1] that $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(X)$, $\mathbb{Z}S^{\mathcal{D}}_{\mathrm{sub}}(X)$, and $\mathsf{C}_*(X)$ are chain homotopy equivalent.

3.3 Diffeological coverings

The notion of a diffeological fiber bundle is a generalization of that of a locally trivial fiber bundle, and is defined by local triviality of the pullback along any plot; see [Iglesias-Zemmour 2013, 8.9]. A diffeological fiber bundle with discrete fiber is called a *diffeological covering*.

Similarly, a simplicial fiber bundle is defined by triviality of the pullback along any map from $\Delta[p]$ $(p \ge 0)$; see [May 1992, Definition 11.8]. A simplicial fiber bundle with discrete fiber is called a *simplicial covering*.

We prove the following result, which is used in the proof of Theorem 1.1.

Proposition 3.7 *The singular functors $S^{\mathcal{D}}$, $S^{\mathcal{D}}_{\mathrm{sub}}$, and $S^{\mathcal{D}}_{\mathrm{aff}}$ transform diffeological coverings with fiber F to simplicial coverings with fiber F. Hence, a diffeological covering $\pi: E \to X$ with fiber F defines the natural morphisms of simplicial coverings with fiber F*

$$\begin{tikzcd}
S^{\mathcal{D}}_{\mathrm{aff}}(E) \arrow[r] \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi)"'] & S^{\mathcal{D}}_{\mathrm{sub}}(E) \arrow[r, hook] \arrow[d, "S^{\mathcal{D}}_{\mathrm{sub}}(\pi)"'] & S^{\mathcal{D}}(E) \arrow[d, "S^{\mathcal{D}}(\pi)"'] \\
S^{\mathcal{D}}_{\mathrm{aff}}(X) \arrow[r] & S^{\mathcal{D}}_{\mathrm{sub}}(X) \arrow[r, hook] & S^{\mathcal{D}}(X).
\end{tikzcd}$$

Proof We prove the result in three steps.

Step 1 We show that $S^{\mathcal{D}}(\pi)\colon S^{\mathcal{D}}(E) \to S^{\mathcal{D}}(X)$ is a simplicial covering with fiber F.

Assume given a map $k\colon \Delta[p] \to S^{\mathcal{D}}(X)$ and let $\kappa\colon \Delta^p \to X$ be the smooth map corresponding to k. Noticing that Δ^p is smoothly contractible (Lemma 3.1(1)), we then have a pullback diagram in $\mathcal{D}$

$$\begin{tikzcd}
\Delta^p \times F \arrow[r] \arrow[d, "\mathrm{proj}"'] & E \arrow[d, "\pi"] \\
\Delta^p \arrow[r, "\kappa"] & X
\end{tikzcd}$$

(see [Iglesias-Zemmour 2013, page 264]). Note that $S^{\mathcal{D}}$ is a right adjoint and consider the commutative diagram in $\mathcal{S}$ consisting of two pullback squares

$$\begin{tikzcd}
\Delta[p] \times S^{\mathcal{D}}(F) \arrow[r] \arrow[d, "\mathrm{proj}"'] & S^{\mathcal{D}}(\Delta^p) \times S^{\mathcal{D}}(F) \arrow[r] \arrow[d, "\mathrm{proj}"'] & S^{\mathcal{D}}(E) \arrow[d, "S^{\mathcal{D}}(\pi)"'] \\
\Delta[p] \arrow[r] & S^{\mathcal{D}}(\Delta^p) \arrow[r, "S^{\mathcal{D}}(\kappa)"] & S^{\mathcal{D}}(X)
\end{tikzcd}$$

where $\Delta[p] \to S^{\mathcal{D}}(\Delta^p)$ is the map corresponding to the p–simplex 1_{Δ^p} of $S^{\mathcal{D}}(\Delta^p)$. Then the outer rectangle gives the desired local triviality of $S^{\mathcal{D}}(\pi)$; see [Mac Lane 1998, Exercise 8 on page 72].

Step 2 Note that Δ^p_{sub} and $\mathbb{A}^p$ are smoothly contractible (Lemma 3.1(1)) and that $S^{\mathcal{D}}_{\mathrm{sub}}$ and $S^{\mathcal{D}}_{\mathrm{aff}}$ are right adjoints (Remark 3.2(1)). Then, by an argument similar to that in Step 1, we can see that $S^{\mathcal{D}}_{\mathrm{sub}}(\pi)$ and $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)$ are also simplicial coverings with fiber F.

Step 3 The natural morphisms of simplicial coverings are defined by Proposition 3.4. □

4 Weak equivalences between smooth singular complexes

In this section, we prove Theorem 1.1 and Corollary 1.2, using results of Section 3.

The main statement of Theorem 1.1 is divided into the following two parts:

(I) The natural map $S^{\mathcal{D}}_{\mathrm{sub}}(X) \hookrightarrow S^{\mathcal{D}}(X)$ is a weak equivalence in $\mathcal{S}$.

(II) The natural map $S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}(X)$ is a weak equivalence in $\mathcal{S}$.

After constructing a fibrant approximation functor for the category of simplicial sets in Section 4.1, we prove parts (I) and (II) in Sections 4.2 and 4.3, respectively. We complete the proofs of Theorem 1.1 and Corollary 1.2 in Section 4.4.

4.1 Fibrant approximation to a simplicial set

The category $\mathcal{S}$ of simplicial sets is a cofibrantly generated model category having

$$\mathcal{J}_{\mathcal{S}} = \{\Lambda_k[p] \hookrightarrow \Delta[p] \mid p > 0, 0 \leq k \leq p\}$$

as a set of generating trivial cofibrations. Applying the infinite gluing construction [Dwyer and Spaliński 1995, pages 104–105] for $\mathcal{J}_S$ to a simplicial map $\varphi\colon K \to L$, we obtain the factorization

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[dr, "\varphi"'] & K' \arrow[d, "p"] \\ & L \end{tikzcd}$$

where i is a trivial cofibration and p is a fibration. However, since every simplicial map to the terminal object $*$ has a right lifting property for $\Lambda_k[1] \hookrightarrow \Delta[1]$ ($k = 0, 1$), we can construct a fibrant approximation $K\hat{}$ of K by applying the infinite gluing construction for

$$\mathcal{J}'_{\mathcal{S}} = \{\Lambda_k[p] \hookrightarrow \Delta[p] \mid p > 1, 0 \leq k \leq p\}$$

to $K \to *$. Let $\mathcal{S}_f$ denote the full subcategory of $\mathcal{S}$ consisting of fibrant objects (ie Kan complexes). Then the functor $\cdot\hat{}\colon \mathcal{S} \to \mathcal{S}_f$ is a fibrant approximation functor, for which $K\hat{}_0 = K_0$ holds. An attachment of $\Delta[2]$ along $\Lambda_k[2]$ adds one nondegenerate 2–simplex and one nondegenerate 1–simplex, which correspond to the basic 2–simplex of $\Delta[2]$ and its k^{th} face respectively.

4.2 Proof of part (I)

We prove part (I) of Theorem 1.1 (see the introduction of this section). Let us begin by reducing the proof to simpler cases. First, consider the decomposition $X = \coprod X_\alpha$ into connected components; see [Iglesias-Zemmour 2013, pages 105–107]. Since

$$S^{\mathcal{D}}_{\text{sub}}(X) = \coprod S^{\mathcal{D}}_{\text{sub}}(X_\alpha) \quad \text{and} \quad S^{\mathcal{D}}(X) = \coprod S^{\mathcal{D}}(X_\alpha),$$

we may assume that X is connected.

Next consider the universal covering $\varpi : Z \to X$; see [Iglesias-Zemmour 2013, page 264]. By Proposition 3.7, we then have the morphism of simplicial coverings with fiber $\pi_1^{\mathcal{D}}(X)$

$$\begin{tikzcd} \pi_1^{\mathcal{D}}(X) \arrow[r, equal] \arrow[d] & \pi_1^{\mathcal{D}}(X) \arrow[d] \\ S^{\mathcal{D}}_{\text{sub}}(Z) \arrow[r, hook] \arrow[d] & S^{\mathcal{D}}(Z) \arrow[d] \\ S^{\mathcal{D}}_{\text{sub}}(X) \arrow[r, hook] & S^{\mathcal{D}}(X) \end{tikzcd}$$

Hence, we may assume that X is 1–connected (note that $S^{\mathcal{D}}_{\text{sub}}(X)$ need not be a Kan complex and use [Gabriel and Zisman 1967, Chapter III, Theorem 4.2]).

Since $S^{\mathcal{D}}(X)$ is Kan (Corollary 2.6(2)), the inclusion $S^{\mathcal{D}}_{\text{sub}}(X) \hookrightarrow S^{\mathcal{D}}(X)$ extends to a map

$$S^{\mathcal{D}}_{\text{sub}}(X)\hat{} \to S^{\mathcal{D}}(X),$$

which induces an isomorphism on the homology (Corollary 3.5). Thus, we have only to show that

$$\pi_1(S^{\mathcal{D}}_{\text{sub}}(X)\hat{}, x_0) = 0$$

for any fixed $x_0 \in X$ (see Theorem 2.7 and [May 1992, Theorem 13.9]).

Recall from [Goerss and Jardine 1999, page 8; May 1992, Lemma 16.3] the following facts concerning the topological realization functor $|\cdot| \colon \mathcal{S} \to \mathcal{C}^0$:

- The topological realization $|K|$ of a simplicial set K is a CW–complex having one n–cell for each nondegenerate n–simplex of K.
- For a pointed Kan complex (K, k_0), the simplicial fundamental group $\pi_1(K, k_0)$ is naturally isomorphic to the topological fundamental group $\pi_1(|K|, k_0)$.

For a simplicial set K, NK_n denotes the set of nondegenerate n–simplices of K. The n–cell of $|K|$ corresponding to $\sigma \in NK_n$ is also denoted by σ. The 1–cell σ of $|K|$ is endowed with the canonical orientation; the 1–cell σ endowed with the reverse orientation is denoted by $\bar{\sigma}$. We also use the standard notation $\text{sk}_n\, K$ for the n–skeleton of K.

From these facts and the construction of the fibrant approximation $K\hat{}$ of K, we see the following:

- $\pi_1(S^{\mathcal{D}}_{\text{sub}}(X)\hat{}, x_0) \cong \pi_1(|\text{sk}_2\, S^{\mathcal{D}}_{\text{sub}}(X)\hat{}|, x_0)$.
- Every element of $\pi_1(|\text{sk}_2\, S^{\mathcal{D}}_{\text{sub}}(X)\hat{}|, x_0)$ can be represented by a continuous map
$$\omega \colon (\Delta^1_{\text{top}}, \dot{\Delta}^1_{\text{top}}) \to (|\text{sk}_1\, S^{\mathcal{D}}_{\text{sub}}(X)\hat{}|, x_0).$$
Further, ω can be chosen as the concatenation of finitely many 1–cells $\tau_1, \dots, \tau_l$, where $\tau_j = \sigma_j$ or $\bar{\sigma}_j$ for some $\sigma_j \in NS^{\mathcal{D}}_{\text{sub}}(X)_1$.

We would like to simplify the expression $\tau_1 \cdots \tau_l$ for ω and show that ω is null homotopic rel $\dot{\Delta}^1_{\text{top}}$.

A smooth 1–simplex $\sigma \colon \Delta^1_{\text{sub}} \to X$ of a diffeological space X is called *tame* if σ is constant near each vertex. By the following lemma, we may assume that each σ_j is tame.

Lemma 4.1 *Let X be a diffeological space and σ a 1–simplex of $S^{\mathcal{D}}_{\text{sub}}(X)$. Then there exists a 2–simplex Σ of $S^{\mathcal{D}}_{\text{sub}}(X)$ such that $d_0\Sigma$ is the constant map to $\sigma((1))$, $d_1\Sigma$ is tame, and $d_2\Sigma = \sigma$.*

Proof We choose a nondecreasing smooth function $\mu \colon [0,1] \to [0,1]$ such that $\mu \equiv 0$ near 0 and $\mu \equiv 1$ near 1, and construct the desired 2–simplex Σ of $S^{\mathcal{D}}_{\text{sub}}(X)$ in two steps.

Step 1: construction of $F:\Delta^2_{\mathrm{sub}}\to\Delta^2_{\mathrm{sub}}$ We construct a smooth map $F:\Delta^2_{\mathrm{sub}}\to\Delta^2_{\mathrm{sub}}$ (ie a 2–simplex F of $S^{\mathcal{D}}_{\mathrm{sub}}(\Delta^2_{\mathrm{sub}})$) satisfying the following condition:

- Each d_iF corestricts to the i^{th} face of Δ^2_{sub} and the corestriction of d_iF is identified with
$$\begin{cases}\mathrm{id} & \text{if } i=0,2,\\ \mu & \text{if } i=1,\end{cases}$$
in a canonical manner.

Set $U=\{(x_0,x_1,x_2)\in\Delta^2_{\mathrm{sub}}\mid 0\le x_1<\frac12\}$. Choose a nonincreasing smooth function $\phi:[0,\frac12]\to[0,1]$ such that $\phi\equiv1$ near 0 and $\phi\equiv0$ near $\frac12$. Under the identification
$$U\xrightarrow{\cong}[0,1]\times[0,\tfrac12),\quad (x_0,x_1,x_2)\mapsto\left(\frac{x_2}{1-x_1},x_1\right),$$
define the self-map $U\xrightarrow{f}U$ by
$$f(x,y)=\big(\phi(y)\mu(x)+(1-\phi(y))x,y\big).$$
Then the desired map $\Delta^2_{\mathrm{sub}}\xrightarrow{F}\Delta^2_{\mathrm{sub}}$ is defined by
$$F=\begin{cases}f & \text{on } U,\\ \mathrm{id} & \text{outside } U.\end{cases}$$

Step 2: construction of $\Sigma:\Delta^2_{\mathrm{sub}}\to X$ The desired 2–simplex Σ of $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ is defined to be the composite
$$\Delta^2_{\mathrm{sub}}\xrightarrow{F}\Delta^2_{\mathrm{sub}}\xrightarrow{s^1}\Delta^1_{\mathrm{sub}}\xrightarrow{\sigma}X,$$
where s^1 is defined by $s^1(x_0,x_1,x_2)=(x_0,x_1+x_2)$. □

Second, let us see that ω can be chosen as the concatenation of $\sigma_1,\dots,\sigma_l$ for some $\sigma_1,\dots,\sigma_l\in NS^{\mathcal{D}}_{\mathrm{sub}}(X)_1$. For this, consider $\Sigma_j\in S^{\mathcal{D}}_{\mathrm{sub}}(X)_2$ defined to be the composite
$$\Delta^2_{\mathrm{sub}}\xrightarrow{s}\Delta^1_{\mathrm{sub}}\xrightarrow{\sigma_j}X,$$
where $s(x_0,x_1,x_2)=(x_0+x_2,x_1)$. Then $d_2\Sigma_j=\sigma_j$, $d_1\Sigma_j$ is constant, and $\sigma'_j:=d_0\Sigma_j$ satisfies $\sigma'_j(t)=\sigma_j(1-t)$. Thus, if $\tau_j=\bar\sigma_j$, then we can replace τ_j with σ'_j. Hence, we may assume that ω is the concatenation of $\sigma_1,\dots,\sigma_l$.

Third, let us see that ω can be chosen as the continuous map corresponding to a single tame 1–simplex σ of $S^{\mathcal{D}}_{\mathrm{sub}}(X)$. For this, we first consider the extension problem in $\mathcal{D}$
$$\begin{tikzcd} \Lambda^2_{1\,\mathrm{sub}} \arrow[r, "\sigma_2+\sigma_1"] \arrow[d, hook] & X \\ \Delta^2_{\mathrm{sub}} \arrow[ur, dashed] & \end{tikzcd}$$

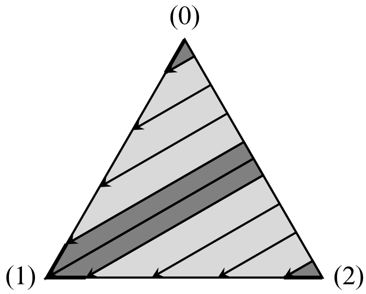


Figure 1: The retraction r.

where $\sigma_2+\sigma_1\colon \Lambda^2_{1\,\mathrm{sub}} \to X$ is defined to be σ_2 on the 0^{th} face and σ_1 on the 2^{nd} face. (the smoothness of $\sigma_2+\sigma_1$ follows from the tameness of σ_1 and σ_2). We define the map $\Sigma\colon \Delta^2_{\mathrm{sub}} \to X$ to be the composite

$$\Delta^2_{\mathrm{sub}} \xrightarrow{r} \Lambda^2_{1\,\mathrm{sub}} \xrightarrow{\sigma_2+\sigma_1} X,$$

where r is the continuous retraction onto $\Lambda^2_{1\,\mathrm{sub}}$ depicted in Figure 1. Noticing that σ_1 and σ_2 are tame, we can easily see that Σ is a solution of the extension problem in $\mathcal{D}$ such that $\eta := d_1\Sigma$ is also tame. Thus, ω can be chosen as the concatenation of $\eta, \sigma_3, \dots, \sigma_l$. By iterating this procedure, we may assume that ω is the continuous map corresponding to a single tame 1–simplex σ of $S^{\mathcal{D}}_{\mathrm{sub}}(X)$.

Last, let us see that ω is null homotopic rel $\dot{\Delta}^1_{\mathrm{top}}$. Since X is 1–connected, the extension problem in $\mathcal{D}$

$$\begin{tikzcd}
\dot{\Delta}^2 \arrow[r, "\sigma+0+0"] \arrow[d, hook] & X \\
\Delta^2 \arrow[ur, dashed]
\end{tikzcd}$$

has a solution Σ, where 0 denotes the constant map to the base point x_0 (see Theorem 2.7).

Now, we recall the smooth map $\psi^2_0\colon \Delta^2 \to \Delta^2$ from [Kihara 2023, Steps 1–3 in the proof of Theorem 8.6]. For $0<\epsilon<\frac{1}{2}$, set $V_i(\epsilon)=\{(x_0,x_1,x_2)\in\Delta^2 \mid x_i > 1-\epsilon\}$. For a given ϵ_0 with $0<\epsilon_0<\frac{1}{2}$, the smooth map

$$\psi^2_0\colon \Delta^2 \to \Delta^2$$

is constructed such that

- ψ^2_0 preserves each closed simplex of Δ^2,
- ψ^2_0 maps each $V_i(\epsilon_0/2)$ to the vertex (i),
- ψ^2_0 coincides with 1_{Δ^2} on $\Delta^2 \backslash \bigcup V_i(\epsilon_0)$.

Thus, we see from Lemma 3.1(3) that $\psi^2_0\colon \Delta^2_{\mathrm{sub}} \to \Delta^2$ is smooth.

Consider the smooth map $\psi^2_0\colon \Delta^2_{\mathrm{sub}} \to \Delta^2$ defined for sufficiently small $\epsilon_0>0$, and define the 2–simplex Σ' of $S^{\mathcal{D}}_{\mathrm{sub}}(X)$ to be the composite

$$\Delta^2_{\mathrm{sub}} \xrightarrow{\psi^2_0} \Delta^2 \xrightarrow{\Sigma} X.$$

Since $\Sigma'|_{\dot{\Delta}^2_{\mathrm{sub}}} = \sigma+0+0$, Σ' yields a homotopy (rel $\dot{\Delta}^1_{\mathrm{top}}$) between ω and 0, which completes the proof.

4.3 Proof of part (II)

We prove part (II) of Theorem 1.1 (see the introduction of this section). By Proposition 3.7 and an argument similar to that in Section 4.2, we may assume that X is 1–connected.

Since $S^{\mathcal{D}}(X)$ is Kan (Corollary 2.6(2)), the canonical map $S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}(X)$ extends to a map

$$S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{} \to S^{\mathcal{D}}(X),$$

which induces an isomorphism on the homology (Corollary 3.5). Thus, we have only to show that

$$\pi_1(S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{}, x_0) = 0$$

for any fixed $x_0 \in X$ (see Theorem 2.7 and [May 1992, Theorem 13.9]).

Similarly to the proof of part (I), we have the following:

- $\pi_1(S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{}, x_0) \cong \pi_1(|\mathrm{sk}_2\, S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{}|, x_0)$.
- Every element of $\pi_1(|\mathrm{sk}_2\, S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{}|, x_0)$ can be represented by a continuous map

  $$\omega\colon (\Delta^1_{\mathrm{top}}, \dot{\Delta}^1_{\mathrm{top}}) \to (|\mathrm{sk}_1\, S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{}|, x_0).$$

  Further, ω can be chosen as the concatenation of finitely many 1–cells $\tau_1, \dots, \tau_l$, where $\tau_j = \sigma_j$ or $\bar{\sigma}_j$ for some $\sigma_j \in NS^{\mathcal{D}}_{\mathrm{aff}}(X)_1$.

We would like to simplify the expression $\tau_1 \cdots \tau_l$ for ω and show that ω is null homotopic rel $\dot{\Delta}^1_{\mathrm{top}}$.

A smooth 1–simplex $\sigma\colon \mathbb{A}^1 \to X$ of a diffeological space X is called *tame* if σ is constant near $(-\infty, 0]$ and near $[1, \infty)$, where $\mathbb{A}^1$ is identified with $\mathbb{R}$ in a canonical manner. By the following analogue of Lemma 4.1, we may assume that each σ_j is tame.

Lemma 4.2 *Let X be a diffeological space and σ a 1–simplex of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$. Then there exists a 2–simplex Σ of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ such that $d_0\Sigma$ is the constant map to $\sigma((1))$, $d_1\Sigma$ is tame, and $d_2\Sigma = \sigma$.*

Proof Set $U = \{(x_0, x_1, x_2) \in \mathbb{A}^2 \mid -\frac{1}{2} < x_1 < \frac{1}{2}\}$. Choose a nondecreasing smooth function $\mu\colon \mathbb{R} \to [0, 1]$ such that $\mu \equiv 0$ near $(-\infty, 0]$ and $\mu \equiv 1$ near $[1, \infty)$, and a smooth function $\phi\colon [-\frac{1}{2}, \frac{1}{2}] \to [0, 1]$ such that $\phi \equiv 1$ near 0 and $\phi \equiv 0$ near $\{-\frac{1}{2}, \frac{1}{2}\}$. Then we can construct the desired 2–simplex Σ in a manner similar to that in the proof of Lemma 4.1. □

Second, let us see that ω can be chosen as the concatenation of $\sigma_1, \dots, \sigma_l$ for some $\sigma_1, \dots, \sigma_l \in NS^{\mathcal{D}}_{\mathrm{aff}}(X)_1$. For this, consider $\Sigma_j \in S^{\mathcal{D}}_{\mathrm{aff}}(X)_2$ defined to be the composite

$$\mathbb{A}^2 \xrightarrow{s} \mathbb{A}^1 \xrightarrow{\sigma_j} X,$$

where $s(x_0, x_1, x_2) = (x_0 + x_2, x_1)$. Then $d_2\Sigma_j = \sigma_j, d_1\Sigma_j$ is constant, and $\sigma'_j := d_0\Sigma_j$ satisfies $\sigma'_j(t) = \sigma_j(1-t)$. Thus, if $\tau_j = \bar{\sigma}_j$, then we can replace τ_j with σ'_j. Hence, we may assume that ω is the concatenation of $\sigma_1, \dots, \sigma_l$.

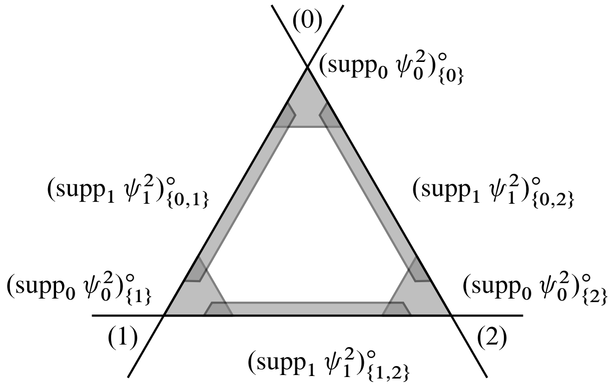


Figure 2

Next we show the following lemma. For $i = 0, 1, 2$, $d^i : \mathbb{A}^1 \to \mathbb{A}^2$ denotes the obvious affine extension of $d^i : \Delta^1 \to \Delta^2$ (see Section 2.2).

Lemma 4.3 *Let X be a diffeological space and γ_0, γ_1, and γ_2 tame 1–simplices of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$ such that $d_0\gamma_2 = d_1\gamma_0$, $d_0\gamma_0 = d_0\gamma_1$, and $d_1\gamma_1 = d_1\gamma_2$. If the extension problem in $\mathcal{D}$*

$$\begin{tikzcd} \dot{\Delta}^2 \arrow[r, "\sum \gamma_i|_{\Delta^1}"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

has a solution, then the extension problem in $\mathcal{D}$

$$\begin{tikzcd} \bigcup d^i\mathbb{A}^1 \arrow[r, "\sum \gamma_i"] \arrow[d, hook] & X \\ \mathbb{A}^2 \arrow[ur, dashed] & \end{tikzcd}$$

also has a solution.

Proof We choose a solution Σ of the first extension problem, and use the smooth map $\psi^2 \colon \Delta^2 \to \Delta^2$ constructed in [Kihara 2023, Steps 1–3 in the proof of Theorem 8.6] to modify and extend Σ.

To describe the basic properties of ψ^2, we adopt the following notation. For a continuous self-map f of Δ^p, we set

$$\mathrm{carr}_k\, f = \{x \in \Delta^p \mid f(x) \neq x,\ f(x) \in \mathrm{sk}_k\, \Delta^p\} \quad \text{and} \quad \mathrm{supp}_k\, f = \overline{\mathrm{carr}_k\, f}.$$

Further, for a subset $\{i_0, \ldots, i_k\}$ of $\{0, \ldots, p\}$, we set

$$V_{\{i_0,\ldots,i_k\}} = \{(x_0, \ldots, x_p) \in \Delta^p \mid x_i > x_j \text{ for } i \in \{i_0, \ldots, i_k\} \text{ and } j \notin \{i_0, \ldots, i_k\}\},$$
$$(\mathrm{supp}_k\, f)^\circ_{\{i_0,\ldots,i_k\}} = (\mathrm{supp}_k\, f)^\circ \cap V_{\{i_0,\ldots,i_k\}}.$$

For a given ϵ_0 with $0 < \epsilon_0 < \frac{1}{2}$, the smooth maps $\psi_k^2 \colon \Delta^2 \to \Delta^2$ ($k = 0, 1$) are defined such that they satisfy the following conditions (see Figure 2):

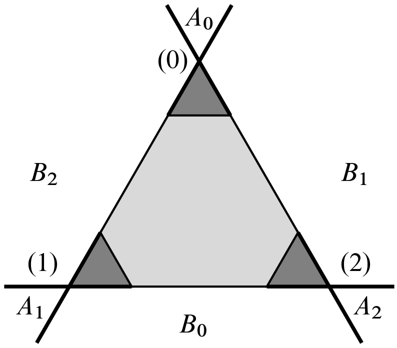


Figure 3

- ψ_k^2 preserves each closed simplex of Δ^2 and $\psi_k^2 = \mathrm{id}$ on $\mathrm{sk}_k\,\Delta^2$ (note that $\mathrm{sk}_0\,\Delta^2 = \{(0),(1),(2)\}$ and that $\mathrm{sk}_1\,\Delta^2 = \dot{\Delta}^2$).
- $(\mathrm{supp}_0\,\psi_0^2)^\circ_{\{i\}} = V_i(\epsilon_0/2)$ and $\psi_0^2 = \mathrm{id}$ on $\Delta^2 \setminus \bigcup V_i(\epsilon_0)$ (see Section 4.2).
- $(\mathrm{supp}_0\,\psi_0^2)^\circ \cup (\mathrm{supp}_1\,\psi_1^2)^\circ$ is a neighborhood of $\dot{\Delta}^2$.
- $(\mathrm{supp}_1\psi_1^2)^\circ = (\mathrm{supp}_1\psi_1^2)^\circ_{\{1,2\}} \coprod (\mathrm{supp}_1\psi_1^2)^\circ_{\{0,2\}} \coprod (\mathrm{supp}_1\psi_1^2)^\circ_{\{0,1\}}$, and ψ_1^2 preserves each $V_i(\epsilon_0/2)$ and maps a point x of $(\mathrm{supp}_1\,\psi_1^2)^\circ_{\{i_0,i_1\}}$ to the intersection of the i^{th} face of Δ^2 and the line through the vertex (i) and x, where $i \neq i_0, i_1$.

The map $\psi^2\colon \Delta^2 \to \Delta^2$ is defined to be the composite

$$\Delta^2 \xrightarrow{\psi_1^2} \Delta^2 \xrightarrow{\psi_0^2} \Delta^2.$$

Consider the smooth map $\psi^2\colon \Delta^2 \to \Delta^2$ for a sufficiently small $\epsilon_0 > 0$ and define Σ' to be the composite

$$\Delta^2 \xrightarrow{\psi^2} \Delta^2 \xrightarrow{\Sigma} X.$$

Then Σ' has the following properties:

- $\Sigma'|_{\dot{\Delta}^2} = \Sigma|_{\dot{\Delta}^2}$.
- $\Sigma'|_{(\mathrm{supp}_0\,\psi_0^2)^\circ_{\{i\}}}$ is constant.
- $\Sigma'|_{(\mathrm{supp}_1\,\psi_1^2)^\circ_{\{i_0,i_1\}}}$ is constant along any ray from the vertex (i) with $i \neq i_0, i_1$.

We thus extend Σ' to $\mathbb{A}^2$ as follows. Define $\Sigma'|_{A_i}$ to be constant for $i = 0, 1, 2$, and define $\Sigma'|_{B_i}$ to be constant along any ray from the vertex (i) (see Figure 3). Then we can easily see that $\Sigma'\colon \mathbb{A}^2 \to X$ is the desired solution of the second extension problem. $\square$

Let us see that ω can be chosen as the continuous map corresponding to a single tame 1–simplex σ of $S^{\mathcal{D}}_{\mathrm{aff}}(X)$. For this, we first consider the extension problem in $\mathcal{D}$

$$\begin{tikzcd} \Lambda_1^2 \arrow[r, "\sigma_2|_{\Delta^1} + \sigma_1|_{\Delta^1}"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

Then we can use the continuous retraction $r\colon \Delta^2 \to \Lambda^2_1$ depicted in Figure 1 to construct a solution $\Sigma\colon \Delta^2 \to X$ such that the composite $\Delta^1 \xrightarrow{d^1} \Delta^2 \xrightarrow{\Sigma} X$ is constant near each vertex. Define the tame 1–simplex η of $S^{\mathscr{D}}_{\mathrm{aff}}(X)$ by $\eta|_{\Delta^1} = \Sigma \circ d^1$ and consider the extension problem in $\mathscr{D}$

$$\begin{tikzcd} \bigcup d^i\mathbb{A}^1 \arrow[r, "\sigma_2+\eta+\sigma_1"] \arrow[d, hook] & X \\ \mathbb{A}^2 \arrow[ur, dashed] & \end{tikzcd}$$

Since this extension problem has a solution (see Lemma 4.3), ω can be chosen as the concatenation of $\eta, \sigma_3, \dots, \sigma_l$. By iterating this procedure, we may assume that ω is the continuous map corresponding to a single tame 1–simplex σ of $S^{\mathscr{D}}_{\mathrm{aff}}(X)$.

Last, let us see that ω is null homotopic rel $\dot{\Delta}^1_{\mathrm{top}}$. Since X is 1–connected, the extension problem in $\mathscr{D}$

$$\begin{tikzcd} \dot{\Delta}^2 \arrow[r, "\sigma|_{\Delta^1}+0+0"] \arrow[d, hook] & X \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

has a solution (see Theorem 2.7). Hence, the extension problem in $\mathscr{D}$

$$\begin{tikzcd} \bigcup d^i\mathbb{A}^1 \arrow[r, "\sigma+0+0"] \arrow[d, hook] & X \\ \mathbb{A}^2 \arrow[ur, dashed] & \end{tikzcd}$$

also has a solution (Lemma 4.3), which shows that ω is null homotopic rel $\dot{\Delta}^1_{\mathrm{top}}$.

4.4 Proofs of Theorem 1.1 and Corollary 1.2

In this subsection, we complete the proofs of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1 The proof of the main statement is given in Sections 4.2 and 4.3. Since $S^{\mathscr{D}}(X)$ is always fibrant (Corollary 2.6(2)), the last statement is obvious. □

Let $\mathscr{S}_*$ denote the category of pointed simplicial sets, and let $\mathscr{S}_{*f}$ denote the full subcategory of $\mathscr{S}_*$ consisting of fibrant objects (ie pointed Kan complexes). Choosing a fibrant approximation functor $R\colon \mathscr{S}_* \to \mathscr{S}_{*f}$, we define the i^{th} homotopy group functor $\pi_i\colon \mathscr{S}_* \to \mathrm{Gr}$ to be the composite

$$\mathscr{S}_* \xrightarrow{R} \mathscr{S}_{*f} \xrightarrow{\pi_i} \mathrm{Gr}.$$

(Strictly speaking, π_0 is defined as a Set_*–valued functor, where Set_* denotes the category of pointed sets.) Then, up to natural isomorphisms, the functor $\pi_i\colon \mathscr{S}_* \to \mathrm{Gr}$ extends the original homotopy group functor $\pi_i\colon \mathscr{S}_{*f} \to \mathrm{Gr}$ and the extension $\pi_i\colon \mathscr{S}_* \to \mathrm{Gr}$ is independent of the choice of R. Further, we can see that if a fibrant approximation $K \to K'$ and a point k of K are given, then $\pi_i(K,k)$ is canonically isomorphic to the i^{th} homotopy group of the pointed Kan complex (K',k).

Proof of Corollary 1.2 The result follows immediately from Theorems 1.1 and 2.7. $\square$

5 Diffeological principal bundles

In this section, we recall the notions of a diffeological principal bundle and a simplicial principal bundle (Section 5.1) and establish Theorem 1.3, which characterizes diffeological principal bundles using the singular functor $S^{\mathcal{D}}_{\mathrm{aff}}$ (Section 5.2).

5.1 Diffeological and simplicial principal bundles

In this subsection, we recall the three notions of principal bundles in $\mathcal{D}$; the weakest notion is due to Iglesias-Zemmour (see Definition 5.1(2)). We also make a brief review on simplicial principal bundles.

Let $\mathscr{C}$ be a category with finite products, and G a group in $\mathscr{C}$. Then $\mathscr{C}G$ denotes the category of right G–objects of $\mathscr{C}$. For $B \in \mathscr{C}$, $\mathscr{C}G/B$ denotes the category of objects of $\mathscr{C}G$ over B, where B is regarded as an object of $\mathscr{C}G$ with trivial G–action.

Definition 5.1 Let G be a diffeological group, and X a diffeological space.

(1) An object $\pi: E \to X$ of $\mathcal{D}G/X$ is called a *locally trivial principal G–bundle* if there exists an open cover $\{U_i\}$ of X such that for each i, a pullback diagram in $\mathcal{D}$

$$\begin{tikzcd} U_i \times G \arrow[r, hook] \arrow[d, "\mathrm{proj}"'] & E \arrow[d, "\pi"] \\ U_i \arrow[r, hook] & X \end{tikzcd}$$

with equivariant upper arrow exists; such an open cover $\{U_i\}$ is called a *trivialization open cover* of $\pi: E \to X$. An object $\pi: E \to X$ of $\mathcal{D}G/X$ is called a *$\mathcal{D}$–numerable principal G–bundle* if π admits a $\mathcal{D}$–numerable trivialization open cover (ie a trivialization open cover $\{U_i\}$ which admits a smooth partition of unity subordinate to it).

(2) An object $\pi: E \to X$ of $\mathcal{D}G/X$ is called a *diffeological principal G–bundle* if for any plot $p: U \to X$, the pullback $p^*E \to U$ is a locally trivial principal G–bundle.

(3) A morphism between locally trivial (or diffeological) principal G–bundles $\pi: E \to X$ and $\pi': E' \to X'$ is a commutative diagram in $\mathcal{D}G$ of the form

$$\begin{tikzcd} E \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \tag{5-1}$$

Note that (5-1) is necessarily a pullback diagram in $\mathcal{D}$; see [Iglesias-Zemmour 2013, 8.13 Note 2]. The categories of locally trivial principal G–bundles, $\mathcal{D}$–numerable principal G–bundles, and diffeological principal G–bundles are denoted by $\mathsf{P}\mathcal{D}G$, $\mathsf{P}\mathcal{D}G_{\mathrm{num}}$, and $\mathsf{P}\mathcal{D}G_{\mathrm{diff}}$, respectively.

We have the obvious fully faithful embeddings

$$\mathsf{P}\mathcal{D}G_{\mathrm{num}} \hookrightarrow \mathsf{P}\mathcal{D}G \hookrightarrow \mathsf{P}\mathcal{D}G_{\mathrm{diff}}.$$

We see from the following examples that the two inclusions are proper (or strict). Recall from [Iglesias-Zemmour 2013, 8.15] that for a diffeological group G and its diffeological subgroup H, the quotient map $\pi : G \to G/H$ is a diffeological principal H–bundle.

Example 5.2 (1) Let $\gamma : \mathbb{Z}^m \to \mathbb{R}^n$ be a monomorphism of abelian groups with $\Gamma := \gamma(\mathbb{Z}^m)$ dense. Then the quotient diffeological group $T_\Gamma = \mathbb{R}^n / \Gamma$ is called an *irrational torus*. Since the underling topology of T_Γ is indiscrete, the diffeological principal $\mathbb{Z}^m$–bundle $\pi : \mathbb{R}^n \to T_\Gamma$ is not locally trivial.

(2) Christensen and Wu constructed a nontrivial locally trivial principal $\mathbb{R}^{>0}$–bundle $\pi : P \to X$ with $X \simeq_{\mathcal{D}} *$; see [Christensen and Wu 2021, Example 3.12]. By [Christensen and Wu 2021, Theorem 5.10], the locally trivial principal $\mathbb{R}^{>0}$–bundle π is not $\mathcal{D}$–numerable.

To study diffeological principal bundles, we also need the notion of a simplicial principal bundle [May 1992, Chapter IV].

Definition 5.3 Let H be a simplicial group, and K a simplicial set.

(1) An object $\pi : E \to K$ of $\mathcal{S}H/K$ is called a *principal H–bundle* if for any map $k : \Delta[p] \to K$, there exists a pullback diagram

$$\begin{tikzcd} \Delta[p] \times H \arrow[r, "\hat{k}"] \arrow[d, "\mathrm{proj}"'] & E \arrow[d] \\ \Delta[p] \arrow[r, "k"] & K \end{tikzcd}$$

with $\hat{k}$ equivariant.

(2) A morphism between principal H–bundles $\pi : E \to K$ and $\pi' : E' \to K'$ is a commutative diagram in $\mathcal{S}H$ of the form

(5-2)
$$\begin{tikzcd} E \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ K \arrow[r, "f"] & K' \end{tikzcd}$$

Note that (5-2) is necessarily a pullback diagram in $\mathcal{S}$. The category of principal H–bundles are denoted by $\mathsf{P}\mathcal{S}H$.

Remark 5.4 An object $\pi\colon E \to K$ of $\mathcal{S}H/K$ is a principal H–bundle if and only if the action of H on E is free and π induces the isomorphism $E/H \xrightarrow[\cong]{\bar{\pi}} K$; see [May 1992, Definition 18.1].

Let $\cdot_0\colon \mathcal{S} \to \mathrm{Set}$ denote the 0^{th} component functor, which is naturally isomorphic to the functor $\mathcal{S}(\Delta[0], \cdot)$. The following simple result is used in the proof of Theorem 1.3.

Lemma 5.5 (1) *The composite*

$$\mathcal{D} \xrightarrow{S^{\mathcal{D}}_{\mathrm{aff}}} \mathcal{S} \xrightarrow{\cdot_0} \mathrm{Set}$$

is naturally isomorphic to the underlying set functor for $\mathcal{D}$.

(2) *The functor $\cdot_0\colon \mathcal{S} \to \mathrm{Set}$ is a right adjoint.*

Proof (1) Obvious.

(2) Define the functor $d\colon \mathrm{Set} \to \mathcal{S}$ to assign to a set A the discrete simplicial set whose 0^{th} component is A. Then we can easily see that $(d, \cdot_0)$ is an adjoint pair. □

For a given set A, the discrete simplicial set dA is usually denoted by A.

5.2 Proof of Theorem 1.3

In this subsection, we prove Theorem 1.3; we begin by proving the "only if" part of (1) and (2), and then prove the "if" part of (1).

Recall that $S^{\mathcal{D}}_{\mathrm{aff}}$ is a right adjoint (Remark 3.2(1)). Then we see that $S^{\mathcal{D}}_{\mathrm{aff}}(G)$ is a simplicial group and that $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)\colon S^{\mathcal{D}}_{\mathrm{aff}}(P) \to S^{\mathcal{D}}_{\mathrm{aff}}(X)$ is an object of $\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)/S^{\mathcal{D}}_{\mathrm{aff}}(X)$.

Proof of the "only if" part of Theorem 1.3(1) Assume given a map $k\colon \Delta[p] \to S^{\mathcal{D}}_{\mathrm{aff}}(X)$ and let $\kappa\colon \mathbb{A}^p \to X$ be the smooth map corresponding to k. Then we have a pullback diagram in $\mathcal{D}$

$$\begin{tikzcd}
\mathbb{A}^p \times G \arrow[r] \arrow[d, "\mathrm{proj}"'] & P \arrow[d, "\pi"] \\
\mathbb{A}^p \arrow[r, "\kappa"] & X
\end{tikzcd}$$

with equivariant upper arrow; see [Iglesias-Zemmour 2013, 8.19]. Note that $S^{\mathcal{D}}_{\mathrm{aff}}$ is a right adjoint and consider the commutative diagram in $\mathcal{S}$ consisting of two pullback squares with equivariant upper arrows

$$\begin{tikzcd}
\Delta[p] \times S^{\mathcal{D}}_{\mathrm{aff}}(G) \arrow[r] \arrow[d, "\mathrm{proj}"'] & S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{A}^p) \times S^{\mathcal{D}}_{\mathrm{aff}}(G) \arrow[r] \arrow[d, "\mathrm{proj}"'] & S^{\mathcal{D}}_{\mathrm{aff}}(P) \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi)"'] \\
\Delta[p] \arrow[r] & S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{A}^p) \arrow[r, "S^{\mathcal{D}}_{\mathrm{aff}}(\kappa)"] & S^{\mathcal{D}}_{\mathrm{aff}}(X)
\end{tikzcd}$$

where $\Delta[p] \to S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{A}^p)$ is the map corresponding to the p–simplex $1_{\mathbb{A}^p}$ of $S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{A}^p)$. Then the outer rectangle gives the desired local triviality of $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)$; see [Mac Lane 1998, Exercise 8 on page 72]. □

Proof of Theorem 1.3(2) Noting that $S^{\mathcal{D}}_{\mathrm{aff}}$ is a right adjoint, we see from part (1) that $S^{\mathcal{D}}_{\mathrm{aff}}$ induces a functor from $\mathsf{P}\mathcal{D}G_{\mathrm{diff}}$ to $\mathsf{P}\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)$. The faithfulness of the functor follows from Lemma 5.5(1). □

Remark 5.6 The functor $S^{\mathcal{D}}_{\mathrm{aff}}\colon \mathsf{P}\mathcal{D}G_{\mathrm{diff}} \to \mathsf{P}\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)$ need not be fully faithful. In fact, let $\pi\colon P \to X$ be the locally trivial principal $\mathbb{R}^{>0}$–bundle in Example 5.2(2), and let $\pi'\colon P' \to X$ be the trivial principal $\mathbb{R}^{>0}$–bundle. Since $X \simeq_{\mathcal{D}} *$, the diagram in $\mathcal{D}$

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "1_X"'] & * \arrow[r, hook, "x"] & X \end{tikzcd}$$

is commutative up to homotopy for $x \in X$. Thus, by Lemma 3.3, the diagram in $\mathcal{S}$

$$\begin{tikzcd} S^{\mathcal{D}}_{\mathrm{aff}}(X) \arrow[r] \arrow[rr, bend right, "1_{S^{\mathcal{D}}_{\mathrm{aff}}(X)}"'] & * \arrow[r, hook, "x"] & S^{\mathcal{D}}_{\mathrm{aff}}(X) \end{tikzcd}$$

is also commutative up to homotopy. Hence, both $S^{\mathcal{D}}_{\mathrm{aff}}(P)$ and $S^{\mathcal{D}}_{\mathrm{aff}}(P')$ are trivial principal $S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{R}^{>0})$–bundles (see [May 1992, Corollary 20.6]), which shows that $S^{\mathcal{D}}_{\mathrm{aff}}\colon \mathsf{P}\mathcal{D}\mathbb{R}^{>0} \to \mathsf{P}\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(\mathbb{R}^{>0})$ is not fully faithful. (From this argument, we also see that the faithful functor $S^{\mathcal{D}}_{\mathrm{aff}}\colon \mathcal{D} \to \mathcal{S}$ is not fully faithful.)

Next we prove the following lemma, which is used in the proof of "if" part of Theorem 1.3(1).

Lemma 5.7 *Let $\pi\colon P \to X$ be an object of $\mathcal{D}G/X$. Then $\pi\colon P \to X$ is a diffeological principal G–bundle if and only if π satisfies the following conditions*:

(i) *G acts on P freely and $\pi\colon P \to X$ induces a bijection $P/G \to X$.*

(ii) *Given a solid arrow diagram in $\mathcal{D}$*

$$\begin{tikzcd} & P \arrow[d, "\pi"] \\ \mathbb{A}^p \arrow[r, "\kappa"] \arrow[ur, dashed] & X \end{tikzcd}$$

there exists a dotted arrow, making the diagram commute.

(iii) *The translation function $\tau\colon P \times_X P \to G$, defined by $u \cdot \tau(u,v) = v$, is smooth.*

Proof We begin with the forward direction.

(i) Obvious.

(ii) By [Iglesias-Zemmour 2013, 8.19],

$$\kappa^* P \cong \mathbb{A}^p \times G \quad \text{in } \mathcal{D}G/\mathbb{A}^p. \tag{5-3}$$

Hence, π satisfies condition (ii).

(iii) We have only to show that $\tau\colon P \times_X P \to G$ preserves global plots.
Assume given a global plot $f\colon \mathbb{A}^p \to P \times_X P$. Since the components f_1 and f_2 of f are global plots of P with $\pi \circ f_1 = \pi \circ f_2$, we only have to show that the composite

$$\kappa^* P \underset{\mathbb{A}^p}{\times} \kappa^* P \to P \underset{X}{\times} P \xrightarrow{\tau} G$$

is smooth, where $\kappa := \pi \circ f_1 = \pi \circ f_2$. By (5-3), we have the identifications

$$\kappa^* P \underset{\mathbb{A}^p}{\times} \kappa^* P \cong (\mathbb{A}^p \times G) \underset{\mathbb{A}^p}{\times} (\mathbb{A}^p \times G) \cong \mathbb{A}^p \times G \times G,$$

under which the composite $\kappa^* P \times_{\mathbb{A}^p} \kappa^* P \to P \times_X P \xrightarrow{\tau} G$ is just the smooth map

$$\mathbb{A}^p \times G \times G \to G$$

sending (x, g, h) to $g^{-1}h$.

For the reverse direction, assume we are given a smooth map $\kappa\colon \mathbb{A}^p \to X$. By condition (ii), we can choose a section σ of the pullback $\kappa^* P \xrightarrow{\hat{\pi}} \mathbb{A}^p$ of $P \xrightarrow{\pi} X$ along κ. Define the maps

$$\mathbb{A}^p \times G \underset{\psi_\kappa}{\overset{\phi_\kappa}{\rightleftarrows}} \kappa^* P$$

by $\phi_\kappa(x, g) = \sigma(x) \cdot g$ and $\psi_\kappa(u) = \big(\hat{\pi}(u), \tau\big(\sigma(\hat{\pi}(u)), u\big)\big)$, respectively. Then we see that ϕ_κ and ψ_κ are mutually inverses in $\mathcal{D}G/\mathbb{A}^p$. □

We give a proof of the "if" part of Theorem 1.3(1), completing the proof of Theorem 1.3.

Proof of the "if" part of Theorem 1.3(1) We only have to show that $\pi\colon P \to X$ satisfies conditions (i)–(iii) in Lemma 5.7. Throughout this proof, bear the following in mind: for a diffeological space Z,

- $S^{\mathcal{D}}_{\mathrm{aff}}(Z)_0$ is just the set Z,
- $S^{\mathcal{D}}_{\mathrm{aff}}(Z)$ can be regarded as the set of global plots of Z.

Recall also that $S^{\mathcal{D}}_{\mathrm{aff}}$ is a right adjoint (see Remark 3.2(1)).

(i) Consider the pullback diagram in $\mathcal{D}$

$$\begin{tikzcd} P_x \arrow[r] \arrow[d] & P \arrow[d, "\pi"] \\ \{x\} \arrow[r] & X \end{tikzcd}$$

for $x \in X$. By applying the singular functor $S^{\mathcal{D}}_{\mathrm{aff}}$, we have the pullback diagram in $\mathcal{S}$

$$\begin{tikzcd} S^{\mathcal{D}}_{\mathrm{aff}}(P_x) \arrow[r] \arrow[d] & S^{\mathcal{D}}_{\mathrm{aff}}(P) \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi)"] \\ \Delta[0] \arrow[r] & S^{\mathcal{D}}_{\mathrm{aff}}(X) \end{tikzcd}$$

Since $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)$ is a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle, $S^{\mathcal{D}}_{\mathrm{aff}}(P_x) \cong S^{\mathcal{D}}_{\mathrm{aff}}(G)$ in $\mathcal{S}S^{\mathcal{D}}_{\mathrm{aff}}(G)$, and hence $P_x \cong G$ in $\mathrm{Set}\,G$ holds (see Lemma 5.5), which shows that π satisfies condition (i).

(ii) Consider the pullback diagram in $\mathscr{D}$

$$\begin{tikzcd} \kappa^* P \arrow[r] \arrow[d] & P \arrow[d, "\pi"] \\ \mathbb{A}^p \arrow[r, "\kappa"] & X \end{tikzcd}$$

and let k denote the simplicial map $\Delta[p] \to S^{\mathscr{D}}_{\mathrm{aff}}(X)$ corresponding to κ. Then we have the commutative diagram in $\mathscr{S}$ consisting of two pullback squares

$$\begin{tikzcd} k^* S^{\mathscr{D}}_{\mathrm{aff}}(P) \arrow[r] \arrow[d] & S^{\mathscr{D}}_{\mathrm{aff}}(\kappa^* P) \arrow[r] \arrow[d] & S^{\mathscr{D}}_{\mathrm{aff}}(P) \arrow[d, "S^{\mathscr{D}}_{\mathrm{aff}}(\pi)"] \\ \Delta[p] \arrow[r] & S^{\mathscr{D}}_{\mathrm{aff}}(\mathbb{A}^p) \arrow[r, "S^{\mathscr{D}}_{\mathrm{aff}}(\kappa)"] & S^{\mathscr{D}}_{\mathrm{aff}}(X) \end{tikzcd}$$

where $\Delta[p] \to S^{\mathscr{D}}_{\mathrm{aff}}(\mathbb{A}^p)$ is the simplicial map corresponding to the p–simplex $1_{\mathbb{A}^p}$ of $S^{\mathscr{D}}_{\mathrm{aff}}(\mathbb{A}^p)$; see [Mac Lane 1998, Exercise 8 on page 72]. Since $S^{\mathscr{D}}_{\mathrm{aff}}(\pi)$ is a simplicial $S^{\mathscr{D}}_{\mathrm{aff}}(G)$–bundle, $k^* S^{\mathscr{D}}_{\mathrm{aff}}(P) \to \Delta[p]$ has a section s. Then the composite

$$\Delta[p] \xrightarrow{s} k^* S^{\mathscr{D}}_{\mathrm{aff}}(P) \to S^{\mathscr{D}}_{\mathrm{aff}}(P)$$

defines the desired lifting of κ along π.

(iii) We show that the map $\tau\colon P \times_X P \to G$ preserves global plots. Assume given a global plot $f = (f_1, f_2)\colon \mathbb{A}^p \to P \times_X P$. Since f_1 and f_2 are global plots of P with $\pi \circ f_1 = \pi \circ f_2$, we set $\kappa = \pi \circ f_1 = \pi \circ f_2$ and let σ_1 and σ_2 denote the sections of $\kappa^* P \to \mathbb{A}^p$ corresponding to f_1 and f_2, respectively. Then σ_1 and σ_2 correspond to sections of $k^* S^{\mathscr{D}}_{\mathrm{aff}}(P) \to \Delta[p]$, which are denoted by s_1 and s_2, respectively (see the verification of condition (ii)). Since the principal $S^{\mathscr{D}}_{\mathrm{aff}}(G)$–bundle $k^* S^{\mathscr{D}}_{\mathrm{aff}}(P) \to \Delta[p]$ is trivial, there exists a unique p–simplex g of $S^{\mathscr{D}}_{\mathrm{aff}}(G)$ such that $s_1 \cdot g = s_2$. We thus see that the composite

$$\mathbb{A}^p \xrightarrow{f} P \underset{X}{\times} P \xrightarrow{\tau} G$$

is just the global plot g. □

Remark 5.8 (1) Recall the notion of a diffeological fiber bundle and that of a simplicial fiber bundle from Section 3.3. We can then use the argument in the proof of the "only if" part of Theorem 1.3(1) to prove the following: If $\pi : E \to X$ is a diffeological fiber bundle, then $S^{\mathscr{D}}_{\mathrm{aff}}(\pi) : S^{\mathscr{D}}_{\mathrm{aff}}(E) \to S^{\mathscr{D}}_{\mathrm{aff}}(X)$ is a simplicial fiber bundle.

This result along with Theorem 1.1 enables us to apply the Serre spectral sequence [May 1992, Section 32] to diffeological fiber bundles (cf [Kihara 2023, Remark 3.8(3)]).

(2) If we restrict ourselves to locally trivial principal G–bundles (resp. locally trivial fiber bundles), then the "only if" part of Theorem 1.3(1) (resp. the result stated in part (1)) remains true for the functor $S^{\mathscr{D}}$ (instead of $S^{\mathscr{D}}_{\mathrm{aff}}$); see [Kihara 2023, Corollary 5.15(1)].

(3) If we restrict ourselves to diffeological coverings, then the result stated in part (1) remains true for the functor $S^{\mathcal{D}}$ (instead of $S^{\mathcal{D}}_{\mathrm{aff}}$); see Proposition 3.7. Similarly, if G is discrete, then Theorem 1.3 remains true for the functor $S^{\mathcal{D}}$ (instead of $S^{\mathcal{D}}_{\mathrm{aff}}$).

6 Characteristic classes of diffeological principal bundles

In this section, we first give a criterion for a simplicial principal bundle to be universal (Section 6.1). We then use this criterion to determine the homotopy type of $S^{\mathcal{D}}(X)$ for a diffeological space X which admits a diffeological principal bundle with contractible total space (Proposition 6.3), applying it to the classifying space BG of a diffeological group G and exceptional diffeological spaces such as irrational tori and $\mathbb{R}/\mathbb{Q}$ (Section 6.2). We use the proof of Proposition 6.3 along with Theorems 1.1 and 1.3 to prove Proposition 1.4 (Section 6.3). We end this section by discussing the sets of characteristic classes for various classes of principal bundles and their relation (Section 6.4).

6.1 Universal simplicial principal bundles

In this subsection, we recall the basics of universal simplicial principal bundles and give a criterion for a simplicial principal bundle to be universal.

Let H be a simplicial group. A principal H–bundle $\varpi\colon E \to L$ is called *universal* if L is Kan (ie fibrant in $\mathcal{S}$) and the natural map

$$[K, L] \to \{\text{isomorphism classes of principal } H\text{–bundles over } K\}, \qquad [f] \mapsto [f^*E],$$

is bijective; the base L of a universal principal H–bundle $\varpi\colon E \to L$ is called a *classifying complex* of H. By a simple argument, a classifying complex of H is unique up to homotopy. Recall that the W–construction $q\colon WH \to \overline{W}H$ is a universal principal H–bundle [Goerss and Jardine 1999, Chapter V, Section 4; May 1992, Section 21] and that WH is contractible [May 1992, Proposition 21.5].

Lemma 6.1 *Let H be a simplicial group, and $\varpi\colon E \to L$ be a principal H–bundle. Then the following are equivalent:*

(i) *$\varpi\colon E \to L$ is universal.*

(ii) *L is Kan and the canonical map $E \to *$ is a weak equivalence.*

(iii) *E is a contractible Kan complex.*

Proof (ii) $\Longleftrightarrow$ (iii) Noticing that H is Kan [May 1992, Theorem 17.1], we see that L is Kan if and only if E is Kan (see [May 1992, Proposition 7.5]), and hence that (ii) $\Longleftrightarrow$ (iii).

(i) $\Longleftrightarrow$ (iii) We have only to prove that under the assumption that L is Kan,

$$\varpi\colon E \to L \text{ is universal} \iff E \text{ is contractible}$$

(see [May 1992, Proposition 7.5]).

Since $q\colon WH \to \overline{W}H$ is universal, we have a morphism of principal H–bundles

(6-1)
$$\begin{tikzcd} E \arrow[r] \arrow[d, "\varpi"'] & WH \arrow[d, "q"] \\ L \arrow[r, "\varphi"] & \overline{W}H \end{tikzcd}$$

Note that H and the four simplicial sets in (6-1) are Kan and consider the morphism between the homotopy exact sequences induced by (6-1). Then we have the equivalences

$$\varpi : E \to L \text{ is universal} \iff \varphi\colon L \to \overline{W}H \text{ is a homotopy equivalence} \iff E \text{ is contractible.} \qquad \square$$

Remark 6.2 Lemma 6.1 can be regarded as a variant of [Goerss and Jardine 1999, Chapter V, Theorem 3.9]. However, we record this lemma along with its proof for the following two reasons: one reason is to avoid using the model structure on $\mathcal{S}G$ (see [Goerss and Jardine 1999, Section V.2]) and the other reason is to emphasize the importance of the fibrancy of the base (cf the proof of Proposition 6.3).

6.2 Diffeological principal bundles with contractible total space

In this subsection, we determine the homotopy type of $S^{\mathcal{D}}(X)$ for a diffeological space X which admits a diffeological principal bundle $\pi : E \to X$ with E weakly contractible. Here, a diffeological space Z is called *weakly contractible* if the canonical map $Z \to *$ is a weak equivalence. We can easily see that

$$Z \text{ is weakly contractible} \quad\iff\quad S^{\mathcal{D}}(Z) \simeq * \quad\iff\quad \pi_*^{\mathcal{D}}(Z, z) = 0 \quad \text{for any } z \in Z$$

(see Remark 2.8(1), Corollary 2.6(2), and Theorem 2.7).

Proposition 6.3 *Let G be a diffeological group and $\pi : E \to X$ a diffeological principal G–bundle with E weakly contractible. Then $S^{\mathcal{D}}(X)$ is a classifying complex of the simplicial group $S^{\mathcal{D}}(G)$.*

Proof By Theorem 1.3, $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)\colon S^{\mathcal{D}}_{\mathrm{aff}}(E) \to S^{\mathcal{D}}_{\mathrm{aff}}(X)$ is a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle. Let us construct a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)'\colon S^{\mathcal{D}}_{\mathrm{aff}}(E)' \to S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{}$ (see Section 4.1) and a morphism of principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundles

$$\begin{tikzcd} S^{\mathcal{D}}_{\mathrm{aff}}(E) \arrow[r, hook] \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi)"'] & S^{\mathcal{D}}_{\mathrm{aff}}(E)' \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi)'"] \\ S^{\mathcal{D}}_{\mathrm{aff}}(X) \arrow[r, hook] & S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{} \end{tikzcd}$$

First, choose a classifying map $\varphi_E : S^{\mathcal{D}}_{\mathrm{aff}}(X) \to \overline{W} S^{\mathcal{D}}_{\mathrm{aff}}(G)$. Then note that $\overline{W} S^{\mathcal{D}}_{\mathrm{aff}}(G)$ is Kan and choose an extension $\varphi'_E : S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{} \to \overline{W} S^{\mathcal{D}}_{\mathrm{aff}}(G)$. By setting $S^{\mathcal{D}}_{\mathrm{aff}}(E)' = {\varphi'_E}^{*} W S^{\mathcal{D}}_{\mathrm{aff}}(G)$, we then obtain the desired diagram.

Thus, we can use [Gabriel and Zisman 1967, Chapter III, Theorem 4.2] to see that $S^{\mathcal{D}}_{\mathrm{aff}}(E) \hookrightarrow S^{\mathcal{D}}_{\mathrm{aff}}(E)'$ is a weak equivalence. Noticing that $S^{\mathcal{D}}_{\mathrm{aff}}(E) \to *$ is a weak equivalence (see Theorem 1.1), we see from Lemma 6.1 that $S^{\mathcal{D}}_{\mathrm{aff}}(\pi)'\colon S^{\mathcal{D}}_{\mathrm{aff}}(E)' \to S^{\mathcal{D}}_{\mathrm{aff}}(X)\hat{}$ is a universal principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle. Hence, $S^{\mathcal{D}}(X)$ is a classifying complex of $S^{\mathcal{D}}_{\mathrm{aff}}(G)$, and hence of $S^{\mathcal{D}}(G)$ (see Theorem 1.1). $\square$

Corollary 6.4 *Let G be a diffeological group. Then the singular complex $S^{\mathcal{D}}(BG)$ of the classifying space BG is a classifying complex of the simplicial group $S^{\mathcal{D}}(G)$.*

Proof Recall from [Christensen and Wu 2021, Corollary 5.5] that EG is smoothly contractible. Then the result is immediate from Proposition 6.3. $\square$

Corollary 6.5 *Suppose that X is a pointed diffeological space which has the weakly contractible universal covering. Then the singular complex $S^{\mathcal{D}}(X)$ is the Eilenberg–Mac Lane complex $K(\pi_1^{\mathcal{D}}(X), 1)$. In particular, the (co)homology of X is just the (co)homology of the group $\pi_1^{\mathcal{D}}(X)$.*

Proof Recall from [Iglesias-Zemmour 2013, 8.26] that the universal covering $\pi: Z \to X$ is a diffeological principal $\pi_1^{\mathcal{D}}(X)$–bundle. Then the result follows from Proposition 6.3. $\square$

Remark 6.6 (1) We can prove Corollary 6.4, using neither the functor $S^{\mathcal{D}}_{\mathrm{aff}}$ nor Theorem 1.1. In fact, by Remark 5.8(2) and Lemma 6.1, $S^{\mathcal{D}}(\pi_G): S^{\mathcal{D}}(EG) \to S^{\mathcal{D}}(BG)$ is a universal principal $S^{\mathcal{D}}(G)$–bundle. However, the construction in the proof of Proposition 6.3 is useful in the proof of Proposition 1.4.

(2) We can also prove Corollary 6.5, using neither the functor $S^{\mathcal{D}}_{\mathrm{aff}}$ nor Theorem 1.1. In fact, Corollary 6.5 follows from Proposition 3.7. Alternatively, Corollary 6.5 follows from [Iglesias-Zemmour 2013, 8.24] and Theorem 2.7.

Corollary 6.5 determines the homotopy type of $S^{\mathcal{D}}(X)$, and hence the (co)homology of X for well-known homogeneous diffeological spaces X such as irrational tori and $\mathbb{R}/\mathbb{Q}$.

Example 6.7 (1) Let $\gamma: \mathbb{Z}^m \to \mathbb{R}^n$ be a monomorphism of abelian groups with $\Gamma := \gamma(\mathbb{Z}^m)$ dense, and consider the irrational torus $T_\Gamma = \mathbb{R}^n/\Gamma$. By Corollary 6.5, the singular complex $S^{\mathcal{D}}(T_\Gamma)$ of T_Γ is just the m–dimensional torus $K(\mathbb{Z}^m, 1)$. Hence, $H^*(T_\Gamma; \mathbb{Z}) \cong \Lambda(\mathbb{Z}^m)$ holds.

(2) The singular complex $S^{\mathcal{D}}(\mathbb{R}/\mathbb{Q})$ of the quotient diffeological group $\mathbb{R}/\mathbb{Q}$ is just the rationalized circle $K(\mathbb{Q}, 1)$, and hence $\widetilde{H}_*(\mathbb{R}/\mathbb{Q}; \mathbb{Z}) = H_1(\mathbb{R}/\mathbb{Q}; \mathbb{Z}) = \mathbb{Q}$. More generally, let A be a countable subgroup of $\mathbb{F}$ $(= \mathbb{R}, \mathbb{C})$. Then the singular complex $S^{\mathcal{D}}(\mathbb{F}/A)$ of the quotient diffeological group $\mathbb{F}/A$ is just $K(A, 1)$.

Remark 6.8 Iglesias-Zemmour [2024, Corollary, page 253] and Kuribayashi [2020, Remark 2.9; 2021, Proposition 3.2] obtained calculational results similar to Example 6.7(1) for other cohomology theories of irrational tori. On the other hand, the de Rham cohomology $H^*_{dR}(T_\Gamma)$ is isomorphic to $\Lambda(\mathbb{R}^n)$ [Iglesias-Zemmour 2013, Exercise 119], which along with Example 6.7(1), shows that the de Rham theorem does not hold for irrational tori. This motivates the study of a forthcoming paper [Kihara $\geq$ 2024].

Next we introduce new aspherical homogeneous diffeological spaces, using Corollary 6.5.

Example 6.9 Let k be a countable subfield of $\mathbb{F}$ $(= \mathbb{R}, \mathbb{C})$ (eg an algebraic number field or a countable extension of $\mathbb{Q}$ such as $\overline{\mathbb{Q}} \cap \mathbb{R}$ or $\overline{\mathbb{Q}}$). For an algebraic group G over k, we can consider the homogeneous diffeological space $G(\mathbb{F})/G(k)$.

If G is a unipotent algebraic group over k, then the exponential map $\exp\colon \mathfrak{g} \to G$ is an isomorphism of algebraic varieties, where $\mathfrak{g}$ is the Lie algebra of G; see [Milne 2017, page 289]. Thus, we have the diffeomorphism

$$\mathfrak{g}(\mathbb{F}) \xrightarrow[\cong]{\exp} G(\mathbb{F})$$

and the universal covering

$$G(\mathbb{F}) \to G(\mathbb{F})/G(k)$$

of $G(\mathbb{F})/G(k)$. Hence, by Corollary 6.5,

$$S^{\mathcal{D}}(G(\mathbb{F})/G(k)) = K(G(k), 1),$$

so the (co)homology of $G(\mathbb{F})/G(k)$ is that of the group $G(k)$. The group $U_n(k)$ of upper triangular unipotent matrices and the Hisenberg group $H_n(k)$ (see [Onishchik and Vinberg 1994, page 54]) are typical examples of unipotent algebraic groups.

Further if G is defined over a subring k_0 of k, then

$$S^{\mathcal{D}}(G(\mathbb{F})/G(k_0)) = K(G(k_0), 1).$$

We are interested in the case where k_0 is the ring $\mathcal{O}_k$ of integers of an algebraic number field k. If k is an algebraic number field of degree n with $\mathbb{Q} \subsetneqq k \subsetneqq \mathbb{R}$, then k_0 $(= \mathcal{O}_k)$ is a finitely generated free $\mathbb{Z}$–module of rank n, and hence is dense in $\mathbb{R}$.

6.3 Proof of Proposition 1.4

In this subsection, we prove Proposition 1.4.

Proof of Proposition 1.4 Let $\pi_G\colon EG \to BG$ denote the universal $\mathcal{D}$–numerable principal G–bundle constructed in [Christensen and Wu 2021]. Then by Theorem 1.3(1), $S^{\mathcal{D}}_{\mathrm{aff}}(\pi_G)\colon S^{\mathcal{D}}_{\mathrm{aff}}(EG) \to S^{\mathcal{D}}_{\mathrm{aff}}(BG)$ is a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle.

We prove the result in two steps.

Step 1: construction of a universal principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle which is an extension of $S^{\mathcal{D}}_{\mathrm{aff}}(\pi_G)$ Recall from [Christensen and Wu 2021, Corollary 5.5] that EG is smoothly contractible. Then, by the proof of Proposition 6.3, we have a universal principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle $S^{\mathcal{D}}_{\mathrm{aff}}(\pi_G)'\colon S^{\mathcal{D}}_{\mathrm{aff}}(EG)' \to S^{\mathcal{D}}_{\mathrm{aff}}(BG)\hat{\ }$ and a morphism of principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundles

$$\begin{tikzcd} S^{\mathcal{D}}_{\mathrm{aff}}(EG) \arrow[r, hook] \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi_G)"'] & S^{\mathcal{D}}_{\mathrm{aff}}(EG)' \arrow[d, "S^{\mathcal{D}}_{\mathrm{aff}}(\pi_G)'"] \\ S^{\mathcal{D}}_{\mathrm{aff}}(BG) \arrow[r, hook] & S^{\mathcal{D}}_{\mathrm{aff}}(BG)\hat{\ } \end{tikzcd}$$

Step 2: definition of $\alpha(P)$ Let $\pi\colon P \to X$ be a diffeological principal G–bundle. Since

$$S^{\mathcal{D}}_{\mathrm{aff}}(\pi)\colon S^{\mathcal{D}}_{\mathrm{aff}}(P) \to S^{\mathcal{D}}_{\mathrm{aff}}(X)$$

is a principal $S^{\mathcal{D}}_{\mathrm{aff}}(G)$–bundle (Theorem 1.3(1)), we have a classifying map $\varphi_P\colon S^{\mathcal{D}}_{\mathrm{aff}}(X) \to S^{\mathcal{D}}_{\mathrm{aff}}(BG)\hat{\ }$.

Note that $H^*(Z;A) := H^*\operatorname{Hom}(\mathbb{Z}S^{\mathcal{D}}(Z), A) \cong H^*\operatorname{Hom}(\mathbb{Z}S^{\mathcal{D}}_{\mathrm{aff}}(Z), A)$ (see Corollary 3.5) and that $H^*\operatorname{Hom}(\mathbb{Z}K, A) \cong H^*\operatorname{Hom}(\mathbb{Z}K\hat{\ }, A)$. Then we can define $\alpha(P) \in H^k(X;A)$ by $\alpha(P) = \varphi_P^*\alpha$. We can use Theorem 1.3 to show that $\alpha(f^*P) = f^*\alpha(P)$, and hence that $\alpha(\cdot)$ defines a characteristic class for diffeological principal G–bundles.

Similarly, we can use Theorem 1.3 to show that $\alpha(\cdot)$ extends the characteristic class $\alpha(\cdot)$ for $\mathcal{D}$–numerable principal G–bundles (see Section 1 for the definition). □

Remark 6.10 The author does not know whether $S^{\mathcal{D}}_{\mathrm{aff}}(BG)$ is always Kan. If $S^{\mathcal{D}}_{\mathrm{aff}}(BG)$ is always Kan, the proof of Proposition 1.4 becomes simpler (see Lemma 6.1).

Let us apply Proposition 1.4 to special cases.

Example 6.11 (1) Let $\pi : Z \to X$ be a Galois covering with structure group Γ; see [Iglesias-Zemmour 2013, page 262]. Then for a given class $\alpha \in H^k(\Gamma; A)$ $(\cong H^k(B\Gamma; A))$, the class $\alpha(Z) \in H^k(X;A)$ is defined by Proposition 1.4.

(2) Let G be a diffeological group and H a diffeological subgroup of G. Then for a given class $\alpha \in H^k(BH;A)$, the class $\alpha(G) \in H^k(G/H;A)$ is defined by Proposition 1.4; see [Iglesias-Zemmour 2013, 8.15].

If a relevant diffeological principal bundle in Example 6.11 happens to be $\mathcal{D}$–numerable, then the class at issue is just the image of α under the homomorphism induced by the classifying map. However, this is not the case in general. See the following example, which specializes both parts (1) and (2) of Example 6.11.

Example 6.12 Let $\gamma : \mathbb{Z}^m \to \mathbb{R}^n$ be a monomorphism of abelian groups with $\Gamma := \gamma(\mathbb{Z}^m)$ dense, and consider the diffeological principal $\mathbb{Z}^m$–bundle $P := \mathbb{R}^n \xrightarrow{\pi} T_\Gamma$ over the irrational torus T_Γ (see Examples 6.7(1) and 6.11(2)); note that T_Γ is a diffeological group and that π is the universal covering of T_Γ.

Since $S^{\mathcal{D}}_{\mathrm{aff}}(T_\Gamma)$ is already Kan (see [Christensen and Wu 2014, Proposition 4.30 or Theorem 4.34]), $S^{\mathcal{D}}_{\mathrm{aff}}(\pi) : S^{\mathcal{D}}_{\mathrm{aff}}(P) \to S^{\mathcal{D}}_{\mathrm{aff}}(T_\Gamma)$ is a universal principal $\mathbb{Z}^m$–bundle (see Step 1 in the proof of Proposition 1.4), and hence, we have a classifying map $\varphi_P : S^{\mathcal{D}}_{\mathrm{aff}}(T_\Gamma) \to S^{\mathcal{D}}_{\mathrm{aff}}(B\mathbb{Z}^m)\hat{\ }$ which is obviously a homotopy equivalence in $\mathcal{S}$.

Since $S^{\mathcal{D}}_{\mathrm{aff}}(B\mathbb{Z}^m)\hat{\ }$ is just the Eilenberg–Mac Lane complex $K(\mathbb{Z}^m, 1)$, $H^*(B\mathbb{Z}^m; A) \cong (\Lambda\mathbb{Z}^m) \otimes A$. Thus, for any $\alpha \in H^*(B\mathbb{Z}^m; A)$, the characteristic class $\alpha(P) \in H^*(T_\Gamma; A)$ is just the image $\varphi_P^*(\alpha)$ under the isomorphism $H^*(T_\Gamma; A) \xleftarrow[\cong]{\varphi_P^*} H^*(B\mathbb{Z}^m; A)$.

On the other hand, since $\pi : P \to T_\Gamma$ is not locally trivial (see Example 5.2(1)), P has no classifying map to $B\mathbb{Z}^m$. Further, every nonzero element $\beta \in \widetilde{H}^*(T_\Gamma; A)$ is not contained in the image of the homomorphism induced by any smooth map $f : T_\Gamma \to B\mathbb{Z}^m$. In fact, we have the commutative diagram

$$\begin{tikzcd} S^{\mathfrak{D}}(T_\Gamma) \arrow[r, "S^{\mathfrak{D}}(f)"] \arrow[d] & S^{\mathfrak{D}}(B\mathbb{Z}^m) \arrow[d] \\ S(\widetilde{T}_\Gamma) \arrow[r, "S(\tilde{f})"] & S(\widetilde{B\mathbb{Z}^m}) \end{tikzcd}$$

(see Section 2.3). Since $S^{\mathfrak{D}}(B\mathbb{Z}^m) \to S(\widetilde{B\mathbb{Z}^m})$ is a homotopy equivalence (see [Kihara 2023, Corollary 5.16]) and $S(\widetilde{T}_\Gamma) \simeq *$, $S^{\mathfrak{D}}(f)$ is homotopic to a constant map. (We actually show that $B\mathbb{Z}^m$ is smoothly homotopy equivalent to the torus T^m, and hence that f is smoothly homotopic to a constant map; see a forthcoming paper.)

6.4 Sets of characteristic classes for the classes $\mathsf{P}\mathfrak{D}G$, $\mathsf{P}\mathfrak{D}G_{\mathrm{num}}$, and $\mathsf{P}\mathfrak{D}G_{\mathrm{diff}}$

In this subsection, we discuss the sets of characteristic classes for the classes (or categories) $\mathsf{P}\mathfrak{D}G$, $\mathsf{P}\mathfrak{D}G_{\mathrm{num}}$, and $\mathsf{P}\mathfrak{D}G_{\mathrm{diff}}$ (see Definition 5.1) and their relation.

Let $\mathcal{P}$ denote one of the categories $\mathsf{P}\mathfrak{D}G$, $\mathsf{P}\mathfrak{D}G_{\mathrm{num}}$, and $\mathsf{P}\mathfrak{D}G_{\mathrm{diff}}$. For an abelian group A, $\mathrm{char}(\mathcal{P}; A)$ denotes the set of characteristic classes with coefficients in A for the class $\mathcal{P}$. Then, by [Christensen and Wu 2021, Theorem 5.10] and Proposition 1.4, we have the natural bijection

$$\mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}}; A) \cong H^*(BG; A)$$

and the retract diagram

$$\begin{tikzcd} \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}}; A) \arrow[r, "\mathrm{ext}"] \arrow[rr, bend right, "1"'] & \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{diff}}; A) \arrow[r, "\mathrm{res}"] & \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}}; A) \end{tikzcd}$$

where res is the obvious restriction map and ext is the extension map introduced in Proposition 1.4.

We can also show that $\mathrm{char}(\mathsf{P}\mathfrak{D}G; A) \cong \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}}; A)$. To prove this, we define the map

$$\mathrm{ext}\colon \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}}; A) \to \mathrm{char}(\mathsf{P}\mathfrak{D}G; A)$$

as follows. Let $\alpha(\cdot)$ be an element of $\mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}}; A)$ corresponding to $\alpha \in H^*(BG; A)$. For a given locally trivial principal G–bundle $\pi : P \to X$, consider the CW–approximation $|S^{\mathfrak{D}}(X)|_{\mathfrak{D}} \xrightarrow{p_X} X$ in $\mathfrak{D}$, which is the counit of the adjoint pair $(|\cdot|_{\mathfrak{D}}, S^{\mathfrak{D}})$; see Remark 2.8(2) and [Kihara 2023, Section 3]. Since we can prove that every CW–complex in $\mathfrak{D}$ is smoothly paracompact (see [Kihara $\geq$ 2024]), the pullback $p_X^* P$ is a $\mathfrak{D}$–numerable principal G–bundle. Thus, we can define the characteristic class $\alpha(P)$ of P by $\alpha(P) = \alpha(p_X^* P)$ under the identification $H^*(X; A) \cong H^*(|S^{\mathfrak{D}}(X)|_{\mathfrak{D}}, A)$. Then it is clear that the map $\mathrm{ext}\colon \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}}; A) \to \mathrm{char}(\mathsf{P}\mathfrak{D}G; A)$ and the obvious restriction map

$$\mathrm{res}\colon \mathrm{char}(\mathsf{P}\mathfrak{D}G; A) \to \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}}; A)$$

are mutually inverses. We can easily see from Theorem 1.3 that $\mathrm{ext}\colon \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}}; A) \to \mathrm{char}(\mathsf{P}\mathfrak{D}G; A)$ is just the corestriction of $\mathrm{ext}\colon \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}}; A) \to \mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{diff}}; A)$. (Recall that the class of locally trivial principal G–bundles also does not have the homotopy invariance property with respect to pullback and hence that it has no classifying space; see [Christensen and Wu 2021, Section 3].)

We end this section by raising a problem on diffeological principal bundles.

Problem Let X be a CW–complex in $\mathscr{D}$ (or more generally, a cofibrant diffeological space); see [Kihara 2023, Section 3.1]. Is every diffeological principal G–bundle over X locally trivial?

This problem asks whether there exists a non-locally-trivial diffeological principal bundle over a nice diffeological space; all the non-locally-trivial diffeological principal bundles the author knows are ones over bad diffeological spaces.

If the problem is solved affirmatively, we can use the CW–approximation $|S^{\mathscr{D}}(X)|_{\mathscr{D}} \xrightarrow{p_X} X$ to directly construct the map

$$\mathrm{char}(\mathsf{P}\mathscr{D}G_{\mathrm{num}}; A) \xrightarrow{\mathrm{ext}} \mathrm{char}(\mathsf{P}\mathscr{D}G_{\mathrm{diff}}; A)$$

which is the inverse of $\mathrm{char}(\mathsf{P}\mathscr{D}G_{\mathrm{diff}}; A) \xrightarrow{\mathrm{res}} \mathrm{char}(\mathsf{P}\mathscr{D}G_{\mathrm{num}}; A)$.

Further, if the problem is solved affirmatively, then we can replace the singular functor $S^{\mathscr{D}}_{\mathrm{aff}}$ with $S^{\mathscr{D}}$ in Theorem 1.3 and Remark 5.8(1).

Remark 6.13 (1) Results similar to those mentioned above hold in the category $\mathscr{T}$ of topological spaces. More precisely, the homotopy invariance property with respect to pullback need not hold for topological principal G–bundles which are not numerable, and hence the class of topological principal G–bundles does not have a classifying space; see [Andrade 2013; Christensen and Wu 2021, Section 3; Goodwillie 2012]. However, we have two ways of extending the characteristic class associated to a cohomology class α of the (topological) classifying space BG; one uses the CW–approximation $|S(X)| \xrightarrow{p_X} X$ of the base and the other uses the theory of simplicial principal bundles. We can easily see that they define the same extension; the resulting map is denoted by

$$\mathrm{char}(\mathsf{P}\mathscr{T}G_{\mathrm{num}}; A) \xrightarrow{\mathrm{ext}} \mathrm{char}(\mathsf{P}\mathscr{T}G; A),$$

where $\mathrm{char}(\mathsf{P}\mathscr{T}G_{\mathrm{num}}; A)$ and $\mathrm{char}(\mathsf{P}\mathscr{T}G; A)$ are defined in a way similar to the diffeological case. We then see that

$$\mathrm{char}(\mathsf{P}\mathscr{T}G_{\mathrm{num}}; A) \cong H^*(BG; A)$$

and that

$$\mathrm{char}(\mathsf{P}\mathscr{T}G_{\mathrm{num}}; A) \underset{\mathrm{res}}{\overset{\mathrm{ext}}{\rightleftarrows}} \mathrm{char}(\mathsf{P}\mathscr{T}G; A)$$

are mutually inverses.

The results here remain true even if $\mathscr{T}$ is replaced with the category $\mathscr{C}^0$ of arc-generated spaces; see [Kihara 2023, Proposition 5.14(1)].

(2) Since the underlying topological space functor $\tilde{\cdot} : \mathscr{D} \to \mathscr{C}^0$ preserves finite products [Kihara 2019, Proposition 2.13], it induces the functor

$$\tilde{\cdot} : \mathsf{P}\mathscr{D}G \to \mathsf{P}\mathscr{C}^0\tilde{G}$$

(see [Kihara 2023, Lemma 5.7 and Remark 5.8]). Thus, we use this functor to study the relation between characteristic classes of smooth principal G–bundles and ones of continuous principal G–bundles.

The natural inclusion $S^{\mathcal{D}}X \hookrightarrow S\widetilde{X}$ (see Section 2.3) induces the natural homomorphism

$$H^*(X;A) \xleftarrow{\psi_X} H^*(\widetilde{X};A),$$

which along with [Kihara 2023, Proposition 5.14], defines the horizontal arrows in the commutative diagram

$$\begin{tikzcd}
H^*(BG;A) \arrow[d, "\wr\|"] & H^*(B\widetilde{G};A) \arrow[l, "\psi_{BG}"'] \arrow[d, "\wr\|"'] \\
\mathrm{char}(\mathsf{P}\mathfrak{D}G_{\mathrm{num}};A) \arrow[d, "\mathrm{ext}"', "\wr\|"] & \mathrm{char}(\mathsf{P}\mathscr{C}^0\widetilde{G}_{\mathrm{num}};A) \arrow[l] \arrow[d, "\wr\|"', "\mathrm{ext}"] \\
\mathrm{char}(\mathsf{P}\mathfrak{D}G;A) & \mathrm{char}(\mathsf{P}\mathscr{C}^0\widetilde{G};A) \arrow[l]
\end{tikzcd}$$

We can easily see that the equality

$$(\psi_{BG}\alpha)(P) = \psi_X(\alpha(\widetilde{P}))$$

holds for $P \in \mathsf{P}\mathfrak{D}G$.

If G is a Lie group (or more generally, in the class $\mathscr{V}_{\mathfrak{D}}$), then $H^*(BG;A) \xleftarrow{\psi_{BG}} H^*(B\widetilde{G};A)$ is an isomorphism (see [Kihara 2023, Theorem 11.2, and Corollaries 1.6 and 5.16]), and hence all the arrows in the above commutative diagram are bijective. (Here, a Lie group is defined to be a group in the category C^∞ of C^∞–manifolds in the sense of [Kriegl and Michor 1997, Section 27]; see [Kihara 2023, Section 2.2].)

Acknowledgements

In the conference *Building-up differentiable homotopy theory* 2020 held at Shinshu University, Prof. J D Christensen gave an excellent talk on smooth classifying spaces. After the talk, Prof. Katsuhiko Kuribayashi asked me whether the "only if" part of Theorem 1.3(1) holds and then we briefly discussed this subject with Prof. Christensen, which led me to Proposition 1.4. I would like to show my greatest appreciation to them.

References

[Andrade 2013] **R Andrade**, *Example of fiber bundle that is not a fibration*, MathOverflow answer (2013) Available at `https://mathoverflow.net/q/126228`

[Bröcker and Jänich 1982] **T Bröcker**, **K Jänich**, *Introduction to differential topology*, Cambridge Univ. Press (1982) MR Zbl

[Bunk 2022] **S Bunk**, *The $\mathbb{R}$–local homotopy theory of smooth spaces*, J. Homotopy Relat. Struct. 17 (2022) 593–650 MR Zbl

[Christensen and Wu 2014] **J D Christensen**, **E Wu**, *The homotopy theory of diffeological spaces*, New York J. Math. 20 (2014) 1269–1303 MR Zbl

[Christensen and Wu 2021] **J D Christensen**, **E Wu**, *Smooth classifying spaces*, Israel J. Math. 241 (2021) 911–954 MR Zbl

[Christensen et al. 2014] **J D Christensen**, **G Sinnamon**, **E Wu**, *The D–topology for diffeological spaces*, Pacific J. Math. 272 (2014) 87–110 MR Zbl

[Dwyer and Spaliński 1995] **W G Dwyer**, **J Spaliński**, *Homotopy theories and model categories*, from "Handbook of algebraic topology" (I M James, editor), North-Holland, Amsterdam (1995) 73–126 MR Zbl

[Eilenberg and Mac Lane 1953] **S Eilenberg**, **S Mac Lane**, *Acyclic models*, Amer. J. Math. 75 (1953) 189–199 MR Zbl

[Frölicher and Kriegl 1988] **A Frölicher**, **A Kriegl**, *Linear spaces and differentiation theory*, Wiley, New York (1988) MR Zbl

[Gabriel and Zisman 1967] **P Gabriel**, **M Zisman**, *Calculus of fractions and homotopy theory*, Ergeb. Math. Grenzgeb. 35, Springer (1967) MR Zbl

[Goerss and Jardine 1999] **P G Goerss**, **J F Jardine**, *Simplicial homotopy theory*, Progress in Mathematics 174, Birkhäuser, Basel (1999) MR Zbl

[Goodwillie 2012] **T Goodwillie**, *Non trivial vector bundle over non-paracompact contractible space*, MathOverflow answer (2012) Available at `https://mathoverflow.net/q/106563`

[Hector 1995] **G Hector**, *Géométrie et topologie des espaces difféologiques*, from "Analysis and geometry in foliated manifolds" (I Bajo, E Sanmartín, editors), World Sci., River Edge, NJ (1995) 55–80 MR Zbl

[Iglesias-Zemmour 2013] **P Iglesias-Zemmour**, *Diffeology*, Mathematical Surveys and Monographs 185, Amer. Math. Soc., Providence, RI (2013) MR Zbl

[Iglesias-Zemmour 2024] **P Iglesias-Zemmour**, *Čech–De Rham bicomplex in diffeology*, Israel J. Math. 259 (2024) 239–276 MR Zbl

[Kihara 2014] **H Kihara**, *Minimal fibrations and the organizing theorem of simplicial homotopy theory*, Ric. Mat. 63 (2014) 79–91 MR Zbl

[Kihara 2019] **H Kihara**, *Model category of diffeological spaces*, J. Homotopy Relat. Struct. 14 (2019) 51–90 MR Zbl

[Kihara 2023] **H Kihara**, *Smooth homotopy of infinite-dimensional C^∞–manifolds*, Mem. Amer. Math. Soc. 1436, Amer. Math. Soc., Providence, RI (2023) MR Zbl

[Kihara $\geq$ 2024] **H Kihara**, *De Rham calculus on diffeological spaces*, in preparation

[Kriegl and Michor 1997] **A Kriegl**, **P W Michor**, *The convenient setting of global analysis*, Math. Surv. Monogr. 53, Amer. Math. Soc., Providence, RI (1997) MR Zbl

[Kuribayashi 2020] **K Kuribayashi**, *Simplicial cochain algebras for diffeological spaces*, Indag. Math. 31 (2020) 934–967 MR Zbl

[Kuribayashi 2021] **K Kuribayashi**, *A comparison between two de Rham complexes in diffeology*, Proc. Amer. Math. Soc. 149 (2021) 4963–4972 MR Zbl

[Mac Lane 1998] **S Mac Lane**, *Categories for the working mathematician*, 2nd edition, Graduate Texts in Math. 5, Springer (1998) MR Zbl

[May 1992] **J P May**, *Simplicial objects in algebraic topology*, Univ. Chicago Press (1992) MR Zbl

[May and Ponto 2012] **J P May**, **K Ponto**, *More concise algebraic topology*: *localization*, *completion*, *and model categories*, Univ. Chicago Press (2012) MR Zbl

[Milne 2017] **J S Milne**, *Algebraic groups: the theory of group schemes of finite type over a field*, Cambridge Studies in Advanced Mathematics 170, Cambridge Univ. Press (2017) MR Zbl

[Onishchik and Vinberg 1994] **A L Onishchik**, **E B Vinberg** (editors), *Lie groups and Lie algebras, III: Structure of Lie groups and Lie algebras*, Encycl. Math. Sci. 41, Springer (1994) MR Zbl

Center for Mathematical Sciences, University of Aizu
Aizuwakamatsu, Japan

kihara@u-aizu.ac.jp

Received: 8 November 2021 Revised: 28 April 2023

Algebraic & Geometric Topology **24**:4 (2024) 1953–2010
DOI: 10.2140/agt.2024.24.1953
Published: 16 July 2024

Natural symmetries of secondary Hochschild homology

DAVID AYALA
JOHN FRANCIS
ADAM HOWARD

We identify the group of framed diffeomorphisms of the torus as a semidirect product of the torus with the braid group on three strands; we also identify the topological monoid of framed local diffeomorphisms of the torus in similar terms. It follows that the framed mapping class group is this braid group. We show that the group of framed diffeomorphisms of the torus acts on twice-iterated Hochschild homology, and explain how this recovers a host of familiar symmetries. In the case of cartesian monoidal structures, we show that this action extends to the monoid of framed local diffeomorphisms of the torus. Based on this, we propose a definition of an unstable secondary cyclotomic structure, and show that iterated Hochschild homology possesses such in the cartesian monoidal setting.

58D05; 16E40, 58D27

Introduction

Here are our five main results, all of which are motivated by the study of *factorization homology* as developed in [Ayala and Francis 2015]. We direct a reader to the body of the paper for definitions of terms and notation, in particular of the highlighted terms, as well as precise statements and proofs.

Regard the 2–torus $\mathbb{T}^2$ as a framed 2–manifold via a translation-invariant framing.

Theorem X(2)(a) *There is an equivalence between continuous groups*

$$\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \xrightarrow{\simeq} \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2).$$

This homomorphism is given as follows:

- Translation in the group $\mathbb{T}^2$ defines a continuous homomorphism $\mathbb{T}^2 \to \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)$.

- Sheering in each coordinate supplies two extensions from semidirect products,

$$\mathbb{T}^2 \rtimes_{U_1} \mathbb{Z} \to \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \leftarrow \mathbb{T}^2 \rtimes_{U_2} \mathbb{Z},$$

where $U_1 = \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$ and $U_2 = \left[\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right]$, thereby resulting in a single extension

$$\mathbb{T}^2 \rtimes \langle U_1, U_2\rangle \to \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2), \tag{0-0-1}$$

involving the free group on the two abstract generators U_1 and U_2.

- As there is an equality of matrices $U_1U_2U_1 = \left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right] = U_2U_1U_2$, the restrictions of (0-0-1) along the two abstractly isomorphic subgroups $\mathbb{T}^2 \rtimes \langle U_1U_2U_1\rangle \cong \mathbb{T}^2 \rtimes \mathbb{Z} \cong \mathbb{T}^2 \rtimes \langle U_2U_1U_2\rangle$ can be identified, thereby supplying a morphism from the coequalizer among continuous groups

$$\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \simeq \mathbb{T}^2 \rtimes \langle U_1, U_2 \mid U_1U_2U_2 = U_2U_1U_2\rangle \to \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2), \tag{0-0-2}$$

involving a standard presentation of the braid group on three strands.

Theorem X(2)(b) *There is an equivalence between continuous monoids*

$$\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \xrightarrow{\simeq} \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2),$$

involving a central extension among monoids

$$\mathbb{Z} \to \widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \to \mathrm{E}_2^+(\mathbb{Z}) := \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}) \mid \det(A) > 0\}.$$

Proposition 0.3.4 *Let $\mathcal{X}$ be an ∞–category. Then the morphism $\widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \to \mathrm{E}_2^+(\mathbb{Z}) \to \mathrm{End}_{\mathrm{Groups}}(\mathbb{T}^2)$ determines an action by $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$ on the ∞–category $\mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2}$ of **finite-genuine $\mathbb{T}^2$–modules** in $\mathcal{X}$. A finite-genuine $\mathbb{T}^2$–module in $\mathcal{X}$ that is coherently invariant with respect to this $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$–action is simply an* $\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}}$*–module in $\mathcal{X}$* (see Remark 0.3.5):

$$\mathrm{Mod}_{\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}}}(\mathcal{X}) \simeq (\mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2})^{\widetilde{\mathrm{E}}_2^+(\mathbb{Z})}.$$

In particular, there is a forgetful functor

$$\mathrm{Mod}_{\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}}}(\mathcal{X}) \to \mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2}.$$

We define an *unstable secondary cyclotomic structure* to be an $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$–invariant finite-genuine $\mathbb{T}^2$–module. (See Remark 0.3.2.)

Theorem Y.1 *Let $\mathcal{V}$ be a symmetric monoidal ∞–category that is $\otimes$–presentable. Let A be a **2–algebra** in $\mathcal{V}$. Via factorization homology, there is a canonical action*

$$\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \simeq \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \curvearrowright \mathrm{HH}^{(2)}(A)$$

on the twice-iterated Hochschild homology of A.

This action is given as follows:

- The action $\mathbb{T}^2 \curvearrowright \mathrm{HH}^{(2)}(A)$ is Connes' cyclic operators.
- For $i = 1, 2$, the extension $\mathbb{T}^2 \rtimes_{U_i} \mathbb{Z} \curvearrowright \mathrm{HH}^{(2)}(A)$ is a canonical sheering action of the Connes cyclic operators.

- There is an identification between the actions $\mathbb{Z} \underset{U_1U_2U_1}{\curvearrowright} \mathrm{HH}^{(2)}(A)$ and $\mathbb{Z} \underset{U_2U_1U_2}{\curvearrowright} \mathrm{HH}^{(2)}(A)$, thereby giving the action $\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \curvearrowright \mathrm{HH}^{(2)}(A)$.

Theorem Y.2 *Let $\mathcal{X}$ be a presentable ∞–category in which products distribute over colimits. Regard $\mathcal{X}$ as a symmetric monoidal ∞–category via its cartesian monoidal structure. Let A be a 2–algebra in $\mathcal{X}$. Via factorization homology, the twice-iterated Hochschild homology of A is canonically endowed with an unstable secondary cyclotomic structure*:

$$\big((\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z}))^{\mathrm{op}} \simeq \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}} \curvearrowright \mathrm{HH}^{(2)}(A)\big) \in (\mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2})^{\widetilde{\mathrm{E}}_2^+(\mathbb{Z})}.$$

In other words, $\mathrm{HH}^{(2)}(A)$ canonically has the structure of an $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$–invariant finite-genuine $\mathbb{T}^2$–module.

The remainder of this introduction contextualizes then restates these results.

Conventions • We work in the ∞–category $\mathcal{S}$paces of spaces, or ∞–groupoids, an object in which is a *space*. This ∞–category can be presented as the ∞–categorical localization of the ordinary category of compactly generated Hausdorff topological spaces that are homotopy equivalent with a CW complex, localized on the weak homotopy equivalences. So we present some objects in $\mathcal{S}$paces by naming a topological space.

• By a pullback square among spaces we mean a pullback square in the ∞–category $\mathcal{S}$paces. Should the square be presented by a homotopy-commutative square among topological spaces, then the canonical map from the initial term in the square to the homotopy pullback is a weak homotopy equivalence.

• By a *continuous group* (resp. *continuous monoid*) we mean a group-object (resp. monoid-object) in $\mathcal{S}$paces. A continuous monoid N determines a pointed $(\infty,1)$–category $\mathfrak{B}N$, which can be presented by the Segal space $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{Bar}_\bullet(N)} \mathcal{S}\mathrm{paces}$, which is the bar construction of N. For $X \in \mathcal{X}$ an object in an ∞–category, and for N a continuous monoid, an *action of N on X*, denoted by $N \curvearrowright X$, is an extension $\langle X\rangle\colon * \to \mathfrak{B}N \xrightarrow{\langle N \curvearrowright X\rangle} \mathcal{X}$. The ∞–category of (*left*) *N–modules in $\mathcal{X}$* is

$$\mathrm{Mod}_N(\mathcal{X}) := \mathrm{Fun}(\mathfrak{B}N, \mathcal{X}).$$

Every continuous group can be strictified to a topological group (ie a group-object in the ordinary category of topological spaces), but maps among such are more flexible (corresponding to maps of loop spaces), as not all topological groups are cofibrant with respect to the usual model structure.

• For $G \curvearrowright X$ an action of a continuous group on a space, the space of *coinvariants* is the colimit

$$X_{/G} := \mathrm{colim}(\mathfrak{B}G \xrightarrow{\langle G \curvearrowright X\rangle} \mathcal{S}\mathrm{paces}) \in \mathcal{S}\mathrm{paces}.$$

Should the action $G \curvearrowright X$ be presented by a continuous action of a topological group on a topological space, then this space of coinvariants can be presented by the homotopy coinvariants.

• We work with ∞–operads, as developed in [Lurie 2017]. As such, they are implicitly symmetric. Some ∞–operads are presented as discrete operads, such as Assoc, while some are presented as topological operads, such as the little 2–disks operad $\mathcal{E}_2$.

0.1 Moduli and isogeny of framed tori

Here we restate our first result, which identifies the entire symmetries of a framed torus.

The *braid group on three strands* can be presented as

$$\mathrm{Braid}_3 \cong \langle \tau_1, \tau_2 \mid \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2 \rangle. \tag{0-1-1}$$

Through this presentation, there is a standard representation

$$\Phi\colon \mathrm{Braid}_3 \xrightarrow{\langle \tau_1 \mapsto U_1, \tau_2 \mapsto U_2 \rangle} \mathrm{GL}_2(\mathbb{Z}) \quad \text{where } U_1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } U_2 := \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}. \tag{0-1-2}$$

The homomorphism Φ defines an action $\mathrm{Braid}_3 \xrightarrow{\Phi} \mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ as a topological group. This action defines a topological group:

$$\mathbb{T}^2 \rtimes \mathrm{Braid}_3.$$

The following result, which is essentially due to Milnor, is our starting point.

Proposition 0.1.1 [Milnor 1971, Section 10] *The image of Φ is the subgroup $\mathrm{SL}_2(\mathbb{Z})$; the kernel of Φ is central, and is freely generated by the element $(\tau_1\tau_2)^6 \in \mathrm{Braid}_3$. Equivalently, Φ fits into a central extension among groups:*

$$1 \to \mathbb{Z} \xrightarrow{\langle (\tau_1\tau_2)^6 \rangle} \mathrm{Braid}_3 \xrightarrow{\Phi} \mathrm{SL}_2(\mathbb{Z}) \to 1. \tag{0-1-3}$$

Furthermore, this central extension (0-1-3) is classified by the element

$$[B\mathrm{SL}_2(\mathbb{Z}) \xrightarrow{B(\mathbb{R}\otimes_{\mathbb{Z}})} B\mathrm{SL}_2(\mathbb{R}) \simeq B^2\mathbb{Z}] \in \mathrm{H}^2(\mathrm{SL}_2(\mathbb{Z}); \mathbb{Z}).$$

That is, there is a canonical top horizontal homomorphism defining a pullback among groups:

$$\begin{tikzcd}
\mathrm{Braid}_3 \arrow[r, dashed] \arrow[d, "\Phi"'] & \widetilde{\mathrm{SL}}_2(\mathbb{R}) \arrow[d, "\text{universal cover}"] \\
\mathrm{SL}_2(\mathbb{Z}) \arrow[r, "\mathbb{R}\otimes_{\mathbb{Z}}", "\text{standard}"'] & \mathrm{SL}_2(\mathbb{R})
\end{tikzcd}$$

Consider the subgroup $\mathrm{GL}_2^+(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$ consisting of those 2×2 matrices with positive determinant — it is the connected component of the identity matrix. Consider the submonoid

$$\mathbb{R}\otimes_{\mathbb{Z}} \colon \mathrm{E}_2^+(\mathbb{Z}) \subset \mathrm{GL}_2^+(\mathbb{R})$$

consisting of those 2×2 matrices with positive determinant whose entries are integers. Consider the pullback[1] among monoids

$$\begin{tikzcd}
\widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \arrow[r] \arrow[d, "\Psi"'] & \widetilde{\mathrm{GL}}_2^+(\mathbb{R}) \arrow[d, "\text{universal cover}"] \\
\mathrm{E}_2^+(\mathbb{Z}) \arrow[r, "\mathbb{R}\otimes_{\mathbb{Z}}"] & \mathrm{GL}_2^+(\mathbb{R})
\end{tikzcd} \tag{0-1-4}$$

[1] See Remark B.2.4 for an explicit description of the monoid $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$.

This morphism Ψ supplies a canonical action $\widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \xrightarrow{\Psi} \mathrm{E}_2^+(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ as a topological group. This action defines a topological monoid

$$\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z}).$$

Convention By way of Section B.1, in particular Corollary B.1.2, we regard all actions of Braid_3 and $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$ as *left*-actions.

For $\varphi\colon \tau_{\mathbb{T}^2} \cong \epsilon^2_{\mathbb{T}^2}$ a framing of the torus, we introduce as Definition 1.3.8 the continuous group of *framed diffeomorphisms*, and the continuous monoid of *framed local diffeomorphisms* of the torus,

$$\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2,\varphi) \quad \text{and} \quad \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2,\varphi).$$

For φ_0 the *standard framing* of $\mathbb{T}^2$, which is invariant with respect to translation in the torus, we simply write

$$\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) := \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2,\varphi_0) \quad \text{and} \quad \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2) := \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2,\varphi_0).$$

Theorem X (1) *The map from the set of homotopy classes of framings of $\mathbb{T}^2$ to the set of framed-diffeomorphism-types of tori*,

$$\pi_0\,\mathrm{Fr}(\mathbb{T}^2) \to \pi_0\mathcal{M}_1^{\mathrm{fr}},$$

is canonically equivalent to the map

$$\mathbb{Z}^2 \times \mathbb{Z}_{/2\mathbb{Z}} \to \mathbb{Z}_{\geq 0} \quad \text{given by} \quad \left(\begin{bmatrix} u \\ v \end{bmatrix}, \sigma\right) \mapsto \gcd(u,v).$$

Furthermore, a framing $\varphi \in \mathrm{Fr}(\mathbb{T}^2)$ is homotopic to one that is translation invariant if and only if it is carried to the 0*–component of $\mathcal{M}_1^{\mathrm{fr}}$.*

(2) *Let $\varphi\colon \tau_{\mathbb{T}^2} \cong \epsilon_{\mathbb{T}^2}$ be a framing of the torus.*

(a) *There is a canonical identification of the continuous group of framed diffeomorphisms of $(\mathbb{T}^2,\varphi)$:*

$$\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2,\varphi) \simeq \begin{cases} \mathbb{T}^2 \rtimes \mathrm{Braid}_3 & \text{if } \varphi \text{ is homotopic to a translation-invariant framing,} \\ (\mathbb{T}^2 \rtimes \mathbb{Z}) \times \mathbb{Z} & \text{if } \varphi \text{ is not homotopic to a translation-invariant framing.} \end{cases}$$

(b) *There is a canonical identification of the continuous monoid of framed local diffeomorphisms of $(\mathbb{T}^2,\varphi)$:*

$$\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2,\varphi) \simeq \begin{cases} \mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z}) & \text{if } \varphi \text{ is homotopic to a translation-invariant framing,} \\ (\mathbb{T}^2 \rtimes (\mathbb{Z} \rtimes \mathbb{N}^\times)) \times \mathbb{Z} & \text{if } \varphi \text{ is not homotopic to a translation-invariant framing.} \end{cases}$$

(*See Notation 1.4.1 for a description of lower semidirect products.*)

Taking path-components, Theorem X(2)(a) has the following immediate consequence:

Corollary 0.1.2 *Let φ be a framing of the torus. There is a canonical identification of the framed mapping class group of $(\mathbb{T}^2,\varphi)$ as a subgroup of the braid group on three strands*:

$$\mathrm{MCG}^{\mathrm{fr}}(\mathbb{T}^2,\varphi) \subset \mathrm{Braid}_3.$$

If φ is homotopic with a translation-invariant framing, this subgroup is entire. If φ is not homotopic with a translation-invariant framing, this subgroup is conjugate with a standard subgroup,

$$\mathrm{MCG}^{\mathrm{fr}}(\mathbb{T}^2, \varphi) \overset{\text{conjugate}}{\cong} \langle \tau_1, (\tau_1\tau_2)^6 \rangle \cong \mathbb{Z} \times \mathbb{Z},$$

which is abstractly isomorphic with $\mathbb{Z} \times \mathbb{Z}$.

Remark 0.1.3 Consider the moduli space $\mathcal{M}_1^{\mathrm{fr}}$ of framed tori. Theorem X(1) and (2)(a) can be phrased as the assertion that $\mathcal{M}_1^{\mathrm{fr}}$ has $\mathbb{Z}_{\geq 0}$–many path-components, with the 0–path-component the space of homotopy coinvariants $(\mathbb{CP}^\infty)^2{}_{/\mathrm{Braid}_3}$ with respect to the action $\mathrm{Braid}_3 \xrightarrow{\Phi} \mathrm{GL}_2(\mathbb{Z}) \curvearrowright B^2\mathbb{Z}^2 \simeq (\mathbb{CP}^\infty)^{\times 2}$, and each other path-component the space $(\mathbb{CP}^\infty)^2{}_{/\mathbb{Z}} \times B\mathbb{Z}$ in which the coinvariants are with respect to the action $\mathbb{Z} \xrightarrow{\langle U_1 \rangle} \mathrm{GL}_2(\mathbb{Z}) \curvearrowright B^2\mathbb{Z}^2 \simeq (\mathbb{CP}^\infty)^{\times 2}$. A neat result of Milnor [1971, Section 10] gives an isomorphism between groups:

$$\mathrm{Braid}_3 \cong \pi_1(\mathbb{S}^3 \smallsetminus \mathrm{Trefoil}).$$

Using that $\mathbb{S}^3 \smallsetminus \mathrm{Trefoil}$ is a path-connected 1–type, this isomorphism reveals that the 0–path-component $(\mathcal{M}_1^{\mathrm{fr}})_0 \subset \mathcal{M}_1^{\mathrm{fr}}$ fits into a fiber sequence of spaces:

$$(\mathbb{CP}^\infty)^2 \to (\mathcal{M}_1^{\mathrm{fr}})_0 \to (\mathbb{S}^3 \smallsetminus \mathrm{Trefoil}).$$

Dehn [1938, Section 6] identified the oriented mapping class group of a punctured torus with parametrized boundary as the braid group on three strands, as it is equipped with a homomorphism to the oriented mapping class group of the torus. Through Corollary 0.1.2, this results in an identification between these mapping class groups. The next result lifts this identification to continuous groups; it is proved in Section 1.4.

Corollary 0.1.4 *Fix a smooth framed embedding from the closed 2–disk $\mathbb{D}^2 \hookrightarrow \mathbb{T}^2$ extending the inclusion $\{0\} \hookrightarrow \mathbb{T}^2$ of the identity element. There are canonical identifications*[2] *among continuous groups over* $\mathrm{Diff}(\mathbb{T}^2)$:

$$\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \operatorname{rel} 0) \simeq \mathrm{Braid}_3 \simeq \mathrm{Diff}(\mathbb{T}^2 \operatorname{rel} \mathbb{D}^2).$$

In particular, there are canonical isomorphisms among groups over $\mathrm{MCG}(\mathbb{T}^2)$:

$$\mathrm{MCG}^{\mathrm{fr}}(\mathbb{T}^2) \cong \mathrm{Braid}_3 \cong \mathrm{MCG}(\mathbb{T}^2 \smallsetminus \mathbb{B}^2 \operatorname{rel} \partial),$$

where $\mathbb{B}^2 \subset \mathbb{D}^2$ is the open 2–ball.

Using Theorem X(2)(a), the presentation (0-1-1) of the braid group Braid_3 lends to a simple (fully homotopy coherent) description of an action by $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)$. We articulate this description as the following result, which is proved at the end of Section 1.5, and requires a bit of setup to state.

[2]This composite equivalence of continuous groups can be witnessed by a span among continuous groups, $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \operatorname{rel} 0) \xleftarrow{\simeq} \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \operatorname{rel} \mathbb{D}^2) \to \mathrm{Diff}(\mathbb{T}^2 \operatorname{rel} \mathbb{D}^2)$, in which the leftward map is an equivalence via routine methods. The more novel aspect of this result can then be rephrased as the rightward map being an equivalence. A quick explanation of this fact is that the space of framings of $\mathbb{T}^2$, fixed at $0 \in \mathbb{T}^2$, has contractible path-components; see Theorem X(1).

Setup Let $\mathcal{X}$ be an ∞–category. Let G be a continuous group. Consider the ∞–category $\mathrm{Mod}_G(\mathcal{X})$ of G–modules in $\mathcal{X}$. Let T be an automorphism of the continuous group G. Via pullback, T determines an automorphism $T^*\colon (G \curvearrowright X) \mapsto (G \xrightarrow{T} G \curvearrowright X)$ of $\mathrm{Mod}_G(\mathcal{X})$. Denote the ∞–category of T–invariant G–modules by $\mathrm{Mod}_G(\mathcal{X})^{\langle T\rangle}$, an object in which is a G–module $(G \curvearrowright X)$ in $\mathcal{X}$ together with an identification $(G \xrightarrow{T} G \curvearrowright X) \simeq (G \curvearrowright X)$ between G–modules in $\mathcal{X}$. Similarly, for S and T automorphisms of G, the ∞–category of G–modules that are both S– and T–invariant is $\mathrm{Mod}_G(\mathcal{X})^{\langle S,T\rangle}$, an object in which is a G–module $(G \curvearrowright X)$ in $\mathcal{X}$ together with identifications $(G \xrightarrow{S} G \curvearrowright X) \overset{\gamma_S}{\simeq} (G \curvearrowright X)$ and $(G \xrightarrow{T} G \curvearrowright X) \overset{\gamma_T}{\simeq} (G \curvearrowright X)$ between G–modules in $\mathcal{X}$.

Now, via the standard homomorphism $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathrm{Groups}}(\mathbb{T}^2)$, regard the matrices

$$U_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

as automorphisms of the continuous group $\mathbb{T}^2$.

Corollary 0.1.5 *Let $\mathcal{X}$ be an ∞–category. There is a pullback diagram among ∞–categories*

$$\begin{tikzcd}
\mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)}(\mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})^{\langle U_1,U_2\rangle} \arrow[d] \\
\mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})^{\langle R\rangle} \arrow[r] & \mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})^{\langle R,R\rangle}
\end{tikzcd}$$

In particular, for $X \in \mathcal{X}$ an object, an action $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \curvearrowright X$ is

(1) *an action $\mathbb{T}^2 \underset{\alpha}{\curvearrowright} X$,*

(2) *an identification $\alpha \circ R \overset{\gamma_R}{\simeq} \alpha$ of this action α with the action $\mathbb{T}^2 \xrightarrow{R} \mathbb{T}^2 \underset{\alpha}{\curvearrowright} X$,*

(3) *for $i = 1, 2$, extensions of γ_R to identifications $\alpha \circ U_i \overset{\gamma_{U_i}}{\simeq} \alpha$.*

A generalization of Smale's conjecture to Haken manifolds, proved by Hatcher [1976; 1983], gives that the standard inclusion is an equivalence between continuous groups:

$$\mathrm{Aff}\colon \mathbb{T}^3 \rtimes \mathrm{GL}_3(\mathbb{Z}) \xrightarrow{\simeq} \mathrm{Diff}(\mathbb{T}^3).$$

In particular, there is an identification of the mapping class group: $\mathrm{MCG}(\mathbb{T}^3) \cong \mathrm{GL}_3(\mathbb{Z})$. Using these identifications, we expect our methods could be used to prove the following:

Conjecture 1 *Consider the 3–torus, $\mathbb{T}^3 \cong \mathbb{R}^3_{/\mathbb{Z}^3}$, as it is equipped with its standard framing. There is a canonical identification between continuous groups*

$$\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^3) \simeq (\mathbb{T}^3 \rtimes \Omega(\mathrm{SL}_3(\mathbb{R})_{/\mathrm{SL}_3(\mathbb{Z})})) \times (\Omega^2\mathbb{S}^3 \times \Omega^3\mathbb{S}^3)^3 \times \Omega^4\mathbb{S}^3,$$

in which the semidirect product is with respect to the action $\Omega(\mathrm{SL}_3(\mathbb{R})_{/\mathrm{SL}_3(\mathbb{Z})}) \xrightarrow{\mathrm{Puppe}} \mathrm{SL}_3(\mathbb{Z}) \curvearrowright \mathbb{T}^3$. In particular, there is a central extension among groups:

$$1 \to \mathbb{Z}^3 \times (\mathbb{Z}_{/2\mathbb{Z}})^2 \to \mathrm{MCG}^{\mathrm{fr}}(\mathbb{T}^3) \to \mathrm{SL}_3(\mathbb{Z}) \to 1.$$

0.2 Natural symmetries of secondary Hochschild homology

0.2.1 Hochschild homology

Notation 0.2.1 In Section 0.2.1 we fix $\mathcal{W}$ to be an $\otimes$–presentable symmetric monoidal ∞–category.

We briefly recall a definition of the Hochschild homology and record its natural symmetries. (See [Loday 1992] for a complete account.) Let $B \in \mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{W})$ be an associative algebra. Via left and right translation, regard the underlying object $B \in \mathcal{W}$ as a (B, B)–bimodule. For M a (B, B)–bimodule for B, the *Hochschild homology* (*of B with coefficients in M*) is

$$\mathrm{HH}(B, M) := B \otimes_{B^{\mathrm{op}} \otimes B} M \simeq \mathrm{colim}(\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{B^{\otimes \bullet} \otimes M} \mathcal{W}),$$

which can be constructed as the colimit of a simplicial object in $\mathcal{W}$ naturally associated to the pair (B, M).

Remark For $0 < i < p$, the i^{th} face map of this simplicial object is

$$B^{\otimes\{1,\dots,p\}} \otimes M \simeq B^{\otimes\{1,\dots,i\}} \otimes B^{\otimes\{i,i+1\}} \otimes B^{\otimes\{i+2,\dots,p\}} \otimes M \xrightarrow{\mathrm{id} \otimes \mu \otimes \mathrm{id} \otimes \mathrm{id}} B^{\otimes\{1,\dots,i\}} \otimes B \otimes B^{\otimes\{i+2,\dots,p\}} \otimes M,$$

where μ is the binary multiplication of A. The 0^{th} face map is

$$B^{\otimes\{1,\dots,p\}} \otimes M \simeq B^{\{1\}} \otimes B^{\otimes\{2,\dots,p\}} \otimes M \simeq B^{\otimes\{2,\dots,p\}} \otimes M \otimes B^{\{1\}} \xrightarrow{\mathrm{id} \otimes \mathrm{r.act}} B^{\otimes\{2,\dots,p\}} \otimes M,$$

where r.act is the right action of B on M. The p^{th} face map is

$$B^{\otimes\{1,\dots,p\}} \otimes M \simeq B^{\otimes\{1,\dots,p-1\}} \otimes B^{\{p\}} \otimes M \xrightarrow{\mathrm{id} \otimes \mathrm{l.act}} B^{\otimes\{1,\dots,p-1\}} \otimes M,$$

where l.act is the left action of B on M.

This is functorial in the (B, B)–bimodule

$$\mathrm{BiMod}_{(B,B)} \xrightarrow{\mathrm{HH}(B,-)} \mathcal{W}.$$

The *Hochschild homology* (*of B*) is the instance in which $M = B$ as a (B, B)–bimodule:

$$\mathrm{HH}(B) := B \otimes_{B^{\mathrm{op}} \otimes B} B =: \mathrm{HH}(B, B) \simeq |\mathrm{Bar}^{\mathrm{cyc}}_{\bullet}(B)|,$$

which can be constructed as a geometric realization of the *cyclic bar complex* of B, as recalled in Section 2.1. Also recalled in Section 2.1 is a canonical action $\mathbb{T} \simeq B\mathbb{Z} \curvearrowright \mathrm{HH}(B)$,

$$\mathbb{T} \xrightarrow{\langle \mathbb{T} \curvearrowright \mathrm{HH}(B) \rangle} \mathrm{Aut}_{\mathcal{W}}(\mathrm{HH}(B)),$$

which is *Connes' cyclic operator* [1983], and this is canonically functorial in the argument B:

$$\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{W}) \to \mathrm{Mod}_{\mathbb{T}}(\mathcal{W}), \quad B \mapsto (\mathbb{T} \curvearrowright \mathrm{HH}(B)). \tag{0-2-1}$$

0.2.2 Secondary Hochschild homology

Notation 0.2.2 In Section 0.2.2 we fix $\mathcal{V}$ to be an $\otimes$–presentable symmetric monoidal ∞–category.

Apply Section 0.2.1 to the case $\mathcal{W} := \mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V})$. For this situation, define the ∞–category

$$\mathrm{Alg}_2(\mathcal{V}) := \mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{W}) = \mathrm{Alg}_{\mathrm{Assoc}}(\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V})),$$

an object in which is a 2*–algebra*[3] (*in* $\mathcal{V}$), which is simply an associative algebra in associative algebras in $\mathcal{V}$. Using that Hochschild homology is symmetric monoidal, the Hochschild homology of the underlying associative algebra of a 2–algebra retains the structure of an associative algebra. For A a 2–algebra in $\mathcal{V}$, the *secondary Hochschild homology* (*of* A) is the value

$$\mathrm{HH}^{(2)}(A) := \mathrm{HH}(\mathrm{HH}(A)). \tag{0-2-2}$$

This is evidently functorial in the 2–algebra, as it is equipped with the *two* Connes cyclic operators:

$$\mathrm{HH}^{(2)} : \mathrm{Alg}_2(\mathcal{V}) := \mathrm{Alg}_{\mathrm{Assoc}}(\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V})) \xrightarrow{\mathrm{Alg}_{\mathrm{Assoc}}(\mathrm{HH})} \mathrm{Mod}_{\mathbb{T}}(\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V})) \xrightarrow{\mathrm{HH}} \mathrm{Mod}_{\mathbb{T}}(\mathrm{Mod}_{\mathbb{T}}(\mathcal{V})) \simeq \mathrm{Mod}_{\mathbb{T}^2}(\mathcal{V}).$$

Remark 0.2.3 In Section 2.5, we show that our definition (0-2-2) of secondary Hochschild homology (see Definition 2.2.8) agrees with factorization homology over a torus: $\mathrm{HH}^{(2)}(A) \simeq \int_{\mathbb{T}^2} A$. As such, our definition of secondary Hochschild homology is fit to receive a *secondary trace* map, which is related to a *secondary Chern character* map, from secondary K–theory. (See [Toën and Vezzosi 2009; Hoyois et al. 2017] and Section 0.4.)

Warning 0.2.4 Our definition of secondary Hochschild homology does not appear to agree with the definition introduced by Staic [2016], and further studied in [Laubacher 2017], where its cohomological version parametrizes certain algebraic deformations. Indeed, their definitions are more akin to factorization homology of a pair $\int_{\mathbb{S}^1 \subset \mathbb{D}^2}(B \to A)$ — see [Corrigan-Salter and Staic 2016], where this is established in the commutative context, in the language of higher-order Hochschild homology introduced by Pirashvili [2000] — which is more similar to factorization homology $\int_{\mathbb{S}^2} B$ over the 2–sphere.

Theorem X(2)(a) has the following consequence, proved in Section 2.5 using factorization homology:

Theorem Y.1 *Let $A \in \mathrm{Alg}_2(\mathcal{V})$ be a 2–algebra in an $\otimes$–presentable symmetric monoidal ∞–category $\mathcal{V}$. There is a canonical action of the continuous group $\mathbb{T}^2 \rtimes \mathrm{Braid}_3$ on secondary Hochschild homology*:

$$\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \curvearrowright \mathrm{HH}^{(2)}(A). \tag{0-2-3}$$

We now explain how Theorem Y.1 extends familiar, or at least expected, symmetries of $\mathrm{HH}^{(2)}(A)$, and how the action can be phrased in terms of these expected symmetries.

Let $\mathcal{W}$ be an $\otimes$–presentable symmetric monoidal ∞–category. Let B be an associative algebra in $\mathcal{W}$. Each endomorphism $B \xrightarrow{\sigma} B$ of the associative algebra B determines a (B, B)–bimodule structure B_σ on the underlying object B, which is characterized by $B \xrightarrow{\mathrm{id}} B$ being equivariant with respect to

[3] Dunn's additivity (see Theorem 0.2.7) supplies a host of examples of 2–algebras. In particular, a commutative algebra canonically determines a 2–algebra.

$(B, B) \xrightarrow{(\mathrm{id},\sigma)} (B, B)$. This assignment $\sigma \mapsto B_\sigma$ canonically assembles as a functor from the space of endomorphisms of B to the ∞–category of (B, B)–bimodules:

$$\mathrm{End}_{\mathrm{Alg}(\mathcal{W})}(B) \to \mathrm{BiMod}_{(B,B)}, \quad \sigma \mapsto B_\sigma.$$

This results in a composite functor

$$\mathrm{End}_{\mathrm{Alg}(\mathcal{W})}(B) \xrightarrow{\sigma\mapsto B_\sigma} \mathrm{BiMod}_{(B,B)} \xrightarrow{\mathrm{HH}(B,-)} \mathcal{W} \quad \text{given by } \sigma \mapsto \mathrm{HH}(B, B_\sigma).$$

This functor restricts to automorphisms of $\mathrm{id} \mapsto \mathrm{HH}(B, B_{\mathrm{id}}) = \mathrm{HH}(B)$ as a morphism between continuous groups:

$$\Omega_{\mathrm{id}} \mathrm{Aut}_{\mathrm{Alg}(\mathcal{W})}(B) = \Omega_{\mathrm{id}} \mathrm{End}_{\mathrm{Alg}(\mathcal{W})}(B) \to \mathrm{Aut}_{\mathcal{W}}(\mathrm{HH}(B)). \tag{0-2-4}$$

Now take $\mathcal{W} = \mathrm{Alg}(\mathcal{V})$ to be the ∞–category of associative algebras in an $\otimes$–presentable symmetric monoidal ∞–category $\mathcal{V}$, and $B = \mathrm{HH}(A)$ to be the Hochschild homology of a 2–algebra $A \in \mathrm{Alg}_2(\mathcal{V}) := \mathrm{Alg}(\mathrm{Alg}(\mathcal{V}))$. The above discussion yields the *sheer symmetry*

$$\mathrm{Sheer}_1 \colon \mathbb{Z} \simeq \Omega_0 \mathbb{T} \xrightarrow{\Omega\langle \mathbb{T} \curvearrowright \mathrm{HH}(A)\rangle} \Omega_{\mathrm{id}} \mathrm{Aut}_{\mathrm{Alg}(\mathcal{V})}(\mathrm{HH}(A)) \xrightarrow{(0\text{-}2\text{-}4)} \mathrm{Aut}_{\mathcal{V}}(\mathrm{HH}^{(2)}(A)). \tag{0-2-5}$$

The functoriality of Connes' cyclic operators yields a $\mathbb{T}^2$–action on secondary Hochschild homology of A:

$$\mathrm{Connes}' \colon \mathbb{T}^2 \xrightarrow{\langle \mathbb{T}^2 \curvearrowright \mathrm{HH}^{(2)}(A)\rangle} \mathrm{Aut}_{\mathcal{V}}(\mathrm{HH}^{(2)}(A)). \tag{0-2-6}$$

Corollary 2.3.3 states that the swapped iteration of Hochschild homology results in the same secondary Hochschild homology. This yields yet another *sheer symmetry*

$$\mathrm{Sheer}_2 \colon \mathbb{Z} \simeq \Omega_0 \mathbb{T} \xrightarrow{\Omega\langle \mathbb{T} \curvearrowright \mathrm{HH}(A)\rangle} \Omega_{\mathrm{id}} \mathrm{Aut}_{\mathrm{Alg}(\mathcal{V})}(\mathrm{HH}(A)) \xrightarrow{(0\text{-}2\text{-}4)} \mathrm{Aut}_{\mathcal{V}}(\mathrm{HH}^{(2)}(A)). \tag{0-2-7}$$

Using Theorem Y.1, the presentation (0-1-1) of the braid group Braid_3 lends to the following result, which is proved in Section 2.5.

Corollary 0.2.5 *Let A be a 2–algebra in $\mathcal{V}$. The sheer actions (0-2-5) and (0-2-7) and Connes' cyclic operators (0-2-6) generate the action*

$$\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \underset{(0\text{-}2\text{-}3)}{\curvearrowright} \mathrm{HH}^{(2)}(A)$$

of Theorem Y.1. More specifically, the sheer actions and Connes' cyclic operators satisfy the following three relations, thereafter drawing the final conclusion.

(1) *Consider the action*[4] *defined by the symmetries Sheer_1 and Sheer_2^{-1},*

$$\mathrm{Sheers} \colon \mathbb{Z} \amalg \mathbb{Z} \curvearrowright \mathrm{HH}^{(2)}(A). \tag{0-2-8}$$

Defining the generators $\langle \tau_1, \tau_2\rangle = \mathbb{Z} \amalg \mathbb{Z}$, consider the two natural actions

$$\mathbb{Z} \underset{\langle \tau_2\tau_1\tau_2\rangle}{\overset{\langle \tau_1\tau_2\tau_1\rangle}{\rightrightarrows}} \mathbb{Z} \amalg \mathbb{Z} \underset{(0\text{-}2\text{-}8)}{\curvearrowright} \mathrm{HH}^{(2)}(A).$$

[4] The pushout appearing here is in the category of groups, where it is often referred to as a *free product*.

These two symmetries are coequalized:[5]

$$\mathrm{Braid}_3 \overset{\text{(0-1-1)}}{\cong} \langle \tau_1, \tau_2 \mid \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2 \rangle \curvearrowright \mathrm{HH}^{(2)}(A).$$

(2) *The actions* $\mathbb{Z} \underset{\mathrm{Sheer}_1}{\curvearrowright} \mathrm{HH}^{(2)}(A)$ *and* $\mathbb{T}^2 \underset{\mathrm{Connes'}}{\curvearrowright} \mathrm{HH}^{(2)}(A)$ *intertwine as an action*

$$\mathbb{T}^2 \rtimes_{U_1} \mathbb{Z} \curvearrowright \mathrm{HH}^{(2)}(A),$$

where this semidirect product is defined by $\mathbb{Z} \xrightarrow{\langle U_1 \rangle} \mathrm{GL}_2(\mathbb{Z}) \simeq \mathrm{Aut}_{\mathrm{Groups}}(\mathbb{T}^2)$ (see (0-1-2)).

(3) *The actions* $\mathbb{Z} \underset{\mathrm{Sheer}_2}{\curvearrowright} \mathrm{HH}^{(2)}(A)$ *and* $\mathbb{T}^2 \underset{\mathrm{Connes'}}{\curvearrowright} \mathrm{HH}^{(2)}(A)$ *intertwine as an action*

$$\mathbb{T}^2 \rtimes_{U_2} \mathbb{Z} \xrightarrow[\cong]{\mathrm{id} \rtimes (-1)} \mathbb{T}^2 \rtimes_{U_2^{-1}} \mathbb{Z} \curvearrowright \mathrm{HH}^{(2)}(A),$$

where this semidirect product is defined by $\mathbb{Z} \xrightarrow{\langle U_2 \rangle} \mathrm{GL}_2(\mathbb{Z}) \simeq \mathrm{Aut}_{\mathrm{Groups}}(\mathbb{T}^2)$ (see (0-1-2)).

Defining $R := U_1U_2U_2 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right] = U_2U_1U_2 \in \mathrm{GL}_2(\mathbb{Z}) \simeq \mathrm{Aut}_{\mathrm{Groups}}(\mathbb{T}^2)$, *the above three points imply the two actions*

$$\mathbb{T}^2 \rtimes_R \mathbb{Z} \underset{\mathrm{id} \rtimes \langle \tau_2\tau_1\tau_2 \rangle}{\overset{\mathrm{id} \rtimes \langle \tau_1\tau_2\tau_1 \rangle}{\rightrightarrows}} \mathbb{T}^2 \rtimes_{U_1,U_2} (\mathbb{Z} \amalg \mathbb{Z}) \underset{\text{(0-2-8)}}{\curvearrowright} \mathrm{HH}^{(2)}(A)$$

are coequalized under $\mathbb{T}^2$, *thus generating the action*

$$\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \overset{\mathrm{id} \rtimes \text{(0-1-1)}}{\cong} \mathbb{T} \rtimes_{U_1,U_2} \langle \tau_1, \tau_2 \mid \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2 \rangle \curvearrowright \mathrm{HH}^{(2)}(A).$$

Next, the short exact sequence (0-1-3) of Proposition 0.1.1 implies an identification between moduli spaces

$$\{\text{extensions of } \mathrm{Braid}_3 \curvearrowright \mathrm{HH}^{(2)}(A) \text{ along } \Phi \text{ to an action } \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathrm{HH}^{(2)}(A)\}$$
$$\simeq \{\text{trivializations of } \mathbb{Z} \cong \mathrm{Ker}(\Phi) \curvearrowright \mathrm{HH}^{(2)}(A)\}.$$

Remark 0.2.6 The action $\mathbb{Z} \cong \mathrm{Ker}(\Phi) \curvearrowright \mathrm{HH}^{(2)}(A)$ is simply an automorphism $\rho \in \mathrm{Aut}_{\mathcal{V}}(\mathrm{HH}^{(2)}(A))$. So an extension of $\mathrm{Braid}_3 \curvearrowright \mathrm{HH}^{(2)}(A)$ along Φ to $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathrm{HH}^{(2)}(A)$ exists if and only if there is an equality in the set of path-components of the space of endomorphisms: $[\mathrm{id}_{\mathrm{HH}^{(2)}(A)}] = [\rho] \in \pi_0(\mathrm{End}_{\mathcal{V}}(\mathrm{HH}^{(2)}(A)))$. In the case that the ambient ∞–category of $\mathcal{V}$ is stable, this set of path-components has the canonical structure of a ring[6] (in which $[\rho]$ is a unit), and so the difference $[\rho] - [\mathrm{id}_{\mathrm{HH}^{(2)}(A)}] \in \pi_0(\mathrm{End}_{\mathcal{V}}(\mathrm{HH}^{(2)}(A)))$ obstructs such an extension to an $\mathrm{SL}_2(\mathbb{Z})$–action.

So we are interested in identifying the action $\mathrm{Ker}(\Phi) \curvearrowright \mathrm{HH}^{(2)}(A)$ in familiar, or at least expected, terms. Corollary 0.2.10 does just this, in terms of the familiar/expected symmetry of secondary Hochschild

[5]Phrased more plainly, there is an identification between automorphisms of $\mathrm{HH}^{(2)}(A)$, namely $\mathrm{Sheer}_1 \circ \mathrm{Sheer}_2^{-1} \circ \mathrm{Sheer}_1 \simeq \mathrm{Sheer}_2^{-1} \circ \mathrm{Sheer}_1 \circ \mathrm{Sheer}_2^{-1}$.

[6]For example, let $\Bbbk$ be a commutative ring and take $\mathcal{V} = (\mathrm{Mod}_{\Bbbk}, \otimes)$, where $\otimes$ is taken over $\Bbbk$. Then $\mathrm{HH}^{(2)}(A)$ may be presented as a projective chain complex over $\Bbbk$; the ring $\pi_0(\mathrm{End}_{\mathcal{V}}(\mathrm{HH}^{(2)}(A))) = \mathrm{H}_0(\underline{\mathrm{End}}^{\Bbbk}(\mathrm{HH}^{(2)}(A)))$ is the 0^{th} homology of the chain complex over $\Bbbk$ of self-maps of a such a presentation of $\mathrm{HH}^{(2)}(A)$.

homology given by *braiding-conjugation*, as we now explain. A starting point for this symmetry is given from the following result, which was essentially due to Dunn. Recall the topological operad $\mathcal{E}_2$ of little 2–disks.

Theorem 0.2.7 [Dunn 1988; Lurie 2017, Theorem 5.1.2.2] *There is a canonical equivalence from the ∞–category of $\mathcal{E}_2$–algebras in $\mathcal{V}$ to that of 2–algebras in $\mathcal{V}$:*

$$\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{V}) \xrightarrow{\simeq} \mathrm{Alg}_2(\mathcal{V}).$$

After Theorem 0.2.7, the standard continuous action $\mathrm{O}(2) \curvearrowright \mathcal{E}_2$ on the topological operad immediately implies the following:

Corollary 0.2.8 *There is a canonical action of the continuous group $\mathrm{O}(2) \curvearrowright \mathrm{Alg}_2(\mathcal{V})$. In particular, for each 2–algebra A in $\mathcal{V}$, the orbit map with respect to this action lends to a canonical symmetry of A:*

$$\beta_A : \mathbb{Z} \simeq \Omega_{\mathbb{1}}\, \mathrm{SO}(2) \xrightarrow{\simeq} \Omega_{\mathbb{1}}\mathrm{O}(2) \xrightarrow{\Omega\, \mathrm{Orbit}_A} \mathrm{Aut}_{\mathrm{Alg}_2(\mathcal{V})}(A).$$

Remark 0.2.9 This symmetry β_A on each 2–algebra A is *braiding-conjugation*. For instance, this symmetry β_A is the identity on the underlying object (so $\beta_A(1) = \mathrm{id}_A$), and for $\mu \in \mathcal{E}_2(2)$ it supplies the commutativity of the diagram in $\mathcal{V}$,

$$\begin{tikzcd} A \otimes A \arrow[r, "\mathrm{id}\otimes\mathrm{id}"] \arrow[d, "\mu_A"'] & A \otimes A \arrow[d, "\mu_A"] \\ A \arrow[r, "\mathrm{id}"] & A \end{tikzcd}$$

given by the point,

$$\beta_A(2) : * \xrightarrow{\langle 1 \rangle} \mathbb{Z} \simeq \Omega_\mu \mathcal{E}_2(2) \to \Omega_{\mu_A} \mathrm{Hom}_{\mathcal{V}}(A \otimes A, A).$$

The next result directly follows from Observation 1.3.10 and inspection of the action $\mathrm{Braid}_3 \curvearrowright \mathrm{HH}^{(2)}(A)$ of Theorem Y.1, proved in Section 2.5.

Corollary 0.2.10 *Let A be a 2–algebra in $\mathcal{V}$. Through the action of Theorem Y.1, the kernel of Φ acts on $\mathrm{HH}^{(2)}(A)$ as β_A. Specifically, there is a canonically commutative diagram among continuous groups:*

$$\begin{tikzcd} \mathbb{Z} \arrow[rr, "\beta_A"] \arrow[d, "\langle(\tau_1\tau_2)^6\rangle"', "\cong"] & & \mathrm{Aut}_{\mathrm{Alg}_2(\mathcal{V})}(A) \arrow[d, "\mathrm{HH}^{(2)}"] \\ \mathrm{Ker}(\Phi) \arrow[r] & \mathrm{Braid}_3 \arrow[r, "\text{Theorem Y.1}"] & \mathrm{Aut}_{\mathcal{V}}(\mathrm{HH}^{(2)}(A)) \end{tikzcd}$$

In particular, there is the following immediate consequence of Proposition 0.1.1.

Corollary 0.2.11 *Let A be a 2–algebra in $\mathcal{V}$. An $\mathrm{SO}(2)$–invariant-structure on $A \in \mathrm{Alg}_2(\mathcal{V})$ determines a trivialization of the action $\mathrm{Ker}(\Phi) \curvearrowright \mathrm{HH}^{(2)}(A)$, and thereafter an extension along Φ of the actions $\mathrm{Braid}_3 \to \mathbb{T}^2 \rtimes \mathrm{Braid}_3 \curvearrowright \mathrm{HH}^{(2)}(A)$ to actions*

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{T}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathrm{HH}^{(2)}(A).$$

Example 0.2.12 The action $\mathrm{Braid}_3 \curvearrowright \mathrm{HH}^{(2)}(A)$ does not generally extend along Φ as an action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathrm{HH}^{(2)}(A)$. As a tautologous case, take $A = \mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{R}^2}$, regarded as a 2–algebra in $\mathrm{Cat}_{\infty/\mathrm{Disk}^{\mathrm{fr}}_2}$. The unstraightening of the functor $\mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{T}^2} \xrightarrow{\text{forget}} \mathrm{Disk}^{\mathrm{fr}}_2 \xrightarrow{A} \mathrm{Cat}_{\infty/\mathrm{Disk}^{\mathrm{fr}}_2}$ is the cocartesian fibration $\mathrm{Ar}(\mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{T}^2}) \xrightarrow{\mathrm{ev}_t} \mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{T}^2}$, as it is equipped with the functor $\mathrm{Ar}(\mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{T}^2}) \xrightarrow{\mathrm{ev}_s} \mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{T}^2}$. This functor ev_s is a localization on the ev_t–cocartesian morphisms. Using that a colimit of a diagram in Cat_∞ is the localization on the cocartesian morphisms of its unstraightening, there is an equivalence in $\mathrm{Cat}_{\infty/\mathrm{Disk}^{\mathrm{fr}}_2}$,

$$\int_{\mathbb{T}^2} \mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{R}^2} := \mathrm{colim}(\mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{T}^2} \xrightarrow{\text{forget}} \mathrm{Disk}^{\mathrm{fr}}_2 \xrightarrow{A} \mathrm{Cat}_{\infty/\mathrm{Disk}^{\mathrm{fr}}_2}) \xrightarrow{\simeq} \mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{T}^2},$$

which is evidently $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)$–equivariant. We therefore wish to show the action $\mathrm{Ker}(\Phi) \curvearrowright \mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{T}^2}$ in $\mathrm{Cat}_{\infty/\mathrm{Disk}^{\mathrm{fr}}_2}$ is not trivializable. Consider the composite functor

$$\mathrm{Cat}_{\infty/\mathrm{Disk}^{\mathrm{fr}}_2} \xrightarrow{\mathrm{Mor}} \mathcal{S}\mathrm{paces}_{/\mathrm{Mor}(\mathrm{Disk}^{\mathrm{fr}}_2)} \xrightarrow{\text{fiber over } \underline{2}\to\underline{1}} \mathcal{S}\mathrm{paces}_{/\mathbb{S}^1},$$

where Mor is given by taking spaces of morphisms, and the last functor is given by taking fibers along $\mathrm{Disk}^{\mathrm{fr}}_2 \xrightarrow{\pi_0} \mathrm{Fin}$ over the morphism $\underline{2} = \{1,2\} \xrightarrow{!} * = \underline{1}$ in Fin, recognizing that $\mathrm{Mor}(\mathrm{Disk}^{\mathrm{fr}}_2)_{|(\underline{2}\to\underline{1})} \simeq \mathbb{S}^1$ is the space of 2–ary operations of the ∞–operad $\mathcal{E}_2$. Note that this composite functor carries the object of interest $\mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{T}^2} \in \mathrm{Cat}_{\infty/\mathrm{Disk}^{\mathrm{fr}}_2}$ to the object in $\mathcal{S}\mathrm{paces}_{/\mathbb{S}^1}$,

$$\mathrm{pr}\colon \mathbb{T}^2 \times \mathbb{S}^1 \simeq \mathbb{S}^{\mathrm{fib}}(\mathsf{T}\mathbb{T}^2) \simeq \mathrm{Mor}(\mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{T}^2})_{|(\underline{2}\to\underline{1})} \to \mathrm{Mor}(\mathrm{Disk}^{\mathrm{fr}}_2)_{|(\underline{2}\to\underline{1})} \simeq \mathbb{S}^1,$$

involving the unit tangent bundle of $\mathbb{T}^2$ and its standard framing, which is simply the projection through this identification; the $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)$–action is the canonical one on the unit tangent bundle $\mathbb{S}^{\mathrm{fib}}(\mathsf{T}\mathbb{T}^2)$ as it maps to $\mathbb{S}^1$. In particular, the restricted $(\mathbb{Z}\cong\mathrm{Ker}(\Phi))$–action is generated by the automorphism of $(\mathbb{T}^2 \times \mathbb{S}^1 \xrightarrow{\mathrm{pr}} \mathbb{S}^1) \in \mathcal{S}\mathrm{paces}_{/\mathbb{S}^1}$ that is the diagram

$$\begin{tikzcd} \mathbb{T}^2 \times \mathbb{S}^1 \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mathrm{pr}"'] & & \mathbb{T}^2 \times \mathbb{S}^1 \arrow[dl, "\mathrm{pr}"] \\ & \mathbb{S}^1 & \end{tikzcd}$$

in which the homotopy witnessing commutativity is the image of $1 \in \mathbb{Z}$ via the map between spaces

$$\mathbb{Z} \simeq \Omega_{\mathrm{id}} \mathrm{Map}(\mathbb{S}^1, \mathbb{S}^1) \xrightarrow{\mathbb{T}^2\times -} \Omega_{\mathrm{pr}} \mathrm{Map}(\mathbb{T}^2 \times \mathbb{S}^1, \mathbb{S}^1).$$

It is routine to verify that this map is a monomorphism. In particular, this action $\mathbb{Z} \curvearrowright (\mathbb{T}^2 \times \mathbb{S}^1) \in \mathcal{S}\mathrm{paces}_{/\mathbb{S}^1}$ is not trivializable. Therefore, the action by $\mathbb{Z} \cong \mathrm{Ker}(\Phi)$ on $\int_{\mathbb{T}^2} \mathrm{Disk}^{\mathrm{fr}}_{2/\mathbb{R}^2} \in \mathrm{Cat}_{/\mathrm{Disk}^{\mathrm{fr}}_2}$ is not trivializable.

0.3 Isogenic symmetries of secondary Hochschild homology

Let $\mathcal{X}$ be an ∞–category. The action $\widetilde{\mathrm{E}}^+_2(\mathbb{Z}) \to \mathrm{E}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ as a topological group determines, via precomposition, an action

$$\widetilde{\mathrm{E}}^+_2(\mathbb{Z}) \overset{(-)^T}{\simeq} \widetilde{\mathrm{E}}^+_2(\mathbb{Z})^{\mathrm{op}} \curvearrowright \mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X}), \tag{0-3-1}$$

where $\overset{(-)^T}{\simeq}$ is from Observation B.1.1. We propose the following. (See [Ayala et al. 2019, Appendix A] for a definition of *left-lax invariance*.)

Definition 0.3.1 The ∞–category of *unstable secondary cyclotomic objects* in an ∞–category $\mathcal{X}$ is that of $\mathbb{T}^2$–modules in $\mathcal{X}$ that are left-laxly invariant with respect to the action (0-3-1):

$$\mathrm{Cyc}^{\mathrm{un}(2)}(\mathcal{X}) := \mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})^{\mathrm{l.lax}\,\widetilde{\mathrm{E}}_2^+(\mathbb{Z})}.$$

Remark 0.3.2 Informally, an unstable secondary cyclotomic object in $\mathcal{X}$ consists of

- a $\mathbb{T}^2$–module $(\mathbb{T}^2 \underset{\alpha}{\curvearrowright} X)$ in $\mathcal{X}$,
- for each $\widetilde{A} \in \widetilde{\mathrm{E}}_2^+(\mathbb{Z})$, a morphism between $\mathbb{T}^2$–modules in $\mathcal{X}$
$$(\widetilde{A}^T)^*(\mathbb{T}^2 \underset{\alpha}{\curvearrowright} X) := (\mathbb{T}^2 \xrightarrow{\Psi(\widetilde{A}^T)} \mathbb{T}^2 \underset{\alpha}{\curvearrowright} X) \xrightarrow{c_{\widetilde{A}}} (\mathbb{T}^2 \underset{\alpha}{\curvearrowright} X),$$
- for each pair $\widetilde{A}, \widetilde{B} \in \widetilde{\mathrm{E}}_2^+(\mathbb{Z})$, a commutative square among $\mathbb{T}^2$–modules in $\mathcal{X}$
$$\begin{tikzcd} (\widetilde{A}^T)^*(\widetilde{B}^T)^*(\mathbb{T}^2 \underset{\alpha}{\curvearrowright} X) \arrow[r, "(\widetilde{A}^T)^* c_{\widetilde{B}}"] \arrow[d, "\simeq"'] & (\widetilde{A}^T)^*(\mathbb{T}^2 \underset{\alpha}{\curvearrowright} X) \arrow[d, "c_{\widetilde{A}}"] \\ ((\widetilde{A}\widetilde{B})^T)^*(\mathbb{T}^2 \underset{\alpha}{\curvearrowright} X) \arrow[r, "c_{\widetilde{A}\widetilde{B}}"] & (\mathbb{T}^2 \underset{\alpha}{\curvearrowright} X) \end{tikzcd}$$
- for each triple $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \widetilde{\mathrm{E}}_2^+(\mathbb{Z})$, a similar commutative cube among $\mathbb{T}^2$–modules in $\mathcal{X}$ whose faces are (possibly pulled back from) the above commutative squares,
- et cetera.

After Corollary A.0.6, which is proved in Appendix A, Theorem X(2)(b) implies the following:

Corollary 0.3.3 *For each ∞–category $\mathcal{X}$ there are canonical equivalences among ∞–categories over $\mathcal{X}$*

$$\mathrm{Cyc}^{\mathrm{un}(2)}(\mathcal{X}) \simeq \mathrm{Mod}_{(\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z}))^{\mathrm{op}}}(\mathcal{X}) \simeq \mathrm{Mod}_{\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}}}(\mathcal{X}),$$

where the equivalences are given by Corollary A.0.6 and Theorem X(2)(*b*), *respectively.*

For $\mathcal{X}$ an ∞–category, the ∞–category of *finite-genuine* $\mathbb{T}^2$*–modules in* $\mathcal{X}$ is

$$\mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2} := \mathrm{Fun}((\mathrm{Orbit}_{\mathbb{T}^2}^{\mathrm{fin}})^{\mathrm{op}}, \mathcal{X}),$$

the ∞–category of functors from the opposite of the ∞–category $\mathrm{Orbit}_{\mathbb{T}^2}^{\mathrm{fin}}$ of transitive $\mathbb{T}^2$–topological spaces with finite isotropy and spaces of $\mathbb{T}^2$–equivariant maps between them. The action $\widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \to \mathrm{E}_2^+(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ as a topological group supplies an action via the equivalence of Observation B.1.1,

$$\widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \simeq \widetilde{\mathrm{E}}_2^+(\mathbb{Z})^{\mathrm{op}} \curvearrowright \mathrm{Orbit}_{\mathbb{T}^2}^{\mathrm{fin}}, \quad A \cdot \mathbb{T}^2_{/C} := \mathbb{T}^2_{/A^{-1}(C)}.$$

Precomposition by this action in turn supplies an action

$$\widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \simeq \widetilde{\mathrm{E}}_2^+(\mathbb{Z})^{\mathrm{op}} \curvearrowright \mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2}. \tag{0-3-2}$$

After Theorem X(2)(b), we have the following immediate consequence of Proposition B.4.1.

Proposition 0.3.4 *For each ∞–category $\mathcal{X}$, the ∞–category of finite-genuine $\mathbb{T}^2$–modules in $\mathcal{X}$ invariant with respect to (0-3-2) is equivalent* (via Corollary 0.3.3) *with unstable secondary cyclotomic objects in $\mathcal{X}$:*

$$\mathrm{Mod}_{\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}}}(\mathcal{X}) \simeq \mathrm{Cyc}^{\mathrm{un}(2)}(\mathcal{X}) \xrightarrow{\simeq} (\mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2})^{\widetilde{\mathrm{E}}_2^+(\mathbb{Z})}.$$

In particular, there is a forgetful functor:

$$\mathrm{Mod}_{\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}}}(\mathcal{X}) \simeq \mathrm{Cyc}^{\mathrm{un}(2)}(\mathcal{X}) \to \mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2}.$$

Remark 0.3.5 Proposition 0.3.4 asserts a significant cancellation of homotopy coherence data.

- A finite-genuine $\mathbb{T}^2$–module V in $\mathcal{X}$ is a specification of its *C–fixed-points* $V^C \in \mathrm{Mod}_{\mathbb{T}^2/C}(\mathcal{X})$ for each finite subgroup $C \subset \mathbb{T}^2$ together with coherent compatibility.
- For V a finite-genuine $\mathbb{T}^2$–module in $\mathcal{X}$, the structure of V being invariant with respect to the action

$$\widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \underset{(0\text{-}3\text{-}2)}{\curvearrowright} \mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2}$$

  is an identification $V^C \simeq V^{A^{-1}(C)}$ for each finite subgroup $C \subset \mathbb{T}^2$ and each element $A \in \widetilde{\mathrm{E}}_2^+(\mathbb{Z})$, coherently compatibly.

So to name an object in $(\mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2})^{\widetilde{\mathrm{E}}_2^+(\mathbb{Z})}$ a priori requires an overwhelming wrangling of coherence data. From this perspective, Proposition 0.3.4 is notable: an object in $(\mathcal{X}^{\mathrm{g.fin}\mathbb{T}^2})^{\widetilde{\mathrm{E}}_2^+(\mathbb{Z})}$ is simply a $\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z})$–module in $\mathcal{X}$ — in particular, no "genuine" structure is present. Theorem Y.2 is an application of this: via the theory of factorization homology, for A a 2–algebra in $\mathcal{X}$, its secondary Hochschild homology $\mathrm{HH}^{(2)}(A)$ easily carries the structure of an $\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}}$–module. Through Proposition 0.3.4, $\mathrm{HH}^{(2)}(A)$ then has the structure of a finite-genuine $\mathbb{T}^2$–module that is $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$–invariant.

Corollary 0.3.3 lends to our last main result, which is proved as Section 2.6.

Theorem Y.2 *Let $\mathcal{X}$ be a presentable ∞–category in which finite products distribute over colimits separately in each variable.*[7] *Regard $\mathcal{X}$ as a symmetric monoidal ∞–category via the cartesian symmetric monoidal structure. For each 2–algebra $A \in \mathrm{Alg}_2(\mathcal{X})$, the action* (0-2-3) *of Theorem Y.1 canonically extends as an unstable secondary cyclotomic structure:*

(0-3-3) $$\big((\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z}))^{\mathrm{op}} \curvearrowright \mathrm{HH}^{(2)}(A)\big) \in \mathrm{Cyc}^{\mathrm{un}(2)}(\mathcal{X}).$$

Remark 0.3.6 We explain a relationship between an unstable secondary cyclotomic structure and an iterated unstable cyclotomic structure. As in the discussion preceding Proposition B.3.1, one can construct a morphism between monoids

(0-3-4) $$\mathbb{N}^{\times} \times \mathbb{N}^{\times} \xrightarrow{\widetilde{\text{diagonals}}} \widetilde{\mathrm{E}}_2^+(\mathbb{Z}),$$

[7] Examples include the ∞–categories $\mathcal{S}\mathrm{paces}$, $\mathrm{Cat}_{(\infty,n)}$, $\mathcal{X}$ and ∞–topos.

lifting the inclusion $\mathbb{N}^\times \times \mathbb{N}^\times \overset{\text{diagonals}}{\subset} \mathrm{E}_2^+(\mathbb{Z})$ as diagonal matrices. With respect to (0-3-4), the product isomorphism $\mathbb{T}\times\mathbb{T} \xrightarrow{\times} \mathbb{T}^2$ is equivariant. For $\mathcal{X}$ an ∞–category, this results in a forgetful functor from unstable secondary cyclotomic objects to iterated unstable cyclotomic objects:

$$\mathrm{Cyc}^{\mathrm{un}(2)}(\mathcal{X}) \to \mathrm{Cyc}^{\mathrm{un}}(\mathrm{Cyc}^{\mathrm{un}}(\mathcal{X})). \tag{0-3-5}$$

This functor is generally not an equivalence.[8]

0.4 Remarks on secondary cyclotomic trace

We see the role of Corollary 0.3.3 as informing an approach to secondary cyclotomic traces.

Let $\Bbbk$ be a commutative ring spectrum. Let $A \in \mathrm{Alg}_2(\mathrm{Mod}_\Bbbk)$. Recall the $\Bbbk$–linear Dennis trace map $\mathrm{K}(A) \xrightarrow{\mathrm{tr}} \mathrm{HH}(A)$; see, for instance, [Bökstedt et al. 1993]. The cyclic trace map is a canonical factorization of this Dennis trace map through *negative cyclic homology* $\mathrm{K}(A) \xrightarrow{\mathrm{tr}^{\mathbb{T}}} \mathrm{HH}^-(A) := \mathrm{HH}(A)^{\mathbb{T}}$; see [Goodwillie 1986]. Iterating this cyclic trace map results in a map between spectra $\mathrm{K}(\mathrm{K}(A)) \xrightarrow{\mathrm{tr}^{\mathbb{T}}(\mathrm{tr}^{\mathbb{T}})} \mathrm{HH}^-(\mathrm{HH}^-(A))$. Work of Toën and Vezzosi [2009], followed up by the work of Hoyois, Scherotzke and Sibilla [Hoyois et al. 2017, Theorem 1.2], suggests (from the commutative context) that this map can be refined as a *secondary Chern character* map between spectra

$$\begin{tikzcd}
\mathrm{K}^{(2)}(A) \arrow[r, dashed] & \mathrm{HH}^{(2)}(A)^{\mathbb{T}^2} \\
\mathrm{K}(\mathrm{K}(A)) \arrow[u] \arrow[r, "\mathrm{tr}^{\mathbb{T}}(\mathrm{tr}^{\mathbb{T}})"] & \mathrm{HH}^-(\mathrm{HH}^-(A)) \arrow[u]
\end{tikzcd}$$

from *secondary K–theory* to the $\mathbb{T}^2$–invariants of secondary Hochschild homology. We expect the work of Mazel-Gee and Stern [2021] (in particular Theorem C (see Section 0.4.4)) on universal properties of secondary K–theory to yield a solution both to this, and the following.

Conjecture 2 *For each 2–algebra A over $\Bbbk$, there is a canonical filler in the diagram among spectra*

$$\begin{tikzcd}
\mathrm{K}^{(2)}(A) \arrow[r, dashed] & \mathrm{HH}^{(2)}(A)^{\mathbb{T}^2 \rtimes \mathrm{Braid}_3} \arrow[r] & \mathrm{HH}^{(2)}(A)^{\mathbb{T}^2} \\
\mathrm{K}(\mathrm{K}(A)) \arrow[u] \arrow[rr, "\mathrm{tr}^{\mathbb{T}}(\mathrm{tr}^{\mathbb{T}})"] & & \mathrm{HH}^-(\mathrm{HH}^-(A)) \arrow[u]
\end{tikzcd}$$

For the case in which $\Bbbk = \mathbb{S}$ is the sphere spectrum, where standard notation is $\mathrm{THH} := \mathrm{HH}$ and referred to as *topological Hochschild homology*, the cyclic trace map factors further as the *cyclotomic trace* map,

$$\mathrm{K}(A) \xrightarrow{\mathrm{tr}^{\mathrm{Cyc}}} \mathrm{TC}(A) := \mathrm{THH}(A)^{\mathrm{Cyc}}, \tag{0-4-1}$$

[8] Suppose $\mathcal{X}$ is an ordinary category. Then the forgetful functor $\mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X}) \xrightarrow{\simeq} \mathcal{X}$ is an equivalence. Using Proposition B.3.1, which identifies the group-completion of the monoid $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$, the functor (0-3-5) can then be identified as restriction $\mathrm{Mod}_{\widetilde{\mathrm{GL}}_2^+(\mathbb{Q})}(\mathcal{X}) \to \mathrm{Mod}_{(\mathbb{Q}^\times_{>0})^2}(\mathcal{X})$ along the inclusion $(\mathbb{Q}^\times_{>0})^2 \hookrightarrow \widetilde{\mathrm{GL}}_2^+(\mathbb{Q})$ between groups.

through the *topological cyclotomic homology* which is the *cyclotomic* invariants with respect to a canonical *cyclotomic structure* on topological Hochschild homology. The fantastic culminating result of [Dundas et al. 2013] articulates a sense in which this cyclotomic trace map (0-4-1) is locally constant (in the algebra A). Iterating this cyclotomic trace map results in a map between spectra $\mathrm{K}(\mathrm{K}(A)) \xrightarrow{\mathrm{tr}^{\mathrm{Cyc}}(\mathrm{tr}^{\mathrm{Cyc}})} \mathrm{TC}(\mathrm{TC}(A))$, which is not locally constant (in the 2–argument A). As above, we expect that this iterated cyclotomic trace map can be refined as a map between spectra:

$$\begin{tikzcd}
\mathrm{K}^{(2)}(A) \arrow[r, dashed] & \mathrm{THH}^{(2)}(A)^{\mathrm{Cyc}\times\mathrm{Cyc}} \\
\mathrm{K}(\mathrm{K}(A)) \arrow[u] \arrow[r, "\mathrm{tr}^{\mathrm{Cyc}}(\mathrm{tr}^{\mathrm{Cyc}})"] & \mathrm{TC}(\mathrm{TC}(A)) \arrow[u]
\end{tikzcd}$$

Following the developments in [Ayala et al. 2017c], we expect Definition 0.3.1 of an unstable cyclotomic object to lend to a definition of a (*stable*) *secondary cyclotomic object*, and that Theorem Y.2 lends a secondary cyclotomic structure on secondary topological Hochschild homology. For secondary topological cyclotomic homology to be the invariants with respect to this structure, $\mathrm{TC}^{(2)}(A) := \mathrm{THH}^{(2)}(A)^{\mathrm{Cyc}^{(2)}}$, we again expect the work of Mazel-Gee and Stern [2021] (in particular Theorem C (see Section 0.4.4)) on secondary K–theory to further lend a secondary cyclotomic trace map, which we state as the following:

Problem 1 Define (stable) secondary cyclotomic structure, and then show that secondary topological Hochschild homology canonically possesses such. Show that the iterated cyclotomic trace map factors through the secondary topological cyclotomic homology, compatibly with the factorization of Conjecture 2:

$$\begin{tikzcd}
\mathrm{K}^{(2)}(A) \arrow[dr, dashed, "\mathrm{tr}^{\mathrm{Cyc}^{(2)}}"] \arrow[ddr, bend right, "\text{Conjecture 2}"'] & & & \mathrm{K}(\mathrm{K}(A)) \arrow[lll] \arrow[d, "\mathrm{tr}^{\mathrm{Cyc}}(\mathrm{tr}^{\mathrm{Cyc}})"] \\
& \mathrm{TC}^{(2)}(A) \arrow[r] \arrow[d] & \mathrm{THH}^{(2)}(A)^{\mathrm{Cyc}\times\mathrm{Cyc}} \arrow[d] & \mathrm{TC}(\mathrm{TC}(A)) \arrow[l] \arrow[d] \\
& \mathrm{THH}^{(2)}(A)^{\mathbb{T}^2\rtimes\mathrm{Braid}_3} \arrow[r] & \mathrm{THH}^{(2)}(A)^{\mathbb{T}^2} & \mathrm{THH}^{-}(\mathrm{THH}^{-}(A)) \arrow[l]
\end{tikzcd}$$

Remark 0.4.1 One might be encouraged by Remark 0.3.6 to expect that the secondary cyclotomic trace map $\mathrm{tr}^{\mathrm{Cyc}^{(2)}}$ of Conjecture 2 is locally constant (in the 2–algebra A), thereby correcting the failure of the iterated cyclotomic trace map $\mathrm{tr}^{\mathrm{Cyc}}(\mathrm{tr}^{\mathrm{Cyc}})$ to be locally constant. However, we do not expect this to be so. Namely, the local constancy of the cyclotomic trace map $\mathrm{K}(A) \xrightarrow{\mathrm{tr}^{\mathrm{Cyc}}} \mathrm{TC}(A)$ relies in an essential way on calculations of Hesselholt [1994] which identify the fiber of the canonical map $\mathrm{TC}(V \rtimes A) \to \mathrm{TC}(A)$ associated to a square-zero extension of A. These calculations in turn rely on the fact that, for each $i \geq 0$, the canonical action $\mathbb{T} \simeq \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}) \curvearrowright \mathrm{Conf}_i(\mathbb{T})_{\Sigma_i}$ on unordered configuration space canonically factors as a $\mathbb{T}_{/C_i}$–torsor. Because the canonical action $\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \simeq \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \curvearrowright \mathrm{Conf}_i(\mathbb{T}^2)_{\Sigma_i}$ does not apparently have any such property, we do not expect the secondary cyclotomic trace map of Problem 1 to be locally constant.

Acknowledgments We thank Oscar Randal-Williams and Sam Nariman for input on earlier drafts of this paper. Ayala was supported by the National Science Foundation under awards 1812055 and 1945639. Francis was supported by the National Science Foundation under award 1812057. This material is based upon work supported by the National Science Foundation under grant DMS-1440140 while Ayala and Francis were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester.

1 Moduli and isogeny of framed tori

1.1 Moduli and isogeny of tori

Vector addition, as well as the standard vector norm, gives $\mathbb{R}^2$ the structure of a topological abelian group. Consider its closed subgroup $\mathbb{Z}^2 \subset \mathbb{R}^2$. The *torus* is the quotient in the short exact sequence of topological abelian groups

$$0 \to \mathbb{Z}^2 \xrightarrow{\text{inclusion}} \mathbb{R}^2 \xrightarrow{\text{quot}} \mathbb{T}^2 \to 0.$$

Because $\mathbb{R}^2$ is connected, and because $\mathbb{Z}^2$ acts cocompactly by translations on $\mathbb{R}^2$, the torus $\mathbb{T}^2$ is connected and compact. The quotient map $\mathbb{R}^2 \xrightarrow{\text{quot}} \mathbb{T}^2$ endows the torus with the structure of a Lie group, and in particular a smooth manifold. Consider the submonoid

$$\mathrm{E}_2(\mathbb{Z}) := \{\mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \mid \det(A) \neq 0\} \subset \mathrm{End}_{\mathrm{Groups}}(\mathbb{Z}^2),$$

consisting of the cofinite endomorphisms of the group $\mathbb{Z}^2$. Using that the smooth map $\mathbb{R}^2 \xrightarrow{\text{quot}} \mathbb{T}^2$ is a covering space and $\mathbb{T}^2$ is connected, there is a canonical continuous action on the topological group:

(1-1-1) $$\mathrm{E}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2, \quad Aq := \mathrm{quot}(A\tilde{q}) \quad \text{for any } \tilde{q} \in \mathrm{quot}^{-1}(q).$$

This action[9] defines a semidirect product topological monoid

$$\mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z})\,.$$

Consider the topological monoid of smooth local diffeomorphisms of the torus,

$$\mathrm{Imm}(\mathbb{T}^2) \subset \mathrm{Map}(\mathbb{T}^2, \mathbb{T}^2),$$

which is endowed with the subspace topology of the C^∞–topology on the set of smooth self-maps of the torus. Notice the morphism between topological monoids

(1-1-2) $$\mathrm{Aff}\colon \mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \to \mathrm{Imm}(\mathbb{T}^2) \quad \text{given by } (p, A) \mapsto (q \mapsto Aq + p).$$

Observation 1.1.1 (1) The standard inclusion $\mathrm{GL}_2(\mathbb{Z}) \hookrightarrow \mathrm{E}_2(\mathbb{Z})$ witnesses the maximal subgroup. It follows that the standard inclusion $\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z}) \hookrightarrow \mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z})$ witnesses the maximal subgroup, both as topological monoids and as continuous monoids.

(2) The standard monomorphism $\mathrm{Diff}(\mathbb{T}^2) \hookrightarrow \mathrm{Imm}(\mathbb{T}^2)$ witnesses the maximal subgroup, both as topological monoids and as continuous monoids.

[9] Note that (1-1-1) indeed does not depend on $\tilde{q} \in \mathrm{quot}^{-1}(q)$.

We record the following classical result.

Lemma 1.1.2 *The morphism (1-1-2) restricts to maximal subgroups as a homotopy equivalence*

$$\mathrm{Aff}\colon \mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z}) \xrightarrow{\simeq} \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \quad \textit{given by } (p, A) \mapsto (q \mapsto Aq + p).$$

Proof Let G be a locally path-connected topological group, which we regard as a continuous group. Denote by $G_{\mathbb{1}} \subset G$ the path-component containing the identity element in G. This subspace $G_{\mathbb{1}} \subset G$ is a normal subgroup, and the sequence of continuous homomorphisms

$$1 \to G_{\mathbb{1}} \xrightarrow{\text{inclusion}} G \xrightarrow{\text{quotient}} \pi_0(G) \to 1$$

is a fiber sequence among continuous groups. This fiber sequence is evidently functorial in the argument G. In particular, there is a commutative diagram among topological groups

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, equal] & \mathbb{T}^2 = (\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z}))_{\mathbb{1}} \arrow[r, "\mathrm{inc}"] \arrow[d, "\mathrm{Aff}_{\mathbb{1}}"'] & \mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z}) \arrow[r, "\mathrm{quot}"] \arrow[d, "\mathrm{Aff}"'] & \pi_0(\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})) = \mathrm{GL}_2(\mathbb{Z}) \arrow[r] \arrow[d, "\pi_0(\mathrm{Aff})"'] & 1 \arrow[d, equal] \\
1 \arrow[r] & \mathrm{Diff}(\mathbb{T}^2)_{\mathbb{1}} \arrow[r, "\mathrm{inc}"] & \mathrm{Diff}(\mathbb{T}^2) \arrow[r, "\mathrm{quot}"] & \pi_0(\mathrm{Diff}(\mathbb{T}^2)) \arrow[r] & 1
\end{tikzcd}$$

in which the horizontal sequences are fiber sequences. By the five lemma applied to homotopy groups, we are reduced to showing the vertical homomorphisms $\mathrm{Aff}_{\mathbb{1}}$ and $\pi_0(\mathrm{Aff})$ are homotopy equivalences.

Theorem 2.D.4 of [Rolfsen 1976], along with Theorem B of [Hatcher 2013], implies $\pi_0(\mathrm{Aff})$ is an isomorphism. So it remains to show $\mathrm{Aff}_{\mathbb{1}}$ is a homotopy equivalence.[10] With respect to the canonical continuous action $\mathrm{Diff}(\mathbb{T}^2)_{\mathbb{1}} \curvearrowright \mathbb{T}^2$, the orbit of the identity element $0 \in \mathbb{T}^2$ is the evaluation map

$$\mathrm{ev}_0\colon \mathrm{Diff}(\mathbb{T}^2)_{\mathbb{1}} \to \mathbb{T}^2.$$

Note that the composition

$$\mathrm{id}\colon \mathbb{T}^2 \xrightarrow{\mathrm{Aff}_{\mathbb{1}}} \mathrm{Diff}(\mathbb{T}^2)_{\mathbb{1}} \xrightarrow{\mathrm{ev}_0} \mathbb{T}^2$$

is the identity map. So it remains to show that the homotopy fiber of ev_0 is weakly contractible. The isotopy-extension theorem implies ev_0 is a Serre fibration. So it is sufficient to show the fiber of ev_0, which is the stabilizer $\mathrm{Stab}_0(\mathrm{Diff}(\mathbb{T}^2)_{\mathbb{1}})$, is weakly contractible. Finally, Theorem 1b of [Earle and Eells 1967] states that this stabilizer is contractible. □

Remark 1.1.3 By the classification of compact surfaces, the moduli space $\mathcal{M}_1$ of smooth tori is path-connected, and as so is

$$\mathcal{M}_1 \simeq \mathrm{BDiff}(\mathbb{T}^2) \simeq B(\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})) \simeq (\mathbb{CP}^\infty)^2{}_{/\mathrm{GL}_2(\mathbb{Z})},$$

in which the equivalence is by Lemma 1.1.2 and the quotient is with respect to the standard action $\mathrm{GL}_2(\mathbb{Z}) \curvearrowright B^2\mathbb{Z}^2 \simeq (\mathbb{CP}^\infty)^2$. In particular, this path-connected moduli space fits into a fiber sequence

$$(\mathbb{CP}^\infty)^2 \to \mathcal{M}_1 \to B\mathrm{GL}_2(\mathbb{Z}).$$

[10] See [Gramain 1973]. We include a proof for the convenience of the reader.

Consider the set $\mathcal{L}(2) := \{\Lambda \overset{\text{cofin}}{\subset} \mathbb{Z}^2\}$ of *cofinite subgroups* of $\mathbb{Z}^2$.

Observation 1.1.4 • The orbit-stabilizer theorem immediately implies the composite map $\mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \xrightarrow{\text{pr}} \mathrm{E}_2(\mathbb{Z}) \xrightarrow{\text{Image}} \mathcal{L}(2)$ witnesses the quotient:

$$(\mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}))_{/\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})} \xrightarrow{\cong} \mathrm{E}_2(\mathbb{Z})_{/\,\mathrm{GL}_2(\mathbb{Z})} \xrightarrow{\cong} \mathcal{L}(2).$$

• Since each finite-sheeted cover over $\mathbb{T}^2$ is diffeomorphic with $\mathbb{T}^2$, the classification of covering spaces implies the map given by taking the image of homology $\mathrm{Imm}(\mathbb{T}^2) \xrightarrow{\text{Image}(\mathrm{H}_1)} \mathcal{L}(2)$ witnesses the quotient

$$\mathrm{Imm}(\mathbb{T}^2)_{/\mathrm{Diff}(\mathbb{T}^2)} \xrightarrow{\cong} \mathcal{L}(2).$$

• The following diagram commutes:

$$\begin{tikzcd}
\mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \arrow[r, "\text{Aff}"] \arrow[d, "\text{pr}"'] & \mathrm{Imm}(\mathbb{T}^2) \arrow[d, "\text{Image}(\mathrm{H}_1)"] \arrow[dl, "\mathrm{H}_1"'] \\
\mathrm{E}_2(\mathbb{Z}) \arrow[r, "\text{Image}"] & \mathcal{L}(2)
\end{tikzcd}$$

Corollary 1.1.5 *The morphism* (1-1-2) *between topological monoids is a homotopy equivalence*:

$$\text{Aff}\colon \mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \xrightarrow{\simeq} \mathrm{Imm}(\mathbb{T}^2).$$

Proof Consider the morphism between fiber sequences in the ∞–category $\mathcal{S}$paces:

$$\begin{tikzcd}
\mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \arrow[r, "\text{quotient"] \arrow[d, "\text{Aff}"'] & (\mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}))_{\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})} \arrow[r] \arrow[d, "\text{Aff}_{\text{Aff}}"] & B(\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})) \arrow[d, "B\text{Aff}"] \\
\mathrm{Imm}(\mathbb{T}^2) \arrow[r, "\text{quotient"] & \mathrm{Imm}(\mathbb{T}^2)_{/\mathrm{Diff}(\mathbb{T}^2)} \arrow[r] & B\mathrm{Diff}(\mathbb{T}^2)
\end{tikzcd}$$

Lemma 1.1.2 implies the right vertical map is an equivalence. Observation 1.1.4 implies the middle vertical map is an equivalence. It follows that the left vertical map is an equivalence, as desired. □

1.2 Framings

A *framing* of the torus is a trivialization of its tangent bundle: $\varphi\colon \tau_{\mathbb{T}^2} \cong \epsilon^2_{\mathbb{T}^2}$. Consider the topological *space of framings* of the torus,

$$\mathrm{Fr}(\mathbb{T}^2) := \mathrm{Iso}_{\mathrm{Bdl}_{\mathbb{T}^2}}(\tau_{\mathbb{T}^2}, \epsilon^2_{\mathbb{T}^2}) \subset \mathrm{Map}(\mathrm{T}\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{R}^2),$$

which is endowed with the subspace topology of the C^∞–topology on the set of smooth maps between total spaces. The quotient map $\mathbb{R}^2 \xrightarrow{\text{quot}} \mathbb{T}^2$ endows the smooth manifold $\mathbb{T}^2$ with a *standard framing* φ_0: for

$$\text{trans}\colon \mathbb{T}^2 \times \mathbb{T}^2 \xrightarrow{(p,q)\mapsto \text{trans}_p(q) := p+q} \mathbb{T}^2,$$

the abelian multiplication rule of the Lie group $\mathbb{T}^2$ is

$$(\varphi_0)^{-1}\colon \epsilon^2_{\mathbb{T}^2} \xrightarrow{\cong} \tau_{\mathbb{T}^2} \quad \text{given by } \mathbb{T}^2 \times \mathbb{R}^2 \ni (p, v) \mapsto (p, \mathrm{D}_0(\text{trans}_p \circ \text{quot})(v)) \in \mathrm{T}\mathbb{T}^2,$$

where D_0 is differentiation at zero.

The next sequence of observations culminates in an identification of this space of framings.

Observation 1.2.1 (1) Postcomposition gives the topological space $\mathrm{Fr}(\mathbb{T}^2)$ the structure of a torsor for the topological group $\mathrm{Iso}_{\mathrm{Bdl}_{\mathbb{T}^2}}(\epsilon^2_{\mathbb{T}^2}, \epsilon^2_{\mathbb{T}^2})$. In particular, the orbit map of a framing $\varphi \in \mathrm{Fr}(\mathbb{T}^2)$ is a homeomorphism

(1-2-1) $$\mathrm{Iso}_{\mathrm{Bdl}_{\mathbb{T}^2}}(\epsilon^2_{\mathbb{T}^2}, \epsilon^2_{\mathbb{T}^2}) \xrightarrow{\cong} \mathrm{Fr}(\mathbb{T}^2) \quad \text{given by } \alpha \mapsto \alpha \circ \varphi.$$

(2) Consider the topological space $\mathrm{Map}(\mathbb{T}^2, \mathrm{GL}_2(\mathbb{R}))$ of smooth maps from the torus to the standard smooth structure on $\mathrm{GL}_2(\mathbb{R})$, which is endowed with the C^∞–topology. The map

(1-2-2) $$\mathrm{Map}(\mathbb{T}^2, \mathrm{GL}_2(\mathbb{R})) \xrightarrow{\cong} \mathrm{Iso}_{\mathrm{Bdl}_{\mathbb{T}^2}}(\epsilon^2_{\mathbb{T}^2}, \epsilon^2_{\mathbb{T}^2}) \quad \text{given by } a \mapsto (\mathbb{T}^2 \times \mathbb{R}^2 \xrightarrow{(p,v)\mapsto(p,a_p(v))} \mathbb{T}^2 \times \mathbb{R}^2)$$

is a homeomorphism.

(3) The map to the product,

(1-2-3) $$\mathrm{Map}(\mathbb{T}^2, \mathrm{GL}_2(\mathbb{R})) \xrightarrow{\cong} \mathrm{Map}\big((0 \in \mathbb{T}^2), (\mathbb{1} \in \mathrm{GL}_2(\mathbb{R}))\big) \times \mathrm{GL}_2(\mathbb{R}), \quad a \mapsto (a(0)^{-1}a, a(0)),$$

is a homeomorphism.

(4) Because both of the spaces $\mathbb{T}^2$ and $\mathrm{GL}_2(\mathbb{R})$ are 1–types with the former path-connected, the map,

$$\pi_1 \colon \mathrm{Map}\big((0 \in \mathbb{T}^2), (\mathbb{1} \in \mathrm{GL}_2(\mathbb{R}))\big) \xrightarrow{\simeq} \mathrm{Hom}\big(\pi_1(0 \in \mathbb{T}^2), \pi_1(\mathbb{1} \in \mathrm{GL}_2(\mathbb{R}))\big),$$

is a homotopy equivalence.

(5) Evaluation on the standard basis for $\pi_1(0 \in \mathbb{T}^2) \xrightarrow{\cong} \pi_1(0 \in \mathbb{T})^2 \cong \mathbb{Z}^2$ defines a homeomorphism

(1-2-4) $$\mathrm{Hom}\big(\pi_1(0 \in \mathbb{T}^2), \pi_1(\mathbb{1} \in \mathrm{GL}_2(\mathbb{R}))\big) \xrightarrow{\cong} \pi_1(\mathbb{1} \in \mathrm{GL}_2(\mathbb{R})^2) \cong \mathbb{Z}^2.$$

Observation 1.2.1, together with the Gram–Schmidt homotopy equivalence $\mathrm{GS} \colon \mathrm{O}(2) \xrightarrow{\simeq} \mathrm{GL}_2(\mathbb{R})$, yields the following.

Corollary 1.2.2 *A framing $\varphi \in \mathrm{Fr}(\mathbb{T}^2)$ determines a composite homotopy equivalence*

$$\mathrm{Fr}(\mathbb{T}^2) \xleftarrow[\simeq]{(1\text{-}2\text{-}2)\circ(1\text{-}2\text{-}1)} \mathrm{Map}(\mathbb{T}^2, \mathrm{GL}_2(\mathbb{R})) \xrightarrow[\simeq]{(1\text{-}2\text{-}3)} \mathrm{Map}\big((0 \in \mathbb{T}^2), (\mathbb{1} \in \mathrm{GL}_2(\mathbb{R}))\big) \times \mathrm{GL}_2(\mathbb{R})$$
$$\xrightarrow[\simeq]{\pi_1 \times \mathrm{id}} \mathrm{Hom}\big(\pi_1(0 \in \mathbb{T}^2), \pi_1(\mathbb{1} \in \mathrm{GL}_2(\mathbb{R}))\big) \times \mathrm{GL}_2(\mathbb{R}) \xrightarrow[\simeq]{(1\text{-}2\text{-}4)\times\mathrm{id}} \mathbb{Z}^2 \times \mathrm{GL}_2(\mathbb{R}) \xleftarrow[\simeq]{\mathrm{id}\times\mathrm{GS}} \mathbb{Z}^2 \times \mathrm{O}(2).$$

Notation 1.2.3 We denote the values of the homotopy equivalence of Corollary 1.2.2 applied to the standard framing $\varphi_0 \in \mathrm{Fr}(\mathbb{T}^2)$ by

$$\mathrm{Fr}(\mathbb{T}^2) \xrightarrow{\simeq} \mathbb{Z}^2 \times \mathrm{GL}_2(\mathbb{R}) \quad \text{given by } \varphi \mapsto (\vec{\varphi}, B_\varphi).$$

1.3 Moduli of framed tori

Consider the map

$$\mathrm{Act} \colon \mathrm{Fr}(\mathbb{T}^2) \times \mathrm{Imm}(\mathbb{T}^2) \to \mathrm{Fr}(\mathbb{T}^2) \quad \text{given by } (\varphi, f) \mapsto (\tau_{\mathbb{T}^2} \xrightarrow[\cong]{\mathrm{D}f} f^*\tau_{\mathbb{T}^2} \xrightarrow[\cong]{f^*\varphi} f^*\epsilon^2_{\mathbb{T}^2} = \epsilon^2_{\mathbb{T}^2}).$$

Lemma 1.3.1 *The map* Act *is a continuous right-action of the topological monoid* $\mathrm{Imm}(\mathbb{T}^2)$ *on the topological space* $\mathrm{Fr}(\mathbb{T}^2)$. *In particular, there is a continuous action of the topological group* $\mathrm{Diff}(\mathbb{T}^2)$ *on the topological space* $\mathrm{Fr}(\mathbb{T}^2)$.

Proof Consider the topological subspace of the topological space of smooth maps between total spaces of tangent bundles, which is endowed with the C^∞–topology,

$$\mathrm{Bdl}^{\mathrm{fw.iso}}(\tau_{\mathbb{T}^2}, \tau_{\mathbb{T}^2}) \subset \mathrm{Map}(\mathrm{T}\mathbb{T}^2, \mathrm{T}\mathbb{T}^2),$$

consisting of the smooth maps between tangent bundles that are fiberwise isomorphisms. The factorization

$$\mathrm{Act}\colon \mathrm{Fr}(\mathbb{T}^2) \times \mathrm{Imm}(\mathbb{T}^2) \xrightarrow{\mathrm{id}\times\mathrm{D}} \mathrm{Fr}(\mathbb{T}^2) \times \mathrm{Bdl}^{\mathrm{fw.iso}}(\tau_{\mathbb{T}^2}, \tau_{\mathbb{T}^2}) \xrightarrow{\circ} \mathrm{Fr}(\mathbb{T}^2)$$

first takes the derivative, then composes bundle morphisms. The definition of the C^∞–topology is such that the first map in this factorization is continuous. The second map in this factorization is continuous because composition is continuous with respect to C^∞–topologies. We conclude that Act is continuous.

We now show that Act is an action. Clearly, for each $\varphi \in \mathrm{Fr}(\mathbb{T}^2)$, there is an equality $\mathrm{Act}(\varphi, \mathrm{id}) = \varphi$. Next, let $g, f \in \mathrm{Imm}(\mathbb{T}^2)$, and let $\varphi \in \mathrm{Fr}(\mathbb{T}^2)$. The chain rule, together with universal properties for pullbacks, gives that the diagram among smooth vector bundles

$$\begin{tikzcd}
\tau_{\mathbb{T}^2} \arrow[r, "\mathrm{D}g"'] \arrow[rrr, bend left, "\mathrm{D}(g\circ f)"] & g^*\tau_{\mathbb{T}^2} \arrow[r, "g^*\mathrm{D}f"'] & f^*g^*\tau_{\mathbb{T}^2} \arrow[r, "\cong"'] \arrow[d, "f^*g^*\varphi"] & (g\circ f)^*\tau_{\mathbb{T}^2} \arrow[d, "(g\circ f)^*\varphi"] \\
\epsilon^2_{\mathbb{T}^2} & g^*\epsilon^2_{\mathbb{T}^2} \arrow[l, "\cong"'] & f^*g^*\epsilon^2_{\mathbb{T}^2} \arrow[l, "\cong"'] & (g\circ f)^*\epsilon^2_{\mathbb{T}^2} \arrow[l, "\cong"'] \arrow[lll, bend left, "\cong"]
\end{tikzcd}$$

commutes. Inspecting the definition of Act, the commutativity of this diagram implies the equality $\mathrm{Act}(\mathrm{Act}(\varphi, g), f) = \mathrm{Act}(\varphi, g\circ f)$, as desired. $\square$

Definition 1.3.2 The *moduli space of framed tori*[11] is the space of homotopy coinvariants with respect to this conjugation action Act:

$$\mathcal{M}^{\mathrm{fr}}_1 := \mathrm{Fr}(\mathbb{T}^2)_{/\mathrm{Diff}(\mathbb{T}^2)}.$$

Observation 1.3.3 Through Corollary 1.2.2 applied to the standard framing $\varphi_0 \in \mathrm{Fr}(\mathbb{T}^2)$, the action Act is compatible with familiar actions. Specifically, Act fits into a commutative diagram among topological spaces:

$$\begin{tikzcd}
\mathrm{Fr}(\mathbb{T}^2)\times\mathrm{Imm}(\mathbb{T}^2) \arrow[rr, "\mathrm{Act}"] & & \mathrm{Fr}(\mathbb{T}^2) \\
\mathrm{Map}(\mathbb{T}^2, \mathrm{GL}_2(\mathbb{R}))\times(\mathbb{T}^2\rtimes\mathrm{E}_2(\mathbb{Z})) \arrow[u, "\text{Corollary 1.2.2}\times\mathrm{Aff}", "\simeq"'] \arrow[r, "\mathrm{id}\times\mathrm{pr}"] \arrow[d, "\text{Corollary 1.2.2}\times\mathrm{id}"', "\simeq"] & \mathrm{Map}(\mathbb{T}^2, \mathrm{GL}_2(\mathbb{R}))\times\mathrm{E}_2(\mathbb{Z}) \arrow[r, "\text{valuewise}", "\text{multiply}"'] \arrow[d, "\text{Corollary 1.2.2}\times\mathrm{id}"', "\simeq"] & \mathrm{Map}(\mathbb{T}^2, \mathrm{GL}_2(\mathbb{R})) \arrow[u, "\text{Corollary 1.2.2}"', "\cong"] \arrow[d, "\text{Corollary 1.2.2}"', "\simeq"] \\
(\mathbb{Z}^2\times\mathrm{GL}_2(\mathbb{R}))\times(\mathbb{T}^2\rtimes\mathrm{E}_2(\mathbb{Z})) \arrow[r, "\mathrm{id}\times\mathrm{pr}"] & (\mathbb{Z}^2\times\mathrm{GL}_2(\mathbb{R}))\times\mathrm{E}_2(\mathbb{Z}) \arrow[r, "{(\vec{v},B;A)\mapsto(A^T\vec{v},BA)}"] & \mathbb{Z}^2\times\mathrm{GL}_2(\mathbb{R})
\end{tikzcd}$$

[11]This definition is a particular case of a general definition of a moduli space of framed manifolds; see, for instance, [Ayala and Francis 2015].

We record the following basic application of group theory.

Observation 1.3.4 For $\vec{v} = \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] \in \mathbb{Z}^2$, consider the subset $T_{\vec{v}} := \{P \mid P\vec{v} = \gcd(p,q)\vec{e}_1\} \subset \mathrm{GL}_2(\mathbb{Z})$.

(1) In the case that $p \geq 0$ and $q = 0$, the set $T_{\vec{v}}$ is identical with the stabilizer subgroup,

$$T_{\vec{v}} = \mathrm{Stab}_{\mathrm{GL}_2}(\mathbb{Z})(\gcd(p,q)\cdot\vec{e}_1) = \begin{cases} \mathrm{GL}_2(\mathbb{Z}) & \text{if } p = 0, \\ \left\{\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}\right\} = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle \cong \mathrm{O}(1) \ltimes \mathbb{Z} & \text{if } p > 0, \end{cases}$$

in which the semidirect product is with respect to the standard action $\mathrm{O}(1) \xrightarrow{\cong} \mathrm{Aut}(\mathbb{Z})$.

(2) The set $T_{\vec{v}}$ is not empty. Left multiplication defines a free transitive action of this stabilizer:

$$\mathrm{GL}_2(\mathbb{Z}) \curvearrowright T_{\vec{v}} \quad \text{for } \vec{v} = \vec{0} \qquad \text{and} \qquad \mathrm{O}(1) \ltimes \mathbb{Z} \curvearrowright T_{\vec{v}} \quad \text{for } \vec{v} \neq \vec{0}.$$

(3) An element $P \in T_{\vec{v}}$ determines an isomorphism between groups:

$$\begin{aligned} \mathrm{Stab}_{\mathrm{GL}_2(\mathbb{Z})}(\vec{v}) &= P^{-1}\,\mathrm{Stab}_{\mathrm{GL}_2(\mathbb{Z})}(\gcd(p,q)\cdot\vec{e}_1)\,P \\ &= \begin{cases} \mathrm{GL}_2(\mathbb{Z}) & \text{if } \vec{v} = \vec{0}, \\ \left\langle P^{-1}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} P,\, P^{-1}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} P \right\rangle \cong \mathrm{O}(1) \ltimes \mathbb{Z} & \text{if } \vec{v} \neq \vec{0}. \end{cases} \end{aligned}$$

(4) An element $P = \left[\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right] \in T_{\vec{v}} \cap \mathrm{SL}_2(\mathbb{Z})$ determines an identification:

$$\mathrm{Stab}_{\mathrm{SL}_2(\mathbb{Z})}(\vec{v}) = \begin{cases} \mathrm{SL}_2(\mathbb{Z}) & \text{if } \vec{v} = \vec{0}, \\ \left\langle \begin{bmatrix} 1 + yz & z^2 \\ -y^2 & 1 - yz \end{bmatrix} \right\rangle = \langle P^{-1} U_1 P \rangle \cong \mathbb{Z} & \text{if } \vec{v} \neq \vec{0}. \end{cases}$$

The next result is phrased in terms of spaces fitting into the diagram in which each of the two squares, and therefore their concatenated larger square, is a pullback:

(1-3-1)
$$\begin{tikzcd} (\mathbb{CP}^\infty)^2{}_{/\mathbb{Z}} \times B\mathbb{Z} \arrow[r] \arrow[d] & (\mathbb{CP}^\infty)^2{}_{/\mathrm{Braid}_3} \arrow[r] \arrow[d] & (\mathbb{CP}^\infty)^2{}_{/\mathrm{GL}_2(\mathbb{Z})} \arrow[d] & \\ B\mathbb{Z} \times B\mathbb{Z} \arrow[r, "{\langle \tau_1, (\tau_1\tau_2)^6 \rangle}"] \arrow[d, "\mathrm{pr}"'] & B\mathrm{Braid}_3 \arrow[r, "\Phi"] & B\mathrm{SL}_2(\mathbb{Z}) \arrow[r] & B\mathrm{GL}_2(\mathbb{Z}) \\ B\mathbb{Z} \arrow[urr, "\langle U_1 \rangle"'] & & & \end{tikzcd}$$

Proposition 1.3.5 (1) *The standard framing $\varphi_0 \in \mathrm{Fr}(\mathbb{T}^2)$ determines an identification between spaces*,

$$\mathcal{M}_1^{\mathrm{fr}} \xrightarrow{\cong} ((\mathbb{CP}^\infty)^2{}_{/\mathrm{Braid}_3}) \amalg ((\mathbb{CP}^\infty)^2{}_{/\mathbb{Z}} \times B\mathbb{Z})^{\amalg \mathbb{N}},$$

through which φ_0 selects the distinguished path-component.

(2) *Furthermore, the resulting map* $\pi_0 \operatorname{Fr}(\mathbb{T}^2) \to \pi_0 \mathcal{M}_1^{\mathrm{fr}} \xrightarrow{\cong} \{0\} \amalg \mathbb{N} = \mathbb{Z}_{\geq 0}$ *factors as a composition*

$$\pi_0 \operatorname{Fr}(\mathbb{T}^2) \to \mathbb{Z}^2 \xrightarrow{\gcd} \mathbb{Z}_{\geq 0}$$

in which the second map takes the ***greatest common divisor****, and the first map is*

$$[\varphi] \mapsto [\mathbb{T} \vee \mathbb{T} = \mathrm{sk}_1(\mathbb{T}^2) \xrightarrow{\varphi \circ \varphi_0^{-1}{}_{|\mathrm{sk}_1(\mathbb{T}^2)}} \mathrm{GL}_2(\mathbb{R})] \in \pi_1(\mathbb{1} \in \mathrm{GL}_2(\mathbb{R}))^2 \cong \mathbb{Z}^2.$$

Proof The result follows from the following sequence of identifications in the ∞–category $\mathcal{S}$paces:

$$\mathcal{M}_1^{\mathrm{fr}} \simeq (\mathbb{Z}^2 \times \mathrm{GL}_2(\mathbb{R}))_{/\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})} \qquad \text{(by Observation 1.3.3)}$$

$$\simeq ((\mathbb{Z}^2 \times \mathrm{GL}_2(\mathbb{R}))_{/\mathbb{T}^2})_{/\mathrm{GL}_2(\mathbb{Z})} \qquad \text{(iterate quotient)} \tag{1-3-2}$$

$$\simeq (\mathbb{Z}^2 \times B\mathbb{T}^2 \times \mathrm{GL}_2(\mathbb{R}))_{/\mathrm{GL}_2(\mathbb{Z})} \qquad \text{(trivial } \mathbb{T}^2\text{–action)} \tag{1-3-3}$$

$$\simeq \mathbb{Z}^2{}_{/\mathrm{GL}_2(\mathbb{Z})} \times_{B\mathrm{GL}_2(\mathbb{Z})} ((\mathbb{CP}^\infty)^2 \times \mathrm{GL}_2(\mathbb{R}))_{/\mathrm{GL}_2(\mathbb{Z})} \qquad \text{(groupoids are effective)} \tag{1-3-4}$$

$$\simeq (B\mathrm{GL}_2(\mathbb{Z}) \amalg B(\mathbb{Z} \rtimes \mathrm{O}(1))^{\amalg \mathbb{N}}) \times_{B\mathrm{GL}_2(\mathbb{Z})} ((\mathbb{CP}^\infty)^2 \times \mathrm{GL}_2(\mathbb{R}))_{/\mathrm{GL}_2(\mathbb{Z})} \qquad \text{(explicit quotient)} \tag{1-3-5}$$

$$\simeq (B\mathrm{GL}_2(\mathbb{Z}) \times_{B\mathrm{GL}_2(\mathbb{Z})} ((\mathbb{CP}^\infty)^2 \times \mathrm{GL}_2(\mathbb{R}))_{/\mathrm{GL}_2(\mathbb{Z})}) \amalg (B(\mathbb{Z} \rtimes \mathrm{O}(1)) \times_{B\mathrm{GL}_2(\mathbb{Z})} ((\mathbb{CP}^\infty)^2 \times \mathrm{GL}_2(\mathbb{R}))_{/\mathrm{GL}_2(\mathbb{Z})})^{\amalg \mathbb{N}} \qquad \text{(distribute } \times \text{ over } \amalg\text{)} \tag{1-3-6}$$

$$\simeq ((\mathbb{CP}^\infty)^2 \times \mathrm{GL}_2(\mathbb{R})_{/\mathrm{GL}_2(\mathbb{Z})}) \amalg ((\mathbb{CP}^\infty)^2 \times \mathrm{GL}_2(\mathbb{R})_{/\mathbb{Z} \rtimes \mathrm{O}(1)})^{\amalg \mathbb{N}} \qquad \text{(base change)} \tag{1-3-7}$$

$$\simeq ((\mathbb{CP}^\infty)^2{}_{/\Omega(\mathrm{GL}_2(\mathbb{R})_{/\mathrm{GL}_2(\mathbb{Z})})}) \amalg ((\mathbb{CP}^\infty)^2{}_{/\Omega(\mathrm{GL}_2(\mathbb{R})_{/\mathbb{Z} \rtimes \mathrm{O}(1)})})^{\amalg \mathbb{N}} \qquad \text{(by Lemma A.0.2)} \tag{1-3-8}$$

$$\simeq ((\mathbb{CP}^\infty)^2{}_{/\mathrm{Braid}_3}) \amalg ((\mathbb{CP}^\infty)^2{}_{/\mathbb{Z}} \times B\mathbb{Z})^{\amalg \mathbb{N}}. \qquad \text{(explicit identifications)} \tag{1-3-9}$$

The bottom horizontal map in Observation 1.3.3 reveals that the action $\mathbb{Z}^2 \times \mathrm{GL}_2(\mathbb{R}) \curvearrowleft \mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$ can be identified as the diagonal action of the action

$$(\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z}))^{\mathrm{op}} \xrightarrow{\mathrm{pr}} \mathrm{GL}_2(\mathbb{Z})^{\mathrm{op}} \xrightarrow{(-)^T} \mathrm{GL}_2(\mathbb{Z}) \underset{\text{standard}}{\curvearrowright} \mathbb{Z}^2 \tag{1-3-10}$$

together with the action

$$(\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z}))^{\mathrm{op}} \xrightarrow{\mathrm{pr}} \mathrm{GL}_2(\mathbb{Z})^{\mathrm{op}} \xrightarrow{\text{include}} \mathrm{GL}_2(\mathbb{R})^{\mathrm{op}} \underset{\text{right mult}}{\curvearrowright} \mathrm{GL}_2(\mathbb{R}).$$

The equivalence (1-3-2) identifies the $\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$–quotient as the $\mathbb{T}^2$–quotient followed by the $\mathrm{GL}_2(\mathbb{Z})$–quotient. The equivalence (1-3-3) is a consequence of the $\mathbb{T}^2$–action being trivial on both factors. The equivalence (1-3-4) is an instance of the general base-change identity $(X \times Y)_{/G} \simeq (X_{/G}) \times_{BG} (Y_{/G})$. The equivalence (1-3-5) is the orbit-stabilizer theorem, as we explain. By Observation 1.3.4, two elements $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right], \left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right] \in \mathbb{Z}^2$ are in the same (1-3-10)–orbit if and only if their greatest common divisors agree: $\gcd(u,v) = \gcd(s,t) \in \mathbb{Z}_{\geq 0}$. In particular, there is a bijection between the set of (1-3-10)–orbits and the subset

$$\mathbb{Z}_{\geq 0} \cong \left\{ \begin{bmatrix} g \\ 0 \end{bmatrix} \right\} \subset \mathbb{Z}^2.$$

Furthermore, the stabilizer of $\left[\begin{smallmatrix} g \\ 0 \end{smallmatrix}\right] \in \mathbb{Z}^2$ with respect to the action $\mathrm{GL}_2(\mathbb{Z})^{\mathrm{op}} \xrightarrow{(-)^T} \mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathbb{Z}^2$ is

$$\mathrm{Stab}_{\mathrm{GL}_2(\mathbb{Z})^{\mathrm{op}}}\left(\begin{bmatrix} g \\ 0 \end{bmatrix}\right) = \begin{cases} \mathrm{GL}_2(\mathbb{Z})^{\mathrm{op}} & \text{if } g = 0, \\ \left\{\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix}\right\}^{\mathrm{op}} \cong (\mathbb{Z} \rtimes \mathrm{O}(1))^{\mathrm{op}} & \text{if } g \neq 0. \end{cases}$$

Therefore,

$$\mathbb{Z}^2{}_{/\mathrm{GL}_2(\mathbb{Z})} \simeq \coprod_{g \in \mathbb{Z}_{\geq 0}} B\,\mathrm{Stab}_{\mathrm{GL}_2(\mathbb{Z})^{\mathrm{op}}}\left(\begin{bmatrix} g \\ 0 \end{bmatrix}\right) \simeq B\mathrm{GL}_2(\mathbb{Z}) \amalg B(\mathbb{Z} \rtimes \mathrm{O}(1))^{\amalg \mathbb{N}}.$$

The equivalence (1-3-6) is the distribution of $\times$ over $\amalg$. The equivalence (1-3-7) is an instance of the general base-change identity $X_{/H} \simeq BH \times_{BG} X_{/G}$. The equivalence (1-3-9) is a direct application of Proposition 0.1.1 for the 0–cofactor, and for each other cofactor it is an application of Proposition 0.1.1, then a consequence of the diagram (1-3-1) of pullbacks among spaces. □

For $\varphi \in \mathrm{Fr}(\mathbb{T}^2)$ a framing of the torus, consider the orbit map of φ for this continuous action of Lemma 1.3.1:

$$\mathrm{Orbit}_\varphi \colon \mathrm{Imm}(\mathbb{T}^2) \xrightarrow{(\mathrm{constant}_\varphi, \mathrm{id})} \mathrm{Fr}(\mathbb{T}^2) \times \mathrm{Imm}(\mathbb{T}^2) \xrightarrow{\mathrm{Act}} \mathrm{Fr}(\mathbb{T}^2), \qquad f \mapsto \mathrm{Act}(\varphi, f).$$

Recall Notation 1.2.3.

Observation 1.3.6 After Observation 1.3.3, for each framing $\varphi \in \mathrm{Fr}(\mathbb{T}^2)$, the orbit map for φ fits into a solid diagram among topological spaces:

$$\begin{tikzcd}
\mathrm{Diff}(\mathbb{T}^2) \arrow[rr] \arrow[dr, dashed, "\mathrm{H}_1"] & & \mathrm{Imm}(\mathbb{T}^2) \arrow[rr, "\mathrm{Orbit}_\varphi"] \arrow[dr, dashed, "\mathrm{H}_1"] & & \mathrm{Fr}(\mathbb{T}^2) \arrow[d, "\simeq"', "\text{Corollary 1.2.2}"] \\
& \mathrm{GL}_2(\mathbb{Z}) \arrow[rr] & & \mathrm{E}_2(\mathbb{Z}) \arrow[r, "A \mapsto (A^T \vec{\varphi}, B_\varphi A)"] & \mathbb{Z}^2 \times \mathrm{GL}_2(\mathbb{R}) \\
\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z}) \arrow[uu, "\mathrm{Aff}"] \arrow[ur, "\mathrm{pr}"'] \arrow[rr] & & \mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \arrow[uu, "\mathrm{Aff}"] \arrow[ur, "\mathrm{pr}"'] & &
\end{tikzcd}$$

The existence of the fillers follows from Observation 1.1.4.

Remark 1.3.7 The point–set fiber of Orbit_φ over φ, which is the point–set stabilizer of the action $\mathrm{Fr}(\mathbb{T}^2) \curvearrowleft \mathrm{Imm}(\mathbb{T}^2)$ of Lemma 1.3.1, consists of those local diffeomorphisms f for which the diagram among vector bundles

$$\begin{tikzcd}
\tau_{\mathbb{T}^2} \arrow[r, "\varphi"] \arrow[d, "\mathrm{D}f"'] & \epsilon^2_{\mathbb{T}^2} \arrow[d, equal] \\
f^*\tau_{\mathbb{T}^2} \arrow[r, "f^*\varphi"] & f^*\epsilon^2_{\mathbb{T}^2}
\end{tikzcd}$$

commutes. For a generic framing φ, a local diffeomorphism f satisfies this rigid condition if and only if $f = \mathrm{id}_{\mathbb{T}^2}$ is the identity diffeomorphism. In the special case of the standard framing φ_0, a local diffeomorphism f satisfies this rigid condition if and only if $f = \mathrm{trans}_{f(0)} \circ \mathrm{quot}$ is translation in the group $\mathbb{T}^2$ after a group-theoretic quotient $\mathbb{T}^2 \xrightarrow{\text{quotient}} \mathbb{T}^2$. In particular, the point–set fiber of $(\mathrm{Orbit}_{\varphi_0})_{|\mathrm{Diff}(\mathbb{T}^2)}$ over φ_0 is $\mathbb{T}^2$, and the homomorphism $\mathbb{T}^2 \hookrightarrow \mathrm{Diff}(\mathbb{T}^2)$ witnesses the inclusion of those diffeomorphisms that *strictly* fix φ_0.

On the other hand, the *homotopy* fiber of $\mathrm{Orbit}_{\varphi_0}$ over φ_0 is more flexible. It consists of pairs (f, γ) in which f is a local diffeomorphism and γ is a homotopy

$$\varphi_0 \overset{\gamma}{\sim} \mathrm{Act}(\varphi_0, f).$$

As we will see, every orientation-preserving local diffeomorphism f admits a lift to this homotopy fiber. In particular, small perturbations of such f, such as multiplication by bump functions in neighborhoods of $\mathbb{T}^2$, can be lifted to this homotopy fiber.

Definition 1.3.8 Let $\varphi \in \mathrm{Fr}(\mathbb{T}^2)$ be a framing of the torus. The space of *framed local diffeomorphisms*, and the space of *framed diffeomorphisms*, of the framed smooth manifold $(\mathbb{T}^2, \varphi)$ are respectively the pullbacks in the ∞–category $\mathcal{S}\mathrm{paces}$

$$\begin{tikzcd} \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2,\varphi) \arrow[r] \arrow[d] & \mathrm{Imm}(\mathbb{T}^2) \arrow[d, "\mathrm{Orbit}_\varphi"] \\ * \arrow[r, "\langle\varphi\rangle"] & \mathrm{Fr}(\mathbb{T}^2) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2,\varphi) \arrow[r] \arrow[d] & \mathrm{Diff}(\mathbb{T}^2) \arrow[d, "\mathrm{Orbit}_\varphi"] \\ * \arrow[r, "\langle\varphi\rangle"] & \mathrm{Fr}(\mathbb{T}^2) \end{tikzcd}$$

In the case that the framing $\varphi = \varphi_0$ is the standard framing, we simply define

$$\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2) := \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2, \varphi_0) \quad \text{and} \quad \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) := \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2, \varphi_0).$$

The next result follows directly from Lemma A.0.1 and Proposition 1.3.5(1).

Corollary 1.3.9 *Let $\varphi \in \mathrm{Fr}(\mathbb{T}^2)$ be a framing. The space $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2, \varphi)$ is canonically endowed with the structure of a continuous group over $\mathrm{Diff}(\mathbb{T}^2)$. With respect to this structure, there is a canonical identification given by Proposition 1.3.5(1) between continuous groups:*

$$\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2, \varphi) \simeq \Omega_{[\varphi]}\mathcal{M}_1^{\mathrm{fr}} \simeq \begin{cases} \Omega((\mathbb{CP}^\infty)^2{}_{/\mathrm{Braid}_3}) \simeq \mathbb{T}^2 \rtimes \mathrm{Braid}_3 & \text{if } \vec{\varphi} = \vec{0}, \\ \Omega((\mathbb{CP}^\infty)^2{}_{/\mathbb{Z}} \times B\mathbb{Z}) \simeq (\mathbb{T}^2 \rtimes \mathbb{Z}) \times \mathbb{Z} & \text{if } \vec{\varphi} \neq \vec{0}. \end{cases}$$

Observation 1.3.10 The kernel of Φ acts by rotating the framing, which is to say there is a canonically commutative diagram among continuous groups

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\simeq"] \arrow[d, "\langle(\tau_1\tau_2)^6\rangle"', "\cong"] & \Omega_{\mathbb{1}}\,\mathrm{GL}_2(\mathbb{R}) \arrow[r, "\Omega(A \mapsto A\cdot\varphi_0)"] & \Omega_{\varphi_0}\,\mathrm{Fr}(\mathbb{T}^2) \arrow[d] \\ \mathrm{Ker}(\Phi) \arrow[r] & \mathrm{Braid}_3 \arrow[r, "\mathrm{Aff}^{\mathrm{fr}}"] & \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \end{tikzcd}$$

Here $\mathrm{Aff}^{\mathrm{fr}}$ is defined in Lemma 1.4.3. Indeed, there is a canonically commutative diagram among spaces, in which each row is an Ω–Puppe sequence,

$$\begin{tikzcd}
\mathrm{Ker}(\Phi) \arrow[r] \arrow[d] & \mathrm{Braid}_3 \arrow[r, "\Phi"] \arrow[d, "\mathrm{Aff}^{\mathrm{fr}}"] & \mathrm{GL}_2(\mathbb{Z}) \arrow[r, "\mathbb{R}\otimes_{\mathbb{Z}}"] \arrow[d, "\mathrm{Aff}"] & \mathrm{GL}_2(\mathbb{R}) \arrow[d, "\text{rotate the framing } \varphi_0"] \\
\Omega_{\varphi_0} \mathrm{Fr}(\mathbb{T}^2) \arrow[r] & \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \arrow[r] & \mathrm{Diff}(\mathbb{T}^2) \arrow[r, "\mathrm{Orbit}_{\varphi_0}"] & \mathrm{Fr}(\mathbb{T}^2)
\end{tikzcd}$$

1.4 Proof of Theorem X and Corollary 0.1.4

Theorem X consists of three statements. Theorem X(1) is implied by Proposition 1.3.5. Theorem X(2)(a) is implied by Corollary 1.3.9. Theorem X(2)(b), as well as Theorem X(2)(a), is implied by Lemma 1.4.3.

Notation 1.4.1 Let $\vec{v} = \left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right] \in \mathbb{Z}^2$ and $r \in \mathbb{Z}$. Define the matrices

$$U_{\vec{v}} := \begin{bmatrix} 1+yz & z^2 \\ -u^2 & 1-yz \end{bmatrix}^T \quad \text{and} \quad D_{\vec{v},r} := \begin{bmatrix} 1+(r-1)xy & -(r-1)xz \\ (r-1)wy & 1+(r-1)wz \end{bmatrix}^T$$

for some $w, z, y, z \in \mathbb{Z}$ that solve

$$wp + xq = \gcd(p,q) \geq 0, \quad yp + zq = 0 \quad \text{and} \quad wz - xy = 1. \tag{1-4-1}$$

Denote the semidirect continuous group and continuous monoid by

$$\mathbb{T}^2 \rtimes_{U_{\vec{v}}} \mathbb{Z} \quad \text{and} \quad \mathbb{T}^2 \rtimes_{D_{\vec{v}}, U_{\vec{v}}} (\mathbb{N}^\times \ltimes \mathbb{Z}),$$

given through the actions on the continuous group $\mathbb{T}^2$

$$\mathbb{Z} \xrightarrow{b \mapsto U_{\vec{v}}^b} \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2 \quad \text{and} \quad \mathbb{Z} \rtimes \mathbb{N}^\times \xrightarrow{(b,d) \mapsto U_{\vec{v}}^b D_{\vec{v},d}} \mathrm{E}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2.$$

Remark 1.4.2 Observation 1.3.4 ensures the existence of a solution to (1-4-1). Observation 1.3.4 also implies, for $U'_{\vec{v}}$ and $D'_{\vec{v},r}$ defined by another choice of solution to (1-4-1), that $U'_{\vec{v}}$ and $D'_{\vec{v},r}$ are respectively canonically conjugate with $U_{\vec{v}}$ and $D_{\vec{v},r}$, and therefore the continuous groups and continuous monoids are respectively canonically identified:

$$\mathbb{T}^2 \rtimes_{U_{\vec{v}}} \mathbb{Z} \simeq \mathbb{T}^2 \rtimes_{U'_{\vec{v}}} \mathbb{Z} \quad \text{and} \quad \mathbb{T}^2 \rtimes_{U_{\vec{v}}, D_{\vec{v}}} (\mathbb{Z} \rtimes \mathbb{N}^\times) \simeq \mathbb{T}^2 \rtimes_{U'_{\vec{v}}, D'_{\vec{v}}} (\mathbb{Z} \rtimes \mathbb{N}^\times).$$

The next result extends Corollary 1.3.9 from an assertion about $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2, \varphi)$ to one about $\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2, \varphi)$. Recall Notation 1.2.3.

Lemma 1.4.3 *Let $\varphi \in \mathrm{Fr}(\mathbb{T}^2)$ be a framing of the torus.*

(1) *If $\vec{\varphi} = \vec{0}$, then there are canonical equivalences in the diagrams among continuous monoids*

$$\begin{tikzcd}
\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \arrow[r, dashed, "\simeq", "\mathrm{Aff}^{\mathrm{fr}}"'] \arrow[d, "\mathrm{id} \rtimes \Psi"'] & \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2, \varphi) \arrow[d, "\text{forget}"] \\
\mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \arrow[r, "\simeq", "\mathrm{Aff}"'] & \mathrm{Imm}(\mathbb{T}^2)
\end{tikzcd} \quad \textit{and} \quad \begin{tikzcd}
\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \arrow[r, dashed, "\simeq", "\mathrm{Aff}^{\mathrm{fr}}"'] \arrow[d, "\mathrm{id} \rtimes \Phi"'] & \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2, \varphi) \arrow[d, "\text{forget}"] \\
\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z}) \arrow[r, "\simeq", "\mathrm{Aff}"'] & \mathrm{Diff}(\mathbb{T}^2)
\end{tikzcd} \tag{1-4-2}$$

(2) *If $\vec{\varphi} \neq \vec{0}$, then there are canonical equivalences in the diagrams among continuous monoids*

$$
\begin{tikzcd}
(\mathbb{T}^2 \rtimes_{U_{\vec{\varphi}},D_{\vec{\varphi}}} (\mathbb{Z} \rtimes \mathbb{N}^{\times})) \times \mathbb{Z} \arrow[r, dashed, "\simeq", "\mathrm{Aff}^{\mathrm{fr}}"'] \arrow[d, "\mathrm{id} \rtimes ((b,d,k) \mapsto U_{\vec{\varphi}}^b D_{\vec{\varphi},d})"'] & \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2, \varphi) \arrow[d, "\mathrm{forget}"] \\
\mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \arrow[r, "\simeq", "\mathrm{Aff}"'] & \mathrm{Imm}(\mathbb{T}^2)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(\mathbb{T}^2 \rtimes_{U_{\vec{\varphi}}} \mathbb{Z}) \times \mathbb{Z} \arrow[r, dashed, "\simeq", "\mathrm{Aff}^{\mathrm{fr}}"'] \arrow[d, "\mathrm{id} \rtimes ((b,k) \mapsto U_{\vec{\varphi}}^b)"'] & \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2, \varphi) \arrow[d, "\mathrm{forget}"] \\
\mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z}) \arrow[r, "\simeq", "\mathrm{Aff}"'] & \mathrm{Diff}(\mathbb{T}^2)
\end{tikzcd}
$$

Proof Using Observation 1.1.1, the canonical equivalences in the commutative diagrams on the right follow from those on the left.

Consider the diagrams in the ∞–category $\mathcal{S}\mathrm{paces}$, which make use of Notation 1.2.3:

(1) For $\vec{\varphi} = \vec{0}$,

$$
\begin{tikzcd}
\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \arrow[r, "\mathrm{pr}"] \arrow[d, "\mathrm{id} \rtimes \Psi"'] & \widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \arrow[r, "!"] \arrow[d, "\Psi"'] & * \arrow[d, "\langle (\vec{\varphi}, B_{\varphi}) \rangle"'] \\
\mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \arrow[r, "\mathrm{pr}"'] \arrow[d, "\mathrm{Aff}"', "\simeq"] & \mathrm{E}_2(\mathbb{Z}) \arrow[r, "A \mapsto (A\vec{\varphi}, B_{\varphi} A)"'] & \mathbb{Z}^2 \times \mathrm{GL}_2(\mathbb{R}) \arrow[d, "\simeq"', "\text{Corollary 1.2.2}"] \\
\mathrm{Imm}(\mathbb{T}^2) \arrow[rr, "\mathrm{Orbit}_{\varphi}"] & & \mathrm{Fr}(\mathbb{T}^2)
\end{tikzcd}
$$

(2) For $\vec{\varphi} \neq \vec{0}$,

$$
\begin{tikzcd}
(\mathbb{T}^2 \rtimes_{U_{\vec{\varphi}},D_{\vec{\varphi}}} (\mathbb{Z} \rtimes \mathbb{N}^{\times})) \times \mathbb{Z} \arrow[r, "\mathrm{pr}"] \arrow[d, "\mathrm{pr}"] & (\mathbb{Z} \rtimes \mathbb{N}^{\times}) \times \mathbb{Z} \arrow[r, "!"] \arrow[d, "\mathrm{pr}"] & * \arrow[d, "\langle (*, B_{\varphi}) \rangle"'] \\
\mathbb{T}^2 \rtimes_{U_{\vec{\varphi}},D_{\vec{\varphi}}} (\mathbb{Z} \rtimes \mathbb{N}^{\times}) \arrow[r, "\mathrm{pr}"] \arrow[d, "\mathrm{id} \rtimes ((b,d) \mapsto U_{\vec{\varphi}}^b D_{\vec{\varphi},d})"] & \mathbb{Z} \rtimes \mathbb{N}^{\times} \arrow[r, "(b,d) \mapsto B_{\varphi} U_{\vec{\varphi}}^b D_{\vec{\varphi},d}"] \arrow[d, "(b,d) \mapsto U_{\vec{\varphi}}^b D_{\vec{\varphi},d}"] & * \times \mathrm{GL}_2(\mathbb{R})_{B_{\varphi}} \arrow[d, "\langle \vec{\varphi} \rangle \times \mathrm{inc}"'] \\
\mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \arrow[r, "\mathrm{pr}"'] \arrow[d, "\mathrm{Aff}"', "\simeq"] & \mathrm{E}_2(\mathbb{Z}) \arrow[r, "A \mapsto (A^T \vec{\varphi}, B_{\varphi} A)"'] & \mathbb{Z}^2 \times \mathrm{GL}_2(\mathbb{R}) \arrow[d, "\simeq"', "\text{Corollary 1.2.2}"] \\
\mathrm{Imm}(\mathbb{T}^2) \arrow[rr, "\mathrm{Orbit}_{\varphi}"] & & \mathrm{Fr}(\mathbb{T}^2)
\end{tikzcd}
$$

where $\mathrm{GL}_2(\mathbb{R})_{B_{\varphi}} \subset \mathrm{GL}_2(\mathbb{R})$ is the path-component containing $B_{\varphi} \in \mathrm{GL}_2(\mathbb{R})$.

By Observation 1.3.6, each bottom rectangle canonically commutes. Lemma 1.1.2 and Corollary 1.2.2 together imply each of these bottom rectangles witnesses a pullback. Each of the top left squares, as well as the middle left square in the lower diagram, is clearly a pullback. Corollary B.2.2 states that the top right square in the upper diagram is a pullback. Provided the top right and middle right squares in the lower diagram are pullbacks, we would then conclude that each of the outer squares witnesses a pullback. The result would then follow by Definition 1.3.8 of $\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2, \varphi)$.

So it remains to show that the top right and middle right squares in the lower diagram are pullbacks. The paths of matrices

$$[0,1] \ni t \mapsto \begin{bmatrix} 1+tcd & tz^2 \\ -ty^2 & 1-tyz \end{bmatrix}^T, \quad \begin{bmatrix} 1+t(r-1)xy & -t(r-1)xz \\ t(r-1)wy & 1+t(r-1)wz \end{bmatrix}^T \in \mathrm{GL}_2(\mathbb{R}),$$

determine an identification of the named map $\mathbb{Z} \rtimes \mathbb{N}^\times \to \mathrm{GL}_2(\mathbb{R})$ with the constant map at B_φ. Together with the standard identification $\mathbb{Z} \simeq \Omega_{B_\varphi} \mathrm{GL}_2(\mathbb{R})$, this shows that the top right square in the lower diagram is a pullback. The middle right square of the lower diagram is a pullback because the map

$$\mathbb{Z} \rtimes (\mathbb{Z} \smallsetminus \{0\}) \to \mathrm{Stab}_{\mathrm{E}_2(\mathbb{Z})^{\mathrm{op}}}(\vec{\varphi}) \quad \text{given by } (b,d) \mapsto \left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}^{-1} \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} \right)^T = U^b_{\vec{\varphi}} D_{\vec{\varphi},d}$$

is an isomorphism between monoids, where $w, x, y, z \in \mathbb{Z}$ are as in Notation 1.4.1. $\square$

By applying the product-preserving functor $\mathcal{S}\mathrm{paces} \xrightarrow{\pi_0} \mathcal{S}\mathrm{ets}$, Lemma 1.4.3 implies the following:

Corollary 1.4.4 *There is a canonical isomorphism in the diagram of groups*

$$\begin{tikzcd} \mathrm{Braid}_3 \arrow[r, dashed, "\cong"] \arrow[d, "\Phi"'] & \mathrm{MCG}^{\mathrm{fr}}(\mathbb{T}^2) \arrow[d, "\text{forget}"] \\ \mathrm{GL}_2(\mathbb{Z}) \arrow[r, "\cong"] & \mathrm{MCG}(\mathbb{T}^2) \end{tikzcd}$$

Remark 1.4.5 Proposition 0.1.1 and Corollary 1.4.4 grant a central extension among groups:

$$1 \to \mathbb{Z} \to \mathrm{MCG}^{\mathrm{fr}}(\mathbb{T}^2) \to \mathrm{MCG}^{\mathrm{or}}(\mathbb{T}^2) \to 1.$$

Proof of Corollary 0.1.4 By construction, the diagram among spaces

$$\begin{tikzcd} \mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \arrow[rr, "\simeq", "\text{Corollary 1.1.5}"'] \arrow[dr, "\mathrm{pr}"'] & & \mathrm{Imm}(\mathbb{T}^2) \arrow[dl, "\mathrm{ev}_0"] \\ & \mathbb{T}^2 & \end{tikzcd}$$

canonically commutes, in which the left diagonal map is projection, and the right diagonal map evaluates at the origin $0 \in \mathbb{T}^2$. Therefore, upon taking fibers over $0 \in \mathbb{T}^2$, the (left) commutative diagram (1-4-2) among continuous monoids determines the commutative diagram among commutative monoids

$$\begin{tikzcd} \widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \arrow[rr, "\simeq"] \arrow[d] & & \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2 \operatorname{rel} 0) \arrow[d] \\ \mathrm{E}_2(\mathbb{Z}) \arrow[rr, "\simeq", "\text{Corollary 1.1.5}"'] \arrow[dr, "\mathbb{R}\otimes_{\mathbb{Z}}"'] & & \mathrm{Imm}(\mathbb{T}^2 \operatorname{rel} 0) \arrow[dl, "\mathrm{D}_0"] \\ & \mathrm{GL}_2(\mathbb{R}) & \end{tikzcd}$$

in which the map $\mathbb{R}\otimes_{\mathbb{Z}}$ is the standard inclusion, and D_0 takes the derivative at the origin $0 \in \mathbb{T}^2$. To finish, Corollary B.2.2 supplies the left pullback square in the following diagram among continuous groups, while the right pullback square is definitional:

$$\begin{tikzcd}
\mathrm{Braid}_3 \arrow[r] \arrow[d] & * \arrow[d] & \mathrm{Diff}(\mathbb{T}^2 \smallsetminus \mathbb{B}^2 \text{ rel } \partial) \arrow[l] \arrow[d] \\
\mathrm{GL}_2(\mathbb{Z}) \arrow[r, "\mathbb{R}\otimes_{\mathbb{Z}}"] & \mathrm{GL}_2(\mathbb{R}) & \mathrm{Diff}(\mathbb{T}^2 \text{ rel } 0) \arrow[l, "\mathrm{D}_0"']
\end{tikzcd}$$ □

1.5 Comparison with sheering

We use Theorem X(2) to show that $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)$ is generated by sheering. We quickly tour through some notions and results, which are routine after the above material.

Notation 1.5.1 It will be convenient to define the projection $\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}$ to be projection *off* of the i^{th} coordinate. So for $\mathbb{T}^2 \ni p = (x_p, y_p)$, we have $\mathrm{pr}_1(p) = y_p$ and $\mathrm{pr}_2(p) = x_p$.

Let $i \in \{1, 2\}$. Consider the topological subgroup and topological submonoid

$$\mathrm{Diff}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \subset \mathrm{Diff}(\mathbb{T}^2) \quad \text{and} \quad \mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \subset \mathrm{Imm}(\mathbb{T}^2),$$

consisting of those (local) diffeomorphisms $\mathbb{T}^2 \xrightarrow{f} \mathbb{T}^2$ that lie over some (local) diffeomorphism $\mathbb{T} \xrightarrow{\bar{f}} \mathbb{T}$:

$$\begin{tikzcd}
\mathbb{T}^2 \arrow[r, "f"] \arrow[d, "\mathrm{pr}_i"'] & \mathbb{T}^2 \arrow[d, "\mathrm{pr}_i"] \\
\mathbb{T} \arrow[r, "\bar{f}"] & \mathbb{T}
\end{tikzcd} \tag{1-5-1}$$

The topological space of *framings* of $\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}$ is the subspace

$$\mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \subset \mathrm{Fr}(\mathbb{T}^2)$$

consisting of those framings $\tau_{\mathbb{T}^2} \xrightarrow{\varphi} \epsilon^2_{\mathbb{T}^2}$ that lie over a framing $\tau_{\mathbb{T}} \xrightarrow{\bar{\varphi}} \epsilon^1_{\mathbb{T}}$:

$$\begin{tikzcd}
\tau_{\mathbb{T}^2} \arrow[r, "\varphi", "\cong"'] \arrow[d, "\mathrm{Dpr}_i"'] & \epsilon^2_{\mathbb{T}^2} \arrow[d, "\mathrm{pr}_i \times \mathrm{pr}_i"] \\
\tau_{\mathbb{T}} \arrow[r, "\bar{\varphi}", "\cong"'] & \epsilon^1_{\mathbb{T}}
\end{tikzcd} \tag{1-5-2}$$

Because pr_i is surjective, for a given φ there is a unique $\bar{\varphi}$ as in (1-5-2), if any. Better, $\varphi \mapsto \bar{\varphi}$ defines a continuous map

$$\mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \to \mathrm{Fr}(\mathbb{T}) \quad \text{given by } \varphi \mapsto \bar{\varphi}. \tag{1-5-3}$$

Notice that the continuous right-action Act of Lemma 1.3.1 evidently restricts as a continuous right-action

$$\mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \curvearrowleft \mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}).$$

Furthermore, (1-5-3) is evidently equivariant with respect to the morphism between topological monoids $\mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \xrightarrow{\text{forget}} \mathrm{Imm}(\mathbb{T})$:

$$(\mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \curvearrowleft \mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T})) \xrightarrow{\text{forget}} (\mathrm{Fr}(\mathbb{T}) \curvearrowleft \mathrm{Imm}(\mathbb{T})), \quad \varphi \mapsto \bar{\varphi}.$$

Now let $\varphi \in \mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T})$ be a framing of the projection. The orbit of φ by this action is the map

$$\mathrm{Orbit}_\varphi \colon \mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \to \mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \quad \text{given by } f \mapsto \mathrm{Act}(\varphi, f).$$

The space of *framed local diffeomorphisms*, and the space of *framed diffeomorphisms*, of $(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}, \varphi)$ are respectively the homotopy pullbacks among spaces

$$\begin{tikzcd} \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}, \varphi) \arrow[r] \arrow[d] & \mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \arrow[d, "\mathrm{Orbit}_\varphi"] \\ * \arrow[r, "\langle\varphi\rangle"] & \mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}, \varphi) \arrow[r] \arrow[d] & \mathrm{Diff}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \arrow[d, "\mathrm{Orbit}_\varphi"] \\ * \arrow[r, "\langle\varphi\rangle"] & \mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \end{tikzcd}$$

As in Observation 1.2.1, the topological space $\mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T})$ is a torsor for the topological group $\mathrm{Map}(\mathbb{T}^2, \mathrm{GL}_{\{i\}\subset 2}(\mathbb{R}))$ of smooth maps from $\mathbb{T}^2$ to the subgroup

$$\mathrm{GL}_{\{i\}\subset 2}(\mathbb{R}) := \{A \mid A\vec{e}_i \in \mathrm{Span}\{\vec{e}_i\}\} \subset \mathrm{GL}_2(\mathbb{R})$$

consisting of those 2×2 matrices that carry the i^{th}–coordinate line to itself. For each $i = 1, 2$, define the intersections in $\mathrm{GL}_2(\mathbb{R})$

$$\begin{tikzcd} \mathrm{SL}_2(\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{GL}_2(\mathbb{Z}) \arrow[d] \\ \mathrm{E}_2^+(\mathbb{Z}) \arrow[r] & \mathrm{E}_2(\mathbb{Z}) \end{tikzcd} \xmapsto{-\cap \mathrm{GL}_{\{i\}\subset 2}(\mathbb{R})} \begin{tikzcd} \mathrm{SL}_{\{i\}\subset 2}(\mathbb{Z}) \arrow[r] \arrow[d] & \mathrm{GL}_{\{i\}\subset 2}(\mathbb{Z}) \arrow[d] \\ \mathrm{E}^+_{\{i\}\subset 2}(\mathbb{Z}) \arrow[r] & \mathrm{E}_{\{i\}\subset 2}(\mathbb{Z}) \end{tikzcd}$$

Lemma 1.5.2 *For each $i = 1, 2$, the homotopy equivalences between continuous monoids of Lemma 1.1.2 and Corollary 1.1.5 restrict as homotopy equivalences between continuous monoids:*

$$\begin{tikzcd} \mathbb{T}^2 \rtimes \mathrm{GL}_{\{i\}\subset 2}(\mathbb{Z}) \arrow[r, dashed, "\mathrm{Aff}_i", "\simeq"'] \arrow[d, "\text{inclusion}"'] & \mathrm{Diff}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \arrow[d, "\text{inclusion}"] \\ \mathbb{T}^2 \rtimes \mathrm{GL}(\mathbb{Z}) \arrow[r, "\mathrm{Aff}", "\simeq"'] & \mathrm{Diff}(\mathbb{T}^2) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathbb{T}^2 \rtimes \mathrm{E}_{\{i\}\subset 2}(\mathbb{Z}) \arrow[r, dashed, "\mathrm{Aff}_i", "\simeq"'] \arrow[d, "\text{inclusion}"'] & \mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \arrow[d, "\text{inclusion}"] \\ \mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z}) \arrow[r, "\mathrm{Aff}", "\simeq"'] & \mathrm{Imm}(\mathbb{T}^2) \end{tikzcd}$$

Proof Via the involution $\Sigma_2 \curvearrowright \mathbb{T}^2$ that swaps coordinates, the case in which $i = 1$ implies the case in which $i = 2$. So we only consider the case in which $i = 1$.

The left homotopy equivalence is obtained from the right homotopy equivalence by restricting to maximal continuous subgroups. So we are reduced to establishing the right homotopy equivalence. Direct inspection reveals the indicated factorization Aff_1 of the restriction of Aff to $\mathbb{T}^2 \rtimes \mathrm{E}_{\{1\}\subset 2}(\mathbb{Z}) \subset \mathbb{T}^2 \rtimes \mathrm{E}_2(\mathbb{Z})$. So we are left to show that Aff_1 is a homotopy equivalence.

Projection to the $(1,1)$–entry defines a morphism between monoids, with kernel $\mathrm{K} := \left\{\left[\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right] \in \mathrm{E}_{\{1\}\subset 2}(\mathbb{Z})\right\}$, which fits into a split short exact sequence of monoids:

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{K} \arrow[r] & \mathrm{E}_{\{1\}\subset 2}(\mathbb{Z}) \arrow[r, "(1,1)\text{–entry}"'] & (\mathbb{Z} \smallsetminus \{0\})^{\times} \arrow[r] \arrow[l, bend right, dashed, "{\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right] \leftarrow\!\shortmid a}"'] & 1 \end{tikzcd}$$

Now, because pr_1 is surjective, for a given $f \in \mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T})$ there is a unique $\bar{f} \in \mathrm{Imm}(\mathbb{T})$ as in (1-5-1). Better, $\mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \ni f \mapsto \bar{f} \in \mathrm{Imm}(\mathbb{T})$ defines a forgetful morphism between topological monoids, whose kernel can be identified as the topological monoid of smooth maps from $\mathbb{T}$ to $\mathrm{Imm}(\mathbb{T})$ with valuewise monoid-structure. This is to say there is a bottom short exact sequence of topological monoids which splits as indicated:

$$(1\text{-}5\text{-}4)\qquad \begin{tikzcd} 1 \arrow[r] & \mathbb{T} \rtimes \mathrm{K} \arrow[r, "(\mathrm{id},\langle 0\rangle)\rtimes\text{inclusion}"] \arrow[d, dotted] & \mathbb{T}^2 \rtimes \mathrm{E}_{\{1\}\subset 2}(\mathbb{Z}) \arrow[r, "\mathrm{pr}_1\rtimes(1,1)\text{–entry}"'] \arrow[d, "\mathrm{Aff}_1"'] & \mathbb{T} \rtimes (\mathbb{Z} \smallsetminus \{0\})^{\times} \arrow[r] \arrow[d, dotted] \arrow[l, bend right, dashed, "{\left((0,z),\left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right]\right) \leftarrow\!\shortmid (z,a)}"'] & 1 \\ 1 \arrow[r] & \mathrm{Map}(\mathbb{T},\mathrm{Imm}(\mathbb{T})) \arrow[r] & \mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_1} \mathbb{T}) \arrow[r, "f\mapsto \bar{f}"'] & \mathrm{Imm}(\mathbb{T}) \arrow[r] \arrow[l, bend right, dashed, "\mathrm{id}_{\mathbb{T}}\times f \leftarrow\!\shortmid f"'] & 1 \end{tikzcd}$$

Direct inspection of the definition of Aff reveals the downward factorizations making the commutative diagram (1-5-4) among topological monoids. By the isotopy-extension theorem, the bottom short exact sequence among topological monoids forgets as a short exact sequence among continuous monoids. Using Lemma A.0.4, the proof is complete upon showing that the left and right downward maps are equivalences between spaces. It is routine to verify that the map $\mathrm{Imm}(\mathbb{T}) \xrightarrow{(\mathrm{ev}_0,\mathrm{H}_1(-))} \mathbb{T} \rtimes (\mathbb{Z} \smallsetminus \{0\})^{\times}$ is a homotopy inverse to the right downward map in (1-5-4).

Now observe that the left downward morphism in (1-5-4) fits into a diagram between short exact sequences of continuous monoids:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z} \arrow[r, "{b\mapsto \langle 0\rangle \rtimes \left[\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right]}"] \arrow[d, dotted] & \mathbb{T} \rtimes \mathrm{K} \arrow[r, "\mathrm{id}\rtimes(2,2)\text{–entry}"'] \arrow[d] & \mathbb{T} \rtimes (\mathbb{Z} \smallsetminus \{0\})^{\times} \arrow[r] \arrow[d, dotted] \arrow[l, bend right, dashed, "{\left(z,\left[\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right]\right) \leftarrow\!\shortmid (z,d)}"'] & 1 \\ 1 \arrow[r] & \mathrm{Map}((0\in\mathbb{T}),(\mathrm{id}\in\mathrm{Imm}(\mathbb{T}))) \arrow[r, "\text{forget}"'] & \mathrm{Map}(\mathbb{T},\mathrm{Imm}(\mathbb{T})) \arrow[r, "\mathrm{ev}_0"'] & \mathrm{Imm}(\mathbb{T}) \arrow[r] \arrow[l, bend right, dashed, "\mathrm{constant}_f \leftarrow\!\shortmid f"'] & 1 \end{tikzcd}$$

The right downward map here is a homotopy equivalence, in the same way the right downward map in (1-5-4) is a homotopy equivalence. Through this right downward identification of $\mathrm{Imm}(\mathbb{T})$, the left downward map is a homotopy equivalence, with inverse given by taking π_1. Using Lemma A.0.4, we conclude that the middle downward map is a homotopy equivalence, as desired. □

The Gram–Schmidt algorithm witnesses a deformation-retraction onto the inclusion from the intersection in $\mathrm{GL}_2(\mathbb{R})$:

$$\mathrm{O}(1)^2 = \mathrm{O}(1) \times \mathrm{O}(1) = \mathrm{O}(2) \cap \mathrm{GL}_{\{i\}\subset 2}(\mathbb{R}) \xhookrightarrow{\simeq} \mathrm{GL}_{\{i\}\subset 2}(\mathbb{R}).$$

Observation 1.5.3 For each $i = 1, 2$, the sequence of homotopy equivalences among topological spaces of Corollary 1.2.2, determined by a framing $\varphi \in \mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T})$ restricts as a sequence of homotopy equivalences among topological spaces:

$$\mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \xleftarrow{\cong} \mathrm{Map}(\mathbb{T}^2, \mathrm{GL}_{\{i\}\subset 2}(\mathbb{R})) \xrightarrow{\simeq} \mathrm{Map}((0 \in \mathbb{T}^2), (\mathbb{1} \in \mathrm{GL}_{\{i\}\subset 2}(\mathbb{R}))) \times \mathrm{GL}_{\{i\}\subset 2}(\mathbb{R}) \xleftarrow{\simeq} \mathrm{Map}\big((0 \in \mathbb{T}^2), (+1 \in \mathrm{O}(1))^2\big) \times \mathrm{O}(1)^2 \simeq \mathrm{O}(1)^2.$$

Observation 1.5.4 For each $i = 1, 2$, and each framing $\varphi \in \mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T})$, the following diagram among topological spaces commutes:

$$\begin{tikzcd}
\mathbb{T}^2 \rtimes \mathrm{E}_{\{i\}\subset 2}(\mathbb{Z}) \arrow[r, "\mathrm{Aff}_i"] \arrow[d, "(\text{sign of }(1,1)\text{–entry},\text{sign of }(2,2)\text{–entry})\circ\mathrm{proj}"'] & \mathrm{Imm}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \arrow[d, "\mathrm{Orbit}_\varphi"] \\
O(1)^2 & \mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}) \arrow[l, "\text{Observation 1.5.3}"']
\end{tikzcd}$$

For each $i = 1, 2$, the action $\mathbb{Z} \xrightarrow{\langle U_i \rangle} \mathrm{E}_{\{i\}\subset 2}(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ as a topological group defines the topological submonoid

$$\mathbb{T}^2 \rtimes_{U_i} \mathbb{Z} \subset \mathbb{T}^2 \rtimes \mathrm{E}_{\{i\}\subset 2}(\mathbb{Z}).$$

After Lemma 1.5.2 and Observation 1.5.3, Observation 1.5.4 implies the following:

Corollary 1.5.5 *For each $i = 1, 2$, and each framing $\varphi \in \mathrm{Fr}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T})$, there are canonical identifications among continuous monoids over the identification Aff_i,*

$$\begin{tikzcd}
\mathbb{T}^2 \rtimes_{U_i} \mathbb{Z} \arrow[r, "\mathrm{Aff}_i^{\mathrm{fr}}", "\simeq"'] \arrow[d, "\mathrm{id}\rtimes\langle\tau_i\rangle"'] & \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}, \varphi) \arrow[d, "\mathrm{forget}"] \\
\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \arrow[r, "\simeq", "\text{Lemma 1.4.3}"'] & \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2, \varphi)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbb{T}^2 \rtimes \mathrm{E}_{\{i\}\subset 2}(\mathbb{Z}) \arrow[r, "\mathrm{Aff}_i^{\mathrm{fr}}", "\simeq"'] \arrow[d, "\mathrm{id}\rtimes\widetilde{\langle\mathrm{inclusion}\rangle}"'] & \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}, \varphi) \arrow[d, "\mathrm{forget}"] \\
\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \arrow[r, "\simeq", "\text{Lemma 1.4.3}"'] & \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2, \varphi)
\end{tikzcd}$$

We now explain how the presentation (0-1-1) of Braid_3 gives a presentation of the continuous group $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)$. Observe the canonically commutative diagram among continuous groups

$$\begin{tikzcd}
\mathbb{T}^2 \arrow[r] \arrow[d] & \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_1} \mathbb{T}) \arrow[d] \\
\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_2} \mathbb{T}) \arrow[r] & \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)
\end{tikzcd}$$

which results in a morphism from the pushout,

$$\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_1} \mathbb{T}) \amalg_{\mathbb{T}^2} \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_2} \mathbb{T}) \to \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2).$$

Recall the element $R \in \mathrm{GL}_2(\mathbb{Z})$ from (B-2-1). The two homomorphisms $\mathbb{Z} \underset{\langle\tau_2\tau_1\tau_2\rangle}{\overset{\langle\tau_1\tau_2\tau_1\rangle}{\rightrightarrows}} \mathbb{Z} \amalg \mathbb{Z}$ determine two morphisms among continuous groups under $\mathbb{T}^2$:

$$\mathbb{T}^2 \rtimes_R \mathbb{Z} \underset{\mathrm{id}\rtimes\langle\tau_2\tau_1\tau_2\rangle}{\overset{\mathrm{id}\rtimes\langle\tau_1\tau_2\tau_1\rangle}{\rightrightarrows}} \mathbb{T}^2 \rtimes_{U_1,U_2} (\mathbb{Z} \amalg \mathbb{Z}) \xrightarrow{\simeq} \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_1} \mathbb{T}) \amalg_{\mathbb{T}^2} \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2 \xrightarrow{\mathrm{pr}_2} \mathbb{T}) \to \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2). \tag{1-5-5}$$

Corollary 1.5.6 *The diagram (1-5-5) among continuous groups under $\mathbb{T}^2$ witnesses a coequalizer.*

Proof The presentation (0-1-1) of Braid_3 gives a coequalizer diagram among groups:

$$\mathbb{Z} \underset{\langle \tau_2\tau_1\tau_2\rangle}{\overset{\langle \tau_1\tau_2\tau_1\rangle}{\rightrightarrows}} \mathbb{Z} \amalg \mathbb{Z} \xrightarrow{\langle \tau_1 \text{ and } \tau_2\rangle} \mathrm{Braid}_3.$$

Taking semidirect products with respect to the action $\mathrm{Braid}_3 \xrightarrow{\Phi} \mathrm{GL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ results in a coequalizer diagram among continuous groups:

$$\mathbb{T}^2 \rtimes_R \mathbb{Z} \underset{\mathrm{id}\rtimes\langle \tau_2\tau_1\tau_2\rangle}{\overset{\mathrm{id}\rtimes\langle \tau_1\tau_2\tau_1\rangle}{\rightrightarrows}} \mathbb{T}^2 \rtimes_{U_1,U_2} (\mathbb{Z} \amalg \mathbb{Z}) \xrightarrow{\mathrm{id}\rtimes\langle \tau_1 \text{ and } \tau_2\rangle} \mathbb{T}^2 \rtimes \mathrm{Braid}_3.$$

The result then follows from Corollary 1.5.5. □

Proof of Corollary 0.1.5 Consider the diagram among ∞–categories

$$\begin{tikzcd}
\mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)}(\mathcal{X}) \arrow[r] \arrow[d, equal] & \mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})^{\langle R\rangle} \arrow[d, "\simeq"', "\text{Proposition A.0.5}"] \\
\mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)}(\mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Mod}_{\mathbb{T}^2\rtimes_R\mathbb{Z}}(\mathcal{X}) \arrow[d, "\text{diagonal}"] \\
\mathrm{Mod}_{\mathbb{T}^2\rtimes_{U_1}\mathbb{Z}}(\mathcal{X}) \times_{\mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})} \mathrm{Mod}_{\mathbb{T}^2\rtimes_{U_2}\mathbb{Z}}(\mathcal{X}) \arrow[r, "(\mathrm{id}\rtimes\langle\tau_1\tau_2\tau_1\rangle)^*\times(\mathrm{id}\rtimes\langle\tau_2\tau_1\tau_2\rangle)^*"] & \mathrm{Mod}_{\mathbb{T}^2\rtimes_R\mathbb{Z}}(\mathcal{X}) \times_{\mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})} \mathrm{Mod}_{\mathbb{T}^2\rtimes_R\mathbb{Z}}(\mathcal{X}) \\
\mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})^{\langle U_1,U_2\rangle} \arrow[r] \arrow[u, "\simeq", "\text{Proposition A.0.5}"'] & \mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})^{\langle R,R\rangle} \arrow[u, "\simeq", "\text{Proposition A.0.5}"']
\end{tikzcd}$$

Corollary 1.5.6 implies the middle square is a pullback. Via Proposition A.0.5, which identifies modules for a semidirect product in terms of invariants, the top and bottom squares are pullbacks. Therefore, the outer square is a pullback, as desired. □

2 Natural symmetries of secondary Hochschild homology

Conventions (1) We fix a symmetric monoidal ∞–category $\mathcal{V}$, and assume it is $\otimes$–presentable (meaning the underlying ∞–category $\mathcal{V}$ is presentable, and $\otimes$ distributes over colimits separately in each variable).

(2) In this section, we apply the results from above only to the case of the standard framing φ_0 of the 2–torus $\mathbb{T}^2$. So we suppress the framing φ_0 from all notation, while regarding $\mathbb{T}^2$ as a framed 2–manifold.

Example 2.0.1 For $\Bbbk$ a commutative ring, take

$$(\mathcal{V}, \otimes) = (\mathrm{Ch}_{\Bbbk}[\{\text{quasi-isos}\}^{-1}], \otimes^{\mathbb{L}}_{\Bbbk})$$

to be the ∞–categorical localization of chain complexes over $\Bbbk$ on quasi-isomorphisms, with derived tensor product over $\Bbbk$ presenting the symmetric monoidal structure. More generally, for R a commutative ring spectrum, take $(\mathcal{V}, \otimes) := (\mathrm{Mod}_R, \wedge_R)$ to be the ∞–category of R–module spectra and smash product over R as the symmetric monoidal structure.

2.1 Hochschild homology of an associative algebra

Let A be an associative algebra in $\mathcal{V}$.

Recall the paracyclic category $\mathbf{\Delta}_{\circlearrowleft}$, introduced by Getzler and Jones. An object is a linearly ordered set I with finite intervals, equipped with an order-preserving action $\mathbb{Z} \curvearrowright I$ with the property that $i < 1 \cdot i$ for each $i \in I$; a morphism is a $\mathbb{Z}$–equivariant map between linearly ordered sets. Here are some standard facts about the paracyclic category; see, for instance, [Lurie 2015, Section 4.2].

(1) There is a canonical equivalence

$$\mathrm{Hom}^{\mathrm{surj}}_{\mathrm{LinOrd}}(-, [1]) \colon {\mathbf{\Delta}_{\circlearrowleft}}^{\mathrm{op}} \xrightarrow{\simeq} \mathbf{\Delta}_{\circlearrowleft},$$

whose value on $(\mathbb{Z} \curvearrowright I)$ is the set of surjective maps between linearly ordered sets from I to $[1]$, equipped with inherited linear order and residual $\mathbb{Z}$–action.

(2) The $\mathbb{Z}$–action on each object in $\mathbf{\Delta}_{\circlearrowleft}$, and the $\mathbb{Z}$–equivariance of each morphism in $\mathbf{\Delta}_{\circlearrowleft}$, assemble as an action

$$B\mathbb{Z} \curvearrowright \mathbf{\Delta}_{\circlearrowleft}.$$

(3) There is a standard functor $\mathbf{\Delta} \xrightarrow{[p] \mapsto [p]^{\star\mathbb{Z}}} \mathbf{\Delta}_{\circlearrowleft}$, whose value on a nonempty finite linearly ordered set is its $\mathbb{Z}$–fold join, as it is equipped with the $\mathbb{Z}$–action given by translating joinands. The resulting functor

$$\mathbf{\Delta}^{\mathrm{op}} \to {\mathbf{\Delta}_{\circlearrowleft}}^{\mathrm{op}} \simeq \mathbf{\Delta}_{\circlearrowleft}$$

is final.

Recall from [Loday 1992] Connes' cyclic category $\mathbf{\Lambda}$ in which an object is a cyclically ordered nonempty finite set, and a morphism is a cyclic order-preserving map. For $(\mathbb{Z} \curvearrowright I) \in \mathbf{\Delta}_{\circlearrowleft}$ an object, the $\mathbb{Z}$–coinvariants of the underlying set $I_{/\mathbb{Z}}$ canonically retain a cyclic order; this association assembles as a functor

$$\mathbf{\Delta}_{\circlearrowleft} \to \mathbf{\Lambda} \quad \text{given by } (\mathbb{Z} \curvearrowright I) \mapsto I_{/\mathbb{Z}}.$$

This functor witnesses the $B\mathbb{Z}$–coinvariants:

$$\mathbf{\Delta}_{\circlearrowleft/B\mathbb{Z}} \xrightarrow{\simeq} \mathbf{\Lambda}.$$

Recall from [Boardman and Vogt 1973] an explicit description of the symmetric monoidal envelope $\mathrm{Env}^{\otimes}(\mathrm{Assoc})$ of the associative operad.[12] There is a canonical functor

$$\mathbf{\Delta}_{\circlearrowleft} \to \mathrm{Env}^{\otimes}(\mathrm{Assoc})$$

[12] Specifically, an object is a finite set; a morphisms from I to J is a map between finite sets $I \xrightarrow{f} J$ together with a linear order on $f^{-1}(j)$ for each $j \in J$; composition is composition of maps between finite sets together with joins of finite sets; the symmetric monoidal structure is given by disjoint unions of finite sets.

whose value on an object $(\mathbb{Z} \curvearrowright I) \in \mathbf{\Delta}_{\circlearrowleft}$ is the quotient set $I_{/ZZ}$, and whose value on a morphism $(\mathbb{Z} \curvearrowright I) \xrightarrow{f} (\mathbb{Z} \curvearrowright J)$ in $\mathbf{\Delta}_{\circlearrowleft}$ is the induced map between quotient sets $I_{/\mathbb{Z}} \xrightarrow{f_{/\mathbb{Z}}} J_{/\mathbb{Z}}$ together with the linear order on $f_{/\mathbb{Z}}^{-1}([j])$ inherited through the canonical bijection $I \supset f^{-1}(j) \xrightarrow{\text{bijection}} f_{/\mathbb{Z}}^{-1}([j])$ for some (any) choice of $j \in [j] \in J_{/\mathbb{Z}}$. Evidently, this functor is canonically $B\mathbb{Z}$ invariant, thus canonically factoring through the $B\mathbb{Z}$–coinvariants:

$$\mathbf{\Delta}_{\circlearrowleft/B\mathbb{Z}} \simeq \mathbf{\Lambda} \to \mathrm{Env}^{\otimes}(\mathrm{Assoc}).$$

In particular, each associative algebra A in $\mathcal{V}$ determines a composite functor

$$\mathrm{Bar}^{\mathrm{cyc}}_{\bullet}(A)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}_{\circlearrowleft} \to \mathbf{\Lambda} \to \mathrm{Env}^{\otimes}(\mathrm{Assoc}) \xrightarrow{A} \mathcal{V},$$

which is the *cyclic bar construction* (*of* A). The *Hochschild homology* (*of* A) (*in* $\mathcal{V}$) is the geometric realization of this simplicial object:

$$\mathrm{HH}(A) := \mathrm{HH}_{\mathcal{V}}(A) := A \otimes_{A^{\mathrm{op}} \otimes A} A \simeq |\mathrm{Bar}^{\mathrm{cyc}}_{\bullet}(A)| \in \mathcal{V}.$$

This construction is evidently functorial in the argument A:

$$\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V}) \xrightarrow{\mathrm{HH}} \mathcal{V}.$$

Using finality of $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}_{\circlearrowleft}$, the action $\mathbb{T} \simeq B\mathbb{Z} \curvearrowright \mathbf{\Delta}_{\circlearrowleft}$ determines an action $\mathbb{T} \curvearrowright \mathrm{HH}(A)$, which is Connes' cyclic operator [1983]. This action is evidently functorial in the argument A:

(2-1-1)
$$\begin{tikzcd} & \mathrm{Mod}_{\mathbb{T}}(\mathcal{V}) \arrow[d, "\mathrm{forget}"] \\ \mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V}) \arrow[ur, dashed, "\mathrm{HH}"] \arrow[r, "\mathrm{HH}"] & \mathcal{V} \end{tikzcd}$$

When working over the sphere spectrum (which is to say $\mathcal{V} = (\mathrm{Spectra}, \wedge)$) so that $\mathrm{HH}_{\mathrm{Spectra}}(A) = \mathrm{THH}(A)$ is *topological Hochschild homology*, Bökstedt, Hsiang and Madsen [Bökstedt et al. 1993] extend this $\mathbb{T}$–action as a *cyclotomic structure* on $\mathrm{THH}(A)$. In [Ayala et al. 2017c] it is demonstrated how this cyclotomic structure on $\mathrm{THH}(A)$ is derived from an action of the continuous monoid $\mathbb{T} \rtimes \mathbb{N}^{\times}$ on the unstable version $\mathrm{HH}_{\mathrm{Spaces}}(A)$.

Below, we prove Theorem Y.1, which constructs a canonical $(\mathbb{T}^2 \rtimes \mathrm{Braid}_3)$–action on $\mathrm{HH}^{(2)}(A)$, which is functorial in the 2–algebra A. We then prove Theorem Y.2, which, in the case that $\mathcal{V} = (\mathcal{S}\mathrm{paces}, \times)$, extends this action to one by the continuous monoid $\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^{+}(\mathbb{Z})$.

2.2 Secondary Hochschild homology of 2–algebras

In order for the Hochschild homology construction to be twice-iterated, we endow the entity $A \in \mathcal{V}$ with an algebra structure among algebras.

Definition 2.2.1 The ∞–category of 2–*algebras* (*in* $\mathcal{V}$) is

$$\mathrm{Alg}_2(\mathcal{V}) := \mathrm{Alg}_{\mathrm{Assoc}}(\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V})).$$

Example 2.2.2 A commutative algebra $A = (A, \mu)$ in $\mathcal{V}$ determines the 2–algebra (A, μ, μ) in $\mathcal{V}$. This association assembles as a functor

$$\mathrm{CAlg}(\mathcal{V}) \to \mathrm{Alg}_2(\mathcal{V}),$$

thus supplying a host of examples of 2–algebras.

Observation 2.2.3 Using that the tensor product of operads is defined by a "hom-tensor" adjunction, there is a canonical equivalence between ∞–categories

$$\mathrm{Alg}_{\mathrm{Assoc}\otimes\mathrm{Assoc}}(\mathcal{V}) \simeq \mathrm{Alg}_2(\mathcal{V}).$$

In particular, swapping the two tensor-factors supplies an involution

$$\Sigma_2 \curvearrowright \mathrm{Alg}_2(\mathcal{V}).$$

Remark 2.2.4 After Observation 2.2.3, a 2–algebra in $\mathcal{V}$ is an object $A \in \mathcal{V}$ together with two associative algebra structures μ_1 and μ_2 on A, and compatibility between them which can be stated as either of the two equivalent structures

- a lift of the morphism $A\otimes A \xrightarrow{\mu_2} A$ in $\mathcal{V}$ to a morphism $(A, \mu_1)\otimes(A, \mu_1) \xrightarrow{\mu_2} (A, \mu_1)$ in $\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V})$,
- a lift of the morphism $A\otimes A \xrightarrow{\mu_1} A$ in $\mathcal{V}$ to a morphism $(A, \mu_2)\otimes(A, \mu_2) \xrightarrow{\mu_1} (A, \mu_2)$ in $\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V})$.

Example 2.2.5 Consider the operad $\mathcal{E}_2$ of little 2–disks. There is a standard morphism between operads $\mathrm{Assoc}\otimes\mathrm{Assoc} \to \mathcal{E}_2$; see [Dunn 1988]. Through Observation 2.2.3, restriction along this morphism defines a functor between ∞–categories

(2-2-1) $$\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{V}) \to \mathrm{Alg}_2(\mathcal{V}),$$

thus supplying some rich examples of 2–algebras. For instance, for $\Bbbk$ a commutative ring, a braided-monoidal $\Bbbk$–linear category R is a 2–algebra in the $(2,1)$–category of $\Bbbk$–linear categories. Specifically, for G a simply connected reductive algebraic group over $\mathbb{C}$, a choice of Killing form on its Lie algebra $\mathfrak{g}$ determines the quantum group $\mathcal{U}_q\mathfrak{g}$, and thereafter the braided-monoidal category $\mathrm{Rep}_q(G)$ (for generic q). (See [Chari and Pressley 1994], for instance.)

Theorem 2.2.6 (Dunn's additivity [1988]; see also [Lurie 2017, Theorem 5.1.2.2]) *The functor (2-2-1) is an equivalence between ∞–categories.*

Remark 2.2.7 The action $\mathrm{O}(2) \curvearrowright \mathrm{Alg}_2(\mathcal{V})$ of Corollary 0.2.8, afforded by Theorem 2.2.6, extends the evident $(\Sigma_2 \wr \mathrm{O}(1))$–action which swaps the two associative algebra structures (as the Σ_2–factor) and takes opposites of the two associative algebra structures (as the two $\mathrm{O}(1)$–factors).

Definition 2.2.8 *Secondary Hochschild homology* is the composite functor, given by twice-iterating Hochschild homology,

$$\mathrm{HH}^{(2)}\colon \mathrm{Alg}_2(\mathcal{V}) := \mathrm{Alg}_{\mathrm{Assoc}}(\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V})) \xrightarrow{\mathrm{HH}} \mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V}) \xrightarrow{\mathrm{HH}} \mathcal{V},$$
$$(A, \mu_1, \mu_1) \mapsto (\mathrm{HH}(A, \mu_1), \mathrm{HH}(\mu_2)) \mapsto \mathrm{HH}(\mathrm{HH}(A, \mu_1), \mathrm{HH}(\mu_2)).$$

The canonical lift (2-1-1) supplies, for each 2–algebra A in $\mathcal{V}$, two commuting actions $\mathbb{T} \curvearrowright \mathrm{HH}^{(2)}(A)$, functorially in the argument A:

(2-2-2)
$$\begin{tikzcd} & \mathrm{Mod}_{\mathbb{T}^2}(\mathcal{V}) \arrow[d] \\ \mathrm{Alg}_2(\mathcal{V}) \arrow[ur, dashed, "\mathrm{HH}^{(2)}"] \arrow[r, "\mathrm{HH}^{(2)}"] & \mathcal{V} \end{tikzcd}$$

2.3 Comparison with factorization homology

Let $n \geq 0$. Recall from [Ayala and Francis 2015] the symmetric monoidal ∞–category $\mathrm{Mfld}_n^{\mathrm{fr}}$ whose objects are (finitary) framed n–manifolds, whose spaces of morphisms are spaces of framed embeddings between them, and whose symmetric monoidal structure is given by disjoint union. Let M be a framed n–manifold. Consider the full ∞–subcategories

$$\mathrm{Disk}_n^{\mathrm{fr}} \xhookrightarrow{\iota} \mathrm{Mfld}_n^{\mathrm{fr}} \hookleftarrow \mathrm{BDiff}^{\mathrm{fr}}(M),$$

respectively consisting of those framed n–manifolds each of whose connected components is equivalent with $\mathbb{R}^n$, and of those framed n–manifolds that are equivalent with M. The left full ∞–subcategory is closed with respect to the symmetric monoidal structure. Restriction along these full ∞–subcategories determines the solid diagram among ∞–categories

(2-3-1)
$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{E}_n}(\mathcal{V}) & \mathrm{Fun}^{\otimes}(\mathrm{Disk}_n^{\mathrm{fr}}, \mathcal{V}) \arrow[l, "\simeq"'] \arrow[r, dashed, bend left, "\int"] & \mathrm{Fun}^{\otimes}(\mathrm{Mfld}_n^{\mathrm{fr}}, \mathcal{V}) \arrow[l, "\mathrm{restrict}"'] \arrow[r, "\mathrm{restrict}"] & \mathrm{Fun}(\mathrm{BDiff}^{\mathrm{fr}}(M), \mathcal{V}) \simeq \mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(M)}(\mathcal{V}). \end{tikzcd}$$

Factorization homology is defined as the left adjoint to the leftward restriction functor, indicated by the dashed arrow; factorization homology over the torus, as it is endowed with a canonical $\mathrm{Diff}^{\mathrm{fr}}(M)$–action, is the rightward composite functor

(2-3-2)
$$\int_M : \mathrm{Alg}_{\mathcal{E}_n}(\mathcal{V}) \to \mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(M)}(\mathcal{V}).$$

Proposition 2.3.1 *There is a canonical equivalence*

$$\mathrm{HH} \simeq \int_{\mathbb{T}} \quad \textit{in } \mathrm{Fun}(\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V}), \mathrm{Mod}_{\mathbb{T}}(\mathcal{V})).$$

Proof Recall from [Ayala and Francis 2015] the functor between ∞–categories $\mathrm{Disk}^{\mathrm{fr}}_{1/\mathbb{S}^1} \xrightarrow{\mathrm{forget}} \mathrm{Disk}_1^{\mathrm{fr}}$. Both of these a priori ∞–categories are ordinary categories. Through [Lurie 2017, Example 5.1.0.7], taking path-components defines an equivalence between ∞–operads $\mathcal{E}_1 \to \mathrm{Assoc}$. Proposition 2.12 of [Ayala et al. 2017b] states an identification between symmetric monoidal ∞–categories $\mathrm{Env}^{\otimes}(\mathcal{E}_1) \xrightarrow{\simeq} \mathrm{Disk}_1^{\mathrm{fr}}$. Consequently, taking path-components of disjoint unions of Euclidean spaces defines an equivalence between symmetric monoidal ∞–categories:

$$\pi_0 : \mathrm{Disk}_1^{\mathrm{fr}} \simeq \mathrm{Env}^{\otimes}(\mathcal{E}_1) \xrightarrow{\simeq} \mathrm{Env}^{\otimes}(\mathrm{Assoc}).$$

Similarly, taking path-components of disjoint unions of Euclidean spaces while remembering cyclic orders from $\mathbb{S}^1$ defines a $(\mathbb{T}\simeq B\mathbb{Z})$–equivariant equivalence between ∞–categories filling the diagram among ∞–categories

$$\begin{tikzcd} & & \mathrm{Disk}^{\mathrm{fr}}_{1/\mathbb{S}^1} \arrow[r, "\text{forget}"] \arrow[d, dashed, "\pi_0"', "\simeq"] & \mathrm{Disk}^{\mathrm{fr}}_1 \arrow[d, "\pi_0"', "\simeq"] \\ \mathbf{\Delta}^{\mathrm{op}} \arrow[r, "\text{final}"] & \mathbf{\Delta}_{\circlearrowleft} \arrow[r, "\text{inclusion}"] & \mathbf{\Delta}_{\circlearrowleft}^{\triangleleft} \arrow[r] & \mathrm{Env}^{\otimes}(\mathrm{Assoc}) \end{tikzcd}$$

In particular, there is a commutative diagram among ∞–categories

$$\begin{tikzcd} (\mathbf{\Delta}_{\circlearrowleft})_{/\mathbb{T}} \arrow[r] \arrow[dr] & (\mathbf{\Delta}_{\circlearrowleft}^{\triangleleft})_{/\mathbb{T}} \arrow[d] & (\mathrm{Disk}^{\mathrm{fr}}_{1/\mathbb{S}^1})_{/\mathbb{T}} \arrow[l, "\simeq"'] \arrow[dl] \\ & \mathrm{Env}^{\otimes}(\mathrm{Assoc}) & \end{tikzcd} \tag{2-3-3}$$

We now explain the diagram among ∞–categories

$$\begin{tikzcd} & & & \mathrm{Fun}(\mathrm{Disk}^{\mathrm{fr}}_{1/\mathbb{S}^1},\mathcal{V})^{\mathbb{T}} \arrow[dr, "\mathrm{colim}"] & \\ \mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{V}) \simeq \mathrm{Fun}^{\otimes}(\mathrm{Env}^{\otimes}(\mathrm{Assoc}),\mathcal{V}) \arrow[r] & \mathrm{Fun}(\mathrm{Env}^{\otimes}(\mathrm{Assoc}),\mathcal{V}) \arrow[r] \arrow[urr] \arrow[drr] & & \mathrm{Fun}(\mathbf{\Delta}_{\circlearrowleft}^{\triangleleft},\mathcal{V})^{\mathbb{T}} \arrow[u, "\simeq"] \arrow[d] \arrow[r, "\mathrm{colim}"] & \mathcal{V}^{\mathbb{T}} \arrow[d, "\simeq"] \\ & & & \mathrm{Fun}(\mathbf{\Delta}_{\circlearrowleft},\mathcal{V})^{\mathbb{T}} \arrow[ur, "\mathrm{colim}"'] & \mathrm{Mod}_{\mathbb{T}}(\mathcal{V}) \end{tikzcd}$$

The rightward functor on the left is the forgetful functor from symmetric monoidal functors to functors between underlying ∞–categories. The equivalence on the left is the universal property of symmetric monoidal envelopes. Restriction along the diagram (2-3-3) defines the two triangles involving unlabeled functors, where the superscript denotes the $\mathbb{T}$–invariants with respect to the action on the domain-argument of each functor ∞–category. The functors labeled by colim are given by taking colimits. The right vertical equivalence is definitional, using that the $\mathbb{T}$–action on $\mathcal{V}$ is understood as trivial. The upper right triangle commutes because the functor $\mathrm{Disk}^{\mathrm{fr}}_{1/\mathbb{S}^1} \xrightarrow{\simeq} \mathbf{\Delta}_{\circlearrowleft}^{\triangleleft}$ is an equivalence, and in particular final. Finality of $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Delta}_{\circlearrowleft}$, together with the fact that $\mathbf{\Delta}$ has a final object, implies the ∞–groupoid-completion of $\mathbf{\Delta}_{\circlearrowleft}$ is contractible. This implies the functor $\mathbf{\Delta}_{\circlearrowleft} \hookrightarrow \mathbf{\Delta}_{\circlearrowleft}^{\triangleleft}$ is final, which proves that the lower triangle commutes.

To finish, the definition of $\int_{\mathbb{T}}$ is the upper composite functor, while the definition of HH is the lower composite functor. □

Corollary 2.3.2 *There is a canonical equivalence*

$$\mathrm{HH}^{(2)} \simeq \int_{\mathbb{T}^2} \quad \textit{in } \mathrm{Fun}(\mathrm{Alg}_2(\mathcal{V}),\mathrm{Mod}_{\mathbb{T}^2}(\mathcal{V})).$$

Proof The sought equivalence is a concatenation of the sequence of equivalences in the ∞–category $\mathrm{Fun}(\mathrm{Alg}_2(\mathcal{V}),\mathrm{Mod}_{\mathbb{T}^2}(\mathcal{V}))$,

$$\mathrm{HH}^{(2)}(-) \simeq \mathrm{HH}(\mathrm{HH}(-)) \simeq \int_{\mathbb{T}}\Big(\int_{\mathbb{T}}(-)\Big) \simeq \int_{\mathbb{T}^2}(-),$$

which we now explain. The first equivalence is the definition of secondary Hochschild homology. The second equivalence is two applications of Proposition 2.3.1. The third equivalence is a consequence of the pushforward formula [Ayala and Francis 2015, Proposition 3.23]. □

Swapping the order of pushforward immediately implies the following:

Corollary 2.3.3 *For $A = (A, \mu_1, \mu_2)$ a 2–algebra in $\mathcal{V}$, the two iterations of Hochschild homology canonically agree:*

$$\mathrm{HH}(\mathrm{HH}(A, \mu_1), \mathrm{HH}(\mu_2)) \simeq \mathrm{HH}(\mathrm{HH}(A, \mu_2), \mathrm{HH}(\mu_1)).$$

2.4 Comparing sheers

Here we show the sheer symmetries of $\mathrm{HH}^{(2)}$ agree.

Consider the composite morphism between continuous groups

$$\langle \tau_1 \rangle : \mathbb{Z} \hookrightarrow \mathbb{T}^2 \rtimes_{U_1} \mathbb{Z} \xrightarrow{\mathrm{Aff}_1} \mathrm{Diff}^{\mathrm{fr}}(\mathrm{pr}_1) \to \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2).$$

Note that the composition $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \to \mathrm{Diff}(\mathbb{T}^2) \xleftarrow{\simeq} \mathbb{T}^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$ carries τ_1 to the sheering matrix $U_1 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{Z})$.

Proposition 2.4.1 *The diagram among ∞–categories*

$$\begin{tikzcd}
\mathrm{Alg}_2(\mathcal{V}) \arrow[d, "\mathrm{HH}^{(2)}", "\mathbb{Z} \curvearrowright_{\mathrm{Sheer}_1}"'] & \mathrm{Alg}_{\mathcal{E}_2}(\mathcal{V}) \arrow[l, "\mathrm{fgt}_1"'] \arrow[r, "\mathrm{fgt}_2"] \arrow[d, "\int_{\mathbb{T}^2}"'] & \mathrm{Alg}_2(\mathcal{V}) \arrow[d, "\mathbb{Z} \curvearrowright_{\mathrm{Sheer}_2^{-1}} \mathrm{HH}^{(2)}"] \\
\mathrm{Mod}_{\mathbb{Z}}(\mathcal{V}) & \mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)}(\mathcal{V}) \arrow[l, "\langle \tau_1 \rangle^*"'] \arrow[r, "\langle \tau_2 \rangle^*"] & \mathrm{Mod}_{\mathbb{Z}}(\mathcal{V})
\end{tikzcd}$$

canonically commutes. In other words, for each $\mathcal{E}_2$–algebra A in $\mathcal{V}$, there are canonical identifications between the two symmetries of $\mathrm{HH}^{(2)}(A)$,

(2-4-1) $$\langle \tau_1 \rangle \simeq \mathrm{Sheer}_1 \quad \textit{and} \quad \langle \tau_2 \rangle \simeq \mathrm{Sheer}_2^{-1},$$

functorially in $A \in \mathrm{Alg}_{\mathcal{E}_2}(\mathcal{V})$.

Proof By swapping the two coordinates of $\mathbb{T}^2$, commutativity of the left square implies commutativity of the right square. So we only establish commutativity of the left square.

Notice that this diagram is functorial in the presentably symmetric monoidal ∞–category $\mathcal{V}$. Therefore, commutativity of this diagram for any presentably symmetric monoidal ∞–category $\mathcal{V}$ is implied by an identification (2-4-1) in the case that the pair $(A, \mathcal{V})$ is initial among presentably symmetric monoidal ∞–categories equipped with an $\mathcal{E}_2$–algebra.

We first identify the initial presentably symmetric monoidal ∞–category equipped with an $\mathcal{E}_2$–algebra. Day convolution supplies a symmetric monoidal structure on the ∞–category $\mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}})$. By construction, this symmetric monoidal ∞–category is $\otimes$–presentable. Also, the Yoneda embedding $\mathrm{Disk}_2^{\mathrm{fr}} \xrightarrow{\text{Yoneda}} \mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}})$ is canonically symmetric monoidal. Via the equivalence $\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{V}) \xleftarrow{\simeq} \mathrm{Fun}^{\otimes}(\mathrm{Disk}_2^{\mathrm{fr}}, \mathcal{V})$, the Yoneda functor is an $\mathcal{E}_2$–algebra in $\mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}})$. Furthermore, it is initial among presentably symmetric monoidal ∞–categories equipped with an $\mathcal{E}_2$–algebra. Indeed, for $A \in \mathrm{Alg}_{\mathcal{E}_2}(\mathcal{V}) \xleftarrow{\simeq} \mathrm{Fun}^{\otimes}(\mathrm{Disk}_2^{\mathrm{fr}}, \mathcal{V})$ an $\mathcal{E}_2$–algebra in $\mathcal{V}$, left Kan extension of A along the Yoneda functor is the unique colimit-preserving symmetric monoidal filler:

$$\begin{tikzcd} & \mathrm{Disk}_2^{\mathrm{fr}} \arrow[dl, "\text{Yoneda}"'] \arrow[dr, "A"] & \\ \mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}}) \arrow[rr, dashed, "\text{LKE}"] & & \mathcal{V} \end{tikzcd}$$

Recall the fully faithful symmetric monoidal functor $\mathrm{Disk}_2^{\mathrm{fr}} \xrightarrow{\iota} \mathrm{Mfld}_2^{\mathrm{fr}}$. The restricted Yoneda functor associated to ι is

$$\mathrm{Mfld}_2^{\mathrm{fr}} \xrightarrow{\text{restricted Yoneda}} \mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}}) \quad \text{given by } M \mapsto \mathrm{Hom}_{\mathrm{Mfld}_2^{\mathrm{fr}}}(\iota, M),$$

which is canonically symmetric monoidal. The definition of factorization homology is such that there is a canonical morphism in $\mathrm{Fun}^{\otimes}(\mathrm{Mfld}_2^{\mathrm{fr}}, \mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}}))$,

$$\int_{-} \text{Yoneda} \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Mfld}_2^{\mathrm{fr}}}(\iota, -). \tag{2-4-2}$$

This morphism is an equivalence. Indeed, unpacking definitions and identifying presheaves with right fibrations via the (un)straightening equivalence, the unstraightening of this morphism is a functor between right fibrations over $\mathrm{Disk}_n^{\mathrm{fr}}$:

$$\int_{-} \mathrm{Disk}^{\mathrm{fr}}_{n/\mathbb{R}^n} \to \mathrm{Disk}^{\mathrm{fr}}_{n/-}.$$

As explained in Example 0.2.12, this functor is an equivalence. In particular, we have a canonical composite equivalence

$$\mathrm{HH}^{(2)}(\text{Yoneda}) \simeq \int_{\mathbb{T}^2} \text{Yoneda} \simeq \mathrm{Hom}_{\mathrm{Mfld}_2^{\mathrm{fr}}}(\iota, \mathbb{T}^2) \quad \text{in } \mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}}), \tag{2-4-3}$$

given by Corollary 2.3.2 and (2-4-2), respectively. Also, the symmetric monoidal functor

$$\mathrm{Hom}_{\mathrm{Mfld}_2^{\mathrm{fr}}}(\iota, \mathbb{T} \times -) \colon \mathrm{Disk}_1^{\mathrm{fr}} \xrightarrow{\mathbb{T}\times -} \mathrm{Mfld}_2^{\mathrm{fr}} \xrightarrow{\text{restricted Yoneda}} \mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}})$$

is the Hochschild homology of the 2–algebra in $\mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}})$ underlying the $\mathcal{E}_2$–algebra ι, as it is equipped with its residual associative algebra structure:

$$\mathrm{HH}(\text{Yoneda}) \simeq \int_{\mathbb{T}\times\mathbb{R}^{\sqcup\bullet}} \text{Yoneda} \simeq \mathrm{Hom}_{\mathrm{Mfld}_2^{\mathrm{fr}}}(\iota, \mathbb{T} \times \mathbb{R}^{\sqcup\bullet}) \quad \text{in } \mathrm{Alg}_{\mathrm{Assoc}}(\mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}})). \tag{2-4-4}$$

Now, taking mapping tori defines a map between pointed spaces $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}) \to B\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)$. Based loops of this map is the morphism between continuous groups

$$\langle \tau_1 \rangle : \mathbb{Z} \xrightarrow{\simeq} \Omega\mathbb{T} \xrightarrow{\simeq} \Omega\,\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}) \xrightarrow{\Omega(\text{mapping torus})} \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2).$$

By construction of the morphism (0-2-4), this fills the diagram among continuous groups

$$\begin{tikzcd}
\mathbb{Z} \arrow[rrr] \arrow[d, "\simeq"'] & & & \mathbb{Z} \arrow[dl] \arrow[ddl, bend left, "\mathrm{Sheer}_1"] \\
\Omega\,\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}) \arrow[r] \arrow[d, dashed, "\Omega(\text{mapping torus})"] & \Omega\,\mathrm{Aut}_{\mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}})}(\mathrm{Hom}_{\mathrm{Mfld}_2^{\mathrm{fr}}}(\iota, \mathbb{T}\times\mathbb{R})) \arrow[r, "\simeq", "\text{(2-4-4)}"'] & \Omega\,\mathrm{Aut}_{\mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}})}(\mathrm{HH}(\iota)) \arrow[d, "\text{(0-2-4)}"'] & \\
\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \arrow[r] & \mathrm{Aut}_{\mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}})}(\mathrm{Hom}_{\mathrm{Mfld}_2^{\mathrm{fr}}}(\iota, \mathbb{T}^2)) \arrow[r, "\simeq", "\text{(2-4-3)}"'] & \mathrm{Aut}_{\mathrm{PShv}(\mathrm{Disk}_2^{\mathrm{fr}})}(\mathrm{HH}^{(2)}(\iota)) &
\end{tikzcd}$$

Commutativity of the outer diagram is the sought identification (2-4-1) in the universal case. □

2.5 Proof of Theorem Y.1 and Corollaries 0.2.5 and 0.2.10

We first explain the following diagram among ∞–categories:

(2-5-1)
$$\begin{tikzcd}
\mathrm{Alg}_2(\mathcal{V}) \arrow[d, "\mathbb{Z} \underset{\mathrm{Sheer}_1}{\curvearrowright} \mathrm{HH}^{(2)}"'] & & \mathrm{Alg}_{\mathcal{E}_2}(\mathcal{V}) \arrow[ll, "\mathrm{fgt}_1", "\simeq"'] \arrow[rr, "\mathrm{fgt}_2", "\simeq"'] \arrow[dl, "\int_{\mathrm{pr}_1}"'] \arrow[d, "\int_{\mathbb{T}^2}"] \arrow[dr, "\int_{\mathrm{pr}_2}"] & & \mathrm{Alg}_2(\mathcal{V}) \arrow[d, "\mathbb{Z} \underset{\mathrm{Sheer}_2^{-1}}{\curvearrowright} \mathrm{HH}^{(2)}"] \\
\mathrm{Mod}_{\mathbb{Z}}(\mathcal{V}) \arrow[d, "\mathrm{f}"'] & \mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(\mathrm{pr}_1)}(\mathcal{V}) \arrow[l, "\mathrm{f}"'] \arrow[dr, "\mathrm{f}"'] & \mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)}(\mathcal{V}) \arrow[l, "\mathrm{f}"'] \arrow[r, "\mathrm{f}"] \arrow[d, "\mathrm{f}"] & \mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(\mathrm{pr}_2)}(\mathcal{V}) \arrow[r, "\mathrm{f}"] \arrow[dl, "\mathrm{f}"] & \mathrm{Mod}_{\mathbb{Z}}(\mathcal{V}) \arrow[d, "\mathrm{f}"] \\
\mathcal{V} & & \mathrm{Mod}_{\mathbb{T}^2}(\mathcal{V}) \arrow[ll, "\mathrm{f}"'] \arrow[rr, "\mathrm{f}"] & & \mathcal{V}
\end{tikzcd}$$

- The functors labeled "f" are restriction along the canonically commutative diagram among continuous groups

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\langle\tau_1\rangle"] & \mathrm{Diff}^{\mathrm{fr}}(\mathrm{pr}_1) \arrow[r] & \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) & \mathrm{Diff}^{\mathrm{fr}}(\mathrm{pr}_2) \arrow[l] & \mathbb{Z} \arrow[l, "\langle\tau_2\rangle"'] \\
\ast \arrow[u] & & \mathbb{T}^2 \arrow[ll] \arrow[rr] \arrow[ul] \arrow[u] \arrow[ur] & & \ast \arrow[u]
\end{tikzcd}$$

in which, for each $i = 1, 2$, the morphism $\langle\tau_i\rangle$ is the composite $\mathbb{Z} \hookrightarrow \mathbb{T}^2 \rtimes_{U_i} \mathbb{Z} \xrightarrow{\mathrm{Aff}_i} \mathrm{Diff}^{\mathrm{fr}}(\mathrm{pr}_i)$. In particular, each of the lower triangles canonically commutes.

- The functor $\int_{\mathbb{T}^2}$ is (2-3-2).

- For $i = 1, 2$, the functor $\int_{\mathrm{pr}_i}$ is factorization homology over the circle $\mathbb{T}$ of the *pushforward* along the projection $\mathbb{T}^2 \xrightarrow{\mathrm{pr}_i} \mathbb{T}$ off of the i^{th} coordinate, as it is endowed with its canonical $\mathrm{Diff}^{\mathrm{fr}}(\mathrm{pr}_i)$–action. The *pushforward formula* $\int_{\mathrm{pr}_i} \simeq \int_{\mathbb{T}}\int_{\mathbb{T}}$ [Ayala and Francis 2015, Proposition 3.23], which is manifestly $\mathrm{Diff}^{\mathrm{fr}}(\mathrm{pr}_i)$–equivariant, supplies commutativity of the upper triangles.

- The functor $\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{V}) \xrightarrow{\mathrm{fgt}_1} \mathrm{Alg}_2(\mathcal{V})$ is restriction along the standard morphism between operads $\mathrm{Assoc} \otimes \mathrm{Assoc} \xrightarrow{\text{standard}} \mathcal{E}_2$. The functor $\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{V}) \xrightarrow{\mathrm{fgt}_2} \mathrm{Alg}_2(\mathcal{V})$ is restriction along the morphism between operads $\mathrm{Assoc} \otimes \mathrm{Assoc} \xrightarrow{\text{swap}} \mathrm{Assoc} \otimes \mathrm{Assoc} \xrightarrow{\text{standard}} \mathcal{E}_2$. Theorem 2.2.6 implies that all of these functors are equivalences.

- For $i = 1, 2$, the outer vertical functors are $\mathrm{HH}^{(2)}$, as it is endowed with its canonical action $\mathbb{Z} \underset{\mathrm{Sheer}_i}{\curvearrowright} \mathrm{HH}^{(2)}(A)$ of (0-2-5) and (0-2-7) from Section 0.2, which is evidently functorial in $A \in \mathrm{Alg}_2(\mathcal{V})$.

- Commutativity of the upper tilted squares is Proposition 2.4.1.

In particular, for each 2–algebra $A \in \mathrm{Alg}_2(\mathcal{V})$, there is a canonical action $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \curvearrowright \mathrm{HH}^{(2)}(A)$. Through Theorem X(2)(a), this is an action $\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \curvearrowright \mathrm{HH}^{(2)}(A)$, which establishes the statement of Theorem Y.1.

After Theorem Y.1, the standard presentation (0-1-1) of the braid group Braid_3 immediately implies Corollary 0.2.5(1). Via the identification

$$\mathbb{T}^2 \rtimes_{U_i} \mathbb{Z} \xrightarrow{\simeq} \mathrm{Diff}^{\mathrm{fr}}(\mathrm{pr}_i)$$

of Corollary 1.5.5, commutativity of the outer squares in (2-5-1) directly implies Corollary 0.2.5(2)(3).

Next, consider the $(\mathrm{O}(2)\simeq\mathrm{GL}_2(\mathbb{R}))$–action on $\mathrm{Mfld}_2^{\mathrm{fr}}$ given by change-of-framing. Observe that this action restricts to one along the full ∞–subcategory $\mathrm{Disk}_2^{\mathrm{fr}} \subset \mathrm{Mfld}_2^{\mathrm{fr}}$. This implies the left adjoint $\int$ is $\mathrm{O}(2)$–equivariant. Therefore, for each $A \in \mathrm{Alg}_2(\mathcal{V})$ and each $(\Sigma, \varphi) \in \mathrm{Mfld}_2^{\mathrm{fr}}$, taking $\mathrm{O}(2)$–orbits of both A and (Σ, φ) defines a canonically commuting diagram among ∞–categories

$$\begin{tikzcd} \mathrm{O}(2) \arrow[r, "\mathrm{Orbit}_A"] \arrow[d, "\mathrm{Orbit}_{(\Sigma,\varphi)}"'] & \mathrm{Alg}_2(\mathcal{V}) \arrow[d, "\int_{(\Sigma,\varphi)}"] \\ \mathrm{Mfld}_2^{\mathrm{fr}} \arrow[r, "\int A"] & \mathcal{V} \end{tikzcd}$$

Through Observation 1.3.10, restricting along $B\mathbb{Z} \simeq B\Omega_{\mathbb{1}}\mathrm{O}(2) \to \mathrm{O}(2)$ gives the commutative diagram asserted in Corollary 0.2.10.

2.6 Proof of Theorem Y.2

After Corollary 0.3.3, to prove Theorem Y.2 we are left to extend the action

$$\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}} \overset{(-)^{-1}}{\simeq} \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2) \xleftarrow[\simeq]{\mathrm{Aff}^{\mathrm{fr}}} \mathbb{T}^2 \rtimes \mathrm{Braid}_3 \curvearrowright \mathrm{HH}^{(2)}(A)$$

to an action $\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}} \curvearrowright \mathrm{HH}^{(2)}(A)$. We do this by extending factorization homology via the developments of [Ayala et al. 2018]. Namely, recall from [loc. cit.] the ∞–category $\mathcal{M}\mathrm{fd}_2^{\mathrm{sfr}}$ of *solidly 2–framed stratified spaces*. Consider the full ∞–subcategory $\mathcal{M}_{=2}^{\mathrm{sfr}} \subset \mathcal{M}\mathrm{fd}_2^{\mathrm{sfr}}$ consisting of those solidly 2–framed stratified spaces each of whose strata is 2–dimensional.

Observation 2.6.1 Inspection of the definition of $\mathcal{M}\mathrm{fd}_2^{\mathrm{sfr}}$ reveals the following.

(1) The moduli space of objects

$$\mathrm{Obj}(\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}) \simeq \coprod_{[\Sigma,\varphi]} \mathrm{BDiff}^{\mathrm{fr}}(\Sigma,\varphi)$$

is that of a framed 2–manifold. That is, there is a canonical bijection between framed-diffeomorphism-types of framed 2–manifolds and equivalence-classes of objects in $\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}$, and for (Σ,φ) a framed 2–manifold, there is a canonical identification between continuous groups:

$$\mathrm{Diff}^{\mathrm{fr}}(\Sigma,\varphi) \simeq \mathrm{Aut}_{\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}}(\Sigma,\varphi).$$

(2) Let (Σ,φ) and (Σ',φ') be framed 2–manifolds. The space of morphisms from (Σ,φ) to (Σ',φ') in $\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}$,

$$\mathrm{Hom}_{\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}}((\Sigma,\varphi),(\Sigma',\varphi')) \simeq \coprod_{[\widetilde{\Sigma}\xrightarrow{\pi}\Sigma]} \mathrm{Emb}^{\mathrm{fr}}((\widetilde{\Sigma},\pi^*\varphi),(\Sigma',\varphi'))_{/\mathrm{Diff}_{/\Sigma}(\widetilde{\Sigma})},$$

is the moduli space of finite-sheeted covers over Σ together with a framed-embedding from its total space to (Σ',φ').

(3) Composition in $\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}$ is given by base change of framed embeddings along finite-sheeted covers, followed by composition of framed-embeddings:

$$\mathrm{Hom}_{\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}}((\Sigma,\varphi),(\Sigma',\varphi'))\times\mathrm{Hom}_{\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}}((\Sigma',\varphi'),(\Sigma'',\varphi''))\xrightarrow{\circ}\mathrm{Hom}_{\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}}((\Sigma,\varphi),(\Sigma'',\varphi'')),$$
$$\big((\Sigma,\varphi)\xleftarrow{\pi}(\widetilde{\Sigma},\pi^*\varphi)\xrightarrow{f}(\Sigma',\varphi'),(\Sigma',\varphi')\xleftarrow{\pi'}(\widetilde{\Sigma}',\pi'^*\varphi')\xrightarrow{g}(\Sigma'',\varphi'')\big)$$
$$\mapsto\big((\Sigma,\varphi)\xleftarrow{\pi\circ\mathrm{pr}_1}(\widetilde{\Sigma}\times_{\Sigma'}\widetilde{\Sigma}',(\mathrm{pr}_1\circ\pi)^*\varphi)\xrightarrow{g\circ\mathrm{pr}_2}(\Sigma'',\varphi'')\big).$$

(4) Evidently, framed embeddings form the left factor in a factorization system on $\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}$, whose right factor is (the opposite of) framed finite-sheeted covers.

(5) Finite products exist in $\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}$, and are implemented by disjoint unions of framed 2–manifolds.

(6) For each framing φ of the 2–torus $\mathbb{T}^2$, there is a canonical identification between continuous monoids:

$$\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2,\varphi)^{\mathrm{op}} \simeq \mathrm{End}_{\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}}(\mathbb{T}^2,\varphi).$$

Define the full ∞–subcategory

$$\iota\colon \boldsymbol{\mathcal{D}}^{\mathrm{sfr}}_{=2}\subset\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2},$$

consisting of those framed 2–manifolds that are equivalent with a finite disjoint union of framed Euclidean spaces. Regard both $\boldsymbol{\mathcal{D}}^{\mathrm{sfr}}_{=2}$ and $\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}$ as symmetric monoidal ∞–categories, via their cartesian monoidal structures.[13] Notice the evident monomorphisms of symmetric monoidal ∞–categories

$$\rho\colon \mathrm{Disk}_2^{\mathrm{fr}}\hookrightarrow\boldsymbol{\mathcal{D}}^{\mathrm{sfr}}_{=2}\quad\text{and}\quad\rho\colon\mathrm{Mfld}_2^{\mathrm{fr}}\hookrightarrow\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2},$$

[13] Indeed, notice that the full ∞–subcategory $\boldsymbol{\mathcal{D}}^{\mathrm{sfr}}_{=2}\subset\boldsymbol{\mathcal{M}}^{\mathrm{sfr}}_{=2}$ is closed under finite products.

each of whose images consists of all objects, yet only those morphisms $((\Sigma,\varphi)\xleftarrow{\pi}(\widetilde{\Sigma},\pi^*\varphi)\xrightarrow{f}(\Sigma',\varphi'))$ in which π is a diffeomorphism.[14]

Let $\mathcal{X}$ be a presentable ∞–category in which products distribute over colimits. Consider the full ∞–subcategory

$$\mathrm{Fun}^{\times}(\mathfrak{D}_2^{\mathrm{sfr}},\mathcal{X})\subset\mathrm{Fun}(\mathfrak{D}_2^{\mathrm{sfr}},\mathcal{X})$$

consisting of those functors that preserve finite products.

Proposition 2.6.2 [Ayala et al. 2017a] *Let $\mathcal{X}$ be a presentable ∞–category in which products distribute over colimits. Restriction along ρ defines an equivalence between ∞–categories*

$$\rho^*\colon\mathrm{Fun}^{\times}(\mathfrak{D}_2^{\mathrm{sfr}},\mathcal{X})\xrightarrow{\simeq}\mathrm{Fun}^{\otimes}(\mathrm{Disk}_2^{\mathrm{fr}},\mathcal{X})\simeq\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{X}).$$

The inverse of restriction along ρ followed by left Kan extension along ι defines a composite functor

$$\widetilde{\int}\colon\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{X})\simeq\mathrm{Fun}^{\otimes}(\mathrm{Disk}_2^{\mathrm{fr}},\mathcal{X})\xrightarrow{(\rho^*)^{-1}}\mathrm{Fun}^{\times}(\mathfrak{D}_{=2}^{\mathrm{sfr}},\mathcal{X})\xrightarrow{\iota_!}\mathrm{Fun}^{\times}(\mathfrak{M}_{=2}^{\mathrm{sfr}},\mathcal{X}).$$

Proposition 2.6.3 *Let $\mathcal{X}$ be a presentable ∞–category in which products distribute over colimits. The following diagram among ∞–categories canonically commutes*:

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{X}) \arrow[r, "\widetilde{\int}"] \arrow[d, "\int"'] & \mathrm{Fun}^{\times}(\mathfrak{M}_{=2}^{\mathrm{sfr}},\mathcal{X}) \arrow[r, "\text{restriction}"] & \mathrm{Fun}(\mathrm{BAut}_{\mathfrak{M}_{=2}^{\mathrm{sfr}}}(\mathbb{T}^2,\varphi_0),\mathcal{X}) \arrow[d, "\simeq"', "\text{Observation 2.6.1(1)}"] \\
\mathrm{Fun}^{\otimes}(\mathrm{Mfld}_2^{\mathrm{fr}},\mathcal{X}) \arrow[r, "\text{restriction}"] & \mathrm{Fun}(\mathrm{BAut}_{\mathrm{Mfld}_2^{\mathrm{fr}}}(\mathbb{T}^2,\varphi_0),\mathcal{X}) \arrow[r, "\simeq"] & \mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2,\varphi_0)}(\mathcal{X})
\end{tikzcd}$$

Proof Let $A\in\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{X})\simeq\mathrm{Fun}^{\otimes}(\mathrm{Disk}_2^{\mathrm{fr}},\mathcal{X})$. Using Proposition 2.6.2, the monomorphism ρ determines a canonical morphism between colimits in $\mathcal{X}$:

(2-6-1) $$\int_{\mathbb{T}^2}A\simeq\mathrm{colim}(\mathrm{Disk}^{\mathrm{fr}}_{2/(\mathbb{T}^2,\varphi_0)}:=\mathrm{Disk}_2^{\mathrm{fr}}\times_{\mathrm{Mfld}_2^{\mathrm{fr}}}\mathrm{Mfld}^{\mathrm{fr}}_{2/(\mathbb{T}^2,\varphi_0)}\xrightarrow{\mathrm{pr}}\mathrm{Disk}_2^{\mathrm{fr}}\xrightarrow{A}\mathcal{X})$$
$$\xrightarrow{\rho}\mathrm{colim}(\mathfrak{D}^{\mathrm{sfr}}_{=2/(\mathbb{T}^2,\varphi_0)}:=\mathfrak{D}^{\mathrm{sfr}}_{=2}\times_{\mathfrak{M}^{\mathrm{sfr}}_{=2}}\mathfrak{M}^{\mathrm{sfr}}_{=2/(\mathbb{T}^2,\varphi_0)}\xrightarrow{\mathrm{pr}}\mathfrak{D}^{\mathrm{sfr}}_{=2}\xrightarrow{\rho^{*-1}(A)}\mathcal{X})\simeq\widetilde{\int}_{\mathbb{T}^2}A.$$

This morphism is manifestly $\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)$–equivariant and functorial in $A\in\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{X})$ as so. So the proposition is proved upon showing this morphism (2-6-1) is an equivalence. The morphism (2-6-1) is an equivalence provided the canonical functor

(2-6-2) $$\mathrm{Disk}^{\mathrm{fr}}_{2/(\mathbb{T}^2,\varphi_0)}\to\mathfrak{D}^{\mathrm{sfr}}_{=2/(\mathbb{T}^2,\varphi_0)}$$

is final. But the factorization system of Observation 2.6.1(4) reveals that this functor (2-6-2) is a right adjoint. Its left adjoint is given by projecting to the right factor of the factorization system:

$$\mathfrak{D}^{\mathrm{sfr}}_{=2/(\mathbb{T}^2,\varphi_0)}\to\mathrm{Disk}^{\mathrm{fr}}_{2/(\mathbb{T}^2,\varphi_0)},\quad(D\xleftarrow{\pi}\widetilde{D}\xrightarrow{f}(\mathbb{T}^2,\varphi_0))\mapsto(\widetilde{D}\xrightarrow{f}(\mathbb{T}^2,\varphi_0)).$$

The sought finality of the functor (2-6-2) follows. □

[14] In other words, ρ is the inclusion of the left factor in the factorization system of Observation 2.6.1(4).

Proposition 2.6.3, together with Observation 2.6.1(6), immediately supplies a filler in the commutative diagram among ∞–categories

$$\begin{tikzcd}
\mathrm{Fun}(\mathfrak{B}\,\mathrm{End}_{\mathcal{M}^{\mathrm{sfr}}_{=2}}(\mathbb{T}^2,\varphi_0),\mathcal{X}) \arrow[r, "\simeq", "\text{Observation 2.6.1(6)}"'] & \mathrm{Mod}_{\mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}}}(\mathcal{X}) \arrow[r, "\simeq", "\text{Corollary 0.3.3}"'] \arrow[d, "\mathrm{forget}"] & \mathrm{Mod}_{(\mathbb{T}^2\rtimes\widetilde{\mathrm{E}}_2^+(\mathbb{Z}))^{\mathrm{op}}}(\mathcal{X}) \arrow[d, "\mathrm{forget}"] \\
\mathrm{Alg}_{\mathcal{E}_2}(\mathcal{X}) \arrow[u, dashed, "\langle \mathrm{Imm}^{\mathrm{fr}}(\mathbb{T}^2)^{\mathrm{op}}\curvearrowright\widetilde{\int}_{\mathbb{T}^2}\rangle"] \arrow[r, "\langle \mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)\curvearrowright\int_{\mathbb{T}^2}\rangle", "\text{(2-3-1)}"'] & \mathrm{Mod}_{\mathrm{Diff}^{\mathrm{fr}}(\mathbb{T}^2)}(\mathcal{X}) \arrow[r, "\simeq", "\text{Theorem X(2)(a)}"'] & \mathrm{Mod}_{\mathbb{T}^2\rtimes\mathrm{Braid}_3}(\mathcal{X})
\end{tikzcd}$$

Theorem Y.2 follows from this commutative diagram, after the commutative diagram (2-5-1).

Appendix A Some facts about continuous monoids

We record some simple formal results concerning continuous monoids.

Lemma A.0.1 *Let $G \curvearrowright X$ be an action of a continuous group on a space. Let $* \xrightarrow{\langle x\rangle} X$ be a point in this space. Consider the stabilizer of x, which is the fiber of the orbit map of x:*

(A-0-1)
$$\begin{tikzcd}
 & \mathrm{Stab}_G(x) \arrow[rr] \arrow[d] & & * \arrow[d, "\langle x\rangle"] \\
 & G\simeq G\times * \arrow[r, "\mathrm{id}\times\langle x\rangle"] \arrow[rr, bend left, "\mathrm{Orbit}_x"] & G\times X \arrow[r, "\mathrm{act}"] & X
\end{tikzcd}$$

There is a canonical identification in $\mathcal{S}$paces between this stabilizer and the based loops at

$$[x]\colon * \xrightarrow{\langle x\rangle} X \xrightarrow{\text{quotient}} X_{/G}$$

of the G–coinvariants,

$$\mathrm{Stab}_G(x)\simeq \Omega_{[x]}(X_{/G}),$$

through which the resulting composite morphism $\Omega_{[x]}(X_{/G})\simeq \mathrm{Stab}_G(x)\to G$ canonically lifts to one between continuous groups.

Proof By definition of a G–action, the orbit map $G \xrightarrow{\mathrm{Orbit}_x} X$ is canonically G–equivariant. Taking G–coinvariants supplies an extension of the commutative diagram (A-0-1) in $\mathcal{S}$paces:

$$\begin{tikzcd}
\mathrm{Stab}_G(x) \arrow[r] \arrow[d] & G \arrow[r, "\text{quotient}"] \arrow[d, "\mathrm{Orbit}_x"] & G_{/G}\simeq * \arrow[d, "(\mathrm{Orbit}_x)_{/G}"] \\
* \arrow[r, "\langle x\rangle"] & X \arrow[r, "\text{quotient}"] & X_{/G}
\end{tikzcd}$$

Through the identification $G_{/G}\simeq *$, the right vertical map is identified as $* \xrightarrow{\langle [x]\rangle} X_{/G}$. Using that groupoids in $\mathcal{S}$paces are effective, the right square is a pullback. Because the left square is defined as a pullback, it follows that the outer square is a pullback. The identification $\mathrm{Stab}_G(x)\simeq\Omega_{[x]}(X_{/G})$ follows. In particular, the space $\mathrm{Stab}_G(x)$ has the canonical structure of a continuous group.

Now, this continuous group $\mathrm{Stab}_G(x)$ is evidently functorial in the argument $G\curvearrowright X\ni x$. In particular, the unique G–equivariant morphism $X \xrightarrow{!} *$ determines a morphism between continuous groups:

$$\mathrm{Stab}_x(X)\to \mathrm{Stab}_*(*)\simeq G. \qquad \square$$

Lemma A.0.2 *Let $H \to G$ be a morphism between continuous groups. Let $H \curvearrowright X$ be an action on a space. There is a canonical map between spaces over $G_{/H}$,*

$$X_{/\Omega(G_{/H})} \to (X \times G)_{/H},$$

from the coinvariants with respect to the action $\Omega(G_{/H}) \xrightarrow{\Omega\text{–Puppe}} H \curvearrowright X$. Furthermore, if the induced map $\pi_0(H) \to \pi_0(G)$ between sets of path-components is surjective, then this map is an equivalence.

Proof The construction of the Ω–Puppe sequence is such that the morphism $\Omega(G_{/H}) \to H$ witnesses the stabilizer of $* \xrightarrow{\text{unit}} G$ with respect to the action $H \to G \underset{\text{left trans}}{\curvearrowright} G$:

$$\begin{tikzcd}
\Omega(G_{/H}) \arrow[r] \arrow[d] & H \arrow[d] \\
* \arrow[r, "\text{unit}"] & G
\end{tikzcd}$$

In particular, there is a canonical $\Omega(G_{/H})$–equivariant map

$$X \simeq X \times * \xrightarrow{\text{id}\times\text{unit}} X \times G.$$

Taking coinvariants lends to a canonically commutative diagram among spaces:

$$\begin{tikzcd}
X_{\Omega(G_{/H})} \arrow[r] \arrow[d] & (X \times G)_{/H} \arrow[r] \arrow[d] & X_{/H} \arrow[d] \\
B\Omega(G_{/H}) \arrow[r] & G_{/H} \arrow[r] & BH
\end{tikzcd} \tag{A-0-2}$$

This proves the first assertion.

We now prove the second assertion. Because groupoid-objects are effective in the ∞–category $\mathcal{S}$paces, the H–coinvariants functor

$$\text{Fun}(BH, \mathcal{S}\text{paces}) \to \mathcal{S}\text{paces}_{/BH} \quad \text{given by } (H \curvearrowright X) \mapsto (X_{/H} \to BH)$$

is an equivalence between ∞–categories. In particular, it preserves products. It follows that the right square in (A-0-2) witnesses a pullback. By definition of coinvariants of the restricted action $\Omega(G_{/H}) \to H \curvearrowright X$, the outer square is a pullback. The connectivity assumption on the morphism $H \to G$ implies the left bottom horizontal map is an equivalence. So the left top horizontal map is also an equivalence, as desired. □

Let $\mathfrak{B}N \xrightarrow{\langle N \curvearrowright M\rangle}$ Monoids be an action of a continuous monoid on a continuous monoid. This action can be codified as unstraightening of the composite functor $\mathfrak{B}N \to \text{Monoids} \xrightarrow{\mathfrak{B}} \text{Cat}^{*/}_{(\infty,1)}$. We denote[15] this unstraightening by

$$(\mathfrak{B}M)_{/^{\text{l.lax}}N} \to \mathfrak{B}N.$$

It is a cocartesian fibration equipped with a section. Because the $(\infty, 1)$–category $\mathfrak{B}N$ is equipped with a functor $* \to \mathfrak{B}N$, the given section supplies the $(\infty, 1)$–category $(\mathfrak{B}M)_{/^{\text{l.lax}}N}$ with a distinguished

[15]The notation here is intended to evoke a *left-lax quotient*. Indeed, for $\mathcal{K} \xrightarrow{F} \text{Cat}_{(\infty,1)}$ a functor from an ∞–category, its *left-lax colimit* is the $(\infty, 1)$–category defined as the domain of the unstraightening of F: $(\text{colim}^{\text{l.lax}}(F) \xrightarrow{\text{colim}^{\text{l.lax}}(!)} \text{colim}^{\text{l.lax}}(*)) := (\text{Un}(F) \to \mathcal{K})$. See [Ayala et al. 2019, Appendix A] for a treatment of lax $(\infty, 1)$–category theory.

point, and so we regard $(\mathfrak{B}M)_{/\mathrm{l.lax}N}$ as a pointed $(\infty,1)$–category. The *semidirect product* (*of* N *by* M) is the continuous monoid

$$M \rtimes N := \mathrm{End}_{(\mathfrak{B}M)_{/\mathrm{l.lax}N}}(*),$$

which is endomorphisms of the point.

Remark The underlying space of this continuous monoid is canonically identified as $M \times N$; the 2–ary monoidal structure $\mu_{M\rtimes N}$ is canonically identified as the composite map between spaces

$$\mu_{M\rtimes N}\colon (M\times N)\times(M\times N) = M\times(N\times M)\times M \xrightarrow{\mathrm{id}_M\times\mathrm{swap}\times\mathrm{id}_N} M\times(M\times N)\times N$$
$$\xrightarrow{\mathrm{id}_M\times(\mathrm{proj_M},\mathrm{action})\times\mathrm{id_N}} M\times(M\times N)\times N = (M\times M)\times(N\times N) \xrightarrow{\mu_M\times\mu_N} M\times N.$$

Note the canonical morphism between monoids $M \rtimes N \to N$ whose kernel is M.

Dually, let $\mathfrak{B}N^{\mathrm{op}} \xrightarrow{\langle M \curvearrowleft N\rangle}$ Monoids be a *right* action. Consider the unstraightening of the composite functor $\mathfrak{B}N^{\mathrm{op}} \to \mathrm{Monoids} \xrightarrow{\mathfrak{B}} \mathrm{Cat}^{*/}_{(\infty,1)}$ as a pointed cartesian fibration $(\mathfrak{B}M)_{/\mathrm{r.lax}N^{\mathrm{op}}} \to \mathfrak{B}N$. The *semidirect product* (*of* N *by* M) is the continuous monoid

$$N \ltimes M := \mathrm{End}_{(\mathfrak{B}M)_{/\mathrm{r.lax}N^{\mathrm{op}}}}(*),$$

which is endomorphisms of the point. Note the canonical morphism between monoids $M \rtimes N \to N$ whose kernel is M.

Observation A.0.3 Let $N \curvearrowright M$ be an action of a continuous monoid on a continuous monoid. There is a canonical identification between continuous monoids under M^{op} and over N^{op}:

$$(M \rtimes N)^{\mathrm{op}} \simeq (N^{\mathrm{op}} \ltimes M^{\mathrm{op}}).$$

The next result is a characterization of semidirect products.

Lemma A.0.4 *Let* $A \underset{r}{\overset{i}{\leftrightarrows}} N$ *be a retraction between continuous monoids* (*so* $r \circ i \simeq \mathrm{id}_N$).

- *If the canonical map between spaces*

$$\mathrm{Ker}(r) \times N \xrightarrow{\mathrm{inclusion}\times i} A \times A \xrightarrow{\mu_A} A \tag{A-0-3}$$

  is an equivalence,[16] *then there is a canonical action*[17] $N \underset{\lambda}{\curvearrowright} \mathrm{Ker}(r)$ *for which there is a canonical equivalence between monoids*

$$\mathrm{Ker}(r) \rtimes_\lambda N \simeq A.$$

- *If the canonical map between spaces*

$$N \times \mathrm{Ker}(r) \xrightarrow{\sigma\times\mathrm{inclusion}} A \times A \xrightarrow{\mu_A} A$$

  is an equivalence,[18] *then there is a canonical action* $\mathrm{Ker}(r) \underset{\rho}{\curvearrowleft} N$ *for which there is a canonical equivalence between monoids*

$$\mathrm{Ker}(r) \rtimes_\rho N \simeq A.$$

[16] Note that this condition is always satisfied if N is a continuous group.

[17] The action map associated to λ can be written as $N \times \mathrm{Ker}(r) \xrightarrow{i\times\mathrm{inclusion}} A \times A \xrightarrow{\mu_A} A \underset{\simeq}{\xleftarrow{\text{(A-0-3)}}} \mathrm{Ker}(r) \times N \xrightarrow{\mathrm{proj}} \mathrm{Ker}(r)$.

[18] Note that this condition is always satisfied if N is a continuous group.

Proof By way of Observation A.0.3, the two assertions imply one another by taking cartesian/cocartesian duals of cocartesian/cartesian fibrations. So we are reduced to proving the first assertion.

Consider the retraction $\mathfrak{B}A \underset{\mathfrak{B}r}{\overset{\mathfrak{B}i}{\leftrightarrows}} \mathfrak{B}N$ among pointed ∞–categories. Note that $\mathfrak{B}i$ is essentially surjective, and that $\mathrm{Ker}(r)$ is the fiber of $\mathfrak{B}r$ over $* \to \mathfrak{B}N$.

Let $c_1 \xrightarrow{\langle n\rangle} \mathfrak{B}N$ be a morphism. Consider the commutative diagram among ∞–categories

$$\begin{tikzcd} c_0 \arrow[r, "\langle * \rangle"] \arrow[d, "s"'] & \mathfrak{B}A \arrow[d, "\mathfrak{B}r"] \\ c_1 \arrow[r, "\langle n\rangle"'] \arrow[ur, dashed, "\langle i(n)\rangle"] & \mathfrak{B}N \end{tikzcd}$$

The assumption on the retraction implies the diagonal filler is initial among all such fillers. This is to say that the morphism $i(n)$ in $\mathfrak{B}A$ is cocartesian over $\mathfrak{B}r$. Because $\mathfrak{B}i$ is essentially surjective, this shows that $\mathfrak{B}r$ is a cocartesian fibration. The result now follows from the definition of the semidirect product $\mathrm{Ker}(r) \rtimes_\lambda N$. □

Proposition A.0.5 *Let $\mathcal{X}$ be an ∞–category. Let $\mathfrak{B}N \xrightarrow{\langle N \curvearrowright M\rangle} \mathrm{Monoids}$ be an action of a continuous monoid N on a continuous monoid M. Consider the precomposition action*

$$\mathfrak{B}N^{\mathrm{op}} \xrightarrow{\langle N \curvearrowright M\rangle^{\mathrm{op}}} \mathrm{Monoids}^{\mathrm{op}} \xrightarrow{\mathrm{Mod}_-(\mathcal{X})} \mathrm{Cat}_{(\infty,1)}.$$

There is a canonical identification over $\mathrm{Mod}_{M^{\mathrm{op}}}(\mathcal{X})$ from the ∞–category of $(M \rtimes N)^{\mathrm{op}}$–modules in $\mathcal{X}$ to that of M^{op}–modules in $\mathcal{X}$ with the structure of being left-laxly invariant with respect to this precomposition N^{op}–action:

$$\mathrm{Mod}_{(M\rtimes N)^{\mathrm{op}}}(\mathcal{X}) \simeq \mathrm{Mod}_{M^{\mathrm{op}}}(\mathcal{X})^{\mathrm{l.lax}\, N^{\mathrm{op}}}.$$

In particular, there is a canonical fully faithful functor from the (strict) N–invariants,

$$\mathrm{Mod}_{M^{\mathrm{op}}}(\mathcal{X})^N \hookrightarrow \mathrm{Mod}_{(M\rtimes N)^{\mathrm{op}}}(\mathcal{X}),$$

which is an equivalence if the continuous monoid N is a continuous group.

Proof The second assertion follows immediately from the first, which is proved upon justifying the following sequence of equivalences among ∞–categories, each of which is evidently over $\mathrm{Mod}_M(\mathcal{X})$:

$$\begin{aligned} \mathrm{Mod}_{(M\rtimes N)^{\mathrm{op}}}(\mathcal{X}) &\overset{(a)}{\simeq} \mathrm{Fun}(\mathfrak{B}(M\rtimes N)^{\mathrm{op}}, \mathcal{X}) \overset{(b)}{\simeq} \mathrm{Fun}(\mathfrak{B}(N^{\mathrm{op}} \ltimes M^{\mathrm{op}}), \mathcal{X}) \\ &\overset{(c)}{\simeq} \mathrm{Fun}_{/\mathfrak{B}N^{\mathrm{op}}}(\mathfrak{B}N^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{rel}}_{\mathfrak{B}N^{\mathrm{op}}}(\mathfrak{B}(N^{\mathrm{op}} \ltimes M^{\mathrm{op}}), \mathcal{X}\times\mathfrak{B}N^{\mathrm{op}})) \\ &\overset{(d)}{\simeq} \mathrm{Fun}_{/\mathfrak{B}N^{\mathrm{op}}}(\mathfrak{B}N^{\mathrm{op}}, \mathrm{Fun}^{\mathrm{rel}}_{\mathfrak{B}N^{\mathrm{op}}}((\mathfrak{B}M^{\mathrm{op}})_{/\mathrm{r.lax}\, N}, \mathcal{X}\times\mathfrak{B}N^{\mathrm{op}})) \\ &\overset{(e)}{\simeq} \mathrm{Fun}_{/\mathfrak{B}N^{\mathrm{op}}}(\mathfrak{B}N^{\mathrm{op}}, \mathrm{Fun}(\mathfrak{B}M^{\mathrm{op}}, \mathcal{X})_{/\mathrm{l.lax}\, N^{\mathrm{op}}}) \\ &\overset{(f)}{\simeq} \mathrm{Fun}_{/\mathfrak{B}N^{\mathrm{op}}}(\mathfrak{B}N^{\mathrm{op}}, \mathrm{Mod}_{M^{\mathrm{op}}}(\mathcal{X})_{/\mathrm{l.lax}\, N^{\mathrm{op}}}) \overset{(g)}{\simeq} \mathrm{Mod}_{M^{\mathrm{op}}}(\mathcal{X})^{\mathrm{l.lax}\, N^{\mathrm{op}}}. \end{aligned}$$

The identifications (a) and (f) are both the definition of ∞–categories of modules for continuous monoids in $\mathcal{X}$. The identification (b) is Observation A.0.3. By definition of semidirect product monoids, the cartesian unstraightening of the composite functor $\mathfrak{B}N \xrightarrow{\langle N \curvearrowright M^{\mathrm{op}}\rangle} \mathrm{Monoids} \xrightarrow{\mathfrak{B}} \mathrm{Cat}_{(\infty,1)}$ is the cartesian fibration

$$\mathfrak{B}(N^{\mathrm{op}} \ltimes M^{\mathrm{op}}) \to \mathfrak{B}N^{\mathrm{op}}.$$

Being a cartesian fibration ensures the existence of the *relative functor ∞–category*; see [Ayala and Francis 2020]. The identification (c) comes directly from the definition of relative functor ∞–categories. Further, there is a definitional identification of the *right-lax coinvariants* $\mathfrak{B}(N^{\mathrm{op}} \ltimes M^{\mathrm{op}}) \simeq (\mathfrak{B}M^{\mathrm{op}})_{/^{\mathrm{r.lax}}N}$ over $\mathfrak{B}N^{\mathrm{op}}$ (see [Ayala et al. 2019, Appendix A]), which determines (d). The identification (e) follows from the codification of the N^{op}–action on $\mathrm{Fun}(\mathfrak{B}M^{\mathrm{op}}, \mathcal{X})$ in the statement of the proposition. The identification (g) is the definition of *left-lax invariants*; see [Ayala et al. 2019, Appendix A]. □

The commutativity of the topological group $\mathbb{T}^2$ determines a canonical identification $\mathbb{T}^2 \cong (\mathbb{T}^2)^{\mathrm{op}}$ between topological groups, and therefore between continuous groups. Together with Observation B.1.1, we have the following consequence of Proposition A.0.5.

Corollary A.0.6 *For $\mathcal{X}$ an ∞–category, there is a canonical identification between ∞–categories over* $\mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})$:

$$\mathrm{Mod}_{(\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z}))^{\mathrm{op}}}(\mathcal{X}) \simeq \mathrm{Mod}_{\mathbb{T}^2}(\mathcal{X})^{\mathrm{l.lax}\, \widetilde{\mathrm{E}}_2^+(\mathbb{Z})}.$$

Appendix B Some facts about the braid group and braid monoid

Here we collect some facts about the braid group on three strands, and the braid monoid on three strands.

B.1 Ambidexterity of $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$

Observation B.1.1 Taking transposes of matrices identifies the nested sequence among monoids with the nested sequence of their opposites:

$$(\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{E}_2^+(\mathbb{Z}) \subset \mathrm{GL}_2^+(\mathbb{R})) \overset{(-)^T}{\cong} (\mathrm{SL}_2(\mathbb{Z})^{\mathrm{op}} \subset \mathrm{E}_2^+(\mathbb{Z})^{\mathrm{op}} \subset \mathrm{GL}_2^+(\mathbb{R})^{\mathrm{op}}).$$

By covering space theory, these identifications canonically lift as identifications between nested sequences among monoids and their opposites:

$$(\mathrm{Braid}_3 \subset \widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \subset \widetilde{\mathrm{GL}}_2^+(\mathbb{R})) \overset{(-)^T}{\cong} (\mathrm{Braid}_3^{\mathrm{op}} \subset \widetilde{\mathrm{E}}_2^+(\mathbb{Z})^{\mathrm{op}} \subset \widetilde{\mathrm{GL}}_2^+(\mathbb{R})^{\mathrm{op}}).$$

Corollary B.1.2 *For each ∞–category $\mathcal{X}$, there are canonical identifications*

$$\mathrm{Mod}_{\mathrm{Braid}_3}(\mathcal{X}) \simeq \mathrm{Mod}_{\mathrm{Braid}_3^{\mathrm{op}}}(\mathcal{X}) \quad \textit{and} \quad \mathrm{Mod}_{\widetilde{\mathrm{E}}_2^+(\mathbb{Z})}(\mathcal{X}) \simeq \mathrm{Mod}_{\widetilde{\mathrm{E}}_2^+(\mathbb{Z})^{\mathrm{op}}}(\mathcal{X})$$

between ∞–categories of (left) modules in $\mathcal{X}$ and those of right-modules in $\mathcal{X}$.

Remark B.1.3 The composite isomorphism $\mathrm{Braid}_3 \xrightarrow[\cong]{(-)^T} \mathrm{Braid}_3^{\mathrm{op}} \xrightarrow[\cong]{(-)^{-1}} \mathrm{Braid}_3$ is the involution of Braid_3 given in terms of the presentation (0-1-1) by exchanging τ_1 and τ_2. Similarly, the involution $\mathrm{SL}_2(\mathbb{Z}) \xrightarrow[\cong]{(-)^T} \mathrm{SL}_2(\mathbb{Z})^{\mathrm{op}} \xrightarrow[\cong]{(-)^{-1}} \mathrm{SL}_2(\mathbb{Z})$ exchanges U_1 and U_2.

B.2 Comments about Braid_3 and $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$

Observation B.2.1 In Braid_3 (recall the presentation of (0-1-1)), there is an identity of the generator of $\mathrm{Ker}(\Phi)$:

$$(\tau_1\tau_2\tau_1)^4 = (\tau_1\tau_2)^6 = (\tau_2\tau_1\tau_2)^4 \in \mathrm{Ker}(\Phi).$$

For that matter, since the matrix

$$R := U_1U_2U_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = U_2U_1U_2 \in \mathrm{GL}_2(\mathbb{Z}) \tag{B-2-1}$$

implements rotation by $-\frac{1}{2}\pi$, we have that $R^4 = \mathbb{1}$ in $\mathrm{GL}_2(\mathbb{Z})$.

The following result is an immediate consequence of how $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$ is defined in (0-1-4), using that the continuous group $\mathrm{GL}_2^+(\mathbb{R})$ is a path-connected 1–type.

Corollary B.2.2 *There are pullbacks among continuous monoids*

$$\begin{tikzcd}
\mathrm{Braid}_3 \arrow[r] \arrow[d, "\Phi"'] & \widetilde{\mathrm{E}}_2^+(\mathbb{Z}) \arrow[r] \arrow[d, "\Psi"'] & * \arrow[d, "\langle \mathbb{1} \rangle"] \\
\mathrm{GL}_2(\mathbb{Z}) \arrow[r] & \mathrm{E}_2(\mathbb{Z}) \arrow[r, "\mathbb{R}\otimes_{\mathbb{Z}}"] & \mathrm{GL}_2(\mathbb{R})
\end{tikzcd}$$

In particular, there is a canonical identification between continuous groups over $\mathrm{GL}_2(\mathbb{Z})$

$$\mathrm{Braid}_3 \simeq \Omega(\mathrm{GL}_2(\mathbb{R})_{/\mathrm{GL}_2(\mathbb{Z})}).$$

Observation B.2.3 The inclusion $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{E}_2^+(\mathbb{Z})$ between submonoids of $\mathrm{GL}_2^+(\mathbb{R})$ determines an inclusion between topological monoids:

$$\mathbb{T}^2 \rtimes \mathrm{Braid}_3 \to \mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}_2^+(\mathbb{Z})\,. \tag{B-2-2}$$

After Observation 1.1.1, this inclusion witnesses the maximal subgroup, both as topological monoids and as monoid-objects in the ∞–category $\mathcal{S}$paces.

Remark B.2.4 We give an explicit description of $\widetilde{\mathrm{E}}_2^+(\mathbb{Z})$. Rawnsley [2012] gives an explicit description for the universal cover of $\mathrm{SP}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ (and goes on to establish the pullback square of Proposition 0.1.1). Following those methods, consider the maps

$$\phi\colon \mathrm{GL}_2(\mathbb{R}) \to \mathbb{S}^1 \quad \text{given by } A \mapsto \frac{(a+d)+i(b-c)}{|(a+d)+i(b-c)|},$$

where $A=\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]$. As in [Rawnsley 2012], consider a map $\eta\colon \mathrm{GL}_2(\mathbb{R})\times \mathrm{GL}_2(\mathbb{R})\to \mathbb{R}$ for which

$$e^{i\eta(A,B)}=\frac{1-\alpha_A\overline{\alpha_{B^{-1}}}}{|1-\alpha_A\overline{\alpha_{B^{-1}}}|},\quad \text{where } \alpha_A=\frac{a^2+c^2-b^2-d^2-2i(ad+bc)}{(a+d)^2+(b-c)^2}.$$

In these terms, the monoid $\widetilde{\mathrm{E}}{}_2^+(\mathbb{Z})$ can be identified as the subset

$$\widetilde{\mathrm{E}}{}_2^+(\mathbb{Z}):=\{(A,s)\mid \phi(A)=e^{is}\}\subset \mathrm{E}_2^+(\mathbb{Z})\times\mathbb{R}\quad \text{with monoid-law } (A,s)\cdot(B,t):=(AB,s+t+\eta(A,B)).$$

B.3 Group-completion of $\widetilde{\mathrm{E}}{}_2^+(\mathbb{Z})$

The continuous group $\mathrm{GL}_2^+(\mathbb{R})$ is path-connected with $\pi_1(\mathrm{GL}_2^+(\mathbb{R}),\mathbb{1})\cong\mathbb{Z}$. Consequently, there is a central extension

$$1\to\mathbb{Z}\to\widetilde{\mathrm{GL}}{}_2^+(\mathbb{R})\xrightarrow{\text{universal cover}}\mathrm{GL}_2^+(\mathbb{R})\to 1. \tag{B-3-1}$$

Consider the inclusion as scalars $\mathbb{R}^\times_{>0}\xhookrightarrow{\text{scalars}}\mathrm{GL}_2^+(\mathbb{R})$. Contractibility of the topological group $\mathbb{R}^\times_{>0}$ implies base change of this central extension (B-3-1) along this inclusion as scalars splits. In particular, for

$$\mathbb{R}\otimes_{\mathbb{Q}}\colon \mathrm{GL}_2^+(\mathbb{Q})\subset\mathrm{GL}_2^+(\mathbb{R})$$

the subgroup with rational coefficients, there are lifts among continuous monoids, in which the squares are pullbacks,

$$\begin{tikzcd}
\mathbb{N}^\times \arrow[r] \arrow[rrr, dashed, bend left] \arrow[drrr, "\text{scalars}"'] & \mathbb{Q}^\times_{>0} \arrow[r] \arrow[rrr, dashed, bend left, "\widetilde{\text{scalars}}"] \arrow[drrr] & \mathbb{R}^\times_{>0} \arrow[rrr, dashed, bend left] \arrow[drrr] & \widetilde{\mathrm{E}}{}_2^+(\mathbb{Z}) \arrow[r, "\widetilde{\mathbb{Q}\otimes_{\mathbb{Z}}}"'] \arrow[d] & \widetilde{\mathrm{GL}}{}_2^+(\mathbb{Q}) \arrow[r, "\widetilde{\mathbb{R}\otimes_{\mathbb{Q}}}"'] \arrow[d] & \widetilde{\mathrm{GL}}{}_2^+(\mathbb{R}) \arrow[d, "\text{universal cover}"] \\
 & & & \mathrm{E}_2^+(\mathbb{Z}) \arrow[r, "\mathbb{Q}\otimes_{\mathbb{Z}}"'] & \mathrm{GL}_2^+(\mathbb{Q}) \arrow[r, "\mathbb{R}\otimes_{\mathbb{Q}}"'] & \mathrm{GL}_2^+(\mathbb{R})
\end{tikzcd}$$

Proposition B.3.1 *Each of the diagrams among continuous monoids*

$$\begin{tikzcd}
\mathbb{N}^\times \arrow[r, "\text{scalars}"] \arrow[d, "\text{inclusion}"'] & \mathrm{E}_2(\mathbb{Z}) \arrow[d, "\mathbb{Q}\otimes_{\mathbb{Z}}"] \\
\mathbb{Q}^\times_{>0} \arrow[r, "\text{scalars}"] & \mathrm{GL}_2(\mathbb{Q})
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathbb{N}^\times \arrow[r, "\widetilde{\text{scalars}}"] \arrow[d, "\text{inclusion}"'] & \widetilde{\mathrm{E}}{}_2^+(\mathbb{Z}) \arrow[d, "\widetilde{\mathbb{Q}\otimes_{\mathbb{Z}}}"] \\
\mathbb{Q}^\times_{>0} \arrow[r, "\widetilde{\text{scalars}}"] & \widetilde{\mathrm{GL}}{}_2^+(\mathbb{Q})
\end{tikzcd}$$

witnesses a pushout. In particular, because $\mathbb{N}^\times\xrightarrow{\text{inclusion}}\mathbb{Q}^\times_{>0}$ witnesses group-completion among continuous monoids, each of the right downward morphisms witnesses group-completion among continuous monoids.

Proof We explain the following commutative diagram among spaces:

$$\begin{tikzcd}
\mathrm{E}_2(\mathbb{Z}) \arrow[rrr, "\mathbb{R}\otimes_{\mathbb{Z}}"] \arrow[dr, dashed, "(a)"'] \arrow[drr] & & & \mathrm{GL}_2(\mathbb{Q}) \arrow[dll, dashed, "(b)"'] \\
 & \operatorname{colim}_{\mathbb{N}^{\mathrm{div}}}\mathrm{E}_2(\mathbb{Z}) \arrow[r, dashed, "(c)"'] & \mathrm{E}_2(\mathbb{Z})[(\mathbb{N}^\times)^{-1}] \arrow[ur, "\overline{\mathbb{R}\otimes_{\mathbb{Z}}}"'] &
\end{tikzcd}$$

The top arrow is the standard inclusion. Here, scalar matrices embed the multiplicative monoid of natural numbers $\mathbb{N}^\times \overset{\text{scalars}}{\subset} \mathrm{E}_2(\mathbb{Z})$. The bottom right term, equipped with the diagonal arrow to it, is the indicated localization (among continuous monoids). The up-rightward arrow is the unique morphism between continuous monoids under $\mathrm{E}_2(\mathbb{Z})$, which exists because the continuous monoid $\mathrm{GL}_2(\mathbb{Q})$ is a continuous group. The solid diagram of spaces is thus forgotten from a diagram among continuous monoids.

Next, the poset $\mathbb{N}^{\text{div}}$ is the natural numbers with partial order given by divisibility: $r \le s$ means r divides s. Consider the functor

$$F_{\mathrm{E}_2(\mathbb{Z})} \colon \mathbb{N}^{\text{div}} \to \mathcal{S}\text{ets} \hookrightarrow \mathcal{S}\text{paces} \quad \text{given by } r \mapsto \mathrm{E}_2(\mathbb{Z}) \text{ and } (r \le s) \mapsto (\mathrm{E}_2(\mathbb{Z}) \xrightarrow{(s/r)\cdot -} \mathrm{E}_2(\mathbb{Z})).$$

The colimit term in the above diagram is $\operatorname{colim}(F_{\mathrm{E}_2(\mathbb{Z})})$, which can be identified as the classifying space of the poset

$$\mathrm{Un}(F_{\mathrm{E}_2(\mathbb{Z})}) = \mathbb{N} \times \mathrm{E}_2(\mathbb{Z}) \quad \text{with partial order } (r, A) \le (s, B) \text{ meaning } r \le s \text{ in } \mathbb{N}^{\text{div}} \text{ and } \frac{s}{r} \cdot A = B.$$

- The dashed arrow (a) is the canonical map from the 1–cofactor of the colimit.
- The dashed arrow (b) is implemented by the map $(\widetilde{b}) \colon \mathrm{GL}_2(\mathbb{Q}) \xrightarrow{A \mapsto (r_A, r_A \cdot A)} \mathbb{N} \times \mathrm{E}_2(\mathbb{Z})$, where $r_A \in \mathbb{N}$ is the smallest natural number for which the matrix $r_A \cdot A \in \mathrm{E}_2(\mathbb{Z})$ has integer coefficients. The triangle with sides (a) and (b) evidently commutes.
- The dashed arrow (c) is implemented by the map $(\widetilde{c}) \colon \mathrm{Un}(F_{\mathrm{E}_2(\mathbb{Z})}) \xrightarrow{(r,A) \mapsto r^{-1}A} \mathrm{E}_2(\mathbb{Z})[(\mathbb{N}^\times)^{-1}]$. The triangle with sides (a) and (c) evidently commutes. We now argue that (c) is an equivalence between spaces.

Observe the identification between continuous monoids

$$\bigoplus_{p \text{ prime}} (\mathbb{Z}_{\ge 0}, +) \xrightarrow{\cong} \mathbb{N}^\times \quad \text{given by } (\{p \text{ prime}\} \xrightarrow{\eta} \mathbb{Z}_{\ge 0}) \mapsto \prod_{p \text{ prime}} p^{\eta(p)},$$

as a direct sum, indexed by the set of prime numbers, of free monoids each on a single generator. For S a set of prime numbers, denote by $\langle S \rangle^\times \subset \mathbb{N}^\times$ the submonoid generated by S. For S a set of primes and for $p \in S$, the above identification as a direct sum of monoids restricts as an identification $(\mathbb{Z}_{\ge 0}, +) \times \langle S \smallsetminus \{p\} \rangle^\times \cong \langle \{p\} \rangle^\times \times \langle S \smallsetminus \{p\} \rangle^\times \cong \langle S \rangle^\times$.

Next, observe an identification of the poset $\mathbb{N}^{\text{div}} \simeq (\mathfrak{B}\mathbb{N}^\times)^{*/}$ as the undercategory of the deloop. Through this identification, the above identification supplies an identification between posets from the direct sum (based at initial objects) indexed by the set of prime numbers:

$$\bigoplus_{p \text{ prime}} (\mathbb{Z}_{\ge 0}, \le) \xrightarrow{\cong} \mathbb{N}^{\text{div}}, \quad (\{p \text{ prime}\} \xrightarrow{\chi} \mathbb{Z}_{\ge 0}) \mapsto \prod_{p \text{ prime}} p^{\chi(p)}.$$

For S a set of prime numbers, denote by $\langle S \rangle^{\text{div}} \subset \mathbb{N}^{\text{div}}$ the full subposet generated by S. For S a set of primes and for $p \in S$, the above identification as a direct sum of posets restricts as an identification $(\mathbb{Z}_{\ge 0}, \le) \times \langle S \smallsetminus \{p\} \rangle^{\text{div}} \cong \langle \{p\} \rangle^{\text{div}} \times \langle S \smallsetminus \{p\} \rangle^{\text{div}} \cong \langle S \rangle^{\text{div}}$. In particular, the standard linear order on the set of prime natural numbers determines the sequence of functors

$$\text{(B-3-2)} \qquad \mathbb{N}^{\text{div}} \xrightarrow{\text{loc}_2} \langle p > 2 \rangle^{\text{div}} \xrightarrow{\text{loc}_3} \langle p > 3 \rangle^{\text{div}} \xrightarrow{\text{loc}_5} \langle p > 5 \rangle^{\text{div}} \xrightarrow{\text{loc}_7} \cdots,$$

each of which is isomorphic with projection off of $(\mathbb{Z}_{\geq 0}, \leq)$. In particular, each projection is a cocartesian fibration, so left Kan extension along each functor is computed as a sequential colimit. Because $\mathbb{N}^\times \overset{\text{scalars}}{\subset} \mathrm{E}_2(\mathbb{Z})$ is (strictly) central, so too is $(\mathbb{Z}_{\geq 0}, +) \cong \langle \{p\} \rangle^\times \subset \mathrm{E}_2(\mathbb{Z})$. The next claim follows from these observations, using induction on the standardly ordered set of primes.

Claim *For each prime q, left Kan extension of $F_{\mathrm{E}_2(\mathbb{Z})}$ along the composite functor $\mathbb{N}^{\mathrm{div}} \xrightarrow{\mathrm{loc}_q} \langle p > q \rangle^{\mathrm{div}}$ is the functor*

$$F_{\mathrm{E}_2(\mathbb{Z})[(\langle p' \leq q \rangle^\times)^{-1}]} \colon \langle p > q \rangle^{\mathrm{div}} \xrightarrow{(\mathrm{loc}_q)_!(\mathrm{E}_2(\mathbb{Z}))} \mathcal{S}\mathrm{paces},$$

given by

$$r \mapsto \mathrm{E}_2(\mathbb{Z})[(\langle p' \leq q \rangle^\times)^{-1}] \quad \textit{and} \quad (r \leq s) \mapsto (\mathrm{E}_2(\mathbb{Z})[(\langle p' \leq q \rangle^\times)^{-1}] \xrightarrow{(s/r)\cdot -} \mathrm{E}_2(\mathbb{Z})[(\langle p' \leq q \rangle^\times)^{-1}]),$$

that evaluates on each r as the localization $\mathrm{E}_2(\mathbb{Z})[(\langle p' \leq q \rangle^\times)^{-1}]$, and on each relation $r \leq s$ in $\mathbb{N}^{\mathrm{div}}$ as scaling by s/r.

Next, the colimit of this sequence (B-3-2) is $\bigcap_{q \text{ prime}} \langle p > q \rangle^{\mathrm{div}} \simeq *$ terminal. Consequently, there is a canonical identification

$$\begin{aligned} \mathrm{colim}(F_{\mathrm{E}_2(\mathbb{Z})}) &\simeq \operatorname*{colim}_{q \in \{2<3<5<\cdots\}} ((\mathrm{loc}_q)_!(F_{\mathrm{E}_2(\mathbb{Z})})) \simeq \operatorname*{colim}_{q \in \{2<3<5<\cdots\}} (F_{\mathrm{E}_2(\mathbb{Z})[(\langle p' \leq q \rangle^\times)^{-1}]}) \\ &\simeq \mathrm{E}_2(\mathbb{Z})\Bigg[\Bigg(\bigcup_{q \in \{2<3<5<\cdots\}} \langle p' \leq q \rangle^\times\Bigg)^{-1}\Bigg] = \mathrm{E}_2(\mathbb{Z})[(\mathbb{N}^\times)^{-1}]. \end{aligned}$$

- By inspection, the resulting self-map of $\mathrm{GL}_2(\mathbb{Q})$ is the identity. Indeed, the natural transformation

$$\begin{tikzcd} \mathrm{Un}(F_{\mathrm{E}_2(\mathbb{Z})}) \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "(\tilde{\mathrm{c}})"'] \arrow[rr, phantom, "\Uparrow"] & & \mathrm{Un}(F_{\mathrm{E}_2(\mathbb{Z})}) \\ \mathrm{E}_2(\mathbb{Z})[(\mathbb{N}^\times)^{-1}] \arrow[rr, "\overline{\mathbb{R} \otimes_{\mathbb{Z}}}"] & & \mathrm{GL}_2(\mathbb{Q}) \arrow[u, "(\tilde{\mathrm{b}})"'] \end{tikzcd}$$

given by, for each $(s, B) \in \mathrm{Un}(F_{\mathrm{E}_2(\mathbb{Z})})$, the relation $(r_{s^{-1}\cdot B}, r_{s^{-1}\cdot B} \cdot (s^{-1} \cdot B)) \leq (s, B)$, witnesses an identification of the resulting self-map of $\mathrm{colim}_{\mathbb{N}^{\mathrm{div}}} \mathrm{E}_2(\mathbb{Z})$ with the identity.

We conclude that the map $\mathrm{E}_2(\mathbb{Z})[(\mathbb{N}^\times)^{-1}] \xrightarrow{\overline{\mathbb{R} \otimes_{\mathbb{Z}}}} \mathrm{GL}_2(\mathbb{Q})$ is an equivalence. It follows that the left square in the statement of the proposition is a pushout because the morphism $\mathbb{N}^\times \xrightarrow{\text{inclusion}} \mathbb{Q}^\times_{>0}$ witnesses a group-completion (among continuous monoids).

The same argument also implies the square

$$\begin{tikzcd} \mathbb{N}^\times \arrow[r, "\text{scalars}"] \arrow[d, "\text{inclusion}"'] & \mathrm{E}_2^+(\mathbb{Z}) \arrow[d, "\mathbb{Q} \otimes_{\mathbb{Z}}"] \\ \mathbb{Q}^\times_{>0} \arrow[r, "\text{scalars}"] & \mathrm{GL}_2^+(\mathbb{Q}) \end{tikzcd}$$

witnesses a pushout among continuous monoids. Base change along the central extension (B-3-1) among continuous groups reveals that the right square is also a pushout among continuous groups. □

B.4 Relationship with the finite orbit category of $\mathbb{T}^2$

Recall the ∞–category $\mathrm{Orbit}^{\mathrm{fin}}_{\mathbb{T}^2}$ of transitive $\mathbb{T}^2$–spaces with finite isotropy, and $\mathbb{T}^2$–equivariant maps between them. Recall that the action $\widetilde{\mathrm{E}}^+_2(\mathbb{Z}) \to \mathrm{E}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ on the topological group determines an action via Observation B.1.1:

$$\widetilde{\mathrm{E}}^+_2(\mathbb{Z}) \simeq \widetilde{\mathrm{E}}^+_2(\mathbb{Z})^{\mathrm{op}} \curvearrowright \mathrm{Orbit}^{\mathrm{fin}}_{\mathbb{T}^2}. \tag{B-4-1}$$

Proposition B.4.1 *There is a canonical identification of the ∞–category of coinvariants with respect to the action (B-4-1):*

$$(\mathrm{Orbit}^{\mathrm{fin}}_{\mathbb{T}^2})_{/\widetilde{\mathrm{E}}^+_2(\mathbb{Z})} \xrightarrow{\simeq} \mathfrak{B}(\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}^+_2(\mathbb{Z})).$$

Proof Recall that $\widetilde{\mathrm{E}}^+_2(\mathbb{Z}) \subset \widetilde{\mathrm{GL}}^+_2(\mathbb{R})$ is defined as a submonoid of a group. As a result, the left-multiplication action by its maximal subgroup, $\widetilde{\mathrm{GL}}^+_2(\mathbb{Z}) \curvearrowright \widetilde{\mathrm{E}}^+_2(\mathbb{Z})$, is free. Consequently, the space of objects $\mathrm{Obj}((\mathfrak{B}\,\widetilde{\mathrm{E}}^+_2(\mathbb{Z}))^{*/}) \simeq \widetilde{\mathrm{E}}^+_2(\mathbb{Z})_{/\widetilde{\mathrm{GL}}^+_2(\mathbb{Z})} \xrightarrow{\cong} \mathrm{E}^+_2(\mathbb{Z})_{/\mathrm{GL}^+_2(\mathbb{Z})}$ is simply the quotient set of $\widetilde{\mathrm{E}}^+_2(\mathbb{Z})$ by its maximal subgroup acting via left-multiplication, which is bijective with the quotient of $\mathrm{E}^+_2(\mathbb{Z})$ by its maximal subgroup via the canonical projection $\widetilde{\mathrm{E}}^+_2(\mathbb{Z}) \to \mathrm{E}^+_2(\mathbb{Z})$. The space of morphisms between objects represented by $A, B \in \mathrm{E}^+_2(\mathbb{Z})$,

$$\mathrm{Hom}_{(\mathfrak{B}\,\widetilde{\mathrm{E}}^+_2(\mathbb{Z}))^{*/}}([A],[B]) \simeq \{X \in \mathrm{E}^+_2(\mathbb{Z}) \mid XA = B\} \subset \mathrm{E}^+_2(\mathbb{Z}),$$

is simply the set of factorizations in $\mathrm{E}^+_2(\mathbb{Z})$ of B by A. In particular, the ∞–category $(\mathfrak{B}\,\widetilde{\mathrm{E}}^+_2(\mathbb{Z}))^{*/}$ is a poset. We now identify this poset essentially through Pontryagin duality.

Consider the poset $\mathrm{P}^{\mathrm{fin}}_{\mathbb{T}^2}$ of finite subgroups of $\mathbb{T}^2$ ordered by inclusion. We now construct mutually inverse functors between posets

$$(\mathfrak{B}\,\widetilde{\mathrm{E}}^+_2(\mathbb{Z}))^{*/} \xrightarrow{[A]\mapsto \mathrm{Ker}(\mathbb{T}^2 \xrightarrow{A} \mathbb{T}^2)} \mathrm{P}^{\mathrm{fin}}_{\mathbb{T}^2} \quad \text{and} \quad \mathrm{P}^{\mathrm{fin}}_{\mathbb{T}^2} \xrightarrow{C \mapsto [\mathbb{Z}^2 \xrightarrow{A_C} \mathbb{Z}^2]} (\mathfrak{B}\,\widetilde{\mathrm{E}}^+_2(\mathbb{Z}))^{*/}. \tag{B-4-2}$$

The first functor assigns to $[A]$ the kernel of the endomorphism of $\mathbb{T}^2$ induced by a representative $A \in \mathrm{E}^+_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$. The second functor assigns to C the endomorphism $(\mathbb{Z}^2 \xrightarrow{A_C} \mathbb{Z}^2) \in \mathrm{E}^+_2(\mathbb{Z})$ defined as follows. The preimage $\mathbb{Z}^2 \subset \mathrm{quot}^{-1}(C) \subset \mathbb{R}^2 \xrightarrow{\mathrm{quot}} \mathbb{R}^2_{/\mathbb{Z}^2} =: \mathbb{T}^2$ by the quotient is a lattice in $\mathbb{R}^2$ that contains the standard lattice cofinitely. There is a unique pair of nonnegative-quadrant vectors $(u_1, u_2) \in (\mathbb{R}_{\geq 0})^2 \times (\mathbb{R}_{\geq 0})^2$ that generate this lattice $\mathrm{quot}^{-1}(C)$ and agree with the standard orientation of $\mathbb{R}^2$. Then $A_C \in \mathrm{E}^+_2(\mathbb{Z})$ is the unique matrix for which $A_C \vec{u}_i = \vec{e}_i$ for $i = 1, 2$. It is straightforward to verify that the two assignments in (B-4-2) indeed respect partial orders, and are mutually inverse to one another. Observe that the action (B-4-1) descends as an action $\widetilde{\mathrm{E}}^+_2(\mathbb{Z})^{\mathrm{op}} \curvearrowright \mathrm{P}^{\mathrm{fin}}_{\mathbb{T}^2}$, with respect to which the equivalences (B-4-2) are $\widetilde{\mathrm{E}}^+_2(\mathbb{Z})^{\mathrm{op}}$–equivariant.

Next, reporting the stabilizer of a transitive $\mathbb{T}^2$–space defines a functor $\mathrm{Orbit}^{\mathrm{fin}}_{\mathbb{T}^2} \xrightarrow{(\mathbb{T}^2 \curvearrowright T) \mapsto \mathrm{Stab}_{\mathbb{T}^2}(t)} \mathrm{P}^{\mathrm{fin}}_{\mathbb{T}^2}$. Evidently, this functor is conservative. Notice also that this functor is a left fibration; its straightening is the composite functor

$$\mathrm{P}^{\mathrm{fin}}_{\mathbb{T}^2} \xrightarrow{C \mapsto \mathbb{T}/C} \mathrm{Groups} \xrightarrow{B} \mathcal{S}\mathrm{paces}. \tag{B-4-3}$$

The result follows upon constructing a canonical filler in the diagram among ∞–categories witnessing a pullback

$$\begin{tikzcd}
\mathrm{Orbit}^{\mathrm{fin}}_{\mathbb{T}^2} \arrow[rrr, dashed] \arrow[d] & & & \mathrm{Ar}(\mathfrak{B}(\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}^+_2(\mathbb{Z}))) \arrow[d, "\mathrm{Ar}(\mathfrak{B}\mathrm{proj})"] \\
\mathrm{P}^{\mathrm{fin}}_{\mathbb{T}^2} \arrow[r, "\simeq", "\text{(B-4-2)}"'] & (\mathfrak{B}\, \widetilde{\mathrm{E}}^+_2(\mathbb{Z}))^{*/} \arrow[rr, "\mathrm{forget}"] & & \mathrm{Ar}(\mathfrak{B}\, \widetilde{\mathrm{E}}^+_2(\mathbb{Z}))
\end{tikzcd}$$

By definition of semidirect products, the canonical functor $\mathfrak{B}(\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}^+_2(\mathbb{Z})) \xrightarrow{\mathfrak{B}\mathrm{proj}} \mathfrak{B}\, \widetilde{\mathrm{E}}^+_2(\mathbb{Z})$ is a cocartesian fibration. Because the ∞–category $\mathfrak{B}\mathbb{T}^2 = B\mathbb{T}^2$ is an ∞–groupoid, this cocartesian fibration is conservative, and therefore a left fibration. Consequently, the functor

$$\mathrm{Ar}(\mathfrak{B}(\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}^+_2(\mathbb{Z}))) \to \mathrm{Ar}(\mathfrak{B}\, \widetilde{\mathrm{E}}^+_2(\mathbb{Z}))$$

is also a left fibration. So the base change of this left fibration along $(\mathfrak{B}\, \widetilde{\mathrm{E}}^+_2(\mathbb{Z}))^{*/} \xrightarrow{\mathrm{forget}} \mathrm{Ar}(\mathfrak{B}\, \widetilde{\mathrm{E}}^+_2(\mathbb{Z}))$ is again a left fibration,

$$\mathrm{Ar}(\mathfrak{B}(\mathbb{T}^2 \rtimes \widetilde{\mathrm{E}}^+_2(\mathbb{Z})))^{|B\mathbb{T}^2} \to (\mathfrak{B}\, \widetilde{\mathrm{E}}^+_2(\mathbb{Z}))^{*/} \simeq \mathrm{P}^{\mathrm{fin}}_{\mathbb{T}^2}, \tag{B-4-4}$$

where the equivalence is by (B-4-2). Direct inspection identifies the straightening of this left fibration (B-4-4) as (B-4-3). □

References

[Ayala and Francis 2015] **D Ayala**, **J Francis**, *Factorization homology of topological manifolds*, J. Topol. 8 (2015) 1045–1084 MR Zbl

[Ayala and Francis 2020] **D Ayala**, **J Francis**, *Fibrations of ∞–categories*, High. Struct. 4 (2020) 168–265 MR Zbl

[Ayala et al. 2017a] **D Ayala**, **J Francis**, **A Mazel-Gee**, **N Rozenblyum**, *Factorization homology of enriched ∞–categories*, preprint (2017) arXiv 1710.06414

[Ayala et al. 2017b] **D Ayala**, **J Francis**, **H L Tanaka**, *Factorization homology of stratified spaces*, Selecta Math. 23 (2017) 293–362 MR Zbl

[Ayala et al. 2017c] **D Ayala**, **A Mazel-Gee**, **N Rozenblyum**, *A naive approach to genuine G–spectra and cyclotomic spectra*, preprint (2017) arXiv 1710.06416

[Ayala et al. 2018] **D Ayala**, **J Francis**, **N Rozenblyum**, *Factorization homology, I: Higher categories*, Adv. Math. 333 (2018) 1042–1177 MR Zbl

[Ayala et al. 2019] **D Ayala**, **A Mazel-Gee**, **N Rozenblyum**, *Stratified noncommutative geometry* (2019) arXiv 1910.14602 To appear in Mem. Amer. Math. Soc.

[Boardman and Vogt 1973] **J M Boardman**, **R M Vogt**, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math. 347, Springer (1973) MR Zbl

[Bökstedt et al. 1993] **M Bökstedt**, **W C Hsiang**, **I Madsen**, *The cyclotomic trace and algebraic K–theory of spaces*, Invent. Math. 111 (1993) 465–539 MR Zbl

[Chari and Pressley 1994] **V Chari**, **A Pressley**, *A guide to quantum groups*, Cambridge Univ. Press (1994) MR Zbl

[Connes 1983] **A Connes**, *Cohomologie cyclique et foncteurs* Ext^n, C. R. Math. Acad. Sci. Paris 296 (1983) 953–958 MR Zbl

[Corrigan-Salter and Staic 2016] **B R Corrigan-Salter**, **M D Staic**, *Higher-order and secondary Hochschild cohomology*, C. R. Math. Acad. Sci. Paris 354 (2016) 1049–1054 MR Zbl

[Dehn 1938] **M Dehn**, *Die Gruppe der Abbildungsklassen: Das arithmetische Feld auf Flächen*, Acta Math. 69 (1938) 135–206 MR Zbl

[Dundas et al. 2013] **B I Dundas**, **T G Goodwillie**, **R McCarthy**, *The local structure of algebraic K–theory*, Algebra Appl. 18, Springer (2013) MR Zbl

[Dunn 1988] **G Dunn**, *Tensor product of operads and iterated loop spaces*, J. Pure Appl. Algebra 50 (1988) 237–258 MR Zbl

[Earle and Eells 1967] **C J Earle**, **J Eells**, *The diffeomorphism group of a compact Riemann surface*, Bull. Amer. Math. Soc. 73 (1967) 557–559 MR Zbl

[Goodwillie 1986] **T G Goodwillie**, *Relative algebraic K–theory and cyclic homology*, Ann. of Math. 124 (1986) 347–402 MR Zbl

[Gramain 1973] **A Gramain**, *Le type d'homotopie du groupe des difféomorphismes d'une surface compacte*, Ann. Sci. École Norm. Sup. 6 (1973) 53–66 MR Zbl

[Hatcher 1976] **A Hatcher**, *Homeomorphisms of sufficiently large* P^2*–irreducible 3–manifolds*, Topology 15 (1976) 343–347 MR Zbl

[Hatcher 1983] **A E Hatcher**, *A proof of the Smale conjecture,* $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$, Ann. of Math. 117 (1983) 553–607 MR Zbl

[Hatcher 2013] **A Hatcher**, *The Kirby torus trick for surfaces*, preprint (2013) arXiv 1312.3518

[Hesselholt 1994] **L Hesselholt**, *Stable topological cyclic homology is topological Hochschild homology*, from "*K*–theory", Astérisque 226, Soc. Math. France, Paris (1994) 175–192 MR Zbl

[Hoyois et al. 2017] **M Hoyois**, **S Scherotzke**, **N Sibilla**, *Higher traces, noncommutative motives, and the categorified Chern character*, Adv. Math. 309 (2017) 97–154 MR Zbl

[Laubacher 2017] **J C Laubacher**, *Secondary Hochschild and cyclic (co)homologies*, PhD thesis, Bowling Green State University (2017) `http://rave.ohiolink.edu/etdc/view?acc_num=bgsu1489422065908758`

[Loday 1992] **J-L Loday**, *Cyclic homology*, Grundl. Math. Wissen. 301, Springer (1992) MR Zbl

[Lurie 2015] **J Lurie**, *Rotation invariance in algebraic K–theory*, preprint (2015) Available at `https://www.math.ias.edu/~lurie/papers/Waldhaus.pdf`

[Lurie 2017] **J Lurie**, *Higher algebra*, preprint (2017) Available at `https://url.msp.org/Lurie-HA`

[Mazel-Gee and Stern 2021] **A Mazel-Gee**, **R Stern**, *A universal characterization of noncommutative motives and secondary algebraic K–theory*, preprint (2021) arXiv 2104.04021

[Milnor 1971] **J Milnor**, *Introduction to algebraic K–theory*, Ann. of Math. Stud. 72, Princeton Univ. Press (1971) MR Zbl

[Pirashvili 2000] **T Pirashvili**, *Hodge decomposition for higher order Hochschild homology*, Ann. Sci. École Norm. Sup. 33 (2000) 151–179 MR Zbl

[Rawnsley 2012] **J Rawnsley**, *On the universal covering group of the real symplectic group*, J. Geom. Phys. 62 (2012) 2044–2058 MR Zbl

[Rolfsen 1976] **D Rolfsen**, *Knots and links*, Math. Lect. Ser. 7, Publish or Perish, Berkeley, CA (1976) MR Zbl

[Staic 2016] **M D Staic**, *Secondary Hochschild cohomology*, Algebr. Represent. Theory 19 (2016) 47–56 MR Zbl

[Toën and Vezzosi 2009] **B Toën**, **G Vezzosi**, *Chern character, loop spaces and derived algebraic geometry*, from "Algebraic topology" (N A Baas, E M Friedlander, B Jahren, P A Østvær, editors), Abel Symp. 4, Springer (2009) 331–354 MR Zbl

Department of Mathematics, Montana State University
Bozeman, MT, United States

Department of Mathematics, Northwestern University
Evanston, IL, United States

Department of Mathematics, Montana State University
Bozeman, MT, United States

david.ayala@montana.edu, jnkf@northwestern.edu, adam.howard1@montana.edu

Received: 15 December 2021 Revised: 13 March 2023

Algebraic & Geometric Topology **24**:4 (2024) 2011–2038
DOI: 10.2140/agt.2024.24.2011
Published: 16 July 2024

The shape of the filling-systole subspace in surface moduli space and critical points of the systole function

YUE GAO

We study the space $X_g \subset \mathcal{M}_g$ consisting of surfaces with filling systoles and its subset, critical points of the systole function. In the first part we obtain a surface with Teichmüller distance $\frac{1}{5}\log\log g$ to X_g, and in the second and third parts prove that most points in $\mathcal{M}_g$ have Teichmüller distance $\frac{1}{5}\log\log g$ and Weil–Petersson distance $0.6521(\sqrt{\log g} - \sqrt{7\log\log g})$ to X_g. So the radius-r neighborhood of X_g cannot cover the thick part of $\mathcal{M}_g$ for any fixed $r > 0$. In the last two parts, we get critical points with small and large (comparable to the diameter of the thick part of $\mathcal{M}_g$) distances.

30F45, 30F60, 57K20

1 Introduction

1.A Motivations

A long-standing and difficult question on the moduli space of Riemann surfaces of genus g (denoted by $\mathcal{M}_g$) is to construct a spine of $\mathcal{M}_g$ (the deformation retract of $\mathcal{M}_g$ with minimal dimension.)[1] This question is equivalent to constructing a mapping class group equivariant deformation retract with the minimal dimension of the Teichmüller space $\mathcal{T}_g$. In an unpublished manuscript, Thurston [1986b] proposed a candidate for the spine of $\mathcal{M}_g$; see Anderson, Parlier and Pettet [Anderson et al. 2016]. This candidate consists of surfaces whose shortest geodesics are filling, and is denoted by X_g (A finite set of

[1] In some papers a deformation retract of $\mathcal{M}_g$ is called a spine of $\mathcal{M}_g$, and the ones with minimal dimension are called minimal (or optimal) spines

essential curves on a surface is filling if the curves cut the surface into polygonal disks.) Thurston outlined a proof that X_g is a deformation retract of $\mathcal{M}_g$, but the proof seems difficult to complete. Recently, some progress on the dimension of X_g has been made; for example, a codimension-2 deformation retract of $\mathcal{M}_g$ containing X_g (see Ji [2014]) and a $(4g-5)$–cell contained in X_g (see Fortier Bourque [2020]). But determining the dimension of X_g still seems very difficult.

Our work mainly concerns the shape of X_g with respect to the Teichmüller and Weil–Petersson metrics on $\mathcal{M}_g$. The shape of X_g was first studied by Anderson, Parlier and Pettet [Anderson et al. 2016], and our work is partly inspired by the notion of the sparseness of subsets in $\mathcal{M}_g$ they raised. Our question is:

Question 1.1 Does there exist a number $R = R(g) > 0$ such that, for most points $p \in \mathcal{M}_g$, $d_{\mathcal{T}}(p, X_g)$ (or $d_{\mathrm{WP}}(p, X_g)$) is larger than $R(g)$?

In other words: is X_g in some sense "sparse" in $\mathcal{M}_g$?

Another motivation to study the shape of X_g is to understand the shape of the critical-point set of the systole function. On each surface $p \in \mathcal{M}_g$, the systole is the length of the shortest geodesics on p. Therefore it can be treated as a function on $\mathcal{M}_g$. Akrout [2003] showed that this function is a topological Morse function; hence the systole function has regular and critical points. The critical-point set of this function is denoted by $\mathrm{Crit}(\mathrm{sys}_g)$. By Schmutz Schaller [1999, Corollary 20], $\mathrm{Crit}(\mathrm{sys}_g) \subset X_g$. Therefore conclusions on the shape of X_g imply corollaries on the shape of $\mathrm{Crit}(\mathrm{sys}_g)$. On the other hand, a natural question is to compare the shape difference between X_g and $\mathrm{Crit}(\mathrm{sys}_g)$. This program is closely related to the question of Mirzakhani as to whether long fingers exist. Details are in the following subsection.

1.B Results and perspectives

Our first result is the construction of an example of a surface in the thick part of $\mathcal{M}_g$ that is distant from X_g.

Proposition 3.6 *When $g \geq 3$ there is a surface S_g with $\mathrm{sys}(S_g) = \operatorname{arccosh} 2$ whose distance to X_g is at least $\frac{1}{4}\log(\log g - K)$, where $K = \log 12$.*

Remark 1.2 If a surface's systole is sufficiently small, then its Teichmüller distance to X_g could be arbitrarily large. But our example has constant systole while it is distant from X_g.

Before stating Theorem 4.3, we make "most points" in Question 1.1 precise.

The Weil–Petersson metric is a mapping class group equivariant Riemannian metric on the Teichmüller space. Therefore the volume of $\mathcal{M}_g$ and Borel subsets of $\mathcal{M}_g$ with respect to this metric is well defined. Mirzakhani [2007] invented the integration formula for geometric functions on $\mathcal{M}_g$ with respect to this volume and then calculated the volume of $\mathcal{M}_g$. She initiated a fast-growing area: random surfaces with respect to the Weil–Petersson metric; see Mirzakhani [2007; 2013].

The random surface theory is based on the probability of Borel sets in $\mathcal{M}_g$. Mirzakhani defined the probability of a Borel set $B \subset \mathcal{M}_g$ as

$$P_{\mathrm{WP}}(B) = \frac{\mathrm{vol}_{\mathrm{WP}}(B)}{\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g)}.$$

Theorem 4.3 $$P_{\mathrm{WP}}\left\{S \in \mathcal{M}_g \mid d_{\mathcal{T}}(S, X_g) < \tfrac{1}{5}\log\log g\right\} \to 0 \quad \text{as } g \to \infty.$$

Remark 1.3 The distance $\frac{1}{5}\log\log g$ is calculated from (3-1) in Lemma 3.2 and the width by Nie, Wu and Xue [Nie et al. 2023, Theorem 2]. Actually, if we replace $\frac{1}{5}$ by any number smaller than $\frac{1}{4}$, this theorem still holds. Besides Lemma 3.2 and [Nie et al. 2023, Theorem 2], Theorem 4.3 also depends on Mirzakhani's Theorem 2.8 in [Mirzakhani and Petri 2019].

Theorem 4.3 gives a positive answer to Question 1.1 with respect to Teichmüller distance. When g is sufficiently large, most points in $\mathcal{M}_g$ have Teichmüller distance at least $\frac{1}{5}\log\log g$ to X_g.

The moduli space $\mathcal{M}_g$ is divided into two parts. The thick part consists of surfaces with systole larger than or equal to ε for some fixed $\varepsilon > 0$, denoted by $\mathcal{M}_g^{\geq \varepsilon}$. This part is compact in $\mathcal{M}_g$, and its diameter with respect to the Teichmüller metric is $C\log(g/\varepsilon)$ for some $C > 0$ by Rafi and Tao [2013]. The complementary part of the thick part is the thin part.

By the collar lemma (see for example Buser [1992, Chapter 4]), X_g is contained in the thick part of $\mathcal{M}_g$ and we have:

Corollary 4.4 $P_{\mathrm{WP}}\left\{d_{\mathcal{T}}(S, X_g) < \frac{1}{5}\log\log g \mid S \text{ lies in the thick part of } \mathcal{M}_g\right\} \to 0 \quad \text{as } g \to \infty.$

From Proposition 3.6 or Corollary 4.4, the Hausdorff distance between the thick part of $\mathcal{M}_g$ and X_g is at least $\frac{1}{5}\log\log g$.

The study of the shape of X_g with respect to the Teichmüller metric was pioneered by Anderson, Parlier and Pettet [Anderson et al. 2016]. By comparing X_g with Y_g, the subset of $\mathcal{M}_g$ with Bers' constant bounded above and below by constants, they obtained the following two results: the diameter of X_g is comparable with the thick part of $\mathcal{M}_g$ [Anderson et al. 2016, Theorem 1.1], and the sparseness of $X_g \cap Y_g$ in Y_g, that is, most points in Y_g have distance at least $\log g$ to $X_g \cap Y_g$ [Anderson et al. 2016, Theorem 1.3].[2]

The distance in Proposition 3.6 and Theorem 4.3 is smaller than that of [Anderson et al. 2016, Theorem 1.3], but we remove the restriction to Y_g and obtain the sparseness of X_g in $\mathcal{M}_g$ and thick part of $\mathcal{M}_g$.

An immediate corollary to Proposition 3.6 or Corollary 4.4 is:

Corollary 1.4 *For any $R > 0$, when g is sufficiently large, the R–neighborhood of X_g does not cover the thick part of $\mathcal{M}_g$. Hence the R–neighborhood of $\mathrm{Crit}(\mathrm{sys}_g)$ does not cover the thick part of $\mathcal{M}_g$.*

For the thick part of $\mathcal{M}_g$, Fletcher, Kahn and Markovic [Fletcher et al. 2013] determined the minimal size of a point set in $\mathcal{M}_g^{\geq \varepsilon}$ whose R neighborhood covers the whole thick part for any $R > 0$. The size

[2]For the meaning of the "most points" and the definition of the distance, see [Anderson et al. 2016].

is $(Cg)^{2g}$ for $C = C(\varepsilon, R) > 0$. Currently the size of $\mathrm{Crit}(\mathrm{sys}_g)$ is not determined, but a known lower bound for $|\mathrm{Crit}(\mathrm{sys}_g)|$ given by the Euler characteristic of $\mathcal{M}_g$ (see [Harer and Zagier 1986]) is quite close to this number. However, by Corollaries 4.4 and 1.4, $\mathrm{Crit}(\mathrm{sys}_g)$ is sparse in $\mathcal{M}_g^{\geq\varepsilon}$.

We also answer Question 1.1 with respect to the Weil–Petersson metric:

Theorem 5.7 $P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid d_{\mathrm{WP}}(S, X_g) < 0.6521(\sqrt{\log g} - \sqrt{7 \log\log g})\} \to 0$ *as* $g \to \infty$.

Besides the tools used in the proof of Theorem 4.3, to prove this theorem we also use Wu's estimate [2022] of lower bounds of Weil–Petersson distance. Using this estimate, Wu [2022, Theorem 1.4] has obtained that the probability of the Weil–Petersson $\sqrt{\log g}$–neighborhood of all surfaces with $o(\log g)$ Bers' constant tends to 0 as g tends to infinity.

After answering Question 1.1, a further question is:

Question 1.5 Is there a critical point $p \in \mathrm{Crit}(\mathrm{sys}_g)$ and a large number $R(g)$ such that $B(p, R(g))$ contains no critical point except p?

This question concerns the distances between the elements of $\mathrm{Crit}(\mathrm{sys}_g)$ and X_g. The radius gives a lower bound for the Hausdorff distance between X_g and $\mathrm{Crit}(\mathrm{sys}_g)$. Moreover, Question 1.5 is very close to but slightly weaker than Mirzakhani's question of whether there exists a long finger (see Fortier Bourque and Rafi [2022]) when the systole has a large local maximum at p.

For such a point p, a component of the level set $\{q \mid \mathrm{sys}(q) > L\}$ that contains p but does not contain any other critical point of the systole function is called a finger. The length of a finger is $\mathrm{sys}(p) - L$. If a finger is long, then the Teichmüller distance from p to other critical points is large (at least $\frac{1}{2}\log(\mathrm{sys}(p)/L)$).

We make the first attempt to compare the difference between X_g and $\mathrm{Crit}(\mathrm{sys}_g)$.

For any $g \geq 2$, we take three surfaces S_g^1, S_g^2 and S_g^3 that were originally constructed by Anderson, Parlier and Pettet [Anderson et al. 2011], Gao and Wang [2023] and Fortier Bourque and Rafi [2022], respectively. The surfaces S_g^1 and S_g^3 are known critical points, and we prove S_g^2 is a critical point by our Proposition 6.3. Then we calculate the distance between the critical points.

Theorem 8.3 *For the surfaces* $S_g^1, S_g^3 \in \mathrm{Crit}(\mathrm{sys}_g)$, *when* $g \geq 13$,

$$d_{\mathcal{T}}(S_g^1, S_g^3) > \tfrac{1}{2}\log(g-6) - K,$$

where $K = \frac{1}{2}\log\bigl(\frac{40}{3}\log((4g+4)/\pi)\bigr)$.

Hence the diameter of $\mathrm{Crit}(\mathrm{sys}_g)$ is comparable with the diameter of X_g and the diameter of the thick part of $\mathcal{M}_g$.

On the other hand, the distance between S_g^1 and S_g^2 is small.

Theorem 7.10 *For any* $g \geq 2$ *and* $S_g^1, S_g^2 \in \mathrm{Crit}(\mathrm{sys}_g)$,

$$d_{\mathcal{T}}(S_g^1, S_g^2) \leq 2.3.$$

It is worth mentioning that to prove the surface Σ_g^2 is a critical point, we use a conclusion (Proposition 6.3) that among all surfaces with a specific symmetry, the surface with maximal systole is a critical point. This proposition is a generalization of Schmutz Schaller [1999, Theorem 37] and Fortier Bourque [2020, Proposition 6.3]. The key point of this generalization is to construct a domain in $\mathcal{M}_g$ containing the point p we consider, and p is the maximal point of the systole function in the domain.

1.C Methods

To prove "most surfaces" are distant from X_g, we avail ourselves of lower bounds of Teichmüller and Weil–Petersson distance (Lemma 3.2 and Wu [2022, Theorem 1.1], respectively). For "most surfaces" there is an embedded cylinder with a large length and large width by Nie, Wu and Xue [Nie et al. 2023] and the systoles of the surfaces are relatively small by a theorem of Mirzakhani [Mirzakhani and Petri 2019, Theorem 2.8]. By the lower bound estimates, surfaces containing such a cylinder are distant from X_g.

Theorem 8.3 is obtained by comparing the diameter of the two surfaces. This method is from Rafi and Tao [2013, Lemma 5.1].

The shapes of S_g^1 and S_g^2 are similar. Then we can construct the deformation from S_g^1 to S_g^2 explicitly. From the deformation we describe in Section 7, we calculate the distance and get Theorem 7.10.

Organization In Section 2, we provide some preliminary knowledge on Teichmüller theory and the systole. Then we prove Proposition 3.6 in Section 3 and Theorem 4.3 in Section 4. On the Weil–Petersson distance, we prove Theorem 5.7 in Section 5. In Section 6, Proposition 6.3 is proved. Then using Proposition 6.3, Theorem 7.10 is proved in Section 7. Finally, Theorem 8.3 is proved in Section 8.

Acknowledgements We acknowledge Prof. Yi Liu for many helpful discussions, comments, suggestions, help, and mentoring. Besides, during the revision of this paper, he gave me helpful suggestions. We acknowledge Prof. Shicheng Wang for the helpful discussion on Remark 7.1 and helpful suggestions. We acknowledge Prof. Yunhui Wu and Yang Shen for suggesting consideration of the Weil–Petersson distance version of Theorem 4.3 and acknowledge Prof. Yunhui Wu for many helpful discussions and comments on Theorem 5.7. We acknowledge Prof. Jiajun Wang for his helpful suggestions. We acknowledge the referee for invaluable suggestions. The author is supported by grant 12301082 of the National Natural Science Foundation of China.

2 Preliminaries

2.A Teichmüller space

We denote by $\mathcal{T}_g$ the Teichmüller space consisting of marked hyperbolic surfaces with genus g, and by $\mathcal{M}_g$ the moduli space consisting of hyperbolic surfaces with genus g. It is known that

$$\mathcal{M}_g \cong \mathcal{T}_g / \Gamma_g.$$

Here Γ_g is the mapping class group of a closed orientable surface of genus g.

The Teichmüller metric is a complete mapping class group equivariant metric on $\mathcal{T}_g$ defined using the dilatation of quasiconformal maps. For $X, Y \in \mathcal{T}_g$, the distance between X and Y is denoted by $d_{\mathcal{T}}(X, Y)$. The formal definition of this metric is deferred to Section 7.C.1 since it is not needed for most of this paper.

2.B Thurston's metric

Thurston [1986a] defined an asymmetric metric on the Teichmüller space. For $X, Y \in \mathcal{T}_g$ and $f : X \to Y$ a Lipschitz homeomorphism between X and Y, we let

$$L(f) = \sup_{\substack{x,y\in X\\ x\neq y}} \frac{d(f(x), f(y))}{d(x, y)}.$$

Then this metric is defined as

$$d_L(X, Y) = \inf_f \{\log L(f) \mid f : X \to Y \text{ is a Lipschitz homeomorphism}\}.$$

Theorem 2.1 [Thurston 1986a] *For $X, Y \in \mathcal{M}_g$,*

$$d_L(X, Y) = \sup_{\alpha\in C(X)} \inf_{f : X\to Y} \log \frac{l_{f(\alpha)}(Y)}{l_\alpha(X)}.$$

Here f is a Lipschitz homeomorphism and $C(X)$ is the set of simple closed curves in X.

For $X, Y \in \mathcal{T}_g$, Rafi and Tao [2013, (2)] have shown that

$$\tfrac{1}{2}\, d_L(X, Y) \le d_{\mathcal{T}}(X, Y). \tag{2-1}$$

2.C The topological Morse function and generalized systole

Definition 2.2 On a topological manifold M^n, a function $f : M^n \to \mathbb{R}$ is a topological Morse function if, at each point $p \in M$, there is a neighborhood U of p and a map $\psi : U \to \mathbb{R}^n$. Here ψ is a homeomorphism between U and its image such that $f \circ \psi^{-1}$ is either a linear function or

$$f \circ \psi^{-1}((x_1, x_2, \dots, x_n)) = f(p) - x_1^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_n^2.$$

In the former case the point p is called a regular point of f, while in the latter case the point p is called a singular point with index j.

On a Riemannian manifold M, $l_\alpha : M \to \mathbb{R}^+$ is a family of smooth functions on M indexed by $\alpha \in I$, called the (*generalized*) *length function*. The length function family is required to satisfy the following condition: for every $p \in M$ there exists a neighborhood U of p and a number $K > 0$ such that the set $\{\alpha \mid l_\alpha(q) \le K \text{ for all } q \in U\}$ is a nonempty finite set. The (*generalized*) *systole function* is defined as

$$\operatorname{sys}(p) := \inf_{\alpha\in I} l_\alpha(p) \quad \text{for all } p \in M.$$

Theorem 2.3 [Akrout 2003] *If, for any $\alpha \in I$, the Hessian of l_α is positively definite, then the generalized systole function is a topological Morse function.*

The critical point of the systole function is also characterized in [Akrout 2003]. A $p \in M$ is a eutactic point if and only if it is a critical point of the systole function.

We assume that, for $p \in M$,

$$S(p) := \{\alpha \in I \mid l_\alpha(p) = \mathrm{sys}(p)\}.$$

Definition 2.4 For $p \in M$, p is eutactic if and only if 0 is contained in the interior of the convex hull of $\{\mathrm{d}l_\alpha|_p \mid \alpha \in S(p)\}$.

An equivalent definition is:

Definition 2.5 $p \in M$ is eutactic if and only if for $v \in T_p M$, if $\mathrm{d}l_\alpha(v) \geq 0$ for all $\alpha \in S(p)$, then $\mathrm{d}l_\alpha(v) = 0$ for all $\alpha \in S(p)$.

2.D Teichmüller space and length function

For a marked hyperbolic surface Σ in the Teichmüller space $\mathcal{T}_g$, $\alpha \subset \Sigma$ is an essential simple closed geodesic. Its length is denoted by $l_\alpha(\Sigma)$. In another point of view, l_α is a function on $\mathcal{T}_g$:

$$l_\alpha \colon \mathcal{T}_g \to \mathbb{R}^+, \quad \Sigma \mapsto l_\alpha(\Sigma).$$

The set of all the shortest geodesics on Σ is denoted by $S(\Sigma)$. For $\alpha \in S(\Sigma)$,

$$l_\alpha(\Sigma) \leq l_\beta(\Sigma) \quad \text{for all simple closed geodesics } \beta \subset \Sigma.$$

The length of the shortest geodesics of Σ is called *systole* of Σ.

Similarly, the systole can be treated as a function on $\mathcal{T}_g$, and we denote it by sys_g or shortly sys. Obviously

$$\mathrm{sys}(\Sigma) = l_\alpha(\Sigma) = \inf_{\text{simple closed geodesics } \beta \subset \Sigma} l_\beta(\Sigma).$$

Remark 2.6 In a small neighborhood U of Σ in $\mathcal{T}_g$, the systole function is realized by the minimum lengths of finitely many simple closed geodesics.

Remark 2.7 Systole function can also be defined as a function on $\mathcal{M}_g$:

$$\mathrm{sys} \colon \mathcal{M}_g \to \mathbb{R}^+, \quad \Sigma \mapsto \mathrm{sys}(\Sigma).$$

However, the length function l_α is not well-defined on $\mathcal{M}_g$ because of the monodromy.

By [Wolpert 1987], the Hessian of l_α is always positive definite for any simple closed geodesic $\alpha \subset \Sigma$ with respect to the Weil–Petersson metric. Therefore:

Corollary 2.8 [Akrout 2003, corollaire, page 2] *The systole function is a topological Morse function on $\mathcal{T}_g$.*

The systole function is also a topological Morse function on $\mathcal{M}_g$, because the systole function is an invariant function on Teichmüller space.

The set of all the critical points of sys_g in $\mathcal{T}_g$ is denoted by $\mathrm{Crit}(\mathrm{sys}_g)$.

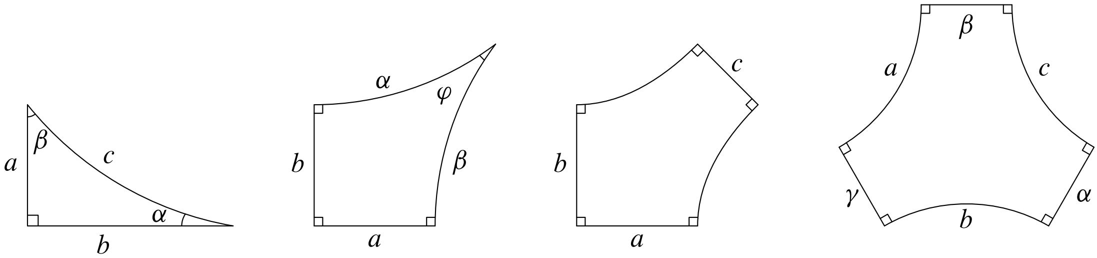


Figure 1: Hyperbolic polygons. The right-angled triangle (left), trirectangle (middle left), right-angled pentagon (middle right) and right-angled hexagon (right).

2.E Hyperbolic trigonometric formulae

The following are from [Buser 1992, page 454] and are pictured in Figure 1:

(2-2) $\cosh c = \cot\alpha \cot\beta.$ (right-angled triangles),

(2-3) $\cos\varphi = \sinh a \sinh b$ (trirectangles),

(2-4) $\cosh c = \sinh a \sinh b$ (right-angled pentagons),

(2-5) $\cosh c = \sinh a \sinh b \cosh\gamma - \cosh a \cosh b$ (right-angled hexagons).

3 The surface S_g

In this section we construct a surface S_g whose Teichmüller distance to X_g is at least $\frac{1}{4}\log(\log g - \log 12)$.

3.A Construction of the surface S_g when $g = 3\cdot 2^{n-1}$

To construct a surface S_g, we first construct a tree $T(n)$ with m vertices. The tree's diameter is required to be comparable with $\log m$.

We define the tree $T(n)$ by the following two properties:

(1) Every vertex, except the leaves of $T(n)$, has degree 3.

(2) There is a vertex O of $T(n)$ such that the combinatorial distance from every leaf of $T(n)$ to O is n.

The tree $T(2)$ is shown in Figure 2.

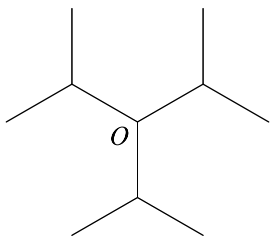


Figure 2: The tree $T(2)$.

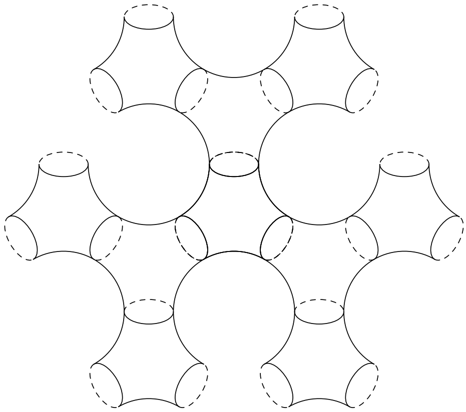


Figure 3: The sphere with $6 \cdot 2^n$ boundary components.

Now we construct the surface S_g from the tree $T(n)$. We pick several isometric pairs of pants as building blocks of S_g. Each pair consists of two regular right-angled hexagons. A boundary component of the pants is called a *cuff*, and an edge of the hexagons in the interior of the pants is called a seam. We glue the pants together according to the tree $T(n)$.

Then we glue together the pants. A vertex of $T(n)$ corresponds to a pair of pants; two pairs of pants are glued together at a cuff if there is an edge that connects the corresponding vertices. Now we get a sphere with $3 \cdot 2^n$ boundary components (Figure 3). For each pair of pants corresponding to a leaf in the tree, we glue together the two cuffs of the pair that are not glued with the other pants. Then we get a closed surface with genus g, where $g = 3 \cdot 2^{n-1}$. At each cuff, we require the gluing to have "no twist". In other words, when gluing two pairs of pants together at a cuff, endpoints of seams from one pair of pants are required to be glued with the endpoints of seams from the other; when gluing two cuffs in the same pair of pants, ends of seams from the two sides of the cuff are required to be glued together. Therefore we construct a unique hyperbolic surface, denoted by S_g.

In S_g, in each one-holed torus (glued from a pair of pants) corresponding to a leaf of the tree, there is a unique simple closed curve consisting of one seam of the pants. We denote this curve by α_k, where $k = 1, 2, \dots, g$. Now we prove that this curve is the shortest in S_g.

Lemma 3.1 *The shortest closed geodesics on S_g are exactly the curves $\alpha_1, \alpha_2, \dots, \alpha_g$, and therefore the systole of S_g is* $\operatorname{arccosh} 2$.

Proof By (2-5), the edge length of regular right-angled hexagons is $\operatorname{arccosh} 2$, and hence the cuff length of the pants is $2 \operatorname{arccosh} 2$ and the seam length is $\operatorname{arccosh} 2$. Therefore the length of α_k is the seam length of the pants, $\operatorname{arccosh} 2$. If a curve in S_g intersects at least three pairs of pants, then this curve is longer than α_k because this curve must pass through two cuffs that belong to one of the three pants.

In a pair of pants, the only simple closed geodesics are the cuffs. The cuff length of the pants is exactly twice the length of α_k.

If a curve is contained in two neighboring pairs of pants, then it intersects the two pants' shared cuff and the seams opposite the cuff. However, by (2-4), the distance between the cuff and the seam is larger than the length of α_k.

Therefore $\{\alpha_k\}_{k=1}^{g}$ is the set of shortest geodesics of S_g. $\square$

3.B Distance between S_g and X_g

The distance between a surface and X_g is estimated below by the following lemma:

Lemma 3.2 *For a surface $S \in \mathcal{M}_g$, let $L > 0$. If, for any filling curve set F in which each pair of curves intersect at most once, F contains a curve longer than L, then*

$$d_{\mathcal{T}}(S, X_g) \geq \tfrac{1}{4}\log\frac{L}{\operatorname{sys}(S)}. \tag{3-1}$$

Proof We let $S \in \mathcal{M}_g$. For any filling curve set $F \subset S$ in which each pair of curves intersects at most once, F contains a curve longer than L.

For any $S' \in X_g$, we assume $F' \subset S'$ is the set of shortest geodesics in S'. Since $S' \in X_g$, F' is filling in S'.

For any Lipschitz homeomorphisms $f : S \to S'$ and $g : S' \to S$, we let $\alpha \subset S$ be a shortest geodesic in S and $\beta \subset S'$ be a shortest geodesic with $l_{g(\beta)(S)} > L$. Then by Theorem 2.1,

$$\exp(d_L(S, S')) \geq \frac{l_{f(\alpha)}(S')}{l_\alpha(S)} \geq \frac{\operatorname{sys}(S')}{\operatorname{sys}(S)}.$$

On the other hand,

$$\exp(d_L(S', S)) \geq \frac{l_{g(\beta)}(S)}{l_\beta(S')} \geq \frac{L}{\operatorname{sys}(S')}.$$

Then, by (2-1), $d_{\mathcal{T}}(S, S') \geq \frac{1}{2} d_L(S, S')$ and $d_{\mathcal{T}}(S, S') \geq \frac{1}{2} d_L(S', S)$. For any $\operatorname{sys}(S') > 0$,

$$\max\left(\frac{\operatorname{sys}(S')}{\operatorname{sys}(S)}, \frac{L}{\operatorname{sys}(S')}\right) \geq \sqrt{\frac{L}{\operatorname{sys}(S)}}.$$

Therefore,

$$d_{\mathcal{T}}(S, S') \geq \tfrac{1}{2}\log\sqrt{\frac{L}{\operatorname{sys}(S)}} = \tfrac{1}{4}\log\frac{L}{\operatorname{sys}(S)}.$$ $\square$

Now we estimate the distance between S_g and X_g using Lemma 3.2.

We let P_k, $k = 1, \ldots, g$ be the one-holed tori corresponding to leaves of the tree $T(n)$. An observation is that $S_g \backslash \{P_k\}_{k=1}^{g}$ is a g–holed sphere.

Immediately we have:

Lemma 3.3 *In S_g, for any filling curve set F in which each pair of curves intersects at most once, any curve in F intersects at least one P_k in $\{P_k\}_{k=1}^{g}$.*

Proof If a curve does not intersect any P_k for $k = 1, 2, \dots, g$, then it is contained in the g–holed sphere $S_g \backslash \{P_k\}_{k=1}^{g}$, and hence is a separating curve. A separating curve cannot intersect any curve once. On the other hand, a curve in a filling set F always intersects other curves in F. □

Lemma 3.4 *In S_g, for any filling curve set F in which each pair of curves intersects at most once, F contains a curve β such that*

$$l_\beta(S_g) > n \operatorname{arccosh} 2,$$

where $g = 3 \cdot 2^{n-1}$.

Proof The construction of S_g gives a natural pants decomposition on S_g. A filling curve set must intersect every pair of pants in this decomposition because filling curve sets cut the surface into disks.

For the pants corresponding to the center vertex O shown in Figure 2, we let β be a curve in F passing through this pair of pants. Then by Lemma 3.3, β intersects some one-holed sphere corresponding to a leaf in the tree $T(n)$. The combinatorial distance between the vertex O and any leaf of the tree is at least n. Then by the construction of S_g, the distance between the corresponding two pairs of pants is at least $n \operatorname{arccosh} 2$, where $\operatorname{arccosh} 2$ is the length of seams of the pairs of pants used to construct S_g.

Therefore $l_\beta(S_g) > n \operatorname{arccosh} 2$. □

By Lemmas 3.4 and 3.2, immediately we have:

Proposition 3.5 *When $g = 3 \cdot 2^{n-1}$ for any positive integer n, the distance between S_g and X_g is larger than*

$$d_{\mathcal{T}}(S_g, X_g) > \tfrac{1}{4} \log n.$$

3.C Construction in general genus

We have proved Proposition 3.6 when $g = 3 \cdot 2^{n-1}$. Now we construct S_g when $3 \cdot 2^{n-1} < g < 3 \cdot 2^n$.

Take a tree T with g leaves, such that $T(n) \subset T \subset T(n+1)$. By the embedding $T(n) \to T$, we define the vertex of O in T as the image of vertex O in $T(n)$. Then in the tree T, the combinatorial distance from O to any leaf of T is larger than n.

Similarly to the construction at the beginning of this section, we can construct a genus-g surface S_g from the tree T. By Lemma 3.2, the distance between S_g and X_g is larger than $\frac{1}{4} \log n$. Since $g < 3 \cdot 2^n$, we have:

Proposition 3.6 *For any $g \geq 3$, the distance from the surface S_g with $\operatorname{sys}(S_g) = \operatorname{arccosh} 2$ to the space X_g is larger than*

$$d_{\mathcal{T}}(S_g, X_g) > \tfrac{1}{4} \log(\log g - \log 12).$$

4 Sparseness of X_g

4.A Two theorems on random surfaces

We list two theorems on random surfaces we need for the proof of Theorem 4.3.

Theorem 4.1 [Mirzakhani and Petri 2019, Theorem 2.8] *There exist $A, B > 0$ such that, for any sequence $\{c_g\}$ of positive numbers with $c_g < A \log g$, we have*

$$P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid \operatorname{sys}(S) > c_g\} < B c_g e^{-c_g}.$$

In a hyperbolic surface, the *half collar* of a simple closed geodesic γ with width w is an embedded cylinder in the surface. One of the boundary curves of the cylinder is the geodesic γ, and this cylinder consists of points with distance at most w to γ on one side of γ.

Theorem 4.2 [Nie et al. 2023, Theorems 1 and 2] *For any $\varepsilon > 0$, consider the following conditions*:

(a) *There is a simple closed curve γ in S that has a half collar with width $\frac{1}{2}\log g - \left(\frac{3}{2}+\varepsilon\right)\log\log g$.*

(b) *The length of the curve γ in* (a) *is larger than $2\log g - 5\log\log g$.*

Then

$$P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid S \text{ satisfies (a) and (b)}\} \to 1$$

as $g \to \infty$.

4.B The sparseness of X_g

Theorem 4.3 $$P_{\mathrm{WP}}\left\{S \in \mathcal{M}_g \mid d_{\mathcal{T}}(S, X_g) < \tfrac{1}{5}\log\log g\right\} \to 0 \quad \text{as } g \to \infty.$$

Proof By Theorem 4.1, if we let $c_g = \frac{1}{5}\log\log g$, then

$$P_{\mathrm{WP}}\left\{S \in \mathcal{M}_g \mid \operatorname{sys}(S) > \tfrac{1}{5}\log\log g\right\} < B\,\frac{\frac{1}{5}\log\log g}{(\log g)^{1/5}}.$$

For $S \in \mathcal{M}_g$ and $\operatorname{sys}(S) \le \frac{1}{5}\log\log g$, if S satisfies Theorem 4.2(a), then for any filling curve set F in S, F contains a curve of length at least $\log g - 2\log\log g$ since in F there must be a curve intersecting the separating curve γ in condition (a). Then by Lemma 3.2, the distance between S and X_g is bounded below by

$$\tfrac{1}{4}\log\frac{\log g - 2\log\log g}{\frac{1}{5}\log\log g} > \tfrac{1}{5}\log\log g.$$

By Theorem 4.2, $P_{\mathrm{WP}}\left\{S \in \mathcal{M}_g \mid d_{\mathcal{T}}(S, X_g) > \frac{1}{5}\log\log g\right\} \to 1$ as $g \to \infty$ and so the theorem holds. □

Recall that X_g is contained in the thick part $\mathcal{M}_g^{\ge\varepsilon}$ in $\mathcal{M}_g$. The thick part $\mathcal{M}_g^{\ge\varepsilon}$ has positive probability in $\mathcal{M}_g$ by [Mirzakhani and Petri 2019, Theorem 4.1]; immediately we have:

Corollary 4.4 $$P_{\mathrm{WP}}\left\{d_{\mathcal{T}}(S, X_g) < \tfrac{1}{5}\log\log g \mid S \text{ lies in the thick part of } \mathcal{M}_g\right\} \to 0 \quad \text{as } g \to \infty.$$

5 The Weil–Petersson distance version of Theorem 4.3

Besides the Teichmüller distance, if we consider the Weil–Petersson distance to X_g, we can prove Theorem 5.7.

5.A Lower bounds on Weil–Petersson distance

The main tools to prove Theorem 5.7 are Theorems 4.1 and 4.2, and the lower bounds on Weil–Petersson distance of Wu [2022].

Before stating Wu's result, we prepare some definitions; for details, see [Wu 2022].

We let $\mathcal{M}$ be the space of complete Riemannian metrics on the topological surface S_g with constant curvature -1. Then by the definition of Teichmüller space, $\mathcal{T}_g = \mathcal{M}/\mathrm{Diff}_0(S_g)$ where $\mathrm{Diff}_0(S_g)$ is the group of diffeomorphism of S_g isotopic to the identity. Let $\pi\colon \mathcal{M} \to \mathcal{T}_g$ be the natural projection. We recall from Rupflin and Topping [2018] that a smooth path $c(t) \subset \mathcal{M}$ is a *horizontal curve* if there exists a holomorphic quadratic differential $q(t)$ on $c(t)$ such that $\partial c(t)/\partial t = \mathrm{Re}\, q(t)$.[3]

On a surface $X \in \mathcal{M}$ for $p \in X$, we let $\mathrm{inj}_X(p)$ be the *injectivity radius* of X at p, namely the half length of shortest essential loop on X passing through p. Then we define

Definition 5.1 On a topological surface $\Sigma_g\,(g \geq 2)$, fix $p \in \Sigma_g$. For any $X, Y \in \mathcal{T}_g$, we define

$$|\sqrt{\mathrm{inj}_X(p)} - \sqrt{\mathrm{inj}_Y(p)}| := \sup_c |\sqrt{\mathrm{inj}_{c(0)}(p)} - \sqrt{\mathrm{inj}_{c(1)}(p)}|,$$

where $c\colon [0,1] \to \mathcal{M}$ runs over all smooth horizontal curves, with $\pi(c(0)) = X$, $\pi(c(1)) = Y$ and $\pi(c([0,1)) \subset \mathcal{T}_g$ the Weil–Petersson geodesic connecting X and Y.

Theorem 5.2 [Wu 2022, Theorem 1.1] *For a topological surface Σ_g with $g \geq 2$, fix a point $p \in S_g$. Then, for any $X, Y \in \mathcal{T}_g$,*

$$|\sqrt{\mathrm{inj}_X(p)} - \sqrt{\mathrm{inj}_Y(p)}| \leq 0.3884 d_{\mathrm{WP}}(X,Y),$$

where $d_{\mathrm{WP}}(X,Y)$ is the Weil–Petersson distance.

A corollary to this theorem is also needed:

Corollary 5.3 [Wu 2022, Corollary 1.2] *For $X, Y \in \mathcal{T}_g$,*

$$|\sqrt{\mathrm{sys}(X)} - \sqrt{\mathrm{sys}(Y)}| \leq 0.5492\, d_{\mathrm{WP}}(X,Y)$$

Remark 5.4 Before this corollary, the function $\sqrt{\mathrm{sys}(\cdot)}$ was proved to be uniformly Lipschitz on $\mathcal{T}_g$ endowed with the Weil–Petersson metric by Wu [2019].

5.B The theorem with respect to Weil–Petersson distance

Now we begin to prove Theorem 5.7. First, we prove the following two lemmas:

Lemma 5.5 *If $S \in \mathcal{T}_g$ satisfies Theorem 4.2*(a)–(b), *then there is a curve $\alpha \subset S$, freely homotopic to the geodesic γ in the conditions* (a) *and* (b), *such that, for any point $p \in \alpha$,*

$$\mathrm{inj}_S(p) \geq \tfrac{1}{4}\log g - \left(\tfrac{3}{4} + \tfrac{\varepsilon}{2}\right)\log\log g.$$

[3] For a hyperbolic metric $g \in \mathcal{M}$, the tangent space of $\mathcal{M}$ can be decomposed as $\{\mathrm{Re}\, q \mid q \text{ is a quadratic differential on } (S,g)\} \oplus \{\mathcal{L}_g \mid X \in \Gamma(TS)\}$. For details, see [Rupflin and Topping 2018].

Proof By conditions (a) and (b), $\gamma \subset S$ is a simple closed geodesic of length $2\log g - 5\log\log g$, having a half collar of width $\frac{1}{2}\log g - \left(\frac{3}{2}+\varepsilon\right)\log\log g$. Then let α be the curve in the half collar of γ consisting of points whose distance to γ is $\frac{1}{4}\log g - \left(\frac{3}{4}+\frac{\varepsilon}{2}\right)\log\log g$. The lemma follows immediately. $\square$

Lemma 5.6 *For any surface $S' \in X_g$, on any essential curve $\alpha' \subset S'$ there is at least one point $p' \in \alpha'$ such that*

$$\mathrm{inj}_{S'}(p') \le \tfrac{1}{2}\,\mathrm{sys}(S').$$

Proof Recall that $S' \in X_g$ means that the shortest geodesics on S' form a filling set of curves. Then any essential curve α' intersects at least one shortest closed geodesic. We pick one of the shortest geodesics that intersects α' and denote it by β'. We let p' be a point in $\alpha' \cap \beta'$. Then $\mathrm{inj}_{S'}(p') \le \frac{1}{2}\, l_{\beta'}(S') = \frac{1}{2}\,\mathrm{sys}(S')$. $\square$

Theorem 5.7 $P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid d_{\mathrm{WP}}(S, X_g) < 0.6521(\sqrt{\log g} - \sqrt{7\log\log g})\} \to 0$ *as* $g \to \infty$.

Proof By Theorem 4.1, if we let $c_g = \log\log g$, then

$$P_{\mathrm{WP}}\{S \in \mathcal{M}_g \mid \mathrm{sys}(S) > \log\log g\} < B\,\frac{\log\log g}{\log g}. \tag{5-1}$$

Let $S \in \mathcal{M}_g$ satisfy Theorem 4.2(a) and (b) and $\mathrm{sys}(S) \le \log\log g$. For any $S' \in X_g$, by Corollary 5.3,

$$0.5492\, d_{\mathrm{WP}}(S,S') \ge |\sqrt{\mathrm{sys}(S')} - \sqrt{\mathrm{sys}(S)}| \ge \sqrt{\mathrm{sys}(S')} - \sqrt{\mathrm{sys}(S)} \ge \sqrt{\mathrm{sys}(S')} - \sqrt{\log\log g}. \tag{5-2}$$

On the other hand, since S satisfies conditions (a) and (b), by Lemma 5.5 there is a curve $\alpha \subset S$ such that, for any $p \in \alpha$,

$$\mathrm{inj}_S(p) \ge \tfrac{1}{4}\log g - \left(\tfrac{3}{4}+\tfrac{\varepsilon}{2}\right)\log\log g. \tag{5-3}$$

We choose an arbitrary horizontal curve $c(t)\colon [0,1] \to \mathcal{M}_{-1}$ with $\pi(c(0)) = S$, $\pi(c(1)) = S'$ and $\pi(c([0,1))$ a Weil–Petersson geodesic connecting S and S'. Then by deforming the metric of S along $c(t)$ to the metric of S', α is also a well-defined essential simple closed curve on S'. By Lemma 5.6, there is a point $p \in \alpha \subset S'$ such that

$$\mathrm{inj}_{S'}(p) \le \tfrac{1}{2}\,\mathrm{sys}(S'). \tag{5-4}$$

Therefore, by Definition 5.1, (5-3) and (5-4),

$$\begin{aligned} 0.3884\, d_{\mathrm{WP}}(S,S') &\ge |\sqrt{\mathrm{inj}_S(p)} - \sqrt{\mathrm{inj}_{S'}(p)}| \ge \sqrt{\mathrm{inj}_S(p)} - \sqrt{\mathrm{inj}_{S'}(p)} \\ &\ge \sqrt{\tfrac{1}{4}\log g - \left(\tfrac{3}{4}+\tfrac{\varepsilon}{2}\right)\log\log g} - \sqrt{\tfrac{1}{2}\,\mathrm{sys}(S')}. \end{aligned} \tag{5-5}$$

Combining (5-2) and (5-5), then eliminating $\mathrm{sys}(S')$, we have

$$d_{\mathrm{WP}}(S,S') \ge 0.6521(\sqrt{\log g} - \sqrt{7\log\log g}).$$

Hence, for any S satisfying (a), (b) and $\mathrm{sys}(S) \le \log\log g$,

$$d_{\mathrm{WP}}(S,X_g) \ge 0.6521(\sqrt{\log g} - \sqrt{7\log\log g}).$$

On the other hand, by Theorem 4.2 and (5-1),

$$P_{\mathrm{WP}}\{S \mid S \text{ satisfies (a), (b) and } \operatorname{sys}(S) \le \log\log g\} \to 1$$

as $g \to \infty$. Therefore,

$$P_{\mathrm{WP}}\{S \mid d_{\mathrm{WP}}(S, X_g) \ge 0.6521(\sqrt{\log g} - \sqrt{7\log\log g})\} \to 1$$

as $g \to \infty$, and the theorem holds. □

6 A criterion for the critical points

This section aims to prove Proposition 6.3: the surface with maximal systole among all the surfaces admitting a specific group action must be a critical point of the systole function.

In Section 6.A, some required knowledge on the tangent space of $\mathcal{T}_g$ for the proof is provided. In Section 6.B, we prove lemmas on local properties of the subspace consisting of surfaces admitting a specific group action. At last, in Section 6.C, we prove the proposition.

6.A Tangent space of the Teichmüller space

This subsection contains some required definitions and conclusions on the tangent space of $\mathcal{T}_g$ for the proof of Proposition 6.3. One may refer to [Imayoshi and Taniguchi 1992; Wolpert 1987; Liu 2023] for details.

For $S \in \mathcal{T}_g$, let Γ be the Fuchsian group that uniformizes S; hence $S \cong \mathbb{H}^2/\Gamma$. The tangent space of $\mathcal{T}_g$ is identified with the space of harmonic Beltrami differentials with respect to Γ, denoted by $\mathrm{HB}(\mathbb{H}^2, \Gamma)$. Here $B(\mathbb{H}^2, \Gamma)$ consists of a Γ–invariant $(-1,1)$–tensor $\mu \in L^\infty(\mathbb{H}^2)$ with $|\mu| < 1$. A Γ–invariant $(-1,1)$–tensor μ satisfies that for any $\gamma \in \Gamma$,

$$\mu = (\mu \circ \gamma)\frac{\overline{\gamma'}}{\gamma'} \quad \text{almost everywhere on } \mathbb{H}^2. \tag{6-1}$$

The map H is a projection from $B(\mathbb{H}^2, \Gamma)$ to itself, depending only on the complex structure of $\mathcal{T}_g$, and $\mathrm{HB}(\mathbb{H}^2, \Gamma)$ is the image of this projection.

There is an exponential map $\Phi\colon \mathrm{HB}(\mathbb{H}^2, \gamma) \to \mathcal{T}_g$, given by associating to $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma)$ the (equivalence class of the marked) surface $\mathbb{H}^2/f^\mu\Gamma(f^\mu)^{-1}$, where f^μ is the quasiconformal map on $\mathbb{H}^2$ satisfying $f^\mu_{\bar z} = \mu f^\mu_z$ and fixing 0, 1 and ∞. Note that Φ is a holomorphic homeomorphism; see [Wolpert 1987].

6.B Symmetric surfaces

For genus-g surface S_g, we assume G is a finite subgroup of $\mathrm{MCG}(S_g)$, and ρ is a marked hyperbolic structure on S_g such that $\Sigma_g = (S_g, \rho) \in \mathcal{T}_g$. Then we define $X_g^G \subset \mathcal{T}_g$, the hyperbolic surfaces admitting a G action:

$$X_g^G = \{\Sigma_g = (S_g, \rho) \in \mathcal{T}_g \mid G \le \mathrm{Aut}(\Sigma_g)\}.$$

Here $\mathrm{Aut}(\Sigma_g)$ is the automorphism group of the hyperbolic surface Σ_g.

The following lemma says that the set of G–invariant tangent vectors at $S \in X_g^G$ is $\mathrm{HB}(\mathbb{H}^2, \Gamma')$ for the Fuchsian group Γ' that uniformizes the orbifold S/G.

Lemma 6.1 *For $S \in X_g^g$, we let S be uniformized by the Fuchsian group Γ, and the orbifold S/G be uniformized by a Fuchsian group denoted by Γ'. Hence $\Gamma \trianglelefteq \Gamma'$ and $G \cong \Gamma'/\Gamma$. Then $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma)$ is a G–invariant tangent vector to $\mathcal{T}_g$ if and only if $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma')$.*

Proof For $g \in \mathrm{Aut}(S)$, since $\mathrm{HB}(\mathbb{H}^2, \Gamma)$ consists of $(-1,1)$–tensors we know g acts on $\mathrm{HB}(\mathbb{H}^2, \Gamma)$ by

$$g_*(\mu) = (\mu \circ \tilde{g}^{-1}) \frac{\overline{(\tilde{g}^{-1})'}}{(\tilde{g}^{-1})'}, \tag{6-2}$$

where $\tilde{g}$ is a lift of g onto $\mathbb{H}^2$.

Since a lift of g is contained in Γ' and $G \cong \Gamma'/\Gamma$, by (6-2), $\mu = g_*(\mu)$ is equivalent to $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma')$. □

For the exponential map Φ, we have:

Lemma 6.2 *For the G–invariant tangent vector $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma')$, $\Phi(\mu) \in X_g^G$.*

Proof The group G, as a subgroup of the mapping class group MCG_g, acts on $\mathcal{T}_g$. To prove $\Phi(\mu) \in X_g^G$ is to prove $\Phi(\mu)$ is a fixed point of this action.

For $g \in G$ and $\Phi(\mu) = \mathbb{H}^2/f^\mu \Gamma (f^\mu)^{-1}$, g acts on $\Phi(\mu)$ by

$$\mathbb{H}^2/f^\mu \Gamma (f^\mu)^{-1} \mapsto \mathbb{H}^2/(\tilde{g})^{-1} f^\mu \Gamma (f^\mu)^{-1} \tilde{g},$$

where $\tilde{g}$ is a lift of g onto $\mathbb{H}^2$.

By the definition of f^μ, $f^\mu \circ (\tilde{g})^{-1} = f^\mu$ if and only if $\mu = (\mu \circ \tilde{g}^{-1}) \overline{(\tilde{g}^{-1})'}/(\tilde{g}^{-1})'$; namely, $\mu = g_*(\mu)$. Therefore, $\Phi(\mu)$ is G–invariant if μ is G–invariant. □

6.C The criterion

Proposition 6.3 *If $R \in X_g^G$ realizes the maximum of the systole function on X_g^G, namely*

$$\mathrm{sys}\, R \geq \mathrm{sys}\, S \quad \text{for all } S \in X_g^G,$$

then R is a critical point of the systole function in $\mathcal{T}_g$.

Proof We assume that R realizes the maximum of sys on X_g^G, $S(R)$ is the set of systoles of R, R is uniformized by the Fuchsian group Γ, and the orbifold R/G is uniformized by the Fuchsian group Γ'.

For $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma)$, if for any $\alpha \in S(R)$ we have $\mathrm{d}l_\alpha(\mu) \geq 0$, we consider $\nu = \sum_{g \in G} g_* \mu$; then by [Fortier Bourque 2020, (6.1)],

$$\mathrm{d}l_\alpha(\nu) = \mathrm{d}l_\alpha\Bigl(\sum_{g \in G} g_*\mu\Bigr) = \sum_{g \in G} \mathrm{d}l_\alpha(g_*\mu) = \sum_{g \in G} \mathrm{d}l_{g(\alpha)}(\mu) \geq \mathrm{d}l_\alpha(\mu) \geq 0. \tag{6-3}$$

The vector $\nu = \sum_{g \in G} g_* \mu$ is in $\mathrm{HB}(\mathbb{H}^2, \Gamma')$. We let ε_0 be a small positive number and consider $U = \{\nu \mid \nu \in \mathrm{HB}(\mathbb{H}^2, \Gamma') \text{ and } \|\nu\|_\infty < \varepsilon_0\}$. Since U is an open neighborhood of 0 in $\mathrm{HB}(\mathbb{H}^2, \Gamma')$, $\Phi(U)$ is an open neighborhood of R in X_g^G. If ε_0 is small enough, then for any $S \in \Phi(U)$ there is at least one curve $\alpha \in S(R)$ such that α is a systole of S. The Hessian of $l_\alpha|_{\Phi(U)}$ is positive definite since the Hessian of l_α is positive definite. Then by Theorem 2.3, $\mathrm{sys}|_{\Phi(U)}$ is a topological Morse function.

Since R realizes the maximum of $\mathrm{sys}|_{X_g^G}$, R realizes the maximum of $\mathrm{sys}|_{\Phi(U)}$ and R is a critical point of $\mathrm{sys}|_{\Phi(U)}$. $\mathrm{HB}(\mathbb{H}^2, \Gamma')$ is the tangent space of $\Phi(U)$ at the basepoint. By Definition 2.5, for $\nu \in \mathrm{HB}(\mathbb{H}^2, \Gamma')$, if $\mathrm{d}l_\alpha(\nu) \geq 0$ for all $\alpha \in S(R)$, then $\mathrm{d}l_\alpha(\nu) = 0$ for all $\alpha \in S(R)$.

Therefore by (6-3), for $\mu \in \mathrm{HB}(\mathbb{H}^2, \Gamma)$, if $\mathrm{d}l_\alpha(\mu) \geq 0$ for all $\alpha \in S(R)$, then $\mathrm{d}l_\alpha(\mu) = 0$ for all $\alpha \in S(R)$. By Definition 2.5 R is a eutactic surface, and therefore a critical point of the systole function. □

7 Small distance

7.A Construction of S_g^1 and S_g^2

The surface S_g^1 was initially constructed in [Anderson et al. 2011], while S_g^2 was initially constructed in [Gao and Wang 2023]. We briefly construct these two surfaces for completeness, which implies how to obtain the Teichmüller distance between the two surfaces.

We first construct a family of genus-g hyperbolic surfaces denoted by $\{S_g(c,t)\}$; each surface in this family is determined by two parameters, c and t for $c > 0$ and $0 \leq t \leq \frac{1}{2}c$. The example S_g^1 is a $S_g(c_1, 0)$–surface for some $c_1 > 0$, while the example S_g^2 is a $S_g(c_2, t_2)$–surface for some $c_2, t_2 > 0$.

Let $n \geq 3$ and pick two isometric right-angled hyperbolic polygons with $2n$ edges admitting an order-n rotation. Two such polygons can be glued to an n–holed sphere admitting the order-n rotation extended from the polygons. By this rotation, all boundary curves of this n–holed sphere have equal length. The geometry of the n–holed sphere is determined by its boundary curves' length (denoted by c), and we denote the corresponding n–holed sphere by $S(c)$. We call the boundary curves of $S(c)$ *cuffs* and the edges of the polygons contained in the interior of $S(c)$ *seams*. By rotational symmetry, all seams also have equal length.

We pick two isometric n–holed spheres and glue them along their cuffs, getting a closed surface. As shown in Figure 4, when gluing the two n–holed spheres, we require that every cuff of one of the n–holed spheres is identified with a cuff in the other n–holed sphere, and every seam of one n–holed sphere is half of a closed curve (denoted by α_k for $k = 1, 2, \ldots, n$) while the other half of α_k is a seam in the other n–holed sphere. This constructed surface has genus $g = n - 1$, and the geometry of this closed surface is determined by the cuff length c. We denote this surface by $S_g(c, 0)$.

For $t > 0$, the surface $S_g(c,t)$ is constructed from $S_g(c,0)$ by conducting a Fenchel–Nielsen deformation of length t simultaneously along each cuff γ_k. Here a *Fenchel–Nielsen deformation* on $X \in \mathcal{T}_g$ along a simple closed geodesic $\alpha \subset X$ with length t is constructed by cutting X along α and then regluing the boundary curves with a left twist of length t.

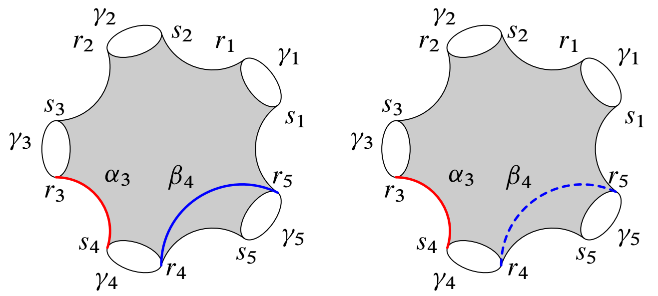


Figure 4

There is a $c_1 > 0$ such that on the surface $S_g(c_1, 0)$, $l(\alpha_k) = l(\gamma_k)$. This surface is the surface S_g^1. The shortest geodesics of S_g^1 consist of α_k and γ_k for $k = 1, 2, \dots, g+1$ by the proof of [Anderson et al. 2011, Theorem 3].

In a surface $S_g(c,t)$, we let β_k be the image of α_k by a Dehn twist along γ_{k+1} (Figure 4). The orientation of this Dehn twist is required to be opposite to the Fenchel–Nielsen deformation.

There is a pair (c_2, t_2) such that on the surface $S_g(c_2, t_2)$, $l(\alpha_k) = l(\beta_k) = l(\gamma_k)$. This surface is the surface S_g^2. The shortest geodesics of S_g^2 consist of α_k, β_k and γ_k for $k = 1, 2, \dots, g+1$ by [Gao and Wang 2023, Proposition 4].

7.B Symmetry on $S(c,t)$

We consider a group G acting isometrically on $S_g(c,t)$, generated by three elements, σ, τ and ς. Here σ is the rotation of order n, τ is the order-2 rotation that exchanges the two n–holed spheres, and ς is the order-2 rotation that is invariant on each n–holed sphere and when restricted to one of the two n–holed spheres exchanges the two $2n$–gons.

On the surface $S_g(c,0)$, there is a reflection ρ extended from the reflection on one of the n–holed spheres exchanging the two polygons of the n–holed sphere. The symmetric group generated by σ, τ, ς and ρ is denoted by $\overline{G}$.

Remark 7.1 A reflection on the n–holed sphere can be extended to the whole surface $S_g(c,t)$ only if $t = 0$ or $t = \frac{1}{2}c$.

The reflection on $S_g\left(c, \frac{1}{2}c\right)$, denoted by $\rho_{\frac{1}{2}c}$, is not conjugate to ρ. This is because their fixed-point sets are different. The fixed points of ρ on $S_g(c,0)$ consist of $g+1$ curves (the β_k curves), while fixed points of $\rho_{\frac{1}{2}c}$ consist of one curve (when g is even) or two curves (when g is odd).

The surface S_g^1 has been proved to be a critical point of the systole function; see [Fortier Bourque 2020, Example 4.2 and Proposition 6.3].

On the other hand, it is proved in [Gao and Wang 2023] that the surface S_g^2 is the surface with the maximal systole among the surfaces admitting the action of G. Then immediately by Proposition 6.3, S_g^2 is a critical point of the systole function.

Hence we have:

Proposition 7.2 *The surfaces S_g^1 and S_g^2 are critical points of the systole function.*

7.C Distance

This subsection aims to bound the Teichmüller distance between S_g^1 and S_g^2.

Recall the parameter of the surfaces $S_g^1 = S_g(c_1, 0)$ and $S_g^2 = S_g(c_2, t_2)$. To get an upper bound of $d_{\mathcal{T}}(S_g^1, S_g^2)$, we need an intermediate surface $S_g(c_2, 0)$. Distance between S_g^1 and S_g^2 is bounded from above by the sum of $d_{\mathcal{T}}(S_g^1, S(c_2, 0))$ and $d_{\mathcal{T}}(S(c_2, 0), S_g^2)$.

7.C.1 Quadratic differential and Teichmüller geodesics Before the calculation, we need some preparations; for details, see [Masur 2009].

For a quasiconformal map $f: X \to Y$ for $X, Y \in \mathcal{T}_g$, the $(-1,1)$–tensor $\mu_f(z) = f_{\bar{z}}/f_z$ is called the *Beltrami differential* of f, where z is a local coordinate of X. We let

$$K(f) = \sup_{z \in X} \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}.$$

Here μ_f is the complex dilatation of f defined in the last subsection.

The Teichmüller distance on $\mathcal{T}_g$ is defined to be

$$d_{\mathcal{T}}(X) = \frac{1}{2} \inf_{f \sim \mathrm{id}} \{\log K(f) \mid f: X \to Y\}.$$

A Teichmüller geodesic ray with respect to Teichmüller distance from $X \in \mathcal{T}_g$ can be induced from a holomorphic quadratic differential q on X. A *holomorphic quadratic differential* is a tensor locally written as $\psi(z)\mathrm{d}z^2$, where $\psi(z)$ is a holomorphic function. We denote the space of quadratic differentials on X by $\mathrm{QD}(X)$. The bundle of quadratic differentials over $\mathcal{T}_g$ is denoted by QD_g.

For $X \in \mathcal{T}_g$ and $q \in \mathrm{QD}(X)$, for any $0 < k < 1$, $\mu_k = k\bar{q}/q$ is a Beltrami coefficient on X. We let f_k be the quasiconformal map induced by μ_k, $f_k: X \to X^{(k)}$. Then f_k is the Teichmüller map from X to $X^{(k)}$, and the Teichmüller geodesic ray induced by (X, q) consists of all the $X^{(k)}$ for all $k \in (0, 1)$.

A nonzero $q \in \mathrm{QD}(X)$ has a canonical coordinate. In this coordinate, q can be locally written as $\mathrm{d}z^2$ in the neighborhood of any nonzero point of q, and q has only finitely many zero points.

The quadratic differential q determines a pair of transverse measured foliations on X, called *horizontal and vertical foliations* for q and denoted by $F_h(q)$ and $F_v(q)$, respectively. In the canonical coordinate of q, the leaves of $F_h(q)$ are given by $y = \text{const}$ and the leaves of $F_v(q)$ are given by $x = \text{const}$. Here $z = x + iy$ is the coordinate. The measures of $F_h(q)$ and $F_v(q)$ are given by $|\mathrm{d}y|$ and $|\mathrm{d}x|$, respectively.

For X_t on the geodesic induced by (X, q) with $d_{\mathcal{T}}(X, X_t) = t$, there is a quadratic differential $q_t \in \mathrm{QD}(X_t)$ as the pushforward of q by f_t. We let $z = x + iy$ be the canonical coordinate of (X, q) and $w = u + iv$ be the canonical coordinate of (X, q). Then

$$u = e^t x \quad \text{and} \quad v = e^{-t} y. \tag{7-1}$$

7.C.2 Extremal length and the Jenkins–Strebel differential A quadratic differential $q \in \mathrm{QD}(X)$ is called a *Jenkins–Strebel differential* if any leaf of $F_h(q)$ and $F_v(q)$ is a simple closed curve, except finitely many leaves that connect zeros of q.

For a Jenkins–Strebel differential $q \in \mathrm{QD}(X)$ and a simple closed leaf α of $F_h(q)$, all simple closed leaves of $F_h(q)$ parallel to α form a cylinder in X. This cylinder is called the *characteristic ring domain* of α and, with respect to the metric $|q|$, is isometric to a Euclidean cylinder

$$R = [0,a] \times (0,b)/((0,t) \sim (a,t), 0 < t < b).$$

We call a the *length* of R and b the *height* of R.

We need the following theorem on the Jenkins–Strebel differential:

Theorem 7.3 [Strebel 1984, Theorem 21.1] *Let $(\gamma_1, \dots, \gamma_p)$ be a finite pairwise-disjoint essential curve system in $X \in \mathcal{T}_g$. For each γ_i, there is a regular neighborhood R'_i of γ_i in X and $R'_1, \dots, R'_p$ are pairwise disjoint. Then for any $(b_1, \dots, b_p) \in \mathbb{R}_+^p$, there is a unique Jenkins–Strebel differential $q \in \mathrm{QD}(X)$ such that*:

- *γ_i is a leaf of $F_h(q)$ and any simple closed leaf of $F_h(q)$ is freely homotopic to a γ_i. Here $i = 1, 2, \dots, p$.*
- *The height of the characteristic ring domain of γ_i is b_i.*

The definition of the *extremal length* of an essential curve α in a Riemann surface X is given by

$$\mathrm{Ext}_\alpha(X) = \sup_\rho \frac{l_\alpha(\rho)^2}{\mathrm{Area}(X,\rho)}.$$

Here the supremum is taken over all metrics ρ conformal to the metric on X, $l_\alpha(\rho)$ is the length of α in the metric ρ and $\mathrm{Area}(X,\rho)$ is the area of X in the metric ρ.

For a Euclidean cylinder with length a and height b, the extremal length of its core curve in the cylinder is a/b; see for example [Ahlfors 1966].

Distance between points on a Teichmüller geodesic can be expressed by extremal lengths of horizontal foliation leaves in their characteristic ring domains. For a Jenkins–Strebel differential $q \in \mathrm{QD}(X)$, we let α be a simple closed leaf of $F_h(q)$ and R be the characteristic ring domain of α with length a and height b. For X_t on the Teichmüller geodesic induced by (X,q) with $d_{\mathcal{T}}(X,X_t) = t$, the characteristic ring $R_t \subset X_t$ corresponding to $R \subset X$ has length $e^t a$ and height $e^{-t} b$. Hence for the simple closed curve α_t corresponding to α, $\mathrm{Ext}_{\alpha_t}(R_t) = e^{2t} a/b$ and

$$d_{\mathcal{T}}(X,X_t) = \frac{1}{2}\left|\log \frac{\mathrm{Ext}_{\alpha_t}(R_t)}{\mathrm{Ext}_\alpha(R)}\right|. \tag{7-2}$$

The last necessary tool for estimating the distance is the comparison between hyperbolic length and extremal length by Maskit.

For a simple closed geodesic α in a hyperbolic surface X, the *collar* of α with width w is an embedded cylinder in X consisting of points with distance at most w to α.

Theorem 7.4 [Maskit 1985] *In hyperbolic surface X, if a simple closed geodesic α has collar C with width* $\operatorname{arccosh}(1/\cos\theta)$ *then*

$$\frac{1}{\pi}\, l_\alpha(X) \le \operatorname{Ext}_\alpha(X) \le \operatorname{Ext}_\alpha(C) \le \frac{1}{2\theta}\, l_\alpha(X). \tag{7-3}$$

7.C.3 The distance between $S_g^1 = S(c_1,0)$ and $S(c_2,0)$ We estimate this distance in two steps:

(1) Prove $\{S(c,0) \mid c > 0\}$ is a Teichmüller geodesic induced by a Jenkins–Strebel differential on some surface $S(c,0)$.

(2) Estimate distance between two points by (7-2) and (7-3).

For $c > 0$, on the surface $S(c,0)$ we consider the cuffs of the n–holed spheres in $S(c,0)$, namely $\{\gamma_k\}_{k=1}^{g+1}$, and assign to each γ_k a positive number b. Then by Theorem 7.3, $\{(\gamma_k, b)\}_{k=1}^{g+1}$ induces a quadratic differential q on $S(c,0)$.

Lemma 7.5 *The quadratic differential $q \in \mathrm{QD}(S_g(c,0))$ is invariant under the action of $\overline{G}$.*

Proof For $g \in \overline{G}$, the quadratic form g^*q is induced by the set $\{(g^{-1}(\gamma_k), b)\}_{k=1}^{g+1}$. By the action of $\overline{G}$ on $S_g(c,0)$, $\{(g^{-1}(\gamma_k), b)\}_{k=1}^{g+1} = \{(\gamma_k, b)\}_{k=1}^{g+1}$. Therefore $g^*q = q$ and q is invariant. □

We consider the Teichmüller geodesic induced by $(S(c,0), q)$.

Lemma 7.6 *We write the Teichmüller geodesic induced by $(S(c,0), q)$ as l. Then the Teichmüller geodesic l coincides with the curve $\{S_g(c,0) \mid c > 0\}$.*

Proof Since q is $\overline{G}$–invariant by Lemma 7.5, for any surface $S' \in l$ the Beltrami coefficient of the Teichmüller map $f\colon S(c,0) \to S'$ is $t\bar{q}/q$ for some $t \in (0,1)$. Hence this Beltrami coefficient is $\overline{G}$–invariant. Then, by Lemma 6.2, $\overline{G}$ isometrically acts on S' by

$$f \circ g \circ f^{-1}\colon S' \to S'$$

for any $g \in \overline{G}$.

Consider the set of cuffs of the n–holed spheres on $S_g(c,0)$, denoted by $\{\gamma_k\}_{k=1}^{g+1}$. Its image $\{f(\gamma_k)\}_{k=1}^{g+1}$ in S' cuts S' into two n–holed spheres. Then $\overline{G}$ isometrically acts on these two n–holed spheres as $\overline{G}$ acts on the two n–holed spheres in $S(c,0)$. Hence S' is a $S(c',0)$–surface, where c' is the length of $f(\gamma_k)$ on S'.

Therefore the Teichmüller geodesic l is contained in the curve $\{S_g(c,0) \mid c > 0\}$. Then by the completeness of Teichmüller geodesics, $\{S_g(c,0) \mid c > 0\}$ coincides with l. □

Now we are ready to estimate:

Proposition 7.7 *For $S_g^1 = S_g(c_1,0)$ and $S_g(c_2,0)$, we have*

$$d_{\mathcal{T}}(S_g^1, S_g(c_2,0)) \le 0.65.$$

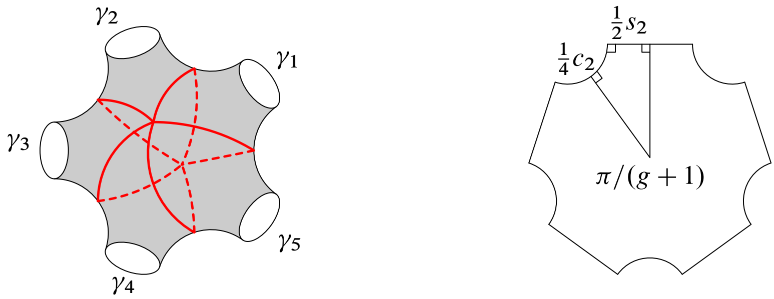


Figure 5: Left: characteristic ring domains. Right: calculate $\frac{1}{2}s_2$.

Proof Recall that c_1 and c_2 are the systoles of S_g^1 and S_g^2, respectively. Then by [Anderson et al. 2011] $c_1 = 4\operatorname{arcsinh}\sqrt{\cos(\pi/(g+1))}$, and c_2 is given by the formula in [Gao and Wang 2023, Theorem 1]. Then we use the following lemma to get the Teichmüller distance:

Lemma 7.8 *The Teichmüller distance between the hyperbolic surfaces $S_g(c_1,0)$ and $S_g(c_2,0)$ with $c_1 < c_2$ is bounded above by*

$$\tfrac{1}{2}\log\frac{\pi c_2}{2\theta c_1},$$

where

$$\cos\theta = \left(1 + \frac{\cos^2(\pi/(g+1))}{\sinh^2(c_2/4)}\right)^{-\frac{1}{2}}.$$

Proof For $i = 1, 2$, we let $\{\gamma_k^{(i)}\}_{k=1}^{g+1}$ be the cuffs in $S_g(c_i,0)$, $q_i \in \mathrm{QD}(S_g(c_i,0))$ be the quadratic differential induced by $\{(\gamma_k^{(i)}, b)\}_{k=1}^{g+1}$ for some $b > 0$, and $R_k^{(i)}$ be the characteristic ring domain of $\gamma_k^{(i)}$. Then, by Theorem 7.4,

$$\mathrm{Ext}_{\gamma_k^{(1)}}(R_k^{(1)}) \geq \mathrm{Ext}_{\gamma_k^{(1)}}(S_g(c_1,0)) \geq \frac{l(\gamma_k^{(1)})}{\pi} = \frac{c_1}{\pi}. \tag{7-4}$$

The set of characteristic ring domains $\{R_k^{(2)}\}_{k=1}^{g+1}$ is invariant under the $\overline{G}$–action. Then by the symmetry of $\overline{G}$, in $S_g(c_2,0)$ the ring domains $R_k^{(2)}$ for $k = 1, \dots, g+1$ are bounded by the hyperbolic geodesics connecting a center of the $2n$–gons and a middle point of the seams (Figure 5, left); otherwise, $\{R_k^{(2)}\}_{k=1}^{g+1}$ is not $\overline{G}$–invariant.

Therefore, if the seam length of n–holed spheres of $S_g(c_2,0)$ is s_2, then the collar C_k of $\gamma_k^{(2)}$ with width s_2/s is contained in the characteristic ring domain $R_k^{(2)}$.

The seam length s_2 is given by the trirectangle formula (2-3):

$$\sinh\left(\tfrac{1}{2}s_2\right)\sinh\left(\tfrac{1}{4}c_2\right) = \cos\frac{\pi}{g+1}. \tag{7-5}$$

See Figure 5, right. Therefore, by Theorem 7.4,

$$\mathrm{Ext}_{\gamma_k^{(2)}}(R_k^{(2)}) \leq \mathrm{Ext}_{\gamma_k^{(2)}}(C_k) \leq \frac{l(\gamma_k^{(2)})}{2\arccos\left(1/\cosh\left(\frac{1}{2}s_2\right)\right)} = \frac{c_2}{2\arccos\left(1/\cosh\left(\frac{1}{2}s_2\right)\right)}. \tag{7-6}$$

By combining (7-2), (7-4), (7-6) and (7-5), this lemma holds. □

Proposition 7.7 follows immediately by Lemma 7.8. □

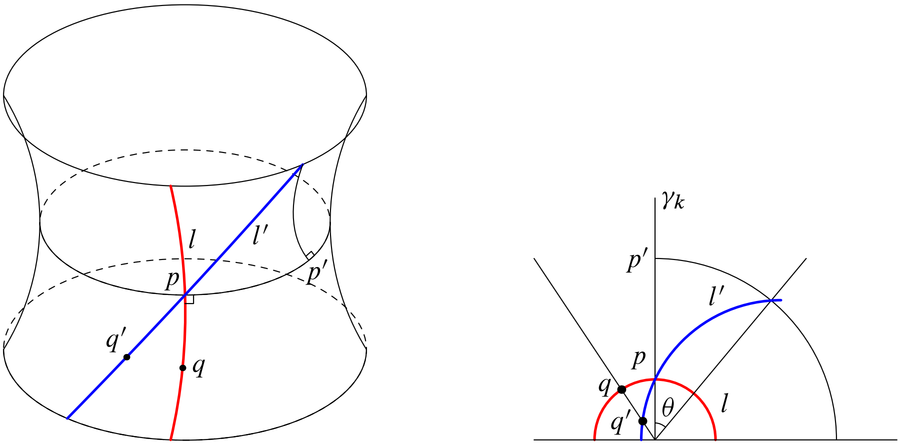


Figure 6: Left: the homeomorphism $h|_{C_k}$. Right: the lift of C_k to $\mathbb{H}^2$.

7.C.4 The distance between $S_g(c_2,0)$ and $S_g^2 = S_g(c_2,t_2)$ Recall that $S_g(c_2,t_2)$ is obtained from $S_g(c_2,0)$ by a Fenchel–Nielsen deformation along the cuffs $\{\gamma_k\}_{k=1}^{g+1}$ in $S(c_2,0)$ with time t_2. For the collar C_k of γ_k, we construct a homeomorphism $h\colon S_g(c_2,0) \to S_g(c_2,t_2)$ such that h is an isometry outside all these collars. Hence the dilatation $K(h)$ is reduced to the dilation restricted to a collar $K(h|_{C_k})$, and the Teichmüller distance between the two surfaces is bounded from above by $\frac{1}{2}\log K(h|_{C_k})$.

Proposition 7.9 *For Σ_g^2 and $\Sigma_g^{1,2}$, we have*

$$d_{\mathcal{T}}(S_g^2, S_g(c_2,0)) \le 1.6450.$$

Proof We proceed by constructing the homeomorphism h and calculating its dilatation on the largest collar of γ_k.

We let C_k be the collar of γ_k with the width $\frac{1}{2}s_2$, where s_2 is the seam length of the n–holed spheres as in Lemma 7.8. The homeomorphism h on C_k is described in Figure 6, left. A geodesic l orthogonal to the core curve γ_k is always mapped to a geodesic l'. The line l is required to intersect l' at a point p on γ_k. The projection of one of the endpoints of l' (denoted by p') is required to have distance $\frac{1}{2}t_2$ to p.

We let h outside the collars be an isometry on this surface of $S_g(c_2,0)$; then the homeomorphism h maps $S_g(c_2,0)$ to $S_g(c_2,t_2)$ by the construction on the collars.

The rest of the proof consists of the calculation of $K(h)$ on the collar C_k. To calculate this dilatation, we lift C_k on the upper half-plane $\mathbb{H}^2$ (Figure 6, right).

We lift γ_k to the y–axis, assuming $p = i$ and $p' = i\,e^{t_2/2}$. The collar of γ_k with width $\frac{1}{2}s_2$ is lifted to a strip $\{re^{i\varphi} \in \mathbb{H}^2 \mid -\theta + \frac{1}{2}\pi < \varphi < \theta + \frac{1}{2}\pi\}$, where

$$\cos\theta = \frac{1}{\cosh\frac{1}{2}s_2}. \tag{7-7}$$

In this strip, l is the unit circle, and l' is the geodesic connecting i and $\exp\bigl(\frac{1}{2}t_2 + i\sin\theta\bigr)$.

The homeomorphism h can be expressed in the form

$$h(re^{i\varphi}) = r\Phi(\varphi)e^{i\varphi}.$$

When $r = 1$, h maps l to l' in Figure 6, right. By this requirement, we can calculate that

$$\Phi(\varphi) = \sinh\left(\tfrac{1}{2}t_2\right)\frac{\cos\varphi}{\sin\theta} + \sqrt{\sinh^2\left(\tfrac{1}{2}t_2\right)\frac{\cos^2\varphi}{\sin^2\theta} + 1}. \tag{7-8}$$

The dilatation $K(h)$ is given by

$$K(h) = \frac{|h_z| + |h_{\bar z}|}{|h_z| - |h_{\bar z}|} = \frac{\sqrt{\Phi^2 + \frac{1}{4}\Phi'^2} + \frac{1}{2}|\Phi'|}{\sqrt{\Phi^2 + \frac{1}{4}\Phi'^2} - \frac{1}{2}|\Phi'|}. \tag{7-9}$$

Here $z = re^{i\varphi}$ and $\bar z = re^{-i\varphi}$.

Combining (7-9), (7-8), (7-7), (7-5) and the formula for (c_2, t_2) in [Gao and Wang 2023, Theorem 1], we obtain $d_{\mathcal{T}}(S_g(c_2, 0), S_g(c_2, t_2)) \le \frac{1}{2}\log K(h) \le 1.6450$. $\square$

Hence by Propositions 7.7 and 7.9, we have:

Theorem 7.10 *For any $g \ge 2$,*

$$d_{\mathcal{T}}(S_g^1, S_g^2) \le 2.3.$$

8 Large distance

8.A The S_g^3 surface

We take the $X(\Gamma)$–surface in [Fortier Bourque and Rafi 2022] when $n = 2$ as the surface S_g^3. We briefly describe this surface for completeness.

We consider the four-holed sphere admitting the order-4 rotation. We pick infinitely many copies of the four-holed sphere $\{P_k\}_{k=-\infty}^{+\infty}$ and glue them together into a surface S_∞ with infinite genus, as shown in Figure 7.

The surface S_∞ admits an isometric action $\psi\colon S_\infty \to S_\infty$ which takes every P_k to P_{k+1}. The surface S_g^3 is the quotient $S_\infty/\langle\psi^{g-1}\rangle$. When $g \ge 13$, S_g^3 is a local maximal point of the systole function.

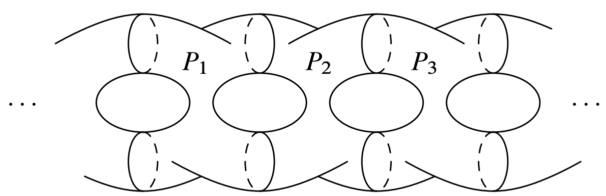


Figure 7: The surface S_∞.

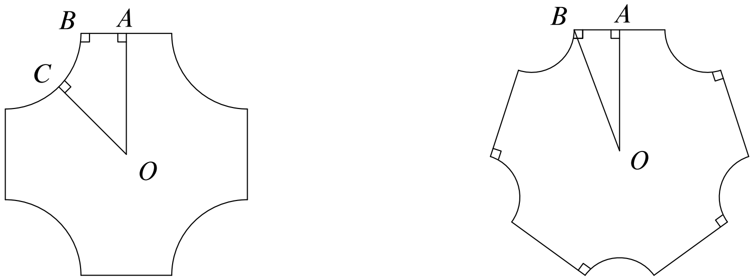


Figure 8: Left: the right-angled octagon. Right: the polygon Q.

8.B The distance between S_g^1 and S_g^3

This distance is obtained from diameter comparison. The diameter of S_g^3 is comparable with g while the diameter of S_g^1 is comparable with $\log g$. Then the distance between these two surfaces is comparable with $\log g$ by the method in the proof of [Rafi and Tao 2013, Lemma 5.1].

Proposition 8.1 *For the diameter of the surface S_g^3, we have*

$$\operatorname{diam}(S_g^3) \geq 0.6\left\lfloor \tfrac{1}{2}(g-5)\right\rfloor.$$

Proof By the construction, the surface S_g^3 consists of $g-1$ four-holed spheres, P_k for $k = 1, 2, \dots, g-1$. When $g \geq 5$, for any $x \in P_k$ and $y \in P_{k+2}$ for some k, a curve connecting x and y must pass through at least one of the four-holed spheres other than P_k or P_{k+2}. Without loss of generality, we assume this curve passes through P_{k+1}; then this curve, if given an orientation, enters P_{k+1} at one cuff and leaves P_{k+1} at another cuff. Therefore, $d(x, y)$ is bounded from below by the distance between neighboring cuffs of P_{k+1}. We denote this distance by d. Then inductively, when $k \leq \frac{1}{2}(g-1)$, distance between $x \in P_1$ and $y \in P_k$ is at least $d\left\lfloor \frac{1}{2}(g-1)-2\right\rfloor$. Hence

$$\operatorname{diam}(S_g^3) \geq d\left\lfloor \tfrac{1}{2}(g-1)-2\right\rfloor.$$

The rest of this proof is to calculate d. The distance d is the seam length of the four-holed spheres. The seam length d is determined by the cuff length (denoted by c) of the four-holed sphere by (8-1). In Figure 8, left, one of the two octagons forming the four-holed sphere, we have

$$\sinh|AB|\sinh|BC| = \cos\angle O, \quad \text{which gives} \quad \sinh\left(\tfrac{1}{4}c\right)\sinh\left(\tfrac{1}{2}d\right) = \cos\left(\tfrac{1}{4}\pi\right). \tag{8-1}$$

According to [Fortier Bourque and Rafi 2022, Lemma 2.5], the cuff length of the four-holed spheres is approximately 6.980. Then by (8-1), this proposition holds. □

For the surface S_g^1, we have:

Proposition 8.2 *The diameter of the surface S_g^1 satisfies*

$$\operatorname{diam}(S_g^1) < 4\log\left(\frac{4g+4}{\pi}\right).$$

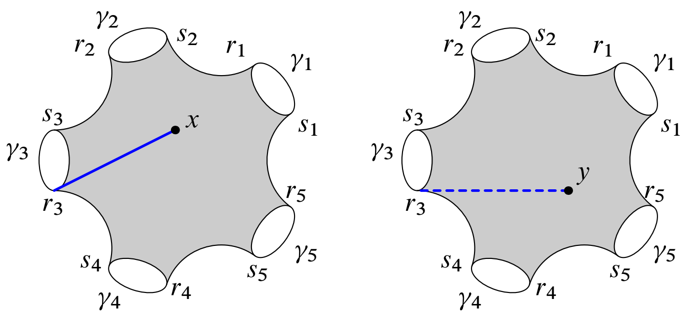


Figure 9: The path between x and y.

Proof Recall that the surface S_g^1 consists of two $(g+1)$–holed spheres, and each of the $(g+1)$–holed spheres consists of two right-angled regular $(2g+2)$–gons. For any $x, y \in S_g^1$, for the two (possibly coinciding) regular $(2g+2)$–gons containing x and y, there is a curve connecting x and y, contained in the union of these two polygons (see Figure 9). Therefore, if we denote one of the four regular $(2g+2)$–gons by Q,

$$\operatorname{diam}(S_g^1) \le 2\operatorname{diam}(Q),$$

The diameter of Q is realized by $2|OB|$ in Figure 8, right. In the triangle $\triangle OAB$, by (2-2),

$$\cosh|OB| = \cot\angle O \cot\angle B,$$

and so

$$\cosh|OB| = \cot\left(\tfrac{1}{4}\pi\right)\cot\frac{\pi}{2g+2} = \cot\frac{\pi}{2g+2} < \frac{2g+2}{\pi}.$$

Therefore,

$$\operatorname{diam}(S_g^1) \le 2\operatorname{diam}(Q) \le 4|OB| < 4\operatorname{arccosh}\left(\frac{2g+2}{\pi}\right) < 4\log\left(\frac{4g+4}{\pi}\right). \qquad \square$$

Theorem 8.3 *When* $g \ge 13$,

$$d_{\mathcal{T}}(S_g^1, S_g^3) > \tfrac{1}{2}\log(g-6) - \tfrac{1}{2}\log\left(\tfrac{40}{3}\log\left(\frac{4g+4}{\pi}\right)\right).$$

Proof The proof here is similar to the proof of [Rafi and Tao 2013, Lemma 5.1].

We let $f\colon S_g^1 \to S_g^3$ be a Lipschitz homeomorphism with $L(f) = d_L(S_g^1, S_g^3)$. (The existence of this homeomorphism is verified in [Thurston 1986a].) By Proposition 8.1, we pick $x, y \in S_g^3$ with $d(x,y) \ge 0.6\left\lfloor \frac{1}{2}(g-5)\right\rfloor$. By Proposition 8.2, $d(f^{-1}(x), f^{-1}(y)) < 4\log((4g+4)/\pi)$. Then

$$L(f) \ge \frac{d(x,y)}{d(f^{-1}(x), f^{-1}(y))} > \frac{0.6\left\lfloor \frac{1}{2}(g-5)\right\rfloor}{4\log((4g+4)/\pi)} > \frac{3(g-6)}{40\log((4g+4)/\pi)}.$$

Hence,

$$d_L(S_g^1, S_g^3) = \log L(f) > \log(g-6) - \log\left(\tfrac{40}{3}\log\left(\frac{4g+4}{\pi}\right)\right).$$

By (2-1),

$$d_{\mathcal{T}}(S_g^1, S_g^3) \ge \tfrac{1}{2}d_L(\Sigma_g^1, \Sigma_g^3) > \tfrac{1}{2}\log(g-6) - \tfrac{1}{2}\log\left(\tfrac{40}{3}\log\left(\frac{4g+4}{\pi}\right)\right). \qquad \square$$

References

[Ahlfors 1966] **L V Ahlfors**, *Lectures on quasiconformal mappings*, Van Nostrand Math. Stud. 10, Van Nostrand, Toronto (1966) MR Zbl

[Akrout 2003] **H Akrout**, *Singularités topologiques des systoles généralisées*, Topology 42 (2003) 291–308 MR Zbl

[Anderson et al. 2011] **J W Anderson**, **H Parlier**, **A Pettet**, *Small filling sets of curves on a surface*, Topology Appl. 158 (2011) 84–92 MR Zbl

[Anderson et al. 2016] **J W Anderson**, **H Parlier**, **A Pettet**, *Relative shapes of thick subsets of moduli space*, Amer. J. Math. 138 (2016) 473–498 MR Zbl

[Buser 1992] **P Buser**, *Geometry and spectra of compact Riemann surfaces*, Progr. Math. 106, Birkhäuser, Boston, MA (1992) MR Zbl

[Fletcher et al. 2013] **A Fletcher**, **J Kahn**, **V Markovic**, *The moduli space of Riemann surfaces of large genus*, Geom. Funct. Anal. 23 (2013) 867–887 MR Zbl

[Fortier Bourque 2020] **M Fortier Bourque**, *Hyperbolic surfaces with sublinearly many systoles that fill*, Comment. Math. Helv. 95 (2020) 515–534 MR Zbl

[Fortier Bourque and Rafi 2022] **M Fortier Bourque**, **K Rafi**, *Local maxima of the systole function*, J. Eur. Math. Soc. 24 (2022) 623–668 MR Zbl

[Gao and Wang 2023] **Y Gao**, **J Wang**, *The maximal systole of hyperbolic surfaces with maximal S^3–extendable abelian symmetry*, Pacific J. Math. 325 (2023) 85–104 MR Zbl

[Harer and Zagier 1986] **J Harer**, **D Zagier**, *The Euler characteristic of the moduli space of curves*, Invent. Math. 85 (1986) 457–485 MR Zbl

[Imayoshi and Taniguchi 1992] **Y Imayoshi**, **M Taniguchi**, *An introduction to Teichmüller spaces*, Springer (1992) MR Zbl

[Ji 2014] **L Ji**, *Well-rounded equivariant deformation retracts of Teichmüller spaces*, Enseign. Math. 60 (2014) 109–129 MR Zbl

[Liu 2023] **Y Liu**, *Virtual homological eigenvalues and the Weil–Petersson translation length*, Sci. China Math. 66 (2023) 2119–2132 MR Zbl

[Maskit 1985] **B Maskit**, *Comparison of hyperbolic and extremal lengths*, Ann. Acad. Sci. Fenn. Ser. A I Math. 10 (1985) 381–386 MR Zbl

[Masur 2009] **H Masur**, *Geometry of Teichmüller space with the Teichmüller metric*, from "Geometry of Riemann surfaces and their moduli spaces" (L Ji, S A Wolpert, S-T Yau, editors), Surv. Differ. Geom. 14, International, Somerville, MA (2009) 295–313 MR Zbl

[Mirzakhani 2007] **M Mirzakhani**, *Simple geodesics and Weil–Petersson volumes of moduli spaces of bordered Riemann surfaces*, Invent. Math. 167 (2007) 179–222 MR Zbl

[Mirzakhani 2013] **M Mirzakhani**, *Growth of Weil–Petersson volumes and random hyperbolic surfaces of large genus*, J. Differential Geom. 94 (2013) 267–300 MR Zbl

[Mirzakhani and Petri 2019] **M Mirzakhani**, **B Petri**, *Lengths of closed geodesics on random surfaces of large genus*, Comment. Math. Helv. 94 (2019) 869–889 MR Zbl

[Nie et al. 2023] **X Nie**, **Y Wu**, **Y Xue**, *Large genus asymptotics for lengths of separating closed geodesics on random surfaces*, J. Topol. 16 (2023) 106–175 MR Zbl

[Rafi and Tao 2013] **K Rafi**, **J Tao**, *The diameter of the thick part of moduli space and simultaneous Whitehead moves*, Duke Math. J. 162 (2013) 1833–1876 MR Zbl

[Rupflin and Topping 2018] **M Rupflin**, **P M Topping**, *Horizontal curves of hyperbolic metrics*, Calc. Var. Partial Differential Equations 57 (2018) art. id. 106 MR Zbl

[Schmutz Schaller 1999] **P Schmutz Schaller**, *Systoles and topological Morse functions for Riemann surfaces*, J. Differential Geom. 52 (1999) 407–452 MR Zbl

[Strebel 1984] **K Strebel**, *Quadratic differentials*, Ergebnisse der Math. 5, Springer (1984) MR Zbl

[Thurston 1986a] **W P Thurston**, *Minimal stretch maps between hyperbolic surfaces*, preprint (1986) arXiv math/9801039 Reprinted as pages 533–585 in his *Collected works*, *I*: *Foliations*, *surfaces and differential geometry*, Amer. Math. Soc., Providence, RI (2022)

[Thurston 1986b] **W P Thurston**, *A spine for Teichmüller space*, preprint (1986)

[Wolpert 1987] **S A Wolpert**, *Geodesic length functions and the Nielsen problem*, J. Differential Geom. 25 (1987) 275–296 MR Zbl

[Wu 2019] **Y Wu**, *Growth of the Weil–Petersson inradius of moduli space*, Ann. Inst. Fourier (Grenoble) 69 (2019) 1309–1346 MR Zbl

[Wu 2022] **Y Wu**, *A new uniform lower bound on Weil–Petersson distance*, Calc. Var. Partial Differential Equations 61 (2022) art. id. 146 MR Zbl

School of Mathematics and Statistics, Anhui Normal University
Wuhu, China

yuegao@ahnu.edu.cn

Received: 7 February 2022 Revised: 17 January 2023

Algebraic & Geometric Topology **24**:4 (2024) 2039–2090
DOI: 10.2140/agt.2024.24.2039
Published: 16 July 2024

Moduli spaces of geometric graphs

Mara Belotti
Antonio Lerario
Andrew Newman

We define a new family of graph invariants, studying the topology of the moduli space of their geometric realizations in Euclidean spaces, using a limiting procedure reminiscent of Floer homology.

Given a labeled graph G on n vertices and $d \geq 1$, let $W_{G,d} \subseteq \mathbb{R}^{d\times n}$ denote the space of nondegenerate realizations of G in $\mathbb{R}^d$. For example, if G is the empty graph, then $W_{G,d}$ is homotopy equivalent to the configuration space of n points in $\mathbb{R}^d$. Questions about when a certain graph G exists as a geometric graph in $\mathbb{R}^d$ have been considered in the literature and in our notation have to do with deciding when $W_{G,d}$ is nonempty. However, $W_{G,d}$ need not be connected, even when it is nonempty, and we refer to the connected components of $W_{G,d}$ as *rigid isotopy classes* of G in $\mathbb{R}^d$. We study the topology of these rigid isotopy classes. First, regarding the connectivity of $W_{G,d}$, we generalize a result of Maehara that $W_{G,d}$ is nonempty for $d \geq n$ to show that $W_{G,d}$ is k–connected for $d \geq n+k+1$, and so $W_{G,\infty}$ is always contractible.

While $\pi_k(W_{G,d}) = 0$ for G, k fixed and d large enough, we also prove that, in spite of this, when $d \to \infty$ the structure of the nonvanishing homology of $W_{G,d}$ exhibits a stabilization phenomenon. The nonzero part of its homology is concentrated in at most $n-1$ equally spaced clusters in degrees between $d-n$ and $(n-1)(d-1)$, and whose structure does not depend on d, for d large enough. This leads to the definition of a family of graph invariants, capturing the asymptotic structure of the homology of the rigid isotopy class. For instance, the sum of the Betti numbers of $W_{G,d}$ does not depend on d for d large enough; we call this number the *Floer number* of the graph G. This terminology comes by analogy with Floer theory, because of the shifting phenomenon in the degrees of positive Betti numbers of $W_{G,d}$ as d tends to infinity.

Finally, we give asymptotic estimates on the number of rigid isotopy classes of $\mathbb{R}^d$–geometric graphs on n vertices for d fixed and n tending to infinity. When $d=1$ we show that asymptotically as $n \to \infty$, each isomorphism class corresponds to a constant number of rigid isotopy classes, on average. For $d > 1$ we prove a similar statement at the logarithmic scale.

14P10, 55R80; 05C30

1 Introduction

Let $P = (p_1, \dots, p_n)$ be a point in $\mathbb{R}^{d\times n}$. The geometric graph associated to P is the labeled graph[1] $G(P)$ whose vertices and edges are, respectively,

$$V(G(P)) = \{(1, p_1), \dots, (n, p_n)\}, \quad E(G(P)) = \{((i, p_i), (j, p_j)) \mid i < j, \|p_i - p_j\|^2 < 1\}.$$

Some readers might be more familiar with the following equivalent definition: $G(P)$ is the 1–skeleton of the Čech complex associated to the covering consisting of open balls of radius $1/\sqrt{2}$ centered at the points $p_1, \dots, p_n$ in $\mathbb{R}^d$. We notice that $G(P)$ *is not* an embedded graph, but it is sometimes useful to visualize it as embedded.

If a graph G on n vertices is isomorphic as a labeled graph to a geometric graph $G(P)$ for some $P \in \mathbb{R}^{d\times n}$, we say it is realizable as an $\mathbb{R}^d$–geometric graph on n vertices. Maehara [23] proved that when $d \geq n$, every graph on n vertices is realizable as an $\mathbb{R}^d$–geometric graph. In particular, denoting by $\#_{d,n}$ the number of isomorphism classes of labeled $\mathbb{R}^d$–geometric graphs on n vertices, for $d \geq n$ we have

$$\#_{d,n} = 2^{\binom{n}{2}}. \tag{1-1}$$

This statement can be rephrased using the theory of *discriminants* from real algebraic geometry. To explain this idea let us first introduce the notion of nondegenerate geometric graph: the $\mathbb{R}^d$–geometric graph $G(P)$ is called *nondegenerate* if there is no pair of indices $1 \leq i < j \leq n$ such that $\|p_i - p_j\|^2 = 1$. Studying nondegenerate graphs is not an actual restriction, since the set of isomorphism classes of labeled nondegenerate $\mathbb{R}^d$–geometric graphs coincides with the set of all possible isomorphism classes of labeled $\mathbb{R}^d$–geometric graphs; see Lemma 16 below. Moreover, nondegenerate geometric graphs are simpler to study, because of their stability under small perturbations of the defining points.

In this setting the discriminant consists of the set of *degenerate* $\mathbb{R}^d$–geometric graphs

$$\Delta_{d,n} = \{P \in \mathbb{R}^{d\times n} \mid \text{there exist } 1 \leq i < j \leq n \text{ such that } \|p_i - p_j\|^2 = 1\} \subset \mathbb{R}^{d\times n}.$$

This discriminant partitions $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ into many disjoint, connected open sets, which we will call *chambers*. If two points P_0 and P_1 belong to the same chamber in $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ then clearly $G(P_0)$ and $G(P_1)$ are isomorphic, but the reverse implication does not hold in general, leading to the following definition.

Definition 1 If two points $P_0, P_1 \in \mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ belong to the same chamber — that is, if there is a continuous curve $P : [0, 1] \to \mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ with $P(0) = P_0$ and $P(1) = P_1$ — we will say that the geometric graphs $G(P_0)$ and $G(P_1)$ are *rigidly isotopic*. We will call the curve $P : [0, 1] \to \mathbb{R}^{d\times n} \setminus \Delta_{n,d}$ a *rigid isotopy*.

As an example of $\mathbb{R}^d$–geometric graphs which are isomorphic but not rigidly isotopic, consider points $P_0 = (-2, 0)$ and $P_1 = (0, -2)$: the $\mathbb{R}$–geometric graphs $G(P_0)$ and $G(P_1)$ are isomorphic; they are

[1] From now on, unless differently specified, the word "graph" stands for "labeled graph".

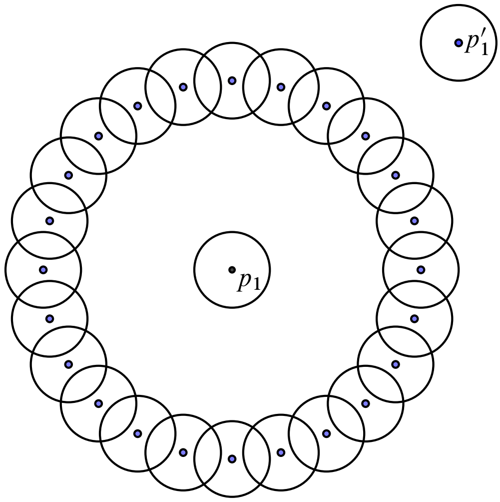


Figure 1: Here we are drawing points in $\mathbb{R}^2$ together with the circles centered at those points with radius $1/\sqrt{2}$. Let us define points P and P' in $\mathbb{R}^{2\times 25}$ in such a way that p_1 is the point inside the big circle and p_1' is the point outside the big circle, while $p_i = p_i'$ for $i > 1$ and they are the points on the big circle. Then the two geometric graphs $G(P)$ and $G(P')$ are isomorphic, but not rigidly isotopic.

both the graph on 2 vertices with no edges, but they are not rigidly isotopic since any curve $P(t) \in \mathbb{R}^{1\times 2}$ with $P(0) = P_0$ and $P(1) = P_1$ must intersect the discriminant. Another example is depicted in Figure 1.

For n and d the number of rigid isotopy classes of geometric graphs on n vertices in $\mathbb{R}^d$ is exactly given by $b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n})$, and we always clearly have

$$b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n}) \geq \#_{d,n}.$$

One natural question therefore is: for what values of n and d do the two notions coincide? Moreover, one could consider higher-dimensional notions of connectivity of $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ and study the higher homology and the homotopy groups of the space of its connected components. As we will see, this study will lead us to a definition of a new graph invariant which reminds of Floer homology, as well as precise asymptotics for the enumeration of rigid isotopy and isomorphism classes of geometric graphs.

Remark 2 Graphs which are realizable as $\mathbb{R}^d$–geometric graphs are often called *d–sphere graphs*, while the minimal dimension d such that a given graph is a d–sphere graph is called its *sphericity*. The result of Maehara [23] mentioned above tells that every graph has finite sphericity; Kang and Müller [20] prove that the problem of deciding, given a graph G, whether G is a d–sphere graph is NP–hard for all $d > 1$. Note that, for every $d > 0$, there are graphs that are not d–sphere graphs. In the particular case of $d = 1$, the 1–sphere graphs are also called *indifference graphs* or *unit interval graphs*; there are many characterizations of such graphs, see for instance Lekkerkerker and Boland [21], Roberts [29], Jackowski [19], Gutierrez and Oubiña [14] and Mertzios [25].

1.1 The case $d \to \infty$

As we will prove in Corollary 47 below, for $d \geq n+1$ the two notions of isomorphic and rigidly isotopic coincide, and $b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n}) = \#_{d,n}$. Therefore, adopting this language we can reformulate[2] the identity in (1-1) as:

$$b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n}) = 2^{\binom{n}{2}},$$

which is true for $d \geq n+1$. The realizability result of Maehara [23] and the fact that for large d "rigid isotopy" and "isomorphism" are the same notion, seem to settle all relevant questions related to the study of the asymptotics for the number of chambers of $b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n})$ for fixed n and large d. However, as we will see, the topology of the chambers of the complement of the discriminant is extremely rich and some unexpected structure emerges as $d \to \infty$.

In order to explain this phenomenon, let us label the chambers of $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ with the corresponding isomorphism class of labeled geometric graphs: given a graph G on n vertices we define

$$W_{G,d} = \{P \in \mathbb{R}^{d\times n} \setminus \Delta_{d,n} \mid G(P) \cong G\} \subset \mathbb{R}^{d\times n}.$$

In other words, $W_{G,d}$ consists of all the points $P \in \mathbb{R}^{d\times n}$ which are not on the discriminant and whose associated geometric graph is isomorphic to G. For small d this set could be a union of several chambers, but for large d it is an actual chamber, that is, a connected open set. This can be rephrased by saying that for every graph G on n vertices and for large enough d, the homotopy group $\pi_0(W_{G,d})$ consists of a single element. In fact, as we will show, the same statement is true for all the homotopy groups, once the group is fixed and d becomes large enough.

Theorem 3 *For every $k \geq 0$ and for $d \geq k+n+1$, we have $\pi_k(W_{G,d}) = 0$.*

Theorem 3 in fact generalizes the result of Maehara [23]. Taking the standard convention that a topological space is said to be (-1)–connected provided it is nonempty, Theorem 3 for $k=-1$ is Maehara's result that every graph on n vertices can be realized as a geometric graph in $\mathbb{R}^d$ for $d \geq n$. Theorem 3 is most likely not sharp for any value of k. Indeed even in the case of $k=-1$, Maehara also shows in [23] that any graph G which is not a clique can be realized as a geometric graph in $(|G|-\omega(G))$–dimensional space. Here $\omega(G)$ denotes the clique number of G, the maximum number of vertices in G that form a complete subgraph. A clique can be realized (with n points in distinct positions) in $\mathbb{R}$ and so for any graph G on at least two vertices, $\pi_{-1}(W_{G,d}) = 0$ for $d \geq n-1$. An interesting open question would be to improve the general lower bound on d in Theorem 3.

Notice that there is a natural sequence of inclusions

$$\cdots \hookrightarrow W_{G,d} \hookrightarrow W_{G,d+1} \hookrightarrow \cdots \tag{1-2}$$

[2] Here and later, for a topological space X we will denote its k^{th} Betti number by $b_k(X) = \dim_{\mathbb{Z}_2}(H^k(X;\mathbb{Z}_2))$ and its total Betti number by $b(X) = \sum_{k=0}^{\infty} b_k(X)$, whenever these numbers are defined. This will happen for all the spaces that we will consider in this paper: they will all be homotopy equivalent to finite CW–complexes.

obtained by simply including $\mathbb{R}^{d\times n}$ into $\mathbb{R}^{(d+1)\times n}$ by appending a list of zeros to the coordinates of P. The proof of Theorem 3 goes through two intermediate steps, which are of independent interest: we first prove that for every $k \geq 0$ and for $d \geq k+n+1$, the inclusion $W_{G,d} \hookrightarrow W_{G,d+1}$ induces an injection on the homotopy classes of maps, then we prove that the inclusion $W_{G,d} \hookrightarrow W_{G,d+n}$ is homotopic to a constant map.

Example 4 (homotopy groups of the configuration space of n points in $\mathbb{R}^d$) Let us consider the graph G consisting of n vertices and no edges. It is easy to see that the corresponding chamber $W_{G,d}$ is homotopy equivalent[3] to the configuration space of n distinct points in $\mathbb{R}^d$:

$$W_{G,d} \sim \mathrm{Conf}_n(\mathbb{R}^d).$$

In this case one can compute exactly the homotopy groups of $W_{G,d}$: for every $k \geq 0$ and for $d \geq 3$ we have (see [11, Chapter 2, Theorem 1.1])

$$\pi_k(W_{G,d}) \simeq \pi_k(\mathrm{Conf}_n(\mathbb{R}^d)) \simeq \bigoplus_{j=1}^{n-1} \pi_k(\underbrace{S^{d-1} \vee \cdots \vee S^{d-1}}_{\text{bouquet of } j \text{ spheres}}).$$

Since $\pi_k(S^{d-1} \vee \cdots \vee S^{d-1}) = 0$ for $d \geq k+2$, in this case we immediately see that also $\pi_k(W_{G,d}) = 0$ for $d \geq k+2$.

It is natural at this point to put the sequence of inclusions (1-2) into the infinite-dimensional space

$$\mathbb{R}^{\infty\times n} = \varinjlim \mathbb{R}^{d\times n}$$

consisting of n–tuples of sequences $(p_1, \ldots, p_n)$ such that for every $j = 1, \ldots, n$ all but finitely many elements in the sequence p_j are zero. The definition of geometric graph and discriminant also makes sense in this infinite-dimensional space; see Section 4.1. The chambers are now defined as follows: for a given graph G on n vertices, we set

$$W_{G,\infty} = \{P = (p_1, \ldots, p_n) \in \mathbb{R}^{\infty\times n} \setminus \Delta_{\infty,n} \mid G(P) \cong G\}.$$

From Theorem 3 we deduce the following.

Theorem 5 *For every graph G, the set $W_{G,\infty} = \varinjlim W_{G,d}$ is contractible.*

1.2 Floer homology of a graph

Summarizing the picture so far: as $d \to \infty$, each $W_{G,d}$ eventually becomes k–connected and its direct limit $W_{G,\infty}$ has no homotopy. Moreover, by the Hurewicz theorem, each fixed reduced Betti number of $W_{G,d}$ vanishes for d large enough: more precisely, for every $k > 0$ there exists $d(k) > 0$ such that

$$b_k(W_{G,d}) = 0 \quad \text{for all } d \geq d(k).$$

[3]Here and below, for two topological spaces X and Y we use the symbol $X \simeq Y$ to denote that they are homeomorphic and $X \sim Y$ to denote that they are homotopy equivalent.

But this is not the whole story. Before we continue let us discuss one more, likely familiar, example.

Example 6 (the infinite-dimensional sphere) Let G be the graph consisting of two disjoint points. Then $W_{G,d} \simeq \mathbb{R}^d \times S^{d-1} \sim S^{d-1}$ and $W_{G,\infty} \simeq S^\infty \times \mathbb{R}^{d\times(n-1)} \sim S^\infty$. In this case (1-2) becomes

$$\cdots \hookrightarrow S^{d-1} \hookrightarrow S^d \hookrightarrow \cdots \hookrightarrow S^\infty = \varinjlim S^d.$$

If we now look at the Betti numbers of S^d, we see that there is a hole in dimension d that moves to infinity as $d \to \infty$, and it disappears when $d = \infty$. The sphere S^∞ has no cohomology except in dimension zero, but it still has cohomology every time we cut it with a finite-dimensional space.

The phenomenon described in Example 6 can be interpreted using some extraordinary cohomology theory, in the context of the Leray–Schauder degree and, more generally, of Floer homology theories; see Szulkin [31], Gęba and Granas [13] and Abbondandolo [1]. This behavior has also been observed for nonholonomic loop spaces in Carnot groups; see Agrachëv, Gentile and Lerario [4]. In all these examples we are dealing with a sequence of spaces X_d whose direct limit X_∞ is contractible, but for every d large enough, each space carries the same amount of cohomology, just shifted in its dimension. A more general family of examples where this occurs is the iterated suspension.

Example 7 (the iterated suspension) Let X_0 be a CW–complex and define $X_d = SX_{d-1}$, where

$$SX = (X \times I)/\sim$$

is the suspension of X, and the equivalence relation $\sim$ is given by $(x_1, 0) \sim (x_2, 0)$ and $(x_1, 1) \sim (x_2, 1)$ for all $x_1, x_2 \in X$. We have a natural sequence of inclusions

$$\cdots \hookrightarrow X_d \hookrightarrow X_{d+1} \hookrightarrow \cdots \tag{1-3}$$

given by mapping $X_d \to SX_d$ homeomorphically to $X_d \times \{\frac{1}{2}\}$. We denote by $X_\infty = \varinjlim X_d$ the direct limit of the sequence of inclusions (1-3). If $X_0 = \{x_1, x_2\}$, then $X_d = S^d$ and $X_\infty = S^\infty$. For every k the space X_d becomes eventually k–connected when d is large enough, and X_∞ is contractible. If one looks at the homology of X_d, this is made by a cluster of holes that shifts to infinity as $d \to \infty$. This can be expressed, for example, by looking at the Poincaré polynomial of X_d,

$$P_{X_d}(t) = 1 + t^d (P_{X_0}(t) - 1).$$

These holes are not present when $d = \infty$, but the sum of the Betti numbers of X_d is constant,

$$b(X_d) = P_{X_d}(1) \equiv P_{X_0}(1) = b(X_0).$$

We will prove that a similar phenomenon happens for all the spaces $W_{G,d}$: their reduced cohomology is made of "clusters of holes" that "shift" to infinity as $d \to \infty$; see Figure 2. In fact we show also that for G on n vertices there are at most $n-1$ such clusters. More precisely, we have the following result.

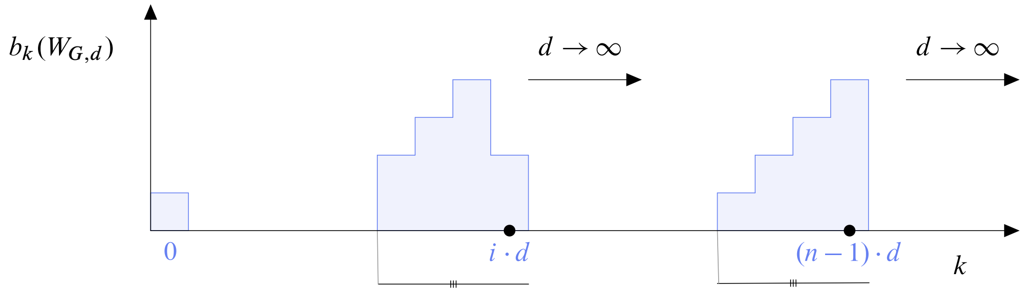


Figure 2: A plot of the Betti numbers of $W_{G,d}$. The width of each nonzero cluster of holes is $\binom{n}{2} + 1$, which is a constant. Each of these clusters is placed at a multiple of d, and as $d \to \infty$ they shift to infinity. The total Betti number of $W_{G,d}$, ie the shaded area, becomes constant for d large enough.

Theorem 8 *For every graph G on n vertices there exist polynomials*[4] $Q_{G,1}, \dots, Q_{G,n-1}$ *each of degree at most $\binom{n}{2} + 1$ such that for $d \geq \binom{n}{2} + 2$, the Poincaré polynomial of $W_{G,d}$ is*

$$P_{W_{G,d}}(t) = 1 + t^{d-\binom{n}{2}-1} Q_{G,1}(t) + \dots + t^{md-\binom{n}{2}-1} Q_{G,m}(t) + \dots + t^{(n-1)d-\binom{n}{2}-1} Q_{G,n-1}(t).$$

In particular, there exists $\beta(G) > 0$ such that

$$b(W_{G,d}) = P_{W_{G,d}}(1) \equiv 1 + \sum_{\ell=1}^{n-1} Q_{G,\ell}(1) = \beta(G) \quad \text{for } d \text{ large enough},$$

ie the sum of the Betti numbers of $W_{G,d}$ becomes a constant, which depends on G only.

Each polynomial $Q_{G,m}$ corresponds to one of the clusters above and keeps track of the Betti numbers $b_k(W_{G,d})$ with $0 \leq dm - k \leq \binom{n}{2}$, ie with index k located "near" dm — remember that $\binom{n}{2}$ is a constant in this asymptotic regime. These clusters are the "Floer homologies" of the graph.

While we show that nonvanishing homology clusters around multiples of d, we also prove the following result that holds for any n and d, in part to determine which multiples of d have to be considered.

Theorem 9 *For every d and n, $\hat{\Delta}_{d,n}$ is $(n+d-3)$–connected, but not $(n+d-2)$–connected. By Alexander–Pontryagin duality this implies that $H^{nd-n-d+1}(\mathbb{R}^{d\times n} \setminus \Delta_{d,n}) \neq 0$, but all higher cohomology groups of $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ vanish.*

The proof of Theorem 8 uses a spectral sequence argument: each $W_{G,d}$ can be described as a system of quadratic inequalities and one can use the technique developed in Agrachëv [3] and Agrachëv and Lerario [5] for the study of its Betti numbers. In this case, as $d \to \infty$, the spectral sequence that we need to consider converges at the second step, ie $E_2 = E_\infty$. One can prove that both E_2 and the second differential d_2 have an asymptotic stable shape: as $d \to \infty$ the nonzero part of the spectral sequence looks like a skinny table where there is no room for higher differentials and the nonzero elements of this table are located near the rows which are labeled by indices which are multiples of d; see Figure 3 later.

[4] These polynomials only depend on G and not on d.

graph	Poincaré polynomial	$\beta(G)$	labeled copies
	$1+6t^{d-1}+11t^{2d-2}+6t^{3d-3}$	24	1
	$1+3t^{d-1}+2t^{2d-2}$	6	6
	$1+t^{d-1}$	2	3
	$1+2t^{d-1}+t^{2d-2}$	4	12
	$1+t^{d-1}$	2	12
	$1+t^{d-2}+t^{d-1}+t^{2d-3}$	4	4
	$1+t^{d-1}$	2	4
	$1+t^{d-1}$	2	12
	$1+t^{d-2}+t^{d-1}+t^{2d-3}$	4	3
	$1+t^{d-1}$	2	6
	1	1	1

Table 1: Poincaré polynomials and Floer numbers for $W_{G,d}$ for G on four vertices. For every graph, "labeled copies" refers to the number of isomorphism classes of labeled graphs with the same unlabeled graph.

An interesting question arising from Theorem 8 is, for a given graph G on n vertices, how to compute

$$\beta(G)=\lim_{d\to\infty} b(W_{G,d}).$$

We call this number the *Floer number* of the graph G. This number is a graph invariant, as well as the polynomials from Theorem 8. Table 1 shows the value of this number for all the possible graphs on four vertices, but the general case is still mysterious. There is more discussion about some of the details of Table 1 in Example 67. Note that the polynomials $Q_{G,m}$ are also invariants of G, but their meaning is even more mysterious.

We can make some observations in a few cases that suggest conjectures about $\beta(G)$ for general graphs. In the case of graphs on four or fewer vertices, we see that the Poincaré polynomial of $W_{G,d}$ has a general form with exponents given in terms of d. Once d is large enough that G has a realization as a geometric graph in $\mathbb{R}^d$, we see that the Poincaré polynomial is determined by the general form. From this we conjecture that for every graph G there is a general form of the Poincaré polynomial of $W_{G,d}$ with exponents given in terms of d, which is valid as long as d is large enough that $W_{G,d}$ is nonempty. This would imply that as soon as G can be realized as a geometric graph in $\mathbb{R}^d$, $b(W_{G,d})=\beta(G)$. In the case of graphs realizable in $\mathbb{R}$, we would have that $\beta(G)$ counts the number of chambers of $W_{G,1}$. To see this recall that in the $d=1$ case, $\mathbb{R}^{1\times n}\setminus\Delta_{1,n}$ is a disjoint union of polyhedra, so every chamber is contractible; therefore homology can only exist in degree zero. A first step toward the proof of this conjecture would be to prove that all the differentials of the spectral sequence that we use in the proof of Theorem 8 are zero for all $d \geq n$.

Example 10 (Betti numbers of the configuration space of n points in $\mathbb{R}^d$) The Poincaré polynomial of $\mathrm{Conf}_n(\mathbb{R}^d)$ for $d \geq 1$ is given (see [11, Chapter V, Corollary 1.4]) by

$$P_{\mathrm{Conf}_n(\mathbb{R}^d)}(t) = \prod_{j=1}^{n-1}(1 + jt^{d-1}).$$

Consequently, $b(\mathrm{Conf}_n(\mathbb{R}^d)) = P_{\mathrm{Conf}_n(\mathbb{R}^d)}(t)(1) \equiv n!$. In other words, for the graph G consisting of n disjoint points, we have $\beta(G) = n!$.

1.3 The case $n \to \infty$

Concerning the other asymptotic regime, the first case of interest is when $d = 1$: here $\Delta_{1,n}$ is a hyperplane arrangement, since each quadric $\{|p_i - p_j|^2 = 1\} \subset \mathbb{R}^{1\times n}$ is the union of the two hyperplanes $\{p_i - p_j = 1\}$ and $\{p_i - p_j = -1\}$. It turns out that the number of chambers of the complement of such a hyperplane arrangement, that is, the number of rigid isotopy classes of $\mathbb{R}$–geometric graphs on n vertices, equals the number of *labeled semiorders* of $[n]$. Using techniques from analytic combinatorics, we will prove the following theorem.

Theorem 11 *The number of rigid isotopy classes of $\mathbb{R}$–geometric graphs on n vertices equals*

$$b_0(\mathbb{R}^{1\times n} \setminus \Delta_{1,n}) = \frac{1}{n} \cdot \sqrt{6 \log \tfrac{4}{3}} \cdot \left(\frac{n}{e \log \frac{4}{3}}\right)^n (1 + O(n^{-1/2})).$$

It is in fact possible also to compute the asymptotics of $\#_{1,n}$ as $n \to \infty$; we do this in Theorem 52. It is remarkable that the two numbers $b_0(\mathbb{R}^{1\times n} \setminus \Delta_{1,n})$ and $\#_{1,n}$ have the same asymptotic, up to a multiplicative constant[5]

$$b_0(\mathbb{R}^{1\times n} \setminus \Delta_{1,n}) = \frac{8}{e^{1/12}} \cdot \#_{1,n}(1 + O(n^{-1/2})). \tag{1-4}$$

The case when $d \geq 2$ is more delicate to handle. This is in large part because the discriminant in higher dimensions is an arrangement of quadrics rather than an arrangement of hyperplanes. For this general case we will prove the following upper and lower bounds for the number of rigid isotopy classes and isomorphism classes.

Theorem 12 *For $d \geq 2$ fixed and for $n \geq 4d + 1$, one has the bounds*

$$\left(\frac{1}{(d+1)e^2}\right)^{dn} n^{dn} \leq \#_{d,n} \leq b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n}) \leq 2dn\left(\frac{3e}{2d}\right)^{dn} n^{dn}$$

[5]The constant $8/e^{1/12}$ is approximately 7.36.

Note that these bounds imply that $\#_{d,n}$ and $b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n})$ become equivalent at the logarithmic scale, giving the following analogue of (1-4):

$$\log b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n}) = (\log \#_{d,n})(1+o(1)) \quad \text{as } n \to \infty.$$

For the proof of the upper bound we will use the fact that $\Delta_{d,n}$ is a real algebraic set: using Alexander–Pontryagin duality, we bound the topology of the complement of the discriminant $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ by studying the topology of the one-point compactification of $\Delta_{d,n}$, which we denote by $\widehat{\Delta}_{d,n}$. This one-point compactification can also be described in an algebraic way and studied using Milnor [27]. Along the way to proving this upper bound we also develop the notation for the spectral sequence that we use to prove Theorem 9.

For the lower bound, our proof is a higher-dimensional version of work of McDiarmid and Müller [24] which showed that, in the case $d = 2$, there exists a constant $\alpha > 0$ such that $\#_{2,n} \geq \alpha^n n^{2n}$.

Acknowledgements Belotti is funded by Deutsche Forschungsgemeinschaft SFB-TRR 195, *Symbolic tools in mathematics and their application*. Newman was supported by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Graduiertenkolleg 2434, *Facets of complexity*.

2 Preliminaries

2.1 Geometric graphs

There are several different notions of geometric graphs in the literature. The type of geometric graph we consider here are also sometimes called intersection graphs or space graphs. Formally the definition for geometric graph we use here is the following.

Definition 13 (geometric graph) Given a point $P \in \mathbb{R}^{d\times n}$ we denote by $G(P)$ the labeled graph whose vertices and edges are, respectively,

$$V(G(P)) = \{(1, p_1), \dots, (n, p_n)\}, \quad E(G(P)) = \{((i, p_i), (j, p_j)) \mid i < j, \|p_i - p_j\|^2 < 1\}.$$

We say that $G(P)$ is an $\mathbb{R}^d$–geometric graph. If a labeled graph G is isomorphic to $G(P)$ for some $P \in \mathbb{R}^{d\times n}$, we say that G is *realizable* as an $\mathbb{R}^d$–geometric graph.

We say that the geometric graph $G(P)$ is nondegenerate if $P \notin \Delta_{d,n}$, where

$$\Delta_{d,n} = \{P \in \mathbb{R}^{d\times n} \mid \text{there exist } 1 \leq i < j \leq n \text{ such that } \|p_i - p_j\|^2 = 1\} \subset \mathbb{R}^{d\times n}.$$

Remark 14 The reason for considering in our definition the list of pairs $\{(1, p_1), \dots, (n, p_n)\}$ as the set of vertices of $G(P)$, instead of the list $\{p_1, \dots, p_n\}$, is just formal. In other settings it may be more natural to take $p_1, \dots, p_n$ to be *distinct* points in $\mathbb{R}^d$ and then to define a graph with vertex set $\{p_1, \dots, p_n\}$ and edges (p_i, p_j) provided that $\|p_i - p_j\|^2 < 1$. This is the approach taken by Maehara [23], who studies the *sphericity* of graphs, the minimum dimension d in which a graph may be realized as a geometric graph in $\mathbb{R}^d$ with vertices given by distinct points. For us it makes sense to associate graphs on n vertices in $\mathbb{R}^d$ to points of $\mathbb{R}^{d\times n}$, therefore the actual points in $\mathbb{R}^d$ may not all be unique from one another. The two-coordinate approach to describe the vertices allows for such repetition and naturally associates each point in $\mathbb{R}^d \setminus \Delta_{d,n}$ to unique labeled graph.

We will consider the following notions of equivalence of geometric graphs.

Definition 15 Let $G(P_0)$ and $G(P_1)$ be two $\mathbb{R}^d$–geometric graphs on n vertices, with $P_0, P_1 \in \mathbb{R}^{d\times n}$. We will say that they are *isomorphic* if they are isomorphic as labeled geometric graphs. Moreover, if they are both nondegenerate, we will say that they are *rigidly isotopic* if there exists a continuous curve $P\colon [0,1] \to \mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ such that $P(0) = P_0$ and $P(1) = P_1$.

Since $\Delta_{d,n}$ is an algebraic set, its complement is a semialgebraic set and its path components are the same as its connected components. Therefore two nondegenerate geometric graphs $G(P_0)$ and $G(P_1)$ are rigidly isotopic if and only if P_0 and P_1 belong to the same connected component of $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$.

Let us introduce the notation

$$\#_{d,n} := \#\{\text{isomorphism classes of geometric graphs on } n \text{ vertices in } \mathbb{R}^d\}. \tag{2-1}$$

In the definition of $\#_{d,n}$ we did not assume the nondegeneracy of the graphs. However, the following lemma proves that isomorphism classes of *nondegenerate* graphs are the same as all isomorphism classes as in (2-1); see also [6, Lemma 2.2] for an analogous statement in the more general context of geometric complexes.

Lemma 16 *For every $P \in \Delta_{d,n}$ there exists $\widetilde{P} \in \mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ such that $G(P)$ and $G(\widetilde{P})$ are isomorphic as labeled graphs.*

Proof Take $P = (p_1, \dots, p_n) \in \Delta_{d,n}$ and for $\epsilon > 0$ small enough, consider

$$\widetilde{P} := (1+\epsilon)P.$$

Then, for ϵ small enough, we have

$$((i,p_i),(j,p_j)) \in E(G(P)) \iff ((i,(1+\epsilon)p_i),(j,(1+\epsilon)p_j)) \in E(G(\widetilde{P})),$$

which proves that $G(P)$ and $G(\widetilde{P})$ are isomorphic as labeled graphs. Moreover, again for ϵ small enough, we have that $\widetilde{P} \notin \Delta_{d,n}$. □

By definition, for nondegenerate graphs we have

$$\text{rigidly isotopic} \implies \text{isomorphic},$$

which means that isomorphism classes are union of rigid isotopy classes. The number of rigid isotopy classes of geometric graphs on n vertices in $\mathbb{R}^d$ is given $b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n})$, and Lemma 16 implies we can compare the number of rigid isotopy classes with the number of isomorphism classes,

$$\#_{d,n} \leq b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n}).$$

Below we will prove, as Corollary 47, that for $d \geq n+1$, two $\mathbb{R}^d$–geometric graphs on n vertices are isomorphic if and only if they are rigidly isotopic, ie that

$$\#_{d,n} = b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n}) \quad \text{for } d \geq n+1.$$

We will deal with the asymptotic of $\#_{d,n}$ and $b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n})$ in the case d fixed and $n \to \infty$ in Section 5.

2.2 Alexander duality and the discriminant

As we are interested in the topology of $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$, the topology of $\Delta_{d,n}$ should play an important role as well, and in some cases it will be easier to study. The key tool for connecting the topology of the two is Alexander duality. Given a compact, locally contractible, nonempty and proper subspace X of the N–dimensional sphere S^N, Alexander duality [16, Corollary 3.45] provides a way to study the topology of X from the topology of $S^N \setminus X$. Namely, for every k we have the following isomorphisms between the homology of X and the cohomology of $S^N \setminus X$:

$$\widetilde{H}_k(X) \cong \widetilde{H}^{N-k-1}(S^N \setminus X).$$

Remark 17 If we are working with $\mathbb{Z}_2$–coefficients, as we will throughout, and with a space $X \subset S^N$ with finitely generated homology, we can relate the Betti numbers of X with those of its complement in the sphere, ie we can freely identify homology and cohomology. When working with compact semialgebraic sets in the sphere, this last requirement will be satisfied thanks to [9, Theorem 9.4.1].

In order to use this duality in the present setting, we work in the one-point compactification of $\Delta_{d,n} \subset \mathbb{R}^{d\times n}$, denoted by $\widehat{\Delta}_{d,n} \subset S^{d\times n}$. Now $\widehat{\Delta}_{d,n}$ contains the point at infinity so $S^{d\times n} \setminus \widehat{\Delta}_{d,n} = \mathbb{R}^{d\times n} \setminus \Delta_{d,n}$.

The discriminant itself is a union of quadratic hypersurfaces of the form

$$\Delta^{i,j}_{d,n} = \{(x_1,\dots,x_n) \in \mathbb{R}^{d\times n} \mid \|x_i - x_j\|^2 = 1\}. \tag{2-2}$$

Each of these quadrics is topologically $S^{d-1} \times \mathbb{R}^{d\times(n-1)}$ and establishing bounds on the top Betti number of $\widehat{\Delta}_{d,n}$ establishes bounds on the number of rigid isotopy classes of $\mathbb{R}^d$–geometric graphs on n vertices. We take such an approach in Section 5.3.

2.3 Semialgebraic triviality

A most useful technical tool that we will use in the paper is Theorem 19, which relates the structure of semialgebraic families and their homotopy.

Definition 18 Let S, T and T' be semialgebraic sets such that $T' \subset T$, and let $f: S \to T$ be a continuous semialgebraic mapping. A semialgebraic trivialization of f over T', with fiber F, is a semialgebraic homeomorphism $\theta: T' \times F \to f^{-1}(T')$ such that $f \circ \theta$ is the projection mapping $\pi: T' \times F \to T'$. We say that the semialgebraic trivialization θ is compatible with a subset S' of S if there is a subset F' of F such that $\theta(T' \times F') = S' \cap f^{-1}(T')$.

Theorem 19 (semialgebraic triviality) *Let S and T be two semialgebraic sets, $f: S \to T$ a semialgebraic mapping, and $(S_j)_{j=1,\dots,q}$ a finite family of semialgebraic subsets of S. There exist a finite partition of T into semialgebraic sets $T = \bigcup_{l=1}^{r} T_l$ and, for each l, a semialgebraic trivialization $\theta_l: T_l \times F_l \to f^{-1}(T_l)$ of f over T_l compatible with S_j for $j = 1, \dots, q$, ie there exists $F_l^j \subset F_l$ such that $\theta_l(T_l \times F_l^j) = S_j \cap f^{-1}(T_l)$.*

Proof This is [9, Theorem 9.3.2]. □

Corollary 20 *Let S be a semialgebraic set and $f: S \to \mathbb{R}$ be a continuous semialgebraic function. Then for $\epsilon > 0$ small enough, the inclusion*

$$\{x \in S \mid f \geq \epsilon\} \hookrightarrow \{x \in S \mid f > 0\}$$

is a homotopy equivalence.

Proof Thanks to semialgebraic triviality, we know that for ϵ sufficiently small there exists T_l such that $(0, \epsilon] \subset T_l$. Then we define a map

$$H: \{f > 0\} \times [0,1] \to \{f > 0\}, \qquad H(x,t) = \begin{cases} x & \text{if } f(x) \notin (0,\epsilon), \\ \theta_l((1-t)\cdot(\pi_1 \circ \theta_l^{-1}) + t\epsilon, \pi_2 \circ \theta_l^{-1}) & \text{if } f(x) \in (0,\epsilon]. \end{cases}$$

This is a continuous function because the two expressions agree on $f^{-1}(\epsilon) \times [0,1]$ and both of them are continuous on closed subsets. Thus the map H is a deformation retraction of $\{f > 0\}$ onto $f \geq \epsilon$. □

Corollary 21 *Let Y be the set of solutions in $\mathbb{R}^d$ of the system*

$$\begin{cases} q_1(x) > 0, \\ \vdots \\ q_r(x) > 0, \end{cases}$$

where the q_i are polynomial functions. There exists $\delta > 0$ such that for all ϵ such that $0 < \epsilon < \delta$, the inclusion of $Y_\epsilon \setminus \{0\}$ in the set $Y \setminus \{0\}$, where Y_ϵ is the set of the solutions in $\mathbb{R}^d$ of the system

$$\begin{cases} q_1(x) \geq \epsilon, \\ \vdots \\ q_r(x) \geq \epsilon, \\ \|x\|^2 \leq \frac{1}{\epsilon}, \end{cases}$$

is a homotopy equivalence.

Proof Define $\alpha\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ by

$$\alpha = \min\left\{\delta, q_1, \dots, q_r, \frac{1}{\|x\|^2}\right\}.$$

Then for $0 < \epsilon < \delta$ we have $Y_\epsilon \setminus \{0\} = \{\alpha \geq \epsilon\}$ and α is a continuous semialgebraic function. By the previous corollary, for ϵ small enough we get that the inclusion $Y_\epsilon = \{\alpha \geq \epsilon\} \hookrightarrow \{\alpha > 0\} = Y \setminus \{0\}$ is a homotopy equivalence. □

2.4 Systems of quadratic inequalities

In this section we recall a general construction from [3; 5] for computing the Betti numbers of the set of solutions of a system of quadratic inequalities.

To start with, let $h\colon \mathbb{R}^{N+1} \to \mathbb{R}^{k+1}$ be a *quadratic map*, ie a map whose components $h = (h_0, \dots, h_k)$ are homogeneous quadratic forms. Let also $K \subseteq \mathbb{R}^{k+1}$ be a closed convex polyhedral cone (centered at the origin). We are interested in the Betti numbers of

(2-3) $$V = h^{-1}(K) \cap S^N \subset \mathbb{R}^{N+1}.$$

Such a set V can be seen as the set of solutions of a system of homogeneous quadratic inequalities on the sphere S^N: in fact, since K is polyhedral, we have

$$K = \{\eta_1 \leq 0, \dots, \eta_\ell \leq 0\}$$

for some linear forms $\eta_1, \dots, \eta_\ell \in (\mathbb{R}^{k+1})^*$ and

$$V = \{\eta_1 h \leq 0, \dots, \eta_\ell h \leq 0\} \cap S^N,$$

which is a system of quadratic inequalities. (Here given a linear form $\eta \in (\mathbb{R}^{k+1})^*$ and a quadratic map $h\colon \mathbb{R}^{N+1} \to \mathbb{R}^{k+1}$, we simply denote by ηh the composition of the two.) Every homogeneous system can be written in this way.

We denote by K° the polar of K, ie $K^\circ = \{\eta \in (\mathbb{R}^{k+1})^* \mid \eta(y) \leq 0 \text{ for all } y \in K\}$, and we set

$$\Omega = K^\circ \cap S^k,$$

where S^k denotes the unit sphere in $(\mathbb{R}^{k+1})^*$ with respect to a fixed scalar product. The scalar product on $\mathbb{R}^{k+1}$ plays no role, but we will also use a scalar product on $\mathbb{R}^{N+1}$ by choosing a positive definite quadratic form g on $\mathbb{R}^{N+1}$. This scalar product will play a role, and we denote it by $\langle \cdot, \cdot \rangle_g$. It is defined by $\langle x, x \rangle_g = g(x)$ for all $x \in \mathbb{R}^{N+1}$. For practical purposes we will omit the g subscripts when not needed.

Once the scalar product on $\mathbb{R}^{N+1}$ has been fixed, we can associate to a quadratic form $q\colon \mathbb{R}^{N+1} \to \mathbb{R}$ a real symmetric matrix, via the equation

(2-4) $$q(x) = \langle x, Qx \rangle_g \quad \text{for all } x \in \mathbb{R}^{N+1}.$$

We will often use small letters for the quadratic form and capital letters for the associated matrices.

Accordingly we can define the eigenvalues of q (with respect to g) as those of Q:

$$\lambda_1(q) \geq \cdots \geq \lambda_{N+1}(q).$$

The eigenvalues (and the eigenvectors) of q depend therefore on the chosen scalar product, but again we will omit this dependence in the notation if not needed.

2.4.1 The index function Using the above notation, we will denote by $\operatorname{ind}^+(q) = \operatorname{ind}^+(Q)$ the positive inertia index, ie the number of positive eigenvalues of the symmetric matrix Q. Note that the index of a quadratic form *does not* depend on the chosen scalar product.

When we are in the situation as above, ie when given a homogeneous quadratic map $h\colon \mathbb{R}^{N+1} \to \mathbb{R}^{k+1}$, for every covector $\eta \in (\mathbb{R}^{k+1})^*$ we can consider the composition ηh, which is a quadratic form. For every natural number $j \geq 0$ we define the sets

$$\Omega^j = \{\omega \in \Omega \mid \operatorname{ind}^+(\omega h) \geq j\}.$$

These sets are open and semialgebraic, as it is easily verified. Moreover, these sets *do not* depend on the choice of the scalar product g.

Over each set $\Omega^j \setminus \Omega^{j+1}$ the function $\operatorname{ind}^+ \equiv j$ is constant, ie the number of positive eigenvalues of the corresponding matrices is j and there exists a natural vector bundle $P^j \subseteq \Omega^j \setminus \Omega^{j+1} \times \mathbb{R}^{N+1}$

$$\begin{tikzcd} \mathbb{R}^j \arrow[r, hook] & P^j \arrow[d] \\ & \Omega^j \setminus \Omega^{j+1} \end{tikzcd} \tag{2-5}$$

whose fiber over a point ω is the positive eigenspace of $\omega h = \omega_0 h_0 + \cdots + \omega_k h_k$. In fact this bundle is the restriction of a more general bundle over the set

$$D_j = \{\omega \mid \lambda_j(\omega h) \neq \lambda_{j+1}(\omega h)\},$$

ie the set where the j^{th} eigenvalue of ωh is distinct from the $(j+1)^{\text{st}}$. We still denote this bundle by $P_j \subset D_j \times \mathbb{R}^{N+1}$:

$$\begin{tikzcd} \mathbb{R}^j \arrow[r, hook] & P^j \arrow[d] \\ & D_j \end{tikzcd} \tag{2-6}$$

Here the fiber over a point $\omega \in D_j$ consists of the eigenspace of ωh associated to the first j eigenvalues (this is well defined); note however that the bundle over D_j *depends* on the choice of the scalar product (since D_j itself depends on this choice).

We denote the first Stiefel–Whitney class of this bundle by

$$\nu_j \in H^1(D_j). \tag{2-7}$$

The following lemma will be useful for us.

Lemma 22 *The cup product with the class ν_j defines a map*

$$(\cdot) \smile \nu_j \colon H^*(\Omega^j, \Omega^{j+1}) \to H^{*+1}(\Omega^j, \Omega^{j+1}). \tag{2-8}$$

Proof To see that the previous cup product is well defined, observe that we can write

$$\Omega^j = A \cup B, \quad A = \Omega^{j+1}, \quad B = \Omega^j \cap D_j.$$

In fact, if a point ω belongs to Ω^j, then either $\omega \in \Omega^{j+1}$, or $\operatorname{ind}^+(\omega h) = j$ and consequently $\omega \in \Omega^j \cap D_j$. Since both A and B are open, by excision we get

$$H^*(\Omega^j, \Omega^{j+1}) \simeq H^*(\Omega^j \cap D_j, \Omega^{j+1} \cap D_j).$$

In particular, in order to see that (2-8) is well defined, it is enough to see that

$$(\cdot) \smile \nu_j \colon H^*(\Omega^j \cap D_j, \Omega^{j+1} \cap D_j) \to H^{*+1}(\Omega^j \cap D_j, \Omega^{j+1} \cap D_j)$$

is well defined. Suppose that $\psi \in C^k(\Omega^j)$ is a singular cochain representing a cohomology class in $H^k(\Omega^j \cap D_j, \Omega^{j+1} \cap D_j)$ and $\phi_j \in C^1(D_j)$ is a cochain representing ν_j. The cup product of ψ and ϕ_j is defined on a singular chain $\sigma \colon [v_0, \dots, v_{k+1}] \to \Omega^j \cap D_j$ in the usual way,

$$(\psi \smile \phi_j)(\sigma) = \psi(\sigma|_{[v_0,\dots,v_k]})\phi_j(\sigma|_{[v_k,v_{k+1}]}),$$

from which we see that $\psi \smile \phi_j$ vanishes on $C_{k+1}(\Omega^{j+1} \cap D_j)$ and consequently defines an element of $H^{k+1}(\Omega^j \cap D_j, \Omega^{j+1} \cap D_j)$. □

Since we have inclusions $\Omega^j \supseteq \Omega^{j+1} \supseteq \Omega^{j+2}$, we also consider the connecting homomorphisms

$$\partial \colon H^*(\Omega^{j+1}, \Omega^{j+2}) \to H^{*+1}(\Omega^j, \Omega^{j+1})$$

of the long exact sequence for the triple $(\Omega^j, \Omega^{j+1}, \Omega^{j+2})$.

We summarize the directions of these homomorphisms in the *noncommutative* diagram of maps

$$\begin{tikzcd}
H^i(\Omega^{j+1}, \Omega^{j+2}) \arrow[r, "\partial"] \arrow[d, "(\cdot)\smile\nu_{j+1}"'] & H^{i+1}(\Omega^j, \Omega^{j+1}) \arrow[d, "(\cdot)\smile\nu_j"] \\
H^{i+1}(\Omega^{j+1}, \Omega^{j+2}) \arrow[r, "\partial"] & H^{i+2}(\Omega^j, \Omega^{j+1})
\end{tikzcd}$$

Remark 23 Let $\partial \colon H^i(X, Y) \to H^{i+1}(Z, X)$ be the boundary operator in the exact sequence of the triple (Z, X, Y), where all spaces are open. Following [16, page 201], thanks to the fact that we are working with $\mathbb{Z}_2$–coefficients, we have that $\partial([\phi]) = [\phi \circ \pi \circ \delta]$, where $\delta \colon C_{i+1}(Z, X) \to C_i(X)$ is the boundary operator and $\pi \colon C_i(X) \to C_i(X, Y)$ is the projection operator. Let us also consider $\widetilde{X} \subset A$ both open and such that $(X \cap \widetilde{X}) \cup Y$ is open. If we take the relative cup product

$$H^i(X, Y) \times H^1(\widetilde{X}) \to H^1(X, Y)$$

as defined in Lemma 22, then this coincides with the cup product

$$H^i(X, Y) \times H^1(A) \to H^1(X, Y)$$

as defined in Lemma 22, meaning that given $a \in H^i(X,Y)$ and $b \in H^1(A)$, then $a \smile b = a \smile r^*(b)$, where r^* is just the restriction. In the same way, if we suppose that $\widetilde{Z} \subset A$, we can repeat a similar reasoning for the cup product $H^i(Z,X) \times H^1(\widetilde{Z}) \to H^{i+1}(Z,X)$. Now, given $[\gamma] \in H^1(A)$, we claim that

$$\partial([a] \smile [\gamma]) = \partial[a] \smile [\gamma]. \tag{2-9}$$

At the level of cochains, if we take $c_{i+1} \in C_{i+1}(Z,X)$ to be a singular chain then $\partial(a \smile \gamma)(c_{i+1}) = (a \smile \gamma)(\pi \circ \delta c_{i+1})$. Reasoning as in [16, proof of Lemma 3.6, page 206] shows $(a \smile \gamma)(\pi \circ \delta c_{i+1}) = \partial a \smile \gamma + (-1)^i a \smile \delta^*(\gamma)$ and (2-9) follows from $\delta^*(\gamma) = 0$.

2.4.2 The spectral sequence For the computation of the Betti numbers of V, defined in (2-3) we will need the following result, which is an adaptation from [3; 5]. Clearly the computation of the cohomology of V is equivalent to that of $S^N \setminus V$, by Alexander duality, and in [3; 5] a spectral sequence is introduced for computing the latter.

The delicate part here is that in [3], the spectral sequence is defined for *nondegenerate* systems of quadrics, ie for systems such that the map h is transversal to the cone K, in the sense of [5]; in [22] the spectral sequence is defined also for degenerate systems, but the second differential is not computed explicitly, and in [5] it is defined also for degenerate systems, and the second differential is explicitly computed, but the solutions are studied in the projective space rather than the sphere. Since in our case the system of quadratic inequalities is always degenerate[6] for $m \geq 2$, we will need to prove the existence of such a spectral sequence and to compute its second differential.

Theorem 24 *Let $V = h^{-1}(K) \cap S^N$ be defined by a system of quadratic inequalities, as above. There exists a cohomology spectral sequence $(E_r, d_r)_{r \geq 1}$ converging to $H^*(S^N \setminus V; \mathbb{Z}_2)$ such that:*

(1) *The second page of the spectral sequence is given, for $j > 0$, by*

$$E_2^{i,j} = H^i(\Omega^{j+1}, \Omega^{j+2}; \mathbb{Z}_2).$$

For $j = 0$, the elements of the second page of the spectral sequence fit into a long exact sequence

$$\cdots \to H^i(\Omega^1; \mathbb{Z}_2) \to E_2^{i,0} \to H^i(\Omega^1, \Omega^2; \mathbb{Z}_2) \xrightarrow{(\cdot) \smile \nu_1} H^{i+1}(\Omega^1; \mathbb{Z}_2) \to \cdots. \tag{2-10}$$

(2) *For $j \geq 1$ the second differential $d_2^{i,j} : H^i(\Omega^{j+1}, \Omega^{j+2}) \to H^{i+2}(\Omega^j, \Omega^{j+1})$ is given by*

$$d_2^{i,j} \xi = \partial(\xi \smile \nu_{j+1}) + \partial\xi \smile \nu_j.$$

Proof The proof proceeds similarly to [5, Theorems 25 and 28], using a regularization process and taking the limit over the regularizing parameter. More precisely, let q_0 be a positive definite quadratic form, chosen as in [5, Lemma 13], and for $t > 0$ consider the set

$$B(t) = \{(\omega, x) \in \Omega \times S^N \mid \omega h(x) - t q_0(x) \geq 0\}.$$

[6]This is a consequence of Lemma 33: in fact, for nondegenerate systems the difference of two nearby values of the index function is ± 1, whereas here it is always greater than 1.

The choice of q_0 as in [5, Lemma 13] makes the map $\omega \mapsto \omega h - t q_0$ nondegenerate with respect to K and will allow us to compute the second differential of our spectral sequence. By semialgebraic triviality, for $t > 0$ small enough the set $B(t)$ is homotopy equivalent to

$$B = \{(\omega, x) \in \Omega \times S^N \mid \omega h(x) > 0\}.$$

Moreover, the projection onto the second factor (ie $p_2\colon \Omega \times S^n \to S^N$) restricts to a homotopy equivalence $B \sim p_2(B) = S^N \setminus V$; see [22, Section 3.2]. Therefore, for $t > 0$ small enough,

$$H^*(S^N \setminus V) \simeq H^*(B(t)).$$

We consider now the Leray spectral sequence $(E_r[t], d_r[t])_{r \geq 0}$ of the map

$$p_t := p_1|_{B(t)}\colon B(t) \to \Omega.$$

This spectral sequence converges to the cohomology of $B(t)$.

For the first part of the statement, the structure of $E_2^{i,j}$ in the case $j > 0$ is proved in [22, Section 3.2], as follows. If $t_1 < t_2$ then $B(t_2) \hookrightarrow B(t_1)$ is a homotopy equivalence and $p_{t_1}|_{B(t_2)} = p_{t_2}$. For $0 < t_1 < t_2 < \delta$ the inclusion defines a morphism of filtered differential graded modules

$$\phi_0(t_1, t_2)\colon (E_0[t_1], d_0[t_1]) \to (E_0[t_2], d_0[t_2])$$

turning $\{E_0[t]\}_t$ into an inverse system and thus $\{(E_r[t], d_r[t])\}_t$ into an inverse system of spectral sequences. Then, we can define a new spectral sequence

$$(E_r, d_r) := \varprojlim_t \{(E_r[t], d_r[t])\}.$$

The proof shows that for $j > 0$ we have $E_2^{i,j}[t] = H^i(\Omega_{n-j}[t], \Omega_{n-j-1}[t]; \mathbb{Z}_2)$, where the sets $\Omega_k[t]$ are defined by

$$\Omega_k[t] := \{w \in \Omega \mid i^-(w \cdot h - tg) \leq k\}.$$

Moreover we also have that for $j > 0$ the isomorphism $\phi_2(t_1, t_2)$ is just the homomorphism induced in cohomology by the inclusion $\Omega_j[t_2] \subseteq \Omega_j[t_1]$, and that

$$E_2^{i,j} = \varprojlim_t E_2^{i,j}[t] = H^i(\Omega^j, \Omega^{j+1}; \mathbb{Z}_2).$$

Thanks to this, by semialgebraic triviality, $\phi_2(t_1, t_2)$ is an isomorphism for $0 < t_1 < t_2 < \delta$ with δ sufficiently small, and therefore also $\phi_\infty(t_1, t_2)$ is an isomorphism, assuring the convergence of (E_r, d_r) to $B(t)$; see also the proof of [5, Theorem 25] for more details on this point. Let us call

$$e_t^*\colon H^*(\Omega^j, \Omega^{j+1}; \mathbb{Z}_2) \to H^*(\Omega_{n-j}(t), \Omega_{n-j-1}(t); \mathbb{Z}_2)$$

the isomorphism induced by the inclusion. From now on we choose our scalar product on $\mathbb{R}^{N+1}$ to be $g = q_0$, in such a way that the matrix associated to q_0 through the polarization identity (2-4) is the identity matrix.

For the case $j = 0$ we know, thanks to [2], that there exists a long exact sequence

$$\text{(2-11)} \quad \cdots \to H^i(\Omega_{n-1}[t];\mathbb{Z}_2) \to E_2^{i,0}[t] \to H^i(\Omega_{n-1}[t],\Omega_{n-2}[t];\mathbb{Z}_2) \xrightarrow{(\cdot)\smile\nu_1} H^{i+1}(\Omega_{n-1}[t];\mathbb{Z}_2) \to \cdots .$$

We can pass to the inverse limit of these long exact sequences respect to t in the obvious way, obtaining a long exact sequence[7]

$$\cdots \to H^i(\Omega^1;\mathbb{Z}_2) \to E_2^{i,0} \to H^i(\Omega^1,\Omega^2;\mathbb{Z}_2) \xrightarrow{(\cdot)\smile\nu_1} H^{i+1}(\Omega^1;\mathbb{Z}_2) \to \cdots ,$$

where we have used the fact that $(e^*)^{-1} \circ ((\cdot) \smile \nu_j) \circ e^* = (\cdot) \smile \nu_j$. We will get back to this point later.

This proves point (1) of the statement. For the point concerning the differential, thanks to [2, Theorem 3] we know that the second differential $d_2[t]$ of the spectral sequence $(E_r[t], d_r[t])$ with

$$d_2^{i,j}[t]\colon H^i(\Omega_{n-j-1}[t],\Omega_{n-j-2}[t];\mathbb{Z}_2) \to H^{i+2}(\Omega_{n-j}[t],\Omega_{n-j-1}[t];\mathbb{Z}_2)$$

has the form

$$d_2^{i,j}[t]\xi = \partial_t (i_t^*)^{-1}(i_t^*\xi \smile \nu_{j+1}) + (i_t^*)^{-1}(i_t^*\partial_t\xi \smile \nu_j),$$

where

$$\partial_t\colon H^i(\Omega_{n-j-1}[t],\Omega_{n-j-2}[t];\mathbb{Z}_2) \to H^{i+1}(\Omega_{n-j}[t],\Omega_{n-j-1}[t];\mathbb{Z}_2)$$

is the connecting homomorphism in the exact sequence of the triple $(\Omega_{n-j}[t], \Omega_{n-j-1}[t], \Omega_{n-j-2}[t])$, and the map

$$i_t^*\colon H^i(\Omega_{n-j}[t],\Omega_{n-j-1}[t];\mathbb{Z}_2) \to H^i(\Omega_{n-j}[t]\cap\mathcal{D}_j,\Omega_{n-j-1}(t)\cap\mathcal{D}_j;\mathbb{Z}_2)$$

is the map induced by the inclusion; this map is an isomorphism by excision.

The second differential for $j > 1$ of our new spectral sequence $(E_r^{i,j}, d_r)$ is $d_2^{i,j} := (e_t^*)^{-1} \circ d_2^{i,j}(t) \circ e_t^*$. More explicitly,

$$d_2^{i,j} = \partial(i_t^* \circ e_t^*)^{-1}((i_t^* \circ e_t^*)\xi \smile \nu_{j+1}) + (i_t^* \circ e_t^*)^{-1}((i_t^* \circ e_t^*)\partial\xi \smile \nu_j),$$

thanks to the naturality of the connecting homomorphism.

Let us now consider the diagram

$$\begin{tikzcd}
& (\Omega^j,\Omega^{j+1}) & \\
(\Omega_{n-j-1}[t]\cap\mathcal{D}_j,\Omega_{n-j-2}[t]\cap\mathcal{D}_j) \arrow[ur, "e_t\circ i_t"] \arrow[rr, "j_t"] & & (\Omega^j\cap\mathcal{D}_j,\Omega^{j+1}\cap\mathcal{D}_j) \arrow[ul, "i"']
\end{tikzcd}$$

where all the maps are inclusions, and all the induced homomorphisms in cohomology are isomorphisms.

[7]In this long exact sequence we are still using ν_1 because we chose our scalar product to be g_0. Same for the definition of $d_2(t)$, where we used ν_j.

We can write

$$d_2^{i,j} = \partial(j_t^* \circ i^*)^{-1}((j_t^* \circ i^*)\xi \smile \nu_{j+1}) + (j_t^* \circ i^*)^{-1}((j_t^* \circ i^*)\partial\xi \smile \nu_j) \tag{2-12}$$
$$= \partial(i^*)^{-1}(j^*)^{-1}(j_t^*(i^*\xi) \smile j_t^* \circ \nu_{j+1}) + (i^*)^{-1}(j_t^*)^{-1}(j_t^*(i^*\partial\xi) \smile j_t^* \circ \nu_j)$$
$$= \partial(i^*)^{-1}(i^*\xi \smile \nu_{j+1}) + (i^*)^{-1}(i^*\partial\xi \smile \nu_j),$$

where the pullback property of the pullback in the third equality holds true because in that case it is just the standard cup product. Because of how we defined the cup product in Lemma 22, we have the claim. □

2.5 Analytic combinatorics

In order to study the asymptotic of the number of isotopy classes of geometric graphs on the real line we will need some tools from analytic combinatorics. For a full introduction to the topic, see [12]. Given a generating function $G(x) = \sum_{n=0}^{\infty} a_n x^n$ of a sequence a_n, we want to study the asymptotics of such a sequence. There are various techniques to do this.

Definition 25 We say that a sequence $\{a_n\}$ is of exponential order K^n, which we abbreviate as $a_n \bowtie K^n$, if and only if $\limsup |a_n|^{1/n} = K$.

If we have $a_n \bowtie K^n$ then $a_n = K^n\theta(n)$ with $\limsup |\theta(n)|^{1/n} = 1$. The term $\theta(n)$ is called the subexponential factor. In order to study the subexponential factor $\theta(n)$ we should look at the singularities of the generating function.

Definition 26 Given two numbers ϕ and R with $R > 1$ and $0 < \phi < \pi/2$, define an open domain

$$D(\phi, R) := \{z \in \mathbb{C} \mid |z| < R, z \neq 1, |\arg(z-1)| > \phi\}.$$

A domain of this type is called D–domain.

Denoting by S the set of all meromorphic functions of the form

$$S := \{(1-z)^{-\alpha} \mid \alpha \in \mathbb{R}, z \in \mathbb{C}\},$$

we recall the next result [12, Theorem VI.4], which we will need in the sequel.

Theorem 27 *Let $G(z)$ be an analytic function at 0 with a singularity at ζ, such that $G(z)$ can be continued to a domain of the form $\zeta \cdot D_0$ for a D–domain D_0, where $\zeta \cdot D_0$ is the image of D_0 by the mapping $z \to \zeta z$. Assume there exist two functions σ and τ, where σ is a finite linear combination of elements in S and $\tau \in S$, such that*

$$G(z) = \sigma\Big(\frac{z}{\zeta}\Big) + O\Big(\tau\Big(\frac{z}{\zeta}\Big)\Big) \quad \text{as } z \to \zeta \text{ in } \zeta \cdot D_0.$$

Then the coefficients of $G(z)$ satisfy the asymptotic estimate

$$a_n = \zeta^{-n}\sigma_n + O(\zeta^{-n}\tau_n^*),$$

where $\sigma(z) = \sum_{n=0}^{\infty} \sigma_n z^n$ and $\tau_n^* = n^{\alpha-1}$ if $\tau(z) = (1-z)^{-\alpha}$.

Remark 28 For later use, we record the following. The Newton binomial series is defined by

$$(1-z)^{-\alpha} = \sum_{n=0}^{\infty} b_n z^n, \quad \text{where } b_n = \binom{n+\alpha-1}{n}.$$

Using

$$\binom{n+\alpha-1}{n} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)},$$

we get the following asymptotics for its coefficients:

$$b_n = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+O\left(\frac{1}{n}\right)\right).$$

3 Homology of the chambers and the Floer number

3.1 Graphs and sign conditions

Recall that given a graph G on n vertices we have defined

$$W_{G,d} = \{P \in \mathbb{R}^{d\times n} \setminus \Delta_{d,n} \mid G(P) \cong G\} \subset \mathbb{R}^{d\times n}.$$

In other words, $W_{G,d}$ consists of all the points $P \in \mathbb{R}^{d\times n}$ not on the discriminant whose corresponding graph is isomorphic to G. For small d this set could be a union of several chambers, but for large d it is an actual chamber (a connected open set).

Now we introduce an alternative notation for labeling the sets $W_{G,d}$. For every $1 \le i < j \le n$ let us denote by $q_{ij} : \mathbb{R}^{d\times n} \to \mathbb{R}$ the quadratic polynomial

$$q_{ij}(x_1,\dots,x_n) = \|x_i - x_j\|^2 - 1, \quad \text{where } (x_1,\dots x_n) \in \mathbb{R}^{d\times n}. \tag{3-1}$$

Notice that the discriminant $\Delta_{d,n}$ is given by

$$\Delta_{d,n} = \left\{(x_1,\dots,x_n) \in \mathbb{R}^{d\times n} \;\middle|\; \prod_{i<j} q_{ij}(x_1,\dots,x_n) = 0\right\} = \bigcup_{i<j} \Delta_{d,n}^{(i,j)}, \tag{3-2}$$

where the sets $\Delta_{d,n}^{(i,j)}$ are as defined in (2-2). We denote by $\binom{[n]}{2}$ the set of all possible pairs (i,j) with $1 \le i < j \le n$, and by $2^{\binom{[n]}{2}}$ the set of all possible choices of signs $\sigma_{ij} \in \{\pm\}$ for elements in $\binom{[n]}{2}$.

Definition 29 (sign condition) For every $\sigma \in 2^{\binom{n}{2}}$, we denote by $W_{\sigma,d} \in \mathbb{R}^{d\times n}$ the open set

$$W_{\sigma,d} = \{x = (x_1,\dots,x_n) \in \mathbb{R}^{d\times n} \mid \operatorname{sign}(q_{ij}(x)) = \sigma_{ij}\}.$$

At this point, what is clear from (3-2) is that $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ can be written as the union of all the possible sign conditions. The following lemma will be useful. It tells us that we can label the chambers of $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ either with a graph or with a sign condition — however, at this point we only prove that the sets $\{W_{G,d}\}$ and $\{W_{\sigma,d}\}$ coincide; the fact that the chambers of $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$, for $d \ge n+1$, are exactly the sign conditions will follow from Corollary 47.

Lemma 30 *For every G graph on n vertices, there is a sign condition $\sigma = \sigma(G)$ such that $W_{G,d} = W_{\sigma,d}$. Conversely, for every σ there exists a $G(\sigma)$ such that $W_{\sigma,d} = W_{G,\sigma}$. In other words, the signs of the family of quadrics $\{q_{ij}\}_{1\le i<j\le n}$ on a point P determine the isomorphism class of $G(P)$ uniquely as a labeled graph.*

Proof Given $P \in \mathbb{R}^{d\times n} \setminus \Delta_{d,n}$, it is clear from the definition of geometric graph that $G(P) \cong G$ if and only if $q_{i,j} < 0$ when $(i,j) \in G$ and $q_{i,j} > 0$ when $(i,j) \notin G$. From this it follows that $W_{G,d} = W_{\sigma,d}$, where $\sigma_{i,j} = 1$ if $(i,j) \in G$, and $\sigma_{i,j} = -1$ if $(i,j) \notin G$. □

3.2 Betti numbers of the chambers

In this section we study the asymptotic distribution of the Betti numbers of the chambers. Before giving the main result, we will need some intermediate steps.

3.2.1 Some preliminary reductions Using Lemma 30 we can immediately switch from the graph labeling to the sign condition, and given G there exists σ such that $W_{G,d} = W_{\sigma,d}$. In this way we describe the chamber we are interested in with a system of quadratic inequalities, and we will take advantage of this description.

From now on we will assume that G is not the complete graph, since for this case $W_{G,d}$ is convex and therefore the study of its topology is complete. Our first step is to replace $W_{\sigma,d}$ with another space which has the same homology and which is compact. To start with, we have

$$W_{\sigma,d} = \{x \in \mathbb{R}^{d\times n} \mid \operatorname{sign}(q_{ij}(x)) = \sigma_{i,j} \text{ for all } 1 \le i < j \le n\},$$

with the quadrics $q_{ij} : \mathbb{R}^{d\times n} \to \mathbb{R}$ defined above. Since σ is fixed, it will be convenient for us to define the new quadrics

$$s_{ij} = \sigma_{ij} q_{ij} \quad \text{and} \quad h_{ij}(x,z) = \sigma_{ij}(\|x_i - x_j\|^2 - z^2).$$

We set $N = nd$, and $k = \binom{n}{2}$ and for every $\epsilon > 0$ consider the set

$$W_{\sigma,d}(\epsilon) = \{[x:z] \in \mathbb{R}\mathrm{P}^N \mid h_{ij}(x,z) \ge \epsilon z^2 \text{ for all } 1 \le i < j \le n,\ \|x\|^2 \le \epsilon^{-1} z^2\} \subseteq \mathbb{R}\mathrm{P}^N.$$

Notice that $W_{\sigma,d}(\epsilon) \cap \{z = 0\} = \varnothing$, because if $z = 0$ then the last inequality defining $W_{\sigma,d}(\epsilon)$ forces $x = 0$. Therefore, in the affine chart $\{z \ne 0\}$ the set $W_{\sigma,d}(\epsilon)$ can be described as

$$W_{\sigma,d}(\epsilon) \cap \{z \ne 0\} = \{x \in \mathbb{R}^N \mid s_{ij}(x) \ge \epsilon \text{ for all } 1 \le i < j \le n,\ \|x\|^2 \le \epsilon^{-1}\},$$

and can be identified with a subset of $W_{\sigma,d}$.

Proposition 31 *For every $d > 0$ there exists a $\epsilon(d)$ such that for all $\epsilon < \epsilon(d)$, the inclusion*

$$W_{\sigma,d}(\epsilon) \hookrightarrow W_{\sigma,d}$$

is a homotopy equivalence.

Proof First of all, let us notice that $W_{\sigma,d}(\epsilon) \cap \{z=0\} = \varnothing$, and we can write

$$W_{\sigma,d}(\epsilon) = \Big\{(p_1,\dots,p_n) \in \mathbb{R}^{d\times n} \;\Big|\; (\|p_i - p_j\|^2 - 1)\sigma^G_{i,j} \geq \epsilon,\ \|P\|^2 \leq \frac{1}{\epsilon}\Big\}.$$

If the sign condition σ is not always negative for every (i,j) — ie the graph G is not complete — we conclude by Corollary 21. $\square$

The set $W_{\sigma,d}(\epsilon)$ is now compact and for $\epsilon < \epsilon(d)$ has the same homology of $W_{\sigma,d}$. For technical reasons, this is not yet the set we will work with. Instead we will work with its double cover

$$V_{\sigma,d}(\epsilon) = \{x \in S^N \mid h_{ij}(x,z) \geq \epsilon z^2 \text{ for all } 1 \leq i < j \leq n,\ \|x\|^2 \leq \epsilon^{-1} z^2\} \subset S^N.$$

This will not be an obstacle for computing the Betti numbers of $W_{\sigma,d}(\epsilon)$, because of the next lemma.

Lemma 32 *For every $\epsilon > 0$, the set $V_{d,\epsilon}(\epsilon) \subset S^N$ consists of two disjoint copies of $W_{\sigma,d}(\epsilon)$. In particular, for all $k \geq 0$,*

$$b_k(W_{\sigma,d}(\epsilon)) = \tfrac{1}{2} b_k(V_{\sigma,d}(\epsilon)).$$

Proof Let $\{z=0\} \simeq S^{N-1}$ be the equator in S^N and observe that $\{z=0\} \cap V_{\sigma,d}(\epsilon) = \varnothing$. This implies that

$$V_{\sigma,d}(\epsilon) = (V_{\sigma,d}(\epsilon) \cap \{z>0\}) \sqcup (V_{\sigma,d}(\epsilon) \cap \{z<0\}).$$

The involution $(x,z) \mapsto (-x-z)$ on the sphere S^N restricts to a homeomorphism between $V_{\sigma,d}(\epsilon) \cap \{z>0\}$ and $V_{\sigma,d}(\epsilon) \cap \{z<0\}$. Each of these sets is homeomorphic to its projection to the projective space $\mathbb{R}\mathrm{P}^N$, which is the set $W_{\sigma,d}(\epsilon)$. $\square$

3.2.2 Systems of quadratic inequalities The set $V_{\sigma,d}(\epsilon)$ defined above is the set of solutions of a system of quadratic inequalities, and we will now use the spectral sequence from Section 2.4 for computing its Betti numbers with $\mathbb{Z}_2$–coefficients.

Let us introduce homogeneous quadrics $h_{ij,\epsilon}, h_{0,\epsilon} \colon \mathbb{R}^{N+1} \to \mathbb{R}^{k+1}$ defined for all $1 \leq i < j \leq n$ by

$$h_{ij,\epsilon}(x,z) = \sigma_{ij}\|x_i - x_j\|^2 - \sigma_{ij} z^2 - \epsilon z^2 \quad \text{and} \quad h_{0,\epsilon}(x,z) = \|x\|^2 - \epsilon^{-1} z^2,$$

in order to reduce to the framework of Section 2.4. These quadrics can be put as the components of a quadratic *map* defined by

$$h_\epsilon = (h_{0,\epsilon}, h_{1,\epsilon}, h_{2,\epsilon}, \dots, h_{k,\epsilon}) \colon \mathbb{R}^{N+1} \to \mathbb{R}^{k+1},$$

where we are using the identification of sets of indices $\{1,2,\dots,k\} = \{(1,2),(1,3),\dots,(n-1,n)\}$. Inside the space $\mathbb{R}^{k+1}$ we can consider the closed convex cone

$$K = \{y_0 \leq 0, y_1 \geq 0, \dots, y_k \geq 0\},$$

so that our original set can be written as

$$V_{\sigma,d}(\epsilon) = h_\epsilon^{-1}(K).$$

In this case the set $\Omega \subset S^k$ is the set

$$\Omega = \{(\omega_0,\dots,\omega_k) \in S^k \mid \omega_0 \geq 0, \omega_1 \leq 0, \dots, \omega_k \leq 0\}.$$

For every point $\omega = (\omega_0, \dots, \omega_k)$ we can consider the quadratic form ωh_ϵ defined by

$$\omega h_\epsilon = \omega_0 h_{0,\epsilon} + \cdots + \omega_k h_{k,\epsilon}.$$

Using this notation, for every $j \geq 0$ we define the sets

$$\Omega^j(\epsilon) = \{(\omega_0, \dots, \omega_k) \in \Omega \mid \operatorname{ind}^+(\omega H_\epsilon) \geq j\}.$$

These are just the sets Ω^j defined in Section 2.4, in the case of the quadratic map h_ϵ.

For every $1 \leq i < j \leq n$ let us also denote by $U_{ij} \in \operatorname{Sym}(n, \mathbb{R})$ the symmetric matrix representing the quadratic form $u_{ij} : \mathbb{R}^n \to \mathbb{R}$ defined by

$$u_{ij}(t_1, \dots, t_n) = \sigma_{ij}(t_i - t_j)^2.$$

Then, if $H_{ij} \in \operatorname{Sym}(dn, \mathbb{R})$ is the matrix representing the quadratic form $x \mapsto \sigma_{ij}\|x_i - x_j\|^2$, we have

$$H_{ij} = U_{ij} \otimes \mathbf{1}_d.$$

Lemma 33 *The index function* $\operatorname{ind}^+ : \Omega \to \mathbb{N}$ *for our family of quadrics can be written as*

$$\operatorname{ind}^+(\omega H_\epsilon) = d \cdot \operatorname{ind}_1^+(\omega) + \operatorname{ind}_{0,\epsilon}^+(\omega),$$

where

$$\operatorname{ind}_1^+(\omega) = \operatorname{ind}^+\Bigl(\omega_0 \mathbf{1}_n + \sum_{i<j} \omega_{ij} U_{ij}\Bigr) \quad \textit{and} \quad \operatorname{ind}_{0,\epsilon}^+(\omega) = \operatorname{ind}^+\Bigl(-\frac{\omega_0}{\epsilon} - \sum_{i<j} \omega_{ij}(\sigma_{ij} + \epsilon)\Bigr).$$

Before giving the proof, observe that none of the functions $\operatorname{ind}_1^+, \operatorname{ind}_{0,\epsilon}^+ : \Omega \to \mathbb{N}$ depends on d and that ind_1^+ does not even depend on ϵ.

Proof Observe that, for $\omega = (\omega_0, \omega_{ij}) \in \Omega$, the matrix ωH_ϵ is a block matrix:

$$\omega H_\epsilon = \left(\begin{array}{c|ccc} -\omega_0/\epsilon - \sum_{i<j} \omega_{ij}(\sigma_{ij} + \epsilon) & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \omega_0 \mathbf{1}_{dn} + \sum_{i<j} \omega_{ij} H_{ij} & \\ 0 & & & \end{array}\right)$$

and, in particular,

$$\operatorname{ind}^+(\omega H_\epsilon) = \operatorname{ind}^+\Bigl(-\frac{\omega_0}{\epsilon} - \sum_{i<j} \omega_{ij}(\sigma_{ij} + \epsilon)\Bigr) + \operatorname{ind}^+\Bigl(\omega_0 \mathbf{1}_{dn} + \sum_{i<j} \omega_{ij} H_{ij}\Bigr).$$

The matrix $\omega_0 \mathbf{1}_{dn} + \sum_{i<j} \omega_{ij} H_{ij}$ is a tensor product of matrices,

$$\omega_0 \mathbf{1}_{dn} + \sum_{i<j} \omega_{ij} H_{ij} = \Bigl(\omega_0 \mathbf{1}_d + \sum_{i<j} \omega_{ij} U_{ij}\Bigr) \otimes \mathbf{1}_n.$$

If a matrix $Q \in \mathrm{Sym}(n, \mathbb{R})$ has eigenvalues $\lambda_1(Q) \geq \cdots \geq \lambda_n(Q)$ (possibly with repetitions), the matrix $Q \otimes \mathbf{1}_d$ has eigenvalues

$$\lambda_{i,j}(Q \otimes \mathbf{1}_d) = \lambda_i(Q) \quad \text{for } i = 1, \dots, n,\ j = 1, \dots, d.$$

In particular,

$$\mathrm{ind}^+(Q \otimes \mathbf{1}_d) = d \cdot \mathrm{ind}^+(Q),$$

and the result now follows. □

Corollary 34 *For $d \geq n+1$, the set $\Omega^{nd}(\epsilon)$ is contractible and $\Omega^{nd+1}(\epsilon)$ is empty.*

Proof Let us first show that $\Omega^{nd+1}(\epsilon) = \varnothing$. To this end, consider the set

$$B(\epsilon) = \{(\omega, [x]) \in \Omega \times \mathbb{R}\mathrm{P}^N \mid \omega h_\epsilon(x) \geq 0\}.$$

By [5, Lemma 24] the projection $\pi = p_2|_{B(\epsilon)}$ on the second factor gives a homotopy equivalence between $B(\epsilon)$ and its image

$$\pi(B(\epsilon)) = \mathbb{R}\mathrm{P}^N \setminus W_{\sigma,d}(\epsilon).$$

Since $W_{\sigma,d}(\epsilon)$ is nonempty, we know that

$$\pi(B(\epsilon)) \neq \mathbb{R}\mathrm{P}^N. \tag{3-3}$$

If now there was $\omega \in \Omega$ such that $\mathrm{ind}^+(\omega) = N+1$, then $\omega h_\epsilon > 0$ and $\{\omega\} \times \mathbb{R}\mathrm{P}^N \subset B(\epsilon)$. This would imply that $\pi(B(\epsilon)) = \mathbb{R}\mathrm{P}^N$, which contradicts (3-3).

Let us now prove that $\Omega^{nd}(\epsilon)$ is contractible. For $d \geq n+1$, since $\Omega^{nd+1}(\epsilon) = \varnothing$, then the set $\Omega^{nd}(\epsilon)$ can be described as

$$\Omega^{nd}(\epsilon) = \{\mathrm{ind}^+ = nd\} = \{d \cdot \mathrm{ind}_1^+ + \mathrm{ind}_{0,\epsilon} \geq nd\} = \{\mathrm{ind}_1^+ = n\} \cap \{\mathrm{ind}_{0,\epsilon} = 0\}.$$

Observe that the point $\omega = (1, 0, \dots, 0) \in \Omega$ belongs to both the sets $\{\mathrm{ind}_1^+ = n\}$ and $\{\mathrm{ind}_{0,\epsilon} = 0\}$, and their intersection is nonempty.

Now, $\{\mathrm{ind}_1^+ = n\}$ and $\{\mathrm{ind}_{0,\epsilon} = 0\}$ are obtained by intersecting a convex set in $\mathbb{R}^{k+1}$ with $\Omega \cap S^k$, as they coincide with the set of the points where the linear families of symmetric matrices $\omega_0 \mathbf{1}_n + \sum_{i<j} \omega_{ij} U_{ij}$ and $-\omega_0/\epsilon - \sum_{i<j} \omega_{ij}(\sigma_{ij} + \epsilon)$ are, respectively, positive definite and negative semidefinite. In other words, $\{\mathrm{ind}_1^+ = n\}$ is the preimage of the positive definite cone under the linear map $\omega \mapsto \omega_0 \mathbf{1}_d + \sum_{i<j} \omega_{ij} U_{ij}$, and $\{\mathrm{ind}_{0,\epsilon} = 0\}$ is the preimage of the negative semidefinite cone under the linear map $\omega \mapsto -\omega_0/\epsilon - \sum_{i<j} \omega_{ij}(\sigma_{ij} + \epsilon)$.

Therefore $\Omega^{nd}(\epsilon)$ is the intersection in $S^k \cap \Omega$ of convex sets, and being Ω itself also convex, this intersection is contractible. □

Recalling the notation of Section 2.4, but making it dependent on ϵ, we have the vector bundle $P^j(\epsilon) \subseteq \Omega^j(\epsilon) \setminus \Omega^{j+1}(\epsilon) \times \mathbb{R}^{N+1}$

$$\begin{tikzcd} \mathbb{R}^j \arrow[r, hook] & P^j(\epsilon) \arrow[d] \\ & \Omega^j(\epsilon) \setminus \Omega^{j+1}(\epsilon) \end{tikzcd} \tag{3-4}$$

whose fiber over a point ω is the positive eigenspace of ωH_ϵ. As above, this bundle is the restriction of a bundle over the set

$$D_j(\epsilon) = \{\omega \mid \lambda_j(\omega H_\epsilon) \neq \lambda_{j+1}(\omega H_\epsilon)\},$$

ie the set where the jth eigenvalue of ωH_ϵ is distinct from the $(j+1)^{\text{st}}$. We still denote this bundle by $P_j(\epsilon) \subset D_j(\epsilon) \times \mathbb{R}^{N+1}$:

$$\begin{tikzcd} \mathbb{R}^j \arrow[r, hook] & P^j(\epsilon) \arrow[d] \\ & D_j(\epsilon) \end{tikzcd} \tag{3-5}$$

Here the fiber over a point $\omega \in D_j(\epsilon)$ consists of the eigenspace of ωH_ϵ associated to the first j eigenvalues. We denote the first Stiefel–Whitney class of this bundle by

$$\nu_j(\epsilon) \in H^1(D_j(\epsilon)). \tag{3-6}$$

Restating Theorem 24 in this setting, we get the following.

Theorem 35 *There exists a cohomology spectral sequence $(E_r(\epsilon), d_r(\epsilon)_{r\geq 1})$ which converges to $H^*(S^N \setminus V_{\sigma,d}(\epsilon); \mathbb{Z}_2)$ and is such that:*

(1) *The second page of the spectral sequence is given, for $j > 0$, by*

$$E_2^{i,j}(\epsilon) = H^i(\Omega^{j+1}(\epsilon), \Omega^{j+2}(\epsilon); \mathbb{Z}_2).$$

For $j = 0$, the elements of the second page of the spectral sequence fit into a long exact sequence

$$\cdots \to H^i(\Omega^1(\epsilon); \mathbb{Z}_2) \to E_2^{i,0}(\epsilon) \to H^i(\Omega^1(\epsilon), \Omega^2(\epsilon); \mathbb{Z}_2) \xrightarrow{(\cdot)\smile \nu_1(\epsilon)} H^{i+1}(\Omega^1(\epsilon); \mathbb{Z}_2) \to \cdots. \tag{3-7}$$

(2) *For $j \geq 1$, the second differential $d_2^{i,j}(\epsilon)\colon H^i(\Omega^{j+1}(\epsilon), \Omega^{j+2}(\epsilon)) \to H^{i+2}(\Omega^j(\epsilon), \Omega^{j+1}(\epsilon))$ is given by*

$$d_2^{i,j}(\epsilon)\xi = \partial(\xi \smile \nu_{j+1}(\epsilon)) + \partial\xi \smile \nu_j(\epsilon).$$

Remark 36 As explained in [5, Introduction], the second differential only depends on the restriction of $\nu_j(\epsilon)$ to the set $\Omega^j(\epsilon) \setminus \Omega^{j+1}(\epsilon)$.

Remark 37 In the previous spectral sequence, the coefficient group for the various cohomologies is the field $\mathbb{Z}_2$. There is an analogous spectral sequence for coefficients in $\mathbb{Z}$, but the description of its differentials is less clear.

3.2.3 The analysis of the spectral sequence and its asymptotic structure We will start by proving the following proposition, which deals with the stabilization of entries of the second page of the spectral sequence of Theorem 35.

Proposition 38 *There exist semialgebraic topological spaces*

$$\Omega = A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \supseteq \cdots \supseteq A_n \supseteq B_n = \varnothing,$$

vector spaces $N^{0,0}, \dots, N^{k,0}$ and $\epsilon_1 > 0$ such that for all $\epsilon \le \epsilon_1$, the second page of the spectral sequence of Theorem 35 has the structure

$$E_2^{i,j}(\epsilon) \simeq \begin{cases} H^i(B_\ell, A_{\ell+1}) & \text{if } j = \ell d, \\ H^i(A_\ell, B_\ell) & \text{if } j = \ell d - 1, \\ N^{i,0} & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3-8}$$

Proof Observe first that the second page of the spectral sequence is zero in the region

$$\{(i,j) \mid i \ge k+1,\ j > 0\},$$

because all the sets $\Omega^j(\epsilon)$ are semialgebraic and of dimension at most k (since they are contained in $\Omega \subset S^k$). The $j = 0$ row of the spectral sequence is also zero for $i \ge k+2$, since for the same reason all the groups in the exact sequence in (3-7) are zero.

Observe now that Lemma 33 implies that the only possible values of the function $\mathrm{ind}^+ \colon \Omega \to \mathbb{N}$ are $0, 1, d, d+1, \dots, nd, nd+1$ and in particular,

$$\begin{aligned} \Omega = \Omega^0(\epsilon) \supseteq \Omega^1(\epsilon) \supseteq \Omega^2(\epsilon) = \Omega^3(\epsilon) = \cdots = \Omega^d(\epsilon) \supseteq \Omega^{d+1}(\epsilon) \supseteq \Omega^{d+2}(\epsilon) = \Omega^{d+3}(\epsilon) \\ = \cdots = \Omega^{nd-1}(\epsilon) = \Omega^{nd}(\epsilon) \supseteq \Omega^{nd+1}(\epsilon) \supseteq \varnothing. \end{aligned} \tag{3-9}$$

In particular, for every $\ell = 0, \dots, n$, we deduce the vanishing of the homology of all the relative pairs:

$$H^*(\Omega^{d\ell+2}(\epsilon), \Omega^{d\ell+3}(\epsilon)) = \cdots = H^*(\Omega^{(\ell+1)d-1}(\epsilon), \Omega^{(\ell+1)d}(\epsilon)) = 0.$$

This proves the "otherwise" part of the claim in (3-8).

We now define the sets $A_\ell(\epsilon) = \{\mathrm{ind}^+ \ge d\ell\}$ and $B_\ell(\epsilon) = \{\mathrm{ind}^+ \ge d\ell + 1\}$ and observe that

$$A_\ell(\epsilon) = \{\mathrm{ind}_1^+ \ge \ell\}, \quad B_\ell(\epsilon) = (\{\mathrm{ind}_1^+ \ge \ell\} \cap \{\mathrm{ind}_{0,\epsilon}^+ = 1\}) \cup \{\mathrm{ind}_1^+ \ge \ell + 1\},$$

where the index functions $\mathrm{ind}_{0,\epsilon}^+, \mathrm{ind}_1^+ \colon \Omega \to \mathbb{N}$ are defined in Lemma 33. Since ind_1^+ does not depend on d nor on ϵ and $\mathrm{ind}_{0,\epsilon}$ does not depend on d, by semialgebraic triviality it follows that there exists $\epsilon_1 > 0$ such that the homotopy of the sequence of inclusions

$$\Omega = A_0(\epsilon) \supseteq B_0(\epsilon) \supseteq A_1(\epsilon) \supseteq B_1(\epsilon) \supseteq \cdots \supseteq A_n(\epsilon) \supseteq B_n(\epsilon) = \varnothing$$

stabilizes for $\epsilon \le \epsilon_1$.

We define $A_\ell = A_\ell(\epsilon_1)$ and $B_\ell = B_\ell(\epsilon_1)$. With this notation we have that the sequence of inclusions (3-9) for $\epsilon \le \epsilon_0$ becomes, up to natural homotopy equivalences,

$$\Omega = A_0 \supseteq B_0 \supseteq A_1 = A_1 = \cdots = A_1 \supseteq B_1 \supseteq A_2 = A_2 = \cdots = A_n = A_n \supseteq B_n \supseteq \varnothing.$$

This proves the statement for the term $E_2^{i,j}(\epsilon)$ of the spectral sequence with $j = \ell d - 1, \ell d$.

In the case $j = 0$, we observe that the dimension of $E_2^{i,0}(\epsilon)$ is determined by the exact sequence

$$0 \to \ker \to H^{i-1}(\Omega^1(\epsilon), \Omega^2(\epsilon)) \to H^i(\Omega^1(\epsilon)) \to E_2^{i,0}(\epsilon) \to H^i(\Omega^1(\epsilon), \Omega^2(\epsilon)) \\ \to H^{i+1}(\Omega^1(\epsilon)) \to \operatorname{coker} \to 0,$$

where ker and coker refer to the map $x \mapsto x \smile \nu_1(\epsilon)$. The homotopy of the first, the third and the fourth element of the above sequence stabilizes for $\epsilon \le \epsilon_1$; moreover (possibly choosing a smaller ϵ_1) also the homotopy of the bundle $P^1(\epsilon) \to D_1(\epsilon)$ from (3-5) stabilizes for $\epsilon \le \epsilon_1$ and therefore the map $x \mapsto x \smile \nu_1(\epsilon))$ stabilizes as well, and consequently the ranks of ker and coker stabilize. This gives the stabilization of $\dim_{\mathbb{Z}_2}(E_2^{i,0}(\epsilon))$ to a finite number for $\epsilon \le \epsilon_1$. We set

$$N^{i,0} := \mathbb{Z}_2^{\dim_{\mathbb{Z}_2}(E_2^{i,0}(\epsilon))} \quad \text{for all } \epsilon \le \epsilon_1. \qquad \square$$

Next we deal with the stabilization of the second differential.

Proposition 39 *The second differential of the spectral sequence (3-8) is zero.*

Proof Observe that the only possible nonzero differential of the spectral sequence is, for $d \ge 2$,

$$d_2^{*,\ell d}(\epsilon)\colon E_2^{*,\ell d}(\epsilon) \to E_2^{*+2,\ell d-1}(\epsilon).$$

Let us recall that we have defined $\omega H_\epsilon = \omega q_1 + \omega q_2$, where $\omega q_1 = (\omega_0 \mathbf{1}_{dn} + \sum_{i<j} \omega_{ij} H_{ij})$ and $\omega q_2 = \bigl(-\omega_0/\epsilon - \sum_{i<j} \omega_{ij}(\sigma_{ij} + \epsilon)\bigr) z^2$. We introduce the vector bundles

$$\begin{array}{ccc} \mathbb{R}^{d\times l} \hookrightarrow N_{\ell d} & & \mathbb{R} \hookrightarrow E(\epsilon) \\ \downarrow & \text{and} & \downarrow \\ \mathcal{D}^1_{\ell d} & & \Omega \end{array}$$

where $\mathcal{D}^1_{\ell d} := \{\omega \in \Omega \mid \lambda_{\ell d}(\omega q_1) \ne \lambda_{ld+1}(\omega q_1)\}$, $N_{\ell d} \subset \mathcal{D}^1_{\ell d} \times \mathbb{R}^{dn}$ is the bundle of the eigenspace of the first ℓd eigenvalues of the upper-left block of ωH_ϵ, and the bundle $E(\epsilon)$ associates to every point of Ω the unique eigenvector of ωq_2.

Observe that $\mathcal{D}_{\ell d}(\epsilon) \subset \mathcal{D}^1_{\ell d}$ and also $\mathcal{D}_{\ell d+1}(\epsilon) \subset \mathcal{D}^1_{\ell d}$.

The vector bundle $P_{\ell d}(\epsilon)$ from (3-5) for $j = \ell d$ has the property that

$$P_{\ell d}(\epsilon) = N_{\ell d}|_{\mathcal{D}_{\ell d}(\epsilon)}$$

when $N_{\ell d}$ is thought as a subbundle of $\mathcal{D}^1_{\ell d} \times \mathbb{R}^{nd+1}$. When $j = \ell d + 1$ we have

$$P_{\ell d+1}(\epsilon) = N_{\ell d}|_{\mathcal{D}_{\ell d+1}(\epsilon)} \oplus E(\epsilon)|_{\mathcal{D}_{\ell d+1}(\epsilon)}$$

because the quadratic form ωq has two diagonal blocks. In particular, denoting by $\gamma_{\ell d}$ and by $\eta(\epsilon)$ the first Stiefel–Whitney class of $N_{\ell d}$ and $E(\epsilon)$, respectively, by naturality of characteristic classes we have the identities

$$\nu_{\ell d} = \gamma_{\ell d}|_{\mathcal{D}_{\ell d}(\epsilon)} \quad \text{and} \quad \nu_{\ell d+1} = \gamma_{\ell d}|_{\mathcal{D}_{\ell d+1}(\epsilon)} + \eta(\epsilon)|_{\mathcal{D}_{\ell d+1}(\epsilon)}.$$

Notice that both $\nu_{\ell d}$ and $\nu_{\ell d+1}$ contain the restriction of the same class $\gamma_{\ell d}$ as a summand. Thanks to Theorem 35, the second differential $d_2^{*,\ell d}(\epsilon)$ can be written as

$$\begin{aligned} d_2^{*,\ell d}(\epsilon)\xi &= \partial(\xi \smile \nu_{\ell d+1}) + \partial\xi \smile \nu_{\ell d} = \partial(\xi \smile (\gamma_{\ell d}|_{\mathcal{D}_{\ell d+1}(\epsilon)} + \eta(\epsilon)|_{\mathcal{D}_{\ell d+1}(\epsilon)})) + \partial\xi \smile \gamma_{\ell d}|_{\mathcal{D}_{\ell d}(\epsilon)} \\ &= \partial(\xi \smile \eta(\epsilon)|_{\mathcal{D}_{\ell d+1}(\epsilon)}) = \partial(\xi \smile \eta(\epsilon)), \end{aligned}$$

where we have used Remark 23 (taking $(Z, X, Y) = (\Omega^{\ell d}(\epsilon), \Omega^{\ell d+1}(\epsilon), \Omega^{\ell d+2}(\epsilon))$ and $(\widetilde{X}, \widetilde{Z}, A) = (\mathcal{D}_{\ell d+1}(\epsilon), \mathcal{D}_{\ell d}(\epsilon), \mathcal{D}^1_{\ell d}))$ and the fact that we are working with $\mathbb{Z}_2$–coefficients.

On the other hand, the bundle $E(\epsilon)$ is trivial, because the space Ω is contractible and the class $\eta(\epsilon)$ is zero. Therefore the differential is zero and this concludes the proof. □

Remark 40 It is actually possible to prove the stabilization of the second differential, up to subsequences, in a simpler way. In fact, $\{d_2(\epsilon)^{*,\ell d} \colon H^*(A_\ell, B_\ell) \to H^{*+2}(B_\ell, A_{\ell+1})\}_{d\ge 0}$ is a sequence of maps between finite-dimensional $\mathbb{Z}_2$–vector spaces, ie

$$d_2(\epsilon)^{*,\ell d} \in \mathrm{Hom}(\mathbb{Z}_2^a, \mathbb{Z}_2^b),$$

where $a = \dim_{\mathbb{Z}_2}(H^*(A_\ell, B_\ell))$ and $b = \dim_{\mathbb{Z}_2}(H^{*+2}(B_\ell, A_{\ell+1}))$. Since $\mathrm{Hom}(\mathbb{Z}_2^a, \mathbb{Z}_2^b) \simeq \mathbb{Z}_2^{a\times b}$ is a finite set, then up to subsequences, $d_2(\epsilon)^{*,\ell d}$ is eventually constant.

3.2.4 The asymptotics for the Betti numbers of the chamber We are now in the position of proving the main theorem of this section, namely Theorem 8.

Proof of Theorem 8 The proof of this theorem is based on the analysis of the structure of the spectral sequence and its last page. First observe that by Proposition 31, for all $k \ge 0$ and for all $\epsilon < \epsilon(d)$ we have

$$b_k(W_{\sigma,d}) = b_k(W_{\sigma,d}(\epsilon)).$$

For the rest of the proof we will take $\epsilon \le \min\{\epsilon(d), \epsilon_2\}$, where $\epsilon_2 \le \epsilon_1$ is given by Proposition 39. Lemma 32 implies now that

$$b_k(W_{\sigma,d}) = \tfrac{1}{2} b_k(V_{\sigma,d}(\epsilon)).$$

On the other hand, since the involved spaces are semialgebraic sets (hence triangulable), the Betti numbers of $V_{\sigma,d}(\epsilon)$ are related to those of $S^N \setminus V_{\sigma,d}(\epsilon)$ through Alexander duality:

$$\widetilde{b}_k(V_{\sigma,d}(\epsilon)) = \widetilde{b}_{N-k-1}(S^N \setminus V_{\sigma,d}(\epsilon)).$$

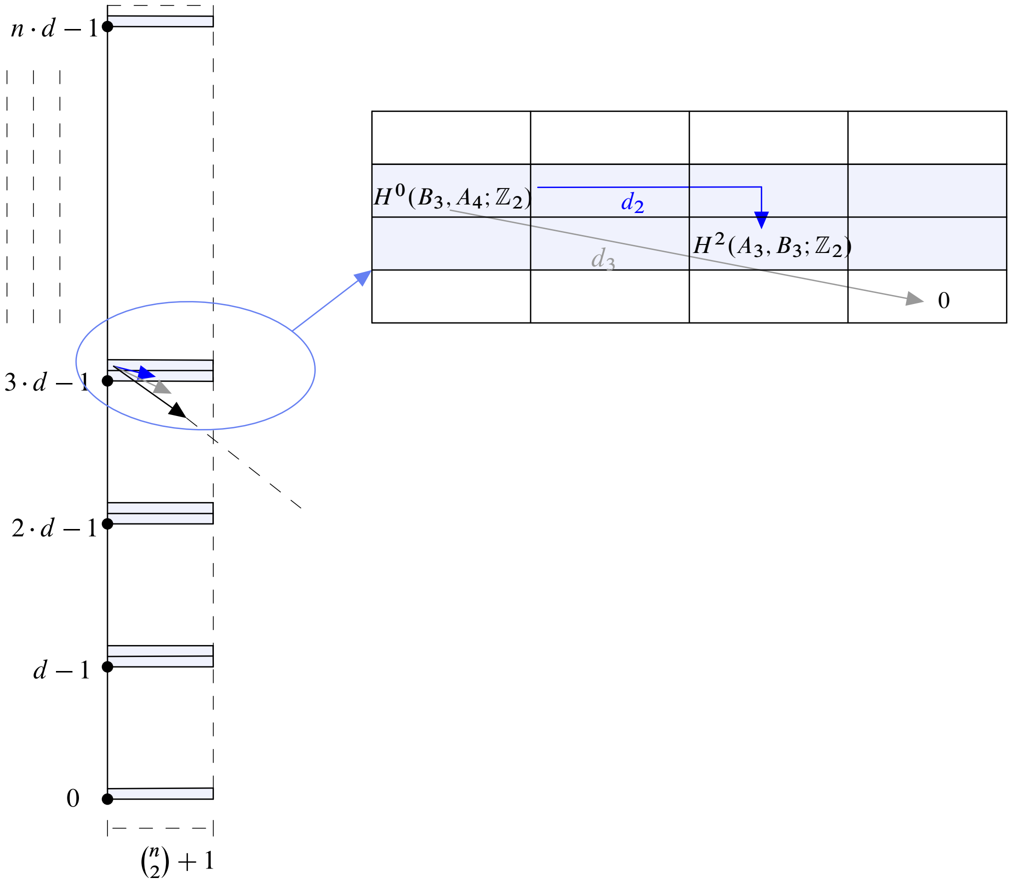


Figure 3: This is a schematic image of the $E_2(\epsilon)$ term of the spectral sequence we are describing. The shaded parts correspond to the elements $E_2^{i,j}(\epsilon)$ of the spectral sequence which are possibly nonzero.

Finally, denoting by $e_\infty^{i,j}(\epsilon)$ the dimension of $E_\infty^{i,j}(\epsilon)$, where $E_\infty(\epsilon)$ is the last page of the spectral sequence from Theorem 35, we have

$$\tilde{b}_{N-k-1}(S^N \setminus V_{\sigma,d}(\epsilon)) = \sum_{i+j=N-k-1} e_\infty^{i,j}(\epsilon).$$

Collecting all this together, for $\epsilon \le \min\{\epsilon(d), \epsilon_2\}$ we have

$$b_k(W_{\sigma,d}) = \frac{1}{2}\begin{cases} 1 + \sum_{i+j=N-1} e_\infty^{i,j}(\epsilon) & \text{if } k = 0, \\ \sum_{i+j=N-k-1} e_\infty^{i,j}(\epsilon) & \text{if } 0 < k < N-1, \\ -1 + e_\infty^{0,0}(\epsilon) & \text{if } k = N-1. \end{cases} \tag{3-10}$$

Observe now that Proposition 38 implies that in the second page of the spectral sequence only the first $\binom{n}{2} + 1$ columns are nonzero (ie those with $0 \le i \le \binom{n}{2}$); moreover, in the second page only the rows with $j = \ell d$ and $j = \ell d - 1$ are potentially nonzero, for $\ell = 0, \dots, n$. Therefore, for $d \ge \binom{n}{2} + 2$ all the higher differentials are zero, and

$$E_\infty(\epsilon) = E_3(\epsilon).$$

On the other hand Proposition 39 implies that $E_2(\epsilon) = E_3(\epsilon) = E_\infty(\epsilon)$, with the last equality for $d \geq \binom{n}{2} + 2$.

Looking now at the top two rows of E_∞, by Corollary 34 we know that

$$E_2^{i,dn}(\epsilon) = 0 \quad \text{for all } i \geq 0, \qquad E_2^{0,dn-1}(\epsilon) \simeq \mathbb{Z}_2, \qquad E_2^{i,dn-1}(\epsilon) = 0 \quad \text{for all } i \geq 1.$$

Thus, for $d \geq \binom{n}{2} + 2$,

$$e_\infty^{i,dn}(\epsilon) = 0 \quad \text{for all } i \geq 0, \qquad e_\infty^{0,dn-1}(\epsilon) = 1, \qquad e_\infty^{i,dn-1}(\epsilon) = 0 \quad \text{for all } i \geq 1.$$

From this we immediately see that for $d \geq \binom{n}{2} + 2$ we have

$$b_0(V_{\sigma,d}(\epsilon)) = 2 \qquad \text{and} \qquad b_k(V_{\sigma,d}(\epsilon)) = 0 \quad \text{for all } 1 \leq k \leq \binom{n}{2}.$$

This already proves

$$b_0(W_{\sigma,d}) = 1 \qquad \text{and} \qquad b_k(W_{\sigma,d}) = 0 \quad \text{for all } 1 \leq k \leq \binom{n}{2}. \tag{3-11}$$

Observe now that the fact that the rows with $j = \ell d$ and $j = \ell d - 1$ in $E_2 = E_\infty$ are the only possibly nonzero rows for $\ell = 0, \dots, n$ influences the Betti numbers $b_k(W_{\sigma,d})$ with

$$k = md, \dots, md - \binom{n}{2} - 1, \tag{3-12}$$

where $m = n - \ell$. We define now, for $m = 1, \dots, n-1$,

$$Q_{G,m}(t) = \frac{1}{2} \cdot \left(e_\infty^{0,(n-m)d-1} t^{\binom{n}{2}+1} + \sum_{i=1}^{\binom{n}{2}} \left(e_\infty^{\binom{n}{2}-i,(n-m)d} + e_\infty^{\binom{n}{2}-i+1,(n-m)d-1}\right) t^i + e_\infty^{\binom{n}{2},(n-m)d} \right).$$

The i^{th} coefficient of the polynomial $Q_{G,m}$ is $b_{md-\binom{n}{2}+i-1}(W_{G,d})$. In principle we would have to consider also the case $m = n$, but Theorem 9 guarantees that there is no homology in dimension greater than $(n-1)d - n + 1$. The proof of Theorem 9 is proved in Section 5 as that is where we describe a *different* spectral sequence which we use to prove it via duality. By (3-12), the conclusion of the theorem follows. □

Remark 41 As we noticed in the introduction, since the polynomials $Q_{G,1}, \dots, Q_{G,n-1}$ do not depend on d, but only on the graph, and since these polynomials are the same for isomorphic graphs, they define a graph invariant. Similarly the same is true for the Floer number $\beta(G)$, which is just the sum of their coefficients. Of course the polynomials are finer invariants, however we do not have a clear interpretation of these quantities.

Remark 42 From (3-11) it immediately follows that for $d \geq \binom{n}{2} + 2$ each sign condition is connected. In particular, if $d \geq \binom{n}{2} + 2$, two $\mathbb{R}^d$–geometric graphs on n vertices are isomorphic if and only if they are rigidly isotopic. We will actually sharpen this in Corollary 47 below.

4 Homotopy groups of the chambers

We turn our attention now to proving Theorem 3, and we start by introducing some notation. For every $0 \le r \le \max\{n, d\}$ let us denote by $(\mathbb{R}^{d\times n})_r$ the set of matrices of rank r,

$$(\mathbb{R}^{d\times n})_r = \{P \in \mathbb{R}^{d\times n} \mid \mathrm{rk}(P) = r\} \subseteq \mathbb{R}^{d\times n}.$$

When $r = n$ we have that $(\mathbb{R}^{d\times n})_n$ deformation retracts the Stiefel manifold of orthonormal n–frames in $\mathbb{R}^n$ (the retraction is given by the Gram–Schmidt procedure; in the case $d = n$ this is simply the deformation retraction of $\mathrm{GL}(n,\mathbb{R})$ onto $O(n)$). We recall that for $k \le d-n-1$ this Stiefel manifold is k–connected (see [16, Example 4.53]), ie

$$\pi_k((\mathbb{R}^{d\times n})_n) = 0 \quad \text{if } k \le d-n-1. \tag{4-1}$$

We will need the next elementary lemma.

Lemma 43 *The complement of $(\mathbb{R}^{d\times n})_n$ can be written as a finite union of smooth submanifolds of codimension at least $d-n+1$.*

Proof Recall that for every $0 \le r \le \max\{n, d\}$, the codimension of $(\mathbb{R}^{d\times n})_r$ in $\mathbb{R}^{d\times n}$ equals $(n-r)(d-r)$ (see [18, Chapter 3, Section 2, Exercise 4]) and, in particular, if $r \le n-1$,

$$\mathrm{codim}_{\mathbb{R}^{d\times n}}(\mathbb{R}^{d\times n})_r \ge d-n+1. \tag{4-2}$$

Now, the complement of $(\mathbb{R}^{d\times n})_n$ in $\mathbb{R}^{d\times n}$ is a semialgebraic set that can be written as

$$\mathbb{R}^{d\times n} \setminus (\mathbb{R}^{d\times n})_n = \coprod_{r=0}^{n-1} (\mathbb{R}^{d\times n})_r,$$

and it is therefore a semialgebraic set of codimension at least $d-n+1$. □

We will now prove a sequence of results on the homotopy groups of the chambers. These results will imply Theorem 3. Since $W_{G,d}$ might not be connected if $d \le \binom{n}{2}+1$, part of these results are formulated using the set $[S^k, W_{G,d}]$ of homotopy classes of continuous maps from S^k to $W_{G,d}$, instead of the homotopy group $\pi_k(W_{G,d})$. As soon as $W_{G,d}$ becomes connected and simply connected, we can endow $[S^k, W_{G,d}]$ with a group structure. To stress this subtlety we will keep both notations.

Proposition 44 *If $d \ge k+n+1$, the inclusion*

$$i : W_{G,d} \cap (\mathbb{R}^{d\times n})_n \hookrightarrow W_{G,d}$$

induces a bijection between $[S^k, W_{G,d} \cap (\mathbb{R}^{d\times n})_n]$ and $[S^k, W_{G,d}]$.

Proof We need to prove that the map $i_* : [S^k, W_{G,d} \cap (\mathbb{R}^{d\times n})_n] \to [S^k, W_{G,d}]$ induced by the inclusion is a bijection if $k \le d-n-1$.

We first prove the surjectivity of i_*. Let $f_0\colon S^k \to W_{G,d}$ be a map representing an element of $[S^k, W_{G,d}]$. Since $W_{G,d}$ is open, up to homotopies we can assume that the map f_0 is smooth. Moreover, by [18, Chapter 3, Theorem 2.5], the map f_0 is homotopic to a map $f_1\colon S^k \to W_{G,d}$ which is transversal to all the strata of the complement of $(\mathbb{R}^{d\times n})_n$. If now $k < d-n+1$, the transversality condition and Lemma 43 imply that the image of f_1 does not intersect these strata; therefore $f_1\colon S^k \to W_{G,d} \cap (\mathbb{R}^{d\times n})_n$ and i_* is surjective. (Notice that for the surjectivity we only need $d \geq k+n$.)

For the injectivity we argue similarly. Let $f_0, f_1\colon S^k \to W_{G,d} \cap (\mathbb{R}^{d\times n})_n$ be two maps such that $i \circ f_0\colon S^k \to W_{G,d}$ is homotopic to $i\circ f_1\colon S^k \to W_{G,d}$. This means there exists a map $F\colon S^k \times I \to W_{G,d}$ such that $F(\,\cdot\,,0) = i \circ f_0$ and $F(\,\cdot\,,1) = i \circ f_1$. Now, we can approximate F with a new map $\widetilde{F}\colon S^k \times I$ which is smooth, homotopic to F and C^0 arbitrarily close to it, and transversal to all the strata of the complement of $(\mathbb{R}^{d\times n})_n$. By Lemma 43, if $k+1 < d-n+1$ this implies that the image of $\widetilde{F}$ does not intersect the complement of $(\mathbb{R}^{d\times n})_n$. In particular we have a homotopy between $\widetilde{F}(\,\cdot\,,0)$ and $\widetilde{F}(\,\cdot\,,1)$ all contained in $W_{G,d} \cap (\mathbb{R}^{d\times n})_n$. On the other hand, since both $F(\,\cdot\,,0)$ and $F(\,\cdot\,,1)$ miss the complement of $(\mathbb{R}^{d\times n})_n$, which is closed, by compactness of S^k, any two maps C^0 sufficiently close to these maps will be homotopic to them and will also miss this complement. In particular, if $\widetilde{F}$ is sufficiently close to F, then $F(\,\cdot\,,0)$ is homotopic to $\widetilde{F}(\,\cdot\,,0)$, and $F(\,\cdot\,,1)$ is homotopic to $\widetilde{F}(\,\cdot\,,1)$; further, these homotopies miss the complement of $(\mathbb{R}^{d\times n})_n$. In this way we have build a homotopy between f_0 and f_1 already in $W_{G,d} \cap (\mathbb{R}^{d\times n})_n$, ie i_* is injective. □

4.0.1 Some useful maps We introduce now some useful maps. First recall the Gram–Schmidt map $\sigma\colon \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$, which orthonormalizes the columns of a matrix $P \in \mathbb{R}^{d\times n}$ and is defined by

$$\sigma(P) = P(P^T P)^{-1/2}.$$

Since the columns of $\sigma(P)$ are orthonormal, it follows that, if $P \in \mathbb{R}^{d\times n}_n$,

$$\sigma(P)^T\sigma(P) = \mathbf{1}_n. \tag{4-3}$$

Moreover, $\sigma(P)\sigma(P)^T$ is the orthogonal projection on the span of the columns of P.

Let now G be a geometric graph on n vertices and $d \geq n$. Our first useful map is

$$W_{G,n} \times (\mathbb{R}^{d\times n})_n \xrightarrow{\alpha_d} W_{G,d}, \quad (Q,P) \mapsto (\sigma(P)Q). \tag{4-4}$$

We need to verify that the isomorphism class of the labeled graph is unchanged, ie that $G(\sigma(P)Q) \simeq G(Q) \simeq G$. This is true because all the relative distances of the points in $\sigma(P)Q$ are the same as the distances of the points in Q. More precisely, write $Q = (q_1,\dots,q_n)$ and $\sigma(P)Q = (p_1',\dots,p_n')$, where $p_i' = \sigma(P)q_i$. Then, using (4-3), we have

$$\|p_i'-p_j'\|^2 = \|\sigma(P)(q_i-q_j)\|^2 = (q_i-q_j)^T\sigma(P)^T\sigma(P)(q_i-q_j) = (q_i-q_j)^T(q_i-q_j) = \|q_i-q_j\|^2.$$

The relative distances between the points are the same, so by definition it follows that $G(\sigma(P)Q) \simeq G(Q)$.

A second useful map is

$$W_{G,d} \cap (\mathbb{R}^{d\times n})_n \xrightarrow{\beta_d} W_{G,n} \times (\mathbb{R}^{d\times n})_n, \quad P \mapsto (\sigma(P)^T P, P).$$

Also for this map we need to check that its first component has target in $W_{G,n}$. Again this follows from the fact that the mutual distances of the corresponding points are preserved. Writing $P = (p_1, \dots, p_n)$, we have

$$\|\sigma(P)^T p_i - \sigma(P)^T p_j\|^2 = (p_i - p_j)^T \sigma(P)\sigma(P)^T (p_i - p_j) = (p_i - p_j)^T (p_i - p_j) = \|p_i - p_j\|^2,$$

where we have used the fact that $\sigma(P)\sigma(P)^T$ is the orthogonal projection onto the span of the columns of P. The claim follows again from Lemma 30.

Observe that it follows immediately from the definition of the maps α and β that

$$\alpha_d \circ \beta_d = i : W_{G,d} \cap (\mathbb{R}^{d\times n})_n \hookrightarrow W_{G,d}. \tag{4-5}$$

4.0.2 Stabilization

Proposition 45 *If $d \geq k+n+1$, the map $j_* : [S^k, W_{G,d}] \to [S^k, W_{G,d+1}]$ induced by the inclusion is injective.*

Proof Let $g_0, g_1 : S^k \to W_{G,d}$ be two continuous maps such that the compositions $j \circ g_0 : S^k \to W_{G,d+1}$ and $j \circ g_1 : S^k \to W_{G,d+1}$ are homotopic. Thanks to Proposition 44 we can assume g_0 and g_1 to be elements of $[S^k, W_{G,d} \cap (\mathbb{R}^{d\times n})_n]$. We want to prove that g_0 and g_1 are homotopic. For a map $f : S^k \to W_{G,d}$, we consider the commutative diagram of maps

$$\begin{tikzcd}
 & (W_{G,n} \cap (\mathbb{R}^{n^2})_n) \times (\mathbb{R}^{dn})_n \arrow[r, hook, "u"] \arrow[d, "\alpha_d"'] & (W_{G,n} \cap (\mathbb{R}^{n^2})_n) \times (\mathbb{R}^{(d+1)n})_n \arrow[d, "\alpha_{d+1}"'] \\
S^k \arrow[ur, "\beta_d \circ f = (f_1, f_2)"] \arrow[r, "f"] & W_{G,d} \cap (\mathbb{R}^{dn})_n \arrow[r, hook, "j"] \arrow[u, bend right, "\beta_d"'] & W_{G,d+1} \cap (\mathbb{R}^{(d+1)n})_n \arrow[u, bend right, "\beta_{d+1}"']
\end{tikzcd}$$

Notice that here the maps α_d and α_{d+1}, defined in (4-4), are restricted to the set of pairs (Q, P) with $\mathrm{rk}(Q) = n$; the values of these maps are in the set of matrices of rank n.

Since $\alpha_d \circ \beta_d = \mathrm{id}$, we can write the map f as

$$f = \alpha_d \circ (\beta_d \circ f) = \alpha_d \circ (f_1, f_2),$$

where (f_1, f_2) are the components of $\beta_d \circ f$. We apply now the diagram to the map $f = g_0$ and $f = g_1$, writing them as

$$g_i = \alpha_d \circ (\beta_d \circ g_i) = \alpha_d \circ (g_{i,1}, g_{i,2}) \quad \text{for } i = 0, 1.$$

We will prove that both components are homotopic $g_{0,1} \sim g_{1,1}$ and $g_{0,2} \sim g_{1,2}$, which implies that g_0 is homotopic to g_1.

Since the map $j \circ g_0$ is homotopic to $j \circ g_1$, then also the first component of $\beta_{d+1} \circ j \circ g_0$ is homotopic to the first component of $\beta_{d+1} \circ j \circ g_1$. But the first component of $\beta_{d+1} \circ j \circ g_0$ equals $g_{0,1}$, the first component of $\beta_d \circ g_0$, and similarly for the first component of $\beta_{d+1} \circ j \circ g_1$. Therefore $g_{0,1} \sim g_{1,1}$.

On the other hand the second components of $\beta_d \circ g_0, \beta_d \circ g_1 \colon S^k \to (\mathbb{R}^{d\times n})_n$ are homotopic simply because $\pi_k((\mathbb{R}^{d\times n})_n) = 0$ for $k \le d-n-1$. □

Proposition 46 *For every $d \ge n$, the inclusion $W_{G,d} \hookrightarrow W_{G,d+n}$ is homotopic to a constant map.*

Before we give the proof, let us observe that, since we do not know if $W_{G,d}$ is path connected, there are *several* constant maps up to homotopy, one for each component; this proposition tells us that all the maps $[S^k, W_{G,d}]$ are mapped to the *same* constant map in $W_{G,d+n}$. This also tells us that $W_{G,d}$ is contained in just one connected component of $W_{G,d+n}$.

Proof Since for $d \ge n$ every graph is realizable as a geometric graph, pick $R = (r_1, \dots, r_n) \in W_{G,n}$ by the previously cited result of Maehara [23].

Consider the homotopy $f_t \colon W_{G,d} \to \mathbb{R}^{(d+n)\times n}$ defined for $t \in [0,1]$ by

$$f_t(P) = \begin{pmatrix} \sqrt{1-t}\,P \\ \sqrt{t}\,R \end{pmatrix}.$$

With this choice,

$$f_0 = i \colon W_{G,d} \hookrightarrow W_{G,d+n} \subset \mathbb{R}^{(d+n)\times n} \quad \text{and} \quad f_1 \equiv \begin{pmatrix} 0 \\ R \end{pmatrix} \in W_{G,d+n}.$$

We only need to prove that $f_t(W_{G,d}) \subseteq W_{G,d+n}$ for all $t \in [0,1]$. To this end, let us write

$$f_t(P) = (p_1(t), \dots, p_n(t)) = \begin{pmatrix} \sqrt{1-t}\,p_1 & \cdots & \sqrt{1-t}\,p_n \\ \sqrt{t}\,r_1 & \cdots & \sqrt{t}\,r_n \end{pmatrix}.$$

Because of Lemma 30, in order to show that $G(f_t(P)) \equiv G$ it is enough to show that the signs of the family of quadrics $\{\|p_i - p_j\|^2 - 1 \colon \mathbb{R}^{(d+n)\times n} \to \mathbb{R}\}$ evaluated on $f_t(P)$ are constants. We have

$$\|p_i(t) - p_j(t)\|^2 = (1-t)\|p_i - p_j\|^2 + t\|r_i - r_j\|^2,$$

and therefore, as

$$\operatorname{sign}(\|p_i - p_j\|^2 - 1) = \operatorname{sign}(\|r_i - r_j\|^2 - 1),$$

it must be the case that

$$\operatorname{sign}(\|p_i(t) - p_j(t)\|^2 - 1) = \operatorname{sign}(\|p_i - p_j\|^2 - 1) = \operatorname{sign}(\|r_i - r_j\|^2 - 1). \qquad \square$$

We are now ready to prove Theorem 3.

Proof of Theorem 3 We first prove that for $d \ge n+1$ the set $W_{G,d}$ is path connected. By Proposition 46 the map $i_* \colon [S^0, W_{G,d}] \to [S^0, W_{G,d+n}]$ is the map that sends everything to the class of a constant map. On the other hand this map factors through the sequence of maps induced by the inclusions $W_{G,d} \hookrightarrow W_{G,d+1}$

$$[S^0, W_{G,d}] \to [S^0, W_{G,d+1}] \to \cdots \to [S^0, W_{G,d+n}].$$

Each map in the previous sequence is an injection for $d \geq n+1$ by Proposition 45, therefore also $i_*\colon [S^0, W_{G,d}] \to [S^0, W_{G,d+n}]$ is an injection, and $[S^0, W_{G,d}]$ consists of only one element. Therefore $W_{G,d}$ is path connected.

We prove now that, for $d \geq n+2$, $W_{G,d}$ is also simply connected. By Proposition 46 the map $i_*\colon [S^1, W_{G,d}] \to [S^1, W_{G,d+n}]$ is the map that sends everything to the class of a constant map. On the other hand this map factors through the sequence of maps induced by the inclusions $W_{G,d} \hookrightarrow W_{G,d+1}$,

$$[S^1, W_{G,d}] \to [S^1, W_{G,d+1}] \to \cdots \to [S^1, W_{G,d+n-1}] \to [S^1, W_{G,d+n}].$$

Each map in the previous sequence is an injection for $d \geq n+2$ by Proposition 45, therefore also $i_*\colon [S^1, W_{G,d}] \to [S^1, W_{G,d+n}]$ is an injection, and $[S^1, W_{G,d}]$ consists of only one element. Recall now that $[S^1, W_{G,d}]$ consists of the set of conjugacy classes in $\pi_1(W_{G,d})$ (we can omit the basepoint because $W_{G,d}$ is path connected): the fact that $[S^1, W_{G,d}]$ consists of one element implies that there is only one conjugacy class in $\pi_1(W_{G,d})$, which means that $W_{G,d}$ is simply connected.

Let now $k \geq 2$ and $d \geq k+n+1$. Since $\pi_1(W_{G,d}) = 0$ for $d \geq n+2$, it follows that

$$[S^k, W_{G,d}] = \pi_k(W_{G,d})/\pi_1(W_{G,d}) = \pi_k(W_{G,d}),$$

by [16, Proposition 4A.2]. By Proposition 46, the map $i_*\colon \pi_k(W_{G,d}) \to \pi_k(W_{G,d+n})$ is the zero map. On the other hand, this map factors through the sequence of maps induced by the inclusions $W_{G,d} \hookrightarrow W_{G,d+1}$,

$$\pi_k(W_{G,d}) \to \pi_k(W_{G,d+1}) \to \cdots \to \pi_k(W_{G,d+n-1}) \to \pi_k(W_{G,d+n}).$$

Each map in the previous sequence is an injection for $d \geq k+n+1$ by Proposition 45, therefore also $i_*\colon \pi_k(W_{G,d}) \to \pi_k(W_{G,d+n})$ is an injection, and $\pi_k(W_{G,d}) = 0$. □

Notice that thanks to this theorem we have the following corollary.

Corollary 47 *For $d \geq n+1$, for each labeled graph G on $[n]$, the isomorphism class $W_{G,d}$ is connected.*

4.1 The infinite-dimensional case

The space $\mathbb{R}^{\infty\times n}$ is a pre-Hilbert space (since it is not complete) with respect to the natural scalar product.[8] The notion of geometric graph and discriminant also makes sense in this infinite-dimensional space. More precisely, given an element $P = (p_1, \dots, p_n) \in \mathbb{R}^{\infty\times n}$, we build the graph $G(P)$ whose vertices and edges are defined as in Definition 13. The discriminant $\Delta_{\infty,n}$ consists of points $P = (p_1, \dots, p_n) \in \mathbb{R}^{\infty\times n}$ such that there exists a pair $1 \leq i < j \leq n$ with $\|p_i - p_j\|^2 = 1$. The chambers are now defined as follows: for a given graph G on n vertices, we set

$$W_{G,\infty} = \{P = (p_1, \dots, p_n) \in \mathbb{R}^{\infty\times n} \setminus \Delta_{\infty,n} \,\big|\, G(P) \simeq G\}.$$

It is easy to see that $W_{G,\infty}$ is the direct limit of the sequence of inclusions in (1-2). In particular, from Theorem 3 we deduce the following.

[8] The completion of $\mathbb{R}^{\infty\times n}$ is $(\ell^2(\mathbb{N}))^n = \{p = (x_1, x_2, \dots) \mid \sum_{k=1}^{\infty} x_k^2 < \infty\}$.

Theorem 48 *For every graph G, the set $W_{G,\infty} = \varinjlim W_{G,d}$ is contractible.*

Proof We first observe that, by Lemma 30, for $d \geq n$ each $W_{G,d}$ is described by a list of quadratic inequalities and therefore it is semialgebraic and it has the homotopy type of a CW–complex. Since $W_{G,\infty} = \varinjlim W_{G,d}$, it follows by [26, Corollary on page 253] that also $W_{G,\infty}$ has the homotopy type of a CW–complex. By Whitehead's theorem [16, Theorem 4.5], in order to prove that $W_{G,\infty}$ is contractible it is enough to prove that all its homotopy groups are zero.

But this is a consequence of Theorem 3: since S^k is compact and $W_{G,d}$ comes with the final topology, any map $f : S^k \to W_{G,\infty}$ will factor through a map $f : S^k \to W_{G,d}$. We can assume d large enough so that $\pi_k(W_{G,d}) = 0$. It follows that the map f is contractible. □

5 Increasing the number of points

5.1 Geometric graphs on the real line

We now want to study the number of possible isotopy classes of geometric graphs on the real line when the number of points is large: this is precisely the case $d = 1$, and n large. If we look at the discriminant $\Delta_{1,n}$, this is an arrangement of hyperplanes, namely

$$\Delta_{1,n} = \{(x_1, \dots, x_n) \in \mathbb{R}^{1\times n} \mid \text{there exist } i, j \text{ such that } |x_i - x_j| = 1\}.$$

Remark 49 There is a way to compute explicitly the number $b_0(\mathbb{R}^{1\times n} \setminus \Delta_{1,n})$ using a generalized version of the Mayer–Vietoris spectral sequence for semialgebraic sets. For $n = 3$ this gives 19, for $n = 4$ it gives 183, and for $n = 5$ it gives 2371. The computations becomes tricky for larger n; however, these numbers are the beginning of a known integer sequence, which is the sequence of *labeled semiorders on* $[n]$.

The remark leads us to an obvious observation. An *interval order for intervals $\{I_i\}_{i=1}^n$ of unit length* (=*semiorder for* $[n]$) is the partial order corresponding to their left-to-right precedence relation, ie one interval I_i is considered less than another I_j if and only if I_i is completely to the left of I_j. In the case $d = 1$, the number of components of the complement of $\Delta_{1,n}$ is exactly the number of possible semiorders for $[n]$. This is because, once we defined the intervals $[p_i - 1, p_i]$ for all i, each component of $\mathbb{R}^{1\times n} \setminus \Delta_{1,n}$ is uniquely determined by whether $p_i < p_j - 1$ or $p_i \not< p_j - 1$; see [30, page 73].

Remark 50 The type of semiorders introduced are usually addressed as semiorders on n labeled items. The number of distinct semiorders on n unlabeled items is given by the Catalan numbers $\{C_n\}_n$.

Let us define $f(n) :=$ *number of labeled semiorders of* $[n]$. There is an explicit generating function for this sequence; see [30, page 78, Corollary 5.12]. We have

$$G(x) := \sum_{n\geq 0} f(n)\frac{x^n}{n!} = C(1 - e^{-x}),$$

where C is the generating function of the known sequence of Catalan numbers $\{C_n\}$. More explicitly,

$$\sum_{n\geq 0} C_n x^n = C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

Theorem 51 *The number of rigid isotopy classes of $\mathbb{R}$–geometric graphs on n vertices is equal to*

$$b_0(\mathbb{R}^{1\times n}\setminus\Delta_{1,n}) = \frac{1}{n}\cdot\sqrt{6\log\tfrac{4}{3}}\cdot\left(\frac{n}{e\log\frac{4}{3}}\right)^n (1+O(n^{-1/2})).$$

Proof First of all let us notice that Theorem 27 can be applied to $G(x)$ since we have only one singularity in $\log\frac{4}{3}$ and we can extend the function to a $\log\frac{4}{3}D$–domain, and actually to the whole of $\mathbb{C}\setminus[\log\frac{4}{3},+\infty)$. The function $C(x)$ has a unique singularity at $x=\frac{1}{4}$. It is easy to see that

$$C(x) = 2-2\sqrt{1-4x}+O(1-4x).$$

By composition we get

$$G(x) = 2-2\sqrt{4e^{-x}-3}+O(4e^{-x}-3),$$

and, from this,

$$G(x) = 2-2\sqrt{3\log\tfrac{4}{3}}\cdot\sqrt{1-\frac{x}{\log\frac{4}{3}}}+O\left(1-\frac{x}{\log\frac{4}{3}}\right) = F\left(\frac{x}{\log\frac{4}{3}}\right)+O\left(1-\frac{x}{\log\frac{4}{3}}\right).$$

We can now apply Theorem 27. We get

$$\frac{f(n)}{n!} = \left(\log\tfrac{4}{3}\right)^{-n}\cdot\sigma_n + O\left(\left(\log\tfrac{4}{3}\right)^{-n}\frac{1}{n^2}\right),$$

where $F(x)=\sum_{n=0}^{\infty}\sigma_n z^n$. Using Remark 28 and $\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi}$ we get

$$\frac{f(n)}{n!} = \left(\log\tfrac{4}{3}\right)^{-n}\cdot\frac{1}{\sqrt{\pi n^3}}\cdot\sqrt{3\log\tfrac{4}{3}} + O\left(\left(\log\tfrac{4}{3}\right)^{-n}\frac{1}{n^2}\right),$$

and, by Stirling's approximation,

$$f(n) = \left(\frac{n}{e\log\frac{4}{3}}\right)^n\cdot\frac{1}{n}\cdot\sqrt{6\log\tfrac{4}{3}} + O\left(\left(\frac{n}{e\log\frac{4}{3}}\right)^n n^{-3/2}\right). \qquad \square$$

With these computations we know asymptotically the number of isotopy classes of geometric graphs on the real line. However, as we discussed before, different isotopy classes can correspond to the same isomorphism class. It is therefore natural to ask for the number $\#_{1,n}$ of isomorphism classes of $\mathbb{R}$–geometric graphs, for n large. In [15], Hanlon computes the exponential generating function for this sequence (and calls the corresponding graphs *labeled unit-interval graphs*).

The exponential generating function for $\{\#_{1,n}\}_n$ is

$$\Lambda(x) = \exp(\Gamma(x))-1,$$

where $\Gamma(x)$ is the generating function for the sequence $\{b_n\}_n$ of isomorphism classes of *connected* $\mathbb{R}$–geometric graphs on n vertices. More explicitly, we have

$$\Gamma(x) = \tfrac{1}{4}(1-2z)-\tfrac{1}{4}\sqrt{\frac{1-3z}{1+z}}, \quad \text{where } z=e^x-1.$$

Reasoning as before, we prove the following theorem.

Theorem 52 *The number of isomorphism classes of $\mathbb{R}$–geometric graphs on n vertices is equal to*

$$\#_{1,n} = \frac{e^{1/12}}{8} \cdot \frac{1}{n} \cdot \sqrt{6 \log \tfrac{4}{3}} \cdot \left(\frac{n}{e \log \frac{4}{3}} \right)^n (1 + O(n^{-1/2})).$$

Proof First of all let us notice that Theorem 27 can be applied to $\Lambda(x)$ since we have only one singularity at $\log \frac{4}{3}$ and we can extend the function to a $\log \frac{4}{3} D$–domain, actually to $\mathbb{C} \setminus \left[\log \frac{4}{3}, +\infty\right)$. We start with

$$1 + \Lambda(x) = \exp\left(\tfrac{1}{4} \cdot (3 - 2e^x)\right) \cdot \exp\left(-\tfrac{1}{4}\sqrt{4e^{-x} - 3}\right).$$

Then

$$1 + \Lambda(x) = \left(e^{1/12} + O\left(1 - \frac{x}{\log \frac{4}{3}}\right)\right) \cdot \left(-\tfrac{1}{4}\sqrt{4e^{-x} - 3} + 1 + O(4e^{-x} - 3)\right),$$

$$1 + \Lambda(x) = e^{1/12} - \tfrac{1}{4} e^{1/12} \sqrt{3 \log \tfrac{4}{3}} \sqrt{1 - \frac{x}{\log \frac{4}{3}}} + O\left(1 - \frac{x}{\log \frac{4}{3}}\right).$$

Finally,

$$\frac{\#_{1,n}}{n!} = \tfrac{1}{8} e^{1/12} \cdot \left(\log \tfrac{4}{3}\right)^{-n} \cdot \frac{1}{\sqrt{\pi n^3}} \cdot \sqrt{3 \log \tfrac{4}{3}} + O\left(\left(\log \tfrac{4}{3}\right)^{-n} \frac{1}{n^2}\right),$$

$$\#_{1,n} = \tfrac{1}{8} e^{1/12} \cdot \left(\frac{n}{e \log \frac{4}{3}}\right)^n \cdot \frac{1}{n} \cdot \sqrt{6 \log \tfrac{4}{3}} + O\left(\left(\frac{n}{e \log \frac{4}{3}}\right)^n n^{-3/2}\right). \qquad \square$$

Even though in the general case we still do not have a clear understanding of the relation between $b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n})$ and $\#_{1,n}$, in the case $d = 1$ we have the following corollary.

Corollary 53 *We have*

$$b_0(\mathbb{R}^{1 \times n} \setminus \Delta_{1,n}) = \frac{8}{e^{1/12}} \cdot \#_{1,n} (1 + O(n^{-1/2})), \quad \textit{where } \frac{8}{e^{1/12}} = 7.3603 \ldots.$$

The number $8/\sqrt[12]{e}$ can be roughly interpreted as the average number of rigid isotopy classes realizing a particular $\mathbb{R}$–geometric graph isomorphism type.

5.2 Asymptotic enumeration in higher dimensions

While the situation for isotopy classes of geometric graphs on the real line is given by the number of semiorders on $[n]$, such a closed-form description apparently does not exist for larger values of d. Nonetheless we are able to obtain reasonable bounds on the asymptotics following methods of McDiarmid and Müller [24], who study asymptotic enumeration of labeled disk graphs in $\mathbb{R}^2$. A disk graph in $\mathbb{R}^2$ is a graph given by an arrangement of open disks in $\mathbb{R}^2$ where the vertices are the disks and there is an edge between a pair of them if and only if the corresponding disks intersect one another. In the case that all the disks have the same radius, this is exactly the setting of our geometric graphs in the case $d = 2$. McDiarmid and Müller show that the number of labeled graphs on n vertices which are unit disk graphs in $\mathbb{R}^2$ — in our notation, $\#_{2,n}$ — is order $\exp(2n\log(n)+\Theta(n))$, and adapting their method we prove the following theorem. In particular we prove that the right asymptotic rate of growth both for $\#_{d,n}$ and for $b_0(\mathbb{R}^{d\times n}\setminus\Delta_{d,n})$ is $\exp(dn\log(n)+\Theta(n))$. We will prove the following theorem, which is a consequence of Theorems 56 and 58.

Theorem 54 *For $d\geq 2$ fixed and $n\geq 4d+1$, one has the bounds*

$$\left(\frac{1}{(d+1)e^2}\right)^{dn} n^{dn}\leq \#_{d,n}\leq b_0(\mathbb{R}^{d\times n}\setminus\Delta_{d,n})\leq 2dn\left(\frac{3e}{2d}\right)^{dn} n^{dn}.$$

In the $d=1$ case we saw that, on average, the number of rigid isotopy classes for an $\mathbb{R}$–geometric graph is a specific constant. Moreover, it is easy to see that there are $n!$ rigid isotopy classes for the empty graph on n vertices in $\mathbb{R}$. In higher dimensions we leave the generalization of these as open questions: On average how many rigid isotopy classes correspond to a particular $\mathbb{R}^d$–geometric graph? Does this average depend on n? What is the maximum for the number of rigid isotopy classes corresponding to an $\mathbb{R}^d$–geometric graph?

5.3 General case: the upper bound

While McDiarmid and Müller are primarily interested in enumerating labeled geometric graphs in the plane, in our notation the number $\#_{2,n}$, their upper bound holds for general d, as they point out in [24]. The key lemma in their proof of their upper bound is the following result of Warren [32].

Theorem 55 [32] *If $P_1,\dots,P_m$ are polynomials of degree at most t in real variables $z_1,\dots,z_k$, then the number of distinct sign patterns*

$$\big(\operatorname{sign}(P_1(\overline{z})),\dots,\operatorname{sign}(P_m(\overline{z}))\big)\in\{-1,1\}^m$$

that occur in $\mathbb{R}^k\setminus\bigcup_{i=1}^m\{\overline{z}\mid P_i(\overline{z})=0\}$ is at most

$$\left(\frac{4etm}{k}\right)^k.$$

Given n,d we take the $\binom{n}{2}$ polynomials in variables $(x_1,\dots,x_n)\in\mathbb{R}^{dn}$ given by $q_{i,j}(x)=\|x_i-x_j\|^2-1$, defined in (3-1). Then each sign pattern of these $\binom{n}{2}$ degree 2 polynomials in dn variables corresponds to a unique isomorphism class of labeled geometric graphs on n vertices in $\mathbb{R}^d$. Therefore we have the bound

$$\#_{d,n}\leq\left(\frac{4e}{d}\right)^{nd} n^{nd}.$$

However, a single sign pattern could be a disjoint union of several rigid isotopy classes, so we need a different argument to bound $b_0(\mathbb{R}^{d\times n}\setminus\Delta_{d,n})$. We prove the following theorem.

Theorem 56 (upper bound) *For fixed d and for $n\geq 4d+1$, we have the bound*

$$b_0(\mathbb{R}^{d\times n}\setminus\Delta_{d,n})\leq 2dn\left(\frac{3e}{2d}\right)^{dn} n^{dn}.$$

Before the proof, we introduce some notation. One defines the unit-distance graph on $\mathbb{R}^d$ to be the graph whose vertex set is all of $\mathbb{R}^d$, and two vertices x and y are connected by an edge if and only if $\|x-y\|=1$. We remark that the unit-distance graph on $\mathbb{R}^d$, especially in the case $d=2$, is studied in the case of the well-known Hadwiger–Nelson problem of establishing the chromatic number of the plane, that is, the chromatic number of the unit-distance graph on $\mathbb{R}^2$.

Given two graphs G and H, a *graph homomorphism* from G to H is a map $\phi\colon V(G)\to V(H)$ such that if (u,v) is an edges of G then $(\phi(u),\phi(v))$ is an edge of H. Thus each quadric $\Delta_{d,n}^{i,j}$ is the space of images of homomorphisms from the graph on $[n]$ with an edge between vertex i and vertex j into the unit-distance graph on $\mathbb{R}^d$. More generally, we denote an intersection of quadrics by $\Delta_{d,n}^G$ for a graph $G=([n],E)$ by

$$\Delta_{d,n}^G := \bigcap_{(i,j)\in E} \Delta_{d,n}^{(i,j)}. \tag{5-1}$$

Then $\Delta_{d,n}^G$ is the space of images of homomorphisms from the graph G to the unit-distance graph on $\mathbb{R}^d$. Putting all of this together, we have that $\Delta_{d,n}$ itself is the set of all points in $\mathbb{R}^{d\times n}$ that are the image of a graph homomorphism for a nonempty graph on n vertices, ie $\Delta_{d,n}$ is the union of the $\Delta_{d,n}^G$ across nonempty graphs G on n vertices.

Remark 57 Let us denote with $\widehat{\Delta}_{d,n}^G$ the one-point compactification of the set $\Delta_{d,n}^G$ defined in (5-1), where G is any graph on $[n]$. This is an algebraic set X of $\mathbb{R}^{nd+1}=(x_1,\dots,x_n,z)$, defined by $k+1$ equations, which are

$$\|x_i-x_j\|^2=(1-z)^2$$

for (i,j) an edge of G, and the equation of the sphere $\|x_1\|^2+\cdots+\|x_n\|^2+z^2=1$. In fact, if we look at the explicit expression of the stereographic projection we get an homeomorphism between $\Delta_{d,n}^G$ and $X\setminus(\{(0,1)\})$, and from this the claim.

Proof By Alexander duality (Section 2.2) $b_0(\mathbb{R}^{d\times n}\setminus\Delta_{d,n})=b_{dn-1}(\widehat{\Delta}_{d,n})+1$, where $\widehat{\Delta}_{d,n}$ is the one-point compactification of the discriminant. Therefore, it is sufficient to bound $b_{dn-1}(\widehat{\Delta}_{d,n})$. Let us consider the Mayer–Vietoris spectral sequence for simplicial complexes; see [7, Section 3.2] for a complete construction. Thanks to the previous remark, $\widehat{\Delta}_{d,n}$ is an algebraic set and we can use the mentioned spectral sequence with respect to the algebraic covering $\{\widehat{\Delta}_{d,n}^G\}_G$, where G varies over nonempty labeled graphs on $[n]$. The E_1 page of the spectral sequence has

$$E_1^{i,j} = \bigoplus_{\substack{G \text{ a graph on } [n] \\ \text{with exactly } i+1 \text{ edges}}} H^j(\widehat{\Delta}_{d,n}^G), \tag{5-2}$$

and we have the bound

$$b_{dn-1}(\widehat{\Delta}_{d,n}) \le \sum_{i=0}^{dn-1} \dim_{\mathbb{Z}_2}(E_1^{i,(dn-1-i)}). \tag{5-3}$$

Using [27, Theorem 2] and the fact that for any labeled graph G on $[n]$ the topological space $\widehat{\Delta}^G_{d,n}$ is an algebraic set defined by equations of degree 2 in $\mathbb{R}^{dn}$, we get that its total Betti number is at most $2(3)^{dn-1}$. Using this, we have

$$b_{dn-1}(\widehat{\Delta}_{d,n}) \le \sum_{k=1}^{dn} \binom{\binom{n}{2}}{k} 2(3)^{dn} \le 2(3)^{dn} \sum_{k=1}^{dn} \binom{\binom{n}{2}}{k} \le 2(3)^{dn} dn \binom{\binom{n}{2}}{dn} \le 2(3)^{dn} dn \Big(\frac{n^2 e}{2dn}\Big)^{dn},$$

where in the third inequality we used $n \ge 4d+1$. □

5.4 General case: the lower bound

For the lower bound on the number of labeled disk graphs, McDiarmid and Müller give a procedure for inductively generating many distinct labeled disk graphs. Here we generalize this procedure to higher dimensions.

For each $k \ge d+1$ we construct a family $U_{k,d}$ of nonisomorphic labeled geometric graphs on k vertices in $\mathbb{R}^d$. If we let $u_{k,d}$ denote the number of graphs in $U_{k,d}$, we show that for $k \ge d+1$,

$$u_{k+1,d} \ge \Big(\Big\lfloor \frac{k}{d+1} \Big\rfloor\Big)^d u_{k,d}.$$

This recursion implies the following result, which we prove in Section 5.4.1.

Theorem 58 (lower bound) *We have for $n > d+1$ that*

$$\Big(\frac{n}{(d+1)e^2}\Big)^{dn} \le \#_{d,n}.$$

For the base of the recursion, we start with the regular d–simplex in $\mathbb{R}^d$ with edges of length 1 and vertices given by $P_1, P_2, \dots, P_{d+1}$. The 1–skeleton of the d–simplex is a geometric graph in $\mathbb{R}^d$; this will be the singleton element of $U_{d+1,d}$. Though this graph is degenerate, it will still contribute to $\#_{d,n}$, which is always a lower bound for $b_0(\mathbb{R}^{d\times n} \setminus \Delta_{d,n})$, by the discussion following Lemma 16.

To construct the families $U_{k,d}$ for $d+1 < k \le n$, we need the following technical lemma, which generalizes [24, Lemma 4.1].

Lemma 59 *There exist constants $\epsilon_0 > 0$ and $C > 0$ such that for all $0 < \epsilon < \epsilon_0$ and all $p_i \in B(P_i, \epsilon)$ for all $i \in [d]$, there exists a unique point*

$$q(p_1, \dots, p_d) \in B(P_{d+1}, C\epsilon)$$

with $\|q - p_i\| = 1$ for all $i \in \{1, \dots, d\}$.

In other words, for ϵ small enough, this lemma tells us that there is a well-defined Lipschitz-continuous function q on $B(P_1,\epsilon) \times B(P_2,\epsilon) \times \cdots \times B(P_d,\epsilon)$, with Lipschitz constant C, mapping $(x_1, \dots, x_d)$ to the unique point of the intersection of sphere $S(x_1,1) \cap S(x_2,1) \cap \cdots \cap S(x_d,1)$ closest to P_{d+1}. The

$d = 2$ case is [24, Lemma 4.1], and is essentially proved directly via a closed form for q in terms of $p_1, p_2 \in B(P_1, \epsilon) \times B(P_2, \epsilon)$, which is well defined and Lipschitz continuous for ϵ small enough. For larger values of d, writing down the closed form of q would be much more complicated. Therefore, we instead describe algorithmically how one would compute q given $p_1, \dots, p_d$ sufficiently close to $P_1, P_2, \dots, P_d$, respectively, and show that q will ultimately be a combination of Lipschitz-continuous functions.

Proof of Lemma 59 We show that for ϵ small enough, the intersection $S(p_1, 1) \cap \cdots \cap S(p_d, 1)$ with $p_i \in B(P_i, \epsilon)$ for all i is two points $q^+(p_1, \dots, p_d)$ and $q^-(p_1, \dots, p_d)$, with $q^+(p_1, \dots, p_d)$ the closer of the two to P_{d+1}, and that $q^+ \colon B(P_1, \epsilon) \times \cdots \times B(P_d, \epsilon) \to \mathbb{R}^d$ is Lipschitz continuous.

We will prove that q^+ is well defined and Lipschitz-continuous close to $P_1, P_2, \dots, P_d$ by describing the algorithm one would use to compute q^+ and show that each step of the algorithm is given by composition or addition of Lipschitz-continuous functions. Given a tuple of points $(p_1, p_2, \dots, p_d) \in B(P_1, \epsilon) \times B(P_2, \epsilon) \times \cdots \times B(P_d, \epsilon)$, with ϵ sufficiently small, one could compute q^+ via the following recursive procedure.

First find the $(d-2)$–dimensional sphere given by the intersection of $S(p_1, 1)$ and $S(p_2, 1)$. Now for any $0 \le k \le d-1$, a k–dimensional sphere in $\mathbb{R}^d$ may be described completely by its center, its radius, and the affine subspace of dimension $k+1$ in which it is contained. In other words a k–dimensional sphere in $\mathbb{R}^d$ is described by a point in $\mathbb{R}^d$, a positive real number, and an element of the Grassmannian $\mathrm{Gr}(k+1, d)$. Given p_1 and p_2 in $\mathbb{R}^d$ with the distance from p_1 to p_2 smaller than 2, the intersection $S(p_1, 1) \cap S(p_2, 1)$ is a $(d-2)$–dimensional sphere. It follows that taking ϵ small enough so that $\|p_1 - p_2\|^2 < 4$ for any $p_1, p_2 \in B(P_1, \epsilon) \times B(P_2, \epsilon)$, we have a continuous function $(C, R, G) \colon B(P_1, \epsilon) \times B(P_2, \epsilon) \to \mathbb{R}^d \times \mathbb{R}^+ \times \mathrm{Gr}(d-1, d)$. This map sends (p_1, p_2) to the $(d-2)$–dimensional sphere $S(p_1, 1) \cap S(p_2, 1)$ with center $C(p_1, p_2)$ and radius $R(p_1, p_2)$ living in the affine hyperplane $C(p_1, p_2) + G(p_1, p_2)$.

Now given $(p_1, \dots, p_d) \in B(P_1, \epsilon) \times \cdots \times B(P_d, \epsilon)$ with ϵ small enough, we have that the intersection of $C(p_1, p_2) + G(p_1, p_2)$ with $S(p_1, 1) \cup S(p_2, 1) \cup \cdots \cup S(p_d, 1) \subseteq \mathbb{R}^d$ gives an arrangement of $d-1$ $(d-2)$–dimensional spheres in the affine hyperplane $C(p_1, p_2) + G(p_1, p_2)$. The center and radii of these spheres will be determined by how $C(p_1, p_2) + G(p_1, p_2)$ intersects each $S(p_i, 1)$. By induction we find the two points of intersection of these $(d-2)$–dimensional spheres in the $(d-1)$–dimensional Euclidean space given by the affine hyperplane $C(p_1, p_2) + G(p_1, p_2)$. Once these two points of intersection have been found, we pick the one that is closest to P_{d+1} to be $q^+(p_1, \dots, p_d)$.

It can be verified routinely that (C, R, G), as defined above, is Lipschitz continuous in each coordinate. Moreover, the intersection of $C(p_1, p_2) + G(p_1, p_2)$ with $S(p_1, 1) \cup S(p_2, 1) \cup \cdots \cup S(p_d, 1) \subseteq \mathbb{R}^d$, when p_i is sufficiently close to P_i for all i, can be described by a $2(d-1)$–tuple of points $(c_2, r_2, \dots, c_d, r_d)$, where each c_i belongs to $C(p_1, p_2) + G(p_1, p_2)$ and $r_i \in (0, 1]$. Here c_i and r_i are, respectively, the center and the radius of the $(d-2)$–dimensional sphere given by $S(p_i, 1) \cap (C(p_1, p_2) + G(p_1, p_2))$ for $i \geq 3$, with c_2 and r_2, respectively, the center and radius of the $(d-2)$–dimensional sphere given by the intersection $S(p_1, 1) \cap S(p_2, 1)$, ie c_2 and r_2 are $C(p_1, p_2)$ and $R(p_1, p_2)$.

By continuity, $C(p_1, p_2)$ can be made arbitrarily close to $C(P_1, P_2)$, $G(p_1, p_2)$ can be made arbitrarily close to $G(P_1, P_2)$, and $R(p_1, p_2)$ can be made arbitrarily close to $R(P_1, P_2)$. From here one can verify that for ϵ small enough, there is a Lipschitz-continuous function

$$\phi : B_{\mathrm{Gr}(d-1,d)}(G(P_1, P_2), \epsilon) \times B_{\mathbb{R}^d}(C(P_1, P_2), \epsilon) \times B_{\mathbb{R}}(R(P_1, P_2), \epsilon) \times B_{\mathbb{R}^d}(P_3, \epsilon) \times \cdots \times B_{\mathbb{R}^d}(P_d, \epsilon) \to (\mathbb{R}^{d-1} \times \mathbb{R})^{d-1}$$

mapping an element of the domain to the arrangement of $(d-1)$ many $(d-2)$–dimensional spheres in the affine hyperplane as described above. By induction we have that the S^0 at the intersection of the arrangement is Lipschitz continuous on the image of ϕ, and finally picking the closest of the two points to the fixed point P_{d+1} is Lipschitz continuous too. Note that the base case for the induction can simply be the $d = 1$ case; given two points in $\mathbb{R}$, picking the one closest to a fixed point is always Lipschitz continuous when the center of the two points lives in some small enough interval around a second fixed point. $\square$

Let us take the sequence $0 < \epsilon_1 < \cdots < \epsilon_n$ defined by $\epsilon_i = \epsilon_0 / C^{n-i}$, where ϵ_0 and C as in the previous lemma and we are assuming $C > 1$. To construct elements of $U_{k,d}$ recursively from $U_{k-1,d}$, we will also use as an inductive hypothesis that all the elements $P = (p_1, \ldots, p_k) \in U_{k,d}$ satisfy the properties

(P1) $\|p_i - P_j\| < \epsilon_i$ with $i \equiv j \mod d + 1$,

(P2) $S(p_{i_1}, 1) \cap \cdots \cap S(p_{i_{d+1}}, 1) = \varnothing$ for all distinct $\{i_1, \ldots, i_{d+1}\}$.

These two conditions hold for $U_{d+1,d}$, whose vertices are $P_1, \ldots, P_{d+1}$.

Condition (P1) above naturally partitions the vertices of $G \in U_{k,d}$ into $d+1$ distinct classes given by the clustering of the vertices of G around the points $P_1, \ldots, P_d, P_{d+1}$. The proof of the claimed recursive lower bound on $u_{k,d}$ will be that if $G \in U_{k-1,d}$ and, without loss of generality, $k \equiv 0 \mod (d+1)$ then picking a transversal $\sigma = \{p_{i_1}, p_{i_2}, \ldots, p_{i_d}\}$ of vertices of G where $i_l \equiv l \mod (d+1)$ for every i, we give a procedure to choose a position for a new vertex p_k to be added to G so that the neighborhood of p_k is unique for each choice of σ and so that (P1) and (P2) are satisfied still satisfied after adding p_k. As p_k will depend on σ and each choice of σ gives a distinct neighborhood for p_k, we have that there are at least as many combinatorially distinct ways to extend G as there are choices for σ.

If we denote by $\mathcal{P}(G)$ for $G \in U_{k-1,d}$ the set of all such transversals, that is, the number of d–tuples $(i_1, \ldots, i_d)$ with $1 \leq i_j < k$ such that $i_j \equiv j \mod d + 1$, we have

$$|\mathcal{P}| \geq \lfloor (k-1)/(d+1) \rfloor^d.$$

Now let $\mathcal{M}_\pi$ be defined as the intersection of the open balls

$$\mathcal{M}_\pi := B(p_{i_1}, 1) \cap \dots \cap B(p_{i_d}, 1).$$

We have the following lemma, which in the 2–dimensional case is [24, Claim 4.3]. The proof, which we omit, is exactly analogous to the 2–dimensional case and relies on the fact that for each $\pi = (i_1, \dots, i_d) \in \mathcal{P}$, $S(p_{i_1}, 1) \cap \dots \cap S(p_{i_d}, 1) \cap B(P_{d+1}, \epsilon_k)$ is a single point due to Lemma 59, and that single point is unique for each choice of π by condition (P2).

Lemma 60 *There exist nonempty open sets $O_\pi \subset \mathcal{M}_\pi$ such that for all $\pi \neq \sigma \in \mathcal{P}$, we have either $O_\pi \cap \mathcal{M}_\sigma = \varnothing$, or $O_\sigma \cap \mathcal{M}_\pi = \varnothing$.*

Now, for each $\pi \in \mathcal{P}$, let us pick an arbitrary

$$q_\pi = O_\pi \setminus \bigcup_{i=1}^{k-1} S(p_i, 1),$$

and we have that the $\mathbb{R}^d$–geometric graph obtained by adding the vertex q_π to G, which we will denote by (G, q_π), satisfies conditions (P1) and (P2). Moreover, for every choice of π we obtain a unique way to extend G, by the following lemma.

Lemma 61 *If $\pi \neq \sigma \in \mathcal{P}(G)$ for $G \in U_{k-1,d}$, then the geometric graphs (G, q_π) and (G, q_σ) are not isomorphic.*

This holds because q_π and q_σ will have different sets of neighbors, generalizing [24, Claim 4.4].

Lemma 62 *If $\pi \neq \sigma \in \mathcal{P}(G)$ for $G \in U_{k-1,d}$, then $N(q_\pi) \neq N(q_\sigma)$, where $N(v)$ denotes the neighbors of a point v in $\mathbb{R}^d$ in the geometric graph (G, v), that is, the vertices of G at distance less than 1 from v.*

Proof For $\pi \neq \sigma$ we have that $\sigma \subseteq N(q_\pi)$ if and only if $q_\pi \in \mathcal{M}_\sigma$. Clearly $\sigma \subseteq N(q_\sigma)$ and $\pi \subseteq N(q_\pi)$, but by Lemma 60 it cannot be the case that both $\sigma \subseteq N(q_\pi)$ and $\pi \subseteq N(q_\sigma)$. □

Now, for each $G \in U_{k-1,d}$ and $\pi \in \mathcal{P}(G)$ we construct a geometric graph (P, q_π) which satisfies conditions (P1) and (P2) and which satisfies Lemma 61. Then

$$u_{k,d} \geq |\mathcal{P}| \cdot u_{k-1,d} \geq \left\lfloor \frac{k-1}{d+1} \right\rfloor^d \cdot u_{k-1,d}.$$

5.4.1 Proof of Theorem 58

We have

$$\#_{n,d} \geq u_{n,d} \geq \left(\prod_{i=d+1}^{n-1} \left\lfloor \frac{i}{d+1} \right\rfloor \right)^d \geq \left(\prod_{i=d+1}^{n-1} \frac{i-d}{d+1} \right)^d \geq \left(\frac{(n-d-1)!}{(d+1)^{n-d-1}} \right)^d.$$

Using the estimate

$$k! \geq \left(\frac{k}{e}\right)^k$$

we get

$$\#_{n,d} \geq \left(\frac{n-d-1}{(d+1)e}\right)^{d(n-d-1)} \geq \left(\frac{n}{(d+1)e}\right)^{dn}\left(\frac{n}{d+1}\right)^{-d(d+1)},$$

where for the last inequality we used

$$\left(1-\frac{d+1}{n}\right)^{d(n-d-1)} \geq e^{-d(d+1)},$$

which derives from

$$\left(1+\frac{d+1}{n-d-1}\right)^{d(n-d-1)} \leq e^{d(d+1)}.$$

We then use $(n/(d+1))^{d(d+1)} \leq (e^d)^n$ to obtain the lower bound.

5.5 The top Betti numbers

The goal of this section will be to finally prove Theorem 9 regarding the low-degree Betti numbers of the one-point compactification of the discriminant. We will prove it using the generalized nerve lemma of Björner.

Theorem 63 (special case of [8, Theorem 6]) *Let X be a regular CW–complex and $(X_i)_{i\in I}$ a family of subcomplexes such that $X=\bigcup_{i\in I} X_i$. Suppose that every nonempty finite intersection $X_{i_1}\cap\cdots\cap X_{i_t}$ is $(k-t+1)$–connected. Then X is k–connected if and only if the nerve $\mathcal{N}(X_i)$ is k–connected.*

Recall that given a CW–complex X and a covering by subcomplexes $(X_i)_{i\in I}$ the nerve of the covering $\mathcal{N}(X_i)$ is the simplicial complex on the vertex set I, where $\sigma=[i_1,\dots,i_t]$ is a face of the nerve if and only if $X_{i_1}\cap\cdots\cap X_{i_t}$ is nonempty.

The covering that we will use for $\widehat{\Delta}_{d,n}$ will be given by $(\widehat{\Delta}^{i,j}_{d,n})_{1\leq i<j\leq n}$, where $\widehat{\Delta}^{i,j}_{d,n}$ denotes the compactification in $\mathbb{R}^{d\times n}$ of the space $\Delta^{i,j}_{d,n}=\{(x_1,\dots,x_n)\in\mathbb{R}^{d\times n}\mid \|x_i-x_j\|^2=1\}$.

Now each intersection of a set of $\Delta^{i,j}_{d,n}$ spaces is naturally associated to a graph G and a space $\Delta^G_{d,n}$ described in (5-1).

To be able to apply the generalized nerve lemma, a key step will be to establish the following about the topology of $\widehat{\Delta}^G_{d,n}$.

Lemma 64 *For any graph G with $\beta_0(G)$ connected components, $\Delta^G_{d,n}$ is a direct product of a compact set $K:=K(G)$ of dimension at most $(d-1)(n-\beta_0)$ and $\mathbb{R}^{d\times\beta_0}$. Therefore $\widehat{\Delta}^G_{d,n}$ is $(d\beta_0-1)$–connected.*

This proof will use the connection to homomorphisms to unit-distance graphs defined in the discussion around (5-1).

Proof of Lemma 64 Let H be a component of G and let T be a spanning tree of H. Clearly $\Delta^H_{d,|V(H)|}\subseteq \Delta^T_{d,|V(H)|}$ as any homomorphism from H to the unit-distance graph on $\mathbb{R}^d$ induces a homomorphism from T to the unit-distance graph on $\mathbb{R}^d$. Moreover, we have that $\Delta^T_{d,|V(H)|}\sim\mathbb{R}^d\times(S^{d-1})^{|E(T)|}$.

Indeed, we may regard T as being a rooted tree and we can map the root of T to any point of $\mathbb{R}^d$ and from there every vertex may live anywhere on the sphere of radius 1 centered at the image of its parent vertex. Now taking $\Delta^H_{d,|V(H)|}$ and modding out by the $\mathbb{R}^d$ factor coming from the choice of image for the root in T, we have a closed subset of the compact set $(S^{d-1})^{|E(T)|} = (S^{d-1})^{|V(H)|-1}$. Thus, without fixing the image of the root, we have that $\Delta^H_{d,|V(H)|}$ is the direct product of $\mathbb{R}^d$ and the compact space $K(H)$ given by the closed subset of $(S^{d-1})^{|V(H)|-1}$. It is clear that we may describe any homomorphism from G to the unit-distance graph on $\mathbb{R}^d$ as a product of graph homomorphisms on the connected components. We have that

$$\Delta^G_{d,n} \cong K(G) \times \mathbb{R}^{d\beta_0},$$

where $K(G)$ is the direct product of $K(H)$ over all connected components H. Thus $K(G)$ is contained in some $(n-\beta_0)$–fold product of $(d-1)$–dimensional spheres, so it is at most $(d-1)(n-\beta_0)$–dimensional.

We now turn our attention to the compactification of $\Delta^G_{d,n}$. By the description of $\Delta^G_{d,n}$, we have that $\Delta^G_{d,n}$ is a $d\beta_0$–ranked vector bundle over a compact CW–complex (since we are working with semialgebraic sets), so its compactification is the Thom space of this vector bundle, which is $(d\beta_0-1)$–connected by [28, Lemma 18.1]. □

Proof of Theorem 9 We apply Theorem 63 to prove that $\widehat{\Delta}_{d,n}$ is $(n+d-3)$–connected. Consider the cover of $\widehat{\Delta}_{d,n}$ by $(\widehat{\Delta}^{i,j}_{d,n})_{1\le i<j\le n}$ as discussed above, so that for any t, the t–fold intersection of complexes in the cover is of the form $\widehat{\Delta}^G_{d,n}$ for some graph G on t edges. Moreover, we observe that the nerve of this covering is just the simplex on $\binom{n}{2}$ vertices, as any intersection of the spaces in the cover at least contains the point at infinity, so in particular, the nerve is $(n+d-3)$–connected. Thus it suffices to check that $\widehat{\Delta}^G_{d,n}$ is $n+d-3-t+1 = n-t+d-2$ connected for any graph G on n vertices with t edges. By Lemma 64, it suffices to verify that, for such a graph, $n-t+d-2 \le d\beta_0(G)-1$. This always holds for $d \ge 1$ because $\beta_0 \ge 1$, $\beta_1 \ge 0$ and $n-t = \beta_0-\beta_1$.

To show that $\widehat{\Delta}_{n,d}$ is not $(n+d-2)$–connected, we use duality and verify that $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ has at least one $dn-1-(n+d-2) = (n-1)(d-1)$ reduced homology class. The path components of $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ are the rigid isotopy classes of the graph on n vertices in $\mathbb{R}^d$. We consider the rigid isotopy classes of the empty graph on n vertices. The rigid isotopy classes of the empty graph on n vertices give all the configurations of n points in $\mathbb{R}^d$ such that the distance between any pair of them is larger than 1. This is homotopy equivalent to $\mathrm{Conf}_n(\mathbb{R}^d)$ (see Examples 4 and 10), which has its top positive reduced Betti number in dimension $(n-1)(d-1)$. □

6 Examples

Here we work out a few examples for computing the Betti numbers of $\mathbb{R}^{d\times n} \setminus \Delta_{d,n}$ for small values of d and n. We start with the case that $d=1$. In the case that $d=1$, computing the topology of $\widehat{\Delta}^G_{1,n}$ across all graphs G on n vertices is sufficient to compute $b_0(\mathbb{R}^{1\times n} \setminus \Delta_{1,n})$. While we have shown the number of rigid isotopy classes of $\mathbb{R}$–geometric graphs on n vertices is given by the number of labeled semiorders on n elements, we work out a computation for $n = 3$ here primarily to show how spectral sequences and Alexander–Pontryagin duality can be used to compute the exact number of rigid isotopy classes of $\mathbb{R}$–geometric graphs.

Example 65 $(n = 3, d = 1)$ By Alexander–Pontryagin duality, it suffices to compute the Betti numbers of $\widehat{\Delta}_{1,3}$. This is a union of three compactified quadrics given by the solutions in $\mathbb{R}$ to $|x_i - x_j|^2 = 1$ for $1 \le i < j \le 3$. Each of these is simply the disjoint union of two hyperplanes in $\mathbb{R}^{1\times 3}$. Thus $\widehat{\Delta}_{1,3}$ is a 2–dimensional cell complex and we know by Theorem 9 that this complex is 1–connected so only the second Betti number is interesting. This is not surprising, as the dual in S^3 of $\widehat{\Delta}_{1,3}$ is $\mathbb{R}^3 \setminus \Delta_{1,3}$, which is a disjoint union of finite intersections of halfspaces, so only its zeroth Betti number is interesting.

Now by computing the E_1 page of the Mayer–Vietoris spectral sequence given by the covering of $\widehat{\Delta}_{1,3}$ by $(\widehat{\Delta}^{i,j}_{1,3})_{1\le i<j\le n}$, we have that the Euler characteristic of $\widehat{\Delta}_{1,3}$ will be given by

$$\chi(\widehat{\Delta}_{1,3}) = \sum_{0\le i\le 2, 0\le j\le 2} (-1)^{i+j} \dim(E_1^{i,j}).$$

On the other hand, $\chi(\widehat{\Delta}_{1,3}) = 1 + b_2(\widehat{\Delta}_{1,3})$, so computing the first page is enough.

Now

$$E_1^{i,j} = \bigoplus_{\substack{G \text{ a graph on } \{1,2,3\} \\ \text{with exactly } i+1 \text{ edges}}} H^j(\widehat{\Delta}^G_{1,3}).$$

So we can compute $\dim(E_1^{i,j})$ for every value of i and j. If $i = 0$ we are looking at $\widehat{\Delta}^G_{1,3}$ for G a graph with three vertices and one edge. Suppose the edge is between vertex 1 and vertex 2. Then we may place vertex 1 anywhere on $\mathbb{R}$ then vertex 2 at either point of the S^0 centered at the location of vertex 1. Next vertex 3 may be mapped into $\mathbb{R}$ arbitrarily. So $\Delta^G_{1,3}$ is given by $S^0 \times \mathbb{R} \times \mathbb{R}$, which compactifies in $\mathbb{R}^3$ to $S^2 \vee S^2$, and there are three graphs with exactly 1 edge. Next we look at graphs with two edges. If G is such a graph then it is easy to see that $\Delta^G_{1,3}$ is $\mathbb{R} \times S^0 \times S^0$, which compactifies to $S^1 \vee S^1 \vee S^1 \vee S^1$; there are three such choices for G. Finally, if G is the triangle, then there is no way to map G into the unit-distance graph on $\mathbb{R}$, so $\Delta^G_{1,3} = \varnothing$, which compactifies to the point at infinity. Therefore the following table stores the values of $\dim(E_1^{i,j})$:

2	6	0	0
1	0	12	0
0	3	3	1
$j \uparrow, i \rightarrow$	0	1	2

From this table we compute the Euler characteristic to be 19, so $b_2(\widehat{\Delta}_{1,3}) = 18$, from which it follows by duality that $b_0(\mathbb{R}^3 \setminus \Delta_{1,3}) = 18 + 1 = 19$, recovering the number of labeled semiorders on $\{1, 2, 3\}$.

We could take the same approach to compute $b_0(\mathbb{R}^{1\times n} \setminus \Delta_{1,n})$ for any n. For any graph G, $\Delta_{1,n}^G$ is always a product of a finite set $X := X(G)$ (which is possibly empty) and $\mathbb{R}^{\beta_0(G)}$, so $\widehat{\Delta}_{1,n}^G$ is a wedge of $|X|$ spheres of dimension $\beta_0(G)$, or the point at infinity if $X = \varnothing$. Moreover, $X(G)$ will be empty if and only if there is no way to map G into the unit-distance graph on the real line, but this means that $X(G)$ is empty if and only if G is not bipartite.

On the other hand, if G is bipartite we can compute $|X(G)|$ exactly, meaning that this approach could be used to compute the first page of the spectral sequence for $d = 1$ and any value of n. However, we should not expect to be able to do so in a reasonable amount of time for large n, assuming $\mathrm{P} \neq \mathrm{NP}$ as we explain below.

We have discussed $\Delta_{d,n}^G$ as the space of graph homomorphisms from G into the unit-distance graph on $\mathbb{R}^d$, but it turns out that given two graphs G and H, it is in general NP–complete to decide if there are any graph homomorphisms from G to H by a result of Hell and Nešetřil [17]. Indeed, [17] shows that for any nonbipartite graph H it is NP–complete to decide whether a graph G admits any homomorphism into H. For us this means that we should not be able to even decide in general if $\Delta_{d,n}^G$ is empty or not if $d \geq 2$, as the unit-distance graph on $\mathbb{R}^d$ for $d \geq 2$ is not bipartite. Now for $d = 1$, $\Delta_{d,n}^G$ will be computable, but a result Dyer and Greenhill [10] shows that the problem of enumerating graph homomorphism into a fixed bipartite graph H is #P–complete unless H is a special type of bipartite graph, which does not include the unit-distance graph on $\mathbb{R}$. Given these results, we should look for other ways to exactly count the number of isotopy classes given d and n, than computing the full Mayer–Vietoris spectral sequence. To have at least some explicit examples for larger d we work out the Betti numbers for $n = 3, 4$ and any $d \geq 2$, though the methods we use are rather ad hoc and don't generalize to higher values of n.

Example 66 ($n = 3$, $d \geq 2$) Here we explicitly compute the topology of each rigid isotopy class on three vertices. The space $W_{G,d}$ for G the complete graph on three vertices is contractible, in fact it is easy to see that it is convex.

For G on two edges we observe that $W_{G,d}$ has the topology of S^{d-1}. To see this we consider G as a path on three vertices, the first vertex on the path can go anywhere, the second vertex can go anywhere in the punctured ball of radius 1 around the first vertex; it must be punctured since only vertices with identical closed neighborhoods can map to the same point. After mapping the first two vertices the final vertex can be moved freely in some contractible subspace of $\mathbb{R}^d$ determined by the position of the first two vertices.

Now, for G on one edge we observe that $W_{G,d}$ has the topology of the configuration space of two points in $\mathbb{R}^d$, which is just S^{d-1}. Finally, if G is the empty graph, then $W_{G,d}$ is homotopy equivalent to the configuration space of three points in $\mathbb{R}^d$, which is known to have $b_0 = 1$, $b_{d-1} = 3$ and $b_{2(d-1)} = 2$.

Putting this all together, we have that the empty graph contributes 1 to b_0, 3 to b_{d-1} and 2 to $b_{2(d-1)}$, the three graphs on one edge each contribute 1 to b_0 and 1 to b_{d-1}, the three graphs on two edges also each contribute 1 to b_0 and 1 to b_{d-1}, and the complete graph contributes 1 to b_0. So we have $8 + 9x^{d-1} + 2x^{2d-2}$ as the Poincaré polynomial for $\mathbb{R}^{d\times 3} \setminus \Delta_{d,3}$.

Example 67 ($n = 4$, $d \geq 2$) The case $n = 4$ is more interesting, but again the computation is rather ad hoc. We examine each graph G on four vertices and compute the Poincaré polynomial for $W_{G,d}$ summarized in Table 1. These Poincaré polynomials are essentially computed by inspection; we don't give the full details for the computations. For example, the graph given by a path on three vertices and an isolated vertex has the homotopy type of $S^{d-1} \times S^{d-1}$. With one vertex of the path fixed, the next vertex is free to go anywhere in the punctured ball around the first vertex. Next the final vertex of the path may be placed freely inside some contractible set. So the path contributes a factor of S^{d-1}. Finally, after the path is placed in $\mathbb{R}^d$, the union of the balls around its vertices gives a contractible space in $\mathbb{R}^d$ and the isolated vertex may be placed anywhere outside of this contractible space, so this contributes the other S^{d-1} factor. From Table 1 we can determine $\beta(G)$ for any graph on four vertices, and determine that the Poincaré Polynomial of $\mathbb{R}^{d\times 4} \setminus \Delta_{d,4}$ is

$$64 + 7x^{d-2} + 92x^{d-1} + 7x^{2d-3} + 35x^{2d-2} + 6x^{3d-3}.$$

In particular, in the case of the plane there are 71 rigid isotopy classes of graphs on four vertices, while there are 64 labeled graphs on four vertices, all of which can be realized as geometric graphs in the plane. We can also recover the number of rigid isotopy classes for four points on the real line with the observation that the 4–cycle and the star cannot be realized as geometric graphs in $\mathbb{R}$. Therefore in the case $d = 1$, the 211 coming from evaluating the Poincaré polynomial at $x = 1$ overcounts by the contribution of 4 for each 4–cycle and for each star, so the overcount is 28, bringing the total number of chambers to $211 - 28 = 183$.

References

[1] **A Abbondandolo**, *A new cohomology for the Morse theory of strongly indefinite functionals on Hilbert spaces*, Topol. Methods Nonlinear Anal. 9 (1997) 325–382 MR Zbl

[2] **A A Agrachëv**, *Homology of the intersections of real quadrics*, Dokl. Akad. Nauk SSSR 299 (1988) 1033–1036 MR Zbl In Russian; translated in Soviet Math. Dokl. 37 (1988) 493–496

[3] **A A Agrachëv**, *The topology of quadratic mappings and Hessians of smooth mappings*, Algebra, Topol. Geom. 26 (1988) 85–124 MR Zbl In Russian; translated in J. Soviet Math. 49 (1990) 990–1013

[4] **A A Agrachëv**, **A Gentile**, **A Lerario**, *Geodesics and horizontal-path spaces in Carnot groups*, Geom. Topol. 19 (2015) 1569–1630 MR Zbl

[5] **A Agrachëv**, **A Lerario**, *Systems of quadratic inequalities*, Proc. Lond. Math. Soc. 105 (2012) 622–660 MR Zbl

[6] **A Auffinger**, **A Lerario**, **E Lundberg**, *Topologies of random geometric complexes on Riemannian manifolds in the thermodynamic limit*, Int. Math. Res. Not. 2021 (2021) 15497–15532 MR Zbl

[7] **S Basu**, *Different bounds on the different Betti numbers of semi-algebraic sets*, Discrete Comput. Geom. 30 (2003) 65–85 MR Zbl

[8] **A Björner**, *Nerves, fibers and homotopy groups*, J. Combin. Theory Ser. A 102 (2003) 88–93 MR Zbl

[9] **J Bochnak**, **M Coste**, **M-F Roy**, *Real algebraic geometry*, Ergebnisse der Math. 36, Springer (1998) MR Zbl

[10] **M Dyer**, **C Greenhill**, *The complexity of counting graph homomorphisms*, Random Structures Algorithms 17 (2000) 260–289 MR Zbl

[11] **E R Fadell**, **S Y Husseini**, *Geometry and topology of configuration spaces*, Springer (2001) MR Zbl

[12] **P Flajolet**, **R Sedgewick**, *Analytic combinatorics*, Cambridge Univ. Press (2009) MR Zbl

[13] **K Gęba**, **A Granas**, *Infinite dimensional cohomology theories*, J. Math. Pures Appl. 52 (1973) 145–270 MR Zbl

[14] **M Gutierrez**, **L Oubiña**, *Metric characterizations of proper interval graphs and tree-clique graphs*, J. Graph Theory 21 (1996) 199–205 MR Zbl

[15] **P Hanlon**, *Counting interval graphs*, Trans. Amer. Math. Soc. 272 (1982) 383–426 MR Zbl

[16] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR Zbl

[17] **P Hell**, **J Nešetřil**, *On the complexity of H–coloring*, J. Combin. Theory Ser. B 48 (1990) 92–110 MR Zbl

[18] **M W Hirsch**, *Differential topology*, Graduate Texts in Math. 33, Springer (1976) MR Zbl

[19] **Z Jackowski**, *A new characterization of proper interval graphs*, Discrete Math. 105 (1992) 103–109 MR Zbl

[20] **R J Kang**, **T Müller**, *Sphere and dot product representations of graphs*, Discrete Comput. Geom. 47 (2012) 548–568 MR Zbl

[21] **C G Lekkerkerker**, **J C Boland**, *Representation of a finite graph by a set of intervals on the real line*, Fund. Math. 51 (1962) 45–64 MR Zbl

[22] **A Lerario**, *Homology invariants of quadratic maps*, PhD thesis, SISSA (2011) Available at `https://www.math.sissa.it/publication/homology-invariants-quadratic-maps`

[23] **H Maehara**, *Space graphs and sphericity*, Discrete Appl. Math. 7 (1984) 55–64 MR Zbl

[24] **C McDiarmid**, **T Müller**, *The number of disk graphs*, European J. Combin. 35 (2014) 413–431 MR Zbl

[25] **G B Mertzios**, *A matrix characterization of interval and proper interval graphs*, Appl. Math. Lett. 21 (2008) 332–337 MR Zbl

[26] **J Milnor**, *Morse theory*, Ann. of Math. Stud. 51, Princeton Univ. Press (1963) MR Zbl

[27] **J Milnor**, *On the Betti numbers of real varieties*, Proc. Amer. Math. Soc. 15 (1964) 275–280 MR Zbl

[28] **J W Milnor**, **J D Stasheff**, *Characteristic classes*, Ann. of Math. Stud. 76, Princeton Univ. Press (1974) MR Zbl

[29] **F S Roberts**, *Indifference graphs*, from "Proof techniques in graph theory" (F Harary, editor), Academic, New York (1969) 139–146 MR Zbl

[30] **R P Stanley**, *An introduction to hyperplane arrangements*, from "Geometric combinatorics" (E Miller, V Reiner, B Sturmfels, editors), IAS/Park City Math. Ser. 13, Amer. Math. Soc., Providence, RI (2007) 389–496 MR Zbl

[31] **A Szulkin**, *Cohomology and Morse theory for strongly indefinite functionals*, Math. Z. 209 (1992) 375–418 MR Zbl

[32] **H E Warren**, *Lower bounds for approximation by nonlinear manifolds*, Trans. Amer. Math. Soc. 133 (1968) 167–178 MR Zbl

SISSA
Trieste, Italy

Current address: *Chair of Discrete Mathematics/Geometry, Technische Universität Berlin*
Berlin, Germany

SISSA
Trieste, Italy

Chair of Discrete Mathematics/Geometry, Technische Universität Berlin
Berlin, Germany

Current address: *Department of Mathematical Sciences, Carnegie Mellon University*
Pittsburgh, PA, United States

belotti@math.tu-berlin.de, lerario@sissa.it, anewman@andrew.cmu.edu

Received: 18 February 2022 Revised: 24 March 2023

Algebraic & Geometric Topology **24**:4 (2024) 2091–2148
DOI: 10.2140/agt.2024.24.2091
Published: 16 July 2024

Classical shadows of stated skein representations at roots of unity

JULIEN KORINMAN
ALEXANDRE QUESNEY

We extend some results of Bonahon, Wong, Bullock and Turaev concerning the skein algebras of closed surfaces to Lê's stated skein algebras associated to open surfaces. We prove that the stated skein algebra with deforming parameter $+1$ embeds canonically into the center of the stated skein algebra whose deforming parameter is an odd root unity. We also construct an isomorphism between the stated skein algebra at $+1$ and the algebra of regular functions of the relative SL_2–character variety of the surface. As a result, we associate to each isomorphism class of irreducible or local representations of the stated skein algebra an invariant which is a point in the relative character variety.

57R56

1 Introduction

A *punctured surface* is a pair $\underline{\Sigma} = (\Sigma, \mathcal{P})$, where Σ is a compact oriented surface and $\mathcal{P}$ is a (possibly empty) finite subset of Σ which intersects nontrivially each boundary component. We write $\Sigma_{\mathcal{P}} := \Sigma \setminus \mathcal{P}$. The set $\partial\Sigma \setminus \mathcal{P}$ consists of a disjoint union of open arcs which we call *boundary arcs*.

Warning In this paper, the punctured surface $\underline{\Sigma}$ will be called open if the surface Σ has nonempty boundary and closed if Σ is closed. This convention differs from the traditional one, where some authors refer to an open punctured surface as a punctured surface $\underline{\Sigma} = (\Sigma, \mathcal{P})$ with Σ closed and $\mathcal{P} \neq \varnothing$ (in which case $\Sigma_{\mathcal{P}}$ is not closed).

We will consider two related objects associated to a punctured surface, namely the Kauffman-bracket skein algebra and the $\mathrm{SL}_2(\mathbb{C})$–character variety. These objects have been well studied in the case where the punctured surface is closed. They were recently generalized to open punctured surfaces in such a way that they have a nice behavior relative to the operation of gluing two boundary arcs together. The goal of this paper is to extend some classical results concerning skein algebras and character varieties to the case of open punctured surfaces. Before we state the main results, let us give a brief historical background.

Historical background *Closed surfaces*: Culler and Shalen [1983] defined the $\mathrm{SL}_2(\mathbb{C})$ character variety $\mathcal{X}_{\mathrm{SL}_2}(M)$ of a manifold M whose fundamental group is finitely generated. This affine variety is closely related to the moduli space of flat connections on a trivial $\mathrm{SL}_2(\mathbb{C})$ bundle over M and, therefore, it is related to Chern–Simons topological quantum field theory, gauge theory and low-dimensional topology; see [Labourie 2013; Marché 2012; 2016] for surveys. If Σ is a closed oriented surface, the smooth part of $\mathcal{X}_{\mathrm{SL}_2}(\Sigma)$ carries a symplectic form, first defined in [Atiyah and Bott 1983] in the context of gauge theory. This symplectic structure was used by Goldman [1986] to equip the algebra of regular functions $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ with a Poisson bracket. A similar Poisson structure for character varieties of punctured closed surfaces was introduced by Fock and Rosly [1999] (see also [Alekseev et al. 2002] for an alternative construction) in the differential geometric context.

Turaev [1988] and Hoste and Przytycki [1992] independently defined the *Kauffman-bracket skein algebra* $\mathcal{S}_A(\underline{\Sigma})$ as a tool to study the Jones polynomial and the SU(2) Witten–Reshetikhin–Turaev TQFTs. Skein algebras are defined for any commutative unital ring $\mathcal{R}$ together with an invertible element $A \in \mathcal{R}^{\times}$ and a closed punctured surface $\underline{\Sigma}$.

Skein algebras are deformations of the algebra of regular functions of character varieties of closed punctured surfaces. In particular, this means that there is an isomorphism of Poisson algebras between $\mathcal{S}_{+1}(\underline{\Sigma})$ and $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$. In more detail, this relies on a (noncanonical) isomorphism from $\mathcal{S}_{+1}(\underline{\Sigma})$ to $\mathcal{S}_{-1}(\underline{\Sigma})$ [Barrett 1999]. The latter algebra carries a natural Poisson bracket (see Section 2.5). An isomorphism of algebras between $\mathcal{S}_{-1}(\underline{\Sigma})$ and $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ was defined by Bullock [1997], assuming that the skein algebra is reduced, ie that its nilradical is null. This latter fact was later proved independently in [Przytycki and Sikora 2000] and [Charles and Marché 2012]. Turaev [1991] showed that Bullock's isomorphism is Poisson.

In TQFT, skein algebras appear through their nontrivial finite-dimensional representations. Skein algebras admit such representations if and only if the parameter A is a root of unity. A recent result of Bonahon and Wong [2016] states, in particular, that when A has odd order, there exists an embedding of $\mathcal{S}_{+1}(\underline{\Sigma})$ into the center of $\mathcal{S}_A(\underline{\Sigma})$. Since each simple representation induces a character on the center of the skein algebra, using Bullock's isomorphism, one can associate to each isomorphism class of simple representation a point in the character variety. This invariant is called *the classical shadow* of the representation.

Open surfaces: Lê [2018] generalized the Kauffman-bracket skein algebras to open punctured surfaces based on the original work of Bonahon and Wong [2011]. We call it *stated skein algebra* and denote it by $\mathcal{S}_\omega(\underline{\Sigma})$. It depends on an invertible element $\omega \in \mathcal{R}^{\times}$. When the surface is closed, it coincides with the classical skein algebra with parameter $A = \omega^{-2}$. An important feature of the stated skein algebra is its behavior under gluing of surfaces. More precisely, let a and b be two boundary arcs of an open punctured surface $\underline{\Sigma}$, and let us denote by $\underline{\Sigma}|_{a\#b}$ the surface obtained from $\underline{\Sigma}$ by gluing a and b. Lê showed that there is an injective algebra morphism

$$i|_{a\#b} : \mathcal{S}_\omega(\underline{\Sigma}|_{a\#b}) \hookrightarrow \mathcal{S}_\omega(\underline{\Sigma}) \tag{1}$$

which is coassociative in that it does not depend on the order we glue the arcs, ie for four distinct boundary arcs a, b, c and d, one has $i|_{a\#b} \circ i|_{c\#d} = i|_{c\#d} \circ i|_{a\#b}$. In particular, for each topological triangulation Δ of $\underline{\Sigma}$, one has an injective morphism of algebras

$$i^{\Delta} : \mathcal{S}_\omega(\underline{\Sigma}) \hookrightarrow \bigotimes_{\mathbb{T} \in F(\Delta)} \mathcal{S}_\omega(\mathbb{T}). \tag{2}$$

Here $\mathbb{T}$ denotes the triangle, ie a disc with three punctures on its boundary. A punctured surface is *triangulable* if it can be obtained from a disjoint union of triangles by gluing some pair of boundary arcs (ie faces of triangles) together. A *topological triangulation* is the data of such a union of triangles together with the pairs of glued boundary arcs. In (2), the tensor product runs over the faces of the triangulation; see Section 2 for precise definitions.

As applications, Lê provided a simple proof that the algebra $\mathcal{S}_\omega(\mathbf{\Sigma})$ has no zero divisor (the case of closed triangulable punctured surfaces was proved earlier in [Bonahon and Wong 2011] using the quantum trace and the case of closed unpunctured surfaces was proved in [Przytycki and Sikora 2019]) and he obtained a simpler formulation of Bonahon and Wong's [2011] quantum trace map.

Motivated by Lê's construction, Korinman [2019] defined a generalization of character varieties to open punctured surfaces. We denote it by $\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})$. This (relative) character variety is a Poisson affine variety which coincides with the classical one when the surface is closed. It shares a similar gluing property to the stated skein algebra; namely, there exist injective Poisson morphisms $i|_{a\#b}\colon \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma}|_{a\#b})] \hookrightarrow \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ and $i^\Delta\colon \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})] \hookrightarrow \bigotimes_{\mathbb{T}\in F(\Delta)} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]$ between the Poisson algebras of regular functions. However, the Poisson structure on $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbf{\Sigma})]$ depends on a choice of an orientation $\mathfrak{o}$ of the boundary arcs of the punctured surface. We denote by $\{\cdot,\cdot\}^{\mathfrak{o}}$ its Poisson bracket.

Main results Let $\mathbf{\Sigma}$ be a punctured surface. Lê's morphism (2) embeds the skein algebra of a triangulated surface into a tensor product of the skein algebras of the triangle. However, it does not provide a full description of the stated skein algebra in terms of these smaller pieces. In a first result we provide such a description; it goes as follows. Note that (1) endows the skein algebra of the bigon $\mathbb{B}$ (ie a disc with two punctures on its boundary) with a bialgebra structure. It is in fact a Hopf algebra and one can show that it is canonically isomorphic to the classical quantum SL_2–algebra $\mathcal{O}_q[\mathrm{SL}_2]$ described in [Chari and Pressley 1994; Kassel 1995] (with $q = \omega^{-4}$). Note also that (1) induces Hopf comodule maps $\Delta_a^L\colon \mathcal{S}_\omega(\mathbf{\Sigma}) \to \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\mathbf{\Sigma})$ and $\Delta_b^R\colon \mathcal{S}_\omega(\mathbf{\Sigma}) \to \mathcal{S}_\omega(\mathbf{\Sigma}) \otimes \mathcal{S}_\omega(\mathbb{B})$ obtained by gluing a bigon on a boundary arc, a or b, of $\mathbf{\Sigma}$; see Section 2.2 for details.

Theorem 1.1 *The sequence*

$$0 \to \mathcal{S}_\omega(\mathbf{\Sigma}|_{a\#b}) \xrightarrow{i|_{a\#b}} \mathcal{S}_\omega(\mathbf{\Sigma}) \xrightarrow{\Delta_a^L - \sigma\circ\Delta_b^R} \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\mathbf{\Sigma})$$

is exact, where $\sigma(x\otimes y) = y \otimes x$.

Theorem 1.1 can be reformulated using co-Hochschild cohomology, whose 0^{th} group (see Definition 2.26 and [Hess et al. 2009]) computes the skein algebra

$$\mathcal{S}_\omega(\mathbf{\Sigma}|_{a\#b}) \cong \mathrm{coHH}^0(\mathcal{O}_q[\mathrm{SL}_2], {}_a\mathcal{S}_\omega(\mathbf{\Sigma})_b),$$

where ${}_a\mathcal{S}_\omega(\mathbf{\Sigma})_b$ is seen as a bicomodule over $\mathcal{O}_q[\mathrm{SL}_2]$ via the comodule maps Δ_a^L and Δ_b^R.

Theorem 1.1 provides, for any topological triangulation Δ of $\underline{\Sigma}$, an isomorphism of algebras

$$\mathcal{S}_\omega(\underline{\Sigma}) \cong \mathrm{coHH}^0\Bigg(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)} \mathcal{O}_q[\mathrm{SL}_2], \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_\omega(\mathbb{T})\Bigg),$$

where the first tensor product runs over the inner edges of the triangulation and the second over the faces of the triangulation. Hence $\mathcal{S}_\omega(\underline{\Sigma})$ is completely determined by the combinatoric of the triangulation together with $\mathcal{S}_\omega(\mathbb{T})$ and its appropriated structures of comodule over $\mathcal{O}_q[\mathrm{SL}_2]$. This is a key feature in the proofs of the next two theorems.

Our second result is a generalization to open punctured surfaces of Bonahon and Wong's [2016] main theorem in the case where the root of unity has odd order. Given $N \geq 1$, denote by $T_N(X)$ the N^{th} Chebyshev polynomial of first kind.

Theorem 1.2 *Suppose that ω is a root of unity of odd order $N \geq 1$. There exists an embedding*

$$j_{\underline{\Sigma}} : \mathcal{S}_{+1}(\underline{\Sigma}) \hookrightarrow \mathcal{Z}(\mathcal{S}_\omega(\underline{\Sigma}))$$

of the (commutative) stated skein algebra with parameter $+1$ into the center of the stated skein algebra with parameter ω. Moreover, the morphism $j_{\underline{\Sigma}}$ is characterized by the property that it sends a closed curve γ to $T_N(\gamma)$ and a stated arc $\alpha_{\varepsilon\varepsilon'}$ to $\alpha^{(N)}_{\varepsilon\varepsilon'}$, where $\alpha^{(N)}_{\varepsilon\varepsilon'}$ is the tangle made by stacking N parallel copies of $\alpha_{\varepsilon\varepsilon'}$ on top of the others.

In Theorem 1.2 we restrict ourselves to roots of unity of odd order for simplicity. Theorem 1.2 should be compared to [Lê and Paprocki 2019, Theorem 8.1]. A marked 3–manifold is a pair $(M, \mathcal{N})$ where M is an oriented 3–manifold and $\mathcal{N} \subset \partial M$ is an oriented submanifold whose connected components are diffeomorphic to $[0, 1]$. To such a pair and $\zeta \in \mathbb{C}^*$, Lê and Paprocki [2019] associate a vector space $\mathcal{S}_\zeta(M, \mathcal{N})$, which generalizes the Muller algebra. And for a root of unity ζ such that ζ^4 has arbitrary order $N > 1$ (not necessary odd), Lê and Paprocki [2019, Theorem 8.1] defined an injective linear map $\Phi_\zeta : \mathcal{S}_{(\zeta)^{N^2}}(M, \mathcal{N}) \hookrightarrow \mathcal{S}_\zeta(M, \mathcal{N})$. If $(\Sigma, \mathcal{P})$ is a punctured surface with no inner punctures and nontrivial boundary, $(M, \mathcal{N}) := (\Sigma \times (0, 1), \mathcal{P} \times (0, 1))$ is a marked 3–manifold and $\mathcal{S}_\zeta(M, \mathcal{N})$ is a subalgebra of the stated skein algebra $\mathcal{S}_\zeta(\Sigma, \mathcal{P})$. If ζ has odd order $N > 1$, the embedding $j_{\underline{\Sigma}}$ of Theorem 1.2 restricts to the embedding Φ_ζ of [Lê and Paprocki 2019, Theorem 8.1]. A generalization of Theorem 1.2 for roots of unity of even order has been recently proved by Bloomquist and Lê [2022, Theorem 1.2] though in this case the source of $j_{\underline{\Sigma}}$ is the skein algebra at $\eta := \omega^{N^2}$ and the image is not always central but rather spanned by $(-1)^{1+N'}$–transparent elements, where $N' := \mathrm{ord}(\omega^4)$ (see [Bloomquist and Lê 2022, Theorem 4.10] for details). Also a generalization of Theorem 1.2 for skein algebras of arbitrary connected reductive groups G and for marked surfaces having 0 or 1 boundary arc was found by Ganev, Jordan and Safronov [Ganev et al. 2024].

In the last result we generalize to open punctured surfaces Bullock's isomorphism [1997] and Turaev's theorem [1991]; we prove that the stated skein algebra is a deformation of the relative character variety. The fundamental result in this direction is as follows.

The $\mathbb{C}[\![\hbar]\!]$–module $\mathscr{S}_{+1}(\underline{\Sigma})[\![\hbar]\!] := \mathscr{S}_{+1}(\underline{\Sigma}) \otimes_{\mathbb{C}} \mathbb{C}[\![\hbar]\!]$ is endowed with a star product $\star_\hbar$. The latter is obtained by pulling back the product of $\mathscr{S}_{+1}(\underline{\Sigma})$ along an isomorphism $\mathscr{S}_{+1}(\underline{\Sigma})[\![\hbar]\!] \xrightarrow{\cong} \mathscr{S}_{\omega_\hbar}(\underline{\Sigma})$ of vector spaces, where $\omega_\hbar := \exp\left(-\frac{1}{4}\hbar\right)$ (see Section 2.7 for details). This equips $\mathscr{S}_{+1}(\underline{\Sigma})$ with a Poisson algebra structure; its Poisson bracket $\{\cdot,\cdot\}^s$ is defined by

$$f \star_\hbar g - g \star_\hbar f = \hbar\{f,g\}^s \pmod{\hbar^2} \quad \text{for all } f,g \in \mathscr{S}_{+1}(\underline{\Sigma}).$$

The superscript s stands for "skein". See Section 2.7.3 for an explicit description.

Theorem 1.3 *Suppose that $\underline{\Sigma}$ has a topological triangulation Δ. Let $\mathfrak{o}_\Delta$ be an orientation of the edges of Δ and $\mathfrak{o}$ be the induced orientation of the boundary arcs of $\underline{\Sigma}$. There exists an isomorphism of Poisson algebras*

$$\Psi^{(\Delta,\mathfrak{o}_\Delta)} : (\mathscr{S}_{+1}(\underline{\Sigma}), \{\cdot,\cdot\}^s) \xrightarrow{\cong} (\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})], \{\cdot,\cdot\}^{\mathfrak{o}}).$$

Moreover, the above isomorphism exists for small punctured surfaces (*see Definition 2.8*), *for which it only depends on $\mathfrak{o}$.*

The isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ induces, by tensoring with $\mathbb{C}[\![\hbar]\!]$, an isomorphism of vector spaces

$$\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})][\![\hbar]\!] \xrightarrow{\cong} \mathscr{S}_{+1}(\underline{\Sigma})[\![\hbar]\!].$$

Denote by $\star_{(\Delta,\mathfrak{o}_\Delta)}$ the product on $\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})][\![\hbar]\!]$ obtained by pulling back the product $\star_\hbar$ by this isomorphism.

Corollary 1.4 *For any triangulable punctured surface $\underline{\Sigma}$, the algebra $\left(\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})][\![\hbar]\!], \star_{(\Delta,\mathfrak{o}_\Delta)}\right)$ is a deformation quantization of the character variety with Poisson structure given by $\mathfrak{o}$.*

Theorems 1.2 and 1.3 allow us to extend Bonahon and Wong's [2016] *classical shadow* to open punctured surfaces. Indeed, suppose that ω is a root of unity of odd order. A finite-dimensional representation $\mathscr{S}_\omega(\underline{\Sigma}) \to \mathrm{End}(V)$ that sends each element of the image of $j_{\underline{\Sigma}} : \mathscr{S}_{+1}(\underline{\Sigma}) \hookrightarrow \mathscr{S}_\omega(\underline{\Sigma})$ to scalar operators, induces a character on the algebra $\mathscr{S}_{+1}(\underline{\Sigma}) \cong \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$, hence defines a point in $\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})$. To sum up, and calling these representations *central*, one has the following.

Corollary 1.5 *When ω is a root of unity of odd order and $\underline{\Sigma}$ is triangulable, to each isomorphism class of central representations of the stated skein algebra $\mathscr{S}_\omega(\underline{\Sigma})$, one can associate an invariant which is a point in the relative character variety $\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})$.*

Central representations include the families of irreducible representations, local representations and representations induced by simple modules of the balanced Chekhov–Fock algebras using the quantum trace map (see Section 3.3 for details).

Soon after the prepublication of this paper on arXiv, Costantino and Lê [2022] prepublished independently some results similar to Theorems 1.1 and 1.3. More precisely, [Costantino and Lê 2022, Theorem 4.7] is identical to Theorem 1.1, and [Costantino and Lê 2022, Theorem 8.12] is closely related, though different, to our Theorem 1.3. Instead of using the generalized character variety $\mathcal{X}_{\mathrm{SL}_2}(\Sigma)$ defined in [Korinman 2019], the authors defined a twisted character variety $\chi(\Sigma)$ (without Poisson structure) and constructed a canonical algebra isomorphism between the stated skein algebra in $+1$ and the algebra of regular functions of $\chi(\Sigma)$, whereas our isomorphism in Theorem 1.3 depends on the noncanonical choice $(\Delta, \mathfrak{o}_\Delta)$ of a triangulation and an orientation of the edges (and is Poisson). Inspired by their enlightening approach, in this new version of the paper we add the following clarification of the isomorphism in Theorem 1.3. As explained before, when the punctured surface is closed, the "standard" isomorphisms between $\mathcal{S}_{+1}(\Sigma)$ and $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\Sigma)]$ are indexed by spin structures. In Section 3.3, we define the notion of *relative spin structure* for punctured surfaces, which coincides with the standard definition when the punctured surface is closed. The motivation for this definition is its good behavior for the operation of gluing boundary arcs together. In particular we associate to each combinatorial data $(\Delta, \mathfrak{o}_\Delta)$, appearing in Theorem 1.3, a relative spin structure and prove:

Theorem 1.6 *The isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ of Theorem 1.3 only depends on the relative spin structure associated to $(\Delta, \mathfrak{o})$.*

In fact, in Theorem 3.20, we provide explicit formulas for the value of $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ on stated arcs and closed curves in terms of the relative spin structure. When the punctured surface is closed, we show that our isomorphism coincides with the standard isomorphism associated to classical spin structures. We also give, in Section 3.3.5, a detailed comparison between the isomorphism in Theorem 1.3 and Costantino and Lê's isomorphism [2022, Theorem 8.12].

Even though our proof of Theorem 1.2 makes uses of triangulations, the theorem is proved for arbitrary punctured surfaces, including (nontriangulable) closed surfaces without punctures, thus providing an alternative proof of the results in [Bonahon and Wong 2016]. However, our proof of Theorem 1.3 only works for triangulable punctured surfaces (and for the bigon), so it does not provide an alternative proof of the result of [Bullock 1997] for closed unpunctured surfaces.

Plan of the paper In the second section we briefly recall from [Lê 2018] the definition and general properties of the stated skein algebra and prove Theorem 1.1. We then use the triangular decomposition to reduce the proof of Theorem 1.2 to the cases of the bigon and the triangle for which the proof is a simple computation. We eventually characterize the Poisson bracket arising in skein theory. In the third section, we briefly recall from [Korinman 2019] the definition of character varieties for open surfaces. Again, using triangular decompositions, we reduce the proof of Theorem 1.3 to the cases of the bigon and the triangles for which the proof is elementary. We then introduce and study the notion of relative spin structure and give in Theorem 3.20 an explicit description of the isomorphism of Theorem 1.3, from which Theorem 1.6 is a straightforward consequence. In the appendix, we prove a technical result needed in the proof of Theorem 1.2 and derive a generalization of the main theorem of [Bonahon 2019].

Acknowledgments The authors thank F Bonahon, F Costantino, L Funar, T Q T Lê and J Toulisse for useful discussions. Korinman also thanks the University of South California and the Federal University of São Carlos for their kind hospitality during the beginning of this work. He acknowledges support from the grant ANR ModGroup, the GDR Tresses, the GDR Platon, CAPES, the GEAR Network, the CNRS, the JSPS and the ERC DerSymp (grant agreement 768679). Quesney was supported by PNPD/CAPES-2013 during the first period of this project, and by the São Paulo Research Foundation (FAPESP) grant 2018/19603-0 during the second period. The authors also thank the referees for valuable corrections and suggestions that improved the quality of the paper. The authors warmly thank T Q T Lê for correcting a mistake in a former version of the paper.

Notation Throughout the paper we reserve the notation $A := \omega^{-2}$ and $q := \omega^{-4}$.

2 Stated skein algebras

2.1 Definitions and general properties of the stated skein algebras

We briefly review from [Lê 2018] the definition and main properties of the stated skein algebras.

Definition 2.1 A *punctured surface* is a pair $\underline{\Sigma} = (\Sigma, \mathcal{P})$ where Σ is a compact oriented surface and $\mathcal{P}$ is a finite subset of Σ which intersects nontrivially each boundary component. A *boundary arc* is a connected component of $\partial\Sigma \setminus \mathcal{P}$. The punctured surface is *open* when $\partial\Sigma \neq \varnothing$ and *closed* otherwise.

Definition of stated skein algebras Let $\underline{\Sigma} = (\Sigma, \mathcal{P})$ be a punctured surface and write $\Sigma_{\mathcal{P}} := \Sigma \setminus \mathcal{P}$. A *tangle* in $\Sigma_{\mathcal{P}} \times (0,1)$ is a compact framed, properly embedded 1–dimensional manifold $T \subset \Sigma_{\mathcal{P}} \times (0,1)$ such that for every point of $\partial T \subset \partial\Sigma_{\mathcal{P}} \times (0,1)$ the framing is parallel to the $(0,1)$ factor and points in the direction of 1. Here, by framing, we refer to a thickening of T to an oriented surface. Define the *height* of a point $(v,h) \in \Sigma_{\mathcal{P}} \times (0,1)$ to be h. If b is a boundary arc and T a tangle, the points of $\partial_b T := \partial T \cap b \times (0,1)$ are totally ordered by their height and we impose that no two points in $\partial_b T$ have the same height. A tangle has *vertical framing* if for each of its points, the framing is parallel to the $(0,1)$ factor and points in the direction of 1. Two tangles are isotopic if they are isotopic through the class of tangles that preserves the partial boundary height orders. By convention, the empty set is a tangle only isotopic to itself.

Every tangle is isotopic to a tangle with vertical framing. We can further isotope a tangle such that it is in general position with the standard projection $\pi \colon \Sigma_{\mathcal{P}} \times (0,1) \to \Sigma_{\mathcal{P}}$ with $\pi(v,h) = v$, that is such that $\pi|_T \colon T \to \Sigma_{\mathcal{P}}$ is an immersion with at most transversal double points in the interior of $\Sigma_{\mathcal{P}}$. We call a *diagram* of T the image $D = \pi(T)$ together with the over/undercrossing information at each double point. An isotopy class of diagram D together with a total order of $\partial_b D = \partial D \cap b$ for each boundary arc b define uniquely an isotopy class of tangle. Here isotopy of diagrams refers to isotopies where endpoints of diagrams are not allowed to cross. When choosing an orientation $\mathfrak{o}(b)$ of a boundary arc b and a diagram D, the set $\partial_b D$ receives a natural total order $\leq_{\mathfrak{o}}$ by setting that the points are increasing when going in the direction of $\mathfrak{o}(b)$. We will represent tangles by drawing a diagram and an orientation (an arrow) for each boundary arc. When a boundary arc b is oriented, $\partial_b D$ is ordered by $\leq_{\mathfrak{o}}$ according to the orientation. The data of an isotopy class of diagram D and a choice $\mathfrak{o}$ of orientations of the boundary arcs define uniquely an isotopy class of tangle T by imposing that for every boundary arc a, for $v, w \in \partial_a D$ such that $v \leq_{\mathfrak{o}} w$, the endpoint of $\partial_a T$ corresponding to w has higher height than the endpoint corresponding to v. A *state* of a tangle is a map $s\colon \partial T \to \{-,+\}$. A pair (T,s) is called a *stated tangle*. We define a *stated diagram* (D,s) in a similar manner.

Let $\mathcal{R}$ be a commutative unital ring and $\omega \in \mathcal{R}^\times$ an invertible element.

Definition 2.2 The *stated skein algebra* $\mathcal{S}_\omega(\underline{\Sigma})$ is the free $\mathcal{R}$–module generated by isotopy classes of stated tangles in $\Sigma_{\mathcal{P}} \times (0,1)$ modulo the relations (3) and (4), which are

- the Kauffman bracket relations

(3) 

- the boundary relations

(4) 

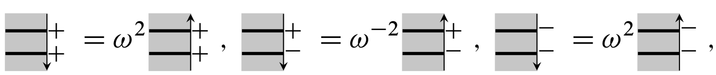
According to our graphical conventions, in these skein relations, the boundary points are consecutive in the height order. The product of two classes of stated tangles $[T_1, s_1]$ and $[T_2, s_2]$ is defined by isotoping T_1 and T_2 in $\Sigma_{\mathcal{P}} \times \left(\frac{1}{2}, 1\right)$ and $\Sigma_{\mathcal{P}} \times \left(0, \frac{1}{2}\right)$, respectively, and then setting $[T_1, s_1] \cdot [T_2, s_2] = [T_1 \cup T_2, s_1 \cup s_2]$.

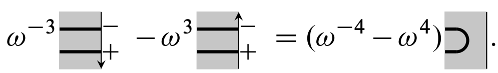
Bases for stated skein algebras A closed component of a diagram D is trivial if it bounds an embedded disc in $\Sigma_{\mathcal{P}}$. An open component of D is trivial if it can be isotoped, relatively to its boundary, inside some boundary arc. A diagram is *simple* if it has neither double points nor trivial component. The empty set is considered as a simple diagram. Let $\mathfrak{o}$ be an orientation of the boundary arcs of $\underline{\Sigma}$ and denote by $\leq_{\mathfrak{o}}$ the total orders induced on each boundary arc. A state $s\colon \partial D \to \{-,+\}$ is $\mathfrak{o}$*–increasing* if for any boundary arc b and any points $x, y \in \partial_b D$, $x <_{\mathfrak{o}} y$ implies $s(x) < s(y)$. Here we choose the convention $- < +$. We denote by $[D,s] \in \mathcal{S}_\omega(\underline{\Sigma})$ the class of the stated tangle associated to (D,s) (note that $[D,s]$ depends on the orientation $\mathfrak{o}$).

Definition 2.3 We denote by $\mathcal{B}^{\mathfrak{o}} \subset \mathcal{S}_\omega(\underline{\Sigma})$ the set of classes $[D,s]$ such that D is simple and s is $\mathfrak{o}$–increasing.

Theorem 2.4 [Lê 2018, Theorem 2.11] *The set $\mathcal{B}^{\mathfrak{o}}$ is an $\mathcal{R}$–module basis of $\mathcal{S}_\omega(\underline{\Sigma})$.*

Height exchange moves Important properties that we will use throughout the paper are the following *height exchange moves* (5) and (6) proved in [Lê 2018, Lemma 2.4]. Note that the formula (20) of Lemma 2.4 of [loc. cit.] contains a misprint. It is corrected here in (6):

(5) 

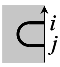
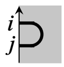
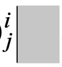
(6) 

Remark 2.5 An important case that we will be led to consider is the stated skein algebra at parameter $\omega = +1$. As shown in [Lê 2018, Corollary 2.5] it is commutative; this is a direct consequence of (3) and the height exchange formulas (5) and (6).

Trivial arcs relations We will also use the following *trivial arcs relations*. Set

$$C = \begin{pmatrix} C_+^+ & C_-^+ \\ C_+^- & C_-^- \end{pmatrix} := \begin{pmatrix} 0 & \omega \\ -\omega^5 & 0 \end{pmatrix} \quad \text{and} \quad C^{-1} = -A^3 C = \begin{pmatrix} 0 & -\omega^{-5} \\ \omega^{-1} & 0 \end{pmatrix}.$$

Lemma 2.6 [Lê 2018, Lemma 2.3] *One has the following* ***trivial arcs relations****:*

(7)  $= C_j^i$  *and*  $= (C^{-1})_j^i$  .

Splitting morphisms Suppose that $\underline{\Sigma}$ has two boundary arcs, say a and b. Let $\underline{\Sigma}|_{a\#b}$ be the punctured surface obtained from $\underline{\Sigma}$ by gluing a and b. Denote by $\pi\colon \Sigma_{\mathcal{P}} \to (\Sigma|_{a\#b})_{\mathcal{P}|_{a\#b}}$ the projection and $c := \pi(a) = \pi(b)$. Let (T_0, s_0) be a stated framed tangle of $\Sigma|_{a\#b\mathcal{P}|_{a\#b}} \times (0,1)$ transversed to $c \times (0,1)$ and such that the heights of the points of $T_0 \cap c \times (0,1)$ are pairwise distinct and such that framings of the points of $c \times (0,1)$ are vertical. Let $T \subset \Sigma_{\mathcal{P}} \times (0,1)$ be the framed tangle obtained by cutting T_0 along c. Using the partition $\partial T = \partial_a T \sqcup \pi^{-1}(\partial T_0) \sqcup \partial_b T$, a state on T can be written (s_a, s, s_b) where s_a, s and s_b are states on $\partial_a T$, ∂T_0 and $\partial_b T$, respectively. Both the sets $\partial_a T$ and $\partial_b T$ are in canonical bijection with the set $T_0 \cap c$ by the map π. Hence the two sets of states s_a and s_b are both in canonical bijection with the set $\mathrm{St}(c) := \{s\colon c \cap T_0 \to \{-,+\}\}$. Let $i|_{a\#b}\colon \mathcal{S}_\omega(\underline{\Sigma}|_{a\#b}) \to \mathcal{S}_\omega(\underline{\Sigma})$ be the linear map given, for any (T_0, s_0) as above, by

$$i|_{a\#b}([T_0, s_0]) := \sum_{s \in \mathrm{St}(c)} [T, (s, s_0, s)].$$

Theorem 2.7 [Lê 2018, Theorem 3.1] *The linear map $i|_{a\#b}$ is an injective morphism of algebras. Moreover the gluing operation is coassociative in the sense that if a, b, c and d are four distinct boundary arcs, then $i|_{a\#b} \circ i|_{c\#d} = i|_{c\#d} \circ i|_{a\#b}$.*

Note that the splitting morphism $i|_{a\#b}$ does not depend on any choice of the boundary arcs.

Triangulations

Definition 2.8 A *small* punctured surface is one of the following four connected punctured surfaces: the sphere with one or two punctures; the disc with only one puncture (on its boundary); and the bigon (disc with two punctures on its boundary).

Definition 2.9 A punctured surface is said to *admit a triangulation* if each of its connected components has at least one puncture and is not small.

Definition 2.10 Suppose $\underline{\Sigma} = (\Sigma, \mathcal{P})$ admits a triangulation. A *topological triangulation* Δ of $\underline{\Sigma}$ is a collection $\mathcal{E}(\Delta)$ of arcs in Σ (named edges) which satisfy the following conditions: the endpoints of the edges belong to $\mathcal{P}$; the interior of the edges are pairwise disjoint and do not intersect $\mathcal{P}$; the edges are not contractible and are pairwise nonisotopic in $\Sigma_{\mathcal{P}}$, if fixed their endpoints; and the boundary arcs of $\underline{\Sigma}$ belong to $\mathcal{E}(\Delta)$. Moreover, the collection $\mathcal{E}(\Delta)$ is required to be maximal for these properties.

Each connected component of $\Sigma \setminus \mathcal{E}(\Delta)$ is called a *face* and the set of faces is denoted by $F(\Delta)$. Given a topological triangulation Δ, the punctured surface is obtained from the disjoint union $\bigsqcup_{\mathbb{T}\in F(\Delta)} \mathbb{T}$ of triangles by gluing the triangles along the boundary arcs corresponding to the edges of the triangulation. Very often, we will let $\mathbb{T}$ be both a face (which is an open contractible space) and the triangle (which is a disc with exactly three punctures on its boundary). We hope that this abuse of notation is harmless. By composing the associated splitting morphisms, one obtains an injective morphism of algebras

$$i^{\Delta} \colon \mathcal{S}_{\omega}(\underline{\Sigma}) \hookrightarrow \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_{\omega}(\mathbb{T}).$$

Filtrations The stated skein algebra has natural filtrations defined as follows. Let $S = \{a_1, \dots, a_n\}$ be a set of boundary arcs of $\underline{\Sigma}$ and fix an orientation $\mathfrak{o}$ of the boundary arcs of $\underline{\Sigma}$. For a basis element $[D, s]$ of $\mathcal{B}^{\mathfrak{o}}$, write $d([D, s]) := \sum_{a\in S} |\partial_a D|$. The map d extends to a map $d \colon \mathcal{S}_{\omega}(\underline{\Sigma}) \to \mathbb{Z}^{\geq 0}$ by the formula $d\big(\sum_i x_i [D_i, s_i]\big) := \max_{i \mid x_i \neq 0} d([D_i, s_i])$. It follows from the relations (3) and (4) that for each $x, y \in \mathcal{S}_{\omega}(\underline{\Sigma})$, we have $d(xy) \leq d(x) + d(y)$. Given $m \geq 0$, denote by $\mathcal{F}_m \subset \mathcal{S}_{\omega}(\underline{\Sigma})$ the subvector space of those vectors x satisfying $d(x) \leq m$. These subspaces satisfy $\mathcal{F}_m \subset \mathcal{F}_{m+1}$, $\mathcal{S}_{\omega}(\underline{\Sigma}) = \bigcup_{m\geq 0} \mathcal{F}_m$ and $\mathcal{F}_{m_1} \cdot \mathcal{F}_{m_2} \subset \mathcal{F}_{m_1+m_2}$; hence they form an algebra filtration of the stated skein algebra.

Definition 2.11 The sequence $(\mathcal{F}_m)_{m\geq 0}$ is called the *filtration* of $\mathcal{S}_{\omega}(\underline{\Sigma})$ associated to the orientation $\mathfrak{o}$ and the set S of boundary arcs. For an element $X = \sum_{i\in I} x_i [D_i, s_i] \in \mathcal{S}_{\omega}(\underline{\Sigma})$, developed in the basis $\mathcal{B}^{\mathfrak{o}}$, we call the *leading term* of X the element

$$\mathrm{lt}(X) := \sum_{\substack{j\in I \\ d([D_j, s_j]) = d(X)}} x_j [D_j, s_j].$$

2.2 Alternative bases

In the next subsection, we will need alternative bases of $\mathcal{S}_{\omega}(\underline{\Sigma})$ which we now introduce. We fix an arbitrary orientation $\mathfrak{o}$ for each boundary arc. Recall that $\mathfrak{o}$ induces a total order $\leq_{\mathfrak{o}}$ on each boundary arc that we use to associate a tangle to a diagram.

Notation 2.12 Let $\mathcal{D}(\underline{\Sigma})$ be the set of isotopy classes of simple diagrams and $\mathcal{CD}(\underline{\Sigma})$ be its subset of classes of connected diagrams. Fix an arbitrary total order $\prec$ on $\mathcal{CD}(\underline{\Sigma})$ and fix an orientation $\mathfrak{o}$ of the boundary arcs of $\underline{\Sigma}$ as before. For $[D] \in \mathcal{CD}(\underline{\Sigma})$, we denote by $[T(D)]$ the isotopy class of the tangle $T(D)$ with vertical framing whose projection is D and such that if $\partial T(D) = \{v_1, v_2\}$ with v_1 and v_2 in

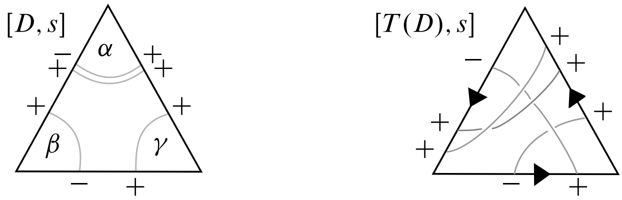


Figure 1: A stated diagram $[D, s]$ in the triangle and its associated stated tangle $[T(D), s]$. Here, we use the order $\gamma \prec \beta \prec \alpha$. Here s is $\mathfrak{o}$–increasing so $[T(D), s] \in \mathcal{TB}^{\mathfrak{o}}$.

the same boundary arc a with $v_1 \leq_{\mathfrak{o}} v_2$, then $h(v_1) < h(v_2)$. For a general class of diagram $[D] \in \mathcal{D}(\underline{\Sigma})$ with connected components $D = \bigsqcup_{i=1}^n D_i$, where $[D_i] \preceq [D_{i+1}]$, we denote by $[T(D)]$ the class of the tangle $T(D) := \bigsqcup_{i=1}^n T(D_i)$ in $\Sigma_{\mathcal{P}} \times (0,1)$, where $T(D_{i+1})$ is on the top of $T(D_i)$ in the height direction. See Figure 1 for an illustration. Let $\nu \colon \partial D \xrightarrow{\cong} \partial T(D)$ be the unique bijection such that, for a a boundary arc, ν restricts to a bijection $\nu|_a \colon \partial_a D \to \partial_a T(D)$ which preserves the order $\leq_{\mathfrak{o}}$ on $\partial_a D$ and the height order on $\partial_a T(D)$. Recall that $\partial_a D = D \cap a$ and that $\partial_a T(D) = T(D) \cap a \times (0,1)$. A state s on D defines a state $s \circ \nu^{-1}$ on $T(D)$ and we denote by $[T(D), s]$ the class of the stated tangle $(T(D), s \circ \nu^{-1})$.

Definition 2.13 We denote by $\mathcal{TB}^{\mathfrak{o}} \subset \mathcal{S}_\omega(\underline{\Sigma})$ the set of classes $[T(D), s]$ with $[D] \in \mathcal{D}(\underline{\Sigma})$ and s an $\mathfrak{o}$–increasing state.

Note that in our pictures the orientation $\mathfrak{o}$ is never represented, the arrows always refer to the height order and not to $\mathfrak{o}$. The following lemma was proved in [Lê 2018], during the proof of Theorem 4.6, in the particular case where $\underline{\Sigma}$ is a triangle.

Proposition 2.14 *The set $\mathcal{TB}^{\mathfrak{o}}$ is a basis of $\mathcal{S}_\omega(\underline{\Sigma})$.*

As an immediate consequence of Proposition 2.14, we get:

Corollary 2.15 *The stated skein algebra is algebraically generated by the classes of closed curves and stated arcs.*

Here by closed curves and stated arcs we mean connected stated diagrams with no crossing which are closed and open, respectively. Obviously, it is sufficient to prove Proposition 2.14 in the case where Σ is connected. If $\partial\Sigma = \varnothing$ or if $\underline{\Sigma}$ is a disc with one puncture on its boundary or a bigon whose boundary arcs points towards the same puncture, then $\mathcal{TB}^{\mathfrak{o}} = \mathcal{B}^{\mathfrak{o}}$ so the proposition follows from Theorem 2.4 in those cases. For the bigon whose boundary arcs point towards distinct punctures, Proposition 2.14 was proved in [Lê 2018, Step 1 of the proof of Theorem 4.1]. So we now assume that $\underline{\Sigma}$ admits a topological triangulation Δ that we fix. The proof of Proposition 2.14 is an easy adaption of Lê's argument from the case of the triangle to the case of a triangulable punctured surface. The key feature is to consider a suitable filtration that we now introduce.

For a diagram D and an edge $e \in \mathscr{E}(\Delta)$, we denote by $i(D,e) \in \mathbb{N}$ the geometric intersection of D with e; that is, the minimal number of intersection points when isotoping D in such a way that it intersects e transversally. We write

$$|D| := \sum_{e \in \mathscr{E}(\Delta)} i(D,e),$$

and, for $i \in \mathbb{N}$, we set

$$\mathscr{F}_i := \mathrm{Span}\{[D,s] \text{ such that } |D| \leq i\}.$$

Lemma 2.16 (1) *One has* $\mathscr{F}_i \cdot \mathscr{F}_j \subset \mathscr{F}_{i+j}$.

(2) *The submodule* $\mathscr{F}_i$ *has basis the set* B_i *of elements* $[D,s] \in \mathscr{B}^{\mathfrak{o}}$ *such that* $|D| \leq i$.

(3) *For* $[D,s] \in \mathscr{B}^{\mathfrak{o}}$, *there exists* $n \in \mathbb{Z}$ *such that*

$$[T(D),s] \equiv A^n[D,s] \pmod{\mathscr{F}_{|D|-2}}.$$

Proof (1) Let $[D_1,s_1]$ and $[D_2,s_2]$ be two classes such that

(i) $D_1 \cup D_2$ has only transversed double intersection points in the interior of $\Sigma_{\mathscr{P}}$ away from the edges of Δ, and

(ii) D_1 and D_2 are transversed to the edges of $\mathscr{E}(\Delta)$ with minimal intersection such that

$$|D_i| = |D_i \cap \mathscr{E}(\Delta)|, \quad i = 1,2.$$

Let D denote the diagram obtained by stacking D_1 on top of D_2 and s the state corresponding to s_1 and s_2 such that $[D,s] = [D_1,s_1][D_2,s_2]$. Then $|D| \leq |D \cap \mathscr{E}(\Delta)| = |D_1| + |D_2|$. Therefore, $[D_1,s_1][D_2,s_2] \in \mathscr{F}_{|D_1|+|D_2|}$ and the first assertion is proved.

(2) To prove the second assertion, first note that since B_i is a subset of $\mathscr{B}^{\mathfrak{o}}$, it is free. We need to show that B_i generates $\mathscr{F}_i$. We proceed in two steps:

Step 1 We first prove that any class of stated diagram $[D,s]$ is a linear combination of elements $[D_i,s_i]$ with $|D_i| = |D|$ and such that D_i has no crossing.

Step 2 We then prove that any $[D,s]$, where D has no crossing, is a linear combination of elements of $B_{|D|}$.

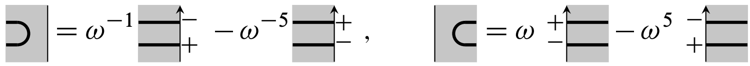
The two steps imply that B_i generates $\mathscr{F}_i$ and conclude the proof of the second assertion.

To prove the first step, fix an arbitrary stated diagram (D,s). A *resolution* of D is a diagram obtained from D by replacing each crossing  by either  (positive resolution of the crossing) or  (negative resolution). Write $\mathrm{Res}(D)$ the set of resolutions and for $D_0 \in \mathrm{Res}(D)$, denote by $n(D_0)$ the difference between the numbers of positive and negative resolution crossings in D_0. Then, by the Kauffman-bracket skein relation (3), one has

$$[D,s] = \sum_{D_i \in \mathrm{Res}(D)} A^{n(D_i)}[D_i,s],$$

where for each resolution D_i, one has $|D_i \cap \mathscr{E}(\Delta)| = |D \cap \mathscr{E}(\Delta)| = |D|$, so $|D_i| = |D|$ and Step 1 is proved.

To prove the second step, consider a stated diagram (D,s) where D has no crossing. If s is $\mathfrak{o}$–increasing, let (D',s) be the stated diagram obtained from (D,s) by removing its trivial components, so $|D'|\le|D|$. Then there exists a scalar c such that $[D,s]=c[D',s]$ and $[D',s]\in B_{|D|}$. Otherwise, we show by induction on the number $m(D,s)$ of pairs of points $v<_{\mathfrak{o}}w$ in ∂D lying in the same boundary arc such that $(s(v),s(w))=(+,-)$, that (D,s) is a linear combination of elements of $B_{|D|}$. Consider such a pair (v,w) of points which are consecutive for $<_{\mathfrak{o}}$, and let s' be the state on D which agrees with s on $\partial D\setminus\{v,w\}$ and such that $(s'(v),s'(w))=(-,+)$. The skein relations



show that there exists $n\in\mathbb{Z}$ such that $[D,s]\equiv\omega^n[D,s']\ (\mathrm{mod}\ \mathcal{F}_{|D|-1})$ (because the stated diagram representing either the term  or  is in $\mathcal{F}_{|D|-1}$). Since $m(D,s')<m(D,s)$, we conclude by decreasing induction on m that $[D,s]$ is a linear combination of elements $[D_i,s_i]$ where D_i has no crossing and s_i is $\mathfrak{o}$–increasing. Now write $[D_i,s_i]=c_i[D'_i,s_i]$, where c_i is a scalar and (D'_i,s_i) is obtained from (D_i,s_i) by removing its trivial components so that $[D'_i,s_i]\in B_{|D|}$. This concludes Step 2 and the proof of the second item.

(3) Let us first make an obvious but useful remark. Let D be a diagram transversed to $\mathcal{E}(\Delta)$. We say that *D contains a returning arc* if there exists a face $\mathbb{T}$ such that $D\cap\mathbb{T}$ contains a connected component that is an arc with both endpoints in the same edge. If D contains a returning arc, then D is not in minimal intersection position with respect to $\mathcal{E}(\Delta)$ so for all states s, $[D,s]\in\mathcal{F}_{|D|-2}$.

Now consider $[D,s]\in\mathcal{B}^{\mathfrak{o}}$ and denote by TD the projection diagram of the tangle $T(D)$ so that $[T(D),s]=[TD,s]$ (think of Figure 1). We further suppose that TD is transversed to $\mathcal{E}(\Delta)$ in minimal position and has its crossings outside $\mathcal{E}(\Delta)$. In the decomposition

$$[TD,s]=\sum_{D_i\in\mathrm{Res}(TD)}A^{n(D_i)}[D_i,s],$$

we claim that there is exactly one resolution $D_0\in\mathrm{Res}(TD)$ such that $D_0=D$ and that any other resolution $D_i\neq D_0$ contains a returning arc, so satisfies $[D_i,s_i]\in\mathcal{F}_{|D|-2}$. Since resolving a crossing is a local operation, it is sufficient to prove the claim in the case of the triangle; this was done by Lê [2018, Lemma 4.7]. Recall that Lê's proof consists noting that if $[T(D),s]$ has two connected components, it has 0 or 1 crossing (after eventually isotoping TD) and when there is one crossing in TD, exactly one of the two resolutions does not contain returning arc. The results then follows by induction on the number of components of $T(D)$ using the fact that the arcs in $T(D)$ are stacked on top of each other.

So we have $[T(D),s]\equiv A^{n(D)}[D,s]\ (\mathrm{mod}\ \mathcal{F}_{|D|-2})$ and the proof is completed. □

Obviously one has $\mathcal{F}_i\subset\mathcal{F}_{i+1}$ and $\bigcup_{i\ge0}\mathcal{F}_i=\mathcal{S}_\omega(\Sigma)$. The first assertion of Lemma 2.16 implies that $(\mathcal{F}_i)_{i\ge0}$ forms an algebra filtration of $\mathcal{S}_\omega(\Sigma)$. Consider the graded algebra $\mathbf{Gr}_\bullet$ associated to the filtration. In other words, we set $\mathbf{Gr}_0:=\mathcal{F}_0$, $\mathbf{Gr}_i:=\mathcal{F}_i\,/\,\mathcal{F}_{i-1}$ for $i\ge1$ and $\mathbf{Gr}_\bullet:=\bigoplus_{i\ge0}\mathbf{Gr}_i$. It follows from the second assertion of Lemma 2.16 that $\mathbf{Gr}_i$ has basis the set $\mathcal{B}^{\mathfrak{o}}_i$ of classes $[D,s]\in\mathcal{B}^{\mathfrak{o}}$ such that $|D|=i$.

Since the set $\{\mathcal{B}_i^{\mathfrak{o}}\}_{i\geq 0}$ forms a partition of $\mathcal{B}^{\mathfrak{o}}$, the natural graded morphism $\psi\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathbf{Gr}_\bullet$ is an isomorphism. To prove Proposition 2.14, we will derive from the third assertion of Lemma 2.16 that the image of $\mathcal{T}\mathcal{B}^{\mathfrak{o}}$ through ψ is a basis of $\mathbf{Gr}_\bullet$.

Proof of Proposition 2.14 As noted previously, if $\underline{\Sigma}$ is closed or if $\underline{\Sigma}$ is bigon or a disc with one puncture on its boundary, then $\mathcal{T}\mathcal{B}^{\mathfrak{o}} = \mathcal{B}^{\mathfrak{o}}$ so the lemma follows from Theorem 2.4. Otherwise, we can consider a topological triangulation and consider the associated graded isomorphism $\psi\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathbf{Gr}_\bullet$. Let $\mathcal{T}\mathcal{B}_i^{\mathfrak{o}} \subset \mathcal{T}\mathcal{B}^{\mathfrak{o}}$ be the subset of elements $[T(D), s]$ such that $|D| = i$. Since $\psi(\mathcal{B}_i^{\mathfrak{o}})$ is a basis of $\mathbf{Gr}_i$, the third assertion of Lemma 2.16 implies that the image $\psi(\mathcal{T}\mathcal{B}_i^{\mathfrak{o}})$ is also a basis of $\mathbf{Gr}_i$. Therefore $\psi(\mathcal{T}\mathcal{B}^{\mathfrak{o}})$ is a basis of $\mathbf{Gr}_\bullet$, so $\mathcal{T}\mathcal{B}^{\mathfrak{o}}$ is a basis of $\mathcal{S}_\omega(\underline{\Sigma})$. □

2.3 Removing a puncture

Let $\underline{\Sigma} = (\Sigma, \mathcal{P})$ and consider a punctured surface $\underline{\Sigma}' := (\Sigma, \mathcal{P} \cup \{p_0\})$ obtained from $\underline{\Sigma}$ by adding a puncture $p_0 \in \Sigma_{\mathcal{P}}$ to $\mathcal{P}$. The goal of this subsection is to define and study a map $\varphi\colon \mathcal{S}_\omega(\underline{\Sigma}') \to \mathcal{S}_\omega(\underline{\Sigma})$. Let $\mathcal{T}(\underline{\Sigma})$ denote the set of stated tangles in $\Sigma_{\mathcal{P}} \times (0,1)$ and denote by $\mathcal{J}(\underline{\Sigma}) \subset \mathcal{R}[\mathcal{T}(\underline{\Sigma})]$ the ideal generated by the skein relations (3) and (4) and by the elements $(T,s) - (T',s)$, where T and T' are isotopic; so by definition, one has $\mathcal{S}_\omega(\underline{\Sigma}) := \mathcal{R}[\mathcal{T}(\underline{\Sigma})] \,/\, \mathcal{J}(\underline{\Sigma})$. The inclusion map $\iota\colon \Sigma_{\mathcal{P}\cup\{p_0\}} \times (0,1) \hookrightarrow \Sigma_{\mathcal{P}} \times (0,1)$ induces a linear map $\underline{\varphi}\colon \mathcal{R}[\mathcal{T}(\underline{\Sigma}')] \to \mathcal{R}[\mathcal{T}(\underline{\Sigma})]$ sending a stated tangle (T,s) to $(\iota(T), s \circ \iota^{-1})$.

First suppose that p_0 is in the interior of $\Sigma_{\mathcal{P}}$. In this case, $\underline{\varphi}$ obviously sends isotopic stated tangles to isotopic stated tangles and skein relations to skein relations, so it sends $\mathcal{J}(\underline{\Sigma}')$ to $\mathcal{J}(\underline{\Sigma})$ and it induces a linear map $\varphi\colon \mathcal{S}_\omega(\underline{\Sigma}') \to \mathcal{S}_\omega(\underline{\Sigma})$ by passing to the quotient. It is clear that φ is a morphism of algebras. Moreover, since any tangle in $\Sigma_{\mathcal{P}} \times (0,1)$ can be isotoped in $\Sigma_{\mathcal{P}\cup\{p_0\}} \times (0,1)$, the map φ is surjective.

When p_0 lies in some boundary arc, say a, of $\underline{\Sigma}$, the construction is more subtle. Denote by b and c the two boundary arcs of $\underline{\Sigma}'$ which are the connected components of $a \setminus \{p_0\}$. The linear map φ still sends skein relations to skein relations; however if (T,s) and (T',s') are two isotopic stated tangles, then $\underline{\varphi}(T,s)$ and $\underline{\varphi}(T',s')$ are no longer necessarily isotopic. Indeed, recall that in our definition of isotopy, for any boundary arc d, the height order of $\partial_d T$ should be preserved. Now if we choose T and T' isotopic in $\Sigma_{\mathcal{P}\cup\{p_0\}} \times (0,1)$, the isotopy relating T to T' preserves the height orders of $\partial_b T$ and $\partial_c T$, but not necessarily the height order of $\partial_a T$, so $\underline{\varphi}(T,s)$ and $\underline{\varphi}(T',s')$ might not be isotopic.

Even worse, T might have two endpoints in $\partial_b T$ and $\partial_c T$ with the same height, so $\iota(T)$ is not a tangle in our sense since it would have two points in $\partial_a \iota(T)$ with the same height.

To remedy this problem, we introduce the subset $\mathcal{T}^0(\underline{\Sigma}') \subset \mathcal{T}(\underline{\Sigma}')$ of stated tangles (T,s) in $\Sigma_{\mathcal{P}\cup\{p_0\}}$ such that for any two points $v \in \partial_b(T)$ and $v' \in \partial_c(T)$, one has $h(v) < h(v')$ (h is the height function). Since any stated tangle $(T,s) \in \mathcal{T}(\underline{\Sigma}')$ is isotopic to a stated tangle in $\mathcal{T}^0(\underline{\Sigma}')$, one has

$$\mathcal{S}_\omega(\underline{\Sigma}') = \mathcal{R}[\mathcal{T}^0(\underline{\Sigma}')] \,/\, \mathcal{J}(\underline{\Sigma}') \cap \mathcal{R}[\mathcal{T}^0(\underline{\Sigma}')].$$

Now, the restriction $\underline{\varphi}^0\colon \mathcal{R}[\mathcal{T}^0(\underline{\Sigma}')] \to \mathcal{R}[\mathcal{T}(\underline{\Sigma})]$ of $\underline{\varphi}^0$ preserves skein relations and (T,s) is isotopic to (T',s') implies that $\underline{\varphi}^0(T,s)$ is isotopic to $\underline{\varphi}^0(T',s')$; therefore φ^0 induces a linear map $\varphi\colon \mathcal{S}_\omega(\underline{\Sigma}') \to \mathcal{S}_\omega(\underline{\Sigma})$ which is obviously an algebra morphism and is surjective.

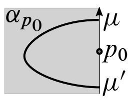
Definition 2.17 The *off-puncture ideal* $\mathcal{I}_{p_0} \subset \mathcal{S}_\omega(\underline{\Sigma}')$ is the ideal generated by

(1) the elements $\gamma - \gamma'$, where γ and γ' are noncontractible simple closed curves in $\Sigma_{\mathcal{P}\cup\{p_0\}}$ which are isotopic in $\Sigma_{\mathcal{P}}$;

(2) the elements $\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}$, where $\alpha_{\varepsilon\varepsilon'}$ and $\alpha'_{\varepsilon\varepsilon'}$ are nontrivial simple stated arcs in $\Sigma_{\mathcal{P}\cup\{p_0\}}$ which are isotopic in $\Sigma_{\mathcal{P}}$;

(3) when p_0 is an inner puncture, the element $\gamma_{p_0} + q + q^{-1}$, where γ_{p_0} is a peripheral curve encircling p_0 (recall that $q = \omega^{-4}$);

(4) when p_0 is on the boundary of $\Sigma_{\mathcal{P}}$, the elements $\alpha_{p_0\,\mu\mu'} - C^{\mu}_{\mu'}$, where α_{p_0} is the trivial arc encircling p_0 as

$$\alpha_{p_0\,\mu\mu'} =$$ 

such that the endpoint with state μ has bigger height than the endpoint with state μ'.

The purpose of this subsection it to prove:

Proposition 2.18 *The following sequence is exact*:

$$0 \to \mathcal{I}_{p_0} \to \mathcal{S}_\omega(\underline{\Sigma}') \xrightarrow{\varphi} \mathcal{S}_\omega(\underline{\Sigma}) \to 0. \tag{8}$$

The surjectivity of φ follows from the preceding discussion and the inclusion $\mathcal{I}_{p_0} \subset \ker(\varphi)$ is an immediate consequence of the definitions and the trivial arcs relations (7) (where the equalities $\varphi(\alpha_{p_0\,\mu\mu'}) = C^{\mu}_{\mu'}$ are proved), so we need to prove the inclusion $\ker(\varphi) \subset \mathcal{I}_{p_0}$.

Notation 2.19

- Let (D,s) be a connected simple stated diagram in $\Sigma_{\mathcal{P}\cup\{p_0\}}$ (so either a closed curve or a stated arc or the empty diagram) and define a scalar $c(D,s) \in \mathcal{R}$ as follows. If $\iota(D)$ is simple in $\Sigma_{\mathcal{P}}$, set $c(D,s) = 1$. If p_0 is an inner puncture and $(D,s) = \gamma_{p_0}$ is a peripheral curve around p_0, write $c(\gamma_{p_0}) = -q - q^{-1}$. If p_0 is on the boundary of $\Sigma_{\mathcal{P}}$ and $\iota(D)$ is a trivial arc encircling p_0, let $c(D,s)$ be the unique element $C^{\mu}_{\mu'}$ or $(C^{-1})^{\mu}_{\mu'}$ such that $\varphi(D,s) = c(D,s)$ (using the trivial arcs relations (7)).
- For a not necessarily connected stated diagram $(D,s) = \bigsqcup_{i\in I}(D_i,s_i)$, where the (D_i,s_i) are its connected components, write $c(D,s) = \prod_{i\in I} c(D_i,s_i)$. Let $J \subset I$ be the subset of indices $j \in I$ such that $\iota(D_j)$ is simple. The *reduction of* D is the simple diagram $D^{\mathrm{red}} := \bigsqcup_{j\in J} D_j$. By definition, one has

$$\varphi\big([T(D),s]\big) = c(D,s)\varphi\big([(T(D^{\mathrm{red}}),s)]\big). \tag{9}$$

Lemma 2.20 *Let M and M' be two free $\mathcal{R}$–modules with respective bases $\mathcal{B}$ and $\mathcal{B}'$. Let $\pi : \mathcal{B}' \to \mathcal{B}$ and $c : \mathcal{B}' \to \mathcal{R}$ two maps and suppose that there exists $\mathcal{B}'^{\text{red}} \subset \mathcal{B}'$ such that the restriction $\pi|_{\mathcal{B}'^{\text{red}}}\mathcal{B}'^{\text{red}} \to \mathcal{B}$ is surjective and such that $c(b'^{\text{red}}) = 1$ for all $b'^{\text{red}} \in \mathcal{B}'^{\text{red}}$. Consider the linear morphism $\varphi : M' \to M$ defined by $\varphi(b') := c(b')\pi(b')$, for $b' \in \mathcal{B}'$. Then*

$$\ker(\varphi) = \operatorname{Span}\{b' - c(b')b'^{\text{red}} \textit{ such that } \pi(b'^{\text{red}}) = \pi(b'),\ b'^{\text{red}} \in \mathcal{B}'^{\text{red}},\ b' \in \mathcal{B}'\}.$$

Proof Let $V \subset M'$ be the submodule linearly spanned by the elements $b' - c(b')b'^{\text{red}}$ with $\pi(b'^{\text{red}}) = \pi(b')$ for $b'^{\text{red}} \in \mathcal{B}'^{\text{red}}$ and $b' \in \mathcal{B}'$. By definition, $\varphi(b' - c(b')b'^{\text{red}}) = c(b')(\pi(b') - \pi(b'^{\text{red}})) = 0$ so the inclusion $V \subset \ker(\varphi)$ is obvious. Conversely, consider an arbitrary element $x = \sum_{b' \in \mathcal{B}'} \alpha_{b'} b' \in \ker(\varphi)$. Fix a right inverse $\iota : \mathcal{B} \to \mathcal{B}'^{\text{red}}$ to π; that is a map such that $\pi \circ \iota = \mathrm{id}$. For $b \in \mathcal{B}$, write $x_b := \sum_{b' \in \pi^{-1}(b)} \alpha_{b'} b'$ so that $x = \sum_{b \in \mathcal{B}} x_b$. Since $\mathcal{B}$ is a basis, the elements $\varphi(x_b)$ are linearly independent so $\varphi(x) = 0$ implies that $\varphi(x_b) = 0$ for all $b \in \mathcal{B}$. Let $b \in \mathcal{B}$ be such that $x_b \neq 0$ and let us prove that $x_b \in V$. Let $b'^{\text{red}} := \iota(b) \in \mathcal{B}'^{\text{red}}$. Since $\varphi(x_b) = 0$, one has $\sum_{b' \in \pi^{-1}(b)} \alpha_{b'} c(b') = 0$. Now

$$\begin{aligned} x_b = \sum_{b' \in \pi^{-1}(b)} \alpha_{b'} b' &= \sum_{b' \in \pi^{-1}(b)} \alpha_{b'}(b' - c(b')b'^{\text{red}}) + \Bigg(\sum_{b' \in \pi^{-1}(b)} \alpha_{b'} c(b')\Bigg) b'^{\text{red}} \\ &= \sum_{b' \in \pi^{-1}(b)} \alpha_{b'}(b' - c(b')b'^{\text{red}}) \in V. \end{aligned}$$ □

Proof of Proposition 2.18 Applying Lemma 2.20 to $M = \mathcal{S}_\omega(\underline{\Sigma})$, $M' = \mathcal{S}_\omega(\underline{\Sigma}')$, $\mathcal{B} = \mathcal{T}\mathcal{B}^{\mathrm{o}}(\underline{\Sigma})$, $\mathcal{B}' = \mathcal{T}\mathcal{B}^{\mathrm{o}}(\underline{\Sigma}')$ and $\mathcal{B}'^{\text{red}}$ the subset of $\mathcal{B}'$ of diagrams $(T(D), s)$ such that $D^{\text{red}} = D$ and π the reduction map, we obtain that $\ker(\varphi)$ is spanned by elements of the form $[T(D), s] - c(D, s)[T(D^{\text{red}}), s]$. By definition, the off-puncture ideal is the ideal generated by the elements $[T(D), s] - c(D, s)[T(D^{\text{red}}), s]$, where D is connected. Let us prove by induction on the number of connected components of D that $[T(D), s] - c(D, s)[T(D^{\text{red}}), s] \in \mathcal{I}_{p_0}$. If D is connected or reduced, this is immediate. Otherwise, (D, s) contains a connected component (D_0, s_0) such that $\iota(D_0)$ is either contractible or a trivial arc. Decompose $(D, s) = (D_1, s_1) \sqcup (D_0, s_0) \sqcup (D_2, s_2)$ so that for any connected component $C_1 \subset D_1$, one has $C_1 \preceq D_0$ and for any connected component $C_2 \subset D_2$ one has $D_0 \preceq C_2$ (recall that $\preceq$ was defined in Section 2.2). By definition, $[T(D), s] = [T(D_2), s_2][T(D_0), s_0][T(D_1), s_1]$ in $\mathcal{S}_\omega(\underline{\Sigma}')$ (this is where working with the basis $\mathcal{T}\mathcal{B}^{\mathrm{o}}$ is important), where s_i are the restriction of s to D_i. Therefore

$$\begin{aligned} &[T(D), s] - c(D, s)[T(D^{\text{red}}), s] \\ &\quad = [T(D_2), s_2]\big([T(D_0), s_0] - c(D_0, s_0)\big)[T(D_1), s_1] \\ &\qquad\qquad + c(D_0, s_0)\big([T(D_2 \cup D_1), s_2 \cup s_1] - c(D_2 \cup D_1, s_2 \cup s_1)[T((D_2 \cup D_1)^{\text{red}}), s]\big) \\ &\quad \equiv c\big([T(D', s')] - c(D', s')[T(D'^{\text{red}}), s]\big) \pmod{\mathcal{I}_{p_0}}, \end{aligned}$$

where $c = c(D_0, s_0)$ and $D' = D_2 \cup D_1$ has one connected component less than D, so we can apply the induction hypothesis to prove that $[T(D), s] - c(D, s)[T(D^{\text{red}}), s] \in \mathcal{I}_{p_0}$. □

2.4 Hopf comodule maps

Recall that the bigon $\mathbb{B}$ is a disc with two punctures on its boundary. It has two boundary arcs, say b_L and b_R. Consider the simple diagram α made of a single arc joining b_L and b_R. For $n \geq 0$, denote by $\alpha^{(n)}$ the diagram made of n parallel copies of α. Denote by $\alpha_{\varepsilon\varepsilon'}$ the class in $\mathcal{S}_\omega(\mathbb{B})$ of the stated diagram (α, s) where $s(\alpha \cap b_L) = \varepsilon$ and $s(\alpha \cap b_R) = \varepsilon'$. It is proved in [Lê 2018, Theorem 4.1] that the stated skein algebra $\mathcal{S}_\omega(\mathbb{B})$ is presented by the four generators $\alpha_{\varepsilon\varepsilon'}$, with $\varepsilon, \varepsilon' = \pm$, and the following relations, where we put $q := \omega^{-4}$:

$$\alpha_{++}\alpha_{+-} = q^{-1}\alpha_{+-}\alpha_{++}, \quad \alpha_{++}\alpha_{-+} = q^{-1}\alpha_{-+}\alpha_{++}, \quad \alpha_{++}\alpha_{--} = 1 + q^{-1}\alpha_{+-}\alpha_{-+},$$
$$\alpha_{--}\alpha_{+-} = q\alpha_{+-}\alpha_{--}, \quad \alpha_{--}\alpha_{-+} = q\alpha_{-+}\alpha_{--}, \quad \alpha_{--}\alpha_{++} = 1 + q\alpha_{+-}\alpha_{-+},$$
$$\alpha_{-+}\alpha_{+-} = \alpha_{+-}\alpha_{-+}.$$

Consider a disjoint union $\mathbb{B} \sqcup \mathbb{B}'$ of two bigons. When gluing the boundary arcs b_R with b'_L, we obtain another bigon. Denote by $\Delta\colon \mathcal{S}_\omega(\mathbb{B}) \to \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\mathbb{B})$ the composition

$$\Delta\colon \mathcal{S}_\omega(\mathbb{B}) \xrightarrow{i|_{b_R \# b'_L}} \mathcal{S}_\omega(\mathbb{B} \sqcup \mathbb{B}') \xrightarrow{\cong} \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\mathbb{B}).$$

The map Δ is characterized by the formula $\Delta(\alpha_{\varepsilon\varepsilon'}) = (\alpha_{\varepsilon+} \otimes \alpha_{+\varepsilon'}) + (\alpha_{\varepsilon-} \otimes \alpha_{-\varepsilon'})$. Define an algebra morphism $\epsilon\colon \mathcal{S}_\omega(\mathbb{B}) \to \mathcal{R}$ and an antialgebra morphism (that is S is linear and $S(xy) = S(y)S(x)$) $S\colon \mathcal{S}_\omega(\mathbb{B}) \to \mathcal{S}_\omega(\mathbb{B})$ by the formulas $\epsilon(\alpha_{\varepsilon\varepsilon'}) = \delta_{\varepsilon\varepsilon'}$, $S(\alpha_{++}) = \alpha_{--}$, $S(\alpha_{--}) = \alpha_{++}$, $S(\alpha_{+-}) = -q\alpha_{+-}$ and $S(\alpha_{-+}) = -q^{-1}\alpha_{-+}$. The coproduct Δ, the counit ϵ and the antipode S endow $\mathcal{S}_\omega(\mathbb{B})$ with the structure of a Hopf algebra. This Hopf algebra is canonically isomorphic to the so-called *quantum* SL_2 Hopf algebra $\mathbb{O}_q[\mathrm{SL}_2]$ as defined in [Brown and Goodearl 2002, Definition I.1.10; Chari and Pressley 1994, Definition 7.1.1; Kassel 1995, Chapter IV Section 6; Manin 1988] where the generators α_{++}, α_{-+}, α_{+-} and α_{--} are denoted by a, b, c and d.

For later use, let us write the coproduct, counit and antipode by the more compact form

$$\begin{pmatrix} \Delta(\alpha_{++}) & \Delta(\alpha_{+-}) \\ \Delta(\alpha_{-+}) & \Delta(\alpha_{--}) \end{pmatrix} = \begin{pmatrix} \alpha_{++} & \alpha_{+-} \\ \alpha_{-+} & \alpha_{--} \end{pmatrix} \otimes \begin{pmatrix} \alpha_{++} & \alpha_{+-} \\ \alpha_{-+} & \alpha_{--} \end{pmatrix},$$

$$\begin{pmatrix} \epsilon(\alpha_{++}) & \epsilon(\alpha_{+-}) \\ \epsilon(\alpha_{-+}) & \epsilon(\alpha_{--}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} S(\alpha_{++}) & S(\alpha_{+-}) \\ S(\alpha_{-+}) & S(\alpha_{--}) \end{pmatrix} = \begin{pmatrix} \alpha_{--} & -q\alpha_{+-} \\ -q^{-1}\alpha_{-+} & \alpha_{++} \end{pmatrix}.$$

Note that when $q = +1$, we recover the Hopf algebra of regular functions of $\mathrm{SL}_2(\mathbb{C})$.

Consider a punctured surface $\underline{\Sigma}$ with boundary arc a. When gluing the boundary a of $\underline{\Sigma}$ with the boundary arc b_L of $\mathbb{B}$ we obtain the same punctured surface $\underline{\Sigma}$. Define a left Hopf comodule map (see eg [Kassel 1995, Definition III.7.1]) $\Delta_a^L\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma})$ as the composition

$$\Delta_a^L\colon \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{i|_{a\# b_L}} \mathcal{S}_\omega(\mathbb{B} \sqcup \underline{\Sigma}) \xrightarrow{\cong} \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma}).$$

Similarly, define a right Hopf comodule map $\Delta_a^R\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathcal{S}_\omega(\underline{\Sigma}) \otimes \mathcal{S}_\omega(\mathbb{B})$ as the composition

$$\Delta_a^R\colon \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{i|_{b_R\# a}} \mathcal{S}_\omega(\underline{\Sigma} \sqcup \mathbb{B}) \xrightarrow{\cong} \mathcal{S}_\omega(\underline{\Sigma}) \otimes \mathcal{S}_\omega(\mathbb{B}).$$

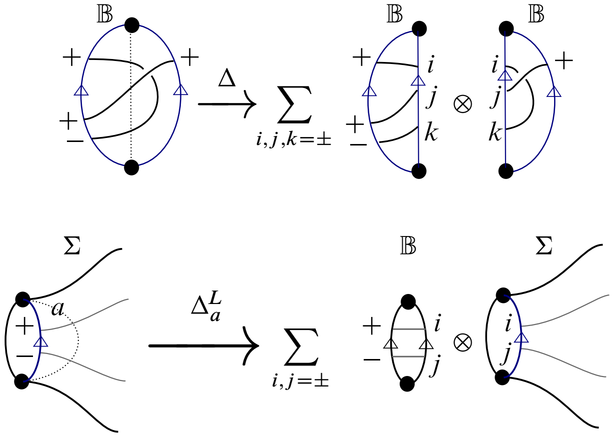


Figure 2: Top: the coproduct in $\mathcal{S}_\omega(\mathbb{B})$. Bottom: the comodule map.

The coassociativity of Δ_a^L and Δ_a^R follows from the coassociativity of the splitting morphisms. Figure 2 illustrates the coproduct and the (left) comodule map.

2.5 The image of the splitting morphism

The goal of this subsection is to prove Theorem 1.1 that we rewrite here for convenience of the reader:

Theorem 2.21 *Let $\underline{\Sigma}$ be a punctured surface, and a and b two distinct boundary arcs. Then the sequence*

$$0 \to \mathcal{S}_\omega(\underline{\Sigma}|_{a\#b}) \xrightarrow{i|_{a\#b}} \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{\Delta_a^L - \sigma\circ\Delta_b^R} \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma})$$

is exact, where $\sigma(x\otimes y) = y \otimes x$.

Throughout this subsection, we fix an orientation $\mathfrak{o}$ of its boundary arcs (though Theorem 2.21 is obviously independent of this choice).

Notation 2.22 For a boundary arc a and a diagram D, we write $n_a(D) := |\partial_a D|$. Given $n \geq 1$, define the set $\mathrm{St}(n) := \{-,+\}^n$ and the subset $\mathrm{St}^{\uparrow}(n) \subset \mathrm{St}(n)$ which consists of n–tuples $(\varepsilon_1, \dots, \varepsilon_n)$ such that $i < j$ implies $\varepsilon_i \leq \varepsilon_j$. If $s = (\varepsilon_1, \dots, \varepsilon_n) \in \mathrm{St}(n)$, denote by $s^{\uparrow} = (\varepsilon'_1, \dots, \varepsilon'_n) \in \mathrm{St}^{\uparrow}(n)$ the unique element such that the number of indices i such that $\varepsilon_i = +$ is equal to the number of indices j such that $\varepsilon'_j = +$. Given $s = (\varepsilon_1, \dots, \varepsilon_n) \in \mathrm{St}(n)$, denote by $k(s)$ the number of pairs (i,j) such that $i < j$ and $\varepsilon_i > \varepsilon_j$. For $s \in \mathrm{St}^{\uparrow}(n)$, let

$$H_s(q) := \sum_{\substack{s' \in \mathrm{St}(n) \\ s'^{\uparrow} = s}} q^{2k(s')}.$$

Let a and b be two boundary arcs of $\underline{\Sigma}$ and consider the filtration associated to $S := \{a,b\}$ and $\mathfrak{o}$ of Definition 2.11.

Lemma 2.23 *Let (D,s) be an $\mathfrak{o}$–oriented simple stated diagram and consider v_1 and v_2 two points which both belong either to $\partial_a D$ or to $\partial_b D$. Suppose that $v_1 <_{\mathfrak{o}} v_2$ and that there is no $v \in \partial D$ such that $v_1 <_{\mathfrak{o}} v <_{\mathfrak{o}} v_2$. Further assume that $s(v_1) = +$ and $s(v_2) = -$. Let s' be the state of D such that $s'(v_1) = -$, $s'(v_2) = +$ and $s'(v) = s(v)$ if $v \in \partial D \setminus \{v_1, v_2\}$. Then one has $\mathrm{lt}([D,s]) = q\,\mathrm{lt}([D,s'])$, where the leading term lt is defined in Definition 2.11.*

Proof This is a straightforward consequence of the boundary relations (4) and the height exchange formulas (5) and (6). $\square$

Let (D,s) be an $\mathfrak{o}$–oriented simple stated diagram of $\underline{\Sigma}$ and write $s = (s_a, s_0, s_b)$ as in the definition of the gluing map before Theorem 2.7. By Lemma 2.23 we have the equality

$$\mathrm{lt}\big([D,(s_a,s_0,s_b)]\big) = q^{k(s_a)+k(s_b)}\,\mathrm{lt}\big([D,(s_a^{\uparrow},s_0,s_b^{\uparrow})]\big).$$

Fix an orientation $\mathfrak{o}_{\mathbb{B}}$ of the left and right boundary arcs of the bigon. Consider the filtration of

$$\mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma}) \cong \mathcal{S}_\omega(\mathbb{B} \sqcup \underline{\Sigma})$$

associated to the set of boundary arcs $S' := \{b_L, b_R, a, b\}$ and the orientations $\mathfrak{o}$ and $\mathfrak{o}_{\mathbb{B}}$, as in Definition 2.11. Given $X' \in \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma})$, we denote by $\mathrm{lt}'(X')$ the associated leading term. By definition of the left comodule map, we have the formula

$$\Delta_a^L\big([D,(s_a,s_0,s_b)]\big) = \sum_{s \in \mathrm{St}(n_a(D))} [\alpha^{(n_a(D))},(s_a,s)] \otimes [D,(s,s_0,s_b)].$$

Lemma 2.24 *Let $[D,(s_a,s_0,s_b)]$ be an element of the basis $\mathcal{B}^{\mathfrak{o}}$. Then*

$$\mathrm{lt}'\big(\Delta_a^L\big([D,(s_a,s_0,s_b)]\big)\big) = \sum_{s \in \mathrm{St}^{\uparrow}(n_a(D))} H_s(q)[\alpha^{(|\partial_a(D)|)},(s_a,s)] \otimes [D,(s,s_0,s_b)],$$

$$\mathrm{lt}'\big(\sigma \circ \Delta_b^R\big([D,(s_a,s_0,s_b)]\big)\big) = \sum_{s \in \mathrm{St}^{\uparrow}(n_b(D))} H_s(q)[\alpha^{(|\partial_b(D)|)},(s,s_b)] \otimes [D,(s_a,s_0,s)],$$

where the summands are written in the basis associated to $(\mathfrak{o},\mathfrak{o}_{\mathbb{B}})$ of $\mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma})$.

Proof This is a straightforward consequence of Lemma 2.23. $\square$

Proof of Theorems 1.1 and 2.21 We want to show that the sequence

$$0 \to \mathcal{S}_\omega(\underline{\Sigma}|_{a\#b}) \xrightarrow{i|_{a\#b}} \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{\Delta_a^L - \sigma\circ\Delta_b^R} \mathcal{S}_\omega(\mathbb{B}) \otimes \mathcal{S}_\omega(\underline{\Sigma})$$

is exact, where $\sigma(x \otimes y) = y \otimes x$. The injectivity of $i|_{a\#b}$ was proved in [Lê 2018]. The inclusion $\mathrm{Im}(i|_{a\#b}) \subset \ker(\Delta_a^L - \sigma\circ\Delta_b^R)$ follows from the coassociativity of the comodule maps. To prove the reverse inclusion, consider an element $X := \sum_{i\in I} x_i[D_i,s_i] \in \ker(\Delta_a^L - \sigma\circ\Delta_b^R)$ developed in the basis $\mathcal{B}^{\mathfrak{o}}$. If $\mathrm{lt}(X) = 0$, then X is a linear combination of diagrams which do not intersect a and b; hence X belongs to the image of $i|_{a\#b}$. Suppose that $\mathrm{lt}(X) > 0$. We will find an element $Y \in \mathcal{S}_\omega(\underline{\Sigma}|_{a\#b})$ such that $\mathrm{lt}(i|_{a\#b}(Y)) = \mathrm{lt}(X)$. Now X belongs to the image of $i|_{a\#b}$ if and only if $Z := X - i|_{a\#b}(Y)$ belongs to this image. Since $d(Z) < d(X)$, the proof will follow by induction on $d(X)$.

Consider the set $\widetilde{\mathfrak{D}}$ of pairs (D, s_0) for which there exists some states s_a and s_b such that the basis element $[D, (s_a, s_0, s_b)]$ appears in the expression of X. Given $\widetilde{D} = (D, s_0) \in \widetilde{\mathfrak{D}}$, denote by $\mathrm{St}_X(\widetilde{D})$ the set of couples (s_a, s_b) such that $[D, (s_a, s_0, s_b)]$ appears in the expression of X. We rewrite the development of X in the basis as

$$X = \sum_{\widetilde{D}=(D,s_0)\in\widetilde{\mathfrak{D}}} \sum_{(s_a,s_b)\in \mathrm{St}_X(\widetilde{D})} x_{[D,(s_a,s_0,s_b)]}[D, (s_a, s_0, s_b)].$$

Consider the subset $\widetilde{\mathfrak{D}}_{\max} \subset \widetilde{\mathfrak{D}}$ of pairs (D, s_0) such that $d(X) = n_a(D) + n_b(D)$. By Lemma 2.24,

$$\begin{aligned}
&\mathrm{lt}'(\Delta^L_a(X)) \\
&\quad= \sum_{(D,s_0)\in\widetilde{\mathfrak{D}}_{\max}} \sum_{(s_a,s_b)\in \mathrm{St}_X((D,s_0))} x_{[D,(s_a,s_0,s_b)]} \sum_{s\in \mathrm{St}^{\uparrow}(n_a(D))} H_s(q)[\alpha^{(n_a(D))}, (s_a, s)] \otimes [D, (s, s_0, s_b)],\\
&\mathrm{lt}'(\sigma \circ \Delta^R_b(X)) \\
&\quad= \sum_{(D,s_0)\in\widetilde{\mathfrak{D}}_{\max}} \sum_{(s_a,s_b)\in \mathrm{St}_X((D,s_0))} x_{[D,(s_a,s_0,s_b)]} \sum_{s'\in \mathrm{St}^{\uparrow}(n_b(D))} H_{s'}(q)[\alpha^{(n_b(D))}, (s', s_b)] \otimes [D, (s_a, s_0, s')].
\end{aligned}$$

From the equality $\mathrm{lt}'(\Delta^L_a(X)) = \mathrm{lt}'(\sigma \circ \Delta^R_b(X))$, we find that for any pair $(D, s_0) \in \widetilde{\mathfrak{D}}_{\max}$, for any pair $(s_a, s_b) \in \mathrm{St}_X((D, s_0))$ and for any state $s \in \mathrm{St}^{\uparrow}(n_a(D))$, there exists a unique pair $(s'_a, s'_b) \in \mathrm{St}_X((D, s_0))$ and a unique state $s' \in \mathrm{St}^{\uparrow}(n_b(D))$ such that

$$\begin{aligned}
&x_{[D,(s_a,s_0,s_b)]} H_s(q)[\alpha^{(n_a(D))}, (s_a, s)] \otimes [D, (s, s_0, s_b)] \\
&\qquad\qquad = x_{[D,(s'_a,s_0,s'_b)]} H_{s'}(q)[\alpha^{(n_b(D))}, (s', s'_b)] \otimes [D, (s'_a, s_0, s')].
\end{aligned}$$

We deduce the following:

- For any $(D, s_0) \in \widetilde{\mathfrak{D}}_{\max}$, we have $n_a(D) = n_b(D) = \frac{1}{2} d(X)$. We will denote by n this integer.
- We have the equalities $s' = s_a = s_b$ and $s = s'_a = s'_b$. Hence for any $(D, s_0) \in \widetilde{\mathfrak{D}}_{\max}$, we have $\mathrm{St}_X((D, s_0)) = \{(s, s), s \in \mathrm{St}^{\uparrow}(n)\}$.
- For any $(D, s_0) \in \widetilde{\mathfrak{D}}_{\max}$ and $s \in \mathrm{St}^{\uparrow}(n)$, the coefficient $x_{[D,(s,s_0,s)]} H_s(q)$ is independent of s. We will denote this coefficient by $x_{(D,s_0)}$.

With the above notation, we rewrite the leading term of X as

$$\mathrm{lt}(X) = \sum_{(D,s_0)\in\widetilde{\mathfrak{D}}_{\max}} x_{(D,s_0)} \sum_{s\in \mathrm{St}^{\uparrow}(n)} [D, (s, s_0, s)].$$

Given $(D, s_0) \in \widetilde{\mathfrak{D}}_{\max}$, since $n_a(D) = n_b(D) = n$, there exists a diagram D_0 of $\underline{\Sigma}|_{a\#b}$ such that D is obtained from D_0 by cutting along the common image in $\Sigma|_{a\#b}$ of a and b by the projection. Define the element

$$Y := \sum_{(D,s_0)\in\widetilde{\mathfrak{D}}_{\max}} x_{(D,s_0)}[D_0, s_0] \in \mathcal{S}_\omega(\underline{\Sigma}).$$

By the above expression, $\mathrm{lt}(X) = \mathrm{lt}(i|_{a\#b}(Y))$. $\square$

Consider a topological triangulation Δ of Σ. The punctured surface $\underline{\Sigma}$ is obtained from the disjoint union $\Sigma_\Delta := \bigsqcup_{\mathbb{T}\in F(\Delta)} \mathbb{T}$ by gluing the triangles along their common edges. Denote by $\mathring{\mathscr{E}}(\Delta) \subset \mathscr{E}(\Delta)$ the subset of edges which are not boundary arcs. Each edge $e \in \mathring{\mathscr{E}}(\Delta)$ lifts in Σ_Δ to two boundary arcs e_L and e_R. By composing all the left comodule maps $\Delta^L_{e_L}$ together (the order does not matter thanks to the coassociativity property in Theorem 2.7) one gets a Hopf comodule map

$$\Delta^L \colon \bigotimes_{\mathbb{T}\in F(\Delta)} \mathscr{S}_\omega(\mathbb{T}) \to \Big(\bigotimes_{e\in\mathring{\mathscr{E}}(\Delta)} \mathscr{S}_\omega(\mathbb{B})\Big) \otimes \Big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathscr{S}_\omega(\mathbb{T})\Big).$$

Similarly, composing all the right comodule maps $\Delta^R_{e_R}$ together gives

$$\Delta^R \colon \bigotimes_{\mathbb{T}\in F(\Delta)} \mathscr{S}_\omega(\mathbb{T}) \to \Big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathscr{S}_\omega(\mathbb{T})\Big) \otimes \Big(\bigotimes_{e\in\mathring{\mathscr{E}}(\Delta)} \mathscr{S}_\omega(\mathbb{B})\Big).$$

Recall the definition of i^Δ in Section 2.1.

Corollary 2.25 *The following sequence is exact*:

$$0 \to \mathscr{S}_\omega(\underline{\Sigma}) \xrightarrow{i^\Delta} \bigotimes_{\mathbb{T}\in F(\Delta)} \mathscr{S}_\omega(\mathbb{T}) \xrightarrow{\Delta^L - \sigma\circ\Delta^R} \Big(\bigotimes_{e\in\mathring{\mathscr{E}}(\Delta)} \mathscr{S}_\omega(\mathbb{B})\Big) \otimes \Big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathscr{S}_\omega(\mathbb{T})\Big).$$

Proof Theorem 1.1 applied to each inner edge provides an isomorphism between $\mathscr{S}_\omega(\underline{\Sigma})$ and the intersection, over the inner edges e, of $\mathrm{Ker}(\Delta^L_{e_L} - \sigma\circ\Delta^R_{e_R})$. We conclude by observing that the latter intersection is $\mathrm{Ker}(\Delta^L - \sigma\circ\Delta^R)$. ◻

We can reformulate the above exact sequence in terms of co-Hochschild cohomology.

Definition 2.26 Given a coalgebra C with a bicomodule M, with comodules maps $\Delta^L \colon M \to C\otimes M$ and $\Delta^R \colon M \to M \otimes C$, the 0^{th} co-Hochschild cohomology group is $\mathrm{coHH}^0(C, M) := \ker(\Delta^L - \sigma\circ\Delta^R)$.

We refer to [Hess et al. 2009] for a self-contained introduction to co-Hochschild (co)homology. The above triangular decomposition of skein algebra can be rewritten as

$$\mathscr{S}_\omega(\underline{\Sigma}) \cong \mathrm{coHH}^0\Big(\bigotimes_{e\in\mathring{\mathscr{E}}(\Delta)} \mathbb{O}_q[\mathrm{SL}_2], \bigotimes_{\mathbb{T}\in F(\Delta)} \mathscr{S}_\omega(\mathbb{T})\Big).$$

2.6 The center of stated skein algebras at odd roots of unity

Here we prove Theorem 1.2. We prove it for the bigon, then for the triangle, and we conclude with the general case. Let us start by the following classical result.

Lemma 2.27 *Let $\mathcal{R}$ be a ring and $q \in \mathcal{R}^\times$ a root of unity of order $N > 1$. Suppose that $\mathscr{A}$ is an $\mathcal{R}$–algebra and $x, y \in \mathscr{A}$ are such that $yx = qxy$. Then $(x+y)^N = x^N + y^N$.*

Proof By [Kassel 1995, Proposition IV.2.2],

$$(x+y)^N = \sum_{k=0}^{N} \binom{N}{k}_q x^k y^{N-k},$$

where

$$\binom{N}{k}_q := \prod_{i=0}^{k-1} \left(\frac{1-q^{N-i}}{1-q^{i+1}} \right).$$

Since $q^N = 1$, the coefficients $\binom{N}{k}_q$ vanish for $1 \le k \le N-1$, and we get the desired formula. □

2.6.1 The case of the bigon Recall from Section 2.2 that the Hopf algebra $\mathcal{S}_\omega(\mathbb{B})$ is canonically isomorphic to $\mathcal{O}_q[\mathrm{SL}_2]$. In this case, Theorem 1.2 is a well-known theorem of Lusztig. More precisely, it is proved in [Lusztig 1990] (see also [Lusztig 1993, Theorem 3.5.1]) that there exists a morphism of braided Hopf algebras $\mathrm{Fr}_* : \dot{U}_q\mathfrak{sl}_2 \to \dot{U}_{+1}\mathfrak{sl}_2$ which induces a braided functor $\mathrm{Fr} : \mathrm{Rep}(\mathrm{SL}_2) \to \mathrm{Rep}_q(\mathrm{SL}_2)$ between the category of finite-rank representations of SL_2 and the category $\mathrm{Rep}_q(\mathrm{SL}_2)$ of finite-rank $\dot{U}_q\mathfrak{sl}_2$ modules. Since $\mathcal{O}_q[\mathrm{SL}_2]$ (resp. $\mathcal{O}[\mathrm{SL}_2]$) is isomorphic to the coend of the forgetful functor $F : \mathrm{Rep}_q(\mathrm{SL}_2) \to \mathrm{Mod}_{\mathcal{R}}$ (resp. of the forgetful functor $\mathrm{Rep}(\mathrm{SL}_2) \to \mathrm{Mod}_{\mathcal{R}}$) the Frobenius functor Fr induces a morphism $j : \mathcal{O}[\mathrm{SL}_2] \to \mathcal{O}_q[\mathrm{SL}_2]$. Moreover, as noticed in [Negron 2021], the image of Fr lies in the Mügen center of $\mathrm{Rep}_q(\mathrm{SL}_2)$ so the image of j is central. We refer to [Negron 2021, Section 5.1] for details on this approach. A down-to-earth construction of j, based on elementary computations using the definition of $\mathcal{O}_q[\mathrm{SL}_2]$ by generators and relations, was described by Brown and Goodearl and goes as follows:

Lemma 2.28 [Brown and Goodearl 2002, Proposition III.3.1] *Suppose that $q := \omega^{-4}$ is a root of unity of odd order $N \ge 1$. There exists a injective morphism of Hopf algebras $j_{\mathbb{B}} : \mathcal{S}_{+1}(\mathbb{B}) \to \mathcal{S}_\omega(\mathbb{B})$ characterized by $j_{\mathbb{B}}(\alpha_{\varepsilon\varepsilon'}) := (\alpha_{\varepsilon\varepsilon'})^N$ whose image lies in the center of $\mathcal{S}_\omega(\mathbb{B})$.*

2.6.2 The case of the triangle Denote by α, β and γ the three arcs of Figure 3 and τ the automorphism of $\mathcal{S}_\omega(\mathbb{T})$ induced by the rotation sending α, β and γ to β, γ and α, respectively. In [Lê 2018, Theorem 4.6], it was proved that the stated skein algebra $\mathcal{S}_\omega(\mathbb{T})$ is presented by the generators $\alpha_{\varepsilon\varepsilon'}$, $\beta_{\varepsilon\varepsilon'}$ and $\gamma_{\varepsilon\varepsilon'}$, and the following relations together with their images through τ and τ^2:

$$\alpha_{-\varepsilon}\alpha_{+\varepsilon'} = A^2\alpha_{+\varepsilon}\alpha_{-\varepsilon'} - \omega^{-5}C^{\varepsilon}_{\varepsilon'}, \tag{10}$$

$$\alpha_{\varepsilon-}\alpha_{\varepsilon'+} = A^2\alpha_{\varepsilon+}\alpha_{\varepsilon'-} - \omega^{-5}C^{\varepsilon}_{\varepsilon'}, \tag{11}$$

$$\beta_{\mu\varepsilon}\alpha_{\mu'\varepsilon'} = A\alpha_{\varepsilon\varepsilon'}\beta_{\mu\mu'} - A^2 C^{\varepsilon}_{\mu'}\gamma_{\varepsilon'\mu}, \tag{12}$$

$$\alpha_{-\varepsilon}\beta_{\varepsilon'+} = A^2\alpha_{+\varepsilon}\beta_{\varepsilon'-} - \omega^{-5}\gamma_{\varepsilon\varepsilon'}, \tag{13}$$

$$\alpha_{\varepsilon-}\gamma_{+\varepsilon'} = A^2\alpha_{\varepsilon+}\gamma_{-\varepsilon'} + \omega\beta_{\varepsilon'\varepsilon}. \tag{14}$$

Here we use the notation $A := \omega^{-2}$, $C^-_- = C^+_+ := 0$, $C^-_+ := -\omega^5$ and $C^+_- := \omega$.

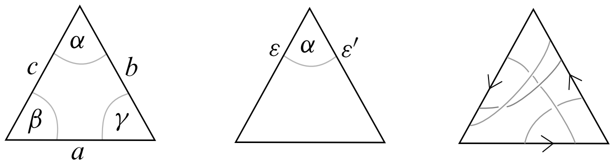


Figure 3: Left: the three diagrams α, β and γ. Middle: the stated diagram representing $\alpha_{\varepsilon\varepsilon'}$. Right: the diagram $\theta^{(2,1,1)}$.

When $\omega = +1$, the algebra $\mathcal{S}_{+1}(\mathbb{T})$ has the following simpler presentation. Consider the commutative unital polynomial algebra $\mathcal{A} := \mathcal{R}[\alpha_{\varepsilon\varepsilon'}, \beta_{\varepsilon\varepsilon'}, \gamma_{\varepsilon\varepsilon'} | \varepsilon, \varepsilon' = \pm]$. Given $\delta \in \{\alpha, \beta, \gamma\}$, denote by M_δ the 2×2 matrix with coefficients in $\mathcal{A}$ defined by

$$M_\delta := \begin{pmatrix} \delta_{++} & \delta_{+-} \\ \delta_{-+} & \delta_{--} \end{pmatrix}$$

and write $C := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mathbb{1} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Lemma 2.29 *The algebra $\mathcal{S}_{+1}(\mathbb{T})$ is isomorphic to*

$$\mathcal{R}[\alpha_{\varepsilon\varepsilon'}, \beta_{\varepsilon\varepsilon'}, \gamma_{\varepsilon\varepsilon'} \mid \varepsilon, \varepsilon' = \pm]/(\det(M_\alpha) = \det(M_\beta) = \det(M_\gamma) = 1,\ M_\gamma C M_\beta C M_\alpha C = \mathbb{1}).$$

Proof That $\mathcal{S}_{+1}(\mathbb{T})$ commutative is a particular case of [Lê 2018, Corollary 2.5]. After setting $\omega = +1$ we see that (10) and (11) coincide; (14) is the image of (13) by rotation, and the latter is a particular case of (12). Moreover, a direct inspection shows that the other part of (10) and of (12) correspond to $\det(M_\alpha) = 1$ and $(M_\gamma C)^{-1} = M_\beta C M_\alpha C$, respectively. □

Lemma 2.30 *Suppose that ω is a root of unity of odd order $N \geq 1$. For every $\varepsilon, \varepsilon', \mu, \mu' \in \{-,+\}$ with $\varepsilon \neq \mu'$, one has*

$$\alpha^N_{\mu'\varepsilon'}\beta^N_{\mu\varepsilon} - \alpha^N_{\varepsilon\varepsilon'}\beta^N_{\mu\mu'} = \gamma^N_{\varepsilon',\mu}.$$

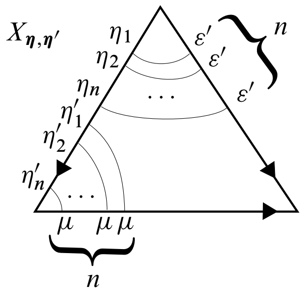
Proof We suppose that $(\varepsilon, \mu') = (-,+)$. The proof in the case where $(\varepsilon, \mu') = (+,-)$ is similar and left to the reader. For $n \geq 0$, let D_n be the simple diagram made of n parallel copies of α and n parallel copies of β and consider the orientation $\mathfrak{o}$ depicted in Figure 4. For $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n) \in \{-,+\}^n$ let $\boldsymbol{\eta}^\vee := \{-\eta_n, \dots, -\eta_1\}$. For $\boldsymbol{\eta}, \boldsymbol{\eta}' \in \{-,+\}^n$, let $s_{\boldsymbol{\eta},\boldsymbol{\eta}'}$ be the state of D_n sending all points of $\partial_b D_n$ to ε', all points of $\partial_a D_n$ to μ and the points $(p_1, \dots, p_n, p'_1, \dots, p'_n)$ of $\partial_c D_n$ ordered by $\mathfrak{o}$, to the states $(\eta_1, \dots, \eta_n, \eta'_1, \dots, \eta'_n)$. Write $X_{\boldsymbol{\eta},\boldsymbol{\eta}'} := [D_n, s_{\boldsymbol{\eta},\boldsymbol{\eta}'}]$.

Using the skein relations (4), as illustrated in Figure 4, we find that

$$X_{\boldsymbol{\eta},\boldsymbol{\eta}'}\gamma_{\varepsilon',\mu} = \omega^{-1} X_{(\boldsymbol{\eta},+),(-,\boldsymbol{\eta}')} - \omega^{-5} X_{(\boldsymbol{\eta},-),(+,\boldsymbol{\eta}')}, \tag{15}$$

where $(\boldsymbol{\eta},+) := (\eta_1, \dots, \eta_n, +)$ and $(-,\boldsymbol{\eta}') := (-,\eta'_1, \dots, \eta'_n)$. Let $n_+(\eta)$ be the number of indices $i \in \{1, \dots, n\}$ such that $\eta_i = +$. Using (15), we prove by induction of n that

$$(\gamma_{\varepsilon'\mu})^n = \sum_{\eta \in \{-,+\}^n} (\omega^{-1})^{n_+(\eta)} (-\omega^{-5})^{n-n_+(\eta)} X_{\eta,\eta^\vee}. \tag{16}$$



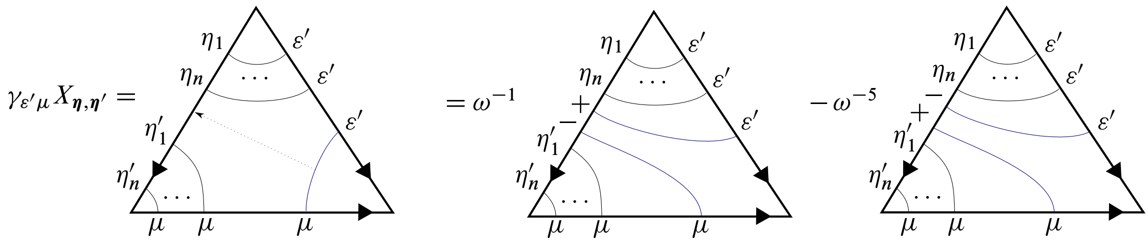


Figure 4: Top: the element $X_{\boldsymbol{\eta},\boldsymbol{\eta}'}$. Bottom: an illustration of (15).

Let $m(\eta) := \#\{1 \le i < j \le n \mid (\eta_i, \eta_j) = (+,-)\}$ and denote by η_+ the unique element of $\{-,+\}^n$ such that $n_+(\eta) = n_+(\eta_+)$ and $m(\eta_+) = 0$. Note that $m(\eta) = m(\eta^\vee)$. Using the skein relation (4), we find that for any $\eta, \eta' \in \{-,+\}^n$,

$$X_{\eta,\eta'} = q^{m(\eta)+m(\eta')} X_{\eta_+,\eta'_+}. \tag{17}$$

For $1 \le k \le N$, let $\eta_+^{(k)} \in \{-,+\}^N$ be the unique element such that $m(\eta_+^{(k)}) = 0$ and $n_+(\eta_+^{(k)}) = k$, ie

$$\eta^{(k)}_{+\,i} = \begin{cases} - & \text{for } 1 \le i \le N-k, \\ + & \text{for } i > N-k. \end{cases}$$

Putting (16) and (17) together, one finds that

$$(\gamma_{\varepsilon'\mu})^N = \sum_{k=0}^{N} (\omega^{-1})^k (-\omega^{-5})^{N-k} \Biggl(\sum_{\substack{\eta \in \{-,+\}^N \\ n_+(\eta)=k}} q^{2m(\eta)} \Biggr) X_{\eta_+^{(k)}, \eta_+^{(k)\vee}}.$$

Now, a simple computation shows that

$$\Biggl(\sum_{\substack{\eta \in \{-,+\}^N \\ n_+(\eta)=k}} q^{2m(\eta)} \Biggr) = q^{2nN-n(n-1)} \sum_{1 \le i_1 < i_2 < \cdots < i_n \le N} q^{2(i_1+\cdots+i_n)} = \begin{cases} 1 & \text{if } k=0 \text{ or } k=N, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$(\gamma_{\varepsilon'\mu})^N = X_{\eta_+^{(N)}, \eta_-^{(N)}} - X_{\eta_-^{(N)}, \eta_+^{(N)}} = \alpha_{+\varepsilon'}^N \beta_{\mu-}^N - \alpha_{-\varepsilon'}^N \beta_{\mu+}^N.$$

Note that we used that $(-1)^N = -1$, so that N is odd. $\square$

Lemma 2.31 *Suppose that ω is a root of unity of odd order $N \geq 1$. There exists an injective morphism of algebras $j_{\mathbb{T}} : \mathscr{S}_{+1}(\mathbb{T}) \to \mathscr{S}_{\omega}(\mathbb{T})$, whose image lies in the center of $\mathscr{S}_{\omega}(\mathbb{T})$, characterized by $j_{\mathbb{T}}(\delta_{\varepsilon\varepsilon'}) := (\delta_{\varepsilon\varepsilon'})^N$ for $\delta \in \{\alpha, \beta, \gamma\}$ and $\varepsilon, \varepsilon' = \pm$. Moreover, if a is a boundary arc of $\mathbb{T}$, the following diagrams commute:*

$$\begin{tikzcd}
\mathscr{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta_a^L"] \arrow[d, hook, "j_{\mathbb{T}}"] & \mathscr{S}_{+1}(\mathbb{B}) \otimes \mathscr{S}_{+1}(\mathbb{T}) \arrow[d, hook, "j_{\mathbb{B}} \otimes j_{\mathbb{T}}"] \\
\mathscr{S}_{\omega}(\mathbb{T}) \arrow[r, "\Delta_a^L"] & \mathscr{S}_{\omega}(\mathbb{B}) \otimes \mathscr{S}_{\omega}(\mathbb{T})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta_a^R"] \arrow[d, hook, "j_{\mathbb{T}}"] & \mathscr{S}_{+1}(\mathbb{T}) \otimes \mathscr{S}_{+1}(\mathbb{B}) \arrow[d, hook, "j_{\mathbb{T}} \otimes j_{\mathbb{B}}"] \\
\mathscr{S}_{\omega}(\mathbb{T}) \arrow[r, "\Delta_a^R"] & \mathscr{S}_{\omega}(\mathbb{T}) \otimes \mathscr{S}_{\omega}(\mathbb{B})
\end{tikzcd}$$

Proof We proceed in a similar way to Lemma 2.28, by showing first that the extension of the assignment $j_{\mathbb{T}}(\delta_{\varepsilon\varepsilon'}) := \delta^N_{\varepsilon\varepsilon'}$ to a morphism of algebras is well defined. In virtue of Lemma 2.29 and by the rotation automorphism, it is enough to show that $\alpha^N_{\varepsilon\varepsilon'}$ lies in the center of $\mathscr{S}_{\omega}(\mathbb{T})$ and that $j_{\mathbb{T}}$ sends $\det(M_\alpha) - 1$ and $M_\gamma C M_\beta C M_\alpha C - \mathbb{1}$ to zero.

First note that the relations (10) and (11) put together coincide with the defining relations of $\mathscr{S}_{\omega}(\mathbb{B})$; hence one has an inclusion of algebras $\phi \colon \mathscr{S}_{\omega}(\mathbb{B}) \hookrightarrow \mathscr{S}_{\omega}(\mathbb{T})$ defined by $\phi(\alpha_{\varepsilon\varepsilon'}) = \alpha_{\varepsilon\varepsilon'}$. By applying Lemma 2.28, one obtains an inclusion $\phi \circ j_{\mathbb{B}} : \mathscr{S}_{+1}(\mathbb{B}) \hookrightarrow \mathscr{S}_{\omega}(\mathbb{T})$ which coincides with $j_{\mathbb{T}}$ on the $\alpha_{\varepsilon\varepsilon'}$'s. It remains to show that the $\alpha^N_{\varepsilon\varepsilon'}$'s commute with the $\beta_{\mu\mu'}$'s and the $\gamma_{\mu\mu'}$'s, and that $j_{\mathbb{T}}$ vanishes on $M_\gamma C M_\beta C M_\alpha C - \mathbb{1}$.

We have $\alpha^N_{\varepsilon\varepsilon'}\beta_{\mu\varepsilon} = A^{-N}\beta_{\mu\varepsilon}\alpha^N_{\varepsilon\varepsilon'} = \beta_{\mu\varepsilon}\alpha^N_{\varepsilon\varepsilon'}$. From

$$\alpha^N_{+\varepsilon}\beta_{\varepsilon'-} = \alpha^{N-1}_{+\varepsilon}(A^{-2}\alpha_{-\varepsilon}\beta_{\varepsilon'+} + \omega^{-1}\gamma_{\varepsilon\varepsilon'}) = (A^{-3N+1}\alpha_{-\varepsilon}\beta_{\varepsilon'+} + \omega^{-1}A^{N-1}\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{+\varepsilon}$$

and

$$\beta_{\varepsilon'-}\alpha^N_{+\varepsilon} = (A\alpha_{-\varepsilon}\beta_{\varepsilon'+} + \omega\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{+\varepsilon},$$

one obtains

$$\alpha^N_{+\varepsilon}\beta_{\varepsilon'-} - \beta_{\varepsilon'-}\alpha^N_{+\varepsilon} = (A(A^{-3N}-1)\alpha_{-\varepsilon}\beta_{\varepsilon'+} + \omega(A^N-1)\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{+\varepsilon} = 0.$$

Similarly, we compute

$$\begin{aligned}
\alpha^N_{-\varepsilon}\beta_{\varepsilon'+} &= \alpha^{N-1}_{-\varepsilon}(A^2\alpha_{+\varepsilon}\beta_{\varepsilon'-} - \omega^{-5}\gamma_{\varepsilon\varepsilon'}) = (A^{N+1}\alpha_{+\varepsilon}\beta_{\varepsilon'-} - \omega^{-3}A^N\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{-\varepsilon}, \\
\beta_{\varepsilon'+}\alpha^N_{-\varepsilon} &= (A\alpha_{+\varepsilon}\beta_{\varepsilon'-} - \omega^{-3}\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{-\varepsilon}.
\end{aligned}$$

Thus we find

$$\alpha^N_{-\varepsilon}\beta_{\varepsilon'+} - \beta_{\varepsilon'+}\alpha^N_{-\varepsilon} = (A(A^N-1)\alpha_{+\varepsilon}\beta_{\varepsilon'-} - \omega^{-3}(A^N-1)\gamma_{\varepsilon\varepsilon'})\alpha^{N-1}_{-\varepsilon} = 0.$$

So we have proven that $\alpha^N_{\varepsilon\varepsilon'}$ commutes with every elements $\beta_{\mu\mu'}$. The commutativity of $\alpha^N_{\varepsilon\varepsilon'}$ with each element $\gamma_{\mu\mu'}$ is shown in a very similar way.

Next, showing that $j_{\mathbb{T}}$ vanishes on $M_\gamma C M_\beta C M_\alpha C - \mathbb{1}$ amounts to showing that

$$\beta^N_{\mu\varepsilon}\alpha^N_{\mu'\varepsilon'} - \alpha^N_{\varepsilon\varepsilon'}\beta^N_{\mu\mu'} = \gamma^N_{\varepsilon',\mu} \quad \text{for } \varepsilon \neq \mu'.$$

This was proved in Lemma 2.30.

Now let us prove that $j_{\mathbb{T}}$ is injective. To this end, let us consider the following basis of $\mathcal{S}_\omega(\mathbb{T})$.

Consider the counterclockwise orientation $\mathfrak{o}$ of the boundary arcs of $\mathbb{T}$ as in Figure 3. Given

$$\boldsymbol{k} = (k_\alpha, k_\beta, k_\gamma) \in (\mathbb{Z}^{\geq 0})^3,$$

denote by $\theta^{\boldsymbol{k}}$ the (not simple) diagram $\alpha^{k_\alpha}\beta^{k_\beta}\gamma^{k_\gamma}$; see Figure 3 for an example. By Proposition 2.14 the set of classes $[\theta^{\boldsymbol{k}}, s]$, where s is $\mathfrak{o}$–increasing, forms a basis of $\mathcal{S}_\omega(\mathbb{T})$.

By construction, $j_{\mathbb{T}}$ sends the elements $[\theta^{\boldsymbol{k}}, s]$ of $\mathcal{S}_{+1}(\mathbb{T})$, where s is $\mathfrak{o}$–increasing, to some basis elements $[\theta^{N\boldsymbol{k}}, s']$, where s' is also $\mathfrak{o}$ increasing, therefore $j_{\mathbb{T}}$ is injective.

It remains to prove that $j_{\mathbb{T}}$ is a morphism of Hopf comodules. To avoid confusion, let us denote by $x_{\varepsilon\varepsilon'}$ the generators of $\mathcal{S}_\omega(\mathbb{B})$ and reserve the notation $\alpha_{\varepsilon\varepsilon'}$ for the element of $\mathcal{S}_\omega(\mathbb{T})$. By definition, we have $\Delta_c^L(\alpha_{\varepsilon\varepsilon'}) = x_{\varepsilon+}\otimes\alpha_{+\varepsilon'} + x_{\varepsilon-}\otimes\alpha_{-\varepsilon'}$. Write $u := x_{\varepsilon+}\otimes\alpha_{+\varepsilon'}$ and $v := x_{\varepsilon-}\otimes\alpha_{-\varepsilon'}$. Since $uv = q^{-2}vu$, by Lemma 2.27 we have $(u+v)^N = u^N + v^N$, so

$$\begin{aligned}\Delta_c^L(j_{\mathbb{B}}(\alpha_{\varepsilon\varepsilon'})) &= (\Delta_c^L(\alpha_{\varepsilon\varepsilon'}))^N = (u+v)^N = u^N + v^N \\ &= x_{\varepsilon+}^N\otimes\alpha_{+\varepsilon'}^N + x_{\varepsilon-}^N\otimes\alpha_{-\varepsilon'}^N = j_{\mathbb{B}}\otimes j_{\mathbb{T}}(\Delta_c^L(\alpha_{\varepsilon\varepsilon'})).\end{aligned}$$

The proof that $\Delta_b^L(j_{\mathbb{B}}(\alpha_{\varepsilon\varepsilon'})) = j_{\mathbb{B}}\otimes j_{\mathbb{T}}(\Delta_b^L(\alpha_{\varepsilon\varepsilon'}))$ is done using a similar computation and the equality $\Delta_a^L(j_{\mathbb{B}}(\alpha_{\varepsilon\varepsilon'})) = j_{\mathbb{B}}\otimes j_{\mathbb{T}}(\Delta_a^L(\alpha_{\varepsilon\varepsilon'}))$ holds since both sides are equal to $1\otimes\alpha_{\varepsilon\varepsilon'}^N$. By symmetry in the generators α, β, γ, we have proved that $j_{\mathbb{B}}$ commutes with the left comodule maps. That it commutes with the right comodule maps is proved similarly. □

2.6.3 The general case: proof of Theorem 1.2

We restate Theorem 1.2 here for the convenience of the reader:

Theorem 2.32 *Suppose that ω is a root of unity of odd order $N \geq 1$ and Σ a punctured surface. There exists an embedding*

$$j_{\Sigma}\colon \mathcal{S}_{+1}(\Sigma) \hookrightarrow \mathcal{Z}(\mathcal{S}_\omega(\Sigma))$$

of the (commutative) stated skein algebra with parameter $+1$ into the center of the stated skein algebra with parameter ω. Moreover, the morphism j_Σ is characterized by the property that it sends a closed curve γ to $T_N(\gamma)$ and a stated arc $\alpha_{\varepsilon\varepsilon'}$ to $\alpha_{\varepsilon\varepsilon'}^{(N)}$, where $\alpha_{\varepsilon\varepsilon'}^{(N)}$ is the tangle made by stacking N parallel copies of $\alpha_{\varepsilon\varepsilon'}$ on top of the others.

Recall from Section 2.2 that closed curves and arcs do not have self-intersection points by definition. We divide the proof in five steps.

In Step 1, we show that the decomposition Theorem 1.1 together with the two previous sections provide an injective morphism of algebras

$$j_{(\Sigma,\Delta)}\colon \mathcal{S}_{+1}(\Sigma) \hookrightarrow \mathcal{S}_\omega(\Sigma), \tag{18}$$

which is central. We study further properties of $j_{(\underline{\Sigma},\Delta)}$ and we show that it is does *not* depend on a topological triangulation Δ. The other steps are devoted to making explicit the morphism $j_{(\underline{\Sigma},\Delta)}$ on arcs and loops. In Steps 2–4, we suppose that the punctured surface has a nondegenerated triangulation (see below); in Step 5 we treat the other punctured surfaces.

In Step 2, we prove that $j_{(\underline{\Sigma},\Delta)}$ sends the stated arcs that have their endpoints on *two different* boundary arcs of Σ, to their N^{th} power.

In Step 3, we prove that $j_{(\underline{\Sigma},\Delta)}$ sends some particular closed curves of $\Sigma_{\mathcal{P}}$ to their N^{th} Chebyshev polynomial of first kind.

Step 4 is more involved. We first prove a structural result. Adding a puncture on a surface $\underline{\Sigma}$ gives rise to a surjective map φ from the skein algebra of the new punctured surface to that of the initial one defined in Section 2.3. We show that $j_{(\underline{\Sigma},\Delta)}$ commutes with these surjections (see Lemma 2.40). From this, we deduce the image by $j_{(\underline{\Sigma},\Delta)}$ of stated arcs that have both their endpoints on the *same* boundary arc of Σ and of *any* closed curve of $\Sigma_{\mathcal{P}}$.

In Step 5, we treat the remaining cases of connected punctured surfaces that do not admit a nondegenerate topological triangulation (including those with no puncture). The proof consists, again, in adding a puncture and using the previous study.

These five steps prove Theorem 1.2.

Throughout this section, $\underline{\Sigma}$ is a punctured surface, Δ a topological triangulation $\underline{\Sigma}$ and ω a root of unity of odd order $N \geq 1$. Except for Steps 1 and 5, the triangulation Δ is required to be *nondegenerate*, that is, such that each of its inner edges separates two distinct faces.

Step 1: formal definition Assume that $\underline{\Sigma}$ admits a (possibly degenerate) triangulation Δ. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, "i^{\Delta}"] \arrow[d, hook, dotted, "j_{(\underline{\Sigma},\Delta)}"', "\exists!"] & \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta^L - \sigma\circ\Delta^R"] \arrow[d, hook, "\otimes_{\mathbb{T}} j_{\mathbb{T}}"] & \big(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)} \mathcal{S}_{+1}(\mathbb{B})\big) \otimes \big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_{+1}(\mathbb{T})\big) \arrow[d, hook, "(\otimes_e j_{\mathbb{B}})\otimes(\otimes_{\mathbb{T}} j_{\mathbb{T}})"] \\ 0 \arrow[r] & \mathcal{S}_{\omega}(\underline{\Sigma}) \arrow[r, "i^{\Delta}"] & \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_{\omega}(\mathbb{T}) \arrow[r, "\Delta^L - \sigma\circ\Delta^R"] & \big(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)} \mathcal{S}_{\omega}(\mathbb{B})\big) \otimes \big(\bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_{\omega}(\mathbb{T})\big) \end{tikzcd} \tag{19}$$

where both lines are exact by Theorem 1.1 and the vertical maps are given by Lemmas 2.28 and 2.31.

The existence of an injective morphism $j_{(\underline{\Sigma},\Delta)}\colon \mathcal{S}_{+1}(\underline{\Sigma}) \hookrightarrow \mathcal{S}_{\omega}(\underline{\Sigma})$ follows from the exactness of the lines and the injectivity of $\bigotimes_{\mathbb{T}\in F(\Delta)} j_{\mathbb{T}}$ (and the fact that all maps involved in the diagram are algebra morphisms). Moreover, since $j_{\mathbb{T}}$ is central, so is $j_{(\underline{\Sigma},\Delta)}$.

Let us show that $j_{(\underline{\Sigma},\Delta)}$ is compatible with the gluing maps.

Lemma 2.33 *If a and b are two boundary arcs of $\underline{\Sigma}$, the following diagram commutes*:

$$\begin{tikzcd} \mathcal{S}_{+1}(\underline{\Sigma}|_{a\#b}) \arrow[r, hook, "j_{\underline{\Sigma}_{a\#b}}"] \arrow[d, hook, "i|_{a\#b}"] & \mathcal{S}_{\omega}(\underline{\Sigma}|_{a\#b}) \arrow[d, hook, "i|_{a\#b}"] \\ \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, hook, "j_{\underline{\Sigma}}"] & \mathcal{S}_{\omega}(\underline{\Sigma}) \end{tikzcd}$$

Proof Let $\Delta_{a\#b}$ the topological triangulation of $\underline{\Sigma}|_{a\#b}$ that is induced by Δ. Let us consider the diagram

$$\begin{tikzcd} \mathcal{S}_{+1}(\underline{\Sigma}|_{a\#b}) \arrow[r, hook, "i|_{a\#b}"] \arrow[d, hook, "j_{(\underline{\Sigma}|_{a\#b},\Delta_{a\#b})}"] \arrow[rr, bend left, "i^{\Delta_{a\#b}}"] & \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, hook, "i^{\Delta}"] \arrow[d, hook, "j_{(\underline{\Sigma},\Delta)}"] & \otimes_{\mathbb{T}}\mathcal{S}_{+1}(\mathbb{T}) \arrow[d, hook, "\otimes_{\mathbb{T}} j_{\mathbb{T}}"] \\ \mathcal{S}_{\omega}(\underline{\Sigma}|_{a\#b}) \arrow[r, hook, "i|_{a\#b}"] \arrow[rr, bend right, "i^{\Delta_{a\#b}}"'] & \mathcal{S}_{\omega}(\underline{\Sigma}) \arrow[r, hook, "i^{\Delta}"] & \otimes_{\mathbb{T}}\mathcal{S}_{\omega}(\mathbb{T}) \end{tikzcd}$$

The outer triangles commute by coassociativity of the gluing maps. Two of the three squares commute by diagram (19). Since i^{Δ} is injective, the remaining (left-hand side) square commutes. $\square$

We now prove that the morphism $j_{(\underline{\Sigma},\Delta)}$ does not depend on Δ. We first need a preliminary result.

Lemma 2.34 *Let Q be a square* (*ie a disc with four punctures on its boundary*) *and Δ_Q a topological triangulation of Q. If $\alpha_{\varepsilon\varepsilon'} \in \mathcal{S}_{\omega}(Q)$ is the class of a stated arc*, *then $j_{(Q,\Delta_Q)}(\alpha_{\varepsilon\varepsilon'}) = \alpha_{\varepsilon\varepsilon'}^N$. In particular*, *$j_{(Q,\Delta_Q)}$ does not depend on Δ_Q.*

Proof Let e be the inner edge of Δ_Q which is a common boundary arc of two triangles $\mathbb{T}_1$ and $\mathbb{T}_2$. Make the intersection $\alpha \cap e$ transversal and minimal via an isotopy on α. If the intersection is empty, then α is included in one of the triangles and the lemma follows from Lemma 2.31. If $\alpha \cap e$ is not empty, then it has only one element. Therefore, by letting $\alpha^{\mathbb{T}_i} := \alpha \cap \mathbb{T}_i$ for $i = 1, 2$, one has

$$i^{\Delta_Q}(\alpha_{\varepsilon\varepsilon'}) = \alpha_{\varepsilon+}^{\mathbb{T}_1} \otimes \alpha_{+\varepsilon'}^{\mathbb{T}_2} + \alpha_{\varepsilon-}^{\mathbb{T}_1} \otimes \alpha_{-\varepsilon'}^{\mathbb{T}_2}.$$

Write $x := \alpha_{\varepsilon+}^{\mathbb{T}_1} \otimes \alpha_{+\varepsilon'}^{\mathbb{T}_2}$ and $y := \alpha_{\varepsilon-}^{\mathbb{T}_1} \otimes \alpha_{-\varepsilon'}^{\mathbb{T}_2}$ and note that $xy = q^{-2}yx$. By Lemma 2.27,

$$i^{\Delta_Q}(\alpha_{\varepsilon\varepsilon'}^N) = i^{\Delta_Q}(\alpha_{\varepsilon\varepsilon'})^N = (x+y)^N = x^N + y^N = (j_{\mathbb{T}_1} \otimes j_{\mathbb{T}_2}) \circ i^{\Delta_Q}(\alpha_{\varepsilon\varepsilon'}).$$

Hence, $j_{(Q,\Delta_Q)}(\alpha_{\varepsilon\varepsilon'}) = \alpha_{\varepsilon\varepsilon'}^N$. $\square$

Lemma 2.35 *The morphism $j_{(\underline{\Sigma},\Delta)}$ does not depend on Δ.*

Proof Every two triangulations can be related by a finite sequence of flips on the edges. Therefore, it is enough to prove that if Δ' differs from Δ by a flip of one edge, then $j_{(\underline{\Sigma},\Delta)} = j_{(\underline{\Sigma},\Delta')}$.

Let e be an inner edge of Δ that bounds two distinct faces $\mathbb{T}_1$ and $\mathbb{T}_2$. Consider the topological triangulation Δ' obtained from Δ by flipping the edge e inside the square $Q = \mathbb{T}_1 \cup \mathbb{T}_2$. Let

$$i : \mathcal{S}_{\omega}(\underline{\Sigma}) \hookrightarrow \mathcal{S}_{\omega}(\underline{\Sigma} \setminus Q) \otimes \mathcal{S}_{\omega}(Q)$$

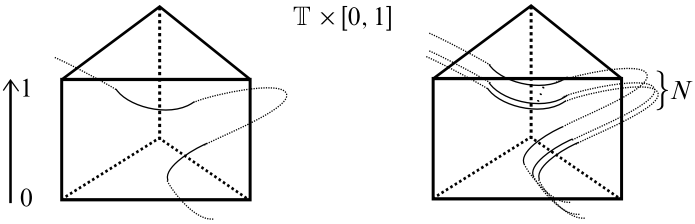


Figure 5: Instance of tangles $T_{\mathbb{T}}$ and $T_{\mathbb{T}}^{(N)}$.

be the gluing morphism. By Lemma 2.34, the morphism $j_Q : \mathcal{S}_{+1}(Q) \hookrightarrow \mathcal{S}_\omega(Q)$ does not depend on the triangulation of Q. Therefore, by Lemma 2.33, both the morphisms $j_{(\underline{\Sigma},\Delta)}$ and $j_{(\underline{\Sigma},\Delta')}$ make the diagram

$$\begin{tikzcd}
\mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, hook, "i"] \arrow[d, hook, shift right, "j_{(\underline{\Sigma},\Delta')}"'] \arrow[d, hook, shift left, "j_{(\underline{\Sigma},\Delta)}"] & \mathcal{S}_{+1}(\underline{\Sigma} \setminus Q) \otimes \mathcal{S}_{+1}(Q) \arrow[d, hook, "j_{(\underline{\Sigma} \setminus Q, \Delta_{\underline{\Sigma} \setminus Q})} \otimes j_Q"] \\
\mathcal{S}_\omega(\underline{\Sigma}) \arrow[r, hook, "i"] & \mathcal{S}_\omega(\underline{\Sigma} \setminus Q) \otimes \mathcal{S}_\omega(Q)
\end{tikzcd}$$

commutative. This proves that $j_{(\underline{\Sigma},\Delta)} = j_{(\underline{\Sigma},\Delta')}$. □

Step 2: arcs with endpoints in distinct boundary arcs We now assume that the triangulation Δ is nondegenerate.

Lemma 2.36 *If $\alpha_{\varepsilon\varepsilon'} \in \mathcal{S}_\omega(\underline{\Sigma})$ is the class of a stated arc such that its endpoints lie on two different boundary arcs, then $j_{\underline{\Sigma}}(\alpha_{\varepsilon\varepsilon'}) = \alpha_{\varepsilon\varepsilon'}^N$.*

Proof By the defining property of $j_{\underline{\Sigma}}$, as depicted in diagram (19), it is enough to prove that

$$i^\Delta(\alpha_{\varepsilon\varepsilon'}^N) = \Big(\bigotimes_{\mathbb{T} \in F(\Delta)} j_{\mathbb{T}}\Big) i^\Delta(\alpha_{\varepsilon\varepsilon'}). \tag{20}$$

Without lost of generality, we suppose that the arc α is in minimal and transverse position with the edges of Δ. Let T be a (vertical framed) tangle of $\Sigma_{\mathcal{P}} \times (0,1)$ that projects on α and such that its height projection is an injective map (this is possible since α is an arc). Note that for each $\mathbb{T} \in F(\Delta)$, the tangle $T_{\mathbb{T}} := T \cap (\mathbb{T} \times (0,1))$ may have various connected components; since the height projection is injective, these components are ordered by height. Let $T^{(N)}$ be a tangle of N parallel copies of T obtained by stacking N copies of T, but close enough to have the following property. For each $\mathbb{T} \in F(\Delta)$, if T_1 and T_2 are two connected components of $T_{\mathbb{T}}$ such that T_1 is below T_2, then, in $T_{\mathbb{T}}^{(N)} := T^{(N)} \cap (\mathbb{T} \times (0,1))$, each copy of T_1 is below all the copies of T_2. See Figure 5 for an illustration. Note that since α is an arc with boundary points at two distinct boundary arcs, the tangle $T^{(N)}$ is a representative of the N^{th} product of $\alpha_{\varepsilon\varepsilon'}$ in $\mathcal{S}_\omega(\underline{\Sigma})$; otherwise it may not be true.

The left-hand term of (20) can be described as the cutting of $T^{(N)}$ along each edge of the triangulation, and summing the result over all possible states at each edge. More formally, it is described as follows.

Let K be a subset of edges of Δ that intersect α. We let $\mathrm{St}_K(\alpha)$ be the set of maps

$$s\colon T\cap(K\times(0,1))\to\{-,+\}.$$

We identify $\mathrm{St}_K(\alpha)$ with $\bigsqcup_{e\in K}\mathrm{St}_{\{e\}}(\alpha)$, which allows us to write $s\in\mathrm{St}_K(\alpha)$ as $\sqcup s_e$. We will only consider the two sets K: the set E of all the *internal* edges of Δ that intersect α, and the set $K=\{e\}$ for an edge e.

For $s\in\mathrm{St}_E(\alpha)$, write $s^{(N)}:=(s,\dots,s)\in\mathrm{St}_E(\alpha)^{\times N}$. We denote by s_0 the state of $\alpha_{\varepsilon\varepsilon'}$ (so $\alpha_{\varepsilon\varepsilon'}=[T,s_0]$).

For $\boldsymbol{s}=(s_1,\dots,s_N)\in\mathrm{St}_E(\alpha)^{\times N}$, we let

$$\alpha(\boldsymbol{s}):=\bigotimes_{\mathbb{T}\in F(\Delta)}[T_{\mathbb{T}}^{(N)},(\boldsymbol{s}\sqcup s_0^{(N)})|_{\partial\mathbb{T}}]\in\bigotimes_{\mathbb{T}\in F(\Delta)}\mathcal{S}_\omega(\mathbb{T}),$$

where we associate, to the k^{th} copy of $T_{\mathbb{T}}^{(N)}$, the restriction of the state s_k. With this notation, the left-hand term of (20) can be written as

$$i^\Delta(\alpha_{\varepsilon\varepsilon'}^N)=\sum_{\boldsymbol{s}\in\mathrm{St}_E(\alpha)^{\times N}}\alpha(\boldsymbol{s}). \tag{21}$$

Now, let us describe the right-hand term of (20). Note that the construction of $T^{(N)}$ ensures that, for each triangle $\mathbb{T}$ and each state s of $T_{\mathbb{T}}$, one has $j_{\mathbb{T}}([T_{\mathbb{T}},s])=[T_{\mathbb{T}}^{(N)},s^{(N)}]$. Therefore, using that $j_{\mathbb{T}}$ is an algebra morphism,

$$\Big(\bigotimes_{\mathbb{T}\in F(\Delta)}j_{\mathbb{T}}\Big)i^\Delta(\alpha_{\varepsilon\varepsilon'})=\sum_{s\in\mathrm{St}_E(\alpha)}\alpha(s^{(N)}). \tag{22}$$

Let Y be the set of nondiagonal states $\mathrm{St}_E(\alpha)^{\times N}\setminus\{(s,\dots,s)\mid s\in\mathrm{St}_E(\alpha)\}$. The sum in (21) and in (22) differ by the sum of $\alpha(\boldsymbol{s})$ for $\boldsymbol{s}\in Y$.

Let us fix an edge e of E and let us split Y into $J\sqcup Y_e$ where Y_e is the set of N–tuples of states at e, that is, $Y_e=\{\boldsymbol{s}\in Y\mid \boldsymbol{s}\colon T^{(N)}\cap(e\times(0,1))\to\{-,+\}\}$. Therefore, showing (20) amounts to showing that

$$\sum_{\boldsymbol{s}'\in J}\sum_{\boldsymbol{s}\in Y_e}\alpha(\boldsymbol{s}'\sqcup\boldsymbol{s})=0.$$

In fact, let us show that, for each $\boldsymbol{s}'\in J$, one has $\sum_{\boldsymbol{s}\in Y_e}\alpha(\boldsymbol{s}'\sqcup\boldsymbol{s})=0$.

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be the two triangles adjoining e (they are distinct since Δ is assumed nondegenerate) and let $Q\subset\Sigma_{\mathcal{P}}$ be the resulting square. Denote by $i_Q\colon\mathcal{S}_\omega(Q)\hookrightarrow\bigotimes_{\mathbb{T}\in F(\Delta)}\mathcal{S}_\omega(\mathbb{T})$ the corresponding embedding and write $T_Q:=T\cap(Q\times(0,1))$. For each $\boldsymbol{s}'\in J$,

$$\sum_{\boldsymbol{s}\in Y_e}\alpha(\boldsymbol{s}'\sqcup\boldsymbol{s})=\Big(\bigotimes_{\mathbb{T}\neq\mathbb{T}_1,\mathbb{T}_2}[T_{\mathbb{T}}^{(N)},\boldsymbol{s}'|_{\partial\mathbb{T}}]\Big)\otimes\big(i_Q([T_Q^{(N)},\boldsymbol{s}'|_{\partial Q}])-(j_{\mathbb{T}_1}\otimes j_{\mathbb{T}_2})\circ i_Q([T_Q^{(N)},\boldsymbol{s}'|_Q])\big).$$

The last term is zero by Lemma 2.34 and the commutativity of the diagrams in Lemma 2.31. ◻

Step 3: closed curves that intersect Δ nicely

Definition 2.37 The N^{th} Chebyshev polynomial of first kind is the polynomial $T_N(X) \in \mathbb{Z}[X]$ defined by the recursive formulas $T_0(X) = 2$, $T_1(X) = X$ and $T_{n+2}(X) = XT_{n+1}(X) - T_n(X)$ for $n \geq 0$.

The following proposition is at the heart of (our proof of) the so-called "miraculous cancellations" from [Bonahon and Wong 2016]. We postpone its proof to the appendix.

Proposition 2.38 *If ω is a root of unity of odd order $N \geq 1$, then in $\mathcal{S}_\omega(\mathbb{B})$,*

$$T_N(\alpha_{++} + \alpha_{--}) = \alpha_{++}^N + \alpha_{--}^N.$$

Recall that we suppose that the triangulation is nondegenerate.

Lemma 2.39 *Let $\gamma \in \mathcal{S}_\omega(\underline{\Sigma})$ be the class of a closed curve. If the closed curve can be chosen such that it intersects an edge of Δ once and only once, then $j_{\underline{\Sigma}}(\gamma) = T_N(\gamma)$.*

Proof Consider the punctured surface $\underline{\Sigma}(e)$ obtained from $\underline{\Sigma}$ by replacing e by two arcs e' and e'' parallel to e with the same endpoints and removing the bigon between e' and e''. Consider the injective morphism $i|_{e' \# e''} : \mathcal{S}_\omega(\underline{\Sigma}) \hookrightarrow \mathcal{S}_\omega(\underline{\Sigma}(e))$. By Lemma 2.33, the following diagram commutes:

$$\begin{tikzcd} \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, hook, "j_{\underline{\Sigma}}"] \arrow[d, hook, "i|_{e'\#e''}"] & \mathcal{S}_\omega(\underline{\Sigma}) \arrow[d, hook, "i|_{e'\#e''}"] \\ \mathcal{S}_{+1}(\underline{\Sigma}(e)) \arrow[r, hook, "j_{\underline{\Sigma}(e)}"] & \mathcal{S}_\omega(\underline{\Sigma}(e)) \end{tikzcd}$$

By cutting γ along e, we get an arc $\beta \subset \Sigma(e)$ such that, by the hypothesis, $i|_{e'\#e''}(\gamma) = \beta_{++} + \beta_{--}$. Consider the algebra morphism $\varphi : \mathcal{S}_\omega(\mathbb{B}) \to \mathcal{S}_\omega(\underline{\Sigma}(e))$ sending $\alpha_{\varepsilon\varepsilon'}$ to $\beta_{\varepsilon\varepsilon'}$. One has

$$\begin{aligned} j_{\underline{\Sigma}(e)} \circ i|_{e'\#e''}(\gamma) &= j_{\underline{\Sigma}(e)}(\beta_{++} + \beta_{--}) \\ &= \varphi(\alpha_{++}^N + \alpha_{--}^N) && \text{(by Lemma 2.36)} \\ &= \varphi(T_N(\alpha_{++} + \alpha_{--})) && \text{(by Proposition 2.38)} \\ &= i|_{e'\#e''}(T_N(\gamma)). \end{aligned}$$

Hence, by the above diagram, $j_{\underline{\Sigma}}(\gamma) = T_N(\gamma)$. □

Step 4: adding a puncture Let $\underline{\Sigma}' = (\Sigma, \mathcal{P} \cup \{p_0\})$ be a punctured surface obtained from $\underline{\Sigma} = (\Sigma, \mathcal{P})$ by adding one puncture $p_0 \in \Sigma_{\mathcal{P}}$ and consider the algebra morphism $\varphi : \mathcal{S}_\omega(\underline{\Sigma}') \to \mathcal{S}_\omega(\underline{\Sigma})$ of Section 2.3. We assume that $\underline{\Sigma}$ is equipped with a nondegenerated triangulation.

Lemma 2.40 *The following diagram is commutative*:

$$\begin{tikzcd} \mathcal{S}_{+1}(\underline{\Sigma}') \arrow[r, hook, "j_{\underline{\Sigma}'}"] \arrow[d, two heads, "\varphi"] & \mathcal{S}_\omega(\underline{\Sigma}') \arrow[d, two heads, "\varphi"] \\ \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, hook, "j_{\underline{\Sigma}}"] & \mathcal{S}_\omega(\underline{\Sigma}) \end{tikzcd}$$

Proof First consider the diagram

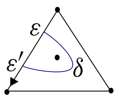
$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{I}^{+1}_{p_0} \arrow[r] \arrow[d, hook, "j_{\underline{\Sigma}'}"] & \mathcal{S}_{+1}(\underline{\Sigma}') \arrow[r, "\varphi"] \arrow[d, hook, "j_{\underline{\Sigma}'}"] & \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r] \arrow[d, hook, "j_{\underline{\Sigma}}"] & 0 \\
0 \arrow[r] & \mathcal{I}_{p_0} \arrow[r] & \mathcal{S}_{\omega}(\underline{\Sigma}') \arrow[r, "\varphi"] & \mathcal{S}_{\omega}(\underline{\Sigma}) \arrow[r] & 0
\end{tikzcd}
\tag{23}
$$

where $\mathcal{I}^{+1}_{p_0} \subset \mathcal{S}_{+1}(\underline{\Sigma}')$ and $\mathcal{I}_{p_0} \subset \mathcal{S}_{\omega}(\underline{\Sigma}')$ denote the off-puncture ideals in $\mathcal{S}_{+1}(\underline{\Sigma}')$ and $\mathcal{S}_{\omega}(\underline{\Sigma}')$, respectively (see Definition 2.17). By Proposition 2.18, both lines are exact so we need to prove the inclusion $j_{\underline{\Sigma}'}(\mathcal{I}^{+1}_{p_0}) \subset \mathcal{I}_{p_0}$ to conclude. We divide the proof in two steps.

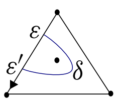
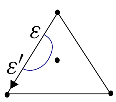
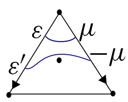
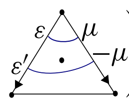
Step 1 We first suppose that $\underline{\Sigma} = \mathbb{T}_0$ is a triangle. In this case, $\mathbb{T}'_0$ is a punctured triangle and we have two possibilities depending whether p_0 is in the boundary or the interior of $\mathbb{T}_0$. Some nondegenerate triangulations Δ'_0 of $\mathbb{T}'_0$ are drawn in Figure 6.

Claim *The off-kernel ideal $\mathcal{I}_{p_0}$ is generated by elements $\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}$ and $\gamma - \gamma'$, where α and α' are arcs isotopic in $\mathbb{T}_0$ whose endpoints lie in distinct boundary arcs and γ and γ' are curves isotopic in $\mathbb{T}_0$ which intersect each edge of Δ'_0 once.*

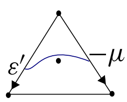
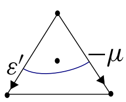
If the claim is proved, then for $\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}$ and $\gamma - \gamma'$ some generators of $\mathcal{I}_{p_0}$, Lemma 2.36 implies that $j_{\mathbb{T}'_0}(\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}) \subset \mathcal{I}_{p_0}$ and Lemma 2.39 implies that $j_{\mathbb{T}'_0}(\gamma - \gamma') = T_N(\gamma) - T_N(\gamma') \in \mathcal{I}_{p_0}$. The claim implies the inclusion $j_{\mathbb{T}'_0}(\mathcal{I}^{+1}_{p_0}) \subset \mathcal{I}_{p_0}$, which concludes the proof in the case of the triangle. To prove the claim, recall from Proposition 2.18 that $\mathcal{I}_{p_0}$ is generated by elements $\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}$ and $\gamma - \gamma'$ with α and α' isotopic in $\mathbb{T}_0$ and γ and γ' isotopic in $\mathbb{T}_0$. First note that when p_0 lies in the boundary of $\mathbb{T}_0$, then $\mathbb{T}'_0$ does not contain any noncontractible simple closed curve and the nontrivial arcs of $\mathbb{T}'_0$ have endpoints in distinct boundary arcs, so the claim is immediate in this case. When p_0 lies in the interior of $\mathbb{T}_0$, there is only one nontrivial simple closed curve (which encircles p_0 once) and this curves intersects each edges of Δ'_0 once. However $\mathbb{T}'_0$ contains three nontrivial arcs with endpoints in the same boundary arcs which are related by a $\frac{2}{3}\pi$ radian rotation. Let δ be one of these arcs and

$\delta_{\varepsilon\varepsilon'} =$ 

Since $x := \delta_{\varepsilon\varepsilon'} - C^{\varepsilon'}_{\varepsilon} \in \mathcal{I}_{p_0}$, we need to show that x belongs to the ideal $\mathcal{J}_{p_0}$ generated by elements $\alpha_{\varepsilon\varepsilon'} - \alpha'_{\varepsilon\varepsilon'}$ with α and α' isotopic in $\mathbb{T}_0$ with distinct endpoints. This is done by a simple application of the skein relation (4):

$x =$  $-$  $= \sum_{\mu=+,-} C^{-\mu}_{\mu} \Big($  $-$  $\Big).$

Therefore x belongs to the ideal generated by elements

 $-$ 

This proves the claim and concludes the proof of the lemma in the case where $\underline{\Sigma} = \mathbb{T}_0$.

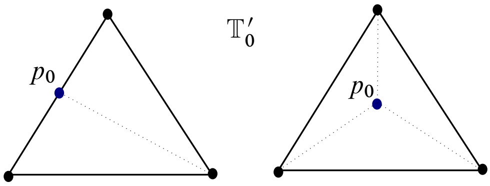


Figure 6: Punctured triangles $\mathbb{T}_0'$ and their nondegenerated triangulations.

Step 2 We consider the general case. Recall that $\underline{\Sigma}$ is equipped with a nondegenerate triangulation Δ and let $\mathbb{T}_0$ be the face containing the point p_0. Let $\underline{\Sigma}_0$ be the (possibly empty) punctured surface made of the faces of Δ distinct from $\mathbb{T}_0$ so that $\underline{\Sigma}$ is obtained from $\mathbb{T}_0 \sqcup \underline{\Sigma}_0$ by gluing some pairs of boundary arcs together and let $i\colon \mathscr{S}_\omega(\underline{\Sigma}) \hookrightarrow \mathscr{S}_\omega(\mathbb{T}_0) \otimes \mathscr{S}_\omega(\underline{\Sigma}_0)$ denote the gluing map. Similarly, let $i'\colon \mathscr{S}_\omega(\underline{\Sigma}') \hookrightarrow \mathscr{S}_\omega(\mathbb{T}_0') \otimes \mathscr{S}_\omega(\underline{\Sigma}_0)$ be the gluing map of $\underline{\Sigma}'$. Consider the diagram

$$\begin{tikzcd}
\mathscr{S}_{+1}(\mathbb{T}_0') \otimes \mathscr{S}_{+1}(\underline{\Sigma}_0) \arrow[rrr, hook, "j_{\mathbb{T}_0'} \otimes j_{\underline{\Sigma}_0}"] \arrow[ddd, two heads, "\varphi_0 \otimes \mathrm{id}"] & & & \mathscr{S}_\omega(\mathbb{T}_0') \otimes \mathscr{S}_\omega(\underline{\Sigma}_0) \arrow[ddd, two heads, "\varphi_0 \otimes \mathrm{id}"] \\
& \mathscr{S}_{+1}(\underline{\Sigma}') \arrow[ul, hook, "i'"] \arrow[r, hook, "j_{\underline{\Sigma}'}"] \arrow[d, two heads, "\varphi"] & \mathscr{S}_\omega(\underline{\Sigma}') \arrow[ur, hook, "i'"] \arrow[d, two heads, "\varphi"] & \\
& \mathscr{S}_\omega(\underline{\Sigma}') \arrow[dl, hook, "i"] \arrow[r, hook, "j_{\underline{\Sigma}}"] & \mathscr{S}_\omega(\underline{\Sigma}) \arrow[dr, hook, "i"] & \\
\mathscr{S}_{+1}(\mathbb{T}_0) \otimes \mathscr{S}_{+1}(\underline{\Sigma}_0) \arrow[rrr, hook, "j_{\mathbb{T}_0} \otimes j_{\underline{\Sigma}_0}"] & & & \mathscr{S}_\omega(\mathbb{T}_0) \otimes \mathscr{S}_\omega(\underline{\Sigma}_0)
\end{tikzcd}$$

In this diagram,

- the outer square commutes by Step 1;
- the squares on the top and bottom commute by Lemma 2.33;
- the squares on the left and right sides commute by definition of φ.

Therefore the innermost square commutes. ◻

Notation 2.41 For $\alpha_{\varepsilon\varepsilon'} \in \mathscr{S}_\omega(\underline{\Sigma})$ the class of a stated arc, we denote by $\alpha_{\varepsilon\varepsilon'}^{(N)}$ be the class of the stated tangle made by stacking N parallel copies of $\alpha_{\varepsilon\varepsilon'}$ on top of the others in the framing direction. More precisely, if both endpoints of α lie in different boundary arcs, then $\alpha_{\varepsilon\varepsilon'}^{(N)} = (\alpha_{\varepsilon\varepsilon'})^N$. If α has its two endpoints, say v and w, in the same boundary arc with $h(v) < h(w)$ such that v has state ε and w has state ε', then $\alpha_{\varepsilon\varepsilon'}^{(N)}$ is the class of the stated tangle $(\alpha^{(N)}, s^{(N)})$ defined as follows. The tangle $\alpha^{(N)}$ is made of N parallel copies $\alpha^{(N)} = \alpha_1 \cup \cdots \cup \alpha_N$ of α such that the height order is given by $h(v_1) < h(v_2) < \cdots < h(v_N) < h(w_1) < \cdots < h(w_N)$. The state $s^{(N)}$ sends the points v_i to ε and the points w_j to ε'.

Lemma 2.42 *If $\alpha_{\varepsilon\varepsilon'} \in \mathscr{S}_\omega(\underline{\Sigma})$ is the class of a stated arc such that its endpoints lie on the same boundary arcs, then $j_{\underline{\Sigma}}(\alpha_{\varepsilon\varepsilon'}) = \alpha_{\varepsilon\varepsilon'}^{(N)}$.*

Proof Since the two endpoints of α lie on the same boundary arc a, we can pick a puncture $p_0 \in a$ that lies between these two endpoints. Denote by $\underline{\Sigma}' = (\Sigma, \mathcal{P} \cup \{p_0\})$ the punctured surface obtained by adding this puncture, and $\varphi \colon \mathcal{S}_\omega(\underline{\Sigma}') \to \mathcal{S}_\omega(\underline{\Sigma})$ the morphism of Section 2.3. With the notation of Section 2.3, the two components of $a \setminus \{p_0\}$ are two boundary arcs b and c of $\underline{\Sigma}'$ and we choose the convention such that $\alpha \in \mathcal{T}^{(0)}(\underline{\Sigma})$. Note that $\alpha^{(N)}$ is in $\mathcal{T}^{(0)}(\underline{\Sigma})$ as well. To avoid confusion, we denote by α' the arc α seen as an arc in $\Sigma_{\mathcal{P}\cup\{p_0\}}$, so that $\iota(\alpha') = \alpha$. By Lemma 2.36, $j_{\underline{\Sigma}'}(\alpha'_{\varepsilon\varepsilon'}) = (\alpha'_{\varepsilon\varepsilon'})^N = {\alpha'}^{(N)}_{\varepsilon\varepsilon'}$. By commutativity of the diagram in Lemma 2.40 and by definition of φ, the image $j_{\underline{\Sigma}}(\alpha_{\varepsilon\varepsilon'})$ is the class in $\mathcal{S}_\omega(\underline{\Sigma})$ of the unique stated tangle in $\mathcal{T}^{(0)}(\underline{\Sigma})$ which is isotopic to ${\alpha'}^{(N)}_{\varepsilon\varepsilon'}$: this is $\alpha^{(N)}_{\varepsilon\varepsilon'}$. □

Lemma 2.43 *If $\gamma \in \mathcal{S}_\omega(\underline{\Sigma})$ is the class of a closed curve, then $j_{\underline{\Sigma}}(\gamma) = T_N(\gamma)$.*

Proof If the closed curve can be chosen such that it intersects an edge of Δ once and only once, then this is Lemma 2.39. Otherwise, we can refine the triangulation by adding an inner puncture in order to have this property. Denote by $\underline{\Sigma}'$ the resulting punctured surface and let $\gamma' \in \mathcal{S}_{+1}(\underline{\Sigma}')$ be such that $\iota(\gamma') = \gamma$. Lemma 2.39 implies that $j_{\underline{\Sigma}'}(\gamma') = T_N(\gamma')$ and Lemma 2.40 implies that $j_{\underline{\Sigma}}(\gamma) = T_N(\gamma)$. □

Step 5: punctured surfaces which do not admit nondegenerate triangulations It remains to prove Theorem 1.2 for connected punctured surfaces which do not admit nondegenerate topological triangulations; that is, for the small punctured surfaces, for the disc with one inner puncture and one puncture on its boundary and for the unpunctured surfaces $\underline{\Sigma} = (\Sigma, \varnothing)$ with empty set of puncture.

The disc with only one puncture (on its boundary) and the sphere with zero or one puncture both have trivial skein algebra, while the sphere with two punctures has a commutative skein algebra. Therefore, Theorem 1.2 holds trivially for them. It remains to prove:

Lemma 2.44 *Theorem 1.2 holds when $\underline{\Sigma}$ is either a disc with one inner puncture and one puncture on its boundary or an unpunctured surface $\underline{\Sigma} = (\Sigma, \varnothing)$ of genus at least one.*

Proof Choose an inner puncture $p_0 \in \mathring{\Sigma}_{\mathcal{P}}$ and consider the punctured surface $\underline{\Sigma}' := (\Sigma, \mathcal{P} \cup \{p_0\})$. Since $\underline{\Sigma}'$ admits a nondegenerate triangulation, our previous study shows the existence of the Chebyshev morphism $j_{\underline{\Sigma}'} \colon \mathcal{S}_{+1}(\underline{\Sigma}') \hookrightarrow \mathcal{Z}(\mathcal{S}_\omega(\underline{\Sigma}'))$. Consider the off-puncture ideals $\mathcal{I}^{+1}_{p_0} \subset \mathcal{S}_{+1}(\underline{\Sigma}')$ and $\mathcal{I}_{p_0} \subset \mathcal{S}_\omega(\underline{\Sigma}')$. Exactly the same argument used in the proof of Lemma 2.40 shows the inclusion $j_{\underline{\Sigma}'}(\mathcal{I}^{+1}_{p_0}) \subset \mathcal{I}_{p_0}$. By Proposition 2.18, both lines in the following diagram are exact:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}^{+1}_{p_0} \arrow[r] \arrow[d, hook, "j_{\underline{\Sigma}'}"] & \mathcal{S}_{+1}(\underline{\Sigma}') \arrow[r, "\varphi"] \arrow[d, hook, "j_{\underline{\Sigma}'}"] & \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r] \arrow[d, dotted, "\exists ! j_{\underline{\Sigma}}"] & 0 \\
0 \arrow[r] & \mathcal{I}_{p_0} \arrow[r] & \mathcal{S}_\omega(\underline{\Sigma}') \arrow[r, "\varphi"] & \mathcal{S}_\omega(\underline{\Sigma}) \arrow[r] & 0
\end{tikzcd}$$

Therefore there exists a unique algebra morphism $j_{\underline{\Sigma}} \colon \mathcal{S}_{+1}(\underline{\Sigma}) \to \mathcal{S}_\omega(\underline{\Sigma})$ which makes the diagram commute. Since $j_{\underline{\Sigma}}$ is obtained from $j_{\underline{\Sigma}'}$ by passing to the quotient, its image is also central and one has the equalities $j_{\underline{\Sigma}}([\gamma]) = T_N([\gamma])$ and $j_{\underline{\Sigma}}(\alpha_{\varepsilon\varepsilon'}) = \alpha^{(N)}_{\varepsilon\varepsilon'}$ for any closed curve γ and any stated arc $\alpha_{\varepsilon\varepsilon'}$. □

2.7 A Poisson bracket on $\mathcal{S}_{+1}(\underline{\Sigma})$

In this section, we define and make explicit a Poisson structure on $\mathcal{S}_{+1}(\underline{\Sigma})$.

2.7.1 Preliminaries We briefly recall some general facts concerning deformation quantization.

Let $\mathcal{A}$ be a complex commutative unital algebra, $\mathbb{C}[\![\hbar]\!]$ be the ring of formal series in a parameter $\hbar$ and $\mathcal{A}[\![\hbar]\!] := \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[\![\hbar]\!]$. A *star product* $\star$ on $\mathcal{A}$ is an associative product on $\mathcal{A}[\![\hbar]\!]$ such that if $f = \sum_i f_i \hbar^i$ and $g = \sum_i g_i \hbar^i$ are elements of $\mathcal{A}[\![\hbar]\!]$, then

$$f \star g = f_0 g_0 \mod \hbar,$$

where $f_0 g_0$ denotes the product of f_0 and g_0 in $\mathcal{A}$. A star product induces a Poisson structure on $\mathcal{A}$ by the formula

$$f \star g - g \star f = \hbar\{f, g\} \mod \hbar^2, \tag{24}$$

for all $f, g \in \mathcal{A}$. The algebra $(\mathcal{A}[\![\hbar]\!], \star)$ is called a *deformation quantization* of the commutative Poisson algebra $(\mathcal{A}, \{\cdot, \cdot\})$. We refer to [Kontsevich 2003; Gutt et al. 2005, II.2] for detailed discussions. A *morphism of star products* between $(\mathcal{A}, \star_{\mathcal{A}})$ and $(\mathcal{B}, \star_{\mathcal{B}})$ is an algebra morphism $\psi : \mathcal{A}[\![\hbar]\!] \to \mathcal{B}[\![\hbar]\!]$ whose restriction to $\mathcal{A} \subset \mathcal{A}[\![\hbar]\!]$ induces a morphism $\phi : \mathcal{A} \to \mathcal{B}$. Note that such a ϕ is, in fact, a morphism of Poisson algebras for the induced Poisson algebra structures. An isomorphism

$$\psi : (\mathcal{A}[\![\hbar]\!], \star_1) \xrightarrow{\cong} (\mathcal{A}[\![\hbar]\!], \star_2)$$

of star products is called a *gauge equivalence* if $\psi(f) = f \pmod{\hbar}$. If two star products are gauge equivalent, they induce the same Poisson bracket on $\mathcal{A}$.

To end this preamble, let us mention that deformation quantization is well behaved with respect to the tensor product. Indeed, if $\mathcal{A}[\![\hbar]\!]$ and $\mathcal{B}[\![\hbar]\!]$ are deformation quantizations of $\mathcal{A}$ and $\mathcal{B}$, respectively, then $\mathcal{A}[\![\hbar]\!] \otimes \mathcal{B}[\![\hbar]\!] \cong (\mathcal{A} \otimes \mathcal{B})[\![\hbar]\!]$ is a deformation quantization of $\mathcal{A} \otimes \mathcal{B}$. Note also that the Poisson structure on $\mathcal{A} \otimes \mathcal{B}$ given by (24) is

$$\{f \otimes g, f' \otimes g'\} = ff' \otimes \{g, g'\} + \{f, f'\} \otimes gg' \tag{25}$$

for $f, f' \in \mathcal{A}$ and $g, g' \in \mathcal{B}$.

2.7.2 Formal definition Let $\underline{\Sigma}$ be a punctured surface and $\mathfrak{o}$ an orientation of its boundary arc. Denote by $\mathcal{S}_{+1}(\underline{\Sigma})$ the stated skein algebra associated to the ring $\mathbb{C}$ with $\omega = +1$ and denote by $\mathcal{S}_{\omega_\hbar}(\underline{\Sigma})$ the stated skein algebra associated to the ring $\mathbb{C}[\![\hbar]\!]$ with $\omega_\hbar := \exp\left(-\frac{1}{4}\hbar\right)$. The convention is chosen so that $q = \exp(\hbar)$. Recall the basis $\mathcal{B}^{\mathfrak{o}}$ from Definition 2.3. Since $\mathcal{B}^{\mathfrak{o}}$ is independent of ω, one has an isomorphism of $\mathbb{C}[\![\hbar]\!]$–modules

$$\psi^{\mathfrak{o}} : \mathcal{S}_{+1}(\underline{\Sigma})[\![\hbar]\!] \xrightarrow{\cong} \mathcal{S}_{\omega_\hbar}(\underline{\Sigma}). \tag{26}$$

Note that $\mathfrak{o}$ tells us how to lift the basis elements $[D, s]$ of $\mathcal{S}_{+1}(\underline{\Sigma})$ (which are independent of the height order) in $\mathcal{S}_{\omega_\hbar}(\underline{\Sigma})$. We emphasize that $\psi^{\mathfrak{o}}$ is not an algebra morphism.

Definition 2.45 Pulling back the product of $\mathcal{S}_{\omega_\hbar}(\underline{\Sigma})$ along $\psi^{\mathfrak{o}}$ gives a star product $\star_\hbar$ on $\mathcal{S}_{+1}(\underline{\Sigma})$. We denote by $\{\cdot,\cdot\}^s$ the resulting Poisson bracket on $\mathcal{S}_{+1}(\underline{\Sigma})$ given by (24).

Here the superscript s stands for "skein".

Remark 2.46 For any two orientations $\mathfrak{o}_1$ and $\mathfrak{o}_2$ of the boundary arcs of $\underline{\Sigma}$, the automorphism $(\psi^{\mathfrak{o}_2})^{-1}\circ \psi^{\mathfrak{o}_1}\colon \mathcal{S}_{+1}(\underline{\Sigma})[\![\hbar]\!] \xrightarrow{\cong} \mathcal{S}_{+1}(\underline{\Sigma})[\![\hbar]\!]$ is a gauge equivalence; hence the Poisson bracket $\{\cdot,\cdot\}^s$ does not depend on $\mathfrak{o}$.

By definition, $(\mathcal{S}_{+1}(\underline{\Sigma})[\![\hbar]\!], \star_\hbar)$ is a quantization deformation of the Poisson algebra $(\mathcal{S}_{+1}(\underline{\Sigma}), \{\cdot,\cdot\}^s)$. Moreover, this structure of Poisson algebra is compatible with decompositions of surfaces. More precisely, one has the following.

Lemma 2.47 *The gluing maps* $i|_{a\#b}\colon \mathcal{S}_{+1}(\underline{\Sigma}|_{a\#b}) \hookrightarrow \mathcal{S}_{+1}(\underline{\Sigma})$, *the maps*

$$i^{\Delta}\colon \mathcal{S}_{+1}(\underline{\Sigma}) \hookrightarrow \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_{+1}(\mathbb{T})$$

and the coproduct maps Δ^L *and* Δ^R *are Poisson morphisms.*

Proof This follows from the fact that each of these morphisms arises from a morphism of star products. $\square$

2.7.3 Explicit formula This section is devoted to making explicit the Poisson bracket $\{\cdot,\cdot\}^s$ on stated diagrams. It will be expressed in terms of *resolutions* of stated diagrams, which are defined at crossings and at points on the boundary arcs.

Throughout this section, $\underline{\Sigma}$ is a punctured surface.

Resolution at a crossing Let (D,s) be a stated diagram and c a crossing of D. Denote by D_+ and D_- the diagrams obtained from D by replacing the crossing c  by its positive  and negative  resolution, respectively. The resolution of (D,s) at the crossing c is defined by

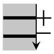
$$\mathrm{Res}_c(D,s) := [D_+,s]-[D_-,s] \in \mathcal{S}_{+1}(\underline{\Sigma}).$$

Resolution at boundary points Let $b_1,\dots,b_k$ be the boundary arcs of $\Sigma_{\mathcal{P}}$.

Definition 2.48 A *height order* on a stated diagram (D,s) of $\Sigma_{\mathcal{P}}$ is a k–tuple $\mathfrak{o}=(\mathfrak{o}_1,\dots,\mathfrak{o}_k)$ of bijections of sets $\mathfrak{o}_i\colon \partial_{b_i}D \to \{1,\dots,|\partial_{b_i}D|\}$.

Note that the product of symmetric groups $\mathbb{S}_{n_1}\times\cdots\times\mathbb{S}_{n_k}$ acts freely and transitively on the set of height orders by left composition.

To a height order $\mathfrak{o}$ on (D,s) corresponds a stated tangle with same height order and which projects to (D,s). Therefore, one can consider the class of $(D,s,\mathfrak{o})$ in $\mathcal{S}_\omega(\underline{\Sigma})$. If $\omega=+1$, the class $[D,s,\mathfrak{o}]\in\mathcal{S}_{+1}(\underline{\Sigma})$ is independent of $\mathfrak{o}$, and we denote it simply by $[D,s]$.

Let us choose a boundary arc b_i and suppose there are two points p_H and p_L of $\partial_{b_i}D$ such that $\mathfrak{o}_i(p_H)=\mathfrak{o}_i(p_L)+1$ (ie p_H is the $\mathfrak{o}_i$–successor of p_L). Let $\tilde{\mathfrak{o}}$ be the order on b_i that is induced by the orientation of Σ. To alleviate notation, we write $p <_{\tilde{\mathfrak{o}}} q$ for $\tilde{\mathfrak{o}}(p) < \tilde{\mathfrak{o}}(q)$. For instance, in the stated diagram , if p_L is the endpoint with $s(p_L) = +$, p_H the endpoint with $s(p_H) = -$ and $\mathfrak{o}$ is the orientation given by the arrow, then $p_L >_{\tilde{\mathfrak{o}}} p_H$ whereas $p_L <_{\mathfrak{o}} p_H$ (because the $\mathfrak{o}$ and $\tilde{\mathfrak{o}}$ orientation of the boundary arc where live p_L and p_H are opposite).

Let $\tau \in \mathbb{S}_{n_i}$ be the transposition that exchanges the $\mathfrak{o}_i$ order of p_H and p_L. The resolution of (D, s) along τ, denoted by $\mathrm{Res}_\tau(D, s, \mathfrak{o}) \in \mathcal{S}_{+1}(\underline{\Sigma})$, is given by

$$\begin{cases} \frac{1}{2}[D,s] & \text{if } s(p_H)=s(p_L) \text{ and } p_L <_{\tilde{\mathfrak{o}}} p_H \text{ or } (s(p_H),s(p_L))=(-,+) \text{ and } p_H <_{\tilde{\mathfrak{o}}} p_L, \\ -\frac{1}{2}[D,s] & \text{if } s(p_H)=s(p_L) \text{ and } p_H <_{\tilde{\mathfrak{o}}} p_L \text{ or } (s(p_H),s(p_L))=(+,-) \text{ and } p_L <_{\tilde{\mathfrak{o}}} p_H, \\ \frac{1}{2}[D,s]-2[D,\tau s] & \text{if } (s(p_H),s(p_L))=(+,-) \text{ and } p_H <_{\tilde{\mathfrak{o}}} p_L, \\ -\frac{1}{2}[D,s]+2[D,\tau s] & \text{if } (s(p_H),s(p_L))=(-,+) \text{ and } p_L <_{\tilde{\mathfrak{o}}} p_H, \end{cases}$$

where τs is the state that differs from s only by exchanging the states of p_H and p_L.

Let us extend the resolution to several points, namely any permutation of the boundary heights on a given boundary component. For two transpositions σ_1 and σ_2 of $\mathfrak{o}$–consecutive points, let

$$\mathrm{Res}_{\sigma_1 \circ \sigma_2}(D, s, \mathfrak{o}) = \mathrm{Res}_{\sigma_1}(D, s, \sigma_2 \circ \mathfrak{o}) + \mathrm{Res}_{\sigma_2}(D, s, \mathfrak{o}). \tag{27}$$

Definition 2.49 For a permutation $\sigma \in \mathbb{S}_{n_1} \times \cdots \times \mathbb{S}_{n_k}$, the resolution $\mathrm{Res}_\sigma(D, s, \mathfrak{o})$ is defined via (27), by considering the decomposition of σ into transpositions of $\mathfrak{o}$–consecutive points. This is clearly independent of the choice of decomposition into transpositions.

Remark 2.50 The resolution $\mathrm{Res}_\sigma(D, s, \mathfrak{o})$ is invariant under isotopy of (D, s). Also, $\mathrm{Res}_{\mathrm{id}}(D, s, \mathfrak{o}) = 0$.

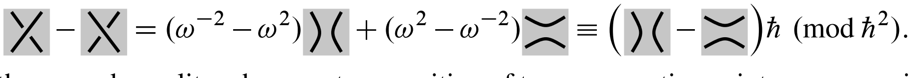
Lemma 2.51 *In the skein algebra $\mathcal{S}_{\omega_\hbar}(\underline{\Sigma})$, the following two statements hold.*

(1) *Let $D_{\times}$ and $D_{\times}$ be two diagrams that differ from each other only by a change of a crossing c. Then*
$$[D_{\times}, s, \mathfrak{o}] - [D_{\times}, s, \mathfrak{o}] = \hbar\, \mathrm{Res}_c(D_{\times}, s) \mod \hbar^2.$$

(2) *Let $(D, s, \mathfrak{o})$ be an $\mathfrak{o}$–ordered stated diagram. For $\pi \in \mathbb{S}_{n_1} \times \cdots \times \mathbb{S}_{n_k}$,*
$$[D, s, \mathfrak{o}] - [D, s, \pi \circ \mathfrak{o}] = \hbar\, \mathrm{Res}_\pi(D, s, \mathfrak{o}) \mod \hbar^2.$$

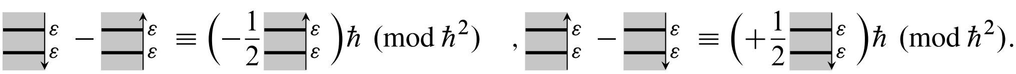
In the two statements, the resolutions Res *are seen in $\mathcal{S}_{\omega_\hbar}(\underline{\Sigma})$ via the isomorphism $\psi^{\tilde{\mathfrak{o}}}$ of (26).*

Proof Recall that $\omega_\hbar = \exp\bigl(-\frac{1}{4}\hbar\bigr) \equiv 1 - \frac{1}{4}\hbar \pmod{\hbar^2}$. The first equality follows from (3):

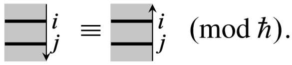


Let us prove the second equality when π a transposition of two consecutive points p_H, p_L with $p_H >_{\mathfrak{o}} p_L$. If $s(p_H) = s(p_L) = \varepsilon$, then (5) gives

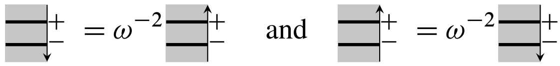
from which we deduce



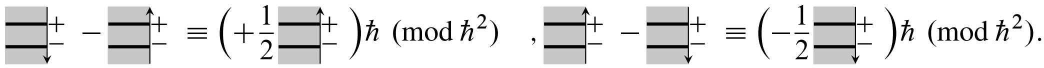
Note that in the stated skein algebra at $\omega = +1$, the height order is irrelevant; said differently, at $\omega_\hbar = \exp\left(-\frac{1}{4}\hbar\right)$, we have the skein relation



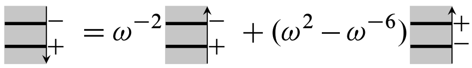
Now, if either $p_H <_{\tilde{\mathfrak{o}}} p_L$ and $(s(p_H), s(p_L)) = (-,+)$ or if $p_L <_{\tilde{\mathfrak{o}}} p_H$ and $(s(p_H), s(p_L)) = (+,-)$ then, using (5),



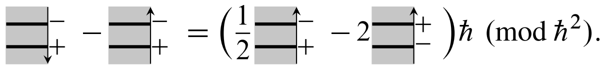
from which we deduce



If $p_H <_{\tilde{\mathfrak{o}}} p_L$ and $(s(p_H), s(p_L)) = (+,-)$, then (6) and (4) imply that



from which we deduce



Eventually the case where $p_L <_{\tilde{\mathfrak{o}}} p_H$ and $(s(p_H), s(p_L)) = (-,+)$ is deduced from this case by taking the opposite of the preceding equality. This concludes the proof of the second equality of the lemma when τ is a transposition. The case of a general permutation π follows by induction on the number of transpositions in a decomposition of π. □

Proposition 2.52 *Let $(D_1, s_2, \mathfrak{o}_1)$ and $(D_2, s_2, \mathfrak{o}_2)$ be two height ordered stated diagrams such that D_1 and D_2 intersect transversally in the interior of $\Sigma_{\mathcal{P}}$. Let $(D_1 D_2, s_1 s_2)$ be the stated diagram obtained by staking D_1 on top of D_2, $\mathfrak{o}_1\mathfrak{o}_2$ the resulting height order and π the permutation sending $\mathfrak{o}_2\mathfrak{o}_1$ to $\mathfrak{o}_1\mathfrak{o}_2$. In $\mathcal{S}_{+1}(\Sigma)$, the Poisson bracket from Definition 2.45 satisfies*

$$\{[D_1, s_1], [D_2, s_2]\}^s = \sum_{c \in D_1 \cap D_2} \mathrm{Res}_c(D_1 D_2, s_1 s_2) + \mathrm{Res}_\pi(D_1 D_2, s_1 s_2, \mathfrak{o}_1\mathfrak{o}_2).$$

Proof In the algebra $\mathcal{S}_{\omega_\hbar}(\Sigma)$, the product gives $[D_1, s_1, \mathfrak{o}_1] \cdot [D_2, s_2, \mathfrak{o}_2] = [D_1 D_2, s_1 s_2, \mathfrak{o}_1\mathfrak{o}_2]$ and $[D_2, s_2, \mathfrak{o}_2] \cdot [D_1, s_1, \mathfrak{o}_1] = [D_2 D_1, s_2 s_1, \mathfrak{o}_2\mathfrak{o}_1]$. We pass from the diagram $D_1 D_2$ to $D_2 D_1$ by changing each crossing in the intersection of the diagrams and changing the height order using π, so the formula is a consequence of Lemma 2.51. □

Remark 2.53 Neither $\{\cdot,\cdot\}^s$ nor the formula in Proposition 2.52 depend on a choice of orientation of the boundary arcs by Remark 2.46. When Σ is a closed surface, we recover Goldman's formula [1986]. When Σ has nontrivial boundary and no inner punctures, the subalgebra of the stated skein algebra generated by tangles with states having only value $+$ is isomorphic to the Muller algebra defined in [Muller 2016] (see also [Lê 2018, Section 6]). The Poisson bracket restricts to the corresponding subalgebra of $\mathcal{S}_{+1}(\Sigma)$ and the resulting Poisson algebra is isomorphic to Yuasa's Poisson algebra [2015].

Example 2.54 The Poisson bracket $\{-,-\}^s$ on the commutative algebra $\mathcal{S}_{+1}(\mathbb{B})$ is given by

$$\begin{aligned} \{\alpha_{++},\alpha_{+-}\}^s &= -\alpha_{+-}\alpha_{++}, & \{\alpha_{++},\alpha_{-+}\}^s &= -\alpha_{-+}\alpha_{++}, \\ \{\alpha_{--},\alpha_{+-}\}^s &= \alpha_{+-}\alpha_{--}, & \{\alpha_{--},\alpha_{-+}\}^s &= \alpha_{-+}\alpha_{--}, \\ \{\alpha_{+-},\alpha_{-+}\}^s &= 0, & \{\alpha_{++},\alpha_{--}\}^s &= -2\alpha_{+-}\alpha_{-+}. \end{aligned}$$

Example 2.55 For the triangle $\mathbb{T}$, the Poisson structure is described by the formulas in Example 2.54 by replacing α by each of the three arcs α, β and γ, together with the following relations and their images through the automorphisms τ and τ^2:

$$\{\gamma_{\varepsilon\mu},\alpha_{\mu'\varepsilon}\}^s = -\tfrac{1}{2}\gamma_{\varepsilon\mu}\alpha_{\mu'\varepsilon}, \quad \{\gamma_{-\mu},\alpha_{\mu'+}\}^s = \tfrac{1}{2}\gamma_{-\mu}\alpha_{\mu'+}, \quad \{\gamma_{+\mu},\alpha_{\mu'-}\}^s = -\tfrac{3}{2}\gamma_{+\mu}\alpha_{\mu'-} + 2\beta_{\mu\mu'}.$$

3 Relative character varieties

3.1 Relative character varieties for open surfaces

In this subsection we briefly recall from [Korinman 2019] the definition and main properties of character varieties for open surfaces.

The character variety of a closed punctured connected surface $\underline{\Sigma}$ is the algebraic quotient (familiar in geometric invariant theory)

$$\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma}) := \mathrm{Hom}(\pi_1(\Sigma_{\mathcal{P}}), \mathrm{SL}_2(\mathbb{C}))/\!\!/\mathrm{SL}_2(\mathbb{C})$$

under the action by conjugation of $\mathrm{SL}_2(\mathbb{C})$. Recall that by "closed", we mean that Σ is closed though in this case $\Sigma_{\mathcal{P}}$ is not closed when $\mathcal{P} \neq \varnothing$. Goldman [1986] defined a Poisson structure on its algebra of regular functions. It follows from [Barrett 1999; Bullock 1997; Przytycki and Sikora 2000; Turaev 1991] that, given a spin structure S on Σ with quadratic form ω_S, there is a Poisson isomorphism

$$\phi^S : (\mathcal{S}_{+1}(\underline{\Sigma}), \{\cdot,\cdot\}^s) \xrightarrow{\cong} \big(\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})], \{\cdot,\cdot\}\big).$$

For each noncontractible closed curve γ, it is given by $\phi^S(\gamma) = (-1)^{\omega_S([\gamma])+1}\tau_\gamma$, where τ_γ is the regular function $\tau_\gamma([\rho]) := \mathrm{Tr}(\rho(\gamma))$.

Korinman [2019] introduced a generalization of the character varieties to punctured surfaces which are not necessarily closed and which is closely related to the construction of Fock and Rosly [1999] and specifies to the constructions in [Alekseev and Malkin 1995; Alekseev et al. 1998; 2002; Guruprasad et al. 1997] when the marked surface is connected and has exactly one boundary arc (see [Korinman 2019] for a precise comparison). We will also denote it by $\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})$.

Notation 3.1 For a topological space X, we let $\Pi_1(X)$ denote its fundamental groupoid: objects are the points in X and morphisms are homotopy classes of oriented paths. We let s and t denote the source and target maps, which for a morphism $\alpha: v_1 \to v_2$ are given by $s(\alpha) = v_1$ and $t(\alpha) = v_2$. By convention, we compose the morphisms from left to right, ie if $\alpha_1\colon v_1\to v_2$ and $\alpha_2\colon v_2\to v_3$ are two paths, their composition is a path $\alpha_1\alpha_2\colon v_1\to v_3$. For $S\subset X$, we denote by $\Pi_1(X,S)$ the full subcategory of $\Pi_1(X)$ whose objects are points in S. For a group G, the set $\mathrm{Hom}(\Pi_1(X,S),G)$ denotes the set of functors $\rho\colon \Pi_1(X,S)\to G$, where G is seen as a category with one element. With our conventions, if $t(\alpha_1)=s(\alpha_2)$, then $\rho(\alpha_1\alpha_2)=\rho(\alpha_1)\rho(\alpha_2)$.

Let $\mathcal{R}_{\mathrm{SL}_2}(\underline{\Sigma})$ be the set of functors $\rho\colon \Pi_1(\Sigma_{\mathcal{P}})\to \mathrm{SL}_2$ whose restriction to $\Pi_1(\partial\Sigma_{\mathcal{P}})\subset\Pi_1(\Sigma_{\mathcal{P}})$ is the constant map with value the neutral element $\mathbb{1}_2\in\mathrm{SL}_2$. Let $\mathcal{G}$ be the group of maps $g\colon\Sigma_{\mathcal{P}}\to\mathrm{SL}_2$ whose restriction to $\partial\Sigma_{\mathcal{P}}$ is constant with value $\mathbb{1}_2$ and with finite support. It acts on $\mathcal{R}_{\mathrm{SL}_2}(\underline{\Sigma})$ by the formula

$$g\cdot\rho(\alpha):=g(s(\alpha))^{-1}\rho(\alpha)g(t(\alpha)),\quad \rho\in\mathcal{R}_{\mathrm{SL}_2}(\underline{\Sigma}),\, g\in\mathcal{G},\,\alpha\in\Pi_1(\Sigma_{\mathcal{P}}).$$

Both $\mathcal{R}_{\mathrm{SL}_2}(\underline{\Sigma})$ and $\mathcal{G}$ have a structure of affine scheme and the action is algebraic so we can define the GIT quotient

$$\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma}):=\mathcal{R}_{\mathrm{SL}_2}(\underline{\Sigma})/\!\!/\mathcal{G}. \tag{28}$$

The character variety turns out to be an affine Poisson variety whose Poisson structure (given by a generalized Goldman formula) depends on a choice of orientation of the boundary arcs. It is proved in [Korinman 2019, Theorem 1.1] that its algebra of regular functions $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ is well behaved under triangular decompositions: for a topological triangulation Δ, there are an injective Poisson morphism $i^{\Delta}\colon\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]\hookrightarrow\bigotimes_{\mathbb{T}\in F(\Delta)}\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]$ and Poisson Hopf comodule maps Δ^L and Δ^R such that the following sequence is exact:

$$0\to\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]\xrightarrow{i^{\Delta}}\bigotimes_{\mathbb{T}\in F(\Delta)}\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]\xrightarrow{\Delta^L-\sigma\circ\Delta^R}\Big(\bigotimes_{e\in\mathring{\mathcal{E}}(\Delta)}\mathbb{C}[\mathrm{SL}_2]\Big)\otimes\Big(\bigotimes_{\mathbb{T}\in F(\Delta)}\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]\Big). \tag{29}$$

In the present paper, we proceed by describing the character variety for the bigon and the triangle, together with the Hopf comodule maps Δ^L and Δ^R. Then, in virtue of the above property, we characterize the Poisson structure of the relative character variety for any triangulated punctured surface as the kernel of $\Delta^L-\sigma\circ\Delta^R$.

First, recall that $\mathfrak{sl}_2$ denotes the Lie algebra of the 2×2 traceless matrices. It has a basis formed by

$$H:=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad E:=\begin{pmatrix}0&1\\0&0\end{pmatrix}\quad\text{and}\quad F:=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

In order to define the Poisson structure, we will need the following.

Definition 3.2 The *classical r–matrices* $r^{\pm}\in\mathfrak{sl}_2^{\otimes 2}$ are the bivectors $r^+:=\frac{1}{2}H\otimes H+2E\otimes F$ and $r^-:=\frac{1}{2}H\otimes H+2F\otimes E$. Their symmetric part $\tau=\frac{1}{2}H\otimes H+E\otimes F+F\otimes E$ is the invariant bivector associated to the (suitably normalized) Killing form and we denote by $\bar{r}^+:=E\otimes F-F\otimes E=:-\bar{r}^-$ their skew-symmetric part.

The classical r–matrices satisfy the classical Yang–Baxter equation (see [Chari and Pressley 1994, Section 2.1; Drinfeld 1983] for details).

Notation 3.3 Given a a boundary arc of Σ, we write $\mathfrak{o}(a) = +$ if the $\mathfrak{o}$–orientation of a coincides with the orientation induced by the orientation of $\Sigma_{\mathcal{P}}$, and write $\mathfrak{o}(a) = -$ if the orientation are opposite.

3.1.1 The bigon Consider the bigon $\mathbb{B}$ and write $\mathfrak{o}(b_L) = \varepsilon_1$ and $\mathfrak{o}(b_R) = \varepsilon_2$.

Definition 3.4 The relative character variety of the bigon is $\mathcal{X}_{\mathrm{SL}_2}(\mathbb{B}) := \mathrm{SL}_2(\mathbb{C})$. Denote by

$$N = \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix}$$

the 2×2 matrix with coefficients in $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{B})]$. The Poisson bracket associated to $\mathfrak{o}$ is defined by

$$\{N \otimes N\}^{\varepsilon_1, \varepsilon_2} := \bar{r}^{\varepsilon_1}(N \otimes N) + (N \otimes N)\bar{r}^{\varepsilon_2}.$$

Here we used the classical notation $\{N \otimes N\}$ to denote the matrix defined by $\{N \otimes N\}_{\varepsilon\varepsilon'\mu\mu'} = \{x_{\varepsilon\varepsilon'}, x_{\mu\mu'}\}$ (see for instance [Chari and Pressley 1994, Section 2.2.A] for details on this notation).

Denote the Poisson variety $(\mathbb{C}[\mathrm{SL}_2], \{\cdot,\cdot\}^{\varepsilon_1,\varepsilon_2})$ by $\mathbb{C}[\mathrm{SL}_2]^{\varepsilon_1,\varepsilon_2}$. Note that $\{\cdot,\cdot\}^{\varepsilon_1,\varepsilon_2} = -\{\cdot,\cdot\}^{-\varepsilon_1,-\varepsilon_2}$. By [Korinman 2019, Lemma 4.1], the coproduct $\Delta\colon \mathbb{C}[\mathrm{SL}_2]^{\varepsilon_1,\varepsilon_2} \to \mathbb{C}[\mathrm{SL}_2]^{\varepsilon_1,\varepsilon} \otimes \mathbb{C}[\mathrm{SL}_2]^{-\varepsilon,\varepsilon_2}$ and the antipode $S\colon \mathbb{C}[\mathrm{SL}_2]^{\varepsilon_1,\varepsilon_2} \to \mathbb{C}[\mathrm{SL}_2]^{-\varepsilon_1,-\varepsilon_2}$ are Poisson morphisms. In particular, the Poisson brackets $\{\cdot,\cdot\}^{-,+}$ and $\{\cdot,\cdot\}^{+,-}$ are the only ones which endow $\mathrm{SL}_2(\mathbb{C})$ with a Poisson–Lie structure.

3.1.2 The triangle Consider the triangle $\mathbb{T}$ and fix an orientation $\mathfrak{o}$ of each of its three boundary arcs a, b and c. We will use the notation $s(\alpha) = t(\beta) := c$, $s(\gamma) = t(\alpha) := b$ and $s(\beta) = t(\gamma) := a$. Here, for instance, we think of α as an oriented path joining a point in $c = s(\alpha)$ (source) to a point in $b = t(\alpha)$ (target).

Definition 3.5 The relative character variety of the triangle is the affine variety

$$\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T}) := \{(M_\alpha, M_\beta, M_\gamma) \in \mathrm{SL}_2(\mathbb{C})^3 \mid M_\gamma M_\beta M_\alpha = \mathbb{1}\}.$$

Given $\delta \in \{\alpha, \beta, \gamma\}$, denote by

$$N_\delta := \begin{pmatrix} \delta(+,+) & \delta(+,-) \\ \delta(-,+) & \delta(-,-) \end{pmatrix}$$

the 2×2 matrix with coefficients in $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]$. The Poisson bracket $\{\cdot,\cdot\}^{\mathfrak{o}}$ is defined by the formulas

$$\begin{aligned}
&\{N_\delta \otimes N_\delta\}^{\mathfrak{o}} := \bar{r}^{\mathfrak{o}(s(\delta))}(N_\delta \otimes N_\delta) + (N_\delta \otimes N_\delta)\bar{r}^{\mathfrak{o}(t(\delta))}, \quad \delta \in \{\alpha, \beta, \gamma\}, \\
&\{N_\alpha \otimes N_\gamma\}^{\mathfrak{o}} := -(N_\alpha \otimes \mathbb{1}) r^{\mathfrak{o}(b)} (\mathbb{1} \otimes N_\gamma), \\
&\{N_\gamma \otimes N_\beta\}^{\mathfrak{o}} := -(N_\gamma \otimes \mathbb{1}) r^{\mathfrak{o}(a)} (\mathbb{1} \otimes N_\beta), \\
&\{N_\beta \otimes N_\alpha\}^{\mathfrak{o}} := -(N_\beta \otimes \mathbb{1}) r^{\mathfrak{o}(c)} (\mathbb{1} \otimes N_\alpha).
\end{aligned}$$

Note that, writing

$$S(N_\delta) := \begin{pmatrix} \delta(-,-) & -\delta(+,-) \\ -\delta(-,+) & \delta(+,+) \end{pmatrix},$$

the last expressions can be rewritten in the form

$$\{N_\alpha \otimes S(N_\gamma)\}^{\mathfrak{o}} = (N_\alpha \otimes S(N_\gamma)) r^{\mathfrak{o}(b)},$$
$$\{N_\gamma \otimes S(N_\beta)\}^{\mathfrak{o}} = (N_\gamma \otimes S(N_\beta)) r^{\mathfrak{o}(a)},$$
$$\{N_\beta \otimes S(N_\alpha)\}^{\mathfrak{o}} = (N_\beta \otimes S(N_\alpha)) r^{\mathfrak{o}(c)}.$$

Given a boundary arc $d \in \{a, b, c\}$, we define a left Hopf-comodule

$$\Delta^L_d \colon \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})] \to \mathbb{C}[\mathrm{SL}_2]^{(+\mathfrak{o}(d), -\mathfrak{o}(d))} \otimes \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})],$$
$$\begin{pmatrix} \Delta^L_d(\delta(+,+)) & \Delta^L_d(\delta(+,-)) \\ \Delta^L_d(\delta(-,+)) & \Delta^L_d(\delta(-,-)) \end{pmatrix} := \begin{cases} \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix} \otimes N_\delta & \text{if } s(\delta) = d, \\ \mathbb{1} \otimes N_\delta & \text{otherwise.} \end{cases}$$

Similarly, define a right Hopf-comodule $\Delta^R_d \colon \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})] \to \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})] \otimes \mathbb{C}[\mathrm{SL}_2]^{(-\mathfrak{o}(d), +\mathfrak{o}(d))}$ by

$$\begin{pmatrix} \Delta^R_d(\delta(+,+)) & \Delta^R_d(\delta(+,-)) \\ \Delta^R_d(\delta(-,+)) & \Delta^R_d(\delta(-,-)) \end{pmatrix} := \begin{cases} N_\delta \otimes \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix} & \text{if } t(\delta) = d, \\ N_\delta \otimes \mathbb{1} & \text{otherwise.} \end{cases}$$

By [Korinman 2019, Lemma 4.6], both Δ^L_d and Δ^R_d are Poisson morphisms.

3.1.3 The general case Let Σ be a punctured surface, Δ a topological triangulation of Σ, and $\mathfrak{o}_\Delta$ an orientation of each edge of Δ. For a face $\mathbb{T} \in F(\Delta)$, let $\mathfrak{o}_{\mathbb{T}}$ be the orientation of its boundary arcs given by $\mathfrak{o}_\Delta$. Equip the algebra $\bigotimes_{\mathbb{T} \in F(\Delta)} \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})]^{\mathfrak{o}_{\mathbb{T}}}$ with the Poisson bracket defined in Definition 3.5. Each inner edge $e \in \mathring{\mathscr{E}}(\Delta)$ lifts to two oriented boundary arcs in $\Sigma_\Delta := \bigsqcup_{\mathbb{T} \in F(\Delta)} \mathbb{T}$. We denote by e_L the lift of e whose orientation coincides with the induced orientation of Σ_Δ and by e_R the other lift. The comodule maps $\Delta^L_{e_L}$ and $\Delta^R_{e_R}$ induce the comodule maps

$$\Delta^L \colon \bigotimes_{\mathbb{T} \in F(\Delta)} \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})]^{\mathfrak{o}_{\mathbb{T}}} \to \Bigl(\bigotimes_{e \in \mathring{\mathscr{E}}(\Delta)} \mathbb{C}[\mathrm{SL}_2]^{-,+} \Bigr) \otimes \Bigl(\bigotimes_{\mathbb{T} \in F(\Delta)} \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})]^{\mathfrak{o}_{\mathbb{T}}} \Bigr),$$
$$\Delta^R \colon \bigotimes_{\mathbb{T} \in F(\Delta)} \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})]^{\mathfrak{o}_{\mathbb{T}}} \to \Bigl(\bigotimes_{\mathbb{T} \in F(\Delta)} \mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\mathbb{T})]^{\mathfrak{o}_{\mathbb{T}}} \Bigr) \otimes \Bigl(\bigotimes_{e \in \mathring{\mathscr{E}}(\Delta)} \mathbb{C}[\mathrm{SL}_2]^{-,+} \Bigr).$$

Definition 3.6 The relative character variety $\mathscr{X}_{\mathrm{SL}_2}(\Sigma)$ is the affine variety whose algebra of regular functions is the kernel of $\Delta^L - \sigma \circ \Delta^R$.

Lemma 3.7 [Korinman 2019, Theorem 1.4] *As a Poisson variety, $\mathscr{X}_{\mathrm{SL}_2}(\Sigma)$ only depends, up to canonical isomorphism, on the marked surface Σ and the orientation $\mathfrak{o}$ of the boundary arcs (so does not depend on the triangulation Δ or on $\mathfrak{o}_\Delta$).*

We denote by $\{\cdot, \cdot\}^{\mathfrak{o}}$ the Poisson bracket on $\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\Sigma)]$. More precisely, in [Korinman 2019], we endow the variety $\mathscr{X}_{\mathrm{SL}_2}(\Sigma) := \mathscr{R}_{\mathrm{SL}_2}(\Sigma) /\!\!/ \mathscr{G}$ (which only depends on Σ) with a Poisson structure, given by a generalization of Goldman formula, which only depends on $\mathfrak{o}$. We then construct a splitting morphism i^Δ and prove in [Korinman 2019, Theorem 1.4] that we have the exact sequence (29), thus $\mathscr{X}_{\mathrm{SL}_2}(\Sigma)$ can be alternatively defined using Definition 3.6.

Moreover when Σ is closed, the Poisson variety $\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})$ is canonically isomorphic to the "classical" (Culler–Shalen) character variety with its Goldman Poisson structure [Korinman 2019, Theorem 1.1].

3.2 Relation between relative character varieties and stated skein algebras

The goal of this subsection is to prove Theorem 1.3 which we recall here for the reader's convenience:

Theorem 3.8 *Suppose that $\underline{\Sigma}$ has a topological triangulation Δ. Let $\mathfrak{o}_\Delta$ be an orientation of the edges of Δ and $\mathfrak{o}$ be the induced orientation of the boundary arcs of $\underline{\Sigma}$. There exists an isomorphism of Poisson algebras*

$$\Psi^{(\Delta,\mathfrak{o}_\Delta)}\colon (\mathcal{S}_{+1}(\underline{\Sigma}), \{\cdot,\cdot\}^s) \xrightarrow{\cong} (\mathbb{C}[\mathscr{X}_{\mathrm{SL}_2}(\underline{\Sigma})], \{\cdot,\cdot\}^{\mathfrak{o}}).$$

Moreover, the above isomorphism exists for small punctured surfaces (see Definition 2.8), for which it only depends on $\mathfrak{o}$.

We first prove this theorem for the bigon and the triangle, then we prove the general case using a topological triangulation.

3.2.1 The case of the bigon Let

$$M := \begin{pmatrix} \alpha_{++} & \alpha_{+-} \\ \alpha_{-+} & \alpha_{--} \end{pmatrix}, \quad N := \begin{pmatrix} x_{++} & x_{+-} \\ x_{-+} & x_{--} \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be three matrices with coefficients in $\mathcal{S}_{+1}(\mathbb{B})$, $\mathbb{C}[\mathrm{SL}_2]$ and $\mathbb{C}$, respectively.

Lemma 3.9 *For $\varepsilon_1, \varepsilon_2 \in \{-,+\}$, there is a Poisson isomorphism*

$$\Psi^{\varepsilon_1,\varepsilon_2}\colon (\mathcal{S}_{+1}(\mathbb{B}), \{\cdot,\cdot\}^s) \xrightarrow{\cong} \mathbb{C}[\mathrm{SL}_2]^{\varepsilon_1,\varepsilon_2}$$

defined by

$$\Psi^{\varepsilon_1,\varepsilon_2}(M) := \begin{cases} N & \text{if } (\varepsilon_1,\varepsilon_2) = (-,+), \\ CNC & \text{if } (\varepsilon_1,\varepsilon_2) = (+,-), \\ -CN & \text{if } (\varepsilon_1,\varepsilon_2) = (+,+), \\ -NC & \text{if } (\varepsilon_1,\varepsilon_2) = (-,-). \end{cases}$$

Proof That $\Psi^{\varepsilon_1,\varepsilon_2}$ is an isomorphism of algebras follows from the fact that $\det(C) = 1$. Let us see the compatibility of $\Psi^{\varepsilon_1,\varepsilon_2}$ with the Poisson structures. For $(\varepsilon_1,\varepsilon_2) = (-,+)$, this follows from a direct comparison of Definition 3.4 and Example 2.54. Indeed,

$$\begin{aligned} \{N \otimes N\}^{-,+} &= \bar{r}^-(N \otimes N) + (N \otimes N)\bar{r}^+ \\ &= (F \otimes E - E \otimes F)(N \otimes N) + (N \otimes N)(E \otimes F - F \otimes E) \\ &= \begin{pmatrix} 0 & x_{++} \\ 0 & x_{-+} \end{pmatrix} \otimes \begin{pmatrix} x_{+-} & 0 \\ x_{--} & 0 \end{pmatrix} - \begin{pmatrix} x_{+-} & 0 \\ x_{--} & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & x_{++} \\ 0 & x_{-+} \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 & 0 \\ x_{++} & x_{+-} \end{pmatrix} \otimes \begin{pmatrix} x_{-+} & x_{--} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} x_{-+} & x_{--} \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ x_{++} & x_{+-} \end{pmatrix}. \end{aligned}$$

We recover the formulas computed in Example 2.54. For $(\varepsilon_1,\varepsilon_2)=(+,+)$, we prove that the isomorphism $\varphi\colon \mathbb{C}[\mathrm{SL}_2]^{-,+}\xrightarrow{\cong}\mathbb{C}[\mathrm{SL}_2]^{+,+}$ given by $\varphi:=\Psi^{+,+}\circ(\Psi^{-,+})^{-1}$, is a Poisson morphism. Note that $\varphi(N)=-CN$ and that $(C\otimes C)\bar{r}^{\varepsilon}=\bar{r}^{-\varepsilon}(C\otimes C)$. It follows that

$$\begin{aligned}\{\varphi(N)\otimes\varphi(N)\}^{+,+}&=\bar{r}^{+}(CN\otimes CN)+(CN\otimes CN)\bar{r}^{+}\\&=(C\otimes C)(\bar{r}^{-}(N\otimes N)+(N\otimes N)\bar{r}^{+})=\varphi^{\otimes 2}(\{N\otimes N\}^{-,+}),\end{aligned}$$

which proves the claim. The two remaining cases for $(\varepsilon_1,\varepsilon_2)$ are proved similarly. □

3.2.2 The case of the triangle

For $\delta\in\{\alpha,\beta,\gamma\}$, let

$$M_\delta:=\begin{pmatrix}\delta_{++} & \delta_{+-}\\ \delta_{-+} & \delta_{--}\end{pmatrix}\quad\text{and}\quad N_\delta:=\begin{pmatrix}\delta(+,+) & \delta(+,-)\\ \delta(-,+) & \delta(-,-)\end{pmatrix}$$

be two matrices with coefficients in $\mathcal{S}_{+1}(\mathbb{T})$ and $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]$, respectively.

Lemma 3.10 *There is a Poisson isomorphism $\Psi^{\mathfrak{o}}\colon(\mathcal{S}_{+1}(\mathbb{T}),\{\cdot,\cdot\}^s)\xrightarrow{\cong}\big(\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})],\{\cdot,\cdot\}^{\mathfrak{o}}\big)$ defined by*

$$\Psi^{\mathfrak{o}}(M_\delta):=\begin{cases}N_\delta & \text{if } (\mathfrak{o}(s(\alpha)),\mathfrak{o}(t(\alpha)))=(-,+),\\ CN_\delta C & \text{if } (\mathfrak{o}(s(\alpha)),\mathfrak{o}(t(\alpha)))=(+,-),\\ -CN_\delta & \text{if } (\mathfrak{o}(s(\alpha)),\mathfrak{o}(t(\alpha)))=(+,+),\\ -N_\delta C & \text{if } (\mathfrak{o}(s(\alpha)),\mathfrak{o}(t(\alpha)))=(-,-),\end{cases}$$

for each $\delta\in\{\alpha,\beta,\gamma\}$. Moreover, if $d\in\{a,b,c\}$ is a boundary arc of $\mathbb{T}$, the following diagrams commute:

$$\begin{tikzcd}\mathcal{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta_d^L"] \arrow[d, "\cong"', "\Psi^{\mathfrak{o}}"] & \mathcal{S}_{+1}(\mathbb{B})\otimes\mathcal{S}_{+1}(\mathbb{T}) \arrow[d, "\cong"', "\Psi^{\mathfrak{o}(d),-\mathfrak{o}(d)}\otimes\Psi^{\mathfrak{o}}"]\\ \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \arrow[r, "\Delta_d^L"] & \mathbb{C}[\mathrm{SL}_2]\otimes\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]\end{tikzcd}\qquad\begin{tikzcd}\mathcal{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta_d^R"] \arrow[d, "\cong"', "\Psi^{\mathfrak{o}}"] & \mathcal{S}_{+1}(\mathbb{T})\otimes\mathcal{S}_{+1}(\mathbb{B}) \arrow[d, "\cong"', "\Psi^{\mathfrak{o}}\otimes\Psi^{-\mathfrak{o}(d),\mathfrak{o}(d)}"]\\ \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \arrow[r, "\Delta_d^R"] & \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]\otimes\mathbb{C}[\mathrm{SL}_2]\end{tikzcd}$$

Proof That $\Psi^{\mathfrak{o}}$ is an algebra morphism follows from Lemma 2.29. For $\delta\in\{\alpha,\beta,\gamma\}$, the equality $(\Psi^{\mathfrak{o}})^{\otimes 2}(\{\delta_{\varepsilon\varepsilon'},\delta_{\mu\mu'}\}^{\mathfrak{o}})=\{\Psi^{\mathfrak{o}}(\delta_{\varepsilon\varepsilon'}),\Psi^{\mathfrak{o}}(\delta_{\mu\mu'})\}^s$ follows from the same computation that the proof of Lemma 3.9. For $\mathfrak{o}(a)=\mathfrak{o}(b)=\mathfrak{o}(c)=+$,

$$\begin{aligned}\{N_\alpha\otimes N_\gamma\}^{\mathfrak{o}}&=-(N_\alpha\otimes\mathbb{1})\Big(\frac{1}{2}H\otimes H+2E\otimes F\Big)(\mathbb{1}\otimes N_\gamma)\\&=-\frac{1}{2}\begin{pmatrix}\alpha(+,+) & -\alpha(+,-)\\ \alpha(-,+) & -\alpha(-,-)\end{pmatrix}\otimes\begin{pmatrix}\gamma(+,+) & \gamma(+,-)\\ -\gamma(-,+) & -\gamma(-,-)\end{pmatrix}\\&\qquad\qquad -2\begin{pmatrix}0 & \alpha(+,+)\\ 0 & \alpha(-,+)\end{pmatrix}\otimes\begin{pmatrix}0 & 0\\ \gamma(+,+) & \gamma(+,-)\end{pmatrix}.\end{aligned}$$

We recover the formulas of Example 2.55; hence $(\Psi^{\mathfrak{o}})^{\otimes 2}(\{\alpha_{\varepsilon\varepsilon'},\gamma_{\mu\mu'}\}^{\mathfrak{o}})=\{\Psi^{\mathfrak{o}}(\alpha_{\varepsilon\varepsilon'}),\Psi^{\mathfrak{o}}(\gamma_{\mu\mu'})\}^s$. We get similar formulas by permuting cyclically the arcs γ, β and α. This proves that $\Psi^{\mathfrak{o}}$ is a Poisson morphism when $\mathfrak{o}(a)=\mathfrak{o}(b)=\mathfrak{o}(c)=+$. For another choice $\mathfrak{o}'$ of orientation of the boundary arcs, we prove that $\Psi^{\mathfrak{o}'}$ is Poisson by showing that the isomorphism $\Psi^{\mathfrak{o}'}\circ(\Psi^{\mathfrak{o}})^{-1}$ is Poisson. This follows from a similar computation to the one in the proof of Lemma 3.9 by using the fact that $(C\otimes C)r^{\varepsilon}=r^{-\varepsilon}(C\otimes C)$. The fact that the two diagrams in the lemma commute follows from a straightforward computation. $\square$

3.2.3 The general case: proof of Theorem 1.3 Consider a topological triangulation Δ of a punctured surface $\underline{\Sigma}$, together with a choice $\mathfrak{o}_\Delta$ of orientation of its edges. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{S}_{+1}(\underline{\Sigma}) \arrow[r, "i^\Delta"] \arrow[d, dotted, "\exists!\Psi^{(\Delta,\mathfrak{o}_\Delta)}"', "\cong"] & \otimes_{\mathbb{T}}\mathcal{S}_{+1}(\mathbb{T}) \arrow[r, "\Delta^L-\sigma\circ\Delta^R"] \arrow[d, "\otimes_{\mathbb{T}}\Psi^{\mathfrak{o}\mathbb{T}}"', "\cong"] & (\otimes_e\mathcal{S}_{+1}(\mathbb{B}))\otimes(\otimes_{\mathbb{T}}\mathcal{S}_{+1}(\mathbb{T})) \arrow[d, "(\otimes_e\Psi^{-,+})\otimes(\otimes_{\mathbb{T}}\Psi^{\mathfrak{o}(\mathbb{T})})"', "\cong"] \\
0 \arrow[r] & \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})] \arrow[r, "i^\Delta"] & \otimes_{\mathbb{T}}\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})] \arrow[r, "\Delta^L-\sigma\circ\Delta^R"] & (\otimes_e\mathbb{C}[\mathrm{SL}_2]^{-,+})\otimes(\otimes_{\mathbb{T}}\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})])
\end{tikzcd}$$

In this diagram, both lines are exact and all morphisms are Poisson by Lemma 2.47 and [Korinman 2019]; hence there exists a unique Poisson isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}\colon(\mathcal{S}_{+1}(\underline{\Sigma}),\{\cdot,\cdot\}^s)\xrightarrow{\cong}(\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})],\{\cdot,\cdot\}^{\mathfrak{o}})$ induced by restriction of $\otimes_{\mathbb{T}}\Psi^{\mathfrak{o}(\mathbb{T})}$. This concludes the proof.

3.3 Relative spin structures and explicit formulas

The goal of this subsection is to give an explicit formula for the morphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$, when evaluated on the generators of $\mathcal{S}_{+1}(\underline{\Sigma})$. A key point is to have a global method to compute some signs that depend on the combinatorial data $(\Delta,\mathfrak{o}_\Delta)$. We provide such a method by introducing the notion of relative spin structure, which gives a geometric interpretation these signs. We end the section by relating the $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ with the morphism of [Costantino and Lê 2022, Theorem 8.12].

3.3.1 Relative spin structures Since the classical identifications between skein algebras of closed punctured surfaces and character varieties are indexed by spin structures, it is natural to expect that the combinatorial data $(\Delta,\mathfrak{o}_\Delta)$ indexing the isomorphism of Theorem 1.3 encode a generalization of the notion of spin structures which would have a good behavior for the operation of gluing boundary arcs together. Before defining this notion, we introduce some notation.

Notation 3.11 (1) In this subsection, $\underline{\Sigma}=(\Sigma,\mathcal{P})$ will denote a triangulable punctured surface, $\mathfrak{o}$ an orientation of its boundary arcs and $(\Delta,\mathfrak{o}_\Delta)$ a combinatorial data, and we equip $\Sigma_{\mathcal{P}}$ with a Riemannian structure compatible with the orientation. For each boundary arc a, we fix a point $v_a\in a$. If $\partial\Sigma\neq\varnothing$, we write $\mathbb{V}:=\{v_a\}_a$ where a runs through the set of boundary arcs. If Σ is closed, we fix an arbitrarily point v_a in each connected component a of $\Sigma_{\mathcal{P}}$ and write $\mathbb{V}:=\{v_a\}_a$.

(2) Let $\pi\colon U\Sigma_{\mathcal{P}}\to\Sigma_{\mathcal{P}}$ denote the unitary tangent bundle. For $\vec{v}=(v,u)\in U\Sigma_{\mathcal{P}}$, we denote by $-\vec{v}=(v,-u)$ the vector with opposite orientation. Let $\theta_{\vec{v}}^{1/2}\colon\vec{v}\to-\vec{v}$ be the class in $\Pi_1(U\Sigma_{\mathcal{P}})$ of a path making a half-twist in the fiber over $\pi(\vec{v})$ in the direction given by the orientation and write $\theta_{\vec{v}}:=\theta_{\vec{v}}^{1/2}\theta_{-\vec{v}}^{1/2}$. For simplicity, for a path $\alpha\colon\vec{v}_1\to\vec{v}_2$, we will write $\theta^{1/2}\alpha$ and $\alpha\theta^{1/2}$ instead of $\theta_{-\vec{v}_1}^{1/2}\alpha$ and $\alpha\theta_{\vec{v}_2}^{1/2}$ with no confusion possible. When $\partial\Sigma\neq\varnothing$, for each boundary arc a, we denote by $\vec{v}_a\in U\Sigma_{\mathcal{P}}$ the lift of v_a pointing in the direction of $\mathfrak{o}$. When Σ is closed, we fix an arbitrarily lift $\vec{v}_a$ of each v_a. We write $\widehat{\mathbb{V}}_+:=\{\vec{v}_a\}_a$ and $\widehat{\mathbb{V}}:=\{\vec{v}_a,-\vec{v}_a\}_a$.

Definition 3.12 A *relative spin structure* on $\underline{\Sigma}$ is a functor $W \in \mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+), \mathbb{Z}/2\mathbb{Z})$ such that $W(\theta_{\vec{v}}) = 1$ for all $\vec{v} \in \widehat{\mathbb{V}}_+$. We denote by $\mathrm{Spin}(\underline{\Sigma})$ the set of relative spin structures on $\underline{\Sigma}$.

Remark 3.13 When Σ is closed and connected, an element $W \in \mathrm{Spin}(\underline{\Sigma})$ is a group morphism $W\colon \pi_1(U\Sigma_{\mathcal{P}}, \vec{v}_0^+) \to \mathbb{Z}/2\mathbb{Z}$ such that $W(\theta_{\vec{v}_0^+}) = 1$. Since $\mathbb{Z}/2\mathbb{Z}$ is abelian, such a morphism is equivalent to a group morphism $\underline{W}\colon \mathrm{H}_1(U\Sigma_{\mathcal{P}}, \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ satisfying $W([\theta]) = 1$. Such a morphism $\underline{W}$ defines a regular double covering $\widetilde{U}$ of $U\Sigma_{\mathcal{P}}$ such that the covering on each fiber is nontrivial. Since $\mathrm{Spin}(2)$ is the only nontrivial double covering of $\mathrm{SO}(2)$, the space $\widetilde{U}$ is the total space of a $\mathrm{Spin}(2)$ fiber bundle over $\Sigma_{\mathcal{P}}$ lifting the bundle of orthogonal frames induced by the metric; hence it defines a spin structure. There is actually a one-to-one correspondence between isomorphism classes of spin structures and such morphisms $\underline{W}$ (see [Milnor 1963] for details). Therefore a relative spin structure is the same as a "standard" spin structure in the closed case. When the surface has nonempty boundary, an element $W \in \mathrm{Spin}(\underline{\Sigma})$ still induces a group morphism $\underline{W}$, thus a spin structure. However, the functor W contains more information than $\underline{W}$ which permits to "glue" relative spin structures together.

Let a and b be two distinct boundary arcs of $\underline{\Sigma}$ and denote by $p\colon \Sigma_{\mathcal{P}} \to \Sigma_{\mathcal{P}}|_{a\#b}$ the projection. Write $c := p(a) = p(b)$. We assume that

(1) the restriction $p\colon \Sigma_{\mathcal{P}} \setminus (a \cup b) \to \Sigma_{\mathcal{P}}|_{a\#b} \setminus c$ is an isometry,

(2) the restriction $p\colon a \to c$ and $p\colon b \to c$ are isometries, and

(3) the orientations $\mathfrak{o}$ of a and b coincide when gluing the arcs and $p(v_a) = p(v_b) =: v_c$.

The projection induces a lift $\vec{v}_c \in U\Sigma_{\mathcal{P}}|_{a\#b}$ of v_c and a functor

$$p_*\colon \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+) \to \Pi_1(U\Sigma_{\mathcal{P}}|_{a\#b}, \widehat{\mathbb{V}}_+^{a\#b} \cup \{\vec{v}_c\}).$$

Lemma 3.14 *For $W \in \mathrm{Spin}(\Sigma)$, there exists a unique $W|_{a\#b} \in \mathrm{Spin}(\Sigma|_{a\#b})$ such that*

$$W|_{a\#b}(p_*(\alpha)) = W(\alpha)$$

for all $\alpha \in \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+)$.

Proof Note that the image of p_* generates the groupoid $\Pi_1(U\Sigma_{\mathcal{P}}|_{a\#b}, \widehat{\mathbb{V}}_+^{a\#b} \cup \{\vec{v}_c\})$ in the sense that any path $\alpha \in \Pi_1(U\Sigma_{\mathcal{P}}|_{a\#b}, \widehat{\mathbb{V}}_+^{a\#b} \cup \{\vec{v}_c\})$ can be written as a composition $\alpha = p_*(\alpha_1) \cdots p_*(\alpha_n)$ for some $\alpha_i \in \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+)$. Hence for $W \in \mathrm{Spin}(\underline{\Sigma})$, there exists a unique functor

$$\widetilde{W}\colon \Pi_1(U\Sigma_{\mathcal{P}}|_{a\#b}, \widehat{\mathbb{V}}_+^{a\#b} \cup \{\vec{v}_c\}) \to \mathbb{Z}/2\mathbb{Z}$$

such that $\widetilde{W}(\pi_*(\alpha)) = W(\alpha)$ for all $\alpha \in \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+)$, and $W|_{a\#b}$ has to be the restriction of $\widetilde{W}$ to the full subcategory $\Pi_1(U\Sigma_{\mathcal{P}}|_{a\#b}, \widehat{\mathbb{V}}_+^{a\#b})$. □

Note that the map $r_{a\#b}\colon \mathrm{Spin}(\underline{\Sigma}) \to \mathrm{Spin}(\underline{\Sigma}|_{a\#b})$ sending W to $W|_{a\#b}$ is surjective but not injective. Indeed when lifting a functor in $\mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+), \mathbb{Z}/2\mathbb{Z})$ to a functor in $\mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+ \cup \{\vec{v}_c\}), \mathbb{Z}/2\mathbb{Z})$ there is a $\mathbb{Z}/2\mathbb{Z}$ ambiguity. Note also that if a, b, c and d are four distinct boundary arcs, one obviously has $r_{a\#b} \circ r_{c\#d} = r_{c\#d} \circ r_{a\#b}$. In particular, once some combinatorial data $(\Delta, \mathfrak{o}_\Delta)$ of $\underline{\Sigma}$ are fixed, any relative spin structure on $\underline{\Sigma}$ can be obtained by gluing some relative spin structure on each face of the triangulation.

3.3.2 Lifts of embedded curves and the function w Let us call *embedded arc* a smooth embedding $\alpha\colon [0,1] \to \Sigma_{\mathcal{P}}$ such that $\alpha(0), \alpha(1) \in \partial\Sigma_{\mathcal{P}}$. To any embedded arc and any simple closed curve, we associate two lifts in $U\Sigma_{\mathcal{P}}$ as follows.

For α an embedded arc oriented from the boundary arc a to the boundary arc b, we isotope α (in the class of embedded arc) such that $\alpha(0) = v_a$, $\alpha(1) = v_b$, the vectors $\alpha'(0)$ and $\alpha'(1)$ are tangent to a and b, and such that $\alpha'(0)$ points in the direction of a opposite to the orientation induced by the orientation of $\Sigma_{\mathcal{P}}$ and $\alpha'(1)$ points in the direction of b induced by the orientation of $\Sigma_{\mathcal{P}}$. The *positive lift* of α is the homotopy class $\hat{\alpha}^+ \in \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}})$ of the continuous map $t \mapsto (\alpha(t), \alpha'(t)/\|\alpha'(t)\|)$.

For v a point in a boundary arc a, we write $\mathfrak{o}(v) = 0$ if the orientation of a agrees with the induced orientation of $\Sigma_{\mathcal{P}}$ and $\mathfrak{o}(v) = 1$ otherwise. The $\mathfrak{o}$–*lift* $\hat{\alpha}^{\mathfrak{o}} \in \Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}_+)$ is defined by the formula

$$\hat{\alpha}^+ = (\theta^{1/2})^{1-\mathfrak{o}(s(\alpha))}\hat{\alpha}^{\mathfrak{o}}(\theta^{1/2})^{\mathfrak{o}(t(\alpha))}. \tag{30}$$

Let γ be a smooth embedded curve and $v \in \mathbb{V}$. We define $\hat{\gamma}_v^+$ as the as the homotopy class of a map $t \mapsto \big(\beta(t), \beta'(t)/\|\beta'(t)\|\big)$ where β is a smooth immersion $\beta\colon [0,1] \to \Sigma_{\mathcal{P}}$ which is isotopic to γ such that $\beta(0) = v = \beta(1)$ and $\beta'(0)$ points in the direction induced by the orientation of the surface for $\hat{\gamma}_v^+$. Similarly, we define $\hat{\gamma}_v^{\mathfrak{o}}$ as the homotopy class of a map $t \mapsto \big(\beta(t), \beta'(t)/\|\beta'(t)\|\big)$ where this time $\beta'(0)$ points in the direction of $\mathfrak{o}$ for $\hat{\gamma}_v^{\mathfrak{o}}$. If Σ is closed and γ is in a connected component b, we impose that $\hat{\gamma}_v^+ = \hat{\gamma}_v^{\mathfrak{o}}$ is defined from an immersion β such that $(\beta(0), \beta'(0)) = v_b$.

Notation 3.15 For $W \in \mathrm{Spin}(\underline{\Sigma})$ and α an embedded arc, we write $w(\alpha) := W(\hat{\alpha}^{\mathfrak{o}}) \in \mathbb{Z}/2\mathbb{Z}$. For γ a closed curve we write $w(\gamma) := W(\hat{\gamma}_v^{\mathfrak{o}})$.

Remark 3.16 The value $w(\gamma)$ associated to a closed curve is obviously independent of the choice of the point v. Moreover, as noted in Remark 3.13, the value $W(\hat{\gamma})$ only depends on the homology class $[\hat{\gamma}^{\mathfrak{o}}] \in \mathrm{H}_1(U\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z})$ and is closely related to the Johnson quadratic form as follows. Let $\{\gamma_i\}_{i=1,\dots,n}$ be a collection of simple closed curves. Johnson [1980, Theorem 1.A] proved that the class

$$y := \sum_{i=1}^{n} [\hat{\gamma}_i^{\mathfrak{o}}] + n[\theta] \in \mathrm{H}_1(U\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z})$$

only depends on the homology class of $x := \sum_{i=1}^n [\gamma_i] \in \mathrm{H}_1(\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z})$; hence the assignation $x \mapsto y$ defines a map (not a morphism) $\mathrm{H}_1(\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}_1(U\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z})$. Moreover, for a (relative) spin structure W, Johnson [1980, Theorem 1.B] proved that the map $\omega\colon \mathrm{H}_1(\Sigma_{\mathcal{P}}; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ defined by $\omega\big(\sum_{i=1}^n [\gamma_i]\big) := n + \sum_{i=1}^n w([\gamma_i]) \pmod 2$ satisfies the relation

$$\omega([\alpha + \beta]) = \omega([\alpha]) + \omega([\beta]) + \langle [\alpha], [\beta] \rangle;$$

hence ω is a quadratic form for $(\mathrm{H}_1(\Sigma_{\mathcal{P}};\mathbb{Z}/2\mathbb{Z}),\langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ represents the intersection form. Thus the values $w(\gamma)$ in Notation 3.15 are related to the Johnson quadratic form of the underlying spin structure by $\omega([\gamma]) = w(\gamma)+1 \pmod 2$.

3.3.3 Relative spin structures associated to combinatorial data In order to assign a relative spin structure to some combinatorial data $(\Delta,\mathfrak{o}_\Delta)$ in a canonical way, we need to assign to each triangle $\mathbb{T}$, equipped with an orientation $\mathfrak{o}_{\mathbb{T}}$ of its boundary arcs, a canonical relative spin structure and then glue the triangles along their faces. Let α, β and γ be the three paths in Figure 3 which generate the groupoid $\Pi_1(\mathbb{T},\mathbb{V})$ with relation $\gamma\beta\alpha=1$. Note that for any choice of $\mathfrak{o}_{\mathbb{T}}$, one has the relation $\hat{\gamma}^{\mathfrak{o}_{\mathbb{T}}}\hat{\beta}^{\mathfrak{o}_{\mathbb{T}}}\hat{\alpha}^{\mathfrak{o}_{\mathbb{T}}}=\theta^{-2}$. Hence a relative spin structure W on $\mathbb{T}$ is described by three elements $W(\hat{\alpha}^{\mathfrak{o}_{\mathbb{T}}}), W(\hat{\beta}^{\mathfrak{o}_{\mathbb{T}}}), W(\hat{\gamma}^{\mathfrak{o}_{\mathbb{T}}})\in\mathbb{Z}/2\mathbb{Z}$ such that $W(\hat{\alpha}^{\mathfrak{o}_{\mathbb{T}}})+W(\hat{\beta}^{\mathfrak{o}_{\mathbb{T}}})+W(\hat{\gamma}^{\mathfrak{o}_{\mathbb{T}}})=0$. Therefore there exist four different relative spin structures on $\mathbb{T}$.

Definition 3.17 The *distinguished* relative spin structure on $\mathbb{T}$ is the relative spin structure W such that $W(\hat{\alpha}^{\mathfrak{o}_{\mathbb{T}}})=W(\hat{\beta}^{\mathfrak{o}_{\mathbb{T}}})=W(\hat{\gamma}^{\mathfrak{o}_{\mathbb{T}}})=0$. For $\underline{\Sigma}$ a punctured surface with combinatorial data $(\Delta,\mathfrak{o}_\Delta)$, we associate a relative spin structure $W^{(\Delta,\mathfrak{o}_\Delta)}\in\mathrm{Spin}(\underline{\Sigma})$ by gluing together the distinguished spin structures on the faces of the triangulation.

Note that the distinguished relative spin structure W on $\mathbb{T}$ satisfies $w(\alpha)=w(\beta)=w(\gamma)=0$ and $w(\alpha^{-1})=w(\beta^{-1})=w(\gamma^{-1})=1$.

Remark 3.18 Since we associate to each face a specific (named distinguished) relative spin structure, there is no reason to believe that every spin structure on $\Sigma_{\mathcal{P}}$ can be associated to some combinatorial data. Moreover we will not investigate under which condition two combinatorial data induce the same relative spin structure. Novak and Runkel [2015] showed that any spin structure on a surface can be encoded by the combinatorial data consisting in a triangulation (with no degenerate face), an orientation of the edges and a choice of distinguished vertex for each face. Moreover they proved that two such combinatorial data induce the same spin structure if and only if they can be related by a sequence of elementary moves. It would be interesting to compare their approach to Definition 3.17.

We now state an explicit formula for the values $w(\alpha)$ associated to a relative spin structure $W^{(\Delta,\mathfrak{o}_\Delta)}$. For each edge $e\in\mathcal{E}(\Delta)$, fix a point $v_e\in e$ and let $\mathbb{V}^\Delta=\{v_e\}_{e\in\mathcal{E}(\Delta)}$. When $\partial\Sigma\neq\varnothing$, we assume that $\mathbb{V}^\Delta\cap\partial\Sigma_{\mathcal{P}}=\mathbb{V}$. When Σ is closed, we assume that $\mathbb{V}\subset\mathbb{V}^\Delta$. Let $\vec{v}_e\in U\Sigma_{\mathcal{P}}$ be the lift of v_e oriented in the direction of $\mathfrak{o}_\Delta$ and set $\widehat{\mathbb{V}}^\Delta_+ := \{\vec{v}_e \mid e\in\mathcal{E}(\Delta)\}$ and $\widehat{\mathbb{V}}^\Delta := \{\vec{v}_e,-\vec{v}_e \mid e\in\mathcal{E}(\Delta)\}$. Note that the set

$$\widehat{\mathbb{G}}^\Delta := \{(\hat{\alpha}^{\mathfrak{o}}_{\mathbb{T}})^{\pm1},(\hat{\beta}^{\mathfrak{o}}_{\mathbb{T}})^{\pm1},(\hat{\gamma}^{\mathfrak{o}}_{\mathbb{T}})^{\pm1} \mid \mathbb{T}\in F(\Delta)\}$$

generates the groupoid $\Pi_1(U\Sigma_{\mathcal{P}},\widehat{\mathbb{V}}^\Delta_+)$. By definition of the gluing operation, the functor $W^{(\Delta,\mathfrak{o}_\Delta)}$ is the restriction of the functor $\widetilde{W}\in\mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}},\widehat{\mathbb{V}}^\Delta_+),\mathbb{Z}/2\mathbb{Z})$ characterized by

$$\widetilde{W}(\hat{\alpha}^{\mathfrak{o}_{\mathbb{T}}}_{\mathbb{T}})=\widetilde{W}(\hat{\beta}^{\mathfrak{o}_{\mathbb{T}}}_{\mathbb{T}})=\widetilde{W}(\hat{\gamma}^{\mathfrak{o}_{\mathbb{T}}}_{\mathbb{T}})=0$$

for every face $\mathbb{T}$ and $\widetilde{W}(\theta_{\vec{v}})=1$ for any $\vec{v}\in\hat{\mathbb{V}}_+^{\Delta}$. Set $\mathbb{G}^{\Delta}:=\pi(\hat{\mathbb{G}}_+^{\Delta})=\{\alpha_{\mathbb{T}}^{\pm 1},\beta_{\mathbb{T}}^{\pm 1},\gamma_{\mathbb{T}}^{\pm 1};\mathbb{T}\in F(\Delta)\}$ and for $\delta\in\mathbb{G}^{\Delta}$ a path in $\mathbb{T}$, write $w(\delta):=\widetilde{W}(\hat{\delta}^{\mathfrak{o}_{\mathbb{T}}})$. Hence $w(\delta)=0$ if $\delta=\alpha_{\mathbb{T}}$, $\beta_{\mathbb{T}}$ or $\gamma_{\mathbb{T}}$ and $w(\delta)=1$ if $\delta=\alpha_{\mathbb{T}}^{-1}$, $\beta_{\mathbb{T}}^{-1}$ or $\gamma_{\mathbb{T}}^{-1}$.

Let α be either an embedded arc or a closed curve and choose a decomposition

$$\alpha=\alpha_1\cdots\alpha_n,\quad \alpha_i\in\mathbb{G}^{\Delta}, \tag{31}$$

such that either α_i and α_{i+1} lie in different faces $\mathbb{T}_i\neq\mathbb{T}_{i+1}$ of Δ, or $\mathbb{T}_i=\mathbb{T}_{i+1}$ is a degenerate triangle, with two boundary arcs glued together to give an arc c in $\Sigma_{\mathcal{P}}$, and $\alpha_i\alpha_{i+1}$ crosses $c=t(\alpha_i)=s(\alpha_{i+1})$ transversally. In the above statement, the indices i are taken in $\mathbb{Z}/n\mathbb{Z}$ when α is a closed curve. Note that such a decomposition is obtained by isotoping α transversally with minimal intersection to the edges of the triangulation, and then cutting α along the edges. For $(\mathbb{T},\mathfrak{o}_{\mathbb{T}})$ a triangle with oriented edges, a an edge and $v_a\in a$, recall that we write $\mathfrak{o}_{\mathbb{T}}(v_a)=0$ if the orientation of a corresponds to the orientation induced by the orientation of $\mathbb{T}$ and write $\mathfrak{o}_{\mathbb{T}}(v_a)=+1$ otherwise.

Lemma 3.19 *The function w associated to the relative spin structure $W^{(\Delta,\mathfrak{o}_\Delta)}$ is characterized by the formula*

$$w(\alpha)=\begin{cases}\sum_{i=1}^n w(\alpha_i)+\sum_{i=1}^{n-1}\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i)) \pmod 2 & \textit{if } \alpha \textit{ is an embedded arc,}\\ \sum_{i=1}^n w(\alpha_i)+\sum_{i=1}^{n}\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i)) \pmod 2 & \textit{if } \alpha \textit{ is a closed curve.}\end{cases}$$

Proof First note that for the positive lifts,

$$\hat{\alpha}^+=\hat{\alpha}_1^+\cdots\hat{\alpha}_n^+.$$

This equality follows from the fact that the embedded curve chosen to represent $\hat{\alpha}^+$ can be isotoped such that it crosses tangentially the edges of Δ in such a way that, when cutting along the edges, one obtains the composition $\hat{\alpha}_1^+\cdots\hat{\alpha}_n^+$. Note also that this equality is essentially [Costantino and Lê 2022, Proposition 8.11]. Recall from (30) that $\hat{\alpha}_i^+=(\theta^{1/2})^{1-\mathfrak{o}(s(\alpha_i))}\hat{\alpha}_i^{\mathfrak{o}}(\theta^{1/2})^{\mathfrak{o}(t(\alpha_i))}$ and note that, since we assume that the faces $\mathbb{T}_i$ and $\mathbb{T}_{i+1}$ are distinct,

$$(1-\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i)))+\mathfrak{o}_{\mathbb{T}_{i+1}}(s(\alpha_{i+1}))=2\mathfrak{o}_{\mathbb{T}_{i+1}}(s(\alpha_i))$$

(where indices are understood modulo n when α is a closed curve). When α is an arc, we thus obtain the equality

$$\hat{\alpha}_1^{\mathfrak{o}_{\mathbb{T}_1}}\cdots\hat{\alpha}_n^{\mathfrak{o}_{\mathbb{T}_n}}=\theta^{\sum_{i=1}^{n-1}\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i))}(\theta^{1/2})^{1-\mathfrak{o}(s(\alpha))}\hat{\alpha}^+(\theta^{1/2})^{\mathfrak{o}(t(\alpha))},$$

from which we deduce that

$$\begin{aligned} w(\alpha):=W(\hat{\alpha}^{\mathfrak{o}})&=W\big((\theta^{-1/2})^{1-\mathfrak{o}(s(\alpha))}\hat{\alpha}^+(\theta^{-1/2})^{\mathfrak{o}(t(\alpha))}\big)\\ &=W\big(\theta^{-\sum_{i=1}^{n-1}\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i))}\hat{\alpha}_1^{\mathfrak{o}_{\mathbb{T}_1}}\cdots\hat{\alpha}_n^{\mathfrak{o}_{\mathbb{T}_n}}\big)\\ &=\sum_{i=1}^{n-1}\mathfrak{o}_{\mathbb{T}_i}(t(\alpha_i))+\sum_{i=1}^{n}w(\alpha_i) \pmod 2.\end{aligned}$$

The computation when α is a closed curve is done in the same manner. □

3.3.4 Explicit formulas for the isomorphism In order to describe the isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ of Theorem 1.3 more explicitly, let us recall from [Korinman 2019] a set of generators for the ring of regular functions of the relative character varieties.

For α an embedded arc, seen as a path in the fundamental groupoid, and $\varepsilon,\varepsilon'=\pm$, the regular function $F_{\alpha_{\varepsilon\varepsilon'}}\in\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ is defined on the class $[\rho]$ of a functor $\rho\in\mathcal{R}_{\mathrm{SL}_2}(\Sigma_{\mathcal{P}})$ by

$$\rho(\alpha)=\begin{pmatrix}F_{\alpha_{++}}(\rho) & F_{\alpha_{+-}}(\rho)\\ F_{\alpha_{-+}}(\rho) & F_{\alpha_{--}}(\rho)\end{pmatrix}.$$

For γ a closed curve, represented by an arbitrary path $\gamma_v\in\Pi_1(\Sigma_{\mathcal{P}},\mathbb{V})$, one defines $F_\gamma\in\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ by $F_\gamma([\rho]):=\mathrm{Tr}(\rho(\gamma_v))$. Since the trace is invariant by conjugacy, the value $\mathrm{Tr}(\rho(\gamma_v))$ does not depend on the choice of base point v nor on the representative ρ in the class $[\rho]$. The functions $F_{\alpha_{\varepsilon\varepsilon'}}$ and F_γ generate the algebra $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$. For α an arc, we set

$$N_\alpha:=\begin{pmatrix}F_{\alpha_{++}} & F_{\alpha_{+-}}\\ F_{\alpha_{-+}} & F_{\alpha_{--}}\end{pmatrix}$$

the 2×2 matrix with coefficients in $\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$. Note that

$$N_{\alpha^{-1}}=\begin{pmatrix}F_{\alpha_{--}} & -F_{\alpha_{+-}}\\ -F_{\alpha_{-+}} & F_{\alpha_{++}}\end{pmatrix}.$$

For α an embedded arc and $\varepsilon,\varepsilon'=\pm$, we denote by $\alpha_{\varepsilon\varepsilon'}\in\mathcal{S}_{+1}(\underline{\Sigma})$ the class of the arc α with state ε at $s(\alpha)$ and ε' at $t(\alpha)$. We write

$$M_\alpha:=\begin{pmatrix}\alpha_{++} & \alpha_{+-}\\ \alpha_{-+} & \alpha_{--}\end{pmatrix}$$

the 2×2 matrix with coefficients in $\mathcal{S}_{+1}(\underline{\Sigma})$. Note that

$$M_{\alpha^{-1}}=(M_\alpha)^{\mathsf{T}}=\begin{pmatrix}\alpha_{++} & \alpha_{-+}\\ \alpha_{+-} & \alpha_{--}\end{pmatrix}.$$

Recall the isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ of Theorem 1.3 and recall that $C^{-1}=\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$.

Theorem 3.20 *For each embedded arc α,*

$$\Psi^{(\Delta,\mathfrak{o}_\Delta)}(M_\alpha)=(-1)^{w(\alpha)}(C^{-1})^{1-\mathfrak{o}(\alpha(0))}N_\alpha(C^{-1})^{\mathfrak{o}(\alpha(1))}. \tag{32}$$

For each closed curve γ,

$$\Psi^{(\Delta,\mathfrak{o}_\Delta)}(\gamma)=(-1)^{w(\gamma)}F_\gamma. \tag{33}$$

Remark 3.21 When Σ is closed, recall from Remarks 3.13 and 3.16 that $W^{(\Delta,\mathfrak{o}_\Delta)}$ is a standard spin structure associated to a quadratic form ω such that $w(\gamma)=\omega([\gamma])+1$. Hence in the closed case, the isomorphism $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ coincides with the "standard" isomorphisms described at the beginning of Section 3.1.

Recall that $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$ is defined by the diagram

$$
\begin{tikzcd}
\mathcal{S}_{+1}(\Sigma) \arrow[r, hook, "i^\Delta"] \arrow[d, "\Psi^{(\Delta,\mathfrak{o}_\Delta)}"', "\cong"] & \otimes_{\mathbb{T}} \mathcal{S}_{+1}(\mathbb{T}) \arrow[d, "\otimes_{\mathbb{T}}\Psi^{\mathfrak{o}_{\mathbb{T}}}"', "\cong"] \\
\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\Sigma)] \arrow[r, "i^\Delta"] & \otimes_{\mathbb{T}} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]
\end{tikzcd}
\tag{34}
$$

For $x \in \mathcal{S}_{+1}(\mathbb{T})$, we still denote by x the element in $\otimes_{\mathbb{T}}\mathcal{S}_{+1}(\mathbb{T})$ having 1 in the factors $\mathcal{S}_{+1}(\mathbb{T}')$ for $\mathbb{T}' \neq \mathbb{T}$ and x in the factor $\mathcal{S}_{+1}(\mathbb{T})$. Hence for $\delta \in \mathbb{G}^\Delta$ a path in $\mathbb{T}$, the matrix M_δ is considered as a 2×2 matrix with coefficients in $\otimes_{\mathbb{T}}\mathcal{S}_{+1}(\mathbb{T})$. Similarly, the matrix N_δ is considered as a 2×2 matrix with coefficients in $\otimes_{\mathbb{T}}\mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\mathbb{T})]$.

Proof We first show that if (32) holds for an arc α, then it holds for α^{-1}. This follows from the fact that $w(\alpha^{-1}) = w(\alpha)+1$, from the equalities $(C^{-1})^{\intercal} = C$ and $A^{-1} = -C^{-1}A^{\intercal}C^{-1}$ for $A \in \mathrm{SL}_2(\mathbb{C})$, and from the computation

$$
\begin{aligned}
\Psi(M_{\alpha^{-1}}) = \Psi(M_\alpha^{\intercal}) &= (-1)^{w(\alpha)}C^{\mathfrak{o}(t(\alpha))}(N_\alpha)^{\intercal}C^{1-\mathfrak{o}(s(\alpha))} \\
&= (-1)^{w(\alpha)+1}(C^{-1})^{1-\mathfrak{o}(s(\alpha^{-1}))}(-C^{-1}N_\alpha^{\intercal}C^{-1})(C^{-1})^{\mathfrak{o}(t(\alpha^{-1}))} \\
&= (-1)^{w(\alpha^{-1})}(C^{-1})^{1-\mathfrak{o}(s(\alpha^{-1}))}N_{\alpha^{-1}}(C^{-1})^{\mathfrak{o}(t(\alpha^{-1}))}.
\end{aligned}
$$

Next let us prove the theorem for the triangle $\mathbb{T}$. The fact that (32) holds for the arcs $\alpha_{\mathbb{T}}$, $\beta_{\mathbb{T}}$ and $\gamma_{\mathbb{T}}$ is an immediate consequence of the definition of $\Psi^{\mathfrak{o}_{\mathbb{T}}}$ in Lemma 3.10 and from the definition of the canonical spin structure in $\mathbb{T}$. By the preceding arguments, (32) also holds for the arcs $\alpha_{\mathbb{T}}^{-1}$, $\beta_{\mathbb{T}}^{-1}$ and $\gamma_{\mathbb{T}}^{-1}$, and the theorem is proved for $\mathbb{T}$.

In the general case, consider an arc α and choose a decomposition

$$\alpha = \alpha_1 \cdots \alpha_n, \quad \alpha_i \in \mathbb{G}^\Delta,$$

as before. By the gluing formula for stated skein algebras [Lê 2018, Theorem 3.1], $i^\Delta(M_\alpha) = M_{\alpha_1}\cdots M_{\alpha_n}$. By definition of the morphism i^Δ in (29), $i^\Delta(N_\alpha) = N_{\alpha_1}\cdots N_{\alpha_n}$. By the preceding case of the triangle,

$$(\otimes_{\mathbb{T}}\Psi^{\mathfrak{o}_{\mathbb{T}}})(M_{\alpha_i}) = (-1)^{w(\alpha_i)}(C^{-1})^{1-\mathfrak{o}_{\mathbb{T}_i}(s(\alpha_i))}N_{\alpha_i}(C^{-1})^{\mathfrak{o}_{\mathbb{T}_i}(t(\alpha))}.$$

Hence, by Lemma 3.19,

$$(\otimes_{\mathbb{T}}\Psi^{\mathfrak{o}_{\mathbb{T}}})\circ i^\Delta(M_\alpha) = i^\Delta\big((-1)^{w(\alpha)}(C^{-1})^{1-\mathfrak{o}(s(\alpha))}N_\alpha(C^{-1})^{\mathfrak{o}(t(\alpha))}\big),$$

and (32) follows from the commutativity of the diagram (34). The proof for a closed curved is done similarly by taking the trace of the above equality. □

3.3.5 Comparison with Costantino and Lê's isomorphism Let Σ be a connected punctured surface with nontrivial boundary. Costantino and Lê [2022] defined the twisted character variety $\chi(\Sigma)$ as the space of functors $\hat{\rho} \in \mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}), \mathrm{SL}_2(\mathbb{C}))$ such that $\hat{\rho}(\theta_{\vec{v}}^{1/2}) = C^{-1}$ for any $\vec{v} \in \widehat{\mathbb{V}}$. Let $\mathcal{S}$

denote the maximal spectrum of $\mathcal{S}_{+1}(\underline{\Sigma})$. For $\chi \in \mathcal{S}$, seen as a character $\chi\colon \mathcal{S}_{+1}(\underline{\Sigma}) \to \mathbb{C}^*$, and for α an oriented arc, write

$$\chi(\alpha) := \begin{pmatrix} \chi(\alpha_{++}) & \chi(\alpha_{+-}) \\ \chi(\alpha_{+-}) & \chi(\alpha_{--}) \end{pmatrix}.$$

Costantino and Lê [2022, Theorem 8.12] defined an affine isomorphism $\Theta\colon \mathcal{S} \xrightarrow{\cong} \chi(\underline{\Sigma})$ sending a character χ to a functor $\hat{\rho}$ such that $\chi(\alpha) = \hat{\rho}(\hat{\alpha}^+)$ for any embedded (even immersed) arc and such that $\chi(\gamma) = \mathrm{Tr}(\hat{\rho}(\hat{\gamma}^+))$ for any closed curve. Composing Θ with the isomorphism induced by $\Psi^{(\Delta,\mathfrak{o}_\Delta)}$, one obtains an isomorphism $\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma}) \cong \chi(\underline{\Sigma})$. By Theorem 3.20, this isomorphism sends a functor $\rho \in \mathrm{Hom}(\Pi_1(\Sigma_{\mathcal{P}}, \mathbb{V}), \mathrm{SL}_2(\mathbb{C}))$ to a functor $\hat{\rho} \in \mathrm{Hom}(\Pi_1(U\Sigma_{\mathcal{P}}, \widehat{\mathbb{V}}), \mathrm{SL}_2(\mathbb{C}))$ characterized by the formulas $\hat{\rho}(\hat{\alpha}^{\mathfrak{o}}) = (-1)^{w(\alpha)}\rho(\alpha)$ for any arc α, $\mathrm{Tr}(\hat{\rho}(\hat{\gamma}^{\mathfrak{o}}) = (-1)^{w(\gamma)}\,\mathrm{Tr}(\rho(\gamma))$ for any closed curve γ and $\hat{\rho}(\theta_{\vec{v}}^{1/2}) = C^{-1}$ for any $\vec{v} \in \widehat{\mathbb{V}}$.

3.4 Classical Shadows

Suppose that $\omega \in \mathbb{C}$ is a root of unity of odd order $N > 1$. A *central representation* of the stated skein algebra is a finite-dimensional representation $r\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathrm{End}(V)$ which sends each element of the image of the morphism j of Theorem 1.2 to scalar operators. Fix a topological triangulation Δ of $\underline{\Sigma}$ and an orientation $\mathfrak{o}_\Delta$ of its edges. Then r induces a character on $\mathcal{S}_{+1}(\underline{\Sigma}) \xrightarrow[\cong]{\Psi^{(\Delta,\mathfrak{o}_\Delta)}} \mathbb{C}[\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})]$ and this character induces a point in the relative character variety $\mathcal{X}_{\mathrm{SL}_2}(\underline{\Sigma})$ that we call the *classical shadow* of r, as in [Bonahon and Wong 2016] in the closed case. By definition, the classical shadow only depends on the isomorphism class of r.

To motivate the results of this paper, we list three families of central representations. First, irreducible representations are obviously central. Then choose for each triangle $\mathbb{T} \in F(\Delta)$ an irreducible representation $r^{\mathbb{T}}: \mathcal{S}_\omega(\mathbb{T}) \to \mathrm{End}(V_{\mathbb{T}})$ and consider the composition

$$r: \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{i^\Delta} \bigotimes_{\mathbb{T}\in F(\Delta)} \mathcal{S}_\omega(\mathbb{T}) \xrightarrow{\otimes_{\mathbb{T}} r^{\mathbb{T}}} \mathrm{End}(\otimes_{\mathbb{T}} V_{\mathbb{T}}).$$

Such a representation is central and were called *local representations* in [Bai et al. 2007]. Eventually, consider the balanced Chekhov–Fock algebra $\mathcal{Z}_\omega(\underline{\Sigma}, \Delta)$ defined in [Bonahon and Wong 2011] after the original construction of [Fock and Chekhov 1999]. Given a triangulated marked surface, Bonahon and Wong [2011] defined an algebra morphism (the quantum trace) $\mathrm{Tr}\colon \mathcal{S}_\omega(\underline{\Sigma}) \to \mathcal{Z}_\omega(\underline{\Sigma}, \Delta)$ (see also [Lê 2018]). One motivation is the fact that the representation theory of the balanced Chekhov–Fock algebra is easier to study than the one of the skein algebras (see [Bonahon and Liu 2007; Bonahon and Wong 2017]). For an irreducible representation $\pi: \mathcal{Z}_\omega(\underline{\Sigma}, \Delta) \to \mathrm{End}\, V$ of the balanced Chekhov–Fock algebra, we call the *quantum Teichmüller representation*, the composition

$$r: \mathcal{S}_\omega(\underline{\Sigma}) \xrightarrow{\mathrm{Tr}} \mathcal{Z}_\omega(\underline{\Sigma}, \Delta) \xrightarrow{\pi} \mathrm{End}(V).$$

Quantum Teichmüller representations are central.

Appendix Proof of Proposition 2.38 and an application

A.1 Proof of Proposition 2.38

We divide the proof of Proposition 2.38 into five lemmas.

Throughout this section, we write $A := \omega^{-2}$. Denote by $\mathbb{A} = ([0,1] \times S^1, \{p, p'\})$ the annulus with punctures $p = \{0\} \times \{1\}$ and $p' = \{1\} \times \{1\}$ in each of its boundary components and let $b = \{0\} \times S^1 \setminus \{p\}$ and $b' = \{1\} \times S^1 \setminus \{p'\}$ be its boundary arcs. Let $\gamma \subset [0,1] \times S^1$ be the curve $\{\frac{1}{2}\} \times S^1$. Let $\delta^{(n)}, \eta^{(n)} \subset [0,1] \times S^1$ be the arcs with endpoints b and b' such that $\delta^{(n)}$ spirals n times in the counterclockwise direction and $\eta^{(n)}$ spirals n times in the clockwise direction while oriented from b' to b. The arcs are drawn in Figure 7. By convention, $\delta^{(0)}$ and $\eta^{(0)}$ represent the empty diagram. Denote by β the arc $[0,1] \times \{-1\}$. By convention, if α is one of the arcs β, $\delta^{(n)}$ or $\eta^{(n)}$, we denote by $\alpha_{\varepsilon\varepsilon'} \in \mathcal{S}_\omega(\mathbb{A})$ the class of the corresponding stated tangle with sign ε in b and ε' in b'. The following lemma and its proof are quite similar, though stated in a different skein algebra, to [Lê 2015, Proposition 2.2].

Lemma A.1 *In $\mathcal{S}_\omega(\mathbb{A})$, the elements $T_N(\gamma)$ and $\beta_{\varepsilon\varepsilon'}$ commute.*

Proof First note that a direct application of the Kauffman bracket skein relations implies that

$$\gamma \cdot \delta^{(n)}_{\varepsilon\varepsilon'} = A\delta^{(n+1)}_{\varepsilon\varepsilon'} + A^{-1}\delta^{(n-1)}_{\varepsilon\varepsilon'} \quad \text{and} \quad \gamma \cdot \eta^{(n)}_{\varepsilon\varepsilon'} = A\eta^{(n-1)}_{\varepsilon\varepsilon'} + A^{-1}\eta^{(n+1)}_{\varepsilon\varepsilon'}$$

when $n \geq 1$. Next we show by induction on $n \geq 0$ that $T_n(\gamma) \cdot \beta_{\varepsilon\varepsilon'} = A^n \delta^{(n)}_{\varepsilon\varepsilon'} + A^{-n}\eta^{(n)}_{\varepsilon\varepsilon'}$. The statements is an immediate consequence of the definitions when $n = 0$ and a direct application of the Kauffman bracket relations when $n = 1$. Suppose that the results holds for n and $n+1$. Then

$$\begin{aligned} T_{n+2}(\gamma)\beta_{\varepsilon\varepsilon'} &= \gamma \cdot T_{n+1}(\gamma) \cdot \beta_{\varepsilon\varepsilon'} - T_n(\gamma) \cdot \beta_{\varepsilon\varepsilon'} \\ &= \gamma \cdot (A^{n+1}\delta^{(n+1)}_{\varepsilon\varepsilon'} + A^{-(n+1)}\eta^{(n+1)}_{\varepsilon\varepsilon'}) - (A^n\delta^{(n)}_{\varepsilon\varepsilon'} + A^{-n}\eta^{(n)}_{\varepsilon\varepsilon'}) \\ &= A^{n+2}\delta^{(n+2)}_{\varepsilon\varepsilon'} + A^{-(n+2)}\eta^{n+2}_{\varepsilon\varepsilon'}, \end{aligned}$$

and the statement follows by induction. Similarly, we show that $\beta_{\varepsilon\varepsilon'} \cdot T_n(\gamma) = A^{-n}\delta^{(n)}_{\varepsilon\varepsilon'} + A^n \eta^{(n)}_{\varepsilon\varepsilon'}$. Hence,

$$T_N(\gamma) \cdot \beta_{\varepsilon\varepsilon'} - \beta_{\varepsilon\varepsilon'} \cdot T_N(\gamma) = (A^N - A^{-N})(\delta^{(N)}_{\varepsilon\varepsilon'} - \eta^{(N)}_{\varepsilon\varepsilon'}) = 0. \qquad \square$$

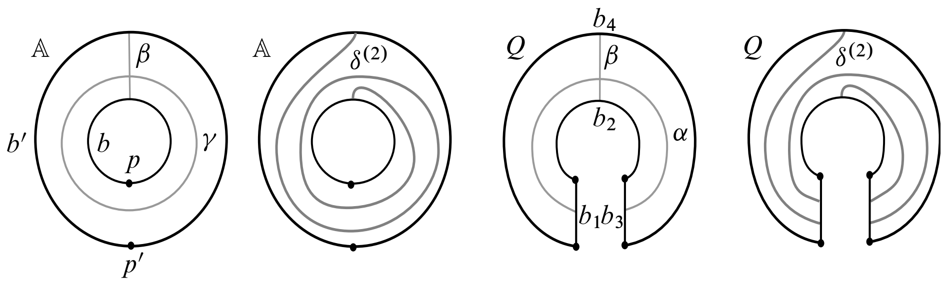


Figure 7: The annulus $\mathbb{A}$, the square Q and some arcs and curves.

Denote by Q the square, ie a disc with four punctures on its boundary. Let $b_1, \dots, b_4$ be its four boundary arcs labeled in the counterclockwise order. When gluing b_1 along b_3, we obtain the annulus with b_2 sent to b and b_4 sent to b'. We denote by $i|_{b_1\# b_3} \colon \mathcal{S}_\omega(\mathbb{A}) \hookrightarrow \mathcal{S}_\omega(Q)$ the gluing morphism. Let $\alpha, \beta, \delta^{(n)}, \eta^{(n)} \subset Q$ be the arcs which are glued together to form γ, β, $\delta^{(n)}$ and $\eta^{(n)}$, respectively, as in Figure 7. Fix an arbitrary orientation $\mathfrak{o}$ of the boundary arcs of Q and consider the filtration $(\mathcal{F}_m)_{m\ge 0}$ associated to $S=\{b_1,b_3\}$ of Definition 2.11. Write $d\colon \mathcal{S}_\omega(Q) \to \mathbb{Z}^{\ge 0}$ the corresponding map and $\mathcal{G}_m := \mathcal{F}_m / \mathcal{F}_{m-1}$ the corresponding graduation.

Lemma A.2 $$\operatorname{lt}((\alpha_{++}+\alpha_{--})^N) = \operatorname{lt}(T_N(\alpha_{++}+\alpha_{--})) = \alpha_{++}^N + \alpha_{--}^N.$$

Proof First note that in $\mathcal{G}_4$, we have $\alpha_{--}\alpha_{++} = q^2\alpha_{++}\alpha_{--}$. So it follows from Lemma 2.27 that in $\mathcal{G}_{2N}$, we have $\operatorname{lt}((\alpha_{++}+\alpha_{--})^N) = \alpha_{++}^N+\alpha_{--}^N$. Since $T_N(X)-X^N$ is a polynomial of degree strictly smaller that N, the degree of $T_N(\alpha_{++}+\alpha_{--})-(\alpha_{++}+\alpha_{--})^N$ is strictly smaller than $2N$; thus $\operatorname{lt}(T_N(\alpha_{++}+\alpha_{--})) = \operatorname{lt}((\alpha_{++}+\alpha_{--})^N)$. □

Let $\alpha^{(n)}$ be the diagram made of n parallel copies of α. Using the identifications $\partial\delta^{(n)} = \partial\eta^{(n)} = \partial\alpha^{(n)} \cup \partial\beta$, we denote by $\delta^{(n)}_{(s,\varepsilon,\varepsilon')}, \eta^{(n)}_{(s,\varepsilon,\varepsilon')} \in \mathcal{S}_\omega(Q)$ the classes of the tangles $\delta^{(n)}$ and $\eta^{(n)}$ with states given by a state s of $\alpha^{(n)}$ and a state $(\varepsilon,\varepsilon')$ of β.

Lemma A.3 *For $0<n<N$ and s a state of $\alpha^{(n)}$,*
$$\operatorname{lt}\big([[\alpha^{(n)},s],\beta_{\varepsilon\varepsilon'}]\big) = (A^n - A^{-n})(\delta^{(n)}_{(s,\varepsilon,\varepsilon')} - \eta^{(n)}_{(s,\varepsilon,\varepsilon')}),$$
where we used the notation $[x,y]=xy-yx$.

Proof The diagram obtained by stacking $\alpha^{(n)}$ on top of β has n crossings and thus 2^n resolutions using the Kauffman bracket relation. We remark that the resolution obtained by replacing each crossing by $\genfrac{}{}{0pt}{}{\smile}{\frown}$ is $A^n\delta^{(n)}_{(s,\varepsilon,\varepsilon')}$ while the resolution obtained by replacing each crossing by $)($ is $A^{-n}\eta^{(n)}_{(s,\varepsilon,\varepsilon')}$. These two resolutions have degree $2n$ and all the others resolutions have degrees strictly smaller; thus
$$\operatorname{lt}([\alpha^{(n)},s]\cdot\beta_{\varepsilon\varepsilon'}) = A^n\delta^{(n)}_{(s,\varepsilon,\varepsilon')} + A^{-n}\eta^{(n)}_{(s,\varepsilon,\varepsilon')}.$$
We similarly prove $\operatorname{lt}(\beta_{\varepsilon\varepsilon'}\cdot[\alpha^{(n)},s]) = A^{-n}\delta^{(n)}_{(s,\varepsilon,\varepsilon')} + A^n\eta^{(n)}_{(s,\varepsilon,\varepsilon')}$ and conclude by taking the difference. □

Lemma A.4 *If $x\in\mathcal{S}_\omega(Q)$ is a polynomial in $\mathcal{S}_\omega(Q)$ in the elements $\alpha_{\varepsilon\varepsilon'}$ such that $d(x)<2N$ and such that x commutes with all elements $\beta_{\mu,\mu'}$, then x is a constant.*

Proof Let $x=\sum_{i\in I} x_i[\alpha^{n_i},s_i]$ be the decomposition in the basis of stated tangles with increasing states s_i and denote by $2n<2N$ its degree. Suppose, for the sake of contradiction, that $n\neq 0$. Let $J=\{j\in I \mid n_i=n\}\subset I$, so $\operatorname{lt}(x)=\sum_{j\in J} x_j[\alpha^n,s_j]$. The hypothesis on x and Lemma A.3 imply that
$$0=\operatorname{lt}([x,\beta_{\varepsilon\varepsilon'}]) = \sum_{j\in J} x_j(A^n-A^{-n})(\delta^{(n)}_{(s_j,\varepsilon,\varepsilon')} - \eta^{(n)}_{(s_j,\varepsilon,\varepsilon')}).$$

Since the elements $\delta^{(n)}_{(s_j,\varepsilon,\varepsilon')}$ and $\eta^{(n)}_{(s_j,\varepsilon,\varepsilon')}$ are linearly independent for $n \geq 1$, we conclude that

$$x_j(A^n - A^{-n}) = 0$$

for all $j \in J$. Since $0 < n < N$ and N is odd, we obtain that $x_j = 0$ for all $j \in J$ thus $\mathrm{lt}(x) = 0$. This gives the contradiction. $\square$

The set $\mathcal{B}' := \{\alpha^a_{-+}\alpha^b_{++}\alpha^c_{+-}, a,b,c \geq 0\} \cup \{\alpha^a_{-+}\alpha^b_{--}\alpha^c_{+-}, a,b,c \geq 0\}$ forms a basis of the algebra $\mathcal{S}_\omega(\mathbb{B})$. This fact is Exercise 7 in Chapter IV, Section 6 of [Kassel 1995], and is proved as follows. Choose an orientation $\mathfrak{o}$ of the boundary arcs of $\mathbb{B}$ such that b_L and b_R points towards different punctures and consider the filtration associated to $S = \{b_L, b_R\}$. For each element of the basis $\mathcal{B}^{\mathfrak{o}}$, there exists exactly one element of $\mathcal{B}'$ which has the same leading term. For $x \in \mathcal{S}_\omega(\mathbb{B})$, denote by $c(x) \in \mathcal{R}$ the coefficient of 1 in the decomposition of the basis $\mathcal{B}'$.

Lemma A.5 $$c(T_N(\alpha_{++} + \alpha_{--})) = 0.$$

Proof Let $n \geq 1$ be an odd integer and let us show that $c((\alpha_{++}+\alpha_{--})^n) = 0$. The proof will then follow from the fact that $T_N(X)$ is an odd polynomial, thus is a linear combination of such elements, and the fact that c is linear. The product $((\alpha_{++}+\alpha_{--})^n)$ develops as a sum of terms of the form $x = x_1 \cdots x_n$ where x_i is either α_{++} or α_{--}. Using the defining relations of $\mathcal{S}_\omega(\mathbb{B})$, we can further develop each term x as a linear combination of terms of the form $\alpha^a_{-+}\alpha^b_{++}\alpha^a_{+-}$ and $\alpha^a_{-+}\alpha^b_{--}\alpha^a_{+-}$ where $2a+b$ has the same parity as n. Since n is odd, each of these summands satisfies $b \neq 0$ so $c(x) = 0$. $\square$

Proof of Proposition 2.38 Consider the element $x := T_N(\alpha_{++}+\alpha_{--}) - \alpha^N_{++} - \alpha^N_{--} \in \mathcal{S}_\omega(Q)$. By Lemma A.2, its degree is strictly smaller that $2N$. By Lemma A.1, in $\mathcal{S}_\omega(\mathbb{A})$ the elements $T_N(\gamma)$ and $\beta_{\varepsilon\varepsilon'}$ commute. The image through the algebra morphism $i|_{b_1\#b_3}\colon \mathcal{S}_\omega(\mathbb{A}) \hookrightarrow \mathcal{S}_\omega(Q)$ of $T_N(\gamma)$ and $\beta_{\varepsilon\varepsilon'}$ are respectively $T_N(\alpha_{++}+\alpha_{--})$ and $\beta_{\varepsilon\varepsilon'}$, thus they commute. By Lemma 2.36, the elements α^N_{++} and α^N_{--} also commute with $\beta_{\varepsilon\varepsilon'}$ so x commutes with each element $\beta_{\varepsilon\varepsilon'}$. Lemma A.4 implies that x is a constant and Lemma A.5 implies that this constant is null. $\square$

A.2 A generalization of a theorem of Bonahon

Proposition 2.38 provides the following generalization of the main theorem of [Bonahon 2019]. Let $\mathcal{A}$ be an $\mathcal{R}$–algebra and $\rho\colon \mathbb{C}_q[\mathrm{SL}_2]^{\otimes k} \to \mathcal{A}$ be a morphism of algebras. Let ρ_i be the i^{th} component of ρ. For $1 \leq i \leq k$, consider the following two matrices with coefficients in $\mathcal{A}$:

$$A_i := \begin{pmatrix} \rho_i(\alpha_{++}) & \rho_i(\alpha_{+-}) \\ \rho_i(\alpha_{-+}) & \rho_i(\alpha_{--}) \end{pmatrix}, \quad A_i^{(N)} := \begin{pmatrix} \rho_i(\alpha_{++})^N & \rho_i(\alpha_{+-})^N \\ \rho_i(\alpha_{-+})^N & \rho_i(\alpha_{--})^N \end{pmatrix}.$$

The following proposition was proved in [Bonahon 2019, Theorem 1] in the particular case where $\rho_i(\alpha_{+-})\rho_i(\alpha_{-+}) = 0$ for each $i \in \{1, \dots, k\}$.

Proposition A.6 *If q is a root of unity of odd order $N > 1$, then*

$$T_N(\mathrm{Tr}(A_1 \cdots A_k)) = \mathrm{Tr}(A_1^{(N)} \cdots A_k^{(N)}).$$

Proof By Proposition 2.38 and using that both ρ and the $(k-1)^{\text{st}}$ coproduct

$$\Delta^{(k-1)}\colon \mathbb{C}_q[\mathrm{SL}_2] \to \mathbb{C}_q[\mathrm{SL}_2]^{\otimes k}$$

are morphisms of algebras,

$$T_N \circ \rho \circ \Delta^{(k-1)}(\alpha_{++} + \alpha_{--}) = \rho \circ \Delta^{(k-1)}(\alpha_{++}^N + \alpha_{--}^N).$$

We conclude by remarking that

$$\rho \circ \Delta^{(k-1)}(\alpha_{++} + \alpha_{--}) = \mathrm{Tr}(A_1 \cdots A_k) \quad \text{and} \quad \rho \circ \Delta^{(k-1)}(\alpha_{++}^N + \alpha_{--}^N) = \mathrm{Tr}(A_1^{(N)} \cdots A_k^{(N)}),$$

where the second equality follows from the fact that $j_{\mathbb{B}}$ is a morphism of Hopf algebras (Lemma 2.28), hence commutes with $\Delta^{(k-1)}$. □

References

[Alekseev and Malkin 1995] **A Y Alekseev**, **A Z Malkin**, *Symplectic structure of the moduli space of flat connection on a Riemann surface*, Comm. Math. Phys. 169 (1995) 99–119 MR Zbl

[Alekseev et al. 1998] **A Alekseev**, **A Malkin**, **E Meinrenken**, *Lie group valued moment maps*, J. Differential Geom. 48 (1998) 445–495 MR Zbl

[Alekseev et al. 2002] **A Alekseev**, **Y Kosmann-Schwarzbach**, **E Meinrenken**, *Quasi-Poisson manifolds*, Canad. J. Math. 54 (2002) 3–29 MR Zbl

[Atiyah and Bott 1983] **M F Atiyah**, **R Bott**, *The Yang–Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. A 308 (1983) 523–615 MR Zbl

[Bai et al. 2007] **H Bai**, **F Bonahon**, **X Liu**, *Local representations of the quantum Teichmüller space*, preprint (2007) arXiv 0707.2151

[Barrett 1999] **J W Barrett**, *Skein spaces and spin structures*, Math. Proc. Cambridge Philos. Soc. 126 (1999) 267–275 MR Zbl

[Bloomquist and Lê 2022] **W Bloomquist**, **T T Q Lê**, *The Chebyshev–Frobenius homomorphism for stated skein modules of 3–manifolds*, Math. Z. 301 (2022) 1063–1105 MR Zbl

[Bonahon 2019] **F Bonahon**, *Miraculous cancellations for quantum* SL_2, Ann. Fac. Sci. Toulouse Math. 28 (2019) 523–557 MR Zbl

[Bonahon and Liu 2007] **F Bonahon**, **X Liu**, *Representations of the quantum Teichmüller space and invariants of surface diffeomorphisms*, Geom. Topol. 11 (2007) 889–937 MR Zbl

[Bonahon and Wong 2011] **F Bonahon**, **H Wong**, *Quantum traces for representations of surface groups in* $\mathrm{SL}_2(\mathbb{C})$, Geom. Topol. 15 (2011) 1569–1615 MR Zbl

[Bonahon and Wong 2016] **F Bonahon**, **H Wong**, *Representations of the Kauffman bracket skein algebra, I: Invariants and miraculous cancellations*, Invent. Math. 204 (2016) 195–243 MR Zbl

[Bonahon and Wong 2017] **F Bonahon**, **H Wong**, *Representations of the Kauffman bracket skein algebra, II: Punctured surfaces*, Algebr. Geom. Topol. 17 (2017) 3399–3434 MR Zbl

[Brown and Goodearl 2002] **K A Brown**, **K R Goodearl**, *Lectures on algebraic quantum groups*, Birkhäuser, Basel (2002) MR Zbl

[Bullock 1997] **D Bullock**, *Rings of* $\mathrm{SL}_2(\mathbb{C})$*–characters and the Kauffman bracket skein module*, Comment. Math. Helv. 72 (1997) 521–542 MR Zbl

[Chari and Pressley 1994] **V Chari**, **A Pressley**, *A guide to quantum groups*, Cambridge Univ. Press (1994) MR Zbl

[Charles and Marché 2012] **L Charles**, **J Marché**, *Multicurves and regular functions on the representation variety of a surface in* $\mathrm{SU}(2)$, Comment. Math. Helv. 87 (2012) 409–431 MR Zbl

[Costantino and Lê 2022] **F Costantino**, **T T Q Lê**, *Stated skein algebras of surfaces*, J. Eur. Math. Soc. 24 (2022) 4063–4142 MR Zbl

[Culler and Shalen 1983] **M Culler**, **P B Shalen**, *Varieties of group representations and splittings of 3–manifolds*, Ann. of Math. 117 (1983) 109–146 MR Zbl

[Drinfeld 1983] **V G Drinfeld**, *Constant quasiclassical solutions of the Yang–Baxter quantum equation*, Dokl. Akad. Nauk SSSR 273 (1983) 531–535 MR Zbl In Russian; translated in Soviet Math. Dokl. 28 (1983) 667–671

[Fock and Chekhov 1999] **V V Fock**, **L O Chekhov**, *Quantum Teichmüller spaces*, Teoret. Mat. Fiz. 120 (1999) 511–528 MR Zbl In Russian; translated in Theoret. Math. Phys. 120 (1999) 1245–1259

[Fock and Rosly 1999] **V V Fock**, **A A Rosly**, *Poisson structure on moduli of flat connections on Riemann surfaces and the* r*–matrix*, from "Moscow Seminar in Mathematical Physics" (A Y Morozov, M A Olshanetsky, editors), Amer. Math. Soc. Transl. Ser. 2 191, Amer. Math. Soc., Providence, RI (1999) 67–86 MR Zbl

[Ganev et al. 2024] **I Ganev**, **D Jordan**, **P Safronov**, *The quantum Frobenius for character varieties and multiplicative quiver varieties*, J. Eur. Math. Soc. (online publication February 2024)

[Goldman 1986] **W M Goldman**, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math. 85 (1986) 263–302 MR Zbl

[Guruprasad et al. 1997] **K Guruprasad**, **J Huebschmann**, **L Jeffrey**, **A Weinstein**, *Group systems, groupoids, and moduli spaces of parabolic bundles*, Duke Math. J. 89 (1997) 377–412 MR Zbl

[Gutt et al. 2005] **S Gutt**, **J Rawnsley**, **D Sternheimer** (editors), *Poisson geometry, deformation quantisation and group representations*, Lond. Math. Soc. Lect. Note Ser. 323, Cambridge Univ. Press (2005) MR Zbl

[Hess et al. 2009] **K Hess**, **P-E Parent**, **J Scott**, *CoHochschild homology of chain coalgebras*, J. Pure Appl. Algebra 213 (2009) 536–556 MR Zbl

[Hoste and Przytycki 1992] **J Hoste**, **J H Przytycki**, *A survey of skein modules of 3–manifolds*, from "Knots '90" (A Kawauchi, editor), de Gruyter, Berlin (1992) 363–379 MR Zbl

[Johnson 1980] **D Johnson**, *Spin structures and quadratic forms on surfaces*, J. Lond. Math. Soc. 22 (1980) 365–373 MR Zbl

[Kassel 1995] **C Kassel**, *Quantum groups*, Graduate Texts in Math. 155, Springer (1995) MR Zbl

[Kontsevich 2003] **M Kontsevich**, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66 (2003) 157–216 MR Zbl

[Korinman 2019] **J Korinman**, *Triangular decomposition of character varieties*, preprint (2019) arXiv 1904.09022

[Labourie 2013] **F Labourie**, *Lectures on representations of surface groups*, Eur. Math. Soc., Zürich (2013) MR Zbl

[Lê 2015] **T T Q Lê**, *On Kauffman bracket skein modules at roots of unity*, Algebr. Geom. Topol. 15 (2015) 1093–1117 MR Zbl

[Lê 2018] **T T Q Lê**, *Triangular decomposition of skein algebras*, Quantum Topol. 9 (2018) 591–632 MR Zbl

[Lê and Paprocki 2019] **T T Q Lê**, **J Paprocki**, *On Kauffman bracket skein modules of marked 3–manifolds and the Chebyshev–Frobenius homomorphism*, Algebr. Geom. Topol. 19 (2019) 3453–3509 MR Zbl

[Lusztig 1990] **G Lusztig**, *Quantum groups at roots of* 1, Geom. Dedicata 35 (1990) 89–113 MR Zbl

[Lusztig 1993] **G Lusztig**, *Introduction to quantum groups*, Progr. Math. 110, Birkhäuser, Boston, MA (1993) MR Zbl

[Manin 1988] **Y I Manin**, *Quantum groups and noncommutative geometry*, Univ. Montreal (1988) MR Zbl

[Marché 2012] **J Marché**, *Geometry of the representation spaces in* SU(2), from "Strasbourg master class on geometry" (A Papadopoulos, F Dal'Bo, F Herrlich, P Korablev, S Matveev, G Link, J Marché, C Petronio, V Schroeder, editors), IRMA Lect. Math. Theor. Phys. 18, Eur. Math. Soc., Zürich (2012) 333–370 MR Zbl

[Marché 2016] **J Marché**, *Character varieties* SL_2 *and Kauffman skein algebras*, from "Topology, geometry and algebra of low-dimensional manifolds", RIMS Kokyuroku 1991, Kyoto Univ. (2016) 27–42

[Milnor 1963] **J Milnor**, *Spin structures on manifolds*, Enseign. Math. 9 (1963) 198–203 MR Zbl

[Muller 2016] **G Muller**, *Skein and cluster algebras of marked surfaces*, Quantum Topol. 7 (2016) 435–503 MR Zbl

[Negron 2021] **C Negron**, *Log-modular quantum groups at even roots of unity and the quantum Frobenius, I*, Comm. Math. Phys. 382 (2021) 773–814 MR Zbl

[Novak and Runkel 2015] **S Novak**, **I Runkel**, *State sum construction of two-dimensional topological quantum field theories on spin surfaces*, J. Knot Theory Ramifications 24 (2015) art. id. 1550028 MR Zbl

[Przytycki and Sikora 2000] **J H Przytycki**, **A S Sikora**, *On skein algebras and* $\mathrm{Sl}_2(\mathbb{C})$*–character varieties*, Topology 39 (2000) 115–148 MR Zbl

[Przytycki and Sikora 2019] **J H Przytycki**, **A S Sikora**, *Skein algebras of surfaces*, Trans. Amer. Math. Soc. 371 (2019) 1309–1332 MR Zbl

[Turaev 1988] **V G Turaev**, *The Conway and Kauffman modules of a solid torus*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 167 (1988) 79–89 MR Zbl In Russian; translated in J. Soviet Math. 52 (1990) 2799–2805

[Turaev 1991] **V G Turaev**, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. École Norm. Sup. 24 (1991) 635–704 MR Zbl

[Yuasa 2015] **W Yuasa**, *Poisson algebras of curves on bordered surfaces and skein quantization*, preprint (2015) arXiv 1504.00174

Institut Montpelliérain Alexandre Grothendieck, Université de Montpellier
Montpellier, France

ETS Ingenieros Informáticos, Universidad Politécnica de Madrid, Campus de Montegancedo
Madrid, Spain

julien.korinman@gmail.com, alexandre.quesney@upm.es

https://sites.google.com/site/homepagejulienkorinman/

Received: 16 March 2022 Revised: 25 March 2023

Algebraic & Geometric Topology **24**:4 (2024) 2149–2170
DOI: 10.2140/agt.2024.24.2149
Published: 16 July 2024

Commensurators of thin normal subgroups and abelian quotients

THOMAS KOBERDA
MAHAN MJ

We give an affirmative answer to many cases of a question due to Shalom, which asks if the commensurator of a thin subgroup of a Lie group is discrete. Let $K < \Gamma < G$ be an infinite normal subgroup of an arithmetic lattice Γ in a rank-one simple Lie group G, such that the quotient $Q = \Gamma/K$ is infinite. We show that the commensurator of K in G is discrete, provided that Q admits a surjective homomorphism to $\mathbb{Z}$. In this case, we also show that the commensurator of K contains the normalizer of K with finite index. We thus vastly generalize a 2021 result of the authors, which showed that many natural normal subgroups of $\mathrm{PSL}_2(\mathbb{Z})$ have discrete commensurator in $\mathrm{PSL}_2(\mathbb{R})$.

22E40; 20F65, 20F67, 57M50

1 Introduction

Let G be a semisimple $\mathbb{Q}$–algebraic group, and let $G(\mathbb{Z})$ denote its group of integer points. Roughly speaking, a subgroup Γ of G is called *arithmetic* if it is *commensurable in a wide sense* with $G(\mathbb{Z})$; see Witte Morris [36]. That is, there is an element $g \in G$ such that the group $G(\mathbb{Z}) \cap \Gamma^g$ has finite index in both $G(\mathbb{Z})$ and Γ^g. In general, if G is an algebraic group and $\Gamma < G$ is a subgroup, we write $\mathrm{Comm}_G(\Gamma)$ for the *commensurator* of Γ in G, ie the subgroup consisting of $g \in G$ such that $\Gamma \cap \Gamma^g$ has finite index in both Γ and Γ^g. The commensurability criterion for arithmeticity due to Margulis [24] (see also Witte Morris [36]) characterizes arithmetic subgroups of algebraic groups via their commensurators. A convention we shall follow throughout in this article: whenever we refer to a semisimple Lie group, we shall mean a connected semisimple real Lie group with no compact factors, unless noted otherwise.

Theorem 1.1 (Margulis) *Let G be a semisimple Lie group with no compact factors and let Γ be an irreducible lattice in G. Then Γ is arithmetic if and only if $\mathrm{Comm}_G(\Gamma)$ is dense in G.*

Here, we are primarily concerned with the discreteness properties of commensurators of *thin groups*, a class of groups which has received a large amount of attention in recent years; see Sarnak [28]. A subgroup $K < G$ is *thin* if K is discrete and Zariski dense in G, and if G/K has infinite volume with respect to the Haar measure on G. Thus, K fails to be a lattice in G only by virtue of having infinite covolume in G. Natural examples of thin groups arise from infinite-index Zariski-dense subgroups of lattices in G.

In the present manuscript, we continue our previous investigations from [21] of the following question due to Shalom (see especially Shalom and Willis [33], wherein the problem has its genesis):

Question 1.2 [22] *Let K be a thin subgroup of a semisimple Lie group G.*

(i) *Is the commensurator $\mathrm{Comm}_G(K)$ of K in G discrete?*

(ii) *In particular, is the normalizer of K in G of finite index in $\mathrm{Comm}_G(K)$?*

For an infinite normal subgroup K of a lattice Γ, the two subquestions of Question 1.2 are equivalent. Indeed, the commensurator of K contains its normalizer, which contains Γ. Since Γ is a lattice, we see that if $\mathrm{Comm}_G(K)$ is discrete then it is a finite-index superlattice of Γ. For the other implication, any such K is discrete and Zariski dense, and thus has a discrete normalizer; cf Lemma 2.1. Since the normalizer of K contains Γ and since Γ has finite covolume, we have that the normalizer of K is itself a lattice. Thus, if the commensurator of K contains the normalizer of K with finite index then the commensurator is discrete.

Positive answers to Question 1.2 are known for all finitely generated thin subgroups K of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{C})$ (see Greenberg [16], Leininger, Long and Reid [22] and Mj [26]), and for thin subgroups of a semisimple Lie groups with limit set a proper subset of the Furstenberg boundary [26]. Here, the *limit set* is a generalization of the limit set occurring in the theory of Kleinian groups, and is a minimal nonempty closed invariant subset of the Furstenberg boundary for a group acting on the corresponding symmetric space; see Benoist [4].

We were thus prompted in [21] to address Question 1.2 when the ambient Lie group is the simplest possible, viz $\mathrm{PSL}_2(\mathbb{R})$, for thin groups whose limit sets consist of the entire Furstenberg boundary, ie $S^1 = \partial\mathbb{H}^2$. More generally, natural examples of thin groups with limit set equal to the Furstenberg boundary come from normal subgroups of rank-one lattices. This general problem provides the context for this paper.

1.1 Main result

Since many rank-one arithmetic lattices surject onto nonabelian free groups, every finitely generated group can be realized as a quotient of an arithmetic lattice. Observe, in particular, that all finitely generated free groups arise as finite-index subgroups of $\Gamma(2)$, the level-two congruence subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, and therefore all infinite, finitely generated groups arise as quotients of a rank-one arithmetic lattice by a thin normal subgroup. This level of generality has led us to impose some natural algebraic conditions on the quotient Q. We will establish the following result, which handles normal subgroups with "nice" quotients.

Theorem 1.3 *Let $\Gamma < G$ be an arithmetic lattice in a rank-one simple Lie group G and let $K < \Gamma$ be an infinite normal subgroup. Write $Q = \Gamma/K$ for the corresponding quotient group. Then the group $\mathrm{Comm}_G(K)$ is discrete, provided that the group Q admits a surjective homomorphism to $\mathbb{Z}$. Under these hypotheses, the commensurator of K in G contains the normalizer of K with finite index.*

The reader is directed to Theorem 5.1 for the context and proof surrounding the main result. Note that the hypotheses of Theorem 1.3 are never satisfied for irreducible lattices in higher rank nor for lattices in the rank-one simple Lie groups $\mathrm{Sp}(n,1)$ for $n \geq 2$, nor in the exceptional group F_4^{-20}. This is because lattices in these Lie groups have Kazhdan's Property (T). Thus, Theorem 1.3 is vacuously true in these cases. Therefore in the course of establishing Theorem 1.3, we pay exclusive attention to $G \in \{\mathrm{SO}(n,1), \mathrm{SU}(n,1)\}_{n\geq 2}$, which give rise to real and complex hyperbolic spaces, respectively, as the associated symmetric spaces of noncompact type.

In [21] we answered Question 1.2 in the special case that K is the commutator subgroup of Γ, where $\Gamma < \mathrm{PSL}_2(\mathbb{Z})$ is a finite-index normal subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ contained in a principal congruence subgroup $\Gamma(k)$ for some $k \geq 2$. We vastly generalize this result, since if $K = [\Gamma, \Gamma]$ has infinite index in Γ, then K falls under the purview of Theorem 1.3.

1.2 Tools and techniques

The main theorems and techniques of [21] are the starting point of this paper.

Preserving lines with holes An important technical tool introduced in [21] was that of a homology pseudoaction. We adapt it here to the notion of *preservation of lines with holes*. Let Γ be a lattice in a rank-one simple Lie group G, let $K < \Gamma$ be a normal subgroup, and let $Q = \Gamma/K$. Quite generally, for $g \in G$ we say that g preserves Q–lines with holes if for all $\gamma \in \Gamma$, there exists $N > 0$ such that

$$\gamma^n \cong (\gamma^n)^g \bmod K \quad \text{for all } n \in N\mathbb{Z}.$$

The terminology arises from thinking of infinite cyclic groups as "lines" and a finite-index subgroup of an infinite cyclic group as a "line with holes". We direct the reader to Section 3 for a detailed discussion.

The usefulness of preserving lines with holes is illustrated by the following purely group-theoretic fact, which provides a rather general criterion for deciding noncommensurability (see Theorem 3.4):

Theorem 1.4 *Let $\Gamma < G$, let $K < \Gamma$ be normal, and let $Q = \Gamma/K$. If*

$$g \in \mathrm{Comm}_G K \cap \mathrm{Comm}_G \Gamma,$$

then $K^g := g^{-1}Kg$ preserves Q–lines with holes.

Harmonic forms and maps The other principal tool used in this paper comes from harmonic forms and harmonic maps via Hodge theory. These include classical Hodge theory and its L^2 analogue for noncompact manifolds. Preservation of Q–lines with holes, or equivalently, lines with holes in Γ modulo the normal subgroup K, can be promoted to something stronger: the harmonic form allows us to convert the "coarse" lines in Γ/K into actual maps to $\mathbb{R}$, ie it allows us to "fill the holes" of coarse lines in a canonical fashion, and thus find canonical G–invariant maps to $\mathbb{R}$.

Discrete patterns Harmonic maps are coupled with the notion of discrete patterns, an idea going back to Schwartz [30], and which was exploited in proving discreteness of commensurators in Leininger, Long and Reid [22] and Mj [26]. Throughout the paper, many of our ideas and methods are inspired by the basic example of arithmetic hyperbolic surfaces as well as the special case $K = [\Gamma, \Gamma]$, and in some places we explicate the underlying geometric intuition. In the context of $\mathrm{PSL}_2(\mathbb{R})$ and hyperbolic surfaces, Teichmüller-theoretic notions such as zeros and saddle connections of abelian differentials provide us the necessary discrete patterns that are preserved by the commensurator when the underlying surface has positive genus *and* lines with holes in the integral homology are preserved. Preservation of such discrete patterns finally ensures that the commensurator is discrete. With the notion of preserving homological lines with holes in place, the discussion for lattices in $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$ splits into uniform and nonuniform cases. For uniform lattices, we use Hodge theory coupled with a Lie-theoretic idea that we learned from Venkataramana [34] and Agol [1]. For nonuniform lattices, we use L^2–Hodge theory along with the fact that preservation of homology lines with holes guarantees the preservation of a discrete pattern given by horoballs. Discreteness of a pattern-preserving subgroup is an essential ingredient in the nonvanishing cuspidal cases: see the proof of Theorem 5.1, especially Claim 5.2 therein.

Relationship with existing literature The previous works [22; 26] on discreteness of commensurators derived discreteness by showing that the commensurator preserves a "discrete geometric subobject" or "pattern" in the sense of Schwartz [29]. These may be regarded as a collection of geometrically defined subspaces of the domain symmetric space X. We refer the reader to the appendix for the material on patterns that will be used in this paper. There is a shift in focus in this paper, as we look at naturally defined dual objects. The canonical nature of harmonic maps ensures that they are preserved by the commensurator. We derive much of our inspiration from Shalom's work [31; 32; 33].

1.3 Structure of the paper

Section 2 contains an account of the general tools from the theory of lattices in Lie groups which we will need. Section 3 describes preservation of lines with holes in detail. Section 4 introduces the notion of a discrete invariant set as it arises from classical and L^2–Hodge theory. In the same section, the commensurator of a form is introduced and the construction of an invariant harmonic form is carried out. Section 5 proves Theorem 1.3.

Remarks on notation Throughout this paper, we will use the notation K to denote a subgroup a discrete group. Usually, this will be a normal subgroup of an arithmetic lattice Γ. In particular, K will generally not denote a maximal compact subgroup of the ambient Lie group G. We will use N to denote a positive integer, as opposed to the more common notation of the unipotent subgroup in the Iwasawa decomposition of a semisimple Lie group. The Iwasawa decomposition will be used briefly in the proof of Claim 5.2, but no confusion will arise. We will use the exponentiation shorthand for conjugation in groups, so that $K^g = g^{-1}Kg$, where K and g are contained in an ambient group. The group G will denote an ambient Lie group, which will typically be $\{\mathrm{SO}(n,1), \mathrm{SU}(n,1)\}_{n\geq 2}$ unless otherwise explicitly noted.

2 Generalities on discrete subgroups of Lie groups

In this section, we gather some general facts about Zariski-dense discrete subgroups of semisimple Lie groups which we will require in this article.

2.1 Zariski-dense subgroups and commensurators

We begin with the following general fact about normalizers of discrete groups. The statement and proof are contained as Lemma 2.1 in [21], and so we omit the proof.

Lemma 2.1 *Let G be a simple Lie group and let $\Gamma < G$ be a discrete Zariski-dense subgroup. Then the normalizer $N_G(\Gamma)$ is again discrete.*

The following well-known fact will be used throughout the paper.

Lemma 2.2 *Let G be a simple real group and let $H < G$ be a Zariski-dense subgroup. If H is not discrete then H is dense.*

Indeed, since H is not discrete, the topological closure $\overline{H}$ of H has the property that the component H^0 of $\overline{H}$ containing the identity is a Zariski-dense subgroup of G which has positive dimension, and therefore must be all of G; indeed the tangent space to H^0 at the identity coincides with the tangent space to G, and so H^0 contains a neighborhood of the identity in G, which generates the identity component of G. We remark that if G is allowed to be a complex group then one must assume that H is not precompact, as can be seen from the Zariski density of the unit complex numbers in $\mathbb{C}$ for instance.

The following lemma generalizes the corresponding statement in [21] for $\mathrm{PSL}_2(\mathbb{R})$.

Lemma 2.3 *Let Γ_0 be a lattice in a noncompact simple Lie group G. Let Γ be a subgroup of G containing Γ_0 such that there exists an $N > 0$ satisfying the property that for all $g \in \Gamma$, we have $g^N \in \Gamma_0$. Then Γ is also discrete.*

Proof We have that G acts by isometries on an associated symmetric space X of noncompact type. Since Γ_0 is a lattice, there exists $\epsilon > 0$ such that any semisimple element of Γ_0 has translation length on X at least ϵ. Since G is simple and Γ is Zariski dense, it follows that Γ is either discrete or dense in G. We argue by contradiction. If Γ is dense, then since the property of being semisimple is an open condition and since translation lengths of semisimple elements of G coincide with $\mathbb{R}_{>0}$, there exists a semisimple element $g \in \Gamma$ such that the translation length of g is less than $\epsilon/2N$. Hence g^N is a semisimple element with translation length at most $\epsilon/2$. In particular, $g^N \notin \Gamma_0$, which yields a contradiction. □

We remark that Lemma 2.3 is false for merely discrete subsets of G, since even the square roots of a fixed matrix can fail to be a discrete set. If G has rank one then one can allow Γ_0 to be a more general subset of G.

Let G be a semisimple Lie group and let $\Gamma < G$ be a subgroup. As usual, we write $\mathrm{Comm}_G(\Gamma)$ to denote its commensurator in G. We shall need the following special case of a general theorem of Borel [8, Theorem 2]; see Zimmer [37, page 123]. This will be the only real use of arithmeticity of the ambient lattice Γ in Theorem 1.3. Strictly speaking, the statement of Proposition 6.2.2 in [37] is for the full group of integral points in an ambient group. The reader will note however that the only salient feature of the group of integral points which is used is its Zariski density. Thus, we obtain the following conclusion:

Proposition 2.4 *Let $\Gamma < G$ be an arithmetic lattice in a semisimple algebraic $\mathbb{Q}$–group and let $K < \Gamma$ be a Zariski-dense subgroup. Then $\mathrm{Comm}_G(K) < \mathrm{Comm}_G(\Gamma)$. Suppose furthermore that the center of G is trivial. Then $\mathrm{Comm}_G(\Gamma)$ coincides with the $\mathbb{Q}$–points of G.*

The hypothesis that G has trivial center in the second part of Proposition 2.4 is crucial. For instance, the commensurator of $\mathrm{SL}_2(\mathbb{Z})$ properly contains $\mathrm{SL}_2(\mathbb{Q})$. The reader will observe that throughout this paper, we will implicitly assume that K is a Zariski-dense subgroup of an arithmetic lattice, though in the statement of Theorem 1.3, we only assume that K is infinite and normal. This latter assumption implies that K is indeed Zariski dense:

Proposition 2.5 *Let $K < \Gamma$ be an infinite normal subgroup of an irreducible lattice in a semisimple algebraic group G. Then K is Zariski dense in G.*

Proof Let Λ denote the limit set of K. Since K is infinite, $\Lambda \neq \varnothing$, since the limit set consists of the limit points of K in the Furstenberg boundary of G. Let $p \in \Lambda$. If $\gamma \in \Gamma$ then $\gamma(p) \in \Lambda$, since K is normal in Γ. It follows that Λ is a closed, nonempty Γ–invariant subset of the Furstenberg boundary. It therefore contains all of the limit set of Γ by the lemma in Section 3.6 of [4]. It follows that Λ is equal to the limit set of Γ.

Since Γ is Zariski dense, so is K. Else, if K were contained in a proper Lie subgroup $H < G$, then Λ would be contained in the Furstenberg boundary of H, which in turn is not Zariski dense in the Furstenberg boundary of G. However, the limit set of Γ is Zariski dense in the boundary: see the remarks at the beginning of Section 3 of [4], especially the lemma in Section 3.6. This is a contradiction. □

The following technical fact will be used several times in this paper, and we extract it for modularity.

Lemma 2.6 *Let $K < G$ be a Zariski-dense subgroup of a simple algebraic group G, and let*

$$K^G = \langle \{K^g \mid g \in \mathrm{Comm}_G(K)\}\rangle$$

be the subgroup of G generated by the conjugates of K by $g \in \mathrm{Comm}_G(K)$. If K^G is a discrete subgroup of G, then $\mathrm{Comm}_G(K)$ is discrete.

It is a trivial though useful observation that $K^G < \mathrm{Comm}_G(K)$.

Proof of Lemma 2.6 We have immediately that $K < \mathrm{Comm}_G(K)$, since K normalizes itself. We therefore conclude that $\mathrm{Comm}_G(K)$ is Zariski dense and hence is either discrete or dense in G. If $\mathrm{Comm}_G(K)$ is dense then there is a sequence $g_i \to 1$ of nontrivial group elements in $\mathrm{Comm}_G(K)$ converging to the identity. We write $K_i = K^{g_i}$, and we observe that $K_i < K^G$ for each i. Choosing finitely many elements $\{k_1, \dots, k_m\} \subset K$ which generate a Zariski-dense subgroup $K_0 < G$, we have that if g_i is nontrivial then it cannot fix the entire collection $\{k_1, \dots, k_m\}$, since then g_i would centralize K_0, contradicting Zariski density of K_0 and the simplicity of G. However, as i tends to infinity, the conjugation action of g_i on $\{k_1, \dots, k_m\}$ tends to the identity. Thus, viewing G as a matrix group, we have that $\{k_1^{g_i}, \dots, k_m^{g_i}\}$ converges to $\{k_1, \dots, k_m\}$ in any matrix norm. Since $K_0^{g_i} < K_i < K^G$, the last of which is discrete, we have that $\{k_1^{g_i}, \dots, k_m^{g_i}\} = \{k_1, \dots, k_m\}$ elementwise for $i \gg 0$, and hence that g_i commutes with K_0 for $i \gg 0$. Again using the fact that K_0 is Zariski dense and G is simple and hence center-free, we conclude that g_i is the identity for $i \gg 0$. This is a contradiction, whence it follows that $\mathrm{Comm}_G(K)$ is discrete. □

The argument in Lemma 2.6 even shows that only the set

$$\bigcup_{g \in \mathrm{Comm}_G(K)} K_0^g$$

need be discrete in order to conclude the discreteness of $\mathrm{Comm}_G(K)$, for an arbitrary Zariski-dense subgroup $K_0 < K$.

3 Preservation of lines with holes

In this section, we develop some ideas which originate in homological algebra and which play a central role in this paper, with the goal of producing a criterion for showing that a particular group element does not commensurate a given subgroup. The historical motivation comes from Chevalley–Weil theory — see Chevalley, Weil and Hecke [12] and Gaschütz [15] — and which we developed in [21] under the name of a *pseudoaction*.

Throughout this section, let $\Gamma < G$, let $K < \Gamma$ be a normal subgroup, and let $g \in \mathrm{Comm}_G(\Gamma)$. We write $Q = \Gamma/K$ for the quotient group. Conjugating by $g \in G$, we obtain groups $K^g < \Gamma^g$ and a corresponding quotient $Q^g := \Gamma^g/K^g$.

For $\gamma \in \Gamma$, we shall refer to the cyclic group $\langle \gamma \rangle$ as a γ*–line in* Γ. Further, any finite-index subgroup $\langle \gamma^N \rangle$ of $\langle \gamma \rangle$ — considered for arbitrary $\gamma \in \Gamma$ and a positive integer N — will be referred to as a Γ*–line with holes*. For any $\gamma \in \Gamma$ and $g \in \mathrm{Comm}_G(\Gamma)$, there exists a positive integer N such that $(\gamma^g)^N \in \Gamma$. Hence, for any $\gamma \in \Gamma$, and $g \in \mathrm{Comm}_G(\Gamma)$, (the conjugation action by) g sends some γ–line with holes to a Γ–line with holes.

Definition 3.1 The element $g \in \mathrm{Comm}_G(\Gamma)$ *preserves Q–lines with holes* if for all $\gamma \in \Gamma$ there exists an integer $N > 0$ such that

$$\gamma^n \equiv (\gamma^n)^g \mod K$$

for all $n \in N\mathbb{Z}$. That is, there exists $N > 0$ such that $x_m = [\gamma^{mN}, g] \in K$ for all $m \in \mathbb{Z}$.

Thus if γ^N and $(\gamma^N)^g$ should both be elements of Γ (which they are after passing to multiples of a sufficiently large N, since g commensurates Γ), then one can compare their images in $Q = \Gamma/K$. If g preserves Q–lines with holes then they must represent the same element of Q. A special case of Definition 3.1 is given by the following:

Definition 3.2 In Definition 3.1, if we specialize to the case where K is the commutator subgroup $[\Gamma, \Gamma]$ (so that in particular $Q = H_1(\Gamma, \mathbb{Z})$), we say that g *preserves homological lines with holes in* Γ.

The usefulness of preservation of homological lines with holes will become apparent when one considers its cohomological consequences in Section 4.1. For now, consider the set of all elements $g \in \mathrm{Comm}_G(\Gamma)$ that preserve Q–lines with holes. It is not difficult to see that this subset of G is actually a monoid. Clearly the identity lies in this set. Moreover, if g and h preserve Q–lines with holes, then for all $\gamma \in \Gamma$, there is an $N = N(g, \gamma)$ such that $[\gamma^N, g] \in K$. Then, $(\gamma^N)^g = \gamma^N \cdot k \in \Gamma$, so there is an $M = M(h, \gamma^N \cdot k)$ such that $[(\gamma^N \cdot k)^M, h] \in K$. This shows that

$$\gamma^{NM} \equiv (\gamma^{NM})^{gh} \mod K,$$

which implies that the set of elements of $\mathrm{Comm}_G(\Gamma)$ which preserve Q–lines with holes does in fact form a monoid. It is not clear that inversion of elements is possible within this set, however. We will not require this monoidal structure in the sequel, though we abstract out the following fact:

Observation 3.3 *Consider the set $C \subset \mathrm{Comm}_G(\Gamma)$ consisting of elements which preserve Q–lines with holes. Then C is closed under multiplication of group elements and contains the identity, and is therefore a monoid. In particular, if $K_1, K_2 \subset C$ are subgroups, then the group*

$$\langle K_1, K_2 \rangle < \mathrm{Comm}_G(\Gamma)$$

is contained in C.

The following is the basic result about preservation of Q–lines with holes.

Theorem 3.4 *Let $\Gamma < G$, let K be a normal subgroup of Γ, and let $Q = \Gamma/K$. Suppose that*

$$g \in \mathrm{Comm}_G \Gamma \cap \mathrm{Comm}_G K.$$

Then K^g preserves Q–lines with holes.

Proof Let $z \in K^g$ and let $\gamma \in \Gamma$ be arbitrary fixed elements. For $N \gg 0$ we have that $\gamma^N \in \Gamma \cap \Gamma^g$ and $(\gamma^N)^z \in \Gamma$. Let $a = (\gamma^N)^z$ and $b = \gamma^N$. We have that $a^m, b^m \in \Gamma$ for all $m \in \mathbb{Z}$.

Since $z \in K^g$ and since K^g is normal in Γ^g, we have that

$$a \equiv b \mod K^g.$$

Hence, for all $m \in \mathbb{Z}$,

$$a^m \equiv b^m \mod K^g.$$

Thus, the commutators

$$x_m := [\gamma^{mN}, z] = a^m b^{-m}$$

have the property that $x_m \in K^g$ for all $m \in \mathbb{Z}$. It is also clear that $x_m \in \Gamma$ for all $m \in \mathbb{Z}$.

Since K and K^g are commensurable, the collection of elements

$$\{x_m = a^m b^{-m}\}_{m \in \mathbb{Z}}$$

has the property that for some $s \neq t$, the elements $x_s = a^s b^{-s}$ and $x_t = a^t b^{-t}$ lie in the same right coset of $K \cap K^g$ in K^g, as follows immediately from the pigeonhole principle.

It follows that there exists an element $k \in K$ such that

$$k a^s b^{-s} = a^t b^{-t}.$$

Therefore, we see that

$$a^{-t} k a^s = b^{s-t},$$

which furnishes an element $k' \in K$ such that $k' a^{s-t} = b^{s-t}$.

Thus, there exists $M = s - t \neq 0$ such that $a^M \equiv b^M \mod K$. In particular, z preserves Q–lines with holes, the desired conclusion. □

In the sequel, we will be interested in specific cases in which Q–lines with holes are preserved, and especially the case where Q is the integral homology of Γ/K.

We now discuss a mild generalization of the notion of preserving homological lines with holes in Definition 3.2. Let $Q = \Gamma/K$ be a quotient group. Clearly, $H_1(Q, \mathbb{Z})$ is a quotient of $H_1(\Gamma, \mathbb{Z})$.

Let $\gamma \in \Gamma$ and let $g \in \mathrm{Comm}_G(\Gamma)$. There is an integer $N > 0$ such that $\{\gamma^n, (\gamma^g)^n\} \subset \Gamma$ for all $n \in N\mathbb{Z}$. We can then compare the homology classes of γ^n and $(\gamma^g)^n$ in $H_1(\Gamma, \mathbb{Z})$, and hence in $H_1(Q, \mathbb{Z})$. As before, we say that g *preserves homological lines with holes* in Q if for all $\gamma \in \Gamma$, there exists an integer $N > 0$ such that for all $n \in N\mathbb{Z}$, the homology classes of γ^n and $(\gamma^g)^n$ in $H_1(Q, \mathbb{Z})$ are equal.

Let Q^{ab} denote the abelianization of Q. Then the condition that g preserves homological lines with holes in Q is equivalent to saying that g preserves Q^{ab}–lines with holes in the sense of Definition 3.1.

When $b_1(Q) > 0$ then Theorem 3.4 above furnishes the following commensurability criterion, whose proof is straightforward now.

Theorem 3.5 *Let $Q = \Gamma/K$, let*

$$g \in \mathrm{Comm}_G \Gamma \cap \mathrm{Comm}_G K.$$

Then K^g preserves homological lines with holes in Q.

Proof Let Q_0 be a quotient of Q, and let $h \in K^g$. Since h preserves Q–lines with holes by Theorem 3.4, it also preserves Q_0–lines with holes. Specializing to $Q_0 = Q^{\mathrm{ab}}$ proves the result. □

In particular, when the commensurator of Γ in G contains the commensurator of K, we have that

$$K^G = \langle K^g \mid g \in \mathrm{Comm}_G(K) \rangle$$

preserves homological lines with holes in Q. We remark that in our applications, $\mathrm{Comm}_G K < \mathrm{Comm}_G \Gamma$ by Proposition 2.4.

4 Homological lines with holes and Hodge theory

The goal of this section is to translate between preservation of lines with holes and the existence of commensuration-invariant harmonic 1–forms. We shall first deduce cohomological consequences of preserving homological lines with holes.

4.1 Preserving homological lines with holes and cohomological consequences

For the purposes of this subsection, let G denote a semisimple Lie group with no compact factors, with associated symmetric space of nonpositive curvature X. Let Γ be a lattice in G and let $g \in \mathrm{Comm}_G(\Gamma)$. We write $S = X/\Gamma$ and $S^g = X/\Gamma^g$. Since $g \in \mathrm{Comm}_G(\Gamma)$, the group $\Gamma \cap \Gamma^g$ is of finite index in both Γ and Γ^g. Let $W = X/(\Gamma \cap \Gamma^g)$ denote the corresponding common cover of S and S^g. We shall refer to S and S^g as *conjugate manifolds* and W as their *minimal common cover*. Here, W depends on g. However, since g will be fixed throughout, we will suppress it from the notation. We will also fix a differential 1–form ω on S. Let $p\colon X \to S$ denote the universal covering map. Note that the 1–form $p^*\omega$ is a 1–form on X. In applications in the sequel, ω will be a harmonic form.

The element $g \in G$ is an isometry of X and hence acts on differential forms on X via pullback. The form $g^* p^* \omega$ is a 1–form on X which is invariant under Γ^g and hence descends to S^g. The resulting 1–form on the quotient manifold S^g is denoted by ω^g. Let $q\colon W \to S$ and $q^g\colon W \to S^g$ denote the natural covering maps. Denote $q^*\omega$ by ω_W and $(q^g)^*\omega^g$ by ω_W^g.

We shall also need to set up notation for g–conjugates of cycles and loops, as basepoints will play an important role in what follows. Let $o \in W$ be a basepoint. By choosing a lift $\widetilde{o} \in X$ and by joining $\widetilde{o}$ to $g.\widetilde{o}$

by a geodesic segment in X and projecting back to W, we obtain a natural geodesic segment $[o, g.o]$ in W, where $g.o$ denotes the image of $g.\widetilde{o}$ under the covering projection. Thus, $g.o$ may be regarded as a new basepoint for integrating chains against a pulled back form.

Now suppose that α is a loop in W representing an element $h \in \pi_1(W)$ such that h^g also belongs to $\pi_1(W)$, where here we have identified $\pi_1(W)$ with $\Gamma \cap \Gamma^g$. Lifting α to a path $\widetilde{\alpha}$ in X, translating by g and quotienting X by $\Gamma \cap \Gamma^g$ we obtain a new loop denoted $g.\alpha$ on W based at $g.o$. Here, we use notation that is similar to the case of a genuine g–action on W, though the action is well-defined only on the universal cover X.

The concatenation $[o, g.o] * g.\alpha * \overline{[o, g.o]}$ gives a loop based at o, where $\overline{[o, g.o]}$ denotes $[o, g.o]$ parametrized in the opposite direction from $g.o$ to o. We denote this loop as α^g:

$$\alpha^g = [o, g.o] * g.\alpha * \overline{[o, g.o]}.$$

Finally, for σ any closed, oriented loop on W, based at o say, the n^{th} power of the loop σ will be the loop which traverses the loop σ a total of n times. The result will be denoted by σ^n.

Remark 4.1 A subtlety in the following lemma needs to be noted. On the one hand, the hypothesis is about preserving homological lines with holes in Γ. The conclusion, on the other hand, is about cohomology classes in the common minimal cover W. The reason for this is that the pullback of ω to X and its pullback by g are both invariant under $\Gamma \cap \Gamma^g$, though not necessarily by Γ nor Γ^g. Thus, ω_W^g is well-defined as a form on W, but does not necessarily live in S.

Lemma 4.2 *Let*

$$\{\Gamma, S, g, S^g, W, \omega_W, \omega_W^g\}$$

be as above, Suppose that g preserves homological lines with holes in Γ. Then we have $[\omega_W] = [\omega_W^g]$ as elements of $H^1(W, \mathbb{R})$.

The importance of Lemma 4.2 will become apparent in Section 4.2, particularly Corollary 4.7. It follows from the Hodge theorem that if ω_W is a harmonic form representing $[\omega_W] \in H^1(W, \mathbb{R})$, then $\omega_W = \omega_W^g$ as forms, and not just as cohomology classes.

Proof We continue with the notation from the discussion before the statement of the lemma. Let σ be any closed loop on W based at o. Since g commensurates Γ, we may choose $n > 0$ such that σ^n and $(\sigma^n)^g$ are both cycles, and so are viewed as loops based at o. Observe that if h denotes the element of $\pi_1(W, o)$ represented by σ^n then the loop $(\sigma^n)^g$ represents the group element $h^g \in \pi_1(W, o)$.

Since g is assumed to preserve homological lines with holes in Γ, there exists an integer $N > 0$ such that σ^N and $(\sigma^N)^g$ represent the same element of $H_1(S, \mathbb{Z})$. Indeed, for any differential 1–form ω on S, we have

$$\int_{q(\sigma^N)} \omega = \int_{q((\sigma^N)^g)} \omega = \int_{q(g.\sigma^N)} \omega, \tag{1}$$

where $q\colon W \to S$ is the covering projection, and where the second inequality holds because the integrals of ω along $[o, g.o]$ and $\overline{[o, g.o]}$ cancel each other. Note that the integrals in equation (1) are over S.

Next, by the definition of the pullback form $\omega_W = q^*\omega$, we have that

$$\int_{\sigma^N} \omega_W = \int_{q(\sigma^N)} \omega \quad \text{and} \quad \int_{(\sigma^N)^g} \omega_W = \int_{q((\sigma^N)^g)} \omega.$$

Combining the equations above, we obtain

$$\int_{\sigma^N} \omega_W = \int_{(\sigma^N)^g} \omega_W = \int_{g.\sigma^N} \omega_W, \tag{2}$$

where all the integrals in equation (2) are over W.

Finally, we observe that by the definition of the pullback ω_W^g, we have

$$\int_{g.(\sigma^N)} \omega_W = \int_{\sigma^N} \omega_W^g, \tag{3}$$

again using the fact that the integrals of ω_W along $[o, g.o]$ and $\overline{[o, g.o]}$ cancel each other.

Putting all these equalities together, we obtain

$$\int_{\sigma^N} \omega_W = \int_{\sigma^N} \omega_W^g. \tag{4}$$

Since

$$\int_{\sigma^N} \omega_W = N \int_{\sigma} \omega_W,$$

we conclude that

$$\int_{\sigma} \omega_W = \int_{\sigma} \omega_W^g \tag{5}$$

for any closed loop σ in W based at o. The forms ω_W and ω_W^g represent well-defined elements of $H^1(W, \mathbb{R})$, by their very definition. By equation (5) above they have the same periods, and since they are both closed differential forms, they are cohomologous. □

The cohomological consequence of preserving homological lines with holes in quotients is the following (cf Remark 4.1):

Lemma 4.3 *Let $Q = \Gamma/K$, let $g \in \mathrm{Comm}_G(\Gamma)$ preserve homological lines with holes in Q, and let $\omega \in H^1(Q, \mathbb{R})$. Then the periods of $[\omega_W]$ and $[\omega_W^g]$ agree, where W is the common minimal cover of $S = X/\Gamma$ and its conjugate manifold $S^g = X/\Gamma^g$, and where ω_W is the pullback of ω to $H^1(W, \mathbb{R})$.*

Proof Let $\omega \in H^1(Q, \mathbb{R})$ be a nontrivial cohomology class. Then the quotient map $q\colon \Gamma \to Q$ induces a pullback form $q^*\omega \in H^1(\Gamma, \mathbb{R})$, which can be viewed as a differential form on $S = X/\Gamma$. The map q also induces a map $q_*\colon H_1(\Gamma, \mathbb{Z}) \to H_1(Q, \mathbb{Z})$. If σ is any 1–cycle on X/Γ then by definition

$$\int_{\sigma} q^*\omega = \omega(q_*\sigma),$$

where the right-hand side denotes the evaluation of ω on $q_*(\sigma)$ (recall ω is a cohomology class of Q).

Writing ω_W for the form on W given by pullback of $q^*\omega$ along the covering map $p\colon W \to S$, we have that ω_W^g and ω_W have the same periods, provided that g preserves homological lines with holes in Q. A justification of this claim is identical to that in the proof of Lemma 4.2. □

We note the following easy observation (cf Observation 3.3 above).

Observation 4.4 *Consider the set $C \subset \mathrm{Comm}_G(\Gamma)$ consisting of elements which preserve homological lines with holes in Q. Then C is closed under multiplication of group elements and contains the identity, and is therefore a monoid. In particular, if $K_1, K_2 \subset C$ are subgroups, then the group $\langle K_1, K_2\rangle$ is contained in C.*

4.2 Hodge theory

Hodge theory will allow us to leverage preservation of homological lines with holes in order to promote equality of cohomology classes to equality of forms. We recall the necessary tools from Hodge theory and L^2–cohomology that we shall need. Let M be a (not necessarily compact) Riemannian manifold. We fix notation: Ω^k will denote the space of smooth k–forms, d will denote the differential on forms, $*$ will denote the Hodge star operator, d^* will denote the adjoint of d, and $\Delta = dd^* + d^*d$ will denote the Laplacian on forms. A form $\omega \in \Omega^k$ is a *harmonic k–form* for the given metric on M if $\Delta\omega = 0$. Harmonic forms are closed and coclosed.

Theorem 4.5 [35, Chapter 6] *Let M be a compact Riemannian manifold. Then for all k and every real cohomology class $[\omega] \in H^k(M,\mathbb{R})$, there exists a unique harmonic form ω_{harm} representing $[\omega]$.*

We shall need a version of Theorem 4.5 for noncompact complete manifolds M. The appropriate cohomology theory used is L^2–cohomology. Let Ω_2^k denote the space of smooth square-integrable k–forms. The reduced L^2–cohomology groups are given by

$$H^k_{(2)}(M) = \ker(d)/\overline{\mathrm{Im}(d)},$$

where $\overline{\mathrm{Im}(d)}$ denotes the closure of the image of d. We refer the reader to [10] for more details. We shall need only the following special case (see [10, Lemma 1.5] due to Gaffney, or [11] for instance):

Theorem 4.6 *Let M be a complete negatively curved manifold of finite volume modeled on $\mathbb{H}^n$ or $\mathbb{CH}^n$. Then for every real cohomology class $[\omega] \in H^1_{(2)}(M,\mathbb{R})$, there exists a unique L^2 harmonic form ω_{harm} representing $[\omega]$.*

Note that a compactly supported cohomology class is an L^2 class. Thus in our context, if X/Γ has nontrivial real cohomology with compact supports, then we can find nontrivial L^2 harmonic forms representing such cohomology classes. In our analysis of the case $b_1(Q) > 0$ for groups arising as quotients of nonuniform lattices Γ, the absence of a nonzero L^2 harmonic 1–form will (roughly) allow us to assume that $H^1_c(X/\Gamma,\mathbb{R}) = 0$. See the proof of Theorem 5.1 below.

We recall the setup of Lemma 4.2 in a slightly restricted setting: we are given a lattice Γ in a group $G \in \{\mathrm{SO}(n,1), \mathrm{SU}(n,1)\}_{n\geq 2}$ with associated symmetric space of noncompact type X, and an element $g \in G$ commensurating Γ. We have an orbifold $S = X/\Gamma$, the conjugate manifold $S^g = X/\Gamma^g$, the common refinement $W = X/(\Gamma \cap \Gamma^g)$ and a cohomology class $\omega \in H^1(S,\mathbb{R})$. We assume the existence of a (possibly L^2) harmonic representative ω_{harm} of ω, whose uniqueness is then guaranteed by Theorems 4.5 and 4.6. Note that such a harmonic representative may not exist only in the case where S is noncompact.

We will also call the resulting harmonic form ω as it will not cause confusion. Recall the notation

$$p\colon X \to S, \quad W, \quad \omega_W, \quad \omega_W^g,$$

from Section 4.1. For convenience, we will denote $p^*\omega$ by ω_X and $g^*\omega_X$ by ω_X^g, where g^* is the action on 1–forms induced by the isometry g of X.

Corollary 4.7 *Assume the above setup, and suppose that g preserves homological lines with holes in Γ. Then the harmonic representatives of ω_W and ω_W^g are equal as differential 1–forms on W. In particular, the harmonic representatives of ω_X and ω_X^g are equal.*

Proof Since g acts on X by an isometry, the pullback of a harmonic form under g is also harmonic; see Section 4 of [14], for example. Thus, ω_W^g is a form on W which is cohomologous to the form ω_W, by Lemma 4.2. Since $\Gamma \cap \Gamma^g$ has finite index in Γ, we have that W still has finite volume and hence the suitable Hodge theorem (Theorem 4.5 or 4.6) applies, whence the harmonic representatives of ω_W and ω_W^g are equal. The equality of forms on X is immediate. □

A part of the remainder of the paper will deal with the case where there is no harmonic form representing a nontrivial homology class, which is to say a complement to Corollary 4.7 adapted to cusped orbifolds.

4.3 The commensurator of a form

The notion of the commensurator of a form will now be introduced. It will be shown that under suitable hypotheses, K^G lies in the commensurator of a harmonic form, as is forced by preservation of homological lines with forms. The rigid nature of the harmonic form will force it to be zero whenever K^G fails to be discrete, which only occurs if $\mathrm{Comm}_G(K)$ is dense. As before, cohomology with compact supports will be denoted by $H_c^*(\cdot)$.

Definition 4.8 Let $\Gamma < G$ be a lattice in a semisimple Lie group G with associated symmetric space X, and let $S = X/\Gamma$. Let ω be a closed form such that $[\omega] \in H^p(S,\mathbb{Q})$ or $[\omega] \in H_c^p(S,\mathbb{Q})$ is a nonzero cohomology class. Let $p\colon X \to S$ denote the universal cover. The *commensurator* $\mathrm{Comm}(\omega)$ of the form ω is defined as

$$\mathrm{Comm}(\omega) = \{h \in G \mid h^* p^* \omega = p^* \omega\}.$$

A subgroup H of G is said to commensurate ω if $H < \mathrm{Comm}(\omega)$. It is immediate the $\mathrm{Comm}(\omega)$ is itself a group.

We have the following general discreteness result that applies to the isometries of real and complex hyperbolic spaces. We will not consider isometries of quaternionic hyperbolic spaces or the Cayley plane; see the remarks following Theorem 1.3. We direct the reader to [34; 1], from which the main idea used in the following proposition is taken.

Proposition 4.9 *Let X be $\mathbb{H}^n$ or $\mathbb{CH}^n$. For Γ a torsion-free lattice, let $S = X/\Gamma$. Let ω be a nonzero harmonic or L^2–harmonic 1–form according to whether S is compact or noncompact. Then $\mathrm{Comm}(\omega)$ is discrete.*

Proof Let $p\colon X \to S$ denote the universal cover. We now argue by contradiction. Suppose that $\mathrm{Comm}(\omega)$ is not discrete. Since the associated Lie group G (ie $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$) is simple, it follows that $\mathrm{Comm}(\omega)$ is dense in G, as $\mathrm{Comm}(\omega)$ contains the Zariski-dense subgroup Γ. Also, since $\mathrm{Comm}(\omega)$ preserves $p^*(\omega)$, we have that G must preserve $p^*(\omega)$, since G is identified with the group of isometries of X. That is, $p^*(\omega)$ is a G–invariant nonzero harmonic 1–form on X. (Note that here, compactness or noncompactness of S is not relevant, as $p^*(\omega)$ being defined on X is all that we are concerned with at this stage.) Hence $p^*(\omega)$ gives a nonzero harmonic differential 1–form ω^* on the compact dual of $\mathbb{H}^n$ or $\mathbb{CH}^n$; see Venkataramana [34] and Agol [1], cf Sections 2 and 3 of Chapter II in [9]. Since the compact duals S^n and $\mathbb{CP}^n$ of $\mathbb{H}^n$ and $\mathbb{CH}^n$ respectively have trivial first cohomology (at least when $n \geq 2$), this is a contradiction. □

From Lemma 4.3, we obtain the following consequence:

Corollary 4.10 *Suppose Γ is torsion-free. Let $Q = \Gamma/K$, and let $C \subset \mathrm{Comm}_G(\Gamma)$ denote the set of elements which preserve homological lines with holes in Q. If there exists a (possibly L^2) harmonic form on $S = X/\Gamma$ representing a pullback of a nonzero cohomology class of Q, then C is discrete.*

Proof Let ω be the harmonic representative of a form on S arising by pullback from Q, and let $g \in C$. Then by Lemma 4.3 and Corollary 4.7, we have that $\omega_W = \omega_W^g$ as forms, by either classical or L^2–Hodge theory, and where here W is the common refinement of S and its conjugate S^g. Pulling back these forms to the universal cover X, we have that $g \in \mathrm{Comm}(\omega)$. By Proposition 4.9, we conclude that C is discrete. □

5 Abelian quotients and harmonic 1–forms

We are now in a position to assemble the pieces to prove Theorem 1.3. The ideas to establish the result naturally bifurcate:

(i) The vanishing cuspidal case, amenable to L^2–cohomology techniques. For $\mathrm{PSL}_2(\mathbb{R})$, this is the case where the underlying hyperbolic surface has genus greater than zero. This part of the argument uses Hodge theory.

(ii) The nonvanishing cuspidal case, where discrete patterns of horoballs are used to obtain discreteness of the commensurator; see the appendix. For $\mathrm{PSL}_2(\mathbb{R})$, this is the case where the underlying hyperbolic surface has genus equal to zero, and compactly supported cohomology vanishes. This part of the argument borrows heavily from the ideas in [21].

5.1 Proof of Theorem 1.3

We now establish part of the main result of this paper:

Theorem 5.1 *Let $\Gamma < G$ be a lattice in a rank-one simple Lie group. Let $K < \Gamma$ be an infinite normal subgroup, and let $Q = \Gamma/K$. If the first Betti number of Q satisfies $b_1(Q) > 0$ then $\mathrm{Comm}_G(K)$ is discrete.*

Here, the lattice may or may not be torsion-free, and may or may not be uniform. As remarked in the introduction, we only consider lattices in $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$.

Proof We begin by passing to a torsion-free finite-index subgroup Γ' of Γ, and by replacing K with the corresponding finite-index subgroup of K given by the corresponding intersection $K \cap \Gamma'$. The resulting subgroup of K is commensurable with K and hence has the same commensurator in G as K. Moreover, by restricting the quotient map $\Gamma \to Q$ to Γ', we get a finite-index subgroup $Q' < Q$ which also has positive first Betti number. Thus without loss of generality, we will assume that $\Gamma = \Gamma'$.

Recall that we write
$$K^G = \langle K^g \mid g \in \mathrm{Comm}_G(K)\rangle$$
for the subgroup generated by the collection $\{K^g\}$, as g ranges over $\mathrm{Comm}_G(K)$. By Proposition 2.4, we have that $\mathrm{Comm}_G(K) < \mathrm{Comm}_G(\Gamma)$. By Theorem 3.5 and Observation 4.4, we have that if $y \in K^G$, then y preserves homological lines with holes in Q.

By hypothesis, we have $H^1(Q,\mathbb{R}) \neq 0$. Writing $S = X/\Gamma$ as usual, we have that $H^1(S,\mathbb{R}) \neq 0$ since Q is a quotient of Γ and since $\Gamma = \pi_1(S)$. We have that S is metrically complete and is either compact or noncompact, which yields two possible cases concerning cohomology:

(i) ***S* is compact** By Theorem 4.5, there is a harmonic form ω on S which represents the pullback of a nontrivial cohomology class of Q.

(ii) ***S* is not compact** This case bifurcates into further possibilities:

(a) The composition
$$H^1(Q,\mathbb{R}) \to H^1(S,\mathbb{R}) \to H^1(\partial S,\mathbb{R})$$
has a nontrivial kernel, where the first map is the pullback along the quotient map $\Gamma \to Q$ and the second map is the pullback along the inclusion map $\partial S \to S$. Note that the first arrow is an injection. Furthermore,
$$H^1((S,\partial S),\,\mathbb{R}) = H^1_c(S,\,\mathbb{R}) = H^1_{(2)}(S,\,\mathbb{R}).$$

See [23, Lemma 1.93]. Hence, by Theorem 4.6, there is a nonzero cohomology class of S that is represented by a nonzero L^2 harmonic form ω such that $[\omega] \in H^1_{(2)}(S, \mathbb{R})$ is the pullback of a cohomology class of Q.

(b) The composition

$$H^1(Q, \mathbb{R}) \to H^1(S, \mathbb{R}) \to H^1(\partial S, \mathbb{R})$$

is injective.

In case (ii), we interpret ∂S in the usual way, ie by removing a small horoball around the cusps of S, whereby the boundary of S becomes the image of the horosphere bounding the horoball.

Suppose first that there exists a nontrivial (possibly L^2) harmonic form on $S = X/\Gamma$ representing a pullback of a nontrivial class in $H^1(Q, \mathbb{Q})$, as in case (i) or (ii)(a) above. Then K^G is discrete by Corollary 4.10. That $\mathrm{Comm}_G(K)$ is discrete now follows from Lemma 2.6.

If no such form exists, then we are in case (ii)(b). Writing $q\colon \Gamma \to Q$ for the quotient map, we have that

$$q_* \circ i_*\colon H_1(\partial S, \mathbb{Q}) \to H_1(Q, \mathbb{Q})$$

is surjective, where $i\colon \partial S \to S$ denotes inclusion. Because $H_1(Q, \mathbb{Q}) \neq 0$ by hypothesis, there exists a finite collection of cusps $\{T_1, \dots, T_k\}$ of S which contain homology classes $z_j \in H_1(T_j, \mathbb{Q})$ for which

$$q_* \circ i_*(z_j) \neq 0.$$

For $1 \leq j \leq k$, let $t_j \in \partial X$ denote the basepoint (at infinity) of a horoball lift of T_j to X. Let $\mathcal{T}_j$ denote the set of the Γ–translates of t_j in ∂X. Also, let $\mathcal{H}_j$ (resp. $\partial\mathcal{H}_j$) denote the collection of horoballs (resp. horospheres) in X that are lifts of T_j (resp. ∂T_j). These are an instance of a *discrete pattern* in the sense of Schwartz [29]; see Definition A.3 below, for instance. Let $\Gamma_j < G$ denote the subgroup preserving the collection $\partial\mathcal{H}_j$. By [27, Propositions 5.3 and 5.4] (see Lemma A.6 for instance), the group Γ_j is a lattice containing Γ as a subgroup of finite index.

We complete the proof assuming Claim 5.2 below. It follows from Claim 5.2 that each element of K^G has a uniformly bounded power contained in the discrete group $\bigcap_{s=1}^k \Gamma_s$. Hence K^G is discrete by Lemma 2.3. Lemma 2.6 now implies that $\mathrm{Comm}_G(K)$ itself is discrete. □

Claim 5.2 *There is an $N > 0$ such that for all $y \in K^G$, we have*

$$y^N \in \bigcap_{s=1}^k \Gamma_s.$$

Proof By Theorem 3.5, we know that K^G preserves homological lines with holes in Q. Choose parabolic subgroups $\{G_1, \dots, G_k\}$ of G, which we use to identify $\pi_1(T_j)$ as a subgroup of $\pi_1(S)$ for $1 \leq j \leq k$, and let $\{x_1, \dots, x_k\} \subset \partial X$ be their respective fixed points. Let $\gamma \in \Gamma$ be a parabolic isometry representing $z_j \in H_1(T_j, \mathbb{Z})$, and such that $q_* \circ i_*(z_j)$ is nonzero. Replacing γ by a conjugate in Γ if necessary,

γ fixes x_j and hence lies in G_j. Let $y \in K^G$. Since y preserves homological lines with holes in Q, there exists a positive integer m such that

$$[(\gamma^m)^y] = [\gamma^m] = m \cdot q_* \circ i_*(z_j),$$

where $[\cdot]$ denotes the corresponding homology class in $H_1(Q, \mathbb{Z})$, and where elements of Γ acquire homology classes in $H_1(Q, \mathbb{Z})$ via q_*. Since $y \in G$, we have that $(\gamma^m)^y$ is also parabolic. Since y commensurates Γ (by Proposition 2.4) and preserves homological lines with holes in Q, we have that there exists $r \in \Gamma$ such that $(\gamma^m)^{yr} \in G_\ell$ for some $1 \le \ell \le k$. Thus, y preserves homological lines with holes in Q but may "change the cusp" which supports a given cuspidal homology class. Since there are only k many cusps of S which contribute to the homology of Q via $q_* \circ i_*$, for $N = k!$ we may assume that $(\gamma^m)^{y^N}$ is conjugate into G_j by an element $r \in \Gamma$. We thus have that $y^N r \in G_j$.

Now, any element of the parabolic subgroup G_j can be decomposed as $A_\lambda N_\lambda$, where A_λ acts on $\partial X \setminus \{x\}$ by a conformal homothety and N_λ acts by an isometry. Here, the metric on $\partial X \setminus \{x\}$ is obtained by identifying it with a reference horosphere in X based at x via projection along geodesics from x.

For $X = \mathbb{H}^n$, these are all Euclidean similarities and for $X = \mathbb{CH}^n$, these are all Heisenberg similarities (see [29, Section 8.1]). In particular, for any j, and for any $g \in G_j$, g scales all distances on the reference horosphere by a fixed $r_g > 0$. We call r_g the *scale factor* of g. Let

$$\hat{g}\colon H_1(T_j) \to H_1(T_j)$$

denote the induced map on $H_1(T_j)$ *thought of as a subset of* $\partial X \setminus \{x\}$. Here, we use the notation $\hat{g}$ in place of g_* to avoid confusing with the action on homology of the cusp per se. Since g scales the length of all elements by r_g, it follows that $\hat{g}(u) = r_g \cdot u$ for all $u \in H_1(T_j)$. Let $A_\lambda(y^N r) > 0$ denote the scale factor of the homothety component of $y^N r$. Write $\mathcal{H}_{x_j} \in \mathcal{H}_j$ for the horoball in X based at x_j.

Since

$$[(\gamma^m)^{y^N r}] = A_\lambda(y^M r)[\gamma^m] \in H_1(Q, \mathbb{Q}),$$

the scale factor $A_\lambda(y^N r)$ must equal one. But $A_\lambda(y^N r) = 1$ if and only if $y^N r$ preserves the horosphere $\partial\mathcal{H}_{x_j}$. Since $r \in \Gamma$ necessarily preserves $\partial\mathcal{H}_j$, it follows that y^N stabilizes $\partial\mathcal{H}_j$, ie $y^N \in \Gamma_j$. Since $y \in K^G$ and $1 \le j \le k$ were arbitrary, and since Γ_j contains $\bigcap_{s=1}^k \Gamma_s$ with finite index (as follows easily from Lemma A.6) this completes the proof of the claim. □

5.2 Applications

We conclude this section by giving three sets of examples to which Theorem 5.1 applies.

Irrational pencils in complex hyperbolic manifolds Many cocompact arithmetic lattices in $\mathrm{SU}(2,1)$ admit irrational pencils, ie $S = X/\Gamma$ admits a holomorphic fibration (with singular fibers) onto a Riemann surface of genus greater than zero. Let F denote the general fiber and $i\colon F \to S$ denote inclusion. Then $K = i_*(\pi_1(F))$ is normal in Γ and $Q = \Gamma/K$ is a surface group. Theorem 5.1 applies to show that $\mathrm{Comm}_G(K)$ is discrete. We note that M Kapovich in unpublished work [19] (see Biswas, Mj and Pancholi [6] for a small generalization) established that K is never finitely presented.

Real hyperbolic manifolds that algebraically fiber Agol [2] shows that hyperbolic 3–manifolds virtually fiber over the circle with surface group fibers. The resulting normal surface subgroups were dealt with in [22] without the arithmeticity hypothesis. However, a new family of examples of finitely generated (but not necessarily finitely presented) normal subgroups of arithmetic hyperbolic n–manifolds has recently been discovered. A classical result of Dodziuk [13] (see also Anghel [3]) shows that the first L^2–Betti number of a hyperbolic manifold of dimension greater than 2 vanishes. Kielak [20] shows that a cubulated hyperbolic group Γ is virtually algebraically fibered (ie Γ admits a virtual surjection to $\mathbb{Z}$ with a finitely generated kernel) if and only if $\beta^1_{(2)}(\Gamma) = 0$. On the other hand, Bergeron, Haglund and Wise [5] show that standard cocompact arithmetic congruence subgroups Γ of $\mathrm{SO}(n,1)$ are cubulated. Thus standard cocompact arithmetic congruence subgroups Γ of $\mathrm{SO}(n,1)$ admit finitely generated normal subgroups K with quotient $\mathbb{Z}$. This furnishes a family of examples K to which Theorem 5.1 applies to show that $\mathrm{Comm}_G(K)$ is discrete (since $b_1(Q) = b_1(\mathbb{Z}) = 1$ in this case).

Uncountably many pairwise nonisomorphic 2–generated groups P Hall produced uncountably many pairwise nonisomorphic quotients of a free group F_2 on two generators; see [17, III.C.40], for instance. Evidently, the free group on two generators can be realized as a lattice in a rank-one simple Lie group. Hall's construction produces uncountable families of 2–generated torsion-free solvable groups, and each of his groups surjects to $\mathbb{Z}$. This furnishes a continuum's worth of thin normal subgroups of lattices to which Theorem 5.1 applies.

Appendix Discrete patterns of horoballs

In the course of the proof of Theorem 5.1, case (ii)(b), we have used the fact that a certain *discrete pattern of horoballs* is preserved by K^g. Since the notion of a discrete pattern also makes its appearance in earlier approaches to Question 1.2, we give a quick account here.

Let G be a rank-one semisimple Lie group and let X be the associated symmetric space. The space X is, in a natural way, a Riemannian manifold endowed with a left-invariant metric [18]. Following [29; 30; 27; 7] we define the following (see [27, Definition 1.6] in particular):

Definition A.3 Let $\Gamma < G$ be a lattice and $S = X/\Gamma$. A Γ–*discrete pattern of points* on X is a nonempty Γ–invariant set $\mathcal{S} \subset X$ such that $\mathcal{S}/\Gamma$ is finite.

Let $\Gamma < G$ be a nonuniform lattice, and let $S = X/\Gamma$. A Γ–*discrete pattern of horoballs* in X is a nonempty Γ–invariant collection $\mathcal{S} \subset X$ of closed horoballs such that $\mathcal{S}/\Gamma$ is a disjoint union of neighborhoods of cusps.

Definition A.4 Let $\Gamma < G$ be a lattice. A subgroup H of G is said to *preserve* a Γ–discrete pattern $\mathcal{S}$ points if $h(\mathcal{S}) \subset \mathcal{S}$ for all $h \in H$.

Propositions 3.5 and 3.7 of [27] show that a subgroup H of G preserving a Γ–discrete pattern $\mathcal{S}$ is closed and totally disconnected. Since any such subgroup of G is necessarily discrete, we have the following:

Lemma A.5 [27, Propositions 3.5 and 3.7] *Let $\Gamma < G$ be a lattice and $\mathcal{S}$ a Γ–discrete pattern (of points or geodesics). Then the subgroup H of G preserving $\mathcal{S}$ is discrete, and $[H : \Gamma] < \infty$.*

Propositions 5.3 and 5.4 of [27] (see also [25, Theorem 3.11]) prove that the subgroup H of G preserving a Γ–discrete pattern of horoballs is closed and totally disconnected. It follows that:

Lemma A.6 [27, Propositions 5.3 and 5.4] *Let $\Gamma < G$ be a nonuniform lattice in a rank-one Lie group and $S = X/\Gamma$, where X is the associated symmetric space. Let $\mathcal{S}$ be a Γ–discrete pattern of horoballs. Then the subgroup H of G preserving $\mathcal{S}$ is discrete, and $[H : \Gamma] < \infty$.*

As an aside, we mention that for lattices in $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SO}(2,1)$, there are more direct ways of understanding discrete patterns, and in particular Proposition 4.9 above, that are inspired by ideas from Teichmüller theory. In this context, one can view the commensurator of a nontrivial harmonic form as explicitly producing a Γ–discrete pattern. Specifically, one can use the fact that a harmonic form is the real part of an abelian differential on the Riemann surface $\mathbb{H}^2/\Gamma$. In the case of a cocompact lattice, one can use the fact that the set of zeros of the form is nonempty and discrete, and preserved by the commensurator of the form. Then Lemma A.5 gives discreteness of the commensurator itself. In the case of a nonuniform lattice, one uses saddle connections in the Baily–Borel–Satake compactification of $\mathbb{H}^2/\Gamma$, and the fact that these are invariant under the commensurator. Again, Lemma A.5 gives discreteness of the commensurator.

Acknowledgements

The authors are grateful to D Witte Morris, T N Venkataramana, S Varma and C S Rajan for helpful discussions and correspondence, in particular for telling us about Proposition 2.4. Special thanks are due to D Fisher and W van Limbeek for many discussions and for catching an error in an earlier version of this paper. We thank D Wise for his lectures on coherence at IMPAN, Warsaw, and for references on L^2–Betti numbers [13]. We thank Y Shalom for his interest and perceptive comments.

Koberda was partially supported by an Alfred P Sloan Foundation Research Fellowship, by NSF grant DMS-1711488, and by NSF grant DMS-2002596 while this research was carried out. Mj is supported by the Department of Atomic Energy, Government of India, under project 12-R&D-TFR-5.01-0500DST, and also in part by a Department of Science and Technology J C Bose Fellowship, and an endowment from the Infosys Foundation. Both authors were partially supported by the grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund (code BCSim-2019-s11).

References

[1] **I Agol**, *Virtual Betti numbers of symmetric spaces*, preprint (2006) arXiv math/0611828

[2] **I Agol**, *The virtual Haken conjecture*, Doc. Math. 18 (2013) 1045–1087 MR Zbl

[3] **N Anghel**, *The L^2–harmonic forms on rotationally symmetric Riemannian manifolds revisited*, Proc. Amer. Math. Soc. 133 (2005) 2461–2467 MR Zbl

[4] **Y Benoist**, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. 7 (1997) 1–47 MR Zbl

[5] **N Bergeron**, **F Haglund**, **D T Wise**, *Hyperplane sections in arithmetic hyperbolic manifolds*, J. Lond. Math. Soc. 83 (2011) 431–448 MR Zbl

[6] **I Biswas**, **M Mj**, **D Pancholi**, *Homotopical height*, Int. J. Math. 25 (2014) art. id. 1450123 MR Zbl

[7] **K Biswas**, **M Mj**, *Pattern rigidity in hyperbolic spaces: duality and PD subgroups*, Groups Geom. Dyn. 6 (2012) 97–123 MR Zbl

[8] **A Borel**, *Density and maximality of arithmetic subgroups*, J. Reine Angew. Math. 224 (1966) 78–89 MR Zbl

[9] **A Borel**, **N Wallach**, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, 2nd edition, Math. Surv. Monogr. 67, Amer. Math. Soc., Providence, RI (2000) MR Zbl

[10] **G Carron**, *L^2 harmonics forms on non compact manifolds*, course notes (2006) arXiv 0704.3194

[11] **J Cheeger**, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. 76 (1979) 2103–2106 MR Zbl

[12] **C Chevalley**, **A Weil**, **E Hecke**, *Über das verhalten der integrale, I: Gattung bei automorphismen des funktionenkörpers*, Abh. Math. Sem. Univ. Hamburg 10 (1934) 358–361 MR Zbl

[13] **J Dodziuk**, *L^2 harmonic forms on rotationally symmetric Riemannian manifolds*, Proc. Amer. Math. Soc. 77 (1979) 395–400 MR Zbl

[14] **J Eells**, **L Lemaire**, *A report on harmonic maps*, Bull. Lond. Math. Soc. 10 (1978) 1–68 MR Zbl

[15] **W Gaschütz**, *Über modulare Darstellungen endlicher Gruppen, die von freien Gruppen induziert werden*, Math. Z. 60 (1954) 274–286 MR Zbl

[16] **L Greenberg**, *Commensurable groups of Moebius transformations*, from "Discontinuous groups and Riemann surfaces" (L Greenberg, editor), Ann. of Math. Stud. 79, Princeton Univ. Press (1974) 227–237 MR Zbl

[17] **P de la Harpe**, *Topics in geometric group theory*, Univ. Chicago Press (2000) MR Zbl

[18] **S Helgason**, *Differential geometry, Lie groups, and symmetric spaces*, Pure Appl. Math. 80, Academic, New York (1978) MR Zbl

[19] **M Kapovich**, *On the normal subgroups in the fundamental groups of complex surfaces*, preprint (1998) Available at `http://www.math.ucdavis.edu/%7Ekapovich/EPR/koda.pdf`

[20] **D Kielak**, *Residually finite rationally solvable groups and virtual fibring*, J. Amer. Math. Soc. 33 (2020) 451–486 MR Zbl

[21] **T Koberda**, **M Mj**, *Commutators, commensurators, and $\mathrm{PSL}_2(\mathbb{Z})$*, J. Topol. 14 (2021) 861–876 MR Zbl

[22] **C Leininger**, **D D Long**, **A W Reid**, *Commensurators of finitely generated nonfree Kleinian groups*, Algebr. Geom. Topol. 11 (2011) 605–624 MR Zbl

[23] **W Lück**, *L^2–invariants: theory and applications to geometry and K–theory*, Ergebnisse der Math. 44, Springer (2002) MR Zbl

[24] **G A Margulis**, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Math. 17, Springer (1991) MR Zbl

[25] **M Mj**, *Relative rigidity, quasiconvexity and C–complexes*, Algebr. Geom. Topol. 8 (2008) 1691–1716 MR Zbl

[26] **M Mj**, *On discreteness of commensurators*, Geom. Topol. 15 (2011) 331–350 MR Zbl

[27] **M Mj**, *Pattern rigidity and the Hilbert–Smith conjecture*, Geom. Topol. 16 (2012) 1205–1246 MR Zbl

[28] **P Sarnak**, *Notes on thin matrix groups*, from "Thin groups and superstrong approximation" (E Breuillard, H Oh, editors), Math. Sci. Res. Inst. Publ. 61, Cambridge Univ. Press (2014) 343–362 MR Zbl

[29] **R E Schwartz**, *The quasi-isometry classification of rank one lattices*, Inst. Hautes Études Sci. Publ. Math. 82 (1995) 133–168 MR Zbl

[30] **R E Schwartz**, *Symmetric patterns of geodesics and automorphisms of surface groups*, Invent. Math. 128 (1997) 177–199 MR Zbl

[31] **Y Shalom**, *Rigidity of commensurators and irreducible lattices*, Invent. Math. 141 (2000) 1–54 MR Zbl

[32] **Y Shalom**, *Rigidity, unitary representations of semisimple groups, and fundamental groups of manifolds with rank one transformation group*, Ann. of Math. 152 (2000) 113–182 MR Zbl

[33] **Y Shalom**, **G A Willis**, *Commensurated subgroups of arithmetic groups, totally disconnected groups and adelic rigidity*, Geom. Funct. Anal. 23 (2013) 1631–1683 MR Zbl

[34] **T N Venkataramana**, *Virtual Betti numbers of compact locally symmetric spaces*, Israel J. Math. 166 (2008) 235–238 MR Zbl

[35] **F W Warner**, *Foundations of differentiable manifolds and Lie groups*, Scott, Foresman, Glenview, IL (1971) MR Zbl

[36] **D Witte Morris**, *Introduction to arithmetic groups*, Deductive Press (2015) MR Zbl

[37] **R J Zimmer**, *Ergodic theory and semisimple groups*, Monogr. Math. 81, Birkhäuser, Basel (1984) MR Zbl

Department of Mathematics, University of Virginia
Charlottesville, VA, United States

School of Mathematics, Tata Institute of Fundamental Research
Mumbai, India

thomas.koberda@gmail.com, mahan@math.tifr.res.in

https://sites.google.com/view/koberdat, http://www.math.tifr.res.in/~mahan

Received: 9 May 2022 Revised: 16 April 2023

Algebraic & Geometric Topology **24**:4 (2024) 2171–2183
DOI: 10.2140/agt.2024.24.2171
Published: 16 July 2024

Pushouts of Dwyer maps are $(\infty, 1)$–categorical

PHILIP HACKNEY
VIKTORIYA OZORNOVA
EMILY RIEHL
MARTINA ROVELLI

The inclusion of 1–categories into $(\infty, 1)$–categories fails to preserve colimits in general, and pushouts in particular. We observe that if one functor in a span of categories belongs to a certain previously identified class of functors, then the 1–categorical pushout is preserved under this inclusion. Dwyer maps, a kind of neighborhood deformation retract of categories, were used by Thomason in the construction of his model structure on 1–categories. Thomason previously observed that the nerves of such pushouts have the correct weak homotopy type. We refine this result and show that the weak homotopical equivalence is a weak categorical equivalence. We also identify a more general class of functors along which 1–categorical pushouts are $(\infty, 1)$–categorical.

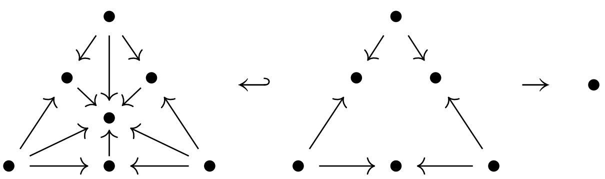
18N60, 55U35

1 Introduction

Classical 1–categories define an important special case of $(\infty, 1)$–categories. The fact that $(\infty, 1)$–category theory restricts to ordinary 1–categories can be understood, in part, by the observation that the inclusion of 1–categories into $(\infty, 1)$–categories is full as an inclusion of $(\infty, 2)$–categories. This full inclusion is reflective — with the left adjoint given by the functor that sends an $(\infty, 1)$–category to its quotient "homotopy category" — but not coreflective and as a consequence colimits of ordinary 1–categories need not be preserved by the passage to $(\infty, 1)$–categories. Indeed there are known examples of colimits of 1–categories that generate nontrivial higher-dimensional structure when the colimit is formed in the category of $(\infty, 1)$–categories.

For example, consider the span of posets

(1.1)



The pushout in 1–categories is the arrow category $\bullet \to \bullet$, while the pushout in $(\infty, 1)$–categories defines an $(\infty, 1)$–category which has the homotopy type of the 2–sphere.

As a second example, let M be the monoid with five elements, e, x_{11}, x_{12}, x_{21} and x_{22}, and multiplication rule given by $x_{ij}x_{k\ell} = x_{i\ell}$. Inverting all elements of M yields the trivial group. That is, if one considers M as a 1–category with a single object, then the pushout of the span $M \leftarrow \coprod_M \mathbb{2} \to \coprod_M \mathbb{I}$ (where $\mathbb{2}$ is the free-living arrow and $\mathbb{I}$ is the free-living isomorphism) in categories is the terminal category $\mathbb{1}$. On the other hand, the pushout of this span in $(\infty, 1)$–categories is the ∞–groupoid S^2 as follows from [Fiedorowicz 2002, Lemma]. The results of [McDuff 1979] imply that this example is generalizable to a vast class of monoids.

More generally, the Gabriel–Zisman category of fractions $\mathscr{C}[\mathscr{W}^{-1}]$ is formed by freely inverting the morphisms in a class of arrows $\mathscr{W}$ in a 1–category $\mathscr{C}$. This can also be constructed as a pushout of 1–categories of the span

$$\mathscr{C} \leftarrow \coprod_{w \in \mathscr{W}} \mathbb{2} \hookrightarrow \coprod_{w \in \mathscr{W}} \mathbb{I}$$

where each arrow in $\mathscr{W}$ is replaced by a free-living isomorphism. By contrast, the $(\infty, 1)$–category defined by this pushout is modeled by the Dwyer–Kan simplicial localization, which has nontrivial higher dimensional structure in many instances [Dwyer and Kan 1980; Joyal 2008, page 168; Stevenson 2017, Lemma 18]. Indeed, all $(\infty, 1)$–categories arise in this way [Barwick and Kan 2012].

As the examples above show, pushouts of 1–categories in particular are problematic. Our aim is to prove that a certain class of pushout diagrams of 1–categories are guaranteed to be $(\infty, 1)$–categorical. The requirement is that one of the two maps in the span that generates the pushout belong to a class of functors between 1–categories first considered by Thomason [1980, Definition 4.1] under the name "Dwyer maps" that feature in a central way in the construction of the Thomason model structure on categories.

Definition 1.2 (Thomason) A full sub-1–category inclusion $I: \mathscr{A} \hookrightarrow \mathscr{B}$ is a *Dwyer map* if the following conditions hold.

(i) The category $\mathscr{A}$ is a *sieve* in $\mathscr{B}$, meaning there is a necessarily unique functor $\chi: \mathscr{B} \to \mathbb{2}$ with $\chi^{-1}(0) = \mathscr{A}$. We write $\mathscr{V} := \chi^{-1}(1)$ for the complementary *cosieve* of $\mathscr{A}$ in $\mathscr{B}$.

(ii) The inclusion $I: \mathscr{A} \hookrightarrow \mathscr{W}$ into the *minimal cosieve*[1] $\mathscr{W} \subset \mathscr{B}$ containing $\mathscr{A}$ admits a right adjoint left inverse $R: \mathscr{W} \to \mathscr{A}$, a right adjoint for which the unit is an identity.

Schwede [2019] describes Dwyer maps as "categorical analogs of the inclusion of a neighborhood deformation retract". In fact, many examples of Dwyer maps are more like deformation retracts, in that the cosieve $\mathscr{W}$ generated by $\mathscr{A}$ is the full codomain category $\mathscr{B}$.

[1]Explicitly $\mathscr{W}$ is the full subcategory of $\mathscr{B}$ containing every object that arises as the codomain of an arrow with domain in $\mathscr{A}$.

Example 1.3 (i) The vertex inclusion $0\colon \mathbb{1} \to \mathbb{2}$ is a Dwyer map, with $!\colon \mathbb{2} \to \mathbb{1}$ the right adjoint left inverse. The other vertex inclusion $1\colon \mathbb{1} \to \mathbb{2}$ is not a Dwyer map.

(ii) Generalizing the previous example, if $\mathscr{A}$ is a category with a terminal object and $\mathscr{A}^{\triangleright}$ is the category which formally adds a new terminal object, then the inclusion $\mathscr{A} \hookrightarrow \mathscr{A}^{\triangleright}$ is a Dwyer map.[2]

Thomason observed that Dwyer maps are stable under pushouts, as we now recall:

Lemma 1.4 [Thomason 1980, Proposition 4.3] *Any pushout of a Dwyer map I defines a Dwyer map J:*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, hook, "I"'] \arrow[dr, phantom, "\ulcorner" very near end] & \mathscr{C} \arrow[d, hook, "J"] \\ \mathscr{B} \arrow[r, "G"'] & \mathscr{D} \end{tikzcd}$$

Note, for example, that Lemma 1.4 explains the Dwyer map of Example 1.3(ii): if $\mathscr{A}$ has a terminal object t, then the pushout

$$\begin{tikzcd} \mathbb{1} \arrow[r, "t"] \arrow[d, hook, "0"'] \arrow[dr, phantom, "\ulcorner" very near end] & \mathscr{A} \arrow[d, hook] \\ \mathbb{2} \arrow[r] & \mathscr{A}^{\triangleright} \end{tikzcd}$$

defines the category $\mathscr{A}^{\triangleright}$.

Our aim is to show that pushouts of categories involving at least one Dwyer map can also be regarded as pushouts of $(\infty, 1)$–categories in the sense made precise by considering the nerve embedding from categories into quasicategories:

Theorem 1.5 *Let*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, hook, "I"'] \arrow[dr, phantom, "\ulcorner" very near end] & \mathscr{C} \arrow[d, hook, "J"] \\ \mathscr{B} \arrow[r, "G"'] & \mathscr{D} \end{tikzcd}$$

be a pushout of categories, and assume that I is a Dwyer map. Then the induced map of simplicial sets

$$N\mathscr{B} \amalg_{N\mathscr{A}} N\mathscr{C} \to N\mathscr{D}$$

is a weak categorical equivalence.

By a weak categorical equivalence, we mean a weak equivalence in Joyal's model structure for quasicategories [Joyal and Tierney 2007, Section 1]. Theorem 1.5 is a refinement of a similar result of Thomason [1980, Proposition 4.3], which proves that the same map is a weak homotopy equivalence.

[2] If $\mathscr{A}$ does not have a terminal object, then $\mathscr{A} \to \mathscr{A}^{\triangleright}$ need not be a Dwyer map. Indeed, if $\mathscr{A} = \mathbb{1} \amalg \mathbb{1}$, the only cosieve containing $\mathscr{A}$ is $\mathscr{A}^{\triangleright}$ itself, and there cannot be a right adjoint $\mathscr{A}^{\triangleright} \to \mathscr{A}$ as $\mathscr{A}$ does not have a terminal object. But see Example 3.5(iii), which explains that this example is *discretely flat*.

While Theorem 1.5 is the natural generalization of Thomason's result, we prove it by considering instead the embedding of 1–categories as discrete simplicially enriched categories, using Bergner's model of $(\infty,1)$–categories. This tactic was suggested by a referee; for our original argument using the quasicategory model see [Hackney et al. 2022]. We show in Proposition 3.3 that a Dwyer map, when considered as a map between discrete simplicial categories, satisfies a certain "flatness" property with respect to the Bergner model structure. Since this discrete embedding of 1–categories into simplicial categories preserves pushouts, unlike the nerve embedding of 1–categories into quasicategories, it is straightforward to prove that:

Theorem 1.6 *The inclusion $\mathsf{Cat}_1 \hookrightarrow \mathsf{Cat}_{(\infty,1)}$ of the $(\infty,1)$–category of 1–categories into the $(\infty,1)$–category of $(\infty,1)$–categories preserves (homotopy) pushouts along Dwyer maps.*

When then deduce Theorem 1.5 as a corollary of this result.

Though the two previous theorems refer to Dwyer maps, they also hold for the pseudo-Dwyer maps introduced by Cisinski [1999], which are retracts of Dwyer maps. In fact, we prove both Theorems 1.5 and 1.6 for more general classes of functors introduced in Definition 3.4 that include the Dwyer maps. The key property of a Dwyer map (or pseudo-Dwyer map) is that it is "discretely flat" as well as a faithful inclusion. For a functor to be *discretely flat* means that pushouts along it, considered as a functor of discrete simplicial categories, preserve Dwyer–Kan equivalences of simplicial categories.

In a companion paper, we give an application of Theorem 1.5 to the theory of $(\infty,2)$–categories. There we prove:

Theorem 1.7 [Hackney et al. 2023, 4.4.2] *The space of composites of any pasting diagram in any $(\infty,2)$–category is contractible.*

To prove this, we make use of Lurie's [2009b] model structure of $(\infty,2)$–categories as categories enriched over quasicategories. In this model, a *pasting diagram* is a simplicially enriched functor out of the free simplicially enriched category defined by gluing together the objects, atomic 1–cells, and atomic 2–cells of a pasting scheme, while the composites of these cells belong to the homotopy coherent diagram indexed by the nerve of the free 2–category generated by the pasting scheme.

This pair of $(\infty,2)$–categories has a common set of objects so the difference lies in their hom-spaces. The essential difference between the procedure of attaching an atomic 2–cell along the bottom of a pasting diagram or along the bottom of the free 2–category it generates is the difference between forming a pushout of hom-categories in the category of $(\infty,1)$–categories or in the category of 1–categories. Since one of the functors in the span that defines the pushout under consideration is a Dwyer map, Theorem 1.5 proves that the resulting $(\infty,2)$–categories are equivalent.

In Section 2, we analyze 1–categorical pushouts of Dwyer maps. In Section 3, we extend these observations to pushouts of simplicial categories involving a Dwyer map between 1–categories as one leg of the span, axiomatize the classes of functors that are well-behaved with respect to simplicial pushouts, and prove Theorem 1.6. In Section 4, we deduce Theorem 1.5 as a corollary and consider a further special case Corollary 4.1, which observes that the canonical comparison between the pushout of nerves of categories and the nerve of the pushout is inner anodyne, provided that one of the functors in the span is a Dwyer map and the other is an injective on objects faithful functor.

2 Dwyer pushouts

We now establish some notation that we will freely reference in the remainder of this paper. By Definition 1.2, a Dwyer map $I\colon \mathcal{A} \hookrightarrow \mathcal{B}$ uniquely determines a functor $\chi\colon \mathcal{B} \to \mathbb{2}$ that classifies the sieve $\mathcal{A} := \chi^{-1}(0)$ and its complementary cosieve $\mathcal{V} := \chi^{-1}(1)$

$$\begin{tikzcd}
\mathcal{V} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "\chi"] & \mathcal{A} \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathbb{1} \arrow[r, hook, "1"'] & \mathbb{2} & \mathbb{1} \arrow[l, hook', "0"]
\end{tikzcd}$$

as well as a right adjoint left inverse adjunction $(I \dashv R, \varepsilon\colon IR \Rightarrow \mathrm{id}_{\mathcal{W}})$ associated to the inclusion of $\mathcal{A}$ into the minimal cosieve $\mathcal{A} \subset \mathcal{W} \subset \mathcal{B}$. This data may be summarized by the diagram

(2.1)
$$\begin{tikzcd}
 & & \varnothing \arrow[dl, hook] \arrow[dr, hook] \arrow[dd, phantom, "\vee", very near start] & \\
 & \mathcal{U} \arrow[dl, hook] \arrow[dr, hook] \arrow[dd, phantom, "\vee", very near start] & & \mathcal{A} \arrow[dl, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dl, phantom, "\top"] \\
\mathcal{V} \arrow[d] \arrow[dr, hook] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathcal{W} \arrow[ur, dashed, bend left, "R"] \arrow[dl, hook'] & \mathbb{1} \arrow[ddll, hook', "0"] \\
\mathbb{1} \arrow[dr, hook, "1"'] & \mathcal{B} \arrow[d] & & \\
 & \mathbb{2} & &
\end{tikzcd}$$

in which $\mathcal{U} := \mathcal{W} \cap \mathcal{V} \cong \mathcal{W} \backslash \mathcal{A}$. Consider the pushout of a Dwyer map along an arbitrary functor $F\colon \mathcal{A} \to \mathcal{C}$:

$$\begin{tikzcd}
 & & \mathcal{A} \arrow[rr, "F"] \arrow[dl, hook', "I"'] \arrow[d] & & \mathcal{C} \arrow[dl, "J"] \arrow[d] \arrow[dl, phantom, "\ulcorner", very near end] \\
\mathcal{V} \arrow[r, hook] \arrow[dd] & \mathcal{B} \arrow[rr, "G"] \arrow[dd, "\chi"'] & \mathbb{1} \arrow[dl, hook', "0"'] \arrow[rr, equal] & \mathcal{D} \arrow[dd, dashed, "\pi"] & \mathbb{1} \arrow[dl, hook', "0"] \\
 & & & & \\
\mathbb{1} \arrow[r, hook, "1"] & \mathbb{2} \arrow[rr, equal] & & \mathbb{2} &
\end{tikzcd}$$

The induced functor $\pi\colon \mathcal{D} \to \mathbb{2}$ partitions the objects of $\mathcal{D}$ into the two fibers $\mathrm{ob}(\pi^{-1}(0)) \cong \mathrm{ob}\,\mathcal{C}$ and $\mathrm{ob}(\pi^{-1}(1)) \cong \mathrm{ob}\,\mathcal{V}$ and prohibits any morphisms from the latter to the former.

The right adjoint left inverse adjunction $(I \dashv R, \varepsilon\colon IR \Rightarrow \mathrm{id}_{\mathscr{W}})$ associated to the inclusion of $\mathscr{A}$ into the minimal cosieve $\mathscr{A} \subset \mathscr{W} \subset \mathscr{B}$ pushes out to define a right adjoint left inverse $(J \dashv S, \nu\colon JS \Rightarrow \mathrm{id}_{\mathscr{Y}})$ to the inclusion of $\mathscr{C}$ into the minimal cosieve $\mathscr{C} \subset \mathscr{Y} \subset \mathscr{D}$:

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, hook, bend right, "I"'] \arrow[d, phantom, "\dashv"] & \mathscr{C} \arrow[d, hook, bend right, "J"'] \arrow[d, phantom, "\dashv"] \\
\mathscr{W} \arrow[r, "G"'] \arrow[u, bend right, "R"'] \arrow[d, hook] \arrow[ur, phantom, "\ulcorner"] & \mathscr{Y} \arrow[u, bend right, "S"'] \arrow[d, hook] \\
\mathscr{B} \arrow[r, "G"'] \arrow[ur, phantom, "\ulcorner"] & \mathscr{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathscr{C} \arrow[d, hook, "J"] \arrow[ddr, equal, bend left] & \\
\mathscr{W} \arrow[r, "G"'] \arrow[dr, "R"'] \arrow[ur, phantom, "\ulcorner"] & \mathscr{Y} \arrow[dr, dashed, "S"] & \\
& \mathscr{A} \arrow[r, "F"'] & \mathscr{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathscr{C} \arrow[d, hook, "J"] \arrow[dr, "\Delta"] & \\
\mathscr{W} \arrow[r, "G"'] \arrow[dr, "\varepsilon"'] \arrow[ur, phantom, "\ulcorner"] & \mathscr{Y} \arrow[dr, dashed, "\nu"] & \mathscr{C}^{\mathbb{2}} \arrow[d, "J^{\mathbb{2}}"] \\
& \mathscr{W}^{\mathbb{2}} \arrow[r, "G^{\mathbb{2}}"'] & \mathscr{Y}^{\mathbb{2}}
\end{tikzcd}$$

These observations explain the closure of Dwyer maps under pushout and furthermore can be used to explicitly describe the structure of the category $\mathscr{D}$ defined by the pushout of a Dwyer map, as proven in [Bohmann et al. 2015, Proof of Lemma 2.5]; cf also [Schwede 2019, Construction 1.2; Ara and Maltsiniotis 2014, Section 7.1].

Proposition 2.2 *The objects in the pushout category $\mathscr{D}$ are given by*

$$\mathrm{ob}\mathscr{C} \amalg \mathrm{ob}\mathscr{V} \xrightarrow{\cong} \mathrm{ob}\mathscr{D}$$

while the hom sets are given by

$$\mathscr{C}(c,c') \cong \mathscr{D}(c,c'), \qquad \mathscr{V}(v,v') \cong \mathscr{B}(v,v') \cong \mathscr{D}(v,v'), \qquad \mathscr{C}(c,Su) \xrightarrow[\cong]{\nu_u \circ (-)} \mathscr{D}(c,u), \quad f \mapsto \hat{f},$$

for all $c, c' \in \mathscr{C}$, $v, v' \in \mathscr{V}$ and $u \in \mathscr{U}$, and are empty otherwise. Functoriality of the inclusions J and G defines the composition on the image of $\mathscr{C}$ and $\mathscr{V}$. For objects $c, c' \in \mathscr{C}$ and $u, u' \in \mathscr{U}$, the composition map

$$\begin{tikzcd}
\mathscr{D}(u,u') \times \mathscr{D}(c,u) \times \mathscr{D}(c',c) \arrow[rr, "\circ"] & & \mathscr{D}(c',u') \\
\mathscr{D}(u,u') \times \mathscr{C}(c,Su) \times \mathscr{C}(c',c) \arrow[u, "\wr\parallel"] \arrow[d, "S\times\mathrm{id}"'] & & \\
\mathscr{C}(Su,Su') \times \mathscr{C}(c,Su) \times \mathscr{C}(c',c) \arrow[rr, "\circ"] & & \mathscr{C}(c',Su') \arrow[uu, "\parallel\wr"']
\end{tikzcd}$$

is the unique map making the diagram commute.[3]

To summarize, J and G define fully faithful inclusions

$$\begin{tikzcd}
\mathscr{V} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{D} \arrow[d, "\pi"] & \mathscr{C} \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathbb{1} \arrow[r, hook, "1"] & \mathbb{2} & \mathbb{1} \arrow[l, hook', "0"']
\end{tikzcd}$$

that are jointly surjective on objects. In particular, we may identify $\mathscr{V}$ with the complementary cosieve of $\mathscr{C}$ in $\mathscr{D}$.

[3] Note if $u \in \mathscr{U}$ and $v \in \mathscr{V} \backslash \mathscr{U}$, then $\mathscr{B}(u,v) = \varnothing$.

3 Pushouts in simplicial categories

By *simplicial category* we always mean *simplicially enriched category* as opposed to a simplicial object in Cat. There is a fully faithful inclusion $\mathsf{sCat} \hookrightarrow \mathsf{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$, identifying a simplicial category with an identity-on-objects simplicial object.

The following model structure on sCat is due to Bergner [2007], though Lurie [2009a, A.3.2.4, A.3.2.25] observed that the Bergner model structure is left proper and combinatorial.

Definition 3.1 (Bergner model structure) The category sCat of simplicially enriched categories admits a proper, combinatorial model structure in which:

- A map $f : \mathscr{C} \to \mathscr{D}$ is a *weak equivalence* just when
  - (W1) for each pair of objects x and y, the map $\mathscr{C}(x, y) \to \mathscr{D}(fx, fy)$ is a weak homotopy equivalence of simplicial sets, and
  - (W2) the functor $\pi_0 f : \pi_0\mathscr{C} \to \pi_0\mathscr{D}$ is essentially surjective.
- A map $f : \mathscr{C} \to \mathscr{D}$ is a *fibration* just when
  - (F1) for each pair of objects x and y, the map $\mathscr{C}(x, y) \to \mathscr{D}(fx, fy)$ is a Kan fibration, and
  - (F2) the functor $\pi_0 f : \pi_0\mathscr{C} \to \pi_0\mathscr{D}$ is an isofibration.

If $\mathscr{C}$ is a simplicial category, then $\pi_0\mathscr{C}$ is the ordinary category obtained by taking path components of each hom-simplicial set. We call simplicial functors satisfying (W1) *fully faithful* (meaning of course in the homotopical sense), and functors satisfying (W2) *essentially surjective*.

The constant diagram functor $\mathsf{Cat} \to \mathsf{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$, given by precomposition with $\mathbf{\Delta}^{\mathrm{op}} \to \mathbb{1}$, factors through the full subcategory inclusion $\mathsf{sCat} \hookrightarrow \mathsf{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$. Write $\mathrm{disc} : \mathsf{Cat} \to \mathsf{sCat}$ for the induced full inclusion, which identifies categories as those simplicial categories with discrete hom-simplicial sets.

Lemma 3.2 *The functor* $\mathrm{disc} : \mathsf{Cat} \to \mathsf{sCat}$ *preserves limits and colimits.*

Proof The precomposition functor $\mathsf{Cat} \to \mathsf{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$ preserves all limits and colimits, while the full inclusion $\mathsf{sCat} \to \mathsf{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$ reflects them. The conclusion follows. □

Our key technical result is the following proposition, which observes that when I is a Dwyer map, the functor $\mathrm{disc}(I)$ of simplicial categories is a *flat map* in the terminology of [Hill et al. 2016, B.9] or an *h–cofibration* in the terminology of [Batanin and Berger 2017, 1.1] relative to the Bergner model structure.

Proposition 3.3 *If* $I : \mathscr{A} \hookrightarrow \mathscr{B}$ *is a Dwyer map, then*

$$\mathrm{disc}\,\mathscr{B} \amalg_{\mathrm{disc}\,\mathscr{A}} (-) : {}^{\mathrm{disc}\,\mathscr{A}/}\mathsf{sCat} \to {}^{\mathrm{disc}\,\mathscr{B}/}\mathsf{sCat}$$

preserves weak equivalences.

Proof Consider a composable pair of simplicial functors $\operatorname{disc}\mathscr{A} \xrightarrow{F} \mathscr{C}' \xrightarrow{M} \mathscr{C}$ and form the following pushouts:

$$\begin{tikzcd} \operatorname{disc}\mathscr{A} \arrow[r, "F"] \arrow[d] & \mathscr{C}' \arrow[r, "M"] \arrow[d] & \mathscr{C} \arrow[d] \\ \operatorname{disc}\mathscr{B} \arrow[r] & \mathscr{D}' \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "H"] & \mathscr{D} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

When M is a weak equivalence in the Bergner model structure, we wish to show that the induced map $H\colon \mathscr{D}' \to \mathscr{D}$ between the pushouts is as well.

As in Proposition 2.2, we identify $\operatorname{ob}\mathscr{D}'$ with $\operatorname{ob}\mathscr{C}' \amalg \operatorname{ob}\mathscr{V}$ and similarly $\operatorname{ob}\mathscr{D} = \operatorname{ob}\mathscr{C} \amalg \operatorname{ob}\mathscr{V}$. We regard the simplicial categories $\mathscr{D}'$ and $\mathscr{D}$ as simplicial objects $\mathscr{D}'_\bullet$ and $\mathscr{D}_\bullet$ via the inclusion $\mathsf{sCat} \hookrightarrow \mathsf{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$. For each n, we have $\mathscr{D}_n = \mathscr{B} \amalg_{\mathscr{A}} \mathscr{C}_n$ and similarly for $\mathscr{D}'_n$. We have already computed the hom sets of these categories in Proposition 2.2, and the descriptions there are functorial in the $\mathscr{C}$–variable. Thus,

$$\begin{tikzcd} \mathscr{C}'(c,c') \arrow[r, "\simeq"] \arrow[d, "\cong"] & \mathscr{C}(Mc, Mc') \arrow[d, "\cong"] \\ \mathscr{D}'(c,c') \arrow[r] & \mathscr{D}(Hc, Hc') \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C}'(c, FRu) \arrow[r, "\simeq"] \arrow[d, "\cong"] & \mathscr{C}(Mc, MFRu) \arrow[d, "\cong"] \\ \mathscr{D}'(c,u) \arrow[r] & \mathscr{D}(Hc, Hu) \end{tikzcd}$$

for $c, c' \in \mathscr{C}'$ and $u \in \mathscr{U}$. Meanwhile, for $v, v' \in \mathscr{V}$ we have that both $\mathscr{D}(v, v')$ and $\mathscr{D}'(v, v')$ are isomorphic to the discrete simplicial set $\mathscr{V}(v, v')$. Finally, the hom-simplicial sets $\mathscr{D}'(c, v')$, $\mathscr{D}(Hc, v')$, $\mathscr{D}(v, c)$ and $\mathscr{D}(v, Hc)$ are all empty for $c \in \mathscr{C}'$, $v \in \mathscr{V}$ and $v' \in \mathscr{V} \backslash \mathscr{U}$. Thus H is fully faithful.

For essential surjectivity of H, notice that we have a commutative square of functors

$$\begin{tikzcd} \mathscr{C}' \amalg \operatorname{disc}\mathscr{V} \arrow[r, "M \amalg \mathrm{id}"] \arrow[d] & \mathscr{C} \amalg \operatorname{disc}\mathscr{V} \arrow[d] \\ \mathscr{D}' \arrow[r, "H"] & \mathscr{D} \end{tikzcd}$$

where the vertical maps are bijective on objects and the top map is essentially surjective. It follows that H is essentially surjective as well. □

Our main results hold not just for Dwyer maps but for arbitrary functors between 1–categories that satisfy the property established in Proposition 3.3 plus some injectivity conditions. The following terminology highlights the required properties.

Definition 3.4 A functor $I\colon \mathscr{A} \to \mathscr{B}$ between 1–categories is *discretely flat* if the simplicial functor $\operatorname{disc}(I)$ is flat, ie if

$$\operatorname{disc}\mathscr{B} \amalg_{\operatorname{disc}\mathscr{A}} (-)\colon {}^{\operatorname{disc}\mathscr{A}/}\mathsf{sCat} \to {}^{\operatorname{disc}\mathscr{B}/}\mathsf{sCat}$$

preserves Bergner weak equivalences. If, in addition, I is injective on objects, we call it a *discretely flat cofibration*, and if it is both injective on objects and faithful, we call it a *discretely flat inclusion*.

Dwyer maps are discretely flat inclusions, but such functors aren't the only examples.

Example 3.5 (i) Since the passage to opposite categories commutes with all of the structures involved in Definition 3.4, *co-Dwyer maps*, whose opposites are Dwyer maps, are also discretely flat inclusions.

(ii) As flat maps are closed under retracts — see [Hill et al. 2016, B.11] or [Batanin and Berger 2017, Lemma 1.3] — Cisinski's [1999] *pseudo-Dwyer maps* are also discretely flat inclusions.

(iii) The inclusion of $\mathbb{1} \amalg \mathbb{1}$ into the cospan category $(\mathbb{1} \amalg \mathbb{1})^{\triangleright}$ is a discretely flat inclusion. Indeed, the hom-simplicial sets of $(\mathbb{1} \amalg \mathbb{1})^{\triangleright} \amalg_{(\mathbb{1} \amalg \mathbb{1})} \mathscr{C}$ are readily computed by hand in terms of those for $\mathscr{C}$, and a variation of the proof of Proposition 3.3 gives the result.

Note not all functors of the form $\mathscr{A} \to \mathscr{A}^{\triangleright}$ are discrete flat inclusions. In light of Theorem 1.6, the left-hand map of (1.1) gives a counterexample. An interesting problem is to characterize the class of discretely flat inclusions.

The fact that Dwyer pushouts are homotopy pushouts now follows from a general fact: in a left proper model category, a pushout where one leg is a flat map is automatically a homotopy pushout; see [Batanin and Berger 2017, Section 1.5].

Proposition 3.6 *Suppose* $I\colon \mathscr{A} \to \mathscr{B}$ *is discretely flat. Then for any functor* $F\colon \operatorname{disc} \mathscr{A} \to \mathscr{C}$ *of simplicial categories, the pushout* $\operatorname{disc} \mathscr{B} \amalg_{\operatorname{disc} \mathscr{A}} \mathscr{C}$ *is a homotopy pushout.*

Proof To form the homotopy pushout of a span in a model category, one replaces it by a cofibrant span as below and then takes the ordinary pushout [Dwyer and Spaliński 1995, 10.4]:

$$\begin{tikzcd}
 & \varnothing \arrow[d, tail] & \\
\mathscr{Y} \arrow[d, "\wr"] & \mathscr{X} \arrow[l, tail] \arrow[r, tail] \arrow[d, "\wr"] & \mathscr{Z} \arrow[d, "\wr"] \\
\operatorname{disc} \mathscr{B} & \operatorname{disc} \mathscr{A} \arrow[l, "\operatorname{disc}(I)"'] \arrow[r, tail] & \mathscr{C}
\end{tikzcd}$$

Thus, we must show that the induced map

$$\mathscr{Y} \amalg_{\mathscr{X}} \mathscr{Z} \to \operatorname{disc} \mathscr{B} \amalg_{\operatorname{disc} \mathscr{A}} \mathscr{C}$$

is a weak equivalence of simplicial categories. Since $\operatorname{disc}(I)$ is flat by assumption and sCat is left proper, the above map is a weak equivalence by [Hill et al. 2016, B.12]. □

Our model-independent statement, that Dwyer pushouts are $(\infty, 1)$–categorical, holds generally for discretely flat cofibrations.

Theorem 1.6 *The inclusion* $\mathsf{Cat}_1 \hookrightarrow \mathsf{Cat}_{(\infty,1)}$ *of the* $(\infty, 1)$*–category of* 1*–categories into the* $(\infty, 1)$*–category of* $(\infty, 1)$*–categories preserves (homotopy) pushouts along discretely flat cofibrations.*

Proof The inclusion $\mathsf{Cat}_1 \hookrightarrow \mathsf{Cat}_{(\infty,1)}$ can be modeled at the point-set level by the right Quillen functor $\operatorname{disc}\colon \mathsf{Cat} \to \mathsf{sCat}$. By hypothesis, a discretely flat cofibration $I\colon \mathscr{A} \to \mathscr{B}$ is injective on objects, hence a cofibration in the canonical model structure on Cat. This model structure is left proper, so ordinary pushouts along such maps are homotopy pushouts by [Hirschhorn 2003, 13.5.4]. Since the functor disc: Cat $\to$ sCat preserves strict pushouts, Proposition 3.6 shows that disc preserves homotopy pushouts along discretely flat cofibrations, so the conclusion follows. $\square$

4 Pushouts in simplicial sets

In this section, we describe the implications of Theorem 1.6 for the Joyal model structure on simplicial sets, proving the results needed in [Hackney et al. 2023]. We utilize the commutative triangle of right Quillen functors

$$\begin{tikzcd} & \mathsf{Cat} \arrow[dl, "N"'] \arrow[dr, "\mathrm{disc}"] & \\ \mathsf{sSet} & & \mathsf{sCat} \arrow[ll, "\simeq", "\mathfrak{N}"'] \end{tikzcd}$$

where $\mathfrak{N}$ is the homotopy coherent nerve, a right Quillen equivalence between the Bergner and Joyal model structures. This diagram commutes up to natural isomorphism since, for any 1–category $\mathscr{A}$,

$$(\mathfrak{N}\,\mathrm{disc}\,\mathscr{A})_n := \hom(\mathfrak{C}[n], \mathrm{disc}\,\mathscr{A}) \cong \hom([n], \mathscr{A}) =: (N\mathscr{A})_n,$$

which holds because the hom simplicial sets of $\mathrm{disc}\,\mathscr{A}$ are discrete.

We first explain how to deduce Theorem 1.5 from Proposition 3.6. In fact, we use the terminology of Definition 3.4 to prove a more general version:

Theorem 1.5 *Let*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathscr{C} \arrow[d, hook, "J"] \\ \mathscr{B} \arrow[r, "G"'] & \mathscr{D} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

be a pushout of categories, and assume I is a discretely flat inclusion. Then the induced map of simplicial sets

$$N\mathscr{B} \amalg_{N\mathscr{A}} N\mathscr{C} \to N\mathscr{D}$$

is a weak categorical equivalence.

Proof We organize the proof into the following commutative square of simplicial sets:

$$\begin{tikzcd} \mathfrak{N}\,\mathrm{disc}\,\mathscr{B} \amalg_{\mathfrak{N}\,\mathrm{disc}\,\mathscr{A}} \mathfrak{N}\,\mathrm{disc}\,\mathscr{C} \arrow[r, "\cong"] \arrow[d] & N\mathscr{B} \amalg_{N\mathscr{A}} N\mathscr{C} \arrow[dd] \\ \mathfrak{N}(\mathrm{disc}\,\mathscr{B} \amalg_{\mathrm{disc}\,\mathscr{A}} \mathrm{disc}\,\mathscr{C}) \arrow[d, "\cong"] & \\ \mathfrak{N}\,\mathrm{disc}(\mathscr{B} \amalg_{\mathscr{A}} \mathscr{C}) \arrow[r, "\cong"] & N(\mathscr{B} \amalg_{\mathscr{A}} \mathscr{C}) \end{tikzcd}$$

The top and bottom isomorphisms are instances of the natural isomorphism $N \cong \mathfrak{N}\,\mathrm{disc}$. The vertical maps are the canonical comparison maps induced by the universal property of the pushouts. The bottom left map is an isomorphism since disc preserves pushouts (Lemma 3.2). It remains to show that the upper left map

$$\mathfrak{N}\,\mathrm{disc}\,\mathscr{B} \amalg_{\mathfrak{N}\,\mathrm{disc}\,\mathscr{A}} \mathfrak{N}\,\mathrm{disc}\,\mathscr{C} \to \mathfrak{N}(\mathrm{disc}\,\mathscr{B} \amalg_{\mathrm{disc}\,\mathscr{A}} \mathrm{disc}\,\mathscr{C})$$

is a weak categorical equivalence, at which point the right map will be a weak categorical equivalence by two-of-three.

Notice that the objects in the top row are homotopy pushouts, since $NI\colon N\mathscr{A} \to N\mathscr{B}$ is a cofibration in sSet by the hypothesis that I is faithful and injective on objects. Let $\boldsymbol{R}\mathfrak{N}$ be the right derived functor of $\mathfrak{N}$. Since discrete simplicial categories are fibrant in the Bergner model structure, the map of simplicial sets above represents the canonical map

$$(\boldsymbol{R}\mathfrak{N})\,\mathrm{disc}\,\mathscr{B} \amalg^h_{(\boldsymbol{R}\mathfrak{N})\,\mathrm{disc}\,\mathscr{A}} (\boldsymbol{R}\mathfrak{N})\,\mathrm{disc}\,\mathscr{C} \to (\boldsymbol{R}\mathfrak{N})(\mathrm{disc}\,\mathscr{B} \amalg^h_{\mathrm{disc}\,\mathscr{A}} \mathrm{disc}\,\mathscr{C}).$$

This is an equivalence since $\boldsymbol{R}\mathfrak{N}$ is an equivalence of $(\infty,1)$–categories, hence preserves homotopy pushouts. We conclude that $N\mathscr{B} \amalg_{N\mathscr{A}} N\mathscr{C} \to N(\mathscr{B} \amalg_{\mathscr{A}} \mathscr{C})$ is a weak categorical equivalence. □

In the case where $F\colon \mathscr{A} \to \mathscr{C}$ is also injective on objects and faithful, as occurs frequently in [Hackney et al. 2023], we are able to strengthen our conclusion and prove that the canonical comparison map is inner anodyne.

Corollary 4.1 *Let*

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, hook, "I"'] & \mathscr{C} \arrow[d, hook, "J"] \\ \mathscr{B} \arrow[r, "G"'] \arrow[ur, phantom, "\ulcorner" very near end] & \mathscr{D} \end{tikzcd}$$

be a pushout of categories, in which I is a discretely flat inclusion and F is faithful and injective on objects. Then the induced inclusion of simplicial sets

$$N\mathscr{B} \amalg_{N\mathscr{A}} N\mathscr{C} \hookrightarrow N\mathscr{D}$$

is inner anodyne.

Proof Observe in this case that the canonical map $j\colon N\mathscr{B} \amalg_{N\mathscr{A}} N\mathscr{C} \hookrightarrow N\mathscr{D}$ is an inclusion and thus, by Theorem 1.5, an acyclic cofibration in the Joyal model structure. This acyclic cofibration is also bijective on 0–simplices and has codomain a quasicategory. By [Stevenson 2018a, 2.19] or [Stevenson 2018b, 5.7] it follows that j is inner anodyne.[4] □

[4] See [Campbell 2020] for related discussion and an example of an acyclic cofibration that is bijective on 0–simplices but whose codomain is not a quasicategory that is not inner anodyne.

Acknowledgements

This material is based upon work supported by the National Science Foundation under grant DMS-1440140, which began while the authors were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2020 semester. The authors learned a lot from fruitful discussions with the other members of the MSRI-based working group on $(\infty, 2)$–categories. This work was supported by a grant from the Simons Foundation (850849, PH). Ozornova thankfully acknowledges the financial support by the DFG grant OZ 91/2-1 with the project 442418934. Riehl is also grateful for support from the NSF via DMS-1652600 and DMS-2204304, from the ARO under MURI grant W911NF-20-1-0082, and by the Johns Hopkins President's Frontier Award program. Rovelli is deeply appreciative of the Mathematical Sciences Institute at the Australian National University for their support during the pandemic year and is grateful for support from the NSF via DMS-2203915. The authors also wish to thank the referee for [Hackney et al. 2023] who pointed out that their originally claimed proof of this result, which originally appeared there, sufficed only to cover the case used there. The referee for this paper suggested a strategy to greatly simplify our original proof of the general result, which allowed us to also prove more general theorems.

References

[Ara and Maltsiniotis 2014] **D Ara**, **G Maltsiniotis**, *Vers une structure de catégorie de modèles à la Thomason sur la catégorie des n–catégories strictes*, Adv. Math. 259 (2014) 557–654 MR Zbl

[Barwick and Kan 2012] **C Barwick**, **D M Kan**, *Relative categories: another model for the homotopy theory of homotopy theories*, Indag. Math. 23 (2012) 42–68 MR Zbl

[Batanin and Berger 2017] **M A Batanin**, **C Berger**, *Homotopy theory for algebras over polynomial monads*, Theory Appl. Categ. 32 (2017) 148–253 MR Zbl

[Bergner 2007] **J E Bergner**, *A model category structure on the category of simplicial categories*, Trans. Amer. Math. Soc. 359 (2007) 2043–2058 MR Zbl

[Bohmann et al. 2015] **A M Bohmann**, **K Mazur**, **A M Osorno**, **V Ozornova**, **K Ponto**, **C Yarnall**, *A model structure on $G\mathscr{C}at$*, from "Women in topology: collaborations in homotopy theory" (M Basterra, K Bauer, K Hess, B Johnson, editors), Contemp. Math. 641, Amer. Math. Soc., Providence, RI (2015) 123–134 MR Zbl

[Campbell 2020] **A Campbell**, *A counterexample in quasi-category theory*, Proc. Amer. Math. Soc. 148 (2020) 37–40 MR Zbl

[Cisinski 1999] **D-C Cisinski**, *La classe des morphismes de Dwyer n'est pas stable par retractes*, Cahiers Topologie Géom. Différentielle Catég. 40 (1999) 227–231 MR Zbl

[Dwyer and Kan 1980] **W G Dwyer**, **D M Kan**, *Simplicial localizations of categories*, J. Pure Appl. Algebra 17 (1980) 267–284 MR Zbl

[Dwyer and Spaliński 1995] **W G Dwyer**, **J Spaliński**, *Homotopy theories and model categories*, from "Handbook of algebraic topology" (I M James, editor), North-Holland, Amsterdam (1995) 73–126 MR Zbl

[Fiedorowicz 2002] **Z Fiedorowicz**, *A counterexample to a group completion conjecture of J C Moore*, Algebr. Geom. Topol. 2 (2002) 33–35 MR Zbl

[Hackney et al. 2022] **P Hackney**, **V Ozornova**, **E Riehl**, **M Rovelli**, *Pushouts of Dwyer maps are* $(\infty, 1)$*–categorical* (2022) arXiv 2205.02353v2

[Hackney et al. 2023] **P Hackney**, **V Ozornova**, **E Riehl**, **M Rovelli**, *An* $(\infty, 2)$*–categorical pasting theorem*, Trans. Amer. Math. Soc. 376 (2023) 555–597 MR Zbl

[Hill et al. 2016] **M A Hill**, **M J Hopkins**, **D C Ravenel**, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. 184 (2016) 1–262 MR Zbl

[Hirschhorn 2003] **P S Hirschhorn**, *Model categories and their localizations*, Math. Surv. Monogr. 99, Amer. Math. Soc., Providence, RI (2003) MR Zbl

[Joyal 2008] **A Joyal**, *The theory of quasi-categories and its applications*, preprint (2008) Available at `http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf`

[Joyal and Tierney 2007] **A Joyal**, **M Tierney**, *Quasi-categories vs Segal spaces*, from "Categories in algebra, geometry and mathematical physics" (A Davydov, M Batanin, M Johnson, S Lack, A Neeman, editors), Contemp. Math. 431, Amer. Math. Soc., Providence, RI (2007) 277–326 MR Zbl

[Lurie 2009a] **J Lurie**, *Higher topos theory*, Ann. of Math. Stud. 170, Princeton Univ. Press (2009) MR Zbl

[Lurie 2009b] **J Lurie**, $(\infty, 2)$*–categories and the Goodwillie calculus, I*, preprint (2009) arXiv 0905.0462

[McDuff 1979] **D McDuff**, *On the classifying spaces of discrete monoids*, Topology 18 (1979) 313–320 MR Zbl

[Schwede 2019] **S Schwede**, *Categories and orbispaces*, Algebr. Geom. Topol. 19 (2019) 3171–3215 MR Zbl

[Stevenson 2017] **D Stevenson**, *Covariant model structures and simplicial localization*, North-West. Eur. J. Math. 3 (2017) 141–203 MR Zbl

[Stevenson 2018a] **D Stevenson**, *Model structures for correspondences and bifibrations*, preprint (2018) arXiv 1807.08226

[Stevenson 2018b] **D Stevenson**, *Notes on the Joyal model structure*, preprint (2018) arXiv 1810.05233

[Thomason 1980] **R W Thomason**, *Cat as a closed model category*, Cahiers Topologie Géom. Différentielle 21 (1980) 305–324 MR Zbl

Department of Mathematics, University of Louisiana at Lafayette
Lafayette, LA, United States

Max Planck Institute for Mathematics
Bonn, Germany

Department of Mathematics, Johns Hopkins University
Baltimore, MD, United States

Department of Mathematics and Statistics, University of Massachusetts at Amherst
Amherst, MA, United States

`philip@phck.net`, `viktoriya.ozornova@mpim-bonn.mpg.de`, `eriehl@jhu.edu`,
`rovelli@math.umass.edu`

Received: 3 June 2022 Revised: 8 March 2023

Algebraic & Geometric Topology **24**:4 (2024) 2185–2208
DOI: 10.2140/agt.2024.24.2185
Published: 16 July 2024

A variant of a Dwyer–Kan theorem for model categories

BORIS CHORNY
DAVID WHITE

If all objects of a simplicial combinatorial model category $\mathcal{A}$ are cofibrant, we construct the homotopy model structure on the category of small functors $\mathcal{S}^{\mathcal{A}}$, where the fibrant objects are the levelwise fibrant homotopy functors, ie functors preserving weak equivalences. When $\mathcal{A}$ fails to have all objects cofibrant, we construct the bifibrant-projective model structure on $\mathcal{S}^{\mathcal{A}}$ and prove that it is an adequate substitute for the homotopy model structure. Next, we generalize a theorem of Dwyer and Kan, characterizing which functors $f : \mathcal{A} \to \mathcal{B}$ induce a Quillen equivalence $\mathcal{S}^{\mathcal{A}} \leftrightarrows \mathcal{S}^{\mathcal{B}}$ with the model structures above. We include an application to Goodwillie calculus, and we prove that the category of small linear functors from simplicial sets to simplicial sets is Quillen equivalent to the category of small linear functors from topological spaces to simplicial sets.

18N40, 55P65

Introduction

Homotopy functors are functors taking weak equivalences to weak equivalences. They have been a central object of interest in algebraic topology from the very beginning of the subject. W G Dwyer and D Kan [18] began the systematic study of the categories of homotopy functors with the theory of (what are nowadays called) relative categories (fully developed by C Barwick and D Kan [1]). In more detail, Dwyer and Kan ask when a map $f : (\mathcal{A}, U) \to (\mathcal{B}, V)$ of relative categories induces a Quillen equivalence $f^* : \mathcal{S}^{\mathcal{B},V} \to \mathcal{S}^{\mathcal{A},U}$ between the categories of homotopy functors (called restricted diagrams in [18]) from the relative categories to the category $\mathcal{S}$ of simplicial sets. Dwyer and Kan prove that f^* is a Quillen equivalence if and only if the induced map of simplicial localizations $Lf : L(\mathcal{A}, U) \to L(\mathcal{B}, V)$ is an r–equivalence of simplicial categories [18, Theorem 2.2]. In the current paper we formulate a version of this theorem for model categories.

Since the concept of r–equivalences is rarely used, especially in comparison to the concept of Dwyer–Kan equivalences, introduced in the same article [18], we recall that a map $f : \mathcal{A} \to \mathcal{B}$ of simplicial categories is an *r–equivalence* if

- for every two objects $A_1, A_2 \in \mathcal{A}$, the induced map
$$\hom_{\mathcal{A}}(A_1, A_2) \to \hom_{\mathcal{B}}(fA_1, fA_2)$$
is a weak equivalence, and
- every object in the "category of components" $\pi_0\mathcal{B}$ is a retract of an object in the image of $\pi_0 f$.

A completely different approach to the study of the category of homotopy functors from spaces to spaces is given by Goodwillie's calculus of functors [20; 21; 22]. It was noticed by W G Dwyer [17] that Goodwillie's polynomial approximation may be interpreted as a homotopical localization. This approach was reworked in terms of model categories by G Biedermann, the first author and O Röndigs [4]; in particular constructing the model category of homotopy functors on the categories of small functors from simplicial sets to simplicial sets and to spectra.

Later on, various generalizations of Goodwillie calculus to other contexts have appeared — see Basterra, Bauer, Beaudry, Eldred, Johnson, Merling and Yeakel [2], Biedermann and Röndigs [5], and Pereira [28] — and, hence, a natural question that arises here is the question of invariance of Goodwillie's calculus under Quillen equivalence. For example, topological spaces $\mathcal{T}\mathrm{op}$ and simplicial sets $\mathcal{S}$ are Quillen equivalent simplicial model categories. Are the model categories of small homotopy (or linear, or n–excisive) functors $\mathcal{S}^{\mathcal{T}\mathrm{op}}$ and $\mathcal{S}^{\mathcal{S}}$ Quillen equivalent?

First we give an analog of the Dwyer–Kan theorem to model categories with all objects cofibrant: in Theorem 3.1 we construct a model category of homotopy functors and in Theorem 5.2 we show that a Quillen equivalence of two combinatorial model categories with all objects cofibrant gives rise to a Quillen equivalence of the categories of small functors into simplicial sets. Unfortunately this approach does not generalize further; we were not able to construct the homotopy model structure for arbitrary model categories.

The purpose of this paper is to develop a context in which the Dwyer–Kan theorem may be formulated for model categories, and then to prove that the categories of what replaces homotopy functors in our setup are equivalent if and only if the domain categories are r–equivalent. We prove this result in Theorem 5.8. In particular, Example 5.6 implies together with Theorem 5.8 that a Quillen equivalence $\mathcal{A} \leftrightarrows \mathcal{B}$ of simplicial combinatorial model categories induces a Quillen equivalence $\mathcal{S}^{\mathcal{A}} \leftrightarrows \mathcal{S}^{\mathcal{B}}$ of the model categories of small homotopy functors.

The absolute version of the Dwyer–Kan theorem states that a map of simplicial categories $f : \mathcal{A} \to \mathcal{B}$ induces a Quillen equivalence $\mathrm{Lan}_f : \mathcal{S}^{\mathcal{A}} \leftrightarrows \mathcal{S}^{\mathcal{B}} : f^*$ if and only if f is an r–equivalence, [18, Theorem 2.1]. Lukáš Vokřínek [31] generalized this result to categories enriched in a closed symmetric monoidal model category. The categories of homotopy functors are not discussed in his work. We give a version of the relative Dwyer–Kan theorem [18, Theorem 2.2] (which generalizes [18, Theorem 2.1]) for model categories in this paper.

As an application, we prove that the categories of small n–excisive functors defined on simplicial sets and on topological spaces are Quillen equivalent. More generally, given a Quillen pair such that the right adjoint preserves homotopy pushouts, we show that the model categories of n–excisive functors defined on this Quillen pair and taking values in simplicial sets, are Quillen equivalent.

The paper is organized as follows. In the preliminary section we characterize which simplicial functors of simplicial combinatorial model categories induce a Quillen adjunction between the categories of small functors into simplicial sets equipped with the projective and the fibrant-projective — see Biedermann and the first author [3] — model structures. In Section 2 we introduce the bifibrant-projective model structure, show its existence and extend the results from the preliminary section to this new setting.

Section 3 is devoted to the study of the homotopy model structures (such that the fibrant objects are the fibrant homotopy functors) on the categories of small functors from a model category to simplicial sets. In order to prove the existence of the homotopy model structure, we require that the domain model category has all objects cofibrant. This is not a major restriction, since, as shown by Ching and Riehl [9] and Dugger [15], every combinatorial model category is Quillen equivalent to one with all objects cofibrant (and in [9] even to one whose objects are objects of the original category). Furthermore, we show that whenever the homotopy model category exists, it is Quillen equivalent to the bifibrant-projective model structure, which exists without the requirement that all objects be cofibrant, and is a suitable replacement. Our comparison of the small functors from topological spaces to simplicial sets with the small functors from simplicial sets to simplicial sets is carried out in Section 4. The homotopy model structure on $\mathcal{S}^{\mathcal{S}}$ was constructed in [4] and it is Quillen equivalent to the fibrant-projective model structure. Because the Quillen model structure on $\mathcal{T}\mathrm{op}$ does not have all objects cofibrant, we do not know if the homotopy model structure on $\mathcal{S}^{\mathcal{T}\mathrm{op}}$ exists, but the cofibrant-projective model structure on $\mathcal{S}^{\mathcal{T}\mathrm{op}}$ is Quillen equivalent to the fibrant-projective model structure on $\mathcal{S}^{\mathcal{S}}$. This means that the bifibrant-projective model structures on both categories produce Quillen equivalent model categories, as we wanted to show.

In Section 5 we prove our main result generalizing the Dwyer–Kan theorem. We first treat the simpler case of the homotopy model structures when all objects in the domain category are cofibrant, and then prove the general case cited above. An application to Goodwillie calculus is given in Section 6. We prove the Quillen equivalence of the categories of the n–excisive functors by localizing the Quillen equivalent categories of small functors equipped with the bifibrant-projective model structure. A tool allowing for such comparison is developed in the appendix and hopefully will be useful in other situations as well. The question of the existence of the n–excisive model structure is not addressed in this work, since it was considered in a number of papers [4; 5; 11], and the methods of localization developed there may be easily applied to the current situation.

Acknowledgements We are grateful to Brooke Shipley and Karol Szumiło for valuable comments. Chorny acknowledges the support of ISF grant 1138/16. White thanks the Center for Mathematics and Scientific Computation for supporting a visit to the University of Haifa at Oranim in 2015, when this work began.

1 Preliminaries

In this section we recall the homotopy theory of small functors and establish some basic properties of various model categories of small functors. We assume the reader is familiar with the basics of model categories and left Bousfield localization, eg [23; 24]. Note that all our model categories and functors between them are simplicial, and $\hom(X, Y)$ denotes the simplicial set of morphisms from X to Y. A model category is combinatorial if it is locally presentable and cofibrantly generated.

Definition 1.1 Let $\mathcal{A}$ be a simplicial category. A functor $F\colon \mathcal{A} \to \mathcal{S}$ is *small* if it is a left Kan extension from some small subcategory. In other words, there exists a small full subcategory $i\colon \mathcal{A}' \hookrightarrow \mathcal{A}$ such that $F = \operatorname{Lan}_i i^* F$. We denote the category of small functors from $\mathcal{A}$ to $\mathcal{S}$ by $\mathcal{S}^{\mathcal{A}}$.

Remark 1.2 In the book by M Kelly [25], small functors are called accessible, which does not correspond to modern terminology (accessible functors are functors of accessible categories preserving λ–filtered colimits for some cardinal λ), though accessible functors are always small and small functors of accessible categories are accessible. A functor is small if and only if it is a small (weighted) colimit of representable functors [25, Proposition 4.83]. Since the category of small functors from $\mathcal{A}$ to $\mathcal{S}$ is cocomplete [25, Proposition 5.34], in particular tensored over $\mathcal{S}$, a colimit of the functor $G\colon (\mathcal{A}')^{\mathrm{op}} \to \mathcal{S}^{\mathcal{A}}$ weighted by the functor $F\colon \mathcal{A}' \to \mathcal{S}$ may be computed using the coend formula: $F \star_{\mathcal{A}'} G = \int^{A \in \mathcal{A}'} FA \otimes GA$ [25, 3.70].

Now we would like to analyze what kind of functors are induced on the categories of small functors by an adjunction of domain categories.

Proposition 1.3 *Let $L\colon \mathcal{A} \to \mathcal{B}$ be a simplicial accessible functor between locally presentable simplicial categories. Then there exists a pair of adjoint functors between the categories of small functors*

$$\operatorname{Lan}_L\colon \mathcal{S}^{\mathcal{A}} \leftrightarrows \mathcal{S}^{\mathcal{B}} : L^*.$$

If in addition L has a right adjoint R, then $\operatorname{Lan}_L = R^*$ is given by the precomposition with R.

Proof Note that every small functor $F \in \mathcal{S}^{\mathcal{A}}$ is a left Kan extension from a full small subcategory $i\colon \mathcal{A}' \hookrightarrow \mathcal{A}$. Then

$$\operatorname{Lan}_L(F) = \operatorname{Lan}_L(\operatorname{Lan}_i i^* F) = \operatorname{Lan}_{Li} i^* F$$

by the transitivity property of the iterated left Kan extensions [25, Theorem 4.47]. Hence, $\operatorname{Lan}_L(F) \in \mathcal{S}^{\mathcal{A}}$ is a small functor.

Given a representable functor $R^B = \hom_{\mathcal{B}}(B, -)$, the functor $L^* R^B = \hom(B, L-)$ is no longer representable, but it is λ–accessible if B is λ–presentable and L is λ–accessible. Hence it is a small functor as an accessible functor of accessible categories.

For any $G \in \mathcal{S}^{\mathcal{B}}$,

$$L^* G = L^*\left(\int^B \hom(B, -) \otimes GB\right) = \int^B \hom(B, L-) \otimes GB$$

is a weighted colimit of small functors, which is again small [25, 5.34].

Suppose now that L has a right adjoint $L \dashv R$. Then, using Yoneda's lemma,

$$\begin{aligned}
\hom(R^*F, G) &= \hom\Big(R^*\Big(\int^{A\in\mathscr{A}'} \hom(A,-)\otimes FA\Big), G\Big) \\
&= \int_A \hom(\hom(A, R(-))\otimes FA, G) \\
&= \int_A \hom(FA, \hom(\hom(LA,-), G)) \\
&= \int_A \hom(FA, G(LA)) \\
&= \int_A \hom(FA, L^*G(A)) \\
&= \int_A \hom(FA, \hom(\hom(A,-), L^*G)) \\
&= \int_A \hom(\hom(A,-)\otimes FA, L^*G) \\
&= \hom\Big(\int^A \hom(A,-)\otimes FA, L^*G\Big) = \hom(F, L^*G).
\end{aligned}$$

In other words $R^* \dashv L^*$; hence $R^* = \mathrm{Lan}_L$. □

We are interested in the homotopy theory of small functors. The projective model structure (where weak equivalences and fibrations are levelwise) on the category of small functors was constructed in [12, Theorem 3.1] for all cocomplete domain categories. The condition of cocompleteness is required to ensure that the category of small functors is complete [14, Corollary 3.9].

Proposition 1.4 *Given a simplicial accessible functor $f : \mathscr{A} \to \mathscr{B}$ of simplicial combinatorial model categories, the adjunction $\mathrm{Lan}_f \dashv f^*$ discussed in Proposition 1.3 is a Quillen pair for the projective model structures on the categories of small functors from $\mathscr{A}$ and $\mathscr{B}$ to $\mathscr{S}$.*

Proof Consider the adjunction

$$\begin{tikzcd} \mathscr{S}^{\mathscr{A}} \arrow[r, bend left, "\mathrm{Lan}_f"] & \mathscr{S}^{\mathscr{B}} \arrow[l, bend left, "f^*"] \end{tikzcd}$$

between the two model categories of small functors equipped with the projective model structure [12]. Let $p\colon F \to G$ be a (trivial) fibration in $\mathscr{S}^{\mathscr{B}}$. Consider the induced map $f^*p\colon f^*F \to f^*G$ in $\mathscr{S}^{\mathscr{A}}$. Let $A \in \mathscr{A}$ be an arbitrary object. Then $p_{fA}\colon F(fA) \to G(fA)$ is a (trivial) fibration by assumption. Furthermore, $f^*p_A = p_{fA}$ is also a (trivial) fibration:

$$\begin{tikzcd} (f^*F)(A) \arrow[r, "f^*p_A"] \arrow[d, equal] & (f^*G)(A) \arrow[d, equal] \\ F(fA) \arrow[r, two heads, "(\sim)", "p_{fA}"'] & G(fA) \end{tikzcd}$$

□

The fibrant-projective model structure on the category of small functors with domain in a combinatorial model category (where weak equivalences and fibrations are levelwise in fibrant objects) was constructed in [3, Definition 3.2]. This is a particular case of the relative model structure [10, Definition 2.2]. In the next proposition we analyze its interaction with the adjunction of Proposition 1.3.

Proposition 1.5 *Given a simplicial accessible functor $f : \mathcal{A} \to \mathcal{B}$ of simplicial combinatorial model categories, the adjunction $\mathrm{Lan}_f \dashv f^*$ discussed in Proposition 1.3 is a Quillen pair for the fibrant-projective model structure on the categories of small functors from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{S}$ if and only if f preserves fibrant objects.*

Proof The "if" direction follows in the same manner as Proposition 1.4 above.

For the "only if" direction, assume that f^* is a right Quillen functor and we need to show that for every fibrant $A \in \mathcal{A}$ the map $p\colon fA \to *$ has the right lifting property with respect to any trivial cofibration $i\colon B_1 \overset{\sim}{\hookrightarrow} B_2$ in $\mathcal{B}$. By [23, Proposition 9.4.3], it suffices to show that (i, p) is a homotopy lifting-extension pair. In other words, it suffices to show that $\hom(B_2, fA) \to \hom(B_1, fA)$ is a trivial fibration of simplicial sets.

For any trivial cofibration $i : B_1 \overset{\sim}{\hookrightarrow} B_2$ in $\mathcal{B}$ the induced map of representable functors

$$i^* : \hom(B_2, -) \to \hom(B_1, -)$$

is a trivial fibration in the fibrant-projective model structure on $\mathcal{S}^{\mathcal{B}}$, by the SM7 axiom [23, Definition 9.1.6]. Since f^* is a right Quillen functor, the map

$$f^* i^* : \hom(B_2, f-) \to \hom(B_1, f-)$$

is a trivial fibration in the fibrant-projective model structure on $\mathcal{S}^{\mathcal{A}}$, ie

$$\hom(B_2, fA) \overset{\sim}{\twoheadrightarrow} \hom(B_1, fA)$$

is a trivial fibration of simplicial sets for all fibrant $A \in \mathcal{A}$. □

2 Bifibrant-projective model structure

Let $\mathcal{A}$ be a simplicial combinatorial model category. By analogy with the fibrant-projective [3, Definition 3.2] and cofibrant-projective [6, Definition 2.1] model structures on the categories of small functors $\mathcal{S}^{\mathcal{A}}$, we introduce the bifibrant-projective model structure on $\mathcal{S}^{\mathcal{A}}$.

Definition 2.1 Let $\mathcal{A}$ be a simplicial combinatorial model category, and let $F, G \in \mathcal{S}^{\mathcal{A}}$ be small functors. A natural transformation $f : F \to G$ is a *bifibrant weak equivalence* (resp. *bifibrant fibration*) if for all bifibrant objects $A \in \mathcal{A}$ (ie objects which are both fibrant and cofibrant), the induced map $f_A : F(A) \to G(A)$ is a weak equivalence (resp. a fibration) of simplicial sets. A natural transformation is a *bifibrant cofibration* if it has the left lifting property with respect to bifibrant trivial fibrations.

Next, we establish the existence of the bifibrant-projective model structure as a particular case of the relative model structure [10, Definition 2.1].

Proposition 2.2 *Let $\mathcal{A}$ be a simplicial combinatorial model category. Then the category of small functors $\mathcal{S}^{\mathcal{A}}$ may be equipped with the bifibrant model structure.*

Proof We will verify the conditions of [10, Proposition 2.8] in order to establish the bifibrant model structure, which is also the bifibrant relative model structure in the terminology of [10].

The condition requiring verification is the local smallness [10, Definition 2.4] of the subcategory of bifibrant objects in the category $\mathcal{A}^{\mathrm{op}}$, or, dually, the solution set condition in $\mathcal{A}$, ie for every object $A \in \mathcal{A}$ we need to find a *set* of bifibrant objects $\mathcal{W}_A$ such that for every bifibrant object B and every map $A \to B$ there exists a (nonunique) object $W \in \mathcal{W}_A$ such that $A \to W \to B$.

For every object $A \in \mathcal{A}$, choose a cardinal κ large enough that A is κ–presentable and $\mathcal{A}$ is κ–combinatorial. Next, look at the set $\mathcal{W}'_A$ of κ–presentable cofibrant objects in $\mathcal{A}$, then put $\mathcal{W}_A = \{\widehat{W} \mid W \in \mathcal{W}'_A\}$, where $\widehat{W}$ denotes fibrant replacement.

The fat small object argument [27, Corollary 5.1] shows that every cofibrant object is a κ–filtered colimit of κ–presentable cofibrant objects in the κ–combinatorial model category $\mathcal{A}$. It follows that every morphism $A \to B$ into a bifibrant object B factors first through some $W_1 \in \mathcal{W}'_A$. Finally, the morphism $W_1 \to B$ factors through the fibrant replacement $W = \widehat{W}_1$ of W_1, since B is fibrant: $W_1 \xhookrightarrow{\sim} W \to B$. □

Now we need to find the conditions on a functor $f : \mathcal{A} \to \mathcal{B}$ between simplicial combinatorial model categories such that the induced adjunction $\mathrm{Lan}_f \dashv f*$ of Proposition 1.3 is a Quillen pair.

Proposition 2.3 *Given a simplicial accessible functor $f : \mathcal{A} \to \mathcal{B}$ of simplicial combinatorial model categories, the adjunction $\mathrm{Lan}_f \dashv f^*$ discussed in Proposition 1.3 is a Quillen pair for the bifibrant-projective model structure on the categories of small functors from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{S}$ if f preserves both fibrant and cofibrant objects.*

Proof Similar to Proposition 1.4. □

Example 2.4 The classical Quillen equivalence $|{-}| : \mathcal{S} \leftrightarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}$ induces the Quillen map of bifibrant-projective model structures

$$\mathrm{Sing}^* : \mathcal{S}^{\mathcal{S}} \leftrightarrows \mathcal{S}^{\mathcal{T}\mathrm{op}} : |{-}|^*,$$

which turn out to be the fibrant-projective and the cofibrant-projective model structures respectively. Of course, this is a very special case when the left Quillen functor preserves fibrant objects. We will have to find a way around this difficulty in order to generalize this example.

3 Homotopy model structure

Let $\mathcal{A}$ be a simplicial combinatorial model category. Recall that *homotopy functors* are functors preserving the weak equivalences. If it exists, the *homotopy model structure* on the category of small functors $\mathcal{S}^{\mathcal{A}}$ is a localization of the projective model structure in such a way that the local objects are the projectively fibrant homotopy functors. We will only construct the homotopy model structure on the category of small presheaves $\mathcal{S}^{\mathcal{A}}$ under the additional assumption that all objects of $\mathcal{A}$ are cofibrant.

3.1 Localization construction

If we localize the projective model structure on the category of small functors $\mathcal{S}^{\mathcal{A}}$ with respect to the class of maps

$$H_{\mathcal{A}} = \{\hom(A_1, -) \to \hom(A_2, -) \mid A_1 \xrightarrow{\sim} A_2 \text{ in } \mathcal{A}\},$$

then the fibrant objects in the new model structure will be precisely the levelwise fibrant homotopy functors. The resulting model structure is the homotopy model structure on $\mathcal{S}^{\mathcal{A}}$. Since the projective model structure is not cofibrantly generated (it has a proper class of generating cofibrations, instead of a small set), and $H_{\mathcal{A}}$ is a proper class of maps, the localization techniques of Smith and Hirschhorn may not be applied.

In the case that all objects of $\mathcal{A}$ are cofibrant, we will use the Bousfield–Friedlander [8, Appendix A] Q–model structure construction further improved by Bousfield [7, Theorem 9.3] in order to obtain the left Bousfield localization of $\mathcal{S}^{\mathcal{A}}$ with respect to $H_{\mathcal{A}}$.

Theorem 3.1 *Let $\mathcal{A}$ be a simplicial combinatorial model category with all objects cofibrant. Then there exists a localization of the projective model structure on $\mathcal{S}^{\mathcal{A}}$, such that the fibrant objects are precisely the levelwise fibrant homotopy functors.*

Proof Since $\mathcal{A}$ is a simplicial combinatorial model category, we can fix a continuous, accessible fibrant replacement functor $\mathrm{Fib}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}$ together with a natural transformation $\varepsilon\colon \mathrm{Id}_{\mathcal{A}} \to \mathrm{Fib}_{\mathcal{A}}$. These properties are required to ensure that the precomposition of $\mathrm{Fib}_{\mathcal{A}}$ with a small functor $F\colon \mathcal{A} \to \mathcal{S}$ produces a small functor again.

We denote the fibrant replacement in $\mathcal{S}$ by $\widehat{(-)}$. In this case the homotopy approximation functor may be constructed very explicitly. Namely, for any small $F\colon \mathcal{A} \to \mathcal{S}$, we can put $\mathcal{H}(F) = \widehat{\mathrm{Fib}_{\mathcal{A}}^* F} = \widehat{F \circ \mathrm{Fib}_{\mathcal{A}}}$. It is equipped with the coaugmentation $\widehat{F\varepsilon}\colon F \to \widehat{\mathrm{Fib}_{\mathcal{A}}^* F}$. This is a homotopy idempotent construction that takes values in homotopy functors, since weak equivalences of objects which are fibrant and cofibrant are simplicial weak equivalences [29, II.2.5] and the latter are preserved by simplicial functors; see [4, Proposition 3.3].

By [30, Proposition 4.3], in any model category $\mathcal{M}$ equipped with a homotopy idempotent functor $L\colon \mathcal{M} \to \mathcal{M}$, the class of L–equivalences (the maps rendered by L into weak equivalences) coincides with the class of the local equivalences (the class of maps simplicially orthogonal to the L–local objects);

therefore $\mathcal{H}$–equivalences are precisely the local equivalences with respect to the fibrant homotopy functors. Since our construction is very simple, we can see immediately that $\mathcal{H}$–equivalences, ie maps rendered into projective weak equivalences by the functor $\mathcal{H}$, are precisely the fibrant-projective weak equivalences of small functors [3, Definition 3.2], ie the natural transformations of functors inducing weak equivalences of fibrant objects.

It remains to verify that our localization construction satisfies the conditions A1–A3 of [7, Theorem 9.3]. The projective model structure on the category $\mathcal{S}^{\mathcal{A}}$ of small functors is proper by [3, Theorem 3.6], since $\mathcal{S}$ is a right proper model category and a strongly left proper monoidal model category [16, Definition 4.5].

Conditions A1 and A2 are satisfied by the construction of $\mathcal{H}$ and the discussion above. To verify A3, consider the pullback of a fibrant-projective weak equivalence along a projective fibration. Since $\mathcal{S}$ is right proper, the base change of a fibrant-projective weak equivalence is a fibrant-projective weak equivalence again.

Hence the left Bousfield localization exists, and defines the $\mathcal{H}$–local model structure on the category of small functors from $\mathcal{A}$ to $\mathcal{S}$. This is the homotopy model structure, since the $\mathcal{H}$–local objects are precisely the projectively fibrant homotopy functors. In other words, $\mathcal{H}$–localization is the localization with respect to $H_{\mathcal{A}}$. □

If we drop the assumption that all objects are cofibrant, we are unable to construct the left Bousfield localization of the projective model category $\mathcal{S}^{\mathcal{A}}$ with local objects being precisely the homotopy functors, but we will show the existence of a homotopy idempotent (nonfunctorial) localization construction Q, such that Q–equivalences are precisely the $H_{\mathcal{A}}$–equivalences.

Proposition 3.2 *Let $\mathcal{A}$ be a simplicial combinatorial model category. Then for each functor $F \in \mathcal{S}^{\mathcal{A}}$ there exists an $H_{\mathcal{A}}$–equivalence $F \to QF$ such that QF is a homotopy functor.*

Proof Let $f \colon \mathcal{A} \to \mathcal{A}$ be a bifibrant replacement functor. Notice that the adjunction $\operatorname{Lan}_f \dashv f^*$ is a Quillen pair for the projective model structure on $\mathcal{S}^{\mathcal{A}}$, since the right adjoint f^* preserves fibrations and trivial fibrations.

For every functor $F \in H_{\mathcal{A}}$ consider the construction

$$\begin{tikzcd}
\tilde{F} \arrow[rr] \arrow[dr, hook] \arrow[dd, two heads, "\wr"'] & & f^* \operatorname{Lan}_f \tilde{F} \\
& Q'F \arrow[ur, two heads, "\sim"] \arrow[d, "\wr"] & \\
F \arrow[r] & QF &
\end{tikzcd}$$

We begin with $F \in H_{\mathcal{A}}$, take its cofibrant replacement $\tilde{F}$ in the projective model structure and factor the unit of the adjunction $\operatorname{Lan}_f \dashv f^*$ into a projective cofibration followed by a projective trivial fibration. Denote the middle term of the factorization by $Q'F$ and put $QF = Q'F \amalg_{\tilde{F}} F$.

Unfortunately this construction is not functorial, since the cofibrant replacement in the projective model structure is not known to be functorial. On the other hand, it naturally extends to morphisms and can be rendered functorial on any small subcategory of $\mathcal{S}^{\mathcal{A}}$.

This construction preserves weak equivalences of functors, since all the stages of the construction do. In particular, Lan_f preserves weak equivalences between cofibrant objects and f^* preserves all projective weak equivalences.

Now, QF is projectively weakly equivalent to $f^* \mathrm{Lan}_f \widetilde{F}$, so in order to show that QF is a homotopy functor it suffices to show that $f^* \mathrm{Lan}_f \widetilde{F}$ is. We will show it by cellular induction, assuming that $\widetilde{F} = \mathrm{colim}_{i<\lambda} F_i$, so that $F_0 = \varnothing$ and F_i is obtained from F_{i-1} by attaching a cell

$$\begin{tikzcd} R^{A_i} \otimes \partial\Delta^n \arrow[r] \arrow[d, hook] & F_{i-1} \arrow[d, hook] \\ R^{A_i} \otimes \Delta^n \arrow[r] & F_i \end{tikzcd}$$

if i is a successor ordinal or $F_i = \mathrm{colim}_{a<i} F_a$ if i is a limit ordinal.

In order to compute $\mathrm{Lan}_f \widetilde{F}$, notice that Lan_f commutes with colimits, and $\mathrm{Lan}_f R^{A_i} = R^{f(A_i)}$. In other words, $\mathrm{Lan}_f \widetilde{F}$ is a cellular complex with cells of type $R^{f(A_i)}$, ie represented in cofibrant objects.

Next, we must show that QF is a homotopy functor, ie it preserves weak equivalences. The following argument proves that a projectively equivalent functor $f^* \mathrm{Lan}_f \widetilde{F}$ is a homotopy functor by cellular induction. First, note that f^* preserves colimits, as a left adjoint to Ran_f, which exists, in turn, by [14], or just because the colimits in the diagrams of functors are computed levelwise. Therefore, $f^* \mathrm{Lan}_f \widetilde{F}$ is a cellular construction, with cells of type $f^* R^{f(A_i)} = \hom(f(A_i), f(-))$, so it is no longer a representable functor, but is a homotopy functor. Hence, assuming for induction that $f^* \mathrm{Lan}_f F_a$ is a homotopy functor for all $a < i$, we obtain that $f^* \mathrm{Lan}_f F_i$ is also a homotopy functor; hence $f^* \mathrm{Lan}_f \widetilde{F}$ is a homotopy functor as a sequential colimit of homotopy functors into $\mathcal{S}$.

The last claim that we need to show is that the map $F \to QF$ is an $H_{\mathcal{A}}$–equivalence. In other words, that our construction is homotopy idempotent. We will show the equivalent statement that the map $F \to QF$ is initial in a suitable sense, ie we will show that in the homotopy category $\mathrm{Ho}(\mathcal{S}^{\mathcal{A}})$ the unit of the derived pair of adjoint functors $\varepsilon\colon [\widetilde{F}] = [F] \to [QF] = \boldsymbol{R} f^* \boldsymbol{L} \mathrm{Lan}_f[\widetilde{F}]$ is initial with respect to maps into homotopy functors.

Let $H \in \mathcal{S}^{\mathcal{A}}$ be a projectively fibrant homotopy functor, and let $g\colon [F] \to [H]$ be a map in the homotopy category. Notice that since H preserves weak equivalences, $[H] = [f^* H] = \boldsymbol{R} f^*[H]$. Then by the universal property of the unit there exists a unique map $h\colon \boldsymbol{R} f^* \boldsymbol{L} \mathrm{Lan}_f \widetilde{F} \to \boldsymbol{R} f^*[H] = [H]$ such that $g = h\varepsilon$.

Therefore, our initial construction $F \to QF$ is a homotopy localization turning every small functor into a homotopy functor, ie localization with respect to $S_{\mathcal{A}}$. $\square$

3.2 Comparison of the homotopy and the bifibrant-projective model structures

We next prove that the homotopy model structure on $\mathcal{S}^{\mathcal{A}}$ is Quillen equivalent to the bifibrant-projective model structure (Definition 2.1), when both model structures exist. This means we can use the bifibrant-projective model structure as a substitute for the homotopy model structure in contexts where the homotopy model structure is not yet known to exist. We conjecture that the homotopy model structure on $\mathcal{S}^{\mathcal{A}}$ exists even if not all objects of $\mathcal{A}$ are cofibrant. We expect that localization of class-combinatorial model categories [13] can be used to prove this conjecture.

For the sake of comparison, we assume in this section that the homotopy model structure exists. We note that the bifibrant-projective model structure exists whenever $\mathcal{A}$ is combinatorial. We show now that these model structures are Quillen equivalent.

Theorem 3.3 *Let $\mathcal{A}$ be a simplicial combinatorial model category. Then the pair of identity functors induces a Quillen equivalence of the homotopy and the bifibrant-projective model structures.*

Proof Consider the pair of adjoint functors

$$\mathrm{Id}\colon \mathcal{S}^{\mathcal{A}}_{\text{bifib-proj}} \rightleftarrows \mathcal{S}^{\mathcal{A}}_{\text{proj}} :\mathrm{Id},$$

where the left adjoint is pointing from left to right.

This is a Quillen pair because the right adjoint obviously preserves fibrations and trivial fibrations. Now we localize the projective model structure and obtain the homotopy model structure on the right-hand side. The identity functors still form an adjoint pair

$$\mathrm{Id}\colon \mathcal{S}^{\mathcal{A}}_{\text{bifib-proj}} \rightleftarrows \mathcal{S}^{\mathcal{A}}_{\text{ho}} :\mathrm{Id},$$

where $\mathcal{S}^{\mathcal{A}}_{\text{ho}}$ denotes the homotopy model structure (which we have assumed to exist). This adjoint pair is still a Quillen pair, as a composition of the previous adjunction with the Quillen pair arising from the left Bousfield localization of the projective model structure. To show that this is a Quillen equivalence we will use [24, Corollary 1.3.16(b)]. The left adjoint reflects weak equivalences between cofibrant objects, since the fibrant approximation in the homotopy model structure (approximation by the levelwise fibrant homotopy functor constructed in Proposition 3.2) can only change the values of a bifibrant-projectively cofibrant functor in fibrant objects up to a weak equivalence.

It remains to show that for every fibrant (homotopy) functor $F \in \mathcal{S}^{\mathcal{A}}_{\text{ho}}$, the cofibrant replacement map $i\colon \widetilde{F} \to F$ in the bifibrant-projective model structure $\mathcal{S}^{\mathcal{A}}_{\text{bifib-proj}}$ is a weak equivalence in the homotopy model structure. In other words, if we apply the homotopy approximation construction Q from Proposition 3.2 we obtain a projective weak equivalence. Indeed, there is a projective weak equivalence $QF \simeq F$, since F is a homotopy functor. Furthermore, $Q\widetilde{F}$ is homotopy functor bifibrant-projectively equivalent to $\widetilde{F}$, hence also to $F \simeq QF$. So, by the 2-out-of-3 property, $Qi\colon Q\widetilde{F} \to QF$ is a bifibrant-projective weak equivalence of homotopy functors and hence is a levelwise weak equivalence, as required. □

4 Motivating example

Before we turn to the proof of the main result, let us consider the example of (Δ–generated [19]) topological spaces $\mathcal{T}\mathrm{op}$ and simplicial sets $\mathcal{S}$, two Quillen equivalent simplicial model categories with very different categories of small functors $\mathcal{S}^{\mathcal{T}\mathrm{op}}$ and $\mathcal{S}^{\mathcal{S}}$. For the category of functors from simplicial sets to simplicial sets, we have both the bifibrant-projective model structure and the homotopy model structure constructed in the previous sections. For the case of functors from topological spaces to simplicial sets, we have several model structures to choose from. The fibrant-projective model structure is not different from the projective model structure, since all objects in $\mathcal{T}\mathrm{op}$ are fibrant. The observation that simplicial functors preserve weak equivalences between cofibrant topological spaces (since every object is fibrant) suggests that for our comparison to $\mathcal{S}^{\mathcal{S}}$ we should establish the cofibrant-projective model structure on $\mathcal{S}^{\mathcal{T}\mathrm{op}}$. We do so below. Such a model structure has been previously established on the category of contravariant functors [6], but for the category of covariant functors it is new.

Proposition 4.1 *The cofibrant-projective model structure on the category of small functors $\mathcal{S}^{\mathcal{T}\mathrm{op}}$ exists, ie weak equivalences (resp. fibrations) are the natural transformations inducing weak equivalences (resp. fibrations) on the values of functors in cofibrant objects.*

Proof The cofibrant-projective model structure is a particular case of the relative model structure [10]. The latter exists if the solution set condition (dual to the local smallness in the case of contravariant functors) for the inclusion functor of cofibrant objects into $\mathcal{T}\mathrm{op}$ is satisfied [10, Proposition 2.8]. For every uncountable regular cardinal κ there exists a set P_κ of κ–presentable cofibrant spaces, such that every cofibrant space is a filtered colimit of the elements of this set [27, Corollary 5.1], since the domains and the codomains of the generating trivial cofibrations are finitely presentable, hence κ–presentable.

The solution set condition readily follows. Given an object $X \in \mathcal{T}\mathrm{op}$, there exists an uncountable regular cardinal κ such that X is κ–presentable. Therefore, every map $X \to A$ with a cofibrant A factors through the set of all possible maps $\{X \to B \mid B \in P_\kappa\}$. □

We now analyze the connection between the newly established cofibrant-projective model structure and the homotopy model structure.

Proposition 4.2 *The Quillen equivalence ($L = |{-}|$, $R = \mathrm{Sing}(-)$) between simplicial sets and topological spaces induces a Quillen equivalence (R^*, L^*) between the categories of small functors $\mathcal{S}^{\mathcal{S}}$ and $\mathcal{S}^{\mathcal{T}\mathrm{op}}$ with the fibrant-projective and the cofibrant-projective model structures, respectively.*

Proof Since L preserves fibrant objects, (R^*, L^*) is a Quillen pair by Proposition 1.5 between the fibrant-projective and the projective model structure, ie R^* takes fibrations (resp. trivial fibrations) in fibrant objects into levelwise (resp. trivial) fibrations, which are also cofibrant-projective (resp. trivial) fibrations.

We will show now that (R^*, L^*) is a Quillen equivalence by verifying [24, Definition 1.3.12].

Given a fibrant-projectively cofibrant $F \in \mathcal{S}^{\mathcal{S}}$ and cofibrant-projectively fibrant $G \in \mathcal{S}^{\mathcal{T}\mathrm{op}}$, consider a cofibrant-projective weak equivalence $f: R^*F \to G$ in $S^{\mathcal{T}\mathrm{op}}$. The corresponding map $g: F \to L^*G$ in $\mathcal{S}^{\mathcal{S}}$ is constructed as a composition of the unit of the adjunction $\eta: F \to L^*R^*F = F(\mathrm{Sing}(|-|)$ with $L^*f: L^*R^*F \to L^*G = G(|-|)$. For every Kan complex $K \in \mathcal{S}$, the natural map $K \to \mathrm{Sing}(|K|)$ is a simplicial homotopy equivalence (as a weak equivalence between cofibrant-fibrant objects). Since F is a simplicial functor, it preserves simplicial homotopy equivalences; therefore η is a fibrant-projective weak equivalence. The second map L^*f is a projective (levelwise) weak equivalence, since f is a cofibrant-projective weak equivalence and $L = |-|$ takes values in cofibrant objects. Therefore $g = L^*f \circ \eta$ is a fibrant-projective weak equivalence.

Conversely, if we start from a fibrant-projective weak equivalence $g: F \to L^*G$, then the adjoint map is a composition of R^*g with the counit $\epsilon: R^*L^*G = G(|\mathrm{Sing}(-)|) \to G$. The first map R^*g is a levelwise weak equivalence since $R = \mathrm{Sing}$ takes values in Kan complexes and the counit ϵ is a cofibrant-projective weak equivalence, since for every (retract of) a CW–complex X the map $|\mathrm{Sing}(X)| \to X$ is a simplicial homotopy equivalence preserved by the simplicial functor G. □

Corollary 4.3 *The homotopy model structure on $\mathcal{S}^{\mathcal{S}}$ is zigzag Quillen equivalent to the cofibrant-projective model structure on $\mathcal{S}^{\mathcal{T}\mathrm{op}}$.*

5 Dwyer–Kan theorem for model categories

In this section, we prove our main result, an extension of [18, Theorem 2.2] to the context of the model structures discussed above. We first prove the case where all objects are cofibrant, and then the general case. Recall that the homotopy model structure is a localization of the projective model structure.

5.1 All objects cofibrant

First we need to show that the adjunction (R^*, L^*) is still a Quillen adjunction after the localization performed in Section 3.

Proposition 5.1 *Consider a Quillen pair of two combinatorial model categories $L: \mathcal{A} \leftrightarrows \mathcal{B} : R$. Then the adjunction (R^*, L^*) constructed in Proposition 1.3 between the categories of small functors equipped with the projective model structure is also a Quillen pair by Proposition 1.4. Assume in addition that all objects of $\mathcal{A}$ and $\mathcal{B}$ are cofibrant. Then the adjunction (R^*, L^*) remains a Quillen pair for the homotopy model structure.*

Proof By Dugger's lemma [23, 8.5.4], it is sufficient to verify that the right adjoint L^* preserves fibrations of fibrant homotopy functors and all trivial fibrations.

Trivial fibrations are preserved since L^* is a right Quillen functor in the nonlocalized model structure and trivial fibrations do not change (since cofibrations do not) under left Bousfield localization.

Given a fibration of two fibrant homotopy functors $f: F \twoheadrightarrow G$ in $\mathcal{S}^{\mathcal{B}}$, the induced map

$$L^* f: F(L-) = L^* F \twoheadrightarrow L^* G = G(L-)$$

is again a levelwise fibration.

Notice that L preserves trivial cofibrations as a left Quillen functor. By Ken Brown's lemma, L preserves weak equivalences between cofibrant objects [23, Corollary 7.7.2]. Since all objects of $\mathcal{A}$ are cofibrant, L preserves weak equivalences.

Then $L^* f$ is a fibration of homotopy functors, since L, G and, hence, $G \circ L$ are homotopy functors, ie $L^* f$ is a fibration in the localized model structure. □

We are ready now to prove the first main result of this section stating that if the Quillen pair (L, R) is a Quillen equivalence of simplicial combinatorial model categories with all objects cofibrant, then the induced Quillen pair (R^*, L^*) between the categories of small functors to simplicial sets, equipped with the homotopy model structure, is also a Quillen equivalence.

Theorem 5.2 *Given a Quillen equivalence $L: \mathcal{A} \leftrightarrows \mathcal{B} : R$ of two model categories with all objects cofibrant, the induced Quillen pair (R^*, L^*) on the categories of small functors equipped with the homotopy model structure (obtained as a localization of the projective model structure) is also a Quillen equivalence.*

Proof We will use the criterion for a Quillen pair to be a Quillen equivalence [24, Corollary 1.3.16(c)].

First we show that the right adjoint L^* reflects weak equivalences of fibrant objects. Given a map of homotopy functors $f: F \to G$, assume that the induced map $L^* f: L^* F \to L^* G$ is a weak equivalence (of homotopy functors, since L preserves weak equivalences).

For every $B \in \mathcal{B}$ consider its fibrant replacement $B \overset{\sim}{\hookrightarrow} \widehat{B}$ and put $A = R\widehat{B} \in \mathcal{A}$. Then $LA \xrightarrow{\sim} \widehat{B}$ is a weak equivalence, since (L, R) is a Quillen equivalence. We obtain the commutative diagram

$$\begin{tikzcd}
F(B) \arrow[r, "f_B"] \arrow[d, "\wr"'] & G(B) \arrow[d, "\wr"] \\
F(\widehat{B}) \arrow[r, "f_{\widehat{B}}"] & G(\widehat{B}) \\
F(LA) \arrow[r] \arrow[u, "\wr"] \arrow[d, equal] & G(LA) \arrow[u, "\wr"'] \arrow[d, equal] \\
L^* F(A) \arrow[r, "\sim"] & L^* G(A)
\end{tikzcd}$$

Therefore, $f_{\widehat{B}}$ is a weak equivalence and hence f_B is a weak equivalence for all $B \in \mathcal{B}$ by the 2-out-of-3 property; hence f is a weak equivalence.

For every cofibrant $F \in \mathcal{S}^{\mathcal{A}}$, the derived unit of the adjunction $R^* \dashv L^*$ from Proposition 1.3 is constructed as an adjoint map to the fibrant approximation in the homotopy model category $R^*F \to \mathcal{H}R^*F$,

$$F \to L^*\mathcal{H}R^*F. \tag{1}$$

It remains to show that it is a weak equivalence in the homotopy model structure.

Note that $L^*\mathcal{H}(R^*F(-)) = L^*\mathcal{H}F(R(-)) = L^*\widehat{F}(R\,\mathrm{Fib}_{\mathcal{B}}(-)) = \widehat{F}(R\,\mathrm{Fib}_{\mathcal{B}}\,L(-))$. Since the pair (L, R) is a Quillen equivalence, for all (cofibrant) $X \in \mathcal{A}$ there is a weak equivalence $X \xrightarrow{\sim} R\,\mathrm{Fib}_{\mathcal{B}}\,L(X)$. Hence, the initial map (1) is a weak equivalence in the homotopy model structure because we can apply $\mathcal{H}$ also to F turning it into a homotopy functor. □

Corollary 5.3 *Assume $\mathcal{A}$ and $\mathcal{B}$ satisfy the conditions of Theorem 5.2, and suppose that the homotopy model structures on $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{B}}$, from Theorem 3.1, exist. Then the fibrant-projective model structures on $\mathcal{S}^{\mathcal{A}}$ and $\mathcal{S}^{\mathcal{B}}$ are Quillen equivalent.*

Proof By Theorem 3.3, the fibrant-projective model structure on $\mathcal{S}^{\mathcal{A}}$ is Quillen equivalent to the homotopy model structure, and the same for $\mathcal{S}^{\mathcal{B}}$. By Theorem 5.2, the homotopy model structures are Quillen equivalent. Hence, the fibrant-projective model structures are Quillen equivalent, via a chain of Quillen equivalences (where the left adjoints are depicted): $\mathcal{S}^{\mathcal{A}}_{\text{fib-proj}} \to \mathcal{S}^{\mathcal{A}}_{\text{ho}} \to \mathcal{S}^{\mathcal{B}}_{\text{ho}} \leftarrow \mathcal{S}^{\mathcal{B}}_{\text{fib-proj}}$. □

5.2 General case

We adapt the definition of r–equivalences [18] for simplicial model categories.

Definition 5.4 A continuous functor $f : \mathcal{A} \to \mathcal{B}$ of simplicial model categories is an r–equivalence if

(1) for every two bifibrant objects $A_1, A_2 \in \mathcal{A}$, the induced map $\hom(A_1, A_2) \to \hom(fA_1, fA_2)$ is a weak equivalence, and

(2) every object in the category of components $\pi_0^{\text{bifib}}\mathcal{B}$ is a retract of an object in the image of $\pi_0^{\text{bifib}} f$, ie for every bifibrant object $B \in \mathcal{B}$ there exists a bifibrant object $A \in \mathcal{A}$ such that B is a retract of $f(A)$, up to homotopy.

Remark 5.5 Note that B is a retract of $f(A)$, up to homotopy, if there are maps $A \xrightarrow{i} B \xrightarrow{r} A$ such that $ri \sim \mathrm{Id}_A$. We do not specify the kind of homotopy relation in Definition 5.4(2), since for maps between bifibrant objects in a simplicial model category left, right, simplicial and strict simplicial homotopy relations coincide and are equivalence relations [23, 9.5.24(2)].

Example 5.6 Let

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

be a Quillen equivalence between simplicial combinatorial model categories. Put $f = \widehat{L}$ the composition of the left adjoint with the fibrant replacement functor in $\mathscr{B}$, and let $\widetilde{R}$ be the composition of R with cofibrant replacement in $\mathscr{A}$. Then f is an r–equivalence:

(1) For all bifibrant $A_1, A_2 \in \mathscr{A}$,

$$\begin{aligned}\hom_{\mathscr{B}}(fA_1, fA_2) = \hom_{\mathscr{B}}(\widehat{L}A_1, fA_2) \simeq \hom_{\mathscr{B}}(LA_1, fA_2) &= \hom_{\mathscr{A}}(A_1, R\widehat{L}A_2)\\ &\simeq \hom_{\mathscr{A}}(A_1, A_2).\end{aligned}$$

(2) For all bifibrant $B \in \mathscr{B}$, factor the weak equivalence $L\widetilde{R}B \xrightarrow{\sim} B$ as a trivial cofibration followed by a fibration (also trivial by the 2-out-of-3 property): $L\widetilde{R}B \overset{\sim}{\hookrightarrow} \widehat{L}\widetilde{R}B \overset{\sim}{\twoheadrightarrow} B$. Therefore, B is a retract of $\widehat{L}\widetilde{R}B$:

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d, hook] & \widehat{L}\widetilde{R}B \arrow[d, two heads, "\wr"] \\ B \arrow[ur, dotted] \arrow[r, equal] & B \end{tikzcd}$$

On the other hand, $\widehat{L}\widetilde{R}B \simeq f(\widetilde{R}B)$, ie B is a retract of $f(\widetilde{R}B)$, up to homotopy.

Lemma 5.7 *Let $A, B \in \mathscr{A}$ be two bifibrant objects in a simplicial model category $\mathscr{A}$ such that A is a retract of B, up to homotopy. Then there exists a bifibrant object $B' \in \mathscr{A}$ such that $B' \simeq B$ and A is a strict retract of B'.*

Proof Suppose that the composition $A \xrightarrow{i} B \xrightarrow{r} A$ is simplicially homotopic to the identity map on A: $ri \sim \mathrm{Id}_A$. Then $A \otimes I$ is a very good cylinder object, ie factors the codiagonal $A \amalg A \to A$ into a cofibration followed by a trivial fibration. Consider the commutative diagram

$$\begin{tikzcd} & A \arrow[rr, "i"] \arrow[d, hook, "i_0"', "\wr"] & & B \arrow[rr, "r"] \arrow[d, hook, dashed, "\wr"] & & A \\ A \arrow[r, hook, "\sim", "i_1"'] & A \otimes I \arrow[rr, dashed, "\iota"'] \arrow[rrrru, bend right=50, "H"'] & & B_1 \arrow[r, hook, dashed, "\sim", "i'"'] & B' \arrow[ur, dashed, two heads, "r'"] & \end{tikzcd}$$

where $Hi_0 = ri$ and $Hi_1 = \mathrm{Id}_A$. Put $B_1 = A \otimes I \amalg_A B$, and, in order to ensure the fibrancy of the intermediate object, we factor the natural map $B_1 \to A$ as a trivial cofibration i' followed by a fibration r'.

Hence, A is a (strict) retract of the bifibrant object B',

$$\begin{tikzcd} A \arrow[r, "i'\iota i_1"] \arrow[rr, bend right, "Hi_1 = \mathrm{Id}_A"'] & B' \arrow[r, "r'"] & A \end{tikzcd}$$

and $B' \simeq B$. ◻

Theorem 5.8 *Let $f: \mathscr{A} \to \mathscr{B}$ be an accessible functor between simplicial combinatorial model categories. Suppose f preserves fibrant and cofibrant objects. Then the Quillen pair*

$$\mathrm{Lan}_f : \mathcal{S}^{\mathscr{A}} \leftrightarrows \mathcal{S}^{\mathscr{B}} : f^*$$

between functor categories equipped with the fibrant-projective model structure is a Quillen equivalence if and only if f is an r–equivalence of simplicial model categories.

Proof Since f preserves both fibrant and cofibrant objects, the induced adjunction $\mathrm{Lan}_f \dashv f^*$ is a Quillen map by Proposition 2.3.

Suppose that f is an r–equivalence. We will use [24, 1.3.16(c)] to show that $\mathrm{Lan}_f \dashv f^*$ is a Quillen equivalence. In other words, we will prove that f^* reflects weak equivalences of bifibrant-projectively fibrant objects and for every cofibrant $F \in \mathcal{S}^{\mathscr{A}}$, the map

$$F \to f^* \widehat{\mathrm{Lan}_f\, F} \tag{2}$$

is a bifibrant-projective weak equivalence.

Consider a natural transformation $p: G \to H$ of bifibrant-projectively fibrant functors in $\mathcal{S}^{\mathscr{B}}$. Assume that $f^* p$ is a weak equivalence. Then for any bifibrant object $A \in \mathscr{A}$ there is a weak equivalence of simplicial sets

$$(f^* p)(A): G(f(A)) = (f^* G)(A) \xrightarrow{\sim} (f^* H)(A) = H(f(A)).$$

Since f is an r–equivalence of model categories, Definition 5.4(2) implies that for any bifibrant object $B \in \mathscr{B}$ there exists a bifibrant object $A \in \mathscr{A}$ such that B is a retract of $f(A)$, up to homotopy. By Lemma 5.7 there exists $B' \in \mathscr{B}$ weakly equivalent to $f(A)$ such that B is a retract of B'. Since $f(A)$ and B' are bifibrant objects, the weak equivalence between them is a simplicial weak equivalence. Since G and H are simplicial functors, they preserve simplicial weak equivalences. Therefore, $p(B')$ is a weak equivalence by the 2-out-of-3 property, because $(f^* p)(A) = p(f(A)): G(f(A)) \xrightarrow{\sim} H(f(A))$ is a weak equivalence by assumption. Hence, $p(B): G(B) \xrightarrow{\sim} H(B)$ is a weak equivalence as a retract of the weak equivalence $p(B'): G(B') \xrightarrow{\sim} H(B')$. Therefore p is a bifibrant-projective weak equivalence.

The retract argument implies that it is sufficient to prove (2) is a weak equivalence for any cellular $F \in \mathcal{S}^{\mathscr{A}}$ with respect to the bifibrant-projective model structure.

For every bifibrant $A \in \mathscr{A}$,

$$f^* \widehat{\mathrm{Lan}_f\, F}(A) = \big(\mathrm{Lan}_f\, F(f(A))\big)^{\mathrm{fib}}$$

since $f(A) \in \mathscr{B}$ is also a bifibrant object and the fibrant replacement in the bifibrant-projective model structure on $\mathcal{S}^{\mathscr{B}}$ applies levelwise to the values of the functor in bifibrant objects. Hence there is a bifibrant-projective weak equivalence

$$f^* \widehat{\mathrm{Lan}_f\, F} \simeq f^* \mathrm{Lan}_f\, F.$$

In other words, it suffices to show that the unit of adjunction

$$F \to f^*\mathrm{Lan}_f\, F$$

is a bifibrant-projective weak equivalence.

We proceed by cellular induction.

Suppose $F = \mathrm{colim}_{i<\lambda} F_i$, so that $F_0 = \varnothing$ and F_{i+1} is obtained from F_i by attaching a cell

$$\begin{tikzcd} R^{A_i} \otimes \partial\Delta^n \arrow[r] \arrow[d, hook] & F_i \arrow[d, hook] \\ R^{A_i} \otimes \Delta^n \arrow[r] & F_{i+1} \end{tikzcd}$$

if $i+1$ is a successor ordinal or $F_i = \mathrm{colim}_{a<i} F_a$ if i is a limit ordinal. Note that $A_i \in \mathcal{A}$ is bifibrant for every $i < \lambda$.

Since both Lan_f and f^* preserve colimits, so does $f^*\,\mathrm{Lan}_f$. Since bifibrant-projective weak equivalences are preserved under sequential colimits, it suffices to show for each $i < \lambda$ that if $F_i \to f^*\mathrm{Lan}_f\, F_i$ is a fibrant-projective weak equivalence, then so is $F_{i+1} \to f^*\mathrm{Lan}_f\, F_{i+1}$. Let us consider the unit of adjunction of the homotopy pushout square above:

$$\begin{tikzcd} f^*R^{f(A_i)} \otimes \partial\Delta^n \arrow[rrr] \arrow[ddd] & & & f^*\,\mathrm{Lan}_f\, F_i \arrow[ddd] \\ & R^{A_i} \otimes \partial\Delta^n \arrow[r] \arrow[d, hook] \arrow[ul, "\sim"'] & F_i \arrow[d, hook] \arrow[ur, "\sim"] & \\ & R^{A_i} \otimes \Delta^n \arrow[r] \arrow[dl, "\sim"] & F_{i+1} \arrow[dr, dashed] & \\ f^*R^{f(A_i)} \otimes \Delta^n \arrow[rrr] & & & f^*\,\mathrm{Lan}_f\, F_{i+1} \end{tikzcd}$$

Then the outer square is also a pushout; moreover, this is a levelwise homotopy pushout.

The slanted map on the right is a weak equivalence by the induction assumption. The slanted maps on the left are bifibrant-projective weak equivalences by Definition 5.4(1), since for every bifibrant $A \in \mathcal{A}$,

$$f^*R^{f(A_i)}(A) = \hom(f(A_i), f(A)).$$

Hence the dashed map is also a bifibrant-projective weak equivalence as an induced map of homotopy pushouts. This completes the cellular induction proving that (2) is a bifibrant-projective weak equivalence, as required.

Conversely, if $f : \mathcal{A} \to \mathcal{B}$ preserves bifibrant objects and induces a Quillen equivalence $\mathrm{Lan}_f \dashv f^*$ between the categories of small functors into simplicial sets, then for any bifibrant object $A_1 \in \mathcal{A}$, the induced map $R^{A_1} \to f^* \operatorname{Lan}_f R^{A_1} = f^* R^{f(A_1)}$ is a bifibrant equivalence, ie evaluating at any bifibrant object $A_2 \in \mathcal{A}$ we obtain a weak equivalence of simplicial sets

$$\hom_{\mathcal{A}}(A_1, A_2) \xrightarrow{\sim} \hom_{\mathcal{B}}(f(A_1), f(A_2)).$$

In other words, f satisfies the first part of Definition 5.4.

It remains to verify Definition 5.4(2), ie that every bifibrant object $B \in \mathcal{B}$ is a retract, up to homotopy, of $f(A)$ for some bifibrant $A \in \mathcal{A}$.

For all (bifibrant projectively) fibrant $G \in \mathcal{S}^{\mathcal{B}}$, let λ be the maximum of the accessibility ranks of G and f. Then

$$G \simeq \operatorname{Lan}_f \widetilde{f^* G} = \operatorname{Lan}_f \int^{A \in \mathcal{A}_\lambda} \widetilde{G(f(A))} \otimes R^A(-) = \int^{A \in \mathcal{A}_\lambda} \widetilde{G(f(A))} \otimes R^{f(A)}(-).$$

Take $G = \hom(B, -) = R^B(-)$. Then

$$\int^{A \in \mathcal{A}_\lambda} \widetilde{\hom(B, f(A))} \otimes \hom(f(A), -) \simeq \hom(B, -).$$

After evaluating at B and passing to connected components, we obtain a bijection (see [31, Theorem 10])

$$\int^{A \in \mathcal{A}_\lambda} \pi_0 \hom(B, f(A)) \times \pi_0 \hom(f(A), B) \cong \pi_0 \hom(B, B).$$

Let $A \in \mathcal{A}_\lambda$ correspond to the identity on the right-hand side. Then B is a retract, up to homotopy, of $f(A)$. □

6 Invariance of Goodwillie calculus under Quillen equivalence

Given a Quillen equivalence f of simplicial combinatorial model categories, consider the model categories of homotopy functors (bifibrant-projective model structure on the categories of small functors) from this Quillen pair to simplicial sets. This is a starting point for Goodwillie's calculus of homotopy functors [22; 26]. Consider the localization of these categories such that the fibrant objects are the fibrant n–excisive functors. Does f induce a Quillen equivalence of these model structures? In other words, is Goodwillie calculus invariant under Quillen equivalence? Under a few additional conditions the answer is "yes".

Theorem 6.1 *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, bend left, "R"] \end{tikzcd}$$

be a Quillen equivalence between simplicial combinatorial model categories. Put $f = \widehat{L}$. Then the Quillen pair of functor categories

$$\operatorname{Lan}_f : \mathcal{S}^{\mathcal{A}} \leftrightarrows \mathcal{S}^{\mathcal{B}} : f^*$$

is a Quillen equivalence. Moreover, if we left Bousfield localize the functor categories so that the local objects are the n–excisive functors in the bifibrant-projective model structures on both sides, then the adjunction $\operatorname{Lan}_f \dashv f^*$ is a Quillen equivalence of the model categories of n–excisive functors.

Proof That the first Quillen pair is a Quillen equivalence follows from Theorem 5.8 and Example 5.6. In order to conclude that the categories of n–excisive functors are Quillen equivalent we will apply Theorem A.1.

First, f^* preserves n–excisive functors. This is because $f = \widehat{L}$ preserves homotopy pushouts of cofibrant objects; hence f preserves strongly cocartesian cubes.

It remains to show that Lan_f commutes with the n^{th} polynomial approximation functor. Let us denote by $P_n^{\mathcal{A}}$ and $P_n^{\mathcal{B}}$ the n–excisive approximations of functors with domain in $\mathcal{A}$ and $\mathcal{B}$ respectively. Then we need to prove that for all cofibrant $F \in \mathcal{S}^{\mathcal{A}}$ there is a bifibrant-projective weak equivalence of functors $\mathrm{Lan}_f\, P_n^{\mathcal{A}} F \simeq P_n^{\mathcal{B}} \mathrm{Lan}_f\, F$.

We next prove that Lan_f takes n–excisive functors to n–excisive functors. We will need the right adjoint R to preserve homotopy pushout squares of bifibrant objects to verify this property. This is true, in turn, since R is a part of a Quillen equivalence; hence its total derived functor is an equivalence of homotopy categories. It follows that there is an equivalence of homotopy categories of diagrams indexed by the category $\bullet \leftarrow \bullet \rightarrow \bullet$. Since the homotopy pushout is a left adjoint to the constant functor, it is preserved by any equivalence of categories, so R preserves homotopy pushout squares.

Let F be a cofibrant functor. Suppose that $P_n^{\mathcal{A}} F$ is a λ–accessible functor. Let $\mathcal{A}_\lambda \subset \mathcal{A}$ be the subcategory of λ–presentable objects. Then

$$\begin{aligned}
\mathrm{Lan}_f\, P_n^{\mathcal{A}} F = \mathrm{Lan}_f \int^{A \in \mathcal{A}_\lambda} \hom(A,-) \otimes P_n^{\mathcal{A}} F(A) &= \int^{A \in \mathcal{A}_\lambda} \mathrm{Lan}_f \hom(A,-) \otimes P_n^{\mathcal{A}} F(A) \\
&= \int^{A \in \mathcal{A}_\lambda} \hom(f(A),-) \otimes P_n^{\mathcal{A}} F(A) \\
&= \int^{A \in \mathcal{A}_\lambda} \hom(\widehat{L}(A),-) \otimes P_n^{\mathcal{A}} F(A) \\
&\simeq \int^{A \in \mathcal{A}_\lambda} \hom(L(A),-) \otimes P_n^{\mathcal{A}} F(A) \\
&= \int^{A \in \mathcal{A}_\lambda} \hom(A,R(-)) \otimes P_n^{\mathcal{A}} F(A) = R^* P_n^{\mathcal{A}} F.
\end{aligned}$$

Since R preserves homotopy pushouts, it preserves strongly cocartesian cubes; hence $R^* P_n^{\mathcal{A}} F \in \mathcal{S}^{\mathcal{B}}$ is an n–excisive functor.

Consider the diagram

$$\begin{tikzcd}
\mathrm{Lan}_f\, F \arrow[r] \arrow[d] & P_n^{\mathcal{B}} \mathrm{Lan}_f\, F \arrow[dl, dotted, bend left] \\
\mathrm{Lan}_f\, P_n^{\mathcal{A}} F \arrow[ur, dashed] &
\end{tikzcd}$$

The dotted arrow exists by the initial, up to homotopy, property of n–excisive approximation in $\mathcal{S}^{\mathcal{B}}$, since $\mathrm{Lan}_f\, P_n^{\mathcal{A}} F$ is n–excisive.

The dashed arrow exists by the same reason in the adjoint diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & f^* P_n^{\mathcal{B}} \operatorname{Lan}_f F \\ P_n^{\mathcal{A}} F \arrow[ur, dashed] & \end{tikzcd}$$

Both maps are unique, up to homotopy, hence mutually homotopy inverse. Therefore, Lan_f commutes with polynomial approximation, up to homotopy.

We have verified the conditions of Theorem A.1, so the Quillen pair $\operatorname{Lan}_f \vdash f^*$ is a Quillen equivalence between the categories of small functors equipped with the n–excisive model structures. □

Example 6.2 For the Quillen equivalence $|-| : \mathcal{S} \leftrightarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}$ the categories of n–excisive functors $\mathcal{S}^{\mathcal{S}}$ and $\mathcal{S}^{\mathcal{T}\mathrm{op}}$ are Quillen equivalent. In other words, simplicial sets and topological spaces have the same calculus of homotopy functors.

Appendix Localization of a Quillen equivalence

Given two Quillen equivalent model categories, consider a left Bousfield localizations of both sides. Under what conditions are the resulting localized categories Quillen equivalent again? We provide a new answer, needed for our Theorem 6.1, in the following theorem. We fix the following notation: for a model category $\mathcal{A}$ equipped with the homotopy localization functor $\mathcal{F}_{\mathcal{A}}^{-1} : \mathcal{A} \to \mathcal{A}$ the left Bousfield localization of $\mathcal{A}$ with respect to $\mathcal{F}_{\mathcal{A}}^{-1}$–equivalences is denoted by $\mathcal{F}_{\mathcal{A}}^{-1}\mathcal{A}$. For all $A \in \mathcal{A}$ we write $\mathcal{F}_{\mathcal{A}}^{-1}(A)$ for the fibrant replacement of A in $\mathcal{F}_{\mathcal{A}}^{-1}\mathcal{A}$ in order to distinguish it from the fibrant replacement in $\mathcal{A}$.

Theorem A.1 *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "R"] \end{tikzcd}$$

be a Quillen equivalence between simplicial model categories. Suppose there exist left Bousfield localizations $\mathcal{F}_{\mathcal{A}}^{-1}\mathcal{A}$ of $\mathcal{A}$ and $\mathcal{F}_{\mathcal{B}}^{-1}\mathcal{B}$ of $\mathcal{B}$ such that:

(1) *R takes $\mathcal{F}_{\mathcal{B}}$–local objects to $\mathcal{F}_{\mathcal{A}}$–local objects.*

(2) *L commutes with the localization, ie for all cofibrant $A \in \mathcal{A}$ the map $L\mathcal{F}_{\mathcal{A}}^{-1}(A) \to \mathcal{F}_{\mathcal{B}}^{-1}(LA)$ is a weak equivalence. The latter map is adjoint to the lift in the commutative square in $\mathcal{F}_{\mathcal{A}}^{-1}\mathcal{A}$:*

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "\wr"] & RLA \arrow[r] & R\mathcal{F}_{\mathcal{B}}^{-1}(LA) \arrow[d, two heads] \\ \mathcal{F}_{\mathcal{A}}^{-1}(A) \arrow[urr, dashed] \arrow[rr] & & * \end{tikzcd}$$

Then there exists a Quillen equivalence of the localized model categories

$$\begin{tikzcd} \mathcal{F}_{\mathcal{A}}^{-1}\mathcal{A} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{F}_{\mathcal{B}}^{-1}\mathcal{B}. \arrow[l, shift left, "R"] \end{tikzcd}$$

Proof L is a left Quillen functor between $\mathscr{A}$ and $\mathscr{B}$; hence it preserves the cofibrations in the localized model structures as well. In order to show that L remains a left Quillen functor after the localization we need to verify that for every cofibration $A_1 \hookrightarrow A_2$, which is also an $\mathscr{F}_{\mathscr{A}}$–local equivalence, the cofibration $LA_1 \hookrightarrow LA_2$ is an $\mathscr{F}_{\mathscr{B}}$–local equivalence. In other words, we need to prove that $\mathscr{F}_{\mathscr{B}}^{-1} LA_1 \to \mathscr{F}_{\mathscr{B}}^{-1} LA_2$ is a weak equivalence. Since L commutes with the localization, it suffices to show that $L\mathscr{F}_{\mathscr{A}}^{-1} A_1 \to L\mathscr{F}_{\mathscr{A}}^{-1} A_2$ is a weak equivalence, which readily follows from the assumption that $A_1 \hookrightarrow A_2$ is an $\mathscr{F}_{\mathscr{A}}$–local equivalence and the fact that L preserves weak equivalences of cofibrant objects.

We will use [24, 1.3.16(c)] to show that $L \dashv R$ is a Quillen equivalence.

(1) R reflects local equivalences of local objects, since these are just weak equivalences in $\mathscr{A}$ and $\mathscr{B}$, and R reflects weak equivalences.

(2) For every cofibrant $A \in \mathscr{A}$ we need to show that the map $A \to R\mathscr{F}_{\mathscr{B}}^{-1} LA$ is an $\mathscr{F}_{\mathscr{A}}^{-1}$–local equivalence. We need to rely on the assumption that L commutes with the localization, ie that $L\mathscr{F}_{\mathscr{A}}^{-1} A \to \mathscr{F}_{\mathscr{B}}^{-1} LA$ is a weak equivalence in $\mathscr{B}$.
Consider the commutative square in the model category $\mathscr{F}_{\mathscr{B}}^{-1}\mathscr{B}$:

$$\begin{tikzcd}
L\mathscr{F}_{\mathscr{A}}^{-1} A \arrow[r, "\sim"] \arrow[d, hook, "\wr"'] & \mathscr{F}_{\mathscr{B}}^{-1} LA \arrow[d, two heads] \\
\hat{L}\mathscr{F}_{\mathscr{A}}^{-1} A \arrow[r] \arrow[ur, dashed, "h"'] & *
\end{tikzcd}$$

The lift h is a weak equivalence of fibrant objects in $\mathscr{F}_{\mathscr{B}}^{-1}\mathscr{B}$. Hence Rh below is a weak equivalence:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook, "\text{loc. equiv.}"'] & R\mathscr{F}_{\mathscr{B}}^{-1} LA \\
\mathscr{F}_{\mathscr{A}}^{-1} A \arrow[r, "\sim"] & R\hat{L}\mathscr{F}_{\mathscr{A}}^{-1} A \arrow[u, "Rh", "\wr"']
\end{tikzcd}$$

In the diagram above the lower horizontal map is a weak equivalence since $L \dashv R$ is a Quillen equivalence between $\mathscr{A}$ and $\mathscr{B}$. Thus, the map $A \to R\mathscr{F}_{\mathscr{B}}^{-1} LA$ is a local equivalence in $\mathscr{F}_{\mathscr{B}}^{-1}\mathscr{A}$. □

Example A.2 Consider the Quillen equivalence $|-| : \mathcal{S} \leftrightarrows \mathcal{T}\mathrm{op} : \mathrm{Sing}$, and consider the localization of both sides with respect to integral homology. Then the conditions of Theorem A.1 are readily verified; hence the category of $H\mathbb{Z}$–local topological spaces is Quillen equivalent to the category of $H\mathbb{Z}$–local simplicial sets.

References

[1] **C Barwick**, **D M Kan**, *Relative categories: another model for the homotopy theory of homotopy theories*, Indag. Math. 23 (2012) 42–68 MR Zbl

[2] **M Basterra**, **K Bauer**, **A Beaudry**, **R Eldred**, **B Johnson**, **M Merling**, **S Yeakel**, *Unbased calculus for functors to chain complexes*, from "Women in topology: collaborations in homotopy theory" (M Basterra, K Bauer, K Hess, B Johnson, editors), Contemp. Math. 641, Amer. Math. Soc., Providence, RI (2015) 29–48 MR Zbl

[3] **G Biedermann**, **B Chorny**, *Duality and small functors*, Algebr. Geom. Topol. 15 (2015) 2609–2657 MR Zbl

[4] **G Biedermann**, **B Chorny**, **O Röndigs**, *Calculus of functors and model categories*, Adv. Math. 214 (2007) 92–115 MR Zbl

[5] **G Biedermann**, **O Röndigs**, *Calculus of functors and model categories, II*, Algebr. Geom. Topol. 14 (2014) 2853–2913 MR Zbl

[6] **D Blanc**, **B Chorny**, *Representability theorems, up to homotopy*, Proc. Amer. Math. Soc. 148 (2020) 1363–1372 MR Zbl

[7] **A K Bousfield**, *On the telescopic homotopy theory of spaces*, Trans. Amer. Math. Soc. 353 (2001) 2391–2426 MR Zbl

[8] **A K Bousfield**, **E M Friedlander**, *Homotopy theory of Γ–spaces, spectra, and bisimplicial sets*, from "Geometric applications of homotopy theory, II" (M G Barratt, M E Mahowald, editors), Lecture Notes in Math. 658, Springer (1978) 80–130 MR Zbl

[9] **M Ching**, **E Riehl**, *Coalgebraic models for combinatorial model categories*, Homology Homotopy Appl. 16 (2014) 171–184 MR Zbl

[10] **B Chorny**, *Homotopy theory of relative simplicial presheaves*, Israel J. Math. 205 (2015) 471–484 MR Zbl

[11] **B Chorny**, *A classification of small linear functors*, Int. Math. Res. Not. 2016 (2016) 5493–5517 MR Zbl

[12] **B Chorny**, **W G Dwyer**, *Homotopy theory of small diagrams over large categories*, Forum Math. 21 (2009) 167–179 MR Zbl

[13] **B Chorny**, **J Rosický**, *Class-combinatorial model categories*, Homology Homotopy Appl. 14 (2012) 263–280 MR Zbl

[14] **B J Day**, **S Lack**, *Limits of small functors*, J. Pure Appl. Algebra 210 (2007) 651–663 MR Zbl

[15] **D Dugger**, *Combinatorial model categories have presentations*, Adv. Math. 164 (2001) 177–201 MR Zbl

[16] **B I Dundas**, **O Röndigs**, **P A Østvær**, *Enriched functors and stable homotopy theory*, Doc. Math. 8 (2003) 409–488 MR Zbl

[17] **W G Dwyer**, *Localizations*, from "Axiomatic, enriched and motivic homotopy theory" (J P C Greenlees, editor), NATO Sci. Ser. II Math. Phys. Chem. 131, Kluwer, Dordrecht (2004) 3–28 MR Zbl

[18] **W G Dwyer**, **D M Kan**, *Equivalences between homotopy theories of diagrams*, from "Algebraic topology and algebraic K–theory" (W Browder, editor), Ann. of Math. Stud. 113, Princeton Univ. Press (1987) 180–205 MR Zbl

[19] **L Fajstrup**, **J Rosický**, *A convenient category for directed homotopy*, Theory Appl. Categ. 21 (2008) 7–20 MR Zbl

[20] **T G Goodwillie**, *Calculus, I: The first derivative of pseudoisotopy theory*, K–Theory 4 (1990) 1–27 MR Zbl

[21] **T G Goodwillie**, *Calculus, II: Analytic functors*, K–Theory 5 (1992) 295–332 MR Zbl

[22] **T G Goodwillie**, *Calculus, III: Taylor series*, Geom. Topol. 7 (2003) 645–711 MR Zbl

[23] **P S Hirschhorn**, *Model categories and their localizations*, Math. Surv. Monogr. 99, Amer. Math. Soc., Providence, RI (2003) MR Zbl

[24] **M Hovey**, *Model categories*, Math. Surv. Monogr. 63, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[25] **G M Kelly**, *Basic concepts of enriched category theory*, Lond. Math. Soc. Lect. Note Ser. 64, Cambridge Univ. Press (1982) MR Zbl

[26] **N J Kuhn**, *Goodwillie towers and chromatic homotopy: an overview*, from "Proceedings of the Nishida Fest" (M Ando, N Minami, J Morava, W S Wilson, editors), Geom. Topol. Monogr. 10, Geom. Topol. Publ., Coventry (2007) 245–279 MR Zbl

[27] **M Makkai**, **J Rosický**, **L Vokřínek**, *On a fat small object argument*, Adv. Math. 254 (2014) 49–68 MR Zbl

[28] **L Pereira**, *A general context for Goodwillie calculus*, preprint (2013) arXiv 1301.2832

[29] **D G Quillen**, *Homotopical algebra*, Lecture Notes in Math. 43, Springer (1967) MR Zbl

[30] **O Raventós**, *The hammock localization preserves homotopies*, Homology Homotopy Appl. 17 (2015) 191–204 MR Zbl

[31] **L Vokřínek**, *Homotopy weighted colimits*, preprint (2012) arXiv 1201.2970

Department of Mathematics, Physics and Computer Science, University of Haifa at Oranim
Tivon, Israel

Department of Mathematics and Computer Science, Denison University
Granville, OH, United States

chorny@math.haifa.ac.il, david.white@denison.edu

Received: 6 June 2022 Revised: 25 January 2023

Algebraic & Geometric Topology **24**:4 (2024) 2209–2244
DOI: 10.2140/agt.2024.24.2209
Published: 16 July 2024

Integral generalized equivariant cohomologies of weighted Grassmann orbifolds

KOUSHIK BRAHMA
SOUMEN SARKAR

We introduce a new definition of weighted Grassmann orbifolds. We study their several invariant q–CW complex structures and the orbifold singularities on the q–cells of these q–CW complexes. We discuss when the integral cohomology of a weighted Grassmann orbifold has no p–torsion. We compute the equivariant K–theory ring of weighted Grassmann orbifolds with rational coefficients. We introduce divisive weighted Grassmann orbifolds and show that they have invariant CW complex structures. We calculate the equivariant cohomology ring, equivariant K–theory ring and equivariant cobordism ring of a divisive weighted Grassmann orbifold with integer coefficients. We discuss how to compute the weighted structure constants for the integral equivariant cohomology ring of a divisive weighted Grassmann orbifold.

14M15, 19L47, 55N91, 57R18, 57R85

1 Introduction

We consider the n–dimensional complex vector space $\mathbb{C}^n$ and a positive integer d satisfying $1 \leq d < n$. Then the set of all d–dimensional vector subspaces of $\mathbb{C}^n$ is called a (complex) Grassmann manifold and denoted by $\mathrm{Gr}(d,n)$. In particular, the space $\mathrm{Gr}(1,n)$ is called the $(n-1)$–dimensional complex projective space. The space $\mathrm{Gr}(d,n)$ has a manifold structure of dimension $d(n-d)$; see Mukherjee [24, Chapter 1]. This is a projective variety via the Plücker embedding. The natural $(\mathbb{C}^*)^n$–action on $\mathbb{C}^n$ induces a $(\mathbb{C}^*)^n$–action on $\mathrm{Gr}(d,n)$. Grassmann manifolds are central objects of study in algebraic geometry, algebraic topology and differential geometry. Several interesting topological and geometrical properties of Grassmann manifolds can be found in Laksov [21], Knutson and Tao [20] and Jiao and Peng [18].

The orbifold version of a complex projective space was introduced in Kawasaki [19] and was called a twisted projective space. Orbifolds, a generalization of manifolds, were introduced by Satake [27; 28] with the name V–manifolds. Later, Thurston [31] used the terminology orbifolds instead. In the past two decades, several developments have appeared to study orbifolds arising in algebraic geometry, differential geometry and string topology. Some cohomology theories, such as de Rham cohomology (see Adem, Leida and Ruan [2, Chapter 2]), singular cohomology (see Hatcher [16]), Dolbeault cohomology (see Baily [5]), the Chen–Ruan cohomology ring [6] and orbifold K–theory [2, Chapter 3] for a class of orbifolds were studied either with rational, real or complex coefficients. One can construct a CW complex structure on an effective orbifold following Goresky [11]. However, in general, the computation of the singular integral cohomology of an orbifold is considerably difficult.

Let G be a topological group and X a G–space. Then the equivariant map $X \to \{\mathrm{pt}\}$ induces a graded $\mathscr{E}^*_G(\{\mathrm{pt}\})$–algebra structure on $\mathscr{E}^*_G(X)$. The readers are referred to May [22] for the definitions and several results on the G–equivariant generalized cohomology theory $\mathscr{E}^*_G$. If $\mathscr{E}^*_G = H^*_G$, then it is known as the equivariant cohomology theory defined by

$$H^*_G(X) := H^*(EG \times_G X).$$

The ring $H^*_G(X)$ is called the Borel equivariant cohomology of X. If $\mathscr{E}^*_G = K^*_G$, then it is known as the equivariant K–theory. If X is compact, then $K^0_G(X)$ is the equivalence classes of G–equivariant complex vector bundles on X; see Segal [29]. If X is a point with trivial action, then $K^*_G(\{\mathrm{pt}\})$ is isomorphic to $R(G)[z, z^{-1}]$, where $R(G)$ is complex representation ring of G and z is the Bott element of cohomological dimension -2. The G–equivariant ring $\mathrm{MU}^*_G(X)$ is known as the equivariant complex cobordism ring; see tom Dieck [9]. Sinha [30] and Hanke [13] have shown several developments on MU^*_G. However, many interesting questions on $\mathrm{MU}^*_G(X)$ remain undetermined. For example, $\mathrm{MU}^*_G(\{\mathrm{pt}\})$ is not completely known for nontrivial groups G.

Corti and Reid [7] introduced the weighted projective analogs of a class of Grassmann manifolds and called them weighted Grassmannians. Then Abe and Matsumura [1] defined weighted Grassmannians explicitly and studied their equivariant cohomology ring of weighted Grassmannians with rational coefficients. The weighted Grassmannians are projective varieties with orbifold singularities. The simplest weighted Grassmannians are the weighted projective spaces. Kawasaki [19] proved that the integral cohomology of weighted projective spaces has no torsion and is concentrated in even degrees. The equivariant cohomology ring of a weighted projective space has been studied in Bahri, Franz and Ray [3] in terms of piecewise polynomials. The equivariant K–theory and equivariant cobordism rings of divisive weighted projective spaces have been discussed in Harada, Holm, Ray and Williams [15] in terms of piecewise Laurent polynomials and piecewise cobordism forms, respectively.

Inspired by the above works, we introduce a different definition of weighted Grassmann orbifolds and study their several topological properties such as torsion in the integral cohomology, equivariant cohomology ring, equivariant K–theory ring and equivariant cobordism ring with integer coefficients. We note that

Abe and Matsumura [1] and Corti and Reid [7] used the name "weighted Grassmannians". However, keeping other names in mind like Milnor manifolds and Seifert manifolds, we prefer to use Grassmann manifolds and weighted Grassmann orbifolds.

The paper is organized as follows. In Section 2, analogously to the definition of Grassmann manifold discussed in Mukherjee [24], we introduce another definition of a weighted Grassmann orbifold $\mathrm{WGr}(d,n)$ for $d < n$, $a \in \mathbb{Z}_{\geq 1}$ and a "weight vector" $W := (w_1, \dots, w_n) \in (\mathbb{Z}_{\geq 0})^n$. Interestingly, this definition is equivalent to the previous one that appeared in Abe and Matsumura [1]. We recall the definition of Schubert symbols for $d < n$ and discuss how to get a total ordering on the Schubert symbols. Using this total order we show that there is an equivariant embedding from a weighted Grassmann orbifold to a weighted projective space; see Lemma 2.5. We describe a q–CW complex structure of $\mathrm{WGr}(d,n)$ in Proposition 2.7. Then we discuss a $(\mathbb{C}^*)^n$–invariant filtration

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = \mathrm{WGr}(d,n)$$

of $\mathrm{WGr}(d,n)$ using the q–CW complex structure, where $m := \binom{n}{d} - 1$. Here, we consider q–CW complex structure in the sense of Poddar and Sarkar [25, Section 4]. We note that one may get different q–CW complex structures depending on the choice of the total orderings on the set of all Schubert symbols for $d < n$. Accordingly, one may obtain different $(\mathbb{C}^*)^n$–invariant filtrations of $\mathrm{WGr}(d,n)$.

In Section 3, first we recall that there is an equivariant homeomorphism from $\mathbb{W}P(rc_0, rc_1, \dots, rc_m)$ to $\mathbb{W}P(c_0, c_1, \dots, c_m)$ for any $1 \leq r \in \mathbb{N}$. Using this technique, we show how the orbifold singularity on a q–cell of some subcomplexes of $\mathrm{WGr}(d,n)$ can be reduced; see Lemma 3.3. Consequently, we get a new q–CW complex structure of these subcomplexes, including $\mathrm{WGr}(d,n)$, possibly with less singularity on each q–cell; see Theorem 3.4. We show in Theorem 3.5 that two weighted Grassmann orbifolds are weakly equivariantly homeomorphic if their weight vectors differ by a permutation $\sigma \in S_n$. We define "admissible permutation" $\sigma \in S_n$ for a prime p and $\mathrm{WGr}(d,n)$; see Definition 3.8. The following result says when $H^*(\mathrm{WGr}(d,n); \mathbb{Z})$ has no p–torsion.

Theorem A (Theorem 3.10) *If there exists an admissible permutation $\sigma \in S_n$ for a prime p and $\mathrm{WGr}(d,n)$, then $H^{\mathrm{odd}}(\mathrm{WGr}(d,n); \mathbb{Z}_p)$ is trivial and $H^*(\mathrm{WGr}(d,n); \mathbb{Z})$ has no p–torsion.*

We introduce "divisive" weighted Grassmann orbifolds. We note that this definition coincides with the concept of divisive weighted projective space of Harada, Holm, Ray and Williams [15] when $1 = d < n$. We prove the following.

Theorem B (Theorem 3.19) *If $\mathrm{WGr}(d,n)$ is a divisive weighted Grassmann orbifold, then it has a $(\mathbb{C}^*)^n$–invariant CW complex structure. Moreover, the $(\mathbb{C}^*)^n$–action on each cell of this CW complex can be described explicitly.*

This result implies that the integral cohomology of a divisive weighted Grassmann orbifold has no torsion and is concentrated in even degrees. We discuss a class of nontrivial examples of divisive weighted

Grassmann orbifolds. We remark that the weighted Grassmann orbifold in Example 3.12 is not divisive. However, its integral cohomology has no torsion.

In Section 4, we show that the $(\mathbb{C}^*)^n$–invariant stratification

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset \cdots \subset X_m = \mathrm{WGr}(d,n)$$

has the following property. The quotient X_i/X_{i-1} is homeomorphic to the Thom space of an orbifold $(\mathbb{C}^*)^n$–bundle

$$\xi^i : \mathbb{C}^{\ell(\lambda^i)}/G_i \to \{\mathrm{pt}\}$$

for some $\ell(\lambda^i) \in \mathbb{Z}_{\geq 1}$ and finite groups G_i for $i = 1, \dots, m$; see Proposition 4.1. Then considering $T^n := (S^1)^n \subset (\mathbb{C}^*)^n$, we compute the equivariant K–theory ring of any weighted Grassmann orbifolds with rational coefficients; see Theorem 4.4. If $\mathrm{WGr}(d,n)$ is divisive then G_i is trivial for $i = 1, \dots, m$. The following result describes the integral equivariant cohomology of certain weighted Grassmann orbifolds.

Theorem C (Theorem 4.7) *Let $\mathrm{WGr}(d,n)$ be a divisive weighted Grassmann orbifold for $d < n$. Then the generalized T^n–equivariant cohomology with integer coefficients $\mathscr{E}^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ can be given by*

$$\left\{ (f_i) \in \bigoplus_{i=0}^{m} \mathscr{E}^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z}) \;\middle|\; e_{T^n}(\xi^{ij}) \text{ divides } f_i - f_j \text{ for } j < i \text{ and } |\lambda^j \cap \lambda^i| = d-1 \right\}$$

for $\mathscr{E}^*_{T^n} = H^*_{T^n}, K^*_{T^n}$ and $\mathrm{MU}^*_{T^n}$.

The computation of $e_{T^n}(\xi^{ij})$ is discussed in (4-4). We compute the equivariant cohomology ring of some weighted Grassmann orbifold with integer coefficients which are not divisive; see Theorem 4.10. For $m \geq 2$, corresponding to each pair of positive integers (n,d) such that $d < n$ and $m+1 = \binom{n}{d}$, we have a T^n–action on $\mathbb{W}P(c_0, c_1, \dots, c_m)$. For each pair (n,d), we discuss the generalized T^n–equivariant cohomology of a divisive $\mathbb{W}P(c_0, c_1, \dots, c_m)$ with integer coefficients; see Theorem 4.11.

In Section 5, we show that there exist equivariant Schubert classes $\{w\widetilde{S}_{\lambda^i}\}_{i=0}^m$ which form a basis for the integral T^n–equivariant cohomology of a divisive weighted Grassmann orbifold; see Proposition 5.3. We study some properties of weighted structure constants; see Lemma 5.5. Then we show the following multiplication rule.

Proposition D (weighted Pieri rule, Proposition 5.7)

$$w\widetilde{S}_{\lambda^1} w\widetilde{S}_{\lambda^j} = (w\widetilde{S}_{\lambda^1}|_{\lambda^j}) w\widetilde{S}_{\lambda^j} + \sum_{\lambda^i \to \lambda^j} \frac{c_0}{c_j} w\widetilde{S}_{\lambda^i}.$$

Moreover, we deduce a recurrence relation which helps to compute the weighted structure constants $\{wc_{ij}^k\}$ corresponding to this Schubert basis $\{w\widetilde{S}_{\lambda^i}\}_{i=0}^m$ with integral coefficients.

Proposition E (Proposition 5.8) *For any three Schubert symbols λ^i, λ^j and λ^k, we have the recurrence relation*

$$(w\widetilde{S}_{\lambda^1}|_{\lambda^k} - w\widetilde{S}_{\lambda^1}|_{\lambda^i})wc_{ij}^k = \Big(\sum_{\lambda^s \to \lambda^i} \frac{c_0}{c_i} wc_{sj}^k - \sum_{\lambda^k \to \lambda^t} \frac{c_0}{c_t} wc_{ij}^t \Big).$$

2 Weighted Grassmann orbifolds and their invariant q–CW complexes

In this section, we introduce another definition of weighted Grassmann orbifold $\mathrm{WGr}(d,n)$, where $d < n$. We recall the definition of a Schubert symbol for $d < n$ and discuss some (total) ordering on the set of Schubert symbols. We show that there is an equivariant embedding from a weighted Grassmann orbifold to a weighted projective space. We show that our definition of weighted Grassmann orbifold is equivalent to the previous one, which appeared in [1]. We study the orbifold and q–CW complex structures of weighted Grassmann orbifolds generalizing the Grassmann manifolds counterpart discussed in [23].

Let $M_d(n,d)$ be the set of all complex $n \times d$ matrices of rank d, and $\mathrm{GL}(d,\mathbb{C})$ the set of all nonsingular complex matrices of order d. We denote a matrix $A \in M_d(n,d)$ by

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1d} \\ a_{21} & a_{22} & \cdots & a_{2d} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nd} \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix}, \quad \text{where } \boldsymbol{a}_i \in \mathbb{C}^d \text{ for } i = 1, \dots, n.$$

Definition 2.1 Let $W := (w_1, w_2, \dots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $a \in \mathbb{Z}_{\geq 1}$. Define an equivalence relation $\sim_w$ on $M_d(n,d)$ by

$$\begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_n \end{pmatrix} \sim_w \begin{pmatrix} t^{w_1}\boldsymbol{a}_1 \\ t^{w_2}\boldsymbol{a}_2 \\ \vdots \\ t^{w_n}\boldsymbol{a}_n \end{pmatrix} T$$

for $T \in \mathrm{GL}(d,\mathbb{C})$ and $t \in \mathbb{C}^*$ such that $t^a = \det(T) \in \mathbb{C}^*$. We denote the identification space by

$$\mathrm{WGr}(d,n) := M_d(n,d)/{\sim_w}.$$

The quotient map

$$\pi_w \colon M_d(n,d) \to \mathrm{WGr}(d,n) \tag{2-1}$$

is defined by $\pi_w(A) = [A]_{\sim_w}$. The topology on $\mathrm{WGr}(d,n)$ is given by the quotient topology via the map π_w.

Remark 2.2 If $W = (0,0,\dots,0)$ and $a = 1$, then $\mathrm{WGr}(d,n)$ is the Grassmann manifold $\mathrm{Gr}(d,n)$. We denote the corresponding quotient map by

$$\pi \colon M_d(n,d) \to \mathrm{Gr}(d,n). \tag{2-2}$$

The space $\mathrm{Gr}(d,n)$ is a $d(n-d)$–dimensional smooth manifold and represents the set of all d–dimensional vector subspaces in $\mathbb{C}^n$. Several basic properties, such as the manifold and CW complex structure of $\mathrm{Gr}(d,n)$, can be found in [23]. In this paper, by dimension we mean complex dimension unless specified explicitly.

Remark 2.3 If $d=1$, then $M_d(n,d)=M_1(n,1)=\mathbb{C}^n\setminus\{0\}$ and $\mathrm{GL}(1,\mathbb{C})=\mathbb{C}^*$. The corresponding $\sim_w$ is given by

$$(z_1,z_2,\dots,z_n)\sim_w (t^{a+w_1}z_1,t^{a+w_2}z_2,\dots,t^{a+w_n}z_n).$$

The quotient space $M_1(n,1)/{\sim_w}$ is called the weighted projective space with weights

$$(a+w_1,a+w_2,\dots,a+w_n),$$

and is denoted by $\mathbb{W}P(c_0,c_1,\dots,c_{n-1})$, where $c_i=a+w_{i+1}$ for $i\in\{0,1,\dots,n-1\}$. For the weighted projective space, we denote $\sim_w$ by $\sim_c$ when $c=(c_0,c_1,\dots,c_{n-1})$. This identification $\sim_c$ is called a weighted $\mathbb{C}^*$–action on $\mathbb{C}^n\setminus\{0\}$ with weights $(c_0,c_1,\dots,c_{n-1})$. In addition, if $W=(0,0,\dots,0)$ and $a=1$, then $c_0=1=c_1=\cdots=c_{n-1}$ and $\mathbb{W}P(c_0,c_1,\dots,c_{n-1})$ is $\mathbb{C}P^{n-1}=\mathrm{Gr}(1,n)$.

A Schubert symbol λ for $d<n$ is a sequence of d integers $(\lambda_1,\lambda_2,\dots,\lambda_d)$ such that $1\le\lambda_1<\lambda_2<\cdots<\lambda_d\le n$. The length $\ell(\lambda)$ of a Schubert symbol $\lambda:=(\lambda_1,\lambda_2,\dots,\lambda_d)$ is defined by

$$\ell(\lambda):=(\lambda_1-1)+(\lambda_2-2)+\cdots+(\lambda_d-d).$$

There are $\binom{n}{d}$ many Schubert symbols for $d<n$. One can define a partial order $\preceq$ on the Schubert symbols for $d<n$ by

$$\lambda\preceq\mu \quad\text{if}\quad \lambda_i\le\mu_i \quad\text{for all } i=1,2,\dots,d. \tag{2-3}$$

Then the set of all Schubert symbols for $d<n$ form a poset with respect to this partial order $\preceq$.

Definition 2.4 Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_d)$ and $\mu=(\mu_1,\mu_2,\dots,\mu_d)$ be two Schubert symbols for $d<n$. We say that $\lambda<\mu$ if $\ell(\lambda)<\ell(\mu)$, otherwise we use the dictionary order if $\ell(\lambda)=\ell(\mu)$.

This gives a total order on the set of all Schubert symbols. Note that the total order $<$ in Definition 2.4 preserves the partial order $\preceq$ in (2-3). That is, for two Schubert symbols λ and μ, if $\lambda\preceq\mu$ then $\lambda\le\mu$, but the converse may not be true in general. Observe that there may exist several other total orders on the set of all Schubert symbols which preserve the partial order $\preceq$. For example, the dictionary order also gives a total order on the Schubert symbols. By a total order on the set of all Schubert symbols for $d<n$, we mean one of these total orders on it. For $m=\binom{n}{d}-1$, let

$$\lambda^0<\lambda^1<\lambda^2<\cdots<\lambda^m \tag{2-4}$$

be a total order on the Schubert symbols for $d<n$.

For $W=(w_1,w_2,\dots,w_n)\in(\mathbb{Z}_{\geq 0})^n$, $a\in\mathbb{Z}_{\geq 1}$ and $i\in\{0,1,\dots,m\}$, let

$$c_i := a+\sum_{j=1}^{d} w_{\lambda^i_j}, \tag{2-5}$$

where $\lambda^i=(\lambda^i_1,\lambda^i_2,\dots,\lambda^i_d)$ is the i^{th} Schubert symbol given in (2-4). Then $c_i\geq 1$ for any $i\in\{0,\dots,m\}$. Therefore, one can define the weighted projective space $\mathbb{W}P(c_0,c_1,\dots,c_m)$ from Remark 2.3. We denote the associated orbit map $\mathbb{C}^{m+1}\setminus\{0\}\to\mathbb{W}P(c_0,c_1,\dots,c_m)$ by π'_c, which can be written as

$$\pi'_c(z_0,z_1,\dots,z_m)=[z_0:z_1:\dots:z_m]_{\sim_c}. \tag{2-6}$$

Note that when $c_0=c_1=\dots=c_m=1$, the corresponding orbit map is denoted by

$$\pi'\colon\mathbb{C}^{m+1}\setminus\{0\}\to\mathbb{C}P^m.$$

Let $(t_1,t_2,\dots,t_n)\in(\mathbb{C}^*)^n$ and $A=(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n)^{\text{tr}}\in M_d(n,d)$. Then $(\mathbb{C}^*)^n$ acts on $M_d(n,d)$ by

$$(t_1,\dots,t_n)(\boldsymbol{a}_1,\boldsymbol{a}_2,\dots,\boldsymbol{a}_n)^{\text{tr}}:=(t_1\boldsymbol{a}_1,t_2\boldsymbol{a}_2,\dots,t_n\boldsymbol{a}_n)^{\text{tr}}. \tag{2-7}$$

This induces a natural $(\mathbb{C}^*)^n$–action on $\text{WGr}(d,n)$ such that the orbit map π_w of (2-1) is $(\mathbb{C}^*)^n$–equivariant.

The standard ordered basis $\{e_1,e_2,\dots,e_n\}$ of $\mathbb{C}^n$ induces an ordered basis $\{e_{\lambda^0},e_{\lambda^1},\dots,e_{\lambda^m}\}$ of $\Lambda^d(\mathbb{C}^n)$, where $e_\lambda=e_{\lambda_1}\wedge\dots\wedge e_{\lambda_d}$ for the Schubert symbol $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_d)$ for $d<n$. Therefore, we can identify $\Lambda^d(\mathbb{C}^n)$ with $\mathbb{C}^{m+1}(=\mathbb{C}\{e_{\lambda^0},e_{\lambda^1},\dots,e_{\lambda^m}\})$. The standard action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ induces an action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^{m+1}\setminus\{0\}$, which is defined by

$$(t_1,t_2,\dots,t_n)\Bigl(\sum_{i=0}^{m}a_ie_{\lambda^i}\Bigr)=\sum_{i=0}^{m}a_it_{\lambda^i}e_{\lambda^i}, \tag{2-8}$$

where $t_\lambda=t_{\lambda_1}\cdots t_{\lambda_d}$ for the Schubert symbol $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_d)$. This induces a $(\mathbb{C}^*)^n$–action on the weighted projective space $\mathbb{W}P(c_0,c_1,\dots,c_m)$ such that the orbit map π'_c in (2-6) is $(\mathbb{C}^*)^n$–equivariant.

For each Schubert symbol $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_d)$, let A_λ be the matrix with row vectors $\boldsymbol{a}_{\lambda_1},\boldsymbol{a}_{\lambda_2},\dots,\boldsymbol{a}_{\lambda_d}$. Define a map $P\colon M_d(n,d)\to\mathbb{C}^{m+1}\setminus\{0\}$ by

$$P(A)=\boldsymbol{v}_1\wedge\boldsymbol{v}_2\wedge\dots\wedge\boldsymbol{v}_d=\sum_{i=0}^{m}\det(A_{\lambda^i})e_{\lambda^i}, \tag{2-9}$$

where $\boldsymbol{v}_1,\boldsymbol{v}_2,\dots,\boldsymbol{v}_d\in\mathbb{C}^n$ are the columns of A. Observe that $P(A)\neq 0$ as $A\in M_d(n,d)$ has rank d.

From (2-9) we have

$$P(DAT)=\sum_{i=0}^{m}\det((DAT)_{\lambda^i})e_{\lambda^i}=\sum_{i=0}^{m}t^{c_i}\det(A_{\lambda^i})e_{\lambda^i},$$

where $T \in \mathrm{GL}(d,\mathbb{C})$, $D = \mathrm{diag}(t^{w_1}, t^{w_2}, \dots, t^{w_n})$ is the diagonal matrix for $t \in \mathbb{C}^*$ such that $t^a = \det(T)$, and c_i is defined in (2-5) for $i = 0, 1, 2, \dots, m$. Therefore, the map P in (2-9) induces a map

$$\mathrm{Pl}_w : \mathrm{WGr}(d,n) \to \mathbb{W}P(c_0, c_1, c_2, \dots, c_m) \tag{2-10}$$

defined by $\mathrm{Pl}_w([A]_{\sim_w}) = [\det(A_{\lambda^0}) : \det(A_{\lambda^1}) : \cdots : \det(A_{\lambda^m})]_{\sim_c}$.

The map Pl_w satisfies the following commutative diagram:

$$\begin{array}{ccc} M_d(n,d) & \xrightarrow{\ P\ } & \mathbb{C}^{m+1} \setminus \{0\} \\ \big\downarrow{\scriptstyle \pi_w} & & \big\downarrow{\scriptstyle \pi'_c} \\ \mathrm{WGr}(d,n) & \xrightarrow{\mathrm{Pl}_w} & \mathbb{W}P(c_0, c_1, \dots, c_m) \end{array}$$

Thus the map Pl_w is continuous, since π_w and π'_c are quotient maps.

Lemma 2.5 *The map* $\mathrm{Pl}_w : \mathrm{WGr}(d,n) \to \mathbb{W}P(c_0, c_1, c_2, \dots, c_m)$ *is an embedding.*

Proof Consider $[A]_{\sim_w} \in \mathrm{WGr}(d,n)$ for some $A \in M_d(n,d)$. There exists a Schubert symbol λ^i such that $\det(A_{\lambda^i}) \neq 0$. Without loss of generality, we can assume that $A_{\lambda^i} = \mathrm{I}_d$, where I_d is the identity matrix of order d. If $A_{\lambda^i} \neq \mathrm{I}_d$ then one can calculate $s \in \mathbb{C}^*$ such that $s^{c_i} = 1/\det(A_{\lambda^i})$. Now we consider the matrices $D = \mathrm{diag}(s^{w_1}, s^{w_2}, \dots, s^{w_n})$ and $T = (D_{\lambda^i} A_{\lambda^i})^{-1}$. Then $\det(T) = s^a$ and $(DAT)_{\lambda^i} = \mathrm{I}_d$. Note that $[DAT]_{\sim_w} = [A]_{\sim_w} \in \mathrm{WGr}(d,n)$.

We prove that Pl_w is injective. Let $[A]_{\sim_w}, [B]_{\sim_w} \in \mathrm{WGr}(d,n)$ be such that $\mathrm{Pl}_w([A]_{\sim_w}) = \mathrm{Pl}_w([B]_{\sim_w})$ for some $A, B \in M_d(n,d)$. Now

$$\mathrm{Pl}_w([A]_{\sim_w}) = \mathrm{Pl}_w([B]_{\sim_w}) \implies \det(A_{\lambda^j}) = t^{c_j} \det(B_{\lambda^j}) \tag{2-11}$$

for some $t \in \mathbb{C}^*$ and for all $j \in \{0, 1, \dots, m\}$. Since $A \in M_d(n,d)$ there exists a Schubert symbol $\lambda^i = (\lambda^i_1, \dots, \lambda^i_d)$ such that $\det(A_{\lambda^i}) \neq 0$. Then using (2-11), $\det(B_{\lambda^i}) \neq 0$. So we can assume $A_{\lambda^i} = B_{\lambda^i} = \mathrm{I}_d$. Then $t^{c_i} = 1$. Consider the matrices $D = \mathrm{diag}(t^{w_1}, t^{w_2}, \dots, t^{w_n})$ and $T = \mathrm{diag}(t^{-w_{\lambda^i_1}}, \dots, t^{-w_{\lambda^i_d}})$. Thus, we have $B_{\lambda^i} = (DAT)_{\lambda^i}$.

For $k \notin (\lambda^i_1, \dots, \lambda^i_d)$ and $1 \le l \le d$, let a_{kl} and b_{kl} be the (k,l) entries of the matrices A and B, respectively. For a fixed l, let λ^j be the Schubert symbol obtained by replacing λ^i_l by k in λ^i and then ordering the latter set. Then $\det(A_{\lambda^j}) = a_{kl}$ and $\det(B_{\lambda^j}) = b_{kl}$. Thus using (2-11), we get

$$b_{kl} = t^{c_j} a_{kl} \implies b_{kl} = t^{c_j - c_i} a_{kl} \implies b_{kl} = t^{w_k - w_{\lambda^i_l}} a_{kl}.$$

The above condition holds for all $1 \le k \le n$ and $1 \le l \le d$. This gives $B = DAT$. Then we have $[A]_{\sim_w} = [B]_{\sim_w}$. Hence, Pl_w is an injective map.

Observe that, if $W = (0, 0, \dots, 0)$ and $a = 1$, then the map Pl_w is the usual Plücker map

$$\mathrm{Pl} : \mathrm{Gr}(d,n) \to \mathbb{C}P^m.$$

It is well known that Pl is an embedding. Moreover, we have the following commutative diagrams:

(2-12)
$$\begin{tikzcd} \mathrm{WGr}(d,n) \arrow[r, "\mathrm{Pl}_w"] & \mathbb{W}P(c_0,c_1,\dots,c_m) \\ M_d(n,d) \arrow[u, "\pi_w"] \arrow[r, "P"] \arrow[d, "\pi"] & \mathbb{C}^{m+1}\setminus\{0\} \arrow[u, "\pi'_c"'] \arrow[d, "\pi'"'] \\ \mathrm{Gr}(d,n) \arrow[r, "\mathrm{Pl}"] & \mathbb{C}P^m \end{tikzcd}$$

Let U be an open subset of $\mathrm{WGr}(d,n)$. Then $\pi_w^{-1}(U)$ is an open subset of $M_d(n,d)$. Since the map π in (2-2) is an orbit map, $\pi(\pi_w^{-1}(U))$ is an open subset of $\mathrm{Gr}(d,n)$. Thus $\mathrm{Pl}(\pi(\pi_w^{-1}(U))) = \pi'(P(\pi_w^{-1}(U)))$ is an open subset of $\mathrm{Pl}(\mathrm{Gr}(d,n))$. Then $P(\pi_w^{-1}(U))$ is an open subset of $P(M_d(n,d))$. Therefore, $\mathrm{Pl}_w(U) = \pi'_c(P(\pi_w^{-1}(U)))$ is an open subset of $\mathrm{Pl}_w(\mathrm{WGr}(d,n))$. Thus Pl_w is an embedding. □

We call the embedding Pl_w the *weighted Plücker embedding*. Note that the actions of $(\mathbb{C}^*)^n$ on $\mathrm{WGr}(d,n)$ and $\mathbb{W}P(c_0,c_1,\dots,c_m)$ imply that the weighted Plücker embedding Pl_w is $(\mathbb{C}^*)^n$–equivariant, and $\mathrm{Pl}_w(\mathrm{WGr}(d,n))$ is a $(\mathbb{C}^*)^n$–invariant subset of $\mathbb{W}P(c_0,c_1,\dots,c_m)$. Thus all the maps in the diagram (2-12) are $(\mathbb{C}^*)^n$–equivariant.

Now we show that Definition 2.1 is equivalent to the definition of a weighted Grassmannian studied in [1]. The algebraic torus $(\mathbb{C}^*)^{n+1}$ acts on $\Lambda^d(\mathbb{C}^n)$ by

$$(t_1,t_2,\dots,t_n,t)\sum_{i=0}^{m} a_{\lambda^i} e_{\lambda^i} = \sum_{i=0}^{m} t\cdot t_{\lambda^i} a_{\lambda^i} e_{\lambda^i},$$

where $t_\lambda = t_{\lambda_1}\cdots t_{\lambda_d}$ for $\lambda = (\lambda_1,\dots,\lambda_d)$. Consider the subgroup WD of $(\mathbb{C}^*)^{n+1}$ defined by

$$\mathrm{WD} := \{(t^{w_1}, t^{w_2}, \dots, t^{w_n}, t^a) \mid t \in \mathbb{C}^*\}.$$

Then the restricted action of WD on $\Lambda^d(\mathbb{C}^n)\setminus\{0\}$ is given by

$$(t^{w_1}, t^{w_2}, \dots, t^{w_n}, t^a)\sum_{i=0}^{m} a_{\lambda^i} e_{\lambda^i} = \sum_{i=0}^{m} t^{c_i} a_{\lambda^i} e_{\lambda^i}.$$

Observe that this action of WD is same as the weighted $\mathbb{C}^*$–action in Remark 2.3. Then we have $\Lambda^d(\mathbb{C}^n)\setminus\{0\}/\mathrm{WD} = \mathbb{W}P(c_0,\dots,c_m)$ and by the commutativity of the diagram (2-12) we have

$$\mathrm{Pl}_w(\mathrm{WGr}(d,n)) = \frac{P(M_d(n,d))}{\mathrm{WD}}.$$

Therefore the topologies on $\mathrm{WGr}(d,n)$ and $P(M_d(n,d))/\mathrm{WD}$ are equivalent. Abe and Matsumura [1] called the quotient $P(M_d(n,d))/\mathrm{WD}$ a weighted Grassmannian and showed that it has an orbifold structure. We call $\mathrm{WGr}(d,n)$ a weighted Grassmann orbifold associated to the pair (W,a).

Next, we recall the Schubert cell decomposition of $\mathrm{Gr}(d,n)$ following [23]. For $k \le n$, we identify

$$\mathbb{C}^k = \{(z_1, z_2, \dots, z_k, 0, \dots, 0) \in \mathbb{C}^n\}.$$

For the Schubert symbol $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$, the Schubert cell $E(\lambda)$ is defined by

$$E(\lambda) := \{X \in \mathrm{Gr}(d,n) \mid \dim(X \cap \mathbb{C}^{\lambda_j}) = j, \dim(X \cap \mathbb{C}^{\lambda_j - 1}) = j-1 \text{ for all } j \in [d]\},$$

where $[d] := \{1, 2, \dots, d\}$. We have the following homeomorphism from [23, Chapter 6]:

$$E(\lambda) \cong \left\{ \begin{bmatrix} * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \\ 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & * \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \;\middle|\; * \in \mathbb{C} \text{ and } e_j \text{ is the } \lambda_j^{\text{th}} \text{ row for } j \in [d] \right\}. \tag{2-13}$$

Note that the j^{th} column in the matrices in (2-13) has λ_j^{th} entry 1 and all subsequent entries of this column are zero for $j \in [d]$. Then $E(\lambda)$ is an open cell of dimension $\ell(\lambda) = (\lambda_1 - 1) + (\lambda_2 - 2) + \cdots + (\lambda_d - d)$.

We recall some basic properties of q–cell and finite q–CW complex from [25; 4]. Let D^n be the open unit disc in $\mathbb{R}^n$ and G a finite group acting on $\bar{D}^n$ such that $\partial \bar{D}^n$ is invariant. Then D^n/G is called a q–cell of real dimension n. Let Y be a space and $\phi\colon \partial \bar{D}^n/G \to Y$ a continuous map. Then the mapping cone

$$X := \left(Y \sqcup \frac{\bar{D}^n}{G} \right) \Big/ \left\{ x \sim \phi(x) \text{ for } x \in \frac{\partial \bar{D}^n}{G} \right\}$$

is obtained from Y by attaching the q–cell D^n/G. As a set, we can write $X = Y \sqcup (D^n/G)$ whenever the attaching map is clear. If a space X is obtained from a finite set by attaching finitely many q–cells, then X is called a finite q–CW complex.

Let k be a positive integer and $G(k)$ the group of k^{th} roots of unity defined by

$$G(k) := \{t \in \mathbb{C}^* \mid t^k = 1\}.$$

Then we have the following.

Lemma 2.6 *Let S be a $\mathbb{C}^*$–space, and suppose that $\mathbb{C}^*$ acts on $S \times \mathbb{C}^*$ by $t \cdot (x, \alpha) = (t \cdot x, t^k \alpha)$. Then*
$$\frac{S \times \mathbb{C}^*}{\mathbb{C}^*} \cong \frac{S}{G(k)},$$
where $G(k)$ acts on S by restriction of the $\mathbb{C}^*$–action.

Proof The inclusion map $S \to S \times \mathbb{C}^*$ defined by $x \to (x, 1)$ induces a map
$$\bar{f} : S \to \frac{S \times \mathbb{C}^*}{\mathbb{C}^*}.$$
Note that every element in the codomain of $\bar{f}$ can be written as $[(u, 1)]$, where $u \in S$. To verify this, consider an element $[(x, t)]$ in the codomain of $\bar{f}$, where $x \in S$ and $t \in \mathbb{C}^*$. Consider $s \in \mathbb{C}^*$ such that $s^k = 1/t$. Then $s \cdot (x, t) = (s \cdot x, 1)$. Hence $[(x, t)] = [(u, 1)]$, where $u = s \cdot x$. Thus $u \in S$ is the preimage of $[u, 1] \in \text{codomain}(f)$ and the map $\bar{f}$ becomes onto.

Now $G(k)$ is a finite subgroup of $\mathbb{C}^*$ acts on S as a restriction of the $\mathbb{C}^*$- action. For any $t \in G(k)$,
$$\bar{f}(t \cdot u) = [(t \cdot u, 1)] = [(t \cdot u, t^k)] = [(u, 1)] = \bar{f}(u).$$
Thus $\bar{f}$ induces an onto map $f : S/G(k) \to S \times \mathbb{C}^*/\mathbb{C}^*$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[rr, "\bar{f}"] \arrow[dr, "\sim G(k)"'] & & \dfrac{S \times \mathbb{C}^*}{\mathbb{C}^*} \\ & \dfrac{S}{G(k)} \arrow[ur, "f"'] & \end{tikzcd} \tag{2-14}$$

To check that f is one-to-one, if $[(x, 1)] = [(y, 1)]$ then $(y, 1) = t \cdot (x, 1) = (t \cdot x, t^k)$. This implies $y = t \cdot x$ for some $t \in G(k)$. Thus $[x] = [y]$ in $S/G(k)$. Therefore,
$$\frac{S \times \mathbb{C}^*}{\mathbb{C}^*} \cong \frac{S}{G(k)}. \qquad \square$$

The next result gives a q–CW complex structure on $\mathrm{WGr}(d, n)$.

Proposition 2.7 $\mathrm{WGr}(d, n)$ *is a finite q–CW complex for $0 < d < n$.*

Proof Consider a total order on the Schubert symbols for $d < n$ as in (2-4), which satisfies the partial order in (2-3). For each $i \in \{0, 1, \dots, m\}$, define $\widetilde{E}(\lambda^i) := \pi^{-1}(E(\lambda^i))$, where the map π is defined in (2-2). The Schubert cell decomposition of $\mathrm{Gr}(d, n)$ gives that $\mathrm{Gr}(d, n) = \bigsqcup_{i=0}^m E(\lambda^i)$. This implies
$$M_d(n, d) = \bigsqcup_{i=0}^{m} \widetilde{E}(\lambda^i), \tag{2-15}$$
since the map π is surjective. Note that
$$\widetilde{E}(\lambda^i) = \{A \in M_d(n, d) \mid \det(A_{\lambda^i}) \neq 0, \det(A_{\lambda^j}) = 0 \text{ for } j > i\}.$$
Let $A \in \widetilde{E}(\lambda^i)$ and $A \sim_w B$ for a matrix $B \in M_d(n, d)$. Then $B \in \widetilde{E}(\lambda^i)$.

Therefore, we have the decomposition of $\mathrm{WGr}(d,n)$

$$\mathrm{WGr}(d,n) = \pi_w(\widetilde{E}(\lambda^0)) \sqcup \pi_w(\widetilde{E}(\lambda^1)) \sqcup \cdots \sqcup \pi_w(\widetilde{E}(\lambda^m)).$$

By the commutativity of the diagram (2-12), we get

$$\mathrm{Pl}_w(\pi_w(\widetilde{E}(\lambda^i))) = \pi'_c(P(\widetilde{E}(\lambda^i))) \quad \text{and} \quad P(\widetilde{E}(\lambda^i)) = (\pi')^{-1}(\mathrm{Pl}(E(\lambda^i))).$$

The map π' is a principal $\mathbb{C}^*$–bundle, and $E(\lambda^i)$ is contractible. So there is a bundle isomorphism

$$\phi_i : P(\widetilde{E}(\lambda^i)) \to E(\lambda^i) \times \mathbb{C}^*.$$

Indeed, this map can be defined by $\phi_i(P(A)) = (\pi(A), \det(A_{\lambda^i}))$. The inverse map is defined by $(\pi(A), s) \mapsto (s(\det(A_{\lambda^i}))^{-1} P(A))$.

Let $\pi(A) \in \mathrm{Gr}(d,n)$ for some $A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)^{\mathrm{tr}} \in M_d(n,d)$ and $t \in \mathbb{C}^*$. There is an action of $\mathbb{C}^*$ on $\mathrm{Gr}(d,n)$ defined by

$$t \cdot \pi(A) = t \cdot \pi((\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)^{\mathrm{tr}}) := \pi((t^{w_1}\boldsymbol{a}_1, t^{w_2}\boldsymbol{a}_2, \dots, t^{w_n}\boldsymbol{a}_n)^{\mathrm{tr}}). \tag{2-16}$$

If $\pi(A) = \pi(B)$, then $A = BT$ if and only if $DA = DBT$ for a diagonal matrix D and $T \in \mathrm{GL}(d,\mathbb{C})$. Thus $t \cdot \pi(A) = t \cdot \pi(B)$. Then ϕ_i becomes $\mathbb{C}^*$–equivariant with the following weighted $\mathbb{C}^*$–action on $E(\lambda^i) \times \mathbb{C}^*$ given by

$$t \cdot (\pi(A), s) = (t \cdot \pi(A), t^{c_i} s),$$

where $t \cdot \pi(A)$ is defined in (2-16) and c_i is defined in (2-5). Thus

$$\pi'_c(P(\widetilde{E}(\lambda^i))) = \frac{P(\widetilde{E}(\lambda^i))}{\text{weighted } \mathbb{C}^*\text{–action}} \cong \frac{E(\lambda^i) \times \mathbb{C}^*}{\text{weighted } \mathbb{C}^*\text{–action}} \cong \frac{E(\lambda^i)}{G(c_i)},$$

where the last identification follows from Lemma 2.6.

Now $E(\lambda^i)/G(c_i)$ is a q–cell of dimension $\ell(\lambda^i)$ as $E(\lambda^i)$ is an open cell of dimension $\ell(\lambda^i)$ and $|G(c_i)| < \infty$.

Let $C(i) := \{[z_0 : z_1 : \cdots : z_{i-1} : 1 : 0 : \cdots : 0] \in \mathbb{W}P(c_0, c_1, \dots, c_m)\}$.

Consider

$$S^{2i-1} = \left\{ (z_0, z_1, \dots, z_{i-1}, 0, \dots, 0) \in \mathbb{C}^{m+1} \;\middle|\; \sum_{j=0}^{i-1} |z_j|^2 = 1 \right\}$$

and the $G(c_i)$–action on S^{2i-1} by $g(z_0, \dots, z_{i-1}, 0, \dots, 0) \mapsto (g^{c_0} z_0, \dots, g^{c_{i-1}} z_{i-1}, 0, \dots, 0)$. The orbit space is called an orbifold Lens space and denoted by $L(c_i; c')$, where $c' = (c_0, \dots, c_{i-1})$. Then $C(i) = \mathbb{C}^i / G(c_i)$ is homeomorphic to the cone $C(L(c_i; c'))$ on $L(c_i; c')$. The space $\mathbb{W}P(c_0, \dots, c_{i-1})$ can be obtained by the weighted S^1–action on S^{2i-1} with the weight vector c'. Thus there is a map

$$\phi_i : \frac{S^{2i-1}}{G(c_i)} = L(c_i; c') \to \frac{S^{2i-1}}{\text{weighted } S^1\text{–action}},$$

which plays the role of the attaching map for the q–cell $C(i)$; see [19].

Note that the set $E(\lambda^i) \cong \{[z_0 : \cdots : z_{i-1} : 1 : 0 : \cdots : 0] \in \mathrm{Pl}(\mathrm{Gr}(d,n))\} \subset \mathbb{C}P^m$. Then $E(\lambda^i) \cong \mathbb{C}^{\ell(\lambda^i)}$ can be considered as a $G(c_i)$–invariant subset of $\mathbb{C}^i$ as $\ell(\lambda^i) < i$. So $S^{2i-1} \cap E(\lambda^i)$ is a $G(c_i)$–invariant sphere of real dimension $2\ell(\lambda^i) - 1$. Thus, we have

$$S\left(\frac{E(\lambda^i)}{G(c_i)}\right) := \frac{S^{2i-1} \cap E(\lambda^i)}{G(c_i)} \hookrightarrow \frac{E(\lambda^i)}{G(c_i)} \hookrightarrow \frac{\mathbb{C}^i}{G(c_i)} = C(i).$$

Therefore, the attaching map for the q–cell $E(\lambda^i)/G(c_i)$ is the restriction on $S(E(\lambda^i)/G(c_i))$ and the following diagram commutes:

$$\begin{tikzcd}
S\left(\frac{E(\lambda^i)}{G(c_i)}\right) \arrow[r, "\psi_i"] \arrow[d, "\mathrm{Pl}_w"] & \{[(z_0 : \cdots : z_{i-1} : 0 : \cdots : 0)] \in \mathrm{Pl}_w(\mathrm{WGr}(d,n))\} \arrow[d] \\
L(c_i, c') \arrow[r, "\phi_i"] & \mathbb{W}P(c_0, c_1, \ldots, c_{i-1})
\end{tikzcd}$$

Therefore, a q–CW complex structure on $\mathrm{WGr}(d,n)$ is given by

$$\mathrm{Pl}_w(\mathrm{WGr}(d,n)) = \frac{E(\lambda^0)}{G(c_0)} \sqcup \frac{E(\lambda^1)}{G(c_1)} \sqcup \frac{E(\lambda^2)}{G(c_2)} \sqcup \cdots \sqcup \frac{E(\lambda^m)}{G(c_m)}. \qquad \square$$

For each $k \in \{0, 1, 2, \ldots, m\}$, let

$$X_k := \bigsqcup_{i=0}^{k} \frac{E(\lambda^i)}{G(c_i)} \subset \mathrm{WGr}(d,n).$$

Here X_k is built inductively by attaching the q–cells $E(\lambda^0)/G(c_0), \ldots, E(\lambda^k)/G(c_k)$ so that X_k remains a subset of $\mathrm{WGr}(d,n)$. Then each X_k is a $(\mathbb{C}^*)^n$–invariant and we have the following filtration of $\mathrm{WGr}(d,n)$:

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = \mathrm{WGr}(d,n). \tag{2-17}$$

We note that the paper [1] discussed a q–CW complex structure of $\mathrm{WGr}(d,n)$. However, our approach is different and helps to study torsions in the integral cohomology of $\mathrm{WGr}(d,n)$.

3 Integral cohomology of certain weighted Grassmann orbifolds

In this section, we study several q–CW complex structures on a weighted Grassmann orbifold. We show how a permutation on the weight vector affects the weighted Grassmann orbifold. We define admissible permutation $\sigma \in S_n$ for a prime p and $\mathrm{WGr}(d,n)$. Then we discuss when $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ has no p–torsion. We introduce the concept of divisive weighted Grassmann orbifolds, which incorporates the divisive weighted projective spaces of [15]. We show that a divisive weighted Grassmann orbifold has a $(\mathbb{C}^*)^n$–invariant CW complex structure. We describe this action on each cell explicitly. As a consequence, we get that the integral cohomology of a divisive weighted Grassmann orbifold has no torsion and is concentrated in even degrees.

The following lemma is well known, but for our purpose we may need its proof.

Lemma 3.1 *The map $\pi'_c\colon \mathbb{C}^{m+1} - \{0\} \to \mathbb{W}P(c_0, c_1, \dots, c_m)$ induces an equivariant homeomorphism $\mathbb{W}P(rc_0, rc_1, \dots, rc_m) \to \mathbb{W}P(c_0, c_1, \dots, c_m)$ for any positive integer r.*

Proof The weighted $\mathbb{C}^*$–action on $\mathbb{C}^{m+1} \setminus \{0\}$ for $\mathbb{W}P(rc_0, rc_1, \dots, rc_m)$ is given by

$$t(z_0, z_1, \dots, z_m) = (t^{rc_0} z_0, t^{rc_1} z_1, \dots, t^{rc_m} z_m).$$

We denote the equivalence class by $[z_0 : z_1 : \dots : z_m]_{\sim_{rc}}$.

One can define a map $f\colon \mathbb{W}P(rc_0, rc_1, \dots, rc_m) \to \mathbb{W}P(c_0, \dots, c_m)$ by

$$f([z_0 : z_1 : \dots : z_m]_{\sim_{rc}}) = [z_0 : z_1 : \dots : z_m]_{\sim_c}$$

and a map $g\colon \mathbb{W}P(c_0, c_1, \dots, c_m) \to \mathbb{W}P(rc_0, rc_1, \dots, rc_m)$ by

$$g([z_0 : z_1 : \dots : z_m]_{\sim_c}) = [z_0 : z_1 : \dots : z_m]_{\sim_{rc}}.$$

Thus the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}^{m+1} \setminus \{0\} \arrow[r, "\mathrm{Id}"] \arrow[d, "\pi'_{rc}"'] & \mathbb{C}^{m+1} \setminus \{0\} \arrow[d, "\pi'_c"] \\
\mathbb{W}P(rc_0, \dots, rc_m) \arrow[r, shift right, "f"'] & \mathbb{W}P(c_0, \dots, c_m) \arrow[l, shift right, "g"']
\end{tikzcd}$$

Observe that, we have $f \circ g = \mathrm{Id}_{\mathbb{W}P(c_0,\dots,c_m)}$ and $g \circ f = \mathrm{Id}_{\mathbb{W}P(rc_0,\dots,rc_m)}$. Thus f is a bijective map with the inverse map g.

Let U be an open subset of $\mathbb{W}P(c_0, \dots, c_m)$ Then $(\pi'_c)^{-1}(U) = (\pi'_{rc})^{-1} \circ f^{-1}(U)$. Since π'_c is a quotient map then $(\pi'_c)^{-1}(U)$ is an open subset of $\mathbb{C}^{m+1} \setminus \{0\}$. Thus $f^{-1}(U)$ is an open subset of $\mathbb{W}P(rc_0, \dots, rc_m)$ as π'_{rc} is a quotient map. Thus f is continuous. By similar arguments, we can show that g is continuous. Hence f is a homeomorphism and also it is equivariant with respect to the $(\mathbb{C}^*)^n$–action on $\mathbb{W}P(c_0, \dots, c_m)$ and $\mathbb{W}P(rc_0, \dots, rc_m)$ defined after (2-8). □

Lemma 3.2 *Let B be a subset of $\mathbb{C}^{m+1} \setminus \{0\}$. Let $B'_c := \pi'_c(B)$ and $B'_{rc} := \pi'_{rc}(B)$. Then the map $f|_{B'_{rc}}\colon B'_{rc} \to B'_c$ is a homeomorphism.*

Proof Consider the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\mathrm{Id}"] \arrow[d, "\pi'_{rc}"] & B \arrow[d, "\pi'_c"] \\
B'_{rc} \arrow[r, "f|_{B'_{rc}}"] & B'_c
\end{tikzcd}$$

The map f is well defined and one-to-one. It follows that $f|_{B'_{rc}}$ is also well defined and one-to-one. Note that $f|_{B'_{rc}}$ is defined by $f|_{B'_{rc}}(\pi'_{rc}(b)) = \pi'_c(b)$. Therefore, $\pi'_{rc}(b) \in B'_{rc}$ is the inverse image of an element $\pi'_c(b) \in B'_c$. So $f|_{B'_{rc}}$ is bijective. Also $(f|_{B'_{rc}})^{-1} = g|_{B'_c}$. To conclude that $f|_{B'_{rc}}$ is a homomorphism, recall that the restriction of a continuous map is also continuous. □

We apply the previous result onto some subsets of $P(M_d(n,d)) \subseteq \mathbb{C}^{m+1} \setminus \{0\}$ for $m+1 = \binom{n}{d}$, where P is defined in (2-9). For all $k \in \{0,1,\dots,m\}$, consider $\widetilde{X}_k \subset M_d(n,d)$ defined by

$$\widetilde{X}_k := \{A \in M_d(n,d) \mid \det(A_{\lambda^j}) = 0 \text{ for } j > k\}.$$

Then $\widetilde{X}_k = \bigsqcup_{i=0}^k \widetilde{E}(\lambda^i) \subset M_d(n,d)$, where $\widetilde{E}(\lambda^i) = \pi^{-1}(E(\lambda^i)$ and

$$P(\widetilde{X}_k) = \bigsqcup_{i=0}^{k} P(\widetilde{E}(\lambda^i)) \subseteq P(M_d(n,d)).$$

Note that $P(\widetilde{X}_k) \subseteq \mathbb{C}^{k+1} \setminus \{0\} \subseteq \mathbb{C}^{m+1} \setminus \{0\}$ for $k \in \{0,1,\dots,m\}$.

One can calculate c_i for all $i \in \{0,1,\dots,m\}$ from (2-5) for a weighted Grassmann orbifold $\mathrm{WGr}(d,n)$. Let $r_k := \gcd\{c_0, c_1, \dots, c_k\}$ for all $k \in \{1,2,\dots m\}$ and $G(r_k)$ be the group of r_k^{th} roots of unity. Since $G(c_i)$ is cyclic, let $G(c_i/r_k)$ be the unique cyclic subgroup of $G(c_i)$ of order c_i/r_k for $i \in \{0,1,2,\dots,k\}$. Also $G(r_k)$ is a subgroup of $G(c_i)$ and $G(c_i)/G(r_k)$ is isomorphic to $G(c_i/r_k)$ for $i \in \{0,1,2,\dots,k\}$. Now $G(c_k)$ acts on $E(\lambda^k)$ as a restriction of the weighted $\mathbb{C}^*$–action. Then we have a restricted $G(c_k/r_k)$–action on $E(\lambda^k)$.

Lemma 3.3 *The space $\pi'_c(P(\widetilde{X}_k))$ is homeomorphic to $\pi'_{c/r_k}(P(\widetilde{X}_k))$. Moreover, $E(\lambda^k)/G(c_k)$ is homeomorphic to $E(\lambda^k)/G(c_k/r_k)$.*

Proof The diagram

$$\begin{tikzcd} P(\widetilde{X}_k) \arrow[r, "\mathrm{Id}"] \arrow[d, "\pi'_c"] & P(\widetilde{X}_k) \arrow[d, "\pi'_{c/r_k}"] \\ \pi'_c(P(\widetilde{X}_k)) \arrow[r, "f|_{\pi'_c(P(\widetilde{X}_k))}"] & \pi'_{c/r_k}(P(\widetilde{X}_k)) \end{tikzcd}$$

is commutative. By Lemma 3.2, the lower horizontal map is a homeomorphism. The second statement of the lemma follows by similar arguments with $P(\widetilde{X}_k)$ is replaced by $P(\widetilde{E}(\lambda^k))$. □

Theorem 3.4 *The collection $\{E(\lambda^i)/G(c_i/r_k)\}_{i=0}^k$ gives a q–CW complex structure of $\pi'_{c/r_k}(P(\widetilde{X}_k))$ for $k = 1,2,\dots,m$. Moreover, $\{E(\lambda^i)/G(c_i/r_i)\}_{i=0}^m$ gives a q–CW complex structure of $\mathrm{WGr}(d,n)$, where $r_0 = c_0$.*

Proof Note that the sets $P(\widetilde{E}(\lambda^i))$ and $P(M_d(n,d)) = \bigsqcup_{i=0}^m P(\widetilde{E}(\lambda^i))$ are invariant under the weighted $\mathbb{C}^*$–action defined in Remark 2.3 for all $i = 0,1,\dots,m$. Then we have the commutative diagram

$$\begin{tikzcd} P(\widetilde{X}_k) \arrow[r, phantom, "\subset"] \arrow[d, "\pi'_{c/r_k}"] & \mathbb{C}^{k+1} \setminus \{0\} \arrow[d, "\pi'_{c/r_k}"] \\ \pi'_{c/r_k}(P(\widetilde{X}_k)) \arrow[r, phantom, "\subset"] & \mathbb{W}P\left(\dfrac{c_0}{r_k}, \dfrac{c_1}{r_k}, \dots, \dfrac{c_k}{r_k}\right) \end{tikzcd}$$

Thus the first part follows from

$$\pi'_{c/r_k}(P(\widetilde{X}_k)) = \pi'_{c/r_k}\left(\bigsqcup_{i=0}^{k} P(\widetilde{E}(\lambda^i))\right) = \bigsqcup_{i=0}^{k} \pi'_{c/r_k}(P(\widetilde{E}(\lambda^i))) = \bigsqcup_{i=0}^{k} \frac{P(\widetilde{E}(\lambda^i))}{\sim_{c/r_k}} \cong \bigsqcup_{i=0}^{k} \frac{E(\lambda^i)}{G(c_i/r_k)}.$$

The second part follows from $\mathrm{WGr}(d,n) \cong \pi'_c(P(\widetilde{X}_m))$ and by applying Lemma 3.3 successively for every $k \in \{1,2,\dots,m\}$. □

We show that two weighted Grassmann orbifolds are weakly equivariantly homeomorphic if the associated weight vectors differ by a permutation $\sigma \in S_n$. Let X and Y be two G–spaces. A map $f: X \to Y$ is called a weakly equivariant homeomorphism if f is a homeomorphism and $f(gx) = \eta(g) f(x)$ for some $\eta \in \mathrm{Aut}(G)$ and for all $(g,x) \in G \times X$. If η is the identity, then f is called an equivariant homeomorphism.

Let $W = (w_1, w_2, \dots, w_n) \in (\mathbb{Z}_{\geq 0})^n$, $0 < a \in \mathbb{Z}$ and $\sigma W := (w_{\sigma_1}, w_{\sigma_2}, \dots, w_{\sigma_n})$ for some $\sigma \in S_n$. Consider two weighted Grassmann orbifolds $\mathrm{WGr}(d,n)$ and $\mathrm{WGr}'(d,n)$ associated to (W,a) and $(\sigma W, a)$, respectively. The group $(\mathbb{C}^*)^n$ acts on $\mathrm{WGr}(d,n)$ described in (2-7). Also, there exists a $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}'(d,n)$ defined by

$$(t_1,\dots,t_n)[(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)^{\mathrm{tr}}] := [(t_{\sigma_1}\boldsymbol{a}_1, t_{\sigma_2}\boldsymbol{a}_2, \dots, t_{\sigma_n}\boldsymbol{a}_n)^{\mathrm{tr}}]. \tag{3-1}$$

Theorem 3.5 *There exists a weakly equivariantly homeomorphism between* $\mathrm{WGr}(d,n)$ *and* $\mathrm{WGr}'(d,n)$*. Moreover, this may induce different q–CW complex structures on* $\mathrm{WGr}(d,n)$ *for different σ.*

Proof The matrix $A = (a_{ij}) \in M_d(n,d)$ if and only if $\sigma A = (a_{\sigma_i j}) \in M_d(n,d)$. Thus the natural weakly equivariant homeomorphism $\bar{f}_\sigma : M_d(n,d) \to M_d(n,d)$ defined by $\bar{f}_\sigma(A) = \sigma A$ induces the commutative diagram

$$\begin{array}{ccc} M_d(n,d) & \xrightarrow{\bar{f}_\sigma} & M_d(n,d) \\ \downarrow{\scriptstyle \pi_w} & & \downarrow{\scriptstyle \pi_{\sigma w}} \\ \mathrm{WGr}(d,n) & \xrightarrow{f_\sigma} & \mathrm{WGr}'(d,n) \end{array} \tag{3-2}$$

Here π_w is the quotient map defined in Definition 2.1. Thus, (3-2) induces a weakly equivariant homeomorphism $f_\sigma : \mathrm{WGr}(d,n) \to \mathrm{WGr}'(d,n)$, where $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}(d,n)$ is defined in (2-7) and the $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}'(d,n)$ is defined in (3-1). Note that $f_\sigma([A]_{\sim w}) = [\sigma A]_{\sim \sigma w}$.

We discuss the effects of the permutation σ on the q–CW complex structure on $\mathrm{WGr}(d,n)$. Consider $\mathbb{C}^i = \{(x_1, x_2, \dots, x_n) \in \mathbb{C}^n \mid x_j = 0 \text{ for } j > i\}$. For $\sigma \in S_n$, define

$$\sigma\mathbb{C}^n := \{(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n})\} \quad \text{and} \quad \sigma\mathbb{C}^i := \{(x_{\sigma_1}, x_{\sigma_2}, \dots, x_{\sigma_n}) \in \sigma\mathbb{C}^n \mid x_{\sigma_j} = 0 \text{ for } \sigma_j > i\}.$$

Let $\lambda = (\lambda_1, \dots, \lambda_d)$ be a Schubert symbol for $d < n$. Then

$$\begin{aligned} \sigma E(\lambda) &= \{\sigma Y \mid Y \in E(\lambda)\} \\ &= \{X \in \mathrm{Gr}(d,n) \mid \dim(X \cap \sigma\mathbb{C}^{\lambda_i}) = i \text{ and } \dim(X \cap \sigma\mathbb{C}^{\lambda_i - 1}) = i-1 \text{ for } i \in [d]\}, \end{aligned}$$

where $[d] = \{1,2,\dots,d\}$. Then $E(\lambda) \cong \sigma E(\lambda)$ and $\dim(\sigma E(\lambda)) = \ell(\lambda)$.

So the permutation of the coordinates in $\mathbb{C}^n$ determines another CW complex structure for $\mathrm{Gr}(d,n)$ given by $\mathrm{Gr}(d,n) = \sigma\,\mathrm{Gr}(d,n) = \bigsqcup_{i=0}^{m} \sigma E(\lambda^i)$. This induces the following decomposition of $M_d(n,d)$, similar to (2-15):

$$M_d(n,d) = \bigsqcup_{i=0}^{m} \sigma \widetilde{E}(\lambda^i) \quad \text{and} \quad P(M_d(n,d)) = \bigsqcup_{i=0}^{m} P(\sigma \widetilde{E}(\lambda^i)).$$

Recall that $\lambda^i = (\lambda^i_1, \dots, \lambda^i_d)$ is a Schubert symbol and c_i is defined in (2-5) for $i = 0, \dots, m$. Then $\sigma\lambda^i := (\sigma(\lambda^i_{i_1}), \dots, \sigma(\lambda^i_{i_d}))$, where $i_1, \dots, i_d \in \{1, \dots, d\}$ and $\sigma(\lambda^i_{i_1}) < \sigma(\lambda^i_{i_2}) < \dots < \sigma(\lambda^i_{i_d})$. Let

$$\sigma c_i := a + \sum_{j=1}^{d} w_{\sigma(\lambda^i_{i_j})}. \tag{3-3}$$

Now from the commutativity of the diagram (2-12), we have

$$\pi_w(\sigma(\widetilde{E}(\lambda^i))) \cong \mathrm{Pl}_w(\pi_w(\sigma \widetilde{E}(\lambda^i))) = \frac{P(\sigma \widetilde{E}(\lambda^i))}{\text{weighted } \mathbb{C}^*\text{–action}}.$$

There exists a homeomorphism

$$P(\sigma \widetilde{E}(\lambda^i) \cong \sigma E(\lambda^i) \times \mathbb{C}^*$$

defined by $P(\sigma A) \to (\pi(\sigma A), \det(A_{\sigma\lambda^i}))$. This is a $\mathbb{C}^*$–equivariant homomorphism, where the weighted $\mathbb{C}^*$–action on the left side is same as the weighted $\mathbb{C}^*$–action on $\mathbb{C}^{m+1} \setminus \{0\}$, and the weighted $\mathbb{C}^*$–action on the right side is defined by

$$t \cdot (\pi(\sigma A), s) = (t \cdot \pi(\sigma A), t^{\sigma c_i} s),$$

where $t \cdot \pi(\sigma A)$ is defined in (2-16). Then using Lemma 2.6, we have

$$\frac{P(\sigma \widetilde{E}(\lambda^i))}{\text{weighted } \mathbb{C}^*\text{–action}} \cong \frac{\sigma E(\lambda^i)}{G(\sigma c_i)}.$$

Then we get a q–CW complex structure of the weighted Grassmann orbifold $\mathrm{WGr}(d,n)$ given by

$$\mathrm{WGr}(d,n) \cong \frac{\sigma E(\lambda^0)}{G(\sigma c_0)} \sqcup \frac{\sigma E(\lambda^1)}{G(\sigma c_1)} \sqcup \dots \sqcup \frac{\sigma E(\lambda^m)}{G(\sigma c_m)}. \qquad \square$$

Remark 3.6 Applying the permutation σ on the rows of the matrices in $E(\lambda)$, we get the matrices of $\sigma E(\lambda)$. That is,

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in E(\lambda) \iff \begin{pmatrix} v_{\sigma_1} \\ v_{\sigma_2} \\ \vdots \\ v_{\sigma_n} \end{pmatrix} \in \sigma E(\lambda).$$

Proposition 3.7 [4, Theorem 1.1] *Let X be a q–CW complex with no odd-dimensional q–cells, and p a prime number. Let $\{\mathrm{pt}\} = X_0 \subseteq X_1 \subseteq \dots \subseteq X_s = X$ be a filtration of X such that X_i is obtained by attaching the q–cell $\mathbb{R}^{2k_i}/G_i$ to X_{i-1} for all $i \in \{1, 2, \dots, s\}$. If $\gcd\{p, |G_i|\} = 1$ for all $i \in \{1, 2, \dots, s\}$, then $H^*(X;\mathbb{Z})$ has no p–torsion and $H^{\mathrm{odd}}(X;\mathbb{Z}_p)$ is trivial.*

Recall σc_i as defined in (3-3) for $\mathrm{WGr}(d,n)$ associated to weight vector $W=(w_1,\dots,w_n)\in(\mathbb{Z}_{\geq 0})^n$ and $1\leq a\in\mathbb{Z}$.

Definition 3.8 A permutation $\sigma\in S_n$ is called admissible for a prime p and $\mathrm{WGr}(d,n)$ if

$$\gcd\left\{p,\frac{\sigma c_i}{d_i}\right\}=1,$$

where σc_i is defined in (3-3) and $d_i=\gcd\{\sigma c_0,\sigma c_1,\dots,\sigma c_i\}$ for $i\in\{1,2,\dots,m\}$.

Some examples of admissible permutations are discussed in Example 3.12.

Remark 3.9 There may not always exist an admissible permutation $\sigma\in S_n$ for a prime p and $\mathrm{WGr}(d,n)$. However if $d=1$, then $m=n-1$ and there always exists an admissible permutation $\sigma\in S_n$ for every prime p. The admissible permutation $\sigma\in S_n$ may not be unique.

The following result says when the integral cohomology of $\mathrm{WGr}(d,n)$ has no p–torsion.

Theorem 3.10 *If there exists an admissible permutation $\sigma\in S_n$ for a prime p and $\mathrm{WGr}(d,n)$, then $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ has no p–torsion and $H^{\mathrm{odd}}(\mathrm{WGr}(d,n);\mathbb{Z}_p)$ is trivial.*

Proof Suppose $\sigma\in S_n$ be an admissible permutation for p and $\mathrm{WGr}(d,n)$. Then

$$\gcd\left\{p,\frac{\sigma c_i}{d_i}\right\}=1$$

by Definition 3.8, where $d_i=\gcd\{\sigma c_0,\sigma c_1,\dots,\sigma c_i\}$ for all $i\in\{1,2,\dots,m\}$. By Theorem 3.5, we have the q–CW complex structure

$$\mathrm{WGr}(d,n)\cong\frac{\sigma E(\lambda^0)}{G(\sigma c_0)}\sqcup\frac{\sigma E(\lambda^1)}{G(\sigma c_1)}\sqcup\cdots\sqcup\frac{\sigma E(\lambda^m)}{G(\sigma c_m)},$$

where $\sigma E(\lambda^i)\cong E(\lambda^i)\cong\mathbb{C}^{\ell(\lambda^i)}$. Let

$$\sigma X_k=\bigsqcup_{i=0}^{k}\frac{\sigma E(\lambda^i)}{G(\sigma c_i)}\subseteq\mathrm{WGr}(d,n)\quad\text{for } k=0,1,\dots,m.$$

Then σX_k is a subcomplex of $\mathrm{WGr}(d,n)$ for $k=0,1,\dots,m$ and $\sigma X_m=\mathrm{WGr}(d,n)$. This gives a filtration

$$\{\mathrm{pt}\}=\sigma X_0\subset\sigma X_1\subset\cdots\subset\sigma X_m=\mathrm{WGr}(d,n)$$

such that $\sigma X_i\setminus\sigma X_{i-1}$ is homeomorphic to $\sigma E(\lambda^i)/G(\sigma c_i)$.

Using Lemma 3.3,

$$\frac{\sigma E(\lambda^i)}{G(\sigma c_i)}\cong\frac{\sigma E(\lambda^i)}{G(\sigma c_i/d_i)}.$$

That is, $\sigma X_i\setminus\sigma X_{i-1}$ is homeomorphic to $\mathbb{C}^{\ell(\lambda^i)}/G(\sigma c_i/d_i)$ for all $i=1,2,\dots,m$. Therefore, by Proposition 3.7, $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ has no p–torsion and the group $H^{\mathrm{odd}}(\mathrm{WGr}(d,n);\mathbb{Z}_p)$ is trivial. This completes the proof. ∎

Corollary 3.11 [19] *$H^*(\mathbb{W}P(c_0, c_1, \dots, c_m); \mathbb{Z})$ has no torsion.*

Proof This follows from Theorem 3.10 and Remarks 2.3 and 3.9. □

Example 3.12 Consider the weighted Grassmann orbifold $\mathrm{WGr}(2,4)$ for weight vector $W = (1,1,3,4)$ and $a = 2$. Here

$$n = 4, \quad d = 2, \quad \binom{n}{d} = 6, \quad m = \binom{n}{d} - 1 = 5.$$

So, in this case, we have six Schubert symbols, which are

$$\lambda^0 = (1,2) < \lambda^1 = (1,3) < \lambda^2 = (1,4) < \lambda^3 = (2,3) < \lambda^4 = (2,4) < \lambda^5 = (3,4),$$

ordered as in Definition 2.4. For the prime $p = 3$, consider the permutation $\sigma \in S_4$ defined by

$$\sigma_1 = 3, \quad \sigma_2 = 4, \quad \sigma_3 = 1, \quad \sigma_4 = 2.$$

Then

$$\sigma c_0 = 9, \quad \sigma c_1 = 6, \quad \sigma c_2 = 6, \quad \sigma c_3 = 7, \quad \sigma c_4 = 7, \quad \sigma c_5 = 4,$$

using (3-3). This σ is admissible for $p = 3$ and $\mathrm{WGr}(2,4)$. Thus $H^*(\mathrm{WGr}(2,4); \mathbb{Z})$ has no 3–torsion by Theorem 3.10.

For the prime $p = 7$, consider the permutation $\sigma \in S_4$ defined by

$$\sigma_1 = 4, \quad \sigma_2 = 2, \quad \sigma_3 = 1, \quad \sigma_4 = 3.$$

Then

$$\sigma c_0 = 7, \quad \sigma c_1 = 7, \quad \sigma c_2 = 9, \quad \sigma c_3 = 4, \quad \sigma c_4 = 6, \quad \sigma c_5 = 6,$$

using (3-3). This σ is admissible for $p = 7$ and $\mathrm{WGr}(2,4)$. Thus $H^*(\mathrm{WGr}(2,4); \mathbb{Z})$ has no 7–torsion by Theorem 3.10.

To compute that it has no 2–torsion, we need to consider a different total order on the Schubert symbols, given by

$$\lambda^0 = (1,2) < \lambda^1 = (1,3) < \lambda^2 = (2,3) < \lambda^3 = (1,4) < \lambda^4 = (2,4) < \lambda^5 = (3,4),$$

which preserves the partial order in (2-3). In this case, using (2-5),

$$c_0 = 4, \quad c_1 = 6, \quad c_2 = 6, \quad c_3 = 7, \quad c_4 = 7, \quad c_5 = 9.$$

The identity permutation in S_4 is admissible for $p = 2$ and this $\mathrm{WGr}(2,4)$. Then $H^*(\mathrm{WGr}(2,4); \mathbb{Z})$ has no 2–torsion by Theorem 3.10.

The only primes which divide the orders of the orbifold singularities of this $\mathrm{WGr}(2,4)$ are 2, 3 and 7. Hence the integral cohomology of $\mathrm{WGr}(2,4)$ of this example has no torsion. □

Remark 3.13 Considering the total order given in Definition 2.4 on the Schubert symbols, there may not exist an admissible permutation σ for a prime. However, one can take another total order on the Schubert symbols for which one can find σ satisfying the hypothesis in Theorem 3.10 for this prime.

The q–CW complex structure in Theorem 3.4 leads us to introduce the following definition, which generalizes the concept of divisive weighted projective spaces of [15].

Definition 3.14 A weighted Grassmann orbifold $\mathrm{WGr}(d,n)$ is called divisive if there exists $\sigma \in S_n$ such that σc_i divides σc_{i-1} for $i = 1, 2, \dots, m$, where σc_i is defined in (3-3).

Example 3.15 Consider the weighted Grassmann orbifold $\mathrm{WGr}(2,4)$ for weight vector $W = (1,6,1,1)$ and $a = 3$. We have the ordering on the six Schubert symbols given by

$$\lambda^0 = (1,2) < \lambda^1 = (1,3) < \lambda^2 = (1,4) < \lambda^3 = (2,3) < \lambda^4 = (2,4) < \lambda^5 = (3,4).$$

Consider the permutation $\sigma \in S_4$ defined by

$$\sigma_1 = 2, \quad \sigma_2 = 1, \quad \sigma_3 = 3, \quad \sigma_4 = 4.$$

Then

$$\sigma c_0 = 10, \quad \sigma c_1 = 10, \quad \sigma c_2 = 10, \quad \sigma c_3 = 5, \quad \sigma c_4 = 5, \quad \sigma c_5 = 5,$$

using (3-3). Thus σc_i divides σc_{i-1} for $i = 1, 2, \dots, 5$. So $\mathrm{WGr}(2,4)$ of this example is divisive. $\square$

Example 3.16 Let α and γ be any two nonnegative integers and β be any positive integer such that $\beta > d\alpha$. Let $\mathrm{WGr}(d,n)$ be the corresponding weighted Grassmann orbifold for $W = (\alpha+\gamma\beta, \alpha, \dots, \alpha) \in (\mathbb{Z}_{\geq 0})^n$ and $a = \beta - d\alpha > 0$. Consider the total order $\{\lambda^0, \lambda^1, \dots, \lambda^m\}$ on the Schubert symbols induced by the dictionary order. Then

$$c_i = \begin{cases} (\gamma+1)\beta & \text{if } i = 0, 1, \dots, \binom{n-1}{d-1} - 1, \\ \beta & \text{if } i = \binom{n-1}{d-1}, \dots, m. \end{cases}$$

Then c_i divides c_{i-1} for all $i = 1, 2, \dots, m$. Therefore this $\mathrm{WGr}(d,n)$ is a divisive weighted Grassmann orbifold. $\square$

Definition 3.17 Let λ be a Schubert symbol for $d < n$. Then a reversal of λ is a pair (k,k') such that $k \in \lambda$, $k' \notin \lambda$ and $k' < k$. We denote the set of all reversals of λ by $\mathrm{rev}(\lambda)$. If $(k,k') \in \mathrm{rev}(\lambda)$ then $(k,k')\lambda$ is the Schubert symbol obtained by replacing k by k' in λ and ordering the later set.

Remark 3.18 If $(k,k') \in \mathrm{rev}(\lambda)$ then $(k,k')\lambda \prec \lambda$ and $\ell(\lambda)$ is the cardinality of the set $\mathrm{rev}(\lambda)$ where $\ell(\lambda)$ is the length of λ. Knutson and Tao [20] and Abe and Matsumura [1] defined an inversion of a Schubert symbol λ as a pair (k,k') such that $k \in \lambda$, $k' \notin \lambda$ and $k < k'$. In some sense, our definition of reversal is dual to the definition of inversion. If $\mathrm{inv}(\lambda)$ is the set of all inversions of λ and $\ell'(\lambda)$ is the cardinality of the set $\mathrm{inv}(\lambda)$, then $\ell(\lambda) + \ell'(\lambda) = d(n-d)$. Also, if $(k,k') \in \mathrm{rev}(\lambda)$ and $(k,k')\lambda = \mu$, then $(k',k) \in \mathrm{inv}(\mu)$ and $(k',k)\mu = \lambda$.

Next, we discuss $(\mathbb{C}^*)^n$–action on some CW complex structure of a divisive weighted Grassmann orbifold. Recall the $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}(d,n)$ which is induced from (2-7). We retain the notation from Section 2.

Theorem 3.19 *If* $\mathrm{WGr}(d,n)$ *is a divisive weighted Grassmann orbifold, then it has a* $(\mathbb{C}^*)^n$*–invariant CW complex structure with cells* $\{\mathbb{C}^{\ell(\lambda^i)} \mid i = 0, 1, \dots, m\}$.

Proof Let $\mathrm{WGr}(d,n)$ be a divisive weighted Grassmann orbifold corresponding to weight vector $W = (w_1, \dots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ and $1 \leq a \in \mathbb{Z}$. Then there exists $\sigma \in S_n$ such that σc_i divides σc_{i-1} for all $i = 1, 2, \dots, m$. Let us assume that $\sigma = \mathrm{Id}$ (the identity permutation in S_n). Then c_i divides c_{i-1} for all $i = 1, 2, \dots, m$. Then $\gcd\{c_0, c_1, \dots, c_i\} = c_i$ for all $i \in \{1, 2, \dots, m\}$. Thus,

$$\pi_w(\widetilde{E}(\lambda^i)) \cong \frac{E(\lambda^i)}{G(c_i)} \cong \frac{E(\lambda^i)}{G(c_i/c_i)} \cong E(\lambda^i) \quad \text{for all } i = 1, 2, \dots, m,$$

by Lemma 3.3. Thus, each element of $\pi_w(\widetilde{E}(\lambda^i))$ can be represented uniquely by the equivalence class of an $n \times d$ matrix defined in (2-13).

Let $\lambda^i = (\lambda_1, \dots, \lambda_d)$ be a Schubert symbol for $d < n$ and let $\boldsymbol{z} \in \mathbb{C}^{\ell(\lambda^i)}$. Since

$$\ell(\lambda^i) = (\lambda_1 - 1) + (\lambda_2 - 2) + \dots + (\lambda_d - d),$$

we can write $\boldsymbol{z} = (\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_d)$, where

$$\boldsymbol{z}_l = (z_1^l, z_2^l, \dots, \widehat{z_{\lambda_1}^l}, \dots, \widehat{z_{\lambda_2}^l}, \dots, \widehat{z_{\lambda_{l-1}}^l}, \dots, z_{\lambda_l - 1}^l) \quad \text{for } l = 1, \dots, d.$$

For $(t_1, \dots, t_n) \in (\mathbb{C}^*)^n$, we define $s \in \mathbb{C}^*$ such that $s^{c_i} = t_{\lambda_1} \cdots t_{\lambda_d}$. Define $T \in \mathrm{GL}(d, \mathbb{C})$ by

$$T = \mathrm{diag}\left(\left(\frac{t_{\lambda_1}}{s^{w_{\lambda_1}}}\right), \left(\frac{t_{\lambda_2}}{s^{w_{\lambda_2}}}\right), \dots, \left(\frac{t_{\lambda_d}}{s^{w_{\lambda_d}}}\right)\right).$$

Then $\det(T) = s^a$.

Define $g_{\lambda^i} : \mathbb{C}^{\ell(\lambda^i)} \to \pi_w(\widetilde{E}(\lambda^i))$ by

$$g_{\lambda^i}(\boldsymbol{z}) := \begin{bmatrix} z_1^1 & z_1^2 & \cdots & z_1^d \\ \vdots & \vdots & & \vdots \\ z_{\lambda_1-1}^1 & z_{\lambda_1-1}^2 & \cdots & z_{\lambda_1-1}^d \\ 1 & 0 & \cdots & 0 \\ 0 & z_{\lambda_1+1}^2 & \cdots & z_{\lambda_1+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & z_{\lambda_2-1}^2 & \cdots & z_{\lambda_2-1}^d \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & z_{\lambda_2+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & z_{\lambda_d-1}^d \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then g_{λ^i} is a homeomorphism. Now we have

$$(t_1,t_2,\dots,t_n)g_{\lambda^i}(z)=\begin{bmatrix} t_1z_1^1 & t_1z_1^2 & \dots & t_1z_1^d \\ \vdots & \vdots & & \vdots \\ t_{\lambda_1-1}z_{\lambda_1-1}^1 & t_{\lambda_1-1}z_{\lambda_1-1}^2 & \dots & t_{\lambda_1-1}z_{\lambda_1-1}^d \\ t_{\lambda_1} & 0 & \dots & 0 \\ 0 & t_{\lambda_1+1}z_{\lambda_1+1}^2 & \dots & t_{\lambda_1+1}z_{\lambda_1+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & t_{\lambda_2-1}z_{\lambda_2-1}^2 & \dots & t_{\lambda_2-1}z_{\lambda_2-1}^d \\ 0 & t_{\lambda_2} & \dots & 0 \\ 0 & 0 & \dots & t_{\lambda_2+1}z_{\lambda_2+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & t_{\lambda_d-1}z_{\lambda_d-1}^d \\ 0 & 0 & \dots & t_{\lambda_d} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then

$$(t_1,t_2,\dots,t_n)g_{\lambda^i}(z)=\begin{bmatrix} \frac{s^{w_{\lambda_1}}}{t_{\lambda_1}}t_1z_1^1 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}}t_1z_1^2 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_1z_1^d \\ \vdots & \vdots & & \vdots \\ \frac{s^{w_{\lambda_1}}}{t_{\lambda_1}}t_{\lambda_1-1}z_{\lambda_1-1}^1 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}}t_{\lambda_1-1}z_{\lambda_1-1}^2 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_1-1}z_{\lambda_1-1}^d \\ \frac{s^{w_{\lambda_1}}}{t_{\lambda_1}}t_{\lambda_1} & 0 & \dots & 0 \\ 0 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}}t_{\lambda_1+1}z_{\lambda_1+1}^2 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_1+1}z_{\lambda_1+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}}t_{\lambda_2-1}z_{\lambda_2-1}^2 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_2-1}z_{\lambda_2-1}^d \\ 0 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2}}t_{\lambda_2} & \dots & 0 \\ 0 & 0 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_2+1}z_{\lambda_2+1}^d \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_d-1}z_{\lambda_d-1}^d \\ 0 & 0 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d}}t_{\lambda_d} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}\times T.$$

Thus, $(t_1, t_2, \dots, t_n) g_{\lambda^i}(z)$ is equal to

$$D \times \begin{bmatrix} \frac{s^{w_{\lambda_1}}}{t_{\lambda_1} s^{w_1}} t_1 z_1^1 & \frac{s^{w_{\lambda_2}}}{t_{\lambda_2} s^{w_1}} t_1 z_1^2 & \dots & \frac{s^{w_{\lambda_d}}}{t_{\lambda_d} s^{w_1}} t_1 z_1^d \\ \vdots & \vdots & & \vdots \\ \frac{s^{w_{\lambda_1}}}{s^{w_{\lambda_1-1}} t_{\lambda_1}} t_{\lambda_1-1} z^1_{\lambda_1-1} & \frac{s^{w_{\lambda_2}}}{s^{w_{\lambda_1-1}} t_{\lambda_2}} t_{\lambda_1-1} z^2_{\lambda_1-1} & \dots & \frac{s^{w_{\lambda_d}}}{s^{w_{\lambda_1-1}} t_{\lambda_d}} t_{\lambda_1-1} z^d_{\lambda_1-1} \\ 1 & 0 & \dots & 0 \\ 0 & \frac{s^{w_{\lambda_2}}}{s^{w_{\lambda_1+1}} t_{\lambda_2}} t_{\lambda_1+1} z^2_{\lambda_1+1} & \dots & \frac{s^{w_{\lambda_d}}}{s^{w_{\lambda_1+1}} t_{\lambda_d}} t_{\lambda_1+1} z^d_{\lambda_1+1} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{s^{w_{\lambda_2}}}{s^{w_{\lambda_2-1}} t_{\lambda_2}} t_{\lambda_2-1} z^2_{\lambda_2-1} & \dots & \frac{s^{w_{\lambda_d}}}{s^{w_{\lambda_2-1}} t_{\lambda_d}} t_{\lambda_2-1} z^d_{\lambda_2-1} \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & \frac{s^{w_{\lambda_d}}}{s^{w_{\lambda_2+1}} t_{\lambda_d}} t_{\lambda_2+1} z^d_{\lambda_2+1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{s^{w_{\lambda_d}}}{s^{w_{\lambda_d-1}} t_{\lambda_d}} t_{\lambda_d-1} z^d_{\lambda_d-1} \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \times T = DMT,$$

where $D = \operatorname{diag}(s^{w_1}, \dots, s^{w_n})$ is a diagonal matrix. So by the equivalence relation $\sim_w$ as in Definition 2.1,

$$(t_1, t_2, \dots, t_n) g_{\lambda^i}(z) = M \in \pi_w(\widetilde{E}(\lambda^i)) \subset \mathrm{WGr}(d, n).$$

Let a_{kl} be the coefficient of z_k^l in the matrix M for $1 \le l \le d$, $1 \le k \le \lambda_l - 1$, $k \ne \lambda_1, \lambda_2, \dots, \lambda_{l-1}$. Then

$$a_{kl} = \frac{s^{w_{\lambda_l}} t_k}{s^{w_k} t_{\lambda_l}}.$$

Now for $1 \le k \le \lambda_l - 1$ with $k \ne \lambda_1, \lambda_2, \dots, \lambda_{l-1}$ we have $(\lambda_l, k) \in \operatorname{rev}(\lambda^i)$. Let $\lambda^j = (\lambda_l, k)\lambda^i$. Note that $\lambda^j < \lambda^i$. Recall c_i from (2-5). So

$$\frac{t_k s^{w_{\lambda_l}}}{s^{w_k} t_{\lambda_l}} = \frac{t_{\lambda^j}}{t_{\lambda^i}} s^{w_{\lambda_l} - w_k} = \frac{t_{\lambda^j}}{t_{\lambda^i}} s^{c_i - c_j} = \frac{t_{\lambda^j}}{t_{\lambda^i}} t_{\lambda^i}^{(c_i - c_j)/c_i} = t_{\lambda^j} (t_{\lambda^i})^{-c_j/c_i},$$

since $s^{c_i} = t_{\lambda_1} \cdots t_{\lambda_d} = t_{\lambda^i}$ and $t_{\lambda^j} = t_{\lambda_1} \cdots t_{\lambda_{l-1}} t_k t_{\lambda_{l+1}} \cdots t_{\lambda_d}$. Since $\mathrm{WGr}(d, n)$ is divisive and $\lambda^j < \lambda^i$, we have that c_i divides c_j.

Define a $(\mathbb{C}^*)^n$–action on $\mathbb{C}^{\ell(\lambda^i)}$ by

$$(t_1, t_2, \dots, t_n)(z_k^l) = (t_{\lambda^j} (t_{\lambda^i})^{-c_j/c_i} z_k^l)$$

for $1 \le l \le d, 1 \le k \le \lambda_l - 1, k \ne \lambda_1, \lambda_2, \dots, \lambda_{l-1}$. With this action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^{\ell(\lambda^i)}$, the map g_{λ^i} becomes $(\mathbb{C}^*)^n$–equivariant.

If $\sigma \neq \mathrm{Id}$, consider the cell

$$\pi_w(\sigma \widetilde{E}(\lambda^i)) \cong \frac{\sigma E(\lambda^i)}{G(\sigma c_i)} \cong \frac{\sigma E(\lambda^i)}{G(\sigma c_i/\sigma c_i)} \cong \sigma E(\lambda^i) \quad \text{for all } i = 1, 2, \dots, m,$$

by Lemma 3.3. Hence, we get the map $\sigma g_{\lambda^i} : \mathbb{C}^{\ell(\lambda^i)} \to \pi_w(\sigma \widetilde{E}(\lambda^i))$ defined by $z \to \sigma g_{\lambda^i}(z)$. Then by similar arguments, we get the $(\mathbb{C}^*)^n$–action on $\mathbb{C}^{\ell(\lambda^i)}$ defined by

$$(t_1, t_2, \dots, t_n)(z_k^l) = (t_{\sigma\lambda^j}(t_{\sigma\lambda^i})^{-\sigma c_j/\sigma c_i} z_k^l). \tag{3-4}$$

□

Corollary 3.20 *If* $\mathrm{WGr}(d,n)$ *is divisive, then* $H^*(\mathrm{WGr}(d,n);\mathbb{Z})$ *has no torsion and is concentrated in even degrees.*

We remark that Corollary 3.20 also follows from the proof of Theorem 3.10 and Definition 3.14. However, Theorem 3.19 describes the representation of the $(\mathbb{C}^*)^n$–action on each invariant cell explicitly. We also get that a divisive weighted Grassmann orbifold is integrally equivariantly formal.

4 Equivariant cohomology, cobordism and K–theory of weighted Grassmann orbifolds

In this section, first we compute the equivariant K–theory ring of any weighted Grassmann orbifold with rational coefficients. Then we compute the equivariant cohomology ring, equivariant K–theory ring and equivariant cobordism ring of a divisive weighted Grassmann orbifold with integer coefficients. We discuss the computation of the equivariant Euler classes for some line bundles on a point. We also compute the integral equivariant cohomology ring of some nondivisive weighted Grassmann orbifolds. We retain the notation of previous sections.

We recall the $(\mathbb{C}^*)^n$–action on $\mathrm{WGr}(d,n)$ which is induced by (2-7). Consider the standard torus $T^n = (S^1)^n \subset (\mathbb{C}^*)^n$. So we have the restricted T^n–action on $\mathrm{WGr}(d,n)$. For each Schubert symbol $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$, consider $C(\lambda) \in M_d(n,d)$ with column vectors given by $e_{\lambda_1}, e_{\lambda_2}, \dots, e_{\lambda_d}$, where $\{e_1, e_2, \dots, e_n\}$ is the standard basis for $\mathbb{C}^n$. Therefore $[C(\lambda)] \in \mathrm{WGr}(d,n)$, and it is a fixed point of the T^n–action on $\mathrm{WGr}(d,n)$.

Proposition 4.1 *Let* $\mathrm{WGr}(d,n)$ *be a weighted Grassmann orbifold corresponding to weight vector* $W = (w_1, w_2, \dots, w_n) \in (\mathbb{Z}_{\geq 0})^n$ *and* $a \geq 1$. *Then there is a* $(\mathbb{C}^*)^n$*–invariant stratification*

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = \mathrm{WGr}(d,n)$$

such that for $i = 1, \dots, m$, *the quotient* X_i/X_{i-1} *is homeomorphic to the Thom space* $\mathrm{Th}(\xi^i)$ *of an orbifold* $(\mathbb{C}^*)^n$*–vector bundle*

$$\xi^i : \mathbb{C}^{\ell(\lambda^i)}/G(c_i) \to [C(\lambda^i)], \tag{4-1}$$

where $G(c_i)$ *is the cyclic group of the* c_i^{th} *roots of unity.*

Proof Recall the $(\mathbb{C}^*)^n$–invariant stratification

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = \mathrm{WGr}(d,n)$$

from (2-17), which is obtained from the q–CW complex structure of $\mathrm{WGr}(d,n)$ as in Proposition 2.7. Note that X_i/X_{i-1} is the one-point compactification of $E(\lambda^i)/G(c_i)$, which is the Thom space of the orbifold $(\mathbb{C}^*)^n$–vector bundle

$$\frac{E(\lambda^i)}{G(c_i)} \to [C(\lambda^i)],$$

where $[C(\lambda^i)]$ is the $(\mathbb{C}^*)^n$–fixed point corresponding to the Schubert symbol λ^i for $i = 1, \dots, m$. It remains to note that $E(\lambda^i)$ is $(\mathbb{C}^*)^n$–equivariantly homeomorphic to $\mathbb{C}^{\ell(\lambda^i)}$; see (2-13). □

Now corresponding to $\mathrm{rev}(\lambda^i)$, one can define a subset of Schubert symbols

$$R(\lambda^i) := \{\lambda^j \mid \lambda^j = (k,k')\lambda^i \text{ for } (k,k') \in \mathrm{rev}(\lambda^i)\}. \tag{4-2}$$

Then the cardinality of the set $R(\lambda^i)$ is $\ell(\lambda^i)$ for every $i \in \{0, 1, \dots, m\}$. Note that the bundle in (4-1) is also an orbifold T^n–bundle.

Proposition 4.2 *The orbifold T^n–bundle in (4-1) has a decomposition*

$$\xi^i : \frac{\mathbb{C}^{\ell(\lambda^i)}}{G(c_i)} \to [C(\lambda^i)] \cong \bigoplus_{j\,:\,\lambda^j \in R(\lambda^i)} \left(\xi^{ij} : \frac{\mathbb{C}_{ij}}{G(c_{ij})} \to [C(\lambda^i)]\right).$$

Proof Observe that

$$X_i \setminus X_{i-1} = \frac{E(\lambda^i)}{G(c_i)} \cong \frac{\mathbb{C}^{\ell(\lambda^i)}}{G(c_i)}.$$

Since T^n is abelian, the T^n action on $E(\lambda^i) \cong \mathbb{C}^{\ell(\lambda^i)}$ determines the decomposition

$$E(\lambda^i) \cong \bigoplus_{j:\lambda^j \in R(\lambda^i)} \mathbb{C}_{ij}$$

for some irreducible representation $\mathbb{C}_{ij}$ of T^n. By [10, Proposition 2.8] there exists a finite covering map $q\colon T^n \to T^n$ such that the projection map $\phi\colon E(\lambda^i) \to E(\lambda^i)/G(c_i)$ is equivariant via the map q, ie $\phi(tx) = q(t)\phi(x)$. Therefore,

$$\frac{E(\lambda^i)}{G(c_i)} \cong \bigoplus_{j:\lambda^j \in R(\lambda^i)} \frac{\mathbb{C}_{ij}}{G(c_{ij})}$$

for some positive integers c_{ij} which divide c_i. Hence the proof follows. □

Remark 4.3 (1) The attaching map $\eta_i : S(\xi^i) \to X_{i-1}$ for the q–CW complex structure in (2-17) satisfies $\eta_i|_{S(\xi^{ij})} = f_{ij} \circ \xi^{ij}$, where $f_{ij} : [C(\lambda^i)] \to [C(\lambda^j)]$ is the constant map.

(2) The equivariant Euler classes $\{e_{T^n}(\xi^{ij}) \mid j < i\}$ are nonzero divisors. They are pairwise prime by [14, Lemma 5.2] and the T^n–action on $E(\lambda^i)$ discussed in the proof of Theorem 3.19.

Theorem 4.4 *Let* $\mathrm{WGr}(d,n)$ *be a weighted Grassmann orbifold for* $d<n$*, corresponding to weight vector* $W=(w_1,w_2,\dots,w_n)\in(\mathbb{Z}_{\geq 0})^n$ *and* $a\geq 1$*. Then the generalized* T^n*–equivariant cohomology* $\mathcal{E}^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Q})$ *can be given by*

$$\left\{(f_i)\in\bigoplus_{i=0}^{m}\mathcal{E}^*_{T^n}(\{\mathrm{pt}\};\mathbb{Q})\;\middle|\; e_{T^n}(\xi^{ij}) \text{ divides } f_i-f_j \text{ for } j<i \text{ and } |\lambda^j\cap\lambda^i|=d-1\right\}$$

for $\mathcal{E}^*_{T^n}=K^*_{T^n},\,H^*_{T^n}$*, where* $e_{T^n}(\xi^{ij})$ *represents the equivariant Euler class of* ξ^{ij}*.*

Proof This follows from [26, Proposition 2.3] using Propositions 4.1 and 4.2, and Remark 4.3. □

We note that equivariant cohomology ring of $\mathrm{WGr}(d,n)$ with rational coefficients is discussed in [1]. In the rest, we give a description of the equivariant cohomology ring, equivariant K–theory ring and equivariant cobordism ring of a divisive weighted Grassmann orbifold with integer coefficients.

Proposition 4.5 *Let* $\mathrm{WGr}(d,n)$ *be a divisive weighted Grassmann orbifold for* $d<n$ *corresponding to* $W=(w_1,w_2,\dots,w_n)\in(\mathbb{Z}_{\geq 0})^n$ *and* $a\geq 1$*. Then there is a* T^n*–invariant stratification*

$$\{\mathrm{pt}\}=X_0\subset X_1\subset\cdots\subset X_m=\mathrm{WGr}(d,n)$$

such that for $i=1,\dots,m$*, the quotient* X_i/X_{i-1} *is homeomorphic to the Thom space* $\mathrm{Th}(\xi^i)$ *of the* T^n*–vector bundle*

$$\xi^i:\mathbb{C}^{\ell(\lambda^i)}\to[C(\lambda^i)].$$

Proof Since $\mathrm{WGr}(d,n)$ is divisive, there exists $\sigma\in S_n$ such that σc_i divides σc_{i-1} for $i=1,2,\dots,m$. Then $\gcd\{\sigma c_0,\sigma c_1,\dots,\sigma c_i\}=\sigma c_i$ for all i. By Theorem 3.5, one can write

$$\mathrm{WGr}(d,n)=\bigsqcup_{i=0}^{m}\frac{\sigma E(\lambda^i)}{G(\sigma c_i)}.$$

By Lemma 3.3, the q–cell $\sigma E(\lambda^i)/G(\sigma c_i)$ is homeomorphic to $\sigma E(\lambda^i)/G(\sigma c_i/\sigma c_i)\cong\mathbb{C}^{\ell(\lambda^i)}$ for $i=1,\dots,m$. Let $X_k=\bigsqcup_{i=0}^{k}\sigma E(\lambda^i)/G(\sigma c_i)$ for $i=0,1,\dots,m$. The rest follows from the proof of Proposition 4.1. □

Remark 4.6 For a divisive weighted Grassmann orbifold, Proposition 4.2 and Remark 4.3 hold with $c_{ij}=1$ for every $j<i$.

Theorem 4.7 *Let* $\mathrm{WGr}(d,n)$ *be a divisive weighted Grassmann orbifold for* $d<n$*. Then the generalized* T^n*–equivariant cohomology* $\mathcal{E}^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ *can be given by*

$$\left\{(f_i)\in\bigoplus_{i=0}^{m}\mathcal{E}^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})\;\middle|\; e_{T^n}(\xi^{ij}) \text{ divides } f_i-f_j \text{ for } j<i \text{ and } |\lambda^j\cap\lambda^i|=d-1\right\}$$

for $\mathcal{E}^*_{T^n}=H^*_{T^n},\,K^*_{T^n}$ *and* $\mathrm{MU}^*_{T^n}$*.*

Proof This follows from Proposition 4.5, Remark 4.6 and [14, Theorem 2.3]. □

Remark 4.8 Let λ^i and λ^j be two Schubert symbols with $j < i$. If $\mathrm{WGr}(d,n)$ is a divisive weighted Grassmann orbifold then there exists a permutation $\sigma \in S_n$ such that σc_i divides σc_j. We write

$$\sigma d_{ij} := \frac{\sigma c_j}{\sigma c_i} \in \mathbb{Z}.$$

Next we discuss how to compute $e_{T^n}(\xi^{ij})$. We recall that

$$H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z}) = H^*(BT^n;\mathbb{Z}) \cong \mathbb{Z}[y_1, y_2, \ldots, y_n],$$

where $y_1, y_2, \ldots, y_n$ be the standard basis of $H^2(BT^n;\mathbb{Z})$. Using (3-4) the character of the one-dimensional representation for the bundle ξ^{ij} is given by

$$(t_1, t_2, \ldots, t_n) \to t_{\sigma\lambda^j}(t_{\sigma\lambda^i})^{-\sigma c_j/\sigma c_i}. \tag{4-3}$$

Also,

$$K^*_{T^n}(\{\mathrm{pt}\}) \cong R(T^n)[z, z^{-1}],$$

where $R(T^n)$ is the complex representation ring of T^n and z is the Bott element in $K^{-2}(\{\mathrm{pt}\})$. Note that the ring $R(T^n)$ is isomorphic to the ring of Laurent polynomials with n variables, ie $R(T^n) \cong \mathbb{Z}[\alpha_1, \ldots, \alpha_n]_{(\alpha_1\cdots\alpha_n)}$, where α_i is the irreducible representation corresponding to the projection on the i^{th} factor; see [17]. Therefore, using (4-3), one has, for $j < i$ and $|\lambda^j \cap \lambda^i| = d-1$,

$$e_{T^n}(\xi^{ij}) = \begin{cases} 1 - \alpha_{\sigma\lambda^j}\alpha_{\sigma\lambda^i}^{-\sigma d_{ij}} & \text{in } K^0_{T^n}(\{\mathrm{pt}\};\mathbb{Z}), \\ e_{T^n}(\alpha_{\sigma\lambda^j}\alpha_{\sigma\lambda^i}^{-\sigma d_{ij}}) & \text{in } \mathrm{MU}^2_{T^n}(\{\mathrm{pt}\};\mathbb{Z}), \\ Y_{\sigma\lambda^j} - \sigma d_{ij} Y_{\sigma\lambda^i} & \text{in } H^2_{T^n}(\{\mathrm{pt}\};\mathbb{Z}), \end{cases} \tag{4-4}$$

where $Y_\lambda := \sum_{i=1}^d y_{\lambda_i}$ and $\alpha_\lambda = \alpha_{\lambda_1}\cdots\alpha_{\lambda_d}$ for a Schubert symbol $\lambda = (\lambda_1, \ldots, \lambda_d)$.

We remark that the structure of $\mathrm{MU}^*_{T^n}(\{\mathrm{pt}\})$ is unknown; however, it is referred to in [15] as the ring of *T^n–cobordism forms*.

Example 4.9 Consider the weighted Grassmann orbifold $\mathrm{WGr}(2,4)$ for $W = (12,2,2,2)$ and $a = 6$. We have the ordering on the six Schubert symbols given by

$$\lambda^0 = (1,2) < \lambda^1 = (1,3) < \lambda^2 = (1,4) < \lambda^3 = (2,3) < \lambda^4 = (2,4) < \lambda^5 = (3,4).$$

Then $c_0 = 20, c_1 = 20, c_2 = 20, c_3 = 10, c_4 = 10, c_5 = 10$ from (2-5). Here c_i divides c_{i-1} for all $i = 1,2,3,4,5$. Thus, $\mathrm{WGr}(2,4)$ is divisive for the identity permutation in S_4. Then $d_{ij} = c_j/c_i$ in Remark 4.8 gives

$$d_{ij} = \begin{cases} 1 & \text{if } j < i \text{ and both } i, j \in \{0,1,2\} \text{ or } \{3,4,5\}, \\ 2 & \text{if } j \in \{0,1,2\} \text{ and } i \in \{3,4,5\}. \end{cases}$$

Then one can calculate the equivariant Euler class $e_{T^n}(\xi^{ij})$ from (4-4). The generalized integral equivariant cohomology ring $\mathscr{E}^*_{T^n}(\mathrm{WGr}(2,4);\mathbb{Z})$ of this divisive weighted Grassmann orbifold $\mathrm{WGr}(2,4)$ can be described by Theorem 4.7. □

The fixed points of the T^n–action on $\mathrm{WGr}(d,n)$ are $V := \{[C(\lambda^i)]\}_{i=0}^m$. Two fixed points $[C(\lambda^i)]$ and $[C(\lambda^j)]$ are connected by a T^n–invariant $\mathbb{W}P(c_i,c_j) \subset \mathrm{WGr}(d,n)$ if and only if $\lambda^j = (k,k')\lambda^i$ for some (k,k'), where $(k,k')\lambda^i$ is described in Definition 3.17. In that case it is said that there is an edge e_{ij} between $[C(\lambda^i)]$ and $[C(\lambda^j)]$. Let $E := \{e_{ij} \mid \lambda^j = (k,k')\lambda^i \text{ for some } (k,k')\}$. Then $\Gamma = (V,E)$ is a $d(n-d)$ valent graph with $(m+1)$–vertices. Consider the connection θ on Γ defined similarly as the GKM–graph of the Grassmann manifold in [12, Theorem 1.11.4, equation (1.34)]. Note that the T^n–action on $\mathbb{W}P(c_i,c_j)$ is given by $(t_1,\dots,t_n)[z_i:z_j] = [t_{\lambda^i}z_i : t_{\lambda^j}z_j]$. This action induces a map

$$\alpha\colon E \to H^*(BT^n;\mathbb{Q}) = \mathbb{Q}[y_1,\dots,y_n]$$

defined by $\alpha(e) := (c_iY_{\lambda^j} - c_jY_{\lambda^i})/c_i$ if e is the oriented edge from $[C(\lambda^i)]$ to $[C(\lambda^j)]$ with $|\lambda^j \cap \lambda^i| = d-1$. Note that if $\overline{e}$ is the edge with the opposite orientation on e then $\alpha(\overline{e}) = (c_jY_{\lambda^i} - c_iY_{\lambda^j})/c_j$. Let $r_e = c_i$ and $r_{\overline{e}} = c_j$. Then

$$r_e\alpha(e) = -r_{\overline{e}}\alpha(\overline{e}) \in H^2(BT^n;\mathbb{Z}). \tag{4-5}$$

Let e and e' be two edges with the same initial vertex. Let e' be the oriented edge from $[C(\lambda^i)]$ to $[C(\lambda^l)]$. Then we have

$$c_jc_l\big(\alpha(\theta_e(e')) - \alpha(e')\big) = 0 \bmod r_e\alpha(e).$$

The map α is called the axial function on Γ. Therefore, (Γ,α,θ) satisfies the definition of orbifold GKM–graph [8, Definition 2.2]. Hence, (Γ,α,θ) is the orbifold GKM–graph for the weighted Grassmann orbifold.

The following result gives equivariant cohomology ring of some nondivisive weighted Grassmann orbifolds with integer coefficients.

Theorem 4.10 *Suppose that* $\mathrm{WGr}(d,n)$ *is a weighted Grassmann orbifold corresponding to the order* $\lambda^0 < \cdots < \lambda^m$ *such that* $r = \gcd\{c_0,c_1\}$ *and* $c_i | c_k$ *for* $k \le i$ *with* $i \ge 2$. *Then the integral equivariant cohomology ring of* $\mathrm{WGr}(d,n)$ *is given by*

$$\begin{aligned} &H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z}) \\ &= \Big\{(f_i) \in \bigoplus_{i=0}^m \mathbb{Z}[y_1,y_2,\dots,y_n] \;\Big|\; (Y_{\lambda^j} - d_{ij}Y_{\lambda^i}) \text{ divides } (f_i - f_j) \text{ if } j < i,\ |\lambda^j \cap \lambda^i| = d-1, \\ &\qquad\qquad (i,j) \neq (0,1) \text{ and } c_1Y_{\lambda^0} - c_0Y_{\lambda^1} \text{ divides } r(f_1 - f_0)\Big\}. \end{aligned}$$

Proof By the given condition $\gcd\{c_0,c_1,\dots,c_i\} = c_i$ for $i \ge 2$. So, by Lemma 3.3, $E(\lambda^i)/G(c_i)$ is homeomorphic to $E(\lambda^i)/G(c_i/c_i) \cong \mathbb{C}^{\ell(\lambda^i)}$ for $i = 1,\dots,m$. When $i = 1$, we have that X_1 is equivariantly homeomorphic to $\mathbb{W}P(c_0,c_1)$. Therefore, $\mathrm{WGr}(d,n)$ has a T^n–invariant CW complex structure. For the edge $e = e_{01}$, the minimum of r_e that satisfies (4-5) is r. Thus, by [8, Definition 2.3 and Theorem 2.9], we get the result. ∎

Next, we discuss the equivariant cohomology ring of the weighted projective space $\mathbb{W}P(b_0, b_1 \dots, b_m)$, where $(b_0, b_1 \dots, b_m) \in (\mathbb{Z}_{\geq 1})^{m+1}$, for several torus actions. By Remark 2.3, $\mathbb{W}P(b_0, b_1 \dots, b_m) = \mathrm{WGr}(1, m+1)$, where the latter is associated to the weight vector $W = (b_0 - 1, \dots, b_m - 1)$ and $a = 1$. The Schubert symbols for $1 < m+1$ are $\{1\}, \dots, \{m\}$ and $\{m+1\}$. Assume that $\mathrm{WGr}(1, m+1)$ is divisive corresponding to this order, ie b_i divides b_{i-1} for $i = 1, 2, \dots, m$. Then

$$E(i+1) \cong \{[(u_0, u_1, \dots, u_{i-1}, 1, 0, \dots, 0)] \in \mathbb{W}P(b_0, b_1 \dots, b_m)\} \cong \mathbb{C}^i \quad \text{for } i = 0, 1, \dots, m.$$

Let (n, d) be such that $d < n$ and $\binom{n}{d} = m+1$. Then (2-8) gives a T^n–action on $\mathbb{W}P(b_0, b_1 \dots, b_m)$. Recall t_{λ^i} from (2-8) for the Schubert symbols $\lambda^0, \lambda^1, \dots, \lambda^m$ corresponding to $d < n$. We have

$$\begin{aligned}(t_1, t_2, \dots, t_n)&[(u_0, u_1, \dots, u_{i-1}, 1, 0, \dots, 0)] \\ &= [(t_{\lambda^0} u_0, t_{\lambda^1} u_1, \dots, t_{\lambda^{i-1}} u_{i-1}, t_{\lambda^i}, 0, \dots, 0)] \\ &= [((t_{\lambda^i})^{-b_0/b_i} t_{\lambda^0} u_0, (t_{\lambda^i})^{-b_1/b_i} t_{\lambda^1} u_1, \dots, (t_{\lambda^i})^{-b_{i-1}/b_i} t_{\lambda^{i-1}} u_{i-1}, 1, 0, \dots, 0)].\end{aligned}$$

Then $E(i+1)$ is T^n–invariant as well as T^{m+1}–invariant. Let

$$X_i := [(u_0, u_1, \dots, u_i, 0, \dots, 0)] \in \mathbb{W}P(b_0, b_1 \dots, b_m)\}.$$

Then X_i gives a filtration

$$\{\mathrm{pt}\} = X_0 \subset X_1 \subset \cdots \subset X_m = \mathbb{W}P(b_0, b_1, \dots, b_m). \tag{4-6}$$

Note that the filtration in (4-6) satisfies Proposition 4.5 and Remark 4.6. Thus in this case

$$\xi^i : E(i+1) \to [e_{i+1}] \cong \bigoplus_{j=0}^{i} (\xi^{ij} : \mathbb{C}_{ij} \to [e_{i+1}])$$

for some irreducible representation $\mathbb{C}_{ij}$. Using the proof of [15, Theorem 2.3] one can get the following result.

Theorem 4.11 *If $\mathbb{W}P(b_0, \dots, b_m)$ is divisive, then the generalized T^n–equivariant cohomology*

$$\mathscr{E}^*_{T^n}(\mathbb{W}P(b_0, \dots, b_m); \mathbb{Z})$$

for $\mathscr{E}^*_{T^n} = H^*_{T^n}, K^*_{T^n}$ and $\mathrm{MU}^*_{T^n}$ can be given by

$$\left\{ (f_i) \in \bigoplus_{i=0}^{m} \mathscr{E}^*_{T^n}(\{\mathrm{pt}\}; \mathbb{Z}) \;\middle|\; e_{T^n}(\xi^{ij}) \text{ divides } f_i - f_j \text{ for all } j < i \right\}.$$

We note that there are several pairs (n, d) such that $d < n$ and $\binom{n}{d} = m+1 > 2$. Now we discuss how to calculate the equivariant Euler class $e_{T^n}(\xi^{ij})$ in Theorem 4.11. The corresponding one-dimensional representation on the bundle ξ^{ij} for $j < i$ is determined by the character

$$(t_1, \dots, t_n) \to (t_{\lambda^i})^{-b_j/b_i} t_{\lambda^j}.$$

Thus, similar to (4-4), one can calculate the equivariant Euler class $e_{T^n}(\xi^{ij})$ of the bundle ξ^{ij} for $j < i$.

Example 4.12 For $m = 2$, we have $\binom{3}{1} = \binom{3}{2} = 3$. Thus, corresponding to two different pairs $(3,1)$ and $(3,2)$, we have two different T^3 actions on $\mathbb{W}P(b_0, b_1, b_2)$. The map $f : T^3 \to T^3$ defined by $(t_1, t_2, t_3) \to (t_1t_2, t_1t_3, t_2t_3)$ is not an automorphism. So these actions are not equivalent. However, using Theorem 4.11, one can calculate the equivariant cohomology of $\mathbb{W}P(b_0, b_1, b_2)$ for both the actions if b_i divides b_{i-1} for $i = 1, 2$. □

5 Equivariant Schubert calculus for divisive weighted Grassmann orbifolds

In this section, we show that there exist equivariant Schubert classes which form a basis for the equivariant cohomology ring of a divisive weighted Grassmann orbifold with integer coefficients. We show some properties of the weighted structure constants. Moreover, we discuss some relations that help to compute the weighted structure constants corresponding to this equivariant Schubert basis with integer coefficients.

For $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n); \mathbb{Z})$, the support of x, denoted by supp(x), is the set of all Schubert symbols λ^i such that $\mathrm{x}|_{\lambda^i} \neq 0$. Recall the partial order $\preceq$ on the Schubert symbols defined in (2-3). We follow this partial order $\preceq$ and we say that an element $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n); \mathbb{Z})$ is supported above by λ^i if $\lambda^i \preceq \lambda^k$ for all $\lambda^k \in \mathrm{supp}(\mathrm{x})$.

Let $\mathrm{WGr}(d,n)$ be a divisive weighted Grassmann orbifold. Then there exists $\sigma \in S_n$ such that

$$\sigma c_i \text{ divides } \sigma c_{i-1} \quad \text{for } i = 1, 2, \ldots, m. \tag{5-1}$$

Using Theorem 3.5, it is sufficient to consider $\sigma = \mathrm{Id}$, the identity permutation on S_n. For $\sigma = \mathrm{Id}$, (5-1) transforms to

$$c_i \text{ divides } c_{i-1} \quad \text{for } i = 1, 2, \ldots, m.$$

Recall the definition of $R(\lambda^i)$ from (4-2). We introduce the following definition.

Definition 5.1 An element $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n); \mathbb{Z})$ is said to be an equivariant Schubert class corresponding to a Schubert symbol λ^i if the following conditions are satisfied:

(1) $\mathrm{x}|_{\lambda^k} \neq 0$ implies $\lambda^i \preceq \lambda^k$. (We say that x is supported above λ^i.)

(2) $\mathrm{x}|_{\lambda^i} = \prod_{\lambda^j \in R(\lambda^i)} (Y_{\lambda^j} - (c_j/c_i) Y_{\lambda^i})$.

(3) $\mathrm{x}|_{\lambda^k}$ is a homogeneous polynomial in $y_1, y_2, \ldots, y_n$ of degree $\ell(\lambda^i)$.

Proposition 5.2 (uniqueness) *For each Schubert symbol λ^i, there is at most one equivariant Schubert class* x *corresponding to λ^i.*

Proof Suppose that there were two distinct equivariant Schubert classes x and x′ corresponding to λ^i. Let λ^j be the minimal Schubert symbol such that $(\mathrm{x} - \mathrm{x}')|_{\lambda^j} \neq 0$. By Definition 5.1(1)–(2), we get $\lambda^i \prec \lambda^j$. Then from the condition in the expression of the equivariant cohomology ring in Theorem 4.7, we get that $(\mathrm{x} - \mathrm{x}')|_{\lambda^j}$ is a multiple of $\prod_{\lambda^k \in R(\lambda^j)} (Y_{\lambda^k} - (c_k/c_j) Y_{\lambda^j})$, which is of degree $\ell(\lambda^j)$. This contradicts the fact that $\mathrm{x} - \mathrm{x}'$ is homogeneous of degree $\ell(\lambda^i) < \ell(\lambda^j)$. □

Let us denote the equivariant Schubert class corresponding to the Schubert symbol λ^i by $w\widetilde{S}_{\lambda^i}$ for $i=0,1,\dots,m$. We remark that the existence of $w\widetilde{S}_{\lambda^i}$ follows from [14, Proposition 4.3] and Theorem 4.7. Geometrically, $w\widetilde{S}_{\lambda^i}$ is the equivariant cohomology class corresponding to the closure of the cell $\sigma_r E(\sigma_r\lambda^i)$, where $\sigma_r \in S_n$ is the permutation defined by

$$\sigma_r := \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix}.$$

Using the arguments in the proof of [20, Proposition 1], one gets the following.

Proposition 5.3 *The equivariant Schubert classes $\{w\widetilde{S}_{\lambda^i}\}_{i=0}^m$ form a basis for $H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ as a module over $H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$. Moreover, any $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ can be written uniquely as an $H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$–linear combination of $w\widetilde{S}_{\lambda^i}$ using only those λ^i such that $\lambda^j \le \lambda^i$ for some $\lambda^j \in \mathrm{supp}(\mathrm{x})$.*

Proof To check that the set $\{w\widetilde{S}_{\lambda^i}\}_{i=0}^m$ is linearly independent, let $\sum_{i=0}^m a_i w\widetilde{S}_{\lambda^i} = 0$ for coefficients $a_i \in H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$ that are not identically zero. Let

$$k = \min\{i \in \{0,1,\dots,m\} \mid a_i \neq 0\}.$$

We also have that $w\widetilde{S}_{\lambda^i}|_{\lambda^k} = 0$ for $i > k$. Thus the restriction $\big(\sum_{i=0}^m a_i w\widetilde{S}_{\lambda^i}\big)|_{\lambda^k} = a_k w\widetilde{S}_{\lambda^k}|_{\lambda^k} \neq 0$, which is a contradiction.

Now, to prove that $\{w\widetilde{S}_{\lambda^i}\}$ spans, consider an element $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$. Let

$$j := \min\{i \in \{0,1,\dots,m\} \mid \lambda^i \in \mathrm{supp}(\mathrm{x})\}.$$

Then $\mathrm{x}|_{\lambda^j} = \beta_j w\widetilde{S}_{\lambda^j}|_{\lambda^j}$ using Theorem 4.7 and (4-4) for some $\beta_j \in \mathbb{Z}[y_1,\dots,y_n]$. Subtracting $\beta_j w\widetilde{S}_{\lambda^j}$, we can inductively reduce support of x upwards until it is empty. This uses only those λ^i such that $\lambda^j \le \lambda^i$ for some $\lambda^j \in \mathrm{supp}(\mathrm{x})$. □

Example 5.4 In Figure 1, we compute the equivariant Schubert class $w\widetilde{S}_{(2,3)} \in H^*_{T^4}(\mathrm{WGr}(2,4);\mathbb{Z})$, where $\mathrm{WGr}(2,4)$ is a divisive weighted Grassmann orbifold for some $W = (\alpha+\gamma\beta, \alpha, \alpha, \alpha) \in (\mathbb{Z}_{\geq 0})^4$ and $a = \beta - 2\alpha \in \mathbb{Z}_{>0}$. Figure 1, left, is the lattice of the Schubert symbols for $2 < 4$. Figure 1, right, gives the equivariant Schubert class corresponding to the Schubert symbol $(2,3)$. □

In the rest of this section, we compute the weighted structure constants for the equivariant cohomology of a divisive weighted Grassmann orbifold. Since the set $\{w\widetilde{S}_{\lambda^i}\}_{i=0}^m$ form a $H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$–basis for $H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$, for any two λ^i and λ^j, one has that

$$w\widetilde{S}_{\lambda^i}\, w\widetilde{S}_{\lambda^j} = \sum_{\lambda^k} wc_{ij}^k\, w\widetilde{S}_{\lambda^k}, \tag{5-2}$$

where $\lambda^k \in \{\lambda^0,\lambda^1,\dots,\lambda^m\}$. The constant $wc_{ij}^k \in H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$ in the formula is called a *weighted structure constant.*

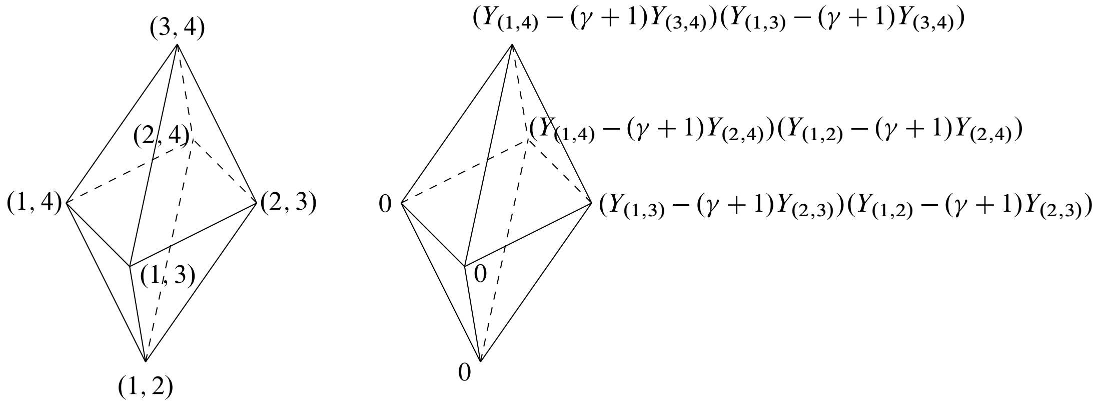


Figure 1

Lemma 5.5 *The weighted structure constants wc_{ij}^k have the following properties.*

(1) *The weighted structure constant wc_{ij}^k has degree $\ell(\lambda^i)+\ell(\lambda^j)-\ell(\lambda^k)$.*

(2) *The constant wc_{ij}^k is 0 unless $\ell(\lambda^k)\le \ell(\lambda^i)+\ell(\lambda^j)$ and $\lambda^k\succeq\lambda^i,\lambda^j$.*

(3) *When $i=k$, we have $wc_{ij}^i = w\widetilde{S}_{\lambda^j}|_{\lambda^i}$.*

Proof (1) The degree of $w\widetilde{S}_{\lambda^i}$ is $\ell(\lambda^i)$. So the degree of the weighted structure constant wc_{ij}^k is given by

$$\deg(wc_{ij}^k)=\deg(w\widetilde{S}_{\lambda^i})+\deg(w\widetilde{S}_{\lambda^j})-\deg(w\widetilde{S}_{\lambda^k})=\ell(\lambda^i)+\ell(\lambda^j)-\ell(\lambda^k).$$

(2) The weighted structure constant $wc_{ij}^k=0$ if $\ell(\lambda^i)+\ell(\lambda^j)-\ell(\lambda^k)<0$. Also,

$$(w\widetilde{S}_{\lambda^i}\ w\widetilde{S}_{\lambda^j})|_{\lambda^s}\neq 0 \implies \lambda^s\succeq\lambda^i,\lambda^j.$$

Thus, by Proposition 5.3, $wc_{ij}^k\neq 0$ implies $\lambda^k\succeq\lambda^i,\lambda^j$.

(3) Comparing the $(\lambda^i)^{\text{th}}$ component of the both sides in (5-2), we get

$$w\widetilde{S}_{\lambda^i}|_{\lambda^i}\ w\widetilde{S}_{\lambda^j}|_{\lambda^i}=wc_{ij}^i\ w\widetilde{S}_{\lambda^i}|_{\lambda^i}+\sum_{k\neq i} wc_{ij}^k\ w\widetilde{S}_{\lambda^k}|_{\lambda^i}.$$

We have that $wc_{ij}^k=0$ unless $\lambda^k\succeq\lambda^i$, but $w\widetilde{S}_{\lambda^k}|_{\lambda^i}=0$ for $\lambda^k\succeq\lambda^i$, and $\lambda^k\neq\lambda^i$. Thus all the terms in the summation vanish. So the claim follows, since $w\widetilde{S}_{\lambda^i}|_{\lambda^i}\neq 0$. □

Now we introduce the equivariant Schubert divisor class. Note that $\ell(\lambda^i)=0$ if and only if $i=0$, and $\ell(\lambda^i)=1$ if and only if $i=1$. The equivariant Schubert class corresponding to the Schubert symbol λ^1 is called the *equivariant Schubert divisor class*.

Lemma 5.6 *The equivariant Schubert divisor class $w\widetilde{S}_{\lambda^1}\in H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ is given by*

$$w\widetilde{S}_{\lambda^1}|_{\lambda^i}=Y_{\lambda^0}-\frac{c_0}{c_i}Y_{\lambda^i}.$$

Proof Consider an element $\mathrm{x} \in \bigoplus_{i=0}^{m} H^*_{T^n}(\{\mathrm{pt}\};\mathbb{Z})$ defined by $\mathrm{x}|_{\lambda^i} = Y_{\lambda^0} - (c_0/c_i)Y_{\lambda^i}$. Let λ^i and λ^j be two Schubert symbols such that $\lambda^j \preceq \lambda^i$. Then

$$\mathrm{x}|_{\lambda^i} - \mathrm{x}|_{\lambda^j} = \frac{c_0}{c_j}\left(Y_{\lambda^j} - \frac{c_j}{c_i}Y_{\lambda^i}\right).$$

Thus $\mathrm{x} \in H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ from Theorem 4.7 and (4-4). Note that $\mathrm{x}|_{\lambda^0} = 0$. If $\mathrm{x}|_{\lambda^k} \neq 0$ then $\lambda^1 \preceq \lambda^k$. Now $R(\lambda^1) = \{\lambda^0\}$ and

$$\mathrm{x}|_{\lambda^1} = Y_{\lambda^0} - \frac{c_0}{c_1}Y_{\lambda^1} = \prod_{\lambda^j \in R(\lambda^1)}\left(Y_{\lambda^j} - \frac{c_j}{c_1}Y_{\lambda^1}\right).$$

Also, $\mathrm{x}|_{\lambda^k}$ is a homogeneous polynomial of degree $1 = \ell(\lambda^1)$. Thus x satisfies all the conditions of Definition 5.1 for $i = 1$. Therefore, by the uniqueness of the equivariant Schubert classes, we have $\mathrm{x} = w\widetilde{S}_{\lambda^1}$. □

For any two Schubert symbols λ^i and λ^j, we write $\lambda^i \to \lambda^j$ if $\ell(\lambda^i) = \ell(\lambda^j) + 1$ and $\lambda^j \preceq \lambda^i$.

Proposition 5.7 (weighted Pieri rule) $\quad w\widetilde{S}_{\lambda^1}\, w\widetilde{S}_{\lambda^j} = (w\widetilde{S}_{\lambda^1}|_{\lambda^j})\, w\widetilde{S}_{\lambda^j} + \sum_{\lambda^i \to \lambda^j} \frac{c_0}{c_j} w\widetilde{S}_{\lambda^i}.$

Proof Using the fact that $\deg(w\widetilde{S}_{\lambda^1}) = 1$, we have

$$w\widetilde{S}_{\lambda^1}\, w\widetilde{S}_{\lambda^j} = (wc^j_{1j})\, w\widetilde{S}_{\lambda^j} + \sum_{\lambda^i \to \lambda^j} (wc^i_{1j})\, w\widetilde{S}_{\lambda^i}.$$

From Lemma 5.5, we get $wc^j_{1j} = w\widetilde{S}_{\lambda^1}|_{\lambda^j}$. Fix λ^i such that $\lambda^i \to \lambda^j$ and compare the $(\lambda^i)^{\text{th}}$ component of both sides; we get

$$\begin{aligned} w\widetilde{S}_{\lambda^1}|_{\lambda^i} w\widetilde{S}_{\lambda^j}|_{\lambda^i} &= (wc^j_{1j}) w\widetilde{S}_{\lambda^j}|_{\lambda^i} + (wc^i_{1j}) w\widetilde{S}_{\lambda^i}|_{\lambda^i} \\ \Longrightarrow \quad (wc^i_{1j}) w\widetilde{S}_{\lambda^i}|_{\lambda^i} &= (w\widetilde{S}_{\lambda^1}|_{\lambda^i} - w\widetilde{S}_{\lambda^1}|_{\lambda^j}) w\widetilde{S}_{\lambda^j}|_{\lambda^i} \\ \Longrightarrow \quad (wc^i_{1j}) w\widetilde{S}_{\lambda^i}|_{\lambda^i} &= \frac{c_0}{c_j}\left(Y_{\lambda^j} - \frac{c_j}{c_i}Y_{\lambda^i}\right) w\widetilde{S}_{\lambda^j}|_{\lambda^i}. \end{aligned}$$

Thus $wc^i_{1j} = c_0/c_j$ if $\lambda^i \to \lambda^j$. □

By applying Proposition 5.7 repeatedly, we can compute the following product, as well as the higher products:

$$\begin{aligned} (w\widetilde{S}_{\lambda^1})^2 w\widetilde{S}_{\lambda^j} &= w\widetilde{S}_{\lambda^1}\Big((w\widetilde{S}_{\lambda^1}|_{\lambda^j}) w\widetilde{S}_{\lambda^j} + \sum_{\lambda^i \to \lambda^j} \frac{c_0}{c_j} w\widetilde{S}_{\lambda^i}\Big) \\ &= (w\widetilde{S}_{\lambda^1}|_{\lambda^j})^2 w\widetilde{S}_{\lambda^j} + \sum_{\lambda^i \to \lambda^j} (w\widetilde{S}_{\lambda^1}|_{\lambda^j}) \frac{c_0}{c_j} w\widetilde{S}_{\lambda^i} + \sum_{\lambda^i \to \lambda^j} \frac{c_0}{c_j} (w\widetilde{S}_{\lambda^1}|_{\lambda^i}) w\widetilde{S}_{\lambda^i} \\ &\qquad + \sum_{\lambda^k \to \lambda^i \to \lambda^j} \frac{c_0}{c_j}\frac{c_0}{c_i} w\widetilde{S}_{\lambda^k}. \end{aligned}$$

Proposition 5.8 *For any three Schubert symbols λ^i, λ^j and λ^k, we have the recurrence relation*

$$(w\widetilde{S}_{\lambda^1}|_{\lambda^k} - w\widetilde{S}_{\lambda^1}|_{\lambda^i})wc^k_{ij} = \sum_{\lambda^s \to \lambda^i} \frac{c_0}{c_i} wc^k_{sj} - \sum_{\lambda^k \to \lambda^t} \frac{c_0}{c_t} wc^t_{ij}.$$

Proof We use the associativity of the multiplication in $H^*_{T^n}(\mathrm{WGr}(d,n);\mathbb{Z})$ and weighted Pieri rule to expand $w\widetilde{S}_{\lambda^1} w\widetilde{S}_{\lambda^i} w\widetilde{S}_{\lambda^j}$ in two different ways:

$$\begin{aligned}(w\widetilde{S}_{\lambda^1} w\widetilde{S}_{\lambda^i})w\widetilde{S}_{\lambda^j} &= ((w\widetilde{S}_{\lambda^1}|_{\lambda^i})w\widetilde{S}_{\lambda^i} + \sum_{\lambda^s \to \lambda^i} \frac{c_0}{c_i} w\widetilde{S}_{\lambda^s})w\widetilde{S}_{\lambda^j}, \\ &= (w\widetilde{S}_{\lambda^1}|_{\lambda^i}) \sum_{\lambda^l} wc^l_{ij} w\widetilde{S}_{\lambda^l} + \sum_{\lambda^s \to \lambda^i} \frac{c_0}{c_i} \sum_{\lambda^l} wc^l_{sj} w\widetilde{S}_{\lambda^l},\end{aligned} \tag{5-3}$$

$$w\widetilde{S}_{\lambda^1}(w\widetilde{S}_{\lambda^i} w\widetilde{S}_{\lambda^j}) = w\widetilde{S}_{\lambda^1} \sum_{\lambda^l} wc^l_{ij} w\widetilde{S}_{\lambda^l} = \sum_{\lambda^l} wc^l_{ij} \Big((w\widetilde{S}_{\lambda^1}|_{\lambda^l})w\widetilde{S}_{\lambda^l} + \sum_{\lambda^r \to \lambda^l} \frac{c_0}{c_l} w\widetilde{S}_{\lambda^r}\Big). \tag{5-4}$$

Comparing the coefficient of $w\widetilde{S}_{\lambda^k}$ in (5-3) and (5-4) we get

$$(w\widetilde{S}_{\lambda^1}|_{\lambda^i})wc^k_{ij} + \sum_{\lambda^s \to \lambda^i} \frac{c_0}{c_i} wc^k_{sj} = wc^k_{ij}(w\widetilde{S}_{\lambda^1}|_{\lambda^k}) + \sum_{\lambda^k \to \lambda^t} \frac{c_0}{c_t} wc^t_{ij}. \qquad \square$$

Acknowledgements The authors would like to thank the referee for several comments and suggestions which helped to improve the manuscript. Brahma thanks the Indian Institute of Technology Madras for a PhD fellowship. Sarkar thanks the Indian Institute of Technology Madras for a SEED research grant.

References

[1] **H Abe**, **T Matsumura**, *Equivariant cohomology of weighted Grassmannians and weighted Schubert classes*, Int. Math. Res. Not. 2015 (2015) 2499–2524 MR Zbl

[2] **A Adem**, **J Leida**, **Y Ruan**, *Orbifolds and stringy topology*, Cambridge Tracts in Math. 171, Cambridge Univ. Press (2007) MR Zbl

[3] **A Bahri**, **M Franz**, **N Ray**, *The equivariant cohomology ring of weighted projective space*, Math. Proc. Cambridge Philos. Soc. 146 (2009) 395–405 MR Zbl

[4] **A Bahri**, **D Notbohm**, **S Sarkar**, **J Song**, *On integral cohomology of certain orbifolds*, Int. Math. Res. Not. 2021 (2021) 4140–4168 MR Zbl

[5] **W L Baily, Jr**, *The decomposition theorem for V–manifolds*, Amer. J. Math. 78 (1956) 862–888 MR Zbl

[6] **W Chen**, **Y Ruan**, *A new cohomology theory of orbifold*, Comm. Math. Phys. 248 (2004) 1–31 MR Zbl

[7] **A Corti**, **M Reid**, *Weighted Grassmannians*, from "Algebraic geometry" (M C Beltrametti, F Catanese, C Ciliberto, A Lanteri, C Pedrini, editors), de Gruyter, Berlin (2002) 141–163 MR Zbl

[8] **A Darby**, **S Kuroki**, **J Song**, *Equivariant cohomology of torus orbifolds*, Canad. J. Math. 74 (2022) 299–328 MR Zbl

[9] **T tom Dieck**, *Bordism of G–manifolds and integrality theorems*, Topology 9 (1970) 345–358 MR Zbl

[10] **F Galaz-García**, **M Kerin**, **M Radeschi**, **M Wiemeler**, *Torus orbifolds, slice-maximal torus actions, and rational ellipticity*, Int. Math. Res. Not. 2018 (2018) 5786–5822 MR Zbl

[11] **R M Goresky**, *Triangulation of stratified objects*, Proc. Amer. Math. Soc. 72 (1978) 193–200 MR Zbl

[12] **V Guillemin**, **C Zara**, *1–skeleta, Betti numbers, and equivariant cohomology*, Duke Math. J. 107 (2001) 283–349 MR Zbl

[13] **B Hanke**, *Geometric versus homotopy theoretic equivariant bordism*, Math. Ann. 332 (2005) 677–696 MR Zbl

[14] **M Harada**, **A Henriques**, **T S Holm**, *Computation of generalized equivariant cohomologies of Kac–Moody flag varieties*, Adv. Math. 197 (2005) 198–221 MR Zbl

[15] **M Harada**, **T S Holm**, **N Ray**, **G Williams**, *The equivariant K–theory and cobordism rings of divisive weighted projective spaces*, Tohoku Math. J. 68 (2016) 487–513 MR Zbl

[16] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR Zbl

[17] **D Husemoller**, *Fibre bundles*, McGraw-Hill, New York (1966) MR Zbl

[18] **X Jiao**, **J Peng**, *Pseudo-holomorphic curves in complex Grassmann manifolds*, Trans. Amer. Math. Soc. 355 (2003) 3715–3726 MR Zbl

[19] **T Kawasaki**, *Cohomology of twisted projective spaces and lens complexes*, Math. Ann. 206 (1973) 243–248 MR Zbl

[20] **A Knutson**, **T Tao**, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. 119 (2003) 221–260 MR Zbl

[21] **D Laksov**, *Algebraic cycles on Grassmann varieties*, Adv. Math. 9 (1972) 267–295 MR Zbl

[22] **J P May**, *Equivariant homotopy and cohomology theory*, CBMS Reg. Conf. Ser. Math. 91, Amer. Math. Soc., Providence, RI (1996) MR Zbl

[23] **J W Milnor**, **J D Stasheff**, *Characteristic classes*, Ann. of Math. Stud. 76, Princeton Univ. Press (1974) MR Zbl

[24] **A Mukherjee**, *Differential topology*, 2nd edition, Hindustan, New Delhi (2015) MR Zbl

[25] **M Poddar**, **S Sarkar**, *On quasitoric orbifolds*, Osaka J. Math. 47 (2010) 1055–1076 MR Zbl

[26] **S Sarkar**, **J Song**, *GKM theory for orbifold stratified spaces and application to singular toric varieties*, Topology Appl. 288 (2021) art. id. 107472 MR Zbl

[27] **I Satake**, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. 42 (1956) 359–363 MR Zbl

[28] **I Satake**, *The Gauss–Bonnet theorem for V–manifolds*, J. Math. Soc. Japan 9 (1957) 464–492 MR Zbl

[29] **G Segal**, *Equivariant K–theory*, Inst. Hautes Études Sci. Publ. Math. 34 (1968) 129–151 MR Zbl

[30] **D P Sinha**, *Computations of complex equivariant bordism rings*, Amer. J. Math. 123 (2001) 577–605 MR Zbl

[31] **W P Thurston**, *The geometry and topology of three-manifolds*, lecture notes, Princeton Univ. (1979) Available at `https://url.msp.org/gt3m`

*Department of Mathematics, Indian Institute of Technology Madras
Chennai, India*

*Department of Mathematics, Indian Institute of Technology Madras
Chennai, India*

koushikbrahma95@gmail.com, soumen@iitm.ac.in

Received: 25 June 2022 Revised: 27 February 2023

Algebraic & Geometric Topology **24**:4 (2024) 2245–2284
DOI: 10.2140/agt.2024.24.2245
Published: 16 July 2024

Projective modules and the homotopy classification of (G,n)–complexes

JOHN NICHOLSON

A (G,n)–complex is an n–dimensional CW–complex with fundamental group G and whose universal cover is $(n-1)$–connected. If G has periodic cohomology then, for appropriate n, we show that there is a one-to-one correspondence between the homotopy types of finite (G,n)–complexes and the orbits of the stable class of a certain projective $\mathbb{Z}G$–module under the action of $\mathrm{Aut}(G)$. We develop techniques to compute this action explicitly and use this to give an example where the action is nontrivial.

55P15; 20C05, 55U15, 57K20

1 Introduction

For a group G and $n \geq 2$, a (G,n)–*complex* is a connected n–dimensional CW–complex X for which $\pi_1(X) \cong G$ and $\widetilde{X}$ is $(n-1)$–connected. Equivalently, it is the n–skeleton of a $K(G,1)$. For example, a finite $(G,2)$–complex is equivalently a finite 2–complex X with $\pi_1(X) \cong G$. An example of a finite $(G,3)$–complex is a closed 3–manifold M with $\pi_1(M) \cong G$ finite. Given a group G and $n \geq 2$, a finite (G,n)–complex exists if and only if G has type F_n in the sense of Wall [1965].

Let $\mathrm{HT}(G,n)$ be the set of homotopy types of finite (G,n)–complexes, which can be viewed as a graph with edges between each X and $X \vee S^n$. It is well known that $\mathrm{HT}(G,n)$ is a tree [Whitehead 1939], i.e. a connected acyclic graph, and has a grading coming from $(-1)^n\chi(X)$ which takes a minimum value $\chi_{\min}(G,n)$. The problem of determining the structure of $\mathrm{HT}(G,n)$ as a tree has a long history which dates back to Cockcroft and Swan [1961] and Dyer and Sieradski [1973; 1975].

In the case of finite abelian groups, the structure of $\mathrm{HT}(G,n)$ has been classified through a series of articles by Metzler [1976], Sieradski and Dyer [1979], Browning [1979] and Linnell [1993]. However, much less is known for nonabelian groups and an important class of examples are the groups with k–periodic cohomology, i.e. finite groups for which the Tate cohomology groups satisfy $\widehat{H}^i(G;\mathbb{Z}) \cong \widehat{H}^{i+k}(G;\mathbb{Z})$ for all $i \in \mathbb{Z}$. For example, if G is finite and n is even, then it was shown by Browning [1978] that $\chi(X) = \chi(Y)$ implies $X \vee S^n \simeq Y \vee S^n$ (see also [Hambleton and Kreck 1993]). However, when n is odd, this is known only when G does not have k–periodic cohomology for $k \mid n+1$ (see Question 7.4).

The aim of this article is to make new progress towards the classification over groups with periodic cohomology, building upon work of Dyer [1976] and Johnson [2003].

1.1 Main results

Let $\mathrm{PHT}(G,n)$ be the tree of polarised homotopy types of finite (G,n)–complexes, i.e. the homotopy types of pairs (X,ρ) where $\rho\colon \pi_1(X)\cong G$.

Let G be a finite group and let $C(\mathbb{Z}G)$ denote the projective class group, i.e. the equivalence classes of finitely generated projective $\mathbb{Z}G$–modules where $P\sim Q$ if $P\oplus\mathbb{Z}G^i\cong Q\oplus\mathbb{Z}G^j$ for some i and j. Note that a class $[P]\in C(\mathbb{Z}G)$ can be viewed as the set of (nonzero) projective $\mathbb{Z}G$–modules P_0 for which $P_0\sim P$, and this has the structure of a graded tree with edges between each P_0 and $P_0\oplus\mathbb{Z}G$. Let $T_G\le C(\mathbb{Z}G)$ denote the Swan subgroup (see Section 3.2). If G has k–periodic cohomology, then the *Swan finiteness obstruction* is an element $\sigma_k(G)\in C(\mathbb{Z}G)/T_G$ which vanishes if and only if there exists a finite CW–complex X with $\pi_1(X)\cong G$ and $\widetilde{X}\simeq S^{k-1}$.

Recall that a finitely presented group G has the *D2 property* if every cohomologically 2–dimensional finite complex X with $\pi_1(X)\cong G$ is homotopy equivalent to a finite 2–complex.

Theorem A *Let G have k–periodic cohomology and let $n=ik$ or $ik-2$ for some $i\ge 1$. Then there is an injective map of graded trees*

$$\Psi\colon \mathrm{PHT}(G,n)\to[P_{(G,n)}]$$

for any projective $\mathbb{Z}G$–module $P_{(G,n)}$ with $\sigma_{ik}(G)=[P_{(G,n)}]\in C(\mathbb{Z}G)/T_G$. Furthermore, Ψ is bijective if and only if $n\ge 3$ or if $n=2$ and G has the D2 *property.*

Remark 1.1 (a) If G satisfies the Eichler condition, then $[P_{(G,n)}]$ has *cancellation* in the sense that $P_1\oplus\mathbb{Z}G\cong P_2\oplus\mathbb{Z}G$ implies $P_1\cong P_2$ for all $P_1,\ P_2\in[P_{(G,n)}]$ (see [Jacobinski 1968]). This implies that $\mathrm{PHT}(G,n)$ and $\mathrm{HT}(G,n)$ have cancellation in the sense that $X\vee S^n\simeq Y\vee S^n$ implies $X\simeq Y$, and recovers the main result of Dyer [1976].

(b) An equivalent statement appeared in [Johnson 2003] in the case $n=2$, though the proof contained a small gap which was patched up in [Nicholson 2021b] using a theorem of Browning [1978].

Our proof is based on the work of Hambleton and Kreck [1993] and is independent of [Browning 1978; Johnson 2003]. After establishing preliminaries in Sections 2 and 3, we will prove general cancellation theorems for chain complexes of projective modules in Section 4. This suffices to prove Theorem A due to the correspondence between $\mathrm{PHT}(G,n)$ and the tree of algebraic n–complexes (see Proposition 5.1). In Theorem 5.3, we give a detailed version of Theorem A which contains an explicit description of the map Ψ.

We then use of this description of Ψ to determine the induced action of $\mathrm{Aut}(G)$ on $[P_{(G,n)}]$ via the bijection $\mathrm{HT}(G,n)\cong\mathrm{PHT}(G,n)/\mathrm{Aut}(G)$. To state the induced action, consider the following two operations for M a (left) projective $\mathbb{Z}G$–module:

(1) If $\theta\in\mathrm{Aut}(G)$, then let M_θ be the $\mathbb{Z}G$–module whose abelian group is that of M but with action $g\cdot x=\theta(g)x$ for $g\in G$ and $x\in M$ (see Lemma 6.1).

(2) If r represents a class in $(\mathbb{Z}/|G|)^\times$ and $I \subseteq \mathbb{Z}G$ is the augmentation ideal, then (I,r) is a projective $\mathbb{Z}G$–module. The tensor product $(I,r)\otimes M$ is a projective $\mathbb{Z}G$–module since (I,r) is a two-sided ideal (see Lemma 4.15).

In Section 6, we will prove the following which is the main result of this article. Note that every projective $\mathbb{Z}G$–module has the form $P\oplus\mathbb{Z}G^r$ where P has rank one and $r\geq 0$ (see Section 3.1).

Theorem B *Let G have k–periodic cohomology and let $n=ik$ or $ik-2$ for some $i\geq 1$. Then Ψ induces an injective map of graded trees*

$$\overline{\Psi}\colon \mathrm{HT}(G,n)\to [P_{(G,n)}]/\mathrm{Aut}(G),$$

where the action by $\theta\in\mathrm{Aut}(G)$ is given by

$$\theta\colon P\oplus\mathbb{Z}G^r\mapsto ((I,\psi_k(\theta)^i)\otimes P_\theta)\oplus\mathbb{Z}G^r,$$

where P has rank one, for some map $\psi_k\colon \mathrm{Aut}(G)\to(\mathbb{Z}/|G|)^\times$ which depends only on G and k. Furthermore, $\overline{\Psi}$ is bijective if and only if $n\geq 3$ or if $n=2$ and G has the D2 *property.*

This reduces the problem of determining when cancellation occurs in the homotopy trees to the purely algebraic problem of determining cancellation for $[P]$ and $[P]/\mathrm{Aut}(G)$ which will be dealt with in [Nicholson 2020].

1.2 Computing the action of Aut(G)

After proving Theorems A and B, the remainder of this article will be devoted to exploring the action of $\mathrm{Aut}(G)$ on $[P_{(G,n)}]$. This includes establishing some general theory in preparation for the more detailed computations in [Nicholson 2020].

First, and perhaps somewhat surprisingly, we could find no example where the $\mathrm{Aut}(G)$–action described in Theorem B does not take the form of the simpler action $P\mapsto P_\theta$. In all examples computed, we had $(I,\psi_k(\theta))\cong\mathbb{Z}G$ which implies that $(I,\psi_k(\theta)^i)\cong\mathbb{Z}G$. If $P\oplus\mathbb{Z}G^r\in[P_{(G,n)}]$ where P has rank one, then this implies that $\theta(P)\cong P_\theta\oplus\mathbb{Z}G^r\cong(P\oplus\mathbb{Z}G^r)_\theta$. In particular, $\theta(P)\cong P_\theta$ for all $P\in[P_{(G,n)}]$. We therefore ask the following:

Question 7.3 *Does there exist G with k–periodic cohomology and $\theta\in\mathrm{Aut}(G)$ for which $(I,\psi_k(\theta))$ is not free?*

There are two approaches to finding examples where $(I,\psi_k(\theta))$ is not free. The first is to find an example where $(I,\psi_k(\theta))$ is not even stably free. It was shown by Dyer [1976, page 276] and Davis [1983] that $(I,\psi_k(\theta))$ is stably free when $\sigma_k(G)=0$. Davis [1983, page 488] asked whether this also holds when $\sigma_k(G)\neq 0$. The second approach is to find an example where $(I,\psi_k(\theta))$ is stably free but not free. This is likely to be difficult since the general question of whether (I,r) can be stably free but not free is still open and dates back to Wall's problems list [1979b, Problem A4].

In Section 8, we develop a general method to compute the action $P \mapsto P_\theta$. We will then use this to give the following example where the action is nontrivial. Let Q_{4n} denote the quaternion group of order $4n$, which has 4–periodic cohomology. Since $\sigma_4(Q_{4n}) = 0$, we can take $[P_{(Q_{4n},2)}] = [\mathbb{Z} Q_{4n}] = \bigcup_{r \geq 1} \mathrm{SF}_r(\mathbb{Z} Q_{4n})$ where $\mathrm{SF}_r(\mathbb{Z} Q_{4n})$ is the set of stably free $\mathbb{Z} Q_{4n}$–modules of rank $r \geq 1$. As above, let $\theta \in \mathrm{Aut}(Q_{4n})$ act on $[\mathbb{Z} Q_{4n}]$ by $\theta\colon P \mapsto (I, \psi_4(\theta)^i) \otimes P_\theta$ for some $i \geq 1$. We show:

Theorem C $\mathrm{Aut}(Q_{24})$ *acts nontrivially on* $[\mathbb{Z} Q_{24}]$. *More specifically, we have* $|\mathrm{SF}_1(\mathbb{Z} Q_{24})| = 3$ *and* $|\mathrm{SF}_1(\mathbb{Z} Q_{24})/\mathrm{Aut}(Q_{24})| = 2$.

This is in contrast to the case Q_{4n} for $2 \leq n \leq 5$, where $|\mathrm{SF}_1(\mathbb{Z} Q_{4n})| = 1$, and the case Q_{28}, where $|\mathrm{SF}_1(\mathbb{Z} Q_{28})| = |\mathrm{SF}_1(\mathbb{Z} Q_{28})/\mathrm{Aut}(Q_{28})| = 2$ (see Table 1).

1.3 Overview of the wider project

This article is the first of a two-part series (followed by [Nicholson 2020]) in which we explore the classification of finite (G, n)–complexes over groups with periodic cohomology. These results are motivated by the following.

Wall's D2 problem for groups with 4–periodic cohomology In the language above, the D2 problem asks whether every finitely presented group G has the D2 property. This dates back to Wall's paper on finiteness conditions [1965] and is currently open. The case where G has 4–periodic cohomology was proposed to contain a counterexample to the D2 problem [Cohen 1977], and has since been studied extensively. In this case, Johnson [2003] proved Theorem A when $n = 2$ and, using results of Swan [1983], he established the D2 property for many new groups. In [Nicholson 2021a; 2021b], we extended these results and determined when $\mathrm{PHT}(G, 2)$ has cancellation.

In the case where $\mathrm{PHT}(G, 2)$ has noncancellation, the D2 property has only been proven for Q_{28} (see [Mannan and Popiel 2021; Nicholson 2021b]). This motivated Theorem B in the case $n = 2$ since one imagines it might be easier to prove that $\overline{\Psi}$ is bijective rather than Ψ. The question of when $\mathrm{HT}(G, 2)$ has cancellation is answered in [Nicholson 2020, Theorem A].

Stable and unstable classification of manifolds If X is a finite (G, n)–complex, then there exists an embedding $i\colon X \hookrightarrow \mathbb{R}^{2n+1}$. The boundary of a smooth regular neighbourhood of i gives a smooth closed $2n$–manifold $M(X)$. If X is determined up to simple homotopy, then $M(X)$ is well defined up to s–cobordism which coincides with homeomorphism in the case where G is finite by work of Freedman. Furthermore, $M(X \vee S^n) \cong M(X) \# (S^n \times S^n)$. This can be found in [Bokor et al. 2021, Section 5].

Kreck and Schafer [1984] used this to construct smooth closed $4n$–manifolds M_1 and M_2 for every $n \geq 1$ such that $M_1 \# (S^{2n} \times S^{2n}) \cong M_2 \# (S^{2n} \times S^{2n})$ are diffeomorphic but $M_1 \not\simeq M_2$. Their examples have the form $M(X_i)$ where the $X_i \in \mathrm{HT}(G, n)$ are the noncancellation examples for G abelian found by Metzler, Sieradski and Dyer [Metzler 1976; Sieradski 1977; Sieradski and Dyer 1979]. Recently, Conway, Crowley, Powell and Sixt constructed examples of both simply connected M_i [Conway et al. 2023] and infinitely many M_i [Conway et al. 2021] for all $n \geq 2$. However, the examples of Kreck and Schafer remain the only known examples in dimension 4. In classifying $\mathrm{HT}(G,n)$ when G has periodic cohomology, we hope to create a second family of examples both in dimension 4 and in higher dimensions.

Conventions

All rings R will be assumed to have a multiplicative identity and all R–modules will be assumed to be finitely generated left R–modules.

Recall that groups with periodic cohomology are necessarily finite. For most of this article, we will therefore restrict to the case where G is a finite group. However, we will briefly consider finitely presented groups more generally at the start of Sections 5 and 6.

Acknowledgements

I would like to thank F E A Johnson for his guidance and many interesting conversations. I would also like to thank Ian Hambleton, Jonathan Hillman and Wajid Mannan for helpful comments and references. Finally, I would like to thank the referee for their careful reading and suggestions which improved the exposition of this article. This work was supported by the UK Engineering and Physical Sciences Research Council (EPSRC) grant EP/N509577/1 as well as the Heilbronn Institute for Mathematical Research.

2 Extensions of modules

Let R be a ring. Recall our convention that all R–modules are assumed to be finitely generated left R–modules. For R–modules A and B, define $\mathrm{Ext}^n_R(A, B)$ to be the set of exact sequences

$$E = (0 \to B \xrightarrow{i} E_{n-1} \xrightarrow{\partial_{n-1}} E_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \xrightarrow{\varepsilon} A \to 0)$$

for R–modules E_i considered up to congruence, i.e. the equivalence relation generated by *elementary congruences* which are chain maps of the form

$$\begin{array}{c} E \\ \downarrow{\scriptstyle\varphi} \\ E' \end{array} \quad = \quad \left(\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{id}"] & E_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \arrow[r] & E_0 \arrow[r] \arrow[d, "\varphi_0"] & A \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & B \arrow[r] & E'_{n-1} \arrow[r] & \cdots \arrow[r] & E'_0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd} \right)$$

That is, two extensions E and E' are *congruent* if there exists extensions $E^{(i)}$ for $0 \leq i \leq n$ such that $E = E^{(0)}$, $E' = E^{(n)}$ and, for $i \leq n-1$, there exists an elementary congruence of the form $\varphi\colon E^{(i)} \to E^{(i+1)}$ or $\varphi\colon E^{(i+1)} \to E^{(i)}$.

We write extensions in $\mathrm{Ext}^n_R(A,B)$ as $E=(E_*,\partial_*)$ where the maps $i\colon B\to E_{n-1}$ and $\varepsilon\colon E_0\to A$ are understood. We will often write $\partial_i=\partial_i^E$, $i=i_E$ and $\varepsilon=\varepsilon_E$ when the need arises to distinguish different extensions.

This is an abelian group under Baer sum, and coincides with the usual definition of $\mathrm{Ext}^n_R(A,B)$ [Weibel 1994, Section 3.4]. We will assume familiarity with the basic operations on extensions such as pullback, pushout and the Yoneda product [Johnson 2003, Section 24].

Worth emphasising however is the operation of stabilisation. If $E=(E_*,\partial_*)\in\mathrm{Ext}^n_R(A,B)$, then define the *stabilised complex* $E\oplus R\in\mathrm{Ext}^n_R(A,B\oplus R)$ by

$$E\oplus R=(0\to B\oplus R\xrightarrow{\cdot\left(\begin{smallmatrix} i & 0\\ 0 & 1\end{smallmatrix}\right)} E_{n-1}\oplus R\xrightarrow{\cdot\left(\begin{smallmatrix} \partial_{n-1}\\ 0\end{smallmatrix}\right)} E_{n-2}\to\cdots\to E_0\to A\to 0).$$

This gives a well-defined map of abelian groups

$$-\oplus R\colon \mathrm{Ext}^n_R(A,B)\to\mathrm{Ext}^n_R(A,B\oplus R).$$

Let $\mathrm{Proj}^n_R(A,B)$ denote the subset of $\mathrm{Ext}^n_R(A,B)$ consisting of extensions (P_*,∂_*) with the P_i projective. This is closed under Baer sum, and so is a subgroup, and is also preserved by pullbacks, pushouts, the Yoneda product and stabilisation. The following is a consequence of the cocycle description of Ext [Wall 1979a, Lemma 1.1].

Lemma 2.1 (shifting) *If A, B, C and D are R–modules, $E\in\mathrm{Proj}^k_R(B,C)$ and $k,n,m\ge 1$, then the Yoneda product induces bijections*

$$-\circ E\colon \mathrm{Ext}^n_R(C,D)\to\mathrm{Ext}^{n+k}_R(B,D),\qquad E\circ -\colon \mathrm{Ext}^m_R(A,B)\to\mathrm{Ext}^{m+k}_R(A,C).$$

This can be viewed as a sort of cancellation theorem for extensions up to congruence in the sense that $F\circ E\cong F'\circ E$ or $E\circ F\cong E\circ F'$ implies that $F\cong F'$.

A simple consequence of this is the following lemma. This can be interpreted as a kind of duality theorem for projective extensions.

Lemma 2.2 (duality) *If A, B and C are R–modules, $F\in\mathrm{Proj}^k_R(A,C)$ and $k>n\ge 1$, then there are bijections*

$$\Psi_F\colon \mathrm{Proj}^n_R(A,B)\to\mathrm{Proj}^{k-n}_R(B,C),\qquad E\mapsto(-\circ E)^{-1}(F),$$
$$\Psi_F^{-1}\colon \mathrm{Proj}^{k-n}_R(B,C)\to\mathrm{Proj}^n_R(A,B),\qquad E'\mapsto(E'\circ -)^{-1}(F).$$

We now turn our attention to an equivalence relation on $\mathrm{Ext}^n_R(A,B)$ which is weaker than congruence. For R–modules A and B, and $E,E'\in\mathrm{Ext}^n_R(A,B)$, a chain map $\varphi\colon E\to E'$ is said to be a *chain homotopy equivalence* if the restriction to the unaugmented chain complexes $\varphi\colon (E_*,\partial_*)_{0\le *<n}\to(E'_*,\partial'_*)_{0\le *<n}$ is a chain homotopy equivalence.

If $E, E' \in \mathrm{Proj}_R^n(A,B)$ then, since a chain map between projective chain complexes is a chain homotopy equivalence if and only if it is a homology equivalence [Johnson 2003, Theorem 46.6], a chain homotopy equivalence $\varphi\colon E \to E'$ can equivalently be defined as a chain map of the form

$$\begin{array}{c} E \\ \big\downarrow{\scriptstyle\varphi} \\ E' \end{array} \;=\; \left(\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "\varphi_B"] & P_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "\varphi_0"] & A \arrow[r] \arrow[d, "\varphi_A"] & 0 \\ 0 \arrow[r] & B \arrow[r] & P'_{n-1} \arrow[r] & \cdots \arrow[r] & P'_0 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}\right)$$

where φ_A and φ_B are R–module isomorphisms. When convenient, we will often abbreviate this to $\varphi = (\varphi_B, \varphi_{n-1}, \dots, \varphi_0, \varphi_A)$. It follows easily that a congruence is a chain homotopy equivalence. We define $\mathrm{hProj}_R^n(A,B)$ to be set of equivalence classes in $\mathrm{Proj}_R^n(A,B)$ up to chain homotopy equivalences, which is an abelian group under Baer sum.

For special choices of modules, the shifting lemma and the duality lemma also hold for chain homotopy equivalences. We define $\mathbb{Z}$ to be the R–module with underlying abelian group $\mathbb{Z}$ and trivial R–action, i.e. $r \cdot n = n$ for all $r \in R$ and $n \in \mathbb{Z}$.

Lemma 2.3 (shifting) *If A and B are R–modules, $F \in \mathrm{Proj}_R^k(\mathbb{Z},\mathbb{Z})$ and $n, m, k \geq 1$, then the Yoneda product induces bijections*

$$-\circ F\colon \mathrm{hProj}_R^n(\mathbb{Z},A) \to \mathrm{hProj}_R^{n+k}(\mathbb{Z},A), \qquad F \circ -\colon \mathrm{hProj}_R^m(B,\mathbb{Z}) \to \mathrm{hProj}_R^{m+k}(B,\mathbb{Z}).$$

Proof First note that $-\circ F$ induces maps on the chain homotopy classes by extending the map to $\pm\,\mathrm{id}$ on F. This is necessarily surjective. To see that it is injective, suppose that there is a chain homotopy equivalence $\varphi\colon E_1 \circ F \to E_2 \circ F$. By considering $-\varphi$ if necessary, we can assume that $\varphi_{\mathbb{Z}} = \mathrm{id}$, so

$$E_2 \circ F \cong (\varphi_A)_*(E_1 \circ F) = (\varphi_A)_*(E_1) \circ F.$$

By Lemma 2.1, this implies that $E_2 \cong (\varphi_A)_*(E_1)$ and so $E_1 \simeq E_2$ as required. □

The proof of the duality lemma in this setting is similar and so will be omitted.

Lemma 2.4 (duality) *If A is an R–module, $F \in \mathrm{Proj}_R^k(\mathbb{Z},\mathbb{Z})$ and $k > n \geq 1$, then there are bijections*

$$\Psi_F\colon \mathrm{hProj}_R^n(\mathbb{Z},A) \to \mathrm{hProj}_R^{k-n}(A,\mathbb{Z}), \qquad E \mapsto (-\circ E)^{-1}(F),$$
$$\Psi_F^{-1}\colon \mathrm{hProj}_R^{k-n}(A,\mathbb{Z}) \to \mathrm{hProj}_R^n(\mathbb{Z},A), \qquad E' \mapsto (E' \circ -)^{-1}(F).$$

We now specialise to the case where the underlying abelian group of R is finitely generated and torsion-free, and where R is a ring with involution, i.e. a ring with an antiautomorphism $r \mapsto \bar{r}$ such that $\bar{\bar{r}} = r$ for all $r \in R$. For example, for a finite group G, the group ring $\mathbb{Z}G$ has underlying abelian group $\mathbb{Z}^{|G|}$ and involution $\sum_{i=1}^n n_i g_i \mapsto \sum_{i=1}^n n_i g_i^{-1}$ where $n_i \in \mathbb{Z}$ and $g_i \in G$. Using this involution, any right R–module A can be viewed as a left R–module under the action $r \cdot x = x \cdot \bar{r}$ for $r \in R$ and $x \in A$. If A is a left R–module, then $A^* = \mathrm{Hom}_R(A,R)$ is a right R–module under the action $(\varphi \cdot r)(x) = \varphi(x)r$ for $\varphi \in A^*$ and $r \in R$. We will view A^* as a left R–module using the involution on R.

Note that $(\cdot)^*$ can be viewed as a functor of R–modules: if $f\colon A_1 \to A_2$ is a map of R–modules, we can define $f^*\colon A_2^* \to A_1^*$ by $\varphi \mapsto \varphi \circ f$. For $E = (P_*, \partial_*) \in \mathrm{Proj}_R^n(A, B)$, define the *dual extension* by

$$E^* = (0 \to A^* \xrightarrow{\varepsilon^*} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\partial_2^*} \cdots \xrightarrow{\partial_{n-2}^*} P_{n-2}^* \xrightarrow{\partial_{n-1}^*} P_{n-1}^* \xrightarrow{i^*} B^* \to 0).$$

The dual of a projective module is projective since $P \oplus Q \cong R^n$ implies that $P^* \oplus Q^* \cong (R^n)^* \cong R^n$. In particular, the P_i^* are projective R–modules.

Whilst E^* is not exact in general, it is true under mild assumptions on the modules involved. We say that an R–module A is an *R–lattice* if its underlying abelian group is finitely generated and torsion-free. For example, if P is a (finitely generated) projective R–module, then P is an R–lattice. This follows from the fact that $P \le R^n$ is an R–submodule for some n and so its underlying abelian group is a subgroup of $\mathbb{Z}^m$ where $m = n \cdot \mathrm{rank}_{\mathbb{Z}}(R)$.

Recall that the *evaluation map* is the map $e_A\colon A \to A^{**}$, defined by $x \mapsto (f \mapsto f(x))$. We say an R–module is *reflexive* if e_A is an R–module isomorphism.

Lemma 2.5 *If A is an R–lattice, then A is reflexive.*

Remark 2.6 Since projective R–modules are R–lattices, this implies that they are reflexive. We note that this is true for arbitrary rings R, not just rings with involution whose underlying abelian group is finitely generated and torsion free.

This follows by noting that, if $A \cong_{\mathrm{Ab}} \mathbb{Z}^k$, then the R–module structure is determined by a map $\rho_A\colon R \to M_k(\mathbb{Z})$. It can be shown that $\rho_{A^*}(r) = \rho_A(\bar{r})^T$ using the induced identification $A^* \cong_{\mathrm{Ab}} \mathbb{Z}^k$, from which the claim follows.

It follows easily from this that the reflexivity property of R–lattices also holds on the level of extensions.

Lemma 2.7 (reflexivity) *If A and B are R–lattices and $n \ge 1$, then dualising gives an isomorphism of abelian groups*

$$*\colon \mathrm{hProj}_R^n(A, B) \to \mathrm{hProj}_R^n(B^*, A^*).$$

If $E \in \mathrm{Proj}_R^n(A, B)$, then there is a chain homotopy equivalence $e\colon E \to E^{**}$ induced by the evaluation maps.

This has the following useful consequence which, in the language of [Johnson 2003, Theorem 28.5], says that projective R–modules are *injective relative to the class of R–lattices*.

Lemma 2.8 *Suppose A, B and E are R–lattices such that $(E, -) \in \mathrm{Ext}_R^1(A, B)$ and P is a projective R–module. Then, for any map $f\colon B \to P$, there exists $\tilde{f}\colon E \to P$ such that $\tilde{f} \circ i = f$, i.e.*

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "i"] \arrow[d, "f"] & E \arrow[r, "\varepsilon"] \arrow[dl, dashed, "\tilde{f}"] & A \arrow[r] & 0 \\ & P & & & \end{tikzcd}$$

We conclude this section by discussing an important invariant of projective extensions. Let $P(R)$ denote the R–module isomorphism classes of (finitely generated) projective R–modules and define the *projective class group* $C(R)$ as the quotient of $P(R)$ by the stable isomorphisms, where $P, Q \in P(R)$ are *stably isomorphic*, written $[P]=[Q]$, if $P \oplus R^i \cong Q \oplus R^j$ for some $i, j \geq 0$. This forms a group under direct sum and coincides with the Grothendieck group of the monoid $P(R)$.

For a projective extension

$$E = (0 \to B \xrightarrow{i} P_{n-1} \xrightarrow{\partial_{n-1}} P_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} A \to 0),$$

we define the *Euler class* $e(E) = \sum_{i=0}^{n-1}(-1)^i[P_i] \in C(R)$. This is known to be a congruence invariant [Wall 1979a, Lemma 1.3]. In fact, more is true:

Lemma 2.9 *If A and B are R–modules, the Euler class defines a map*

$$e\colon \mathrm{hProj}^n_R(A,B) \to C(R),$$

i.e. e is a chain homotopy invariant.

Proof Suppose $E_1, E_2 \in \mathrm{Proj}^n_R(A,B)$ and that $\varphi\colon E_1 \to E_2$ is a chain homotopy equivalence. Then $E_2 \cong (\varphi_A)^*((\varphi_B)_*(E_1))$ and, since e is a congruence invariant, $e(E_2) = e((\varphi_A)^*((\varphi_B)_*(E_1)))$. Since pushout and pullback by automorphisms can be made to not affect the isomorphism classes of the modules in the extension, this implies that $e((\varphi_A)^*((\varphi_B)_*(E_1))) = e(E_1)$ and so e is a chain homotopy invariant. □

The following tells us how the Euler class interacts with the Yoneda product.

Lemma 2.10 *Let A, B and C be R–modules. If $E \in \mathrm{hProj}^n_R(A,B)$ and $F \in \mathrm{hProj}^m_R(B,C)$, then*

$$e(F \circ E) = e(E) + (-1)^n e(F).$$

Proof Let $E = (P_*, \partial_*)_{*=0}^{n-1}$ and let $F = (P_{*+n}, \partial_{*+n})_{*=0}^{m-1}$. Then $F \circ E = (P_*, \partial_*)_{*=0}^{n+m-1}$ and

$$e(F \circ E) = \sum_{i=0}^{n+m-1}(-1)^i[P_i] = \sum_{i=0}^{n-1}(-1)^i[P_i] + \sum_{i=0}^{m-1}(-1)^{i+n}[P_{i+n}] = e(E) + (-1)^n e(F). \qquad \square$$

For a class $\chi \in C(R)$, we define $\mathrm{Proj}^n_R(A,B;\chi)$ to be the subset of $\mathrm{Proj}^n_R(A,B)$ consisting of those extensions with $e(E)=\chi$, and we can define $\mathrm{hProj}^n_R(A,B;\chi)$ similarly as a subset of $\mathrm{hProj}^n_R(A,B)$.

We have the following nice interpretations for the extensions $E \in \mathrm{Proj}^n_R(A,B)$ with $e(E)=0$. This follows easily by repeatedly forming the direct sum with length two extensions $P \xrightarrow{\cong} P$ for various $P \in P(R)$.

Lemma 2.11 *If A and B are R–modules and $n \geq 2$, then every congruence class in $\mathrm{Proj}^n_R(A,B;0)$ has a representative E of the form $E = (F_*, \partial_*)$ with the F_i free.*

This fails in the case $n = 1$, where it is not possible to form the direct sum with length two extensions $R \xrightarrow{\cong} R$ without altering the chain homotopy type. In fact, for a projective extension

$$E = (0 \to B \to P \to A \to 0),$$

we can define the *unstable Euler class* $\hat{e}(E) = P \in P(R)$.

Lemma 2.12 *If A and B are R–modules, the unstable Euler class defines a map*

$$\hat{e}\colon \mathrm{hProj}^1_R(A, B) \to P(R).$$

Proof For $E_1 = (P_1, -), E_2 = (P_2, -) \in \mathrm{Proj}^1_R(A, B)$, recall that a chain map $\varphi\colon E_1 \to E_2$ is a chain homotopy equivalence if it induces a chain homotopy equivalence between the length one chain complexes P_1 and P_2, i.e. if the restriction $\varphi|_{P_1}\colon P_1 \to P_2$ is an isomorphism. □

3 Projective $\mathbb{Z}G$–modules and the Swan finiteness obstruction

Throughout this section, we will let G be a finite group. The results of the previous section apply in the case $R = \mathbb{Z}G$ since $\mathbb{Z}G$ is a ring with involution which is finitely generated and torsion-free as an abelian group. The aim of this section will be to recall some of the special features of projective modules over $\mathbb{Z}G$ and to introduce the Swan finiteness obstruction.

3.1 Preliminaries on projective $\mathbb{Z}G$–modules

We will now summarise the main special properties of (finitely generated) projective $\mathbb{Z}G$–modules in the case where G is finite.

The first was shown by Swan [1960a, Theorem A].

Proposition 3.1 *Let P be a projective $\mathbb{Z}G$–module. Then there is a projective ideal $I \subseteq \mathbb{Z}G$ such that $P \cong I \oplus \mathbb{Z}G^r$ for some $r \geq 0$.*

For a prime p, let $\mathbb{Z}_p$ denote the p–adic integers and let $\mathbb{Z}_{(p)} = \{a/b \mid a, b \in \mathbb{Z},\ p \nmid b\} \leq \mathbb{Q}$ denote the localisation at p. The next property that projective modules over $\mathbb{Z}G$ have is that they are locally free in the following sense (see [Swan 1980, Section 2] for further discussion).

Proposition 3.2 *Let P be a projective $\mathbb{Z}G$–module. There exists $n \geq 0$ such that*

(i) $P \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)}G^n$ *are isomorphic as* $\mathbb{Z}_{(p)}G$*–modules,*

(ii) $P \otimes \mathbb{Q} \cong \mathbb{Q}G^n$ *are isomorphic as* $\mathbb{Q}G$*–modules,*

(iii) $P \otimes \mathbb{Z}_p \cong \mathbb{Z}_pG^n$ *are isomorphic as* $\mathbb{Z}_pG$*–modules,*

(iv) $P \otimes \mathbb{Q}_p \cong \mathbb{Q}_pG^n$ *are isomorphic as* $\mathbb{Q}_pG$*–modules.*

Proof Items (ii) and (iv) each follow from [Swan 1970, Theorem 4.2]. Given this, (i) and (iii) now follow from [Swan 1970, Theorem 2.21]. □

We define the *rank* of P, denoted by $\mathrm{rank}(P)$, to be the $n \geq 0$ in the proposition above. For example, if $I \subseteq \mathbb{Z}G$ is a nonzero projective ideal, then it can be shown that $\mathrm{rank}(I) = 1$; see [Swan 1960a, Section 7].

Let $P(\mathbb{Z}G)$ denote the set of $\mathbb{Z}G$–module isomorphism classes of nonzero projective $\mathbb{Z}G$–modules. This is a monoid under direct sum. Since $\mathrm{rank}(P \oplus Q) = \mathrm{rank}(P) + \mathrm{rank}(Q)$ for all $P, Q \in P(\mathbb{Z}G)$, there is a surjective homomorphism of monoids

$$\mathrm{rank}\colon P(\mathbb{Z}G) \to \mathbb{Z}, \quad P \mapsto \mathrm{rank}(P).$$

Note that $\mathrm{rank}(P) = 0$ if and only if $P = 0$. That is, if P is a nonzero projective $\mathbb{Z}G$–module, then $\mathrm{rank}(P) \geq 1$. This has the following consequence.

Corollary 3.3 *Let P be a nonzero projective $\mathbb{Z}G$–module. Then there exists a surjection $\varphi\colon P \to \mathbb{Z}$.*

Proof Let $n = \mathrm{rank}(P) \geq 1$ and consider the composition

$$P \xhookrightarrow{x \mapsto x \otimes 1} P \otimes \mathbb{Q} \xrightarrow{\cong} \mathbb{Q}G^n \xrightarrow{\pi_1} \mathbb{Q}G \xrightarrow{\varepsilon} \mathbb{Q}$$

where π_1 is projection onto the first coordinate and ε is the augmentation map. Since P is finitely generated, the image of the composition is a finitely generated subgroup of $\mathbb{Q}$ and so is isomorphic to $\mathbb{Z}$. This gives the required surjection. □

3.2 Swan modules

We will now define Swan modules which are a special type of projective module first introduced in [Swan 1960b, Section 6]. Let $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$ denote the augmentation map and let $I = \mathrm{Ker}(\varepsilon) \subseteq \mathbb{Z}G$ denote the augmentation ideal. For any $r \in \mathbb{Z}$ coprime to $|G|$, the ideal $(I, r) \subseteq \mathbb{Z}G$ is projective and depends only on $r \bmod |G|$ up to $\mathbb{Z}G$–isomorphism [Swan 1960b]. Since (I, r) is a nonzero ideal, it has rank one as a projective $\mathbb{Z}G$–module by the remarks in Section 3.1.

The modules (I, r) are known as *Swan modules* and the map

$$S\colon (\mathbb{Z}/|G|)^\times \to C(\mathbb{Z}G)$$

given by $r \mapsto [(I, r)]$ is known as the *Swan map*. This is a well-defined group homomorphism [Swan 1960b], and we define the *Swan subgroup* to be $T_G = \mathrm{Im}(S) \leq C(\mathbb{Z}G)$.

Whilst we will not make explicit use of it in this article, we will briefly mention the closely related ideal $(N, r) \subseteq \mathbb{Z}G$ where $N = \sum_{g \in G} g$ denotes the group norm. Many authors take the (N, r) to be Swan modules instead of the ideals (I, r). In fact, the two notions are equivalent, as the following proposition shows.

Proposition 3.4 *If G is a finite group and $r \in (\mathbb{Z}/|G|)^\times$, then $(I, r) \cong (N, r^{-1})$.*

This is presumably well known, but we will include a detailed proof here since we are not aware that one is currently available in the literature.

Proof By the uniqueness of pullbacks, it will suffice to prove that both (I,r) and (N,r^{-1}) arise as pullbacks of the map $r\colon \mathbb{Z} \to \mathbb{Z}/|G|$ which sends $1 \mapsto r$, and the map $\varepsilon\colon \mathbb{Z}G/(N) \to \mathbb{Z}/|G|$ which sends $x+(N) \mapsto \varepsilon(x)+|G|$.

First let $i\colon I \hookrightarrow (I,r)$ denote inclusion, let $\varphi\colon (I,r) \to \mathbb{Z}G/(N)$ and let $q\colon \mathbb{Z}G \twoheadrightarrow \mathbb{Z}G/(N)$ denote the quotient map. Then there is a diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "i"] \arrow[d, "\mathrm{id}"] & (I,r) \arrow[r, "(1/r)\varepsilon"] \arrow[d, "q"] & \mathbb{Z} \arrow[r] \arrow[d, "r"] & 0 \\ 0 \arrow[r] & I \arrow[r, "j"] & \mathbb{Z}G/(N) \arrow[r, "\varepsilon"] & \mathbb{Z}/|G| \arrow[r] & 0 \end{tikzcd}$$

where q and $(1/r)\varepsilon$ denote the restrictions of these maps to $(I,r) \subseteq \mathbb{Z}G$ and $j = q \circ i$. It can be checked that the diagram commutes and that the rows are exact, and so the right hand square is a pullback.

Now let $s \in \mathbb{Z}$ be such that $s = r^{-1} \in (\mathbb{Z}/|G|)^{\times}$, so that $(N,r^{-1}) \cong (N,s)$. Define $f\colon (N,s) \to \mathbb{Z}G/(N)$ by sending $Nx + sy \mapsto y$. Then consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "s"] \arrow[d, "\mathrm{id}"] & (N,s) \arrow[r, "\varepsilon"] \arrow[d, "f"] & \mathbb{Z} \arrow[r] \arrow[d, "r"] & 0 \\ 0 \arrow[r] & I \arrow[r, "j"] & \mathbb{Z}G/(N) \arrow[r, "\varepsilon"] & \mathbb{Z}/|G| \arrow[r] & 0 \end{tikzcd}$$

Similarly, it can be checked that this commutes and that the rows are exact. □

3.3 Projective extensions

We will now consider the classification of extensions $\mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ for a fixed $\mathbb{Z}G$–module A. The following can be found in [Johnson 2003, Proposition 34.2] and shows that any two elements of $\mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ are related by pullbacks. Note that this isomorphism depends on the choice of E and so only exists when $\mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ is nonempty.

Proposition 3.5 *Let A be a $\mathbb{Z}G$–module and $n \geq 1$. Then, for any $E \in \mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$, there is a bijection*

$$(m_\bullet)^*\colon (\mathbb{Z}/|G|)^{\times} \to \mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$$

given by $r \mapsto (m_r)^*(E)$, where $m_r\colon \mathbb{Z} \to \mathbb{Z}$ denotes multiplication by r.

Remark 3.6 This corresponds to the fact that extensions with fixed ends are determined by their k–invariants; see, for example, [Johnson 2003, Chapter 6].

Let e denote the stable Euler class as defined in Section 2. The next result computes the image of projective extensions under the stable Euler class.

Proposition 3.7 *Let e denote the stable Euler class. Let $n \geq 1$ and let A be a $\mathbb{Z}G$–module such that there exists $E \in \mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$. If $e(E) = [P]$, then*

$$e(\mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)) = [P] + T_G \subseteq C(\mathbb{Z}G).$$

Proof This was proven in [Swan 1960b, Lemmas 7.3 and 7.4] in the case $A = \mathbb{Z}$, and the proof for arbitrary A is analogous. We will outline the steps here for the convenience of the reader.

The first step is to show that, for any $E, E' \in \mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$, we have $e(E') - e(E) \in T_G$. By applying Schanuel's lemma (see [Swan 1960b, Proposition 1.1]) to the duals $E^*, (E')^* \in \mathrm{Proj}^n_{\mathbb{Z}G}(A^*, \mathbb{Z})$, we get an isomorphism $\mathbb{Z} \oplus e(E^*) \cong \mathbb{Z} \oplus e((E')^*)$ and so $e((E')^*) - e(E^*) \in T_G$ by [Swan 1960b, Lemma 6.2]. Since $e(E^*) = e(E)^*$ and projective $\mathbb{Z}G$–modules are reflexive, dualising gives that $e(E') - e(E) \in T_G$.

The second step is to show that, given $E \in \mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$, there exists $E' \in \mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ such that $e(E') - e(E) = [(I, r)]$. This can be constructed in the same way as in [Swan 1960b, Lemma 7.4]. That is, using [Swan 1960b, Remark 2.1]. □

3.4 The Swan finiteness obstruction

We will now specialise further to the case $A = \mathbb{Z}$. Recall that a finite group G is said to have *k–periodic cohomology* if there is an isomorphism of abelian groups $\hat{H}^i(G;\mathbb{Z}) \cong \hat{H}^{i+k}(G;\mathbb{Z})$ for all $i \in \mathbb{Z}$.

Remark 3.8 Many authors define finite groups with periodic cohomology by the a priori stronger condition that there exists a class $u \in \hat{H}^k(G;\mathbb{Z})$ such that cup product induces an isomorphism

$$u \cup -\colon \hat{H}^i(G;\mathbb{Z}) \to \hat{H}^{i+k}(G;\mathbb{Z})$$

for all $i \in \mathbb{Z}$. These definitions are equivalent since, if $\hat{H}^i(G;\mathbb{Z}) \cong \hat{H}^{i+k}(G;\mathbb{Z})$ for all $i \in \mathbb{Z}$, then $\hat{H}^k(G;\mathbb{Z}) \cong \hat{H}^0(G;\mathbb{Z}) \cong \mathbb{Z}/|G|$ which implies that the condition above holds by [Brown 1982, VI.9.1].

The following can be extracted from [Cartan and Eilenberg 1956, Chapter XII].

Proposition 3.9 *Let G be a finite group. Then G has k–periodic cohomology if and only if $\mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ is nonempty.*

If G has k–periodic cohomology then, since $\mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ is nonempty, Proposition 3.7 implies that there exists $P \in P(\mathbb{Z}G)$ for which

$$e(\mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})) = [P] + T_G \subseteq C(\mathbb{Z}G)$$

where $P(\mathbb{Z}G)$ denotes the set of nonzero projective $\mathbb{Z}G$–modules. We can then quotient by T_G to get a unique class in $C(\mathbb{Z}G)/T_G$ which depends only on G and k. The *Swan finiteness obstruction* is defined as

$$\sigma_k(G) = [P] \in C(\mathbb{Z}G)/T_G.$$

Recall that a group G has *free period* k if there exists $E = (F_*, \partial_*) \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ with the F_i free. The following is [Swan 1960b, Proposition 5.1].

Proposition 3.10 *Let G have k–periodic cohomology. Then $\sigma_k(G) = 0$ if and only if G has free period k.*

Remark 3.11 By a construction of Milnor, this is equivalent to the existence of a finite CW–complex X with $\pi_1(X) \cong G$ and $\widetilde{X} \simeq S^{k-1}$ [Swan 1960b, Proposition 3.1]. Examples of groups with $\sigma_k(G) \neq 0$ were found by Milgram [1985].

We will conclude this section by giving a constraint on the projective $\mathbb{Z}G$–modules P which can arise as a representative of the Swan finiteness obstruction.

We would like to compare $[P]$ and $[P^*]$ when $\sigma_k(G) = [P] + T_G$. This is difficult for general projectives since there exists finite groups G and projectives P for which $[P^*] \neq \pm[P]$, even in $C(\mathbb{Z}G)/T_G$. For example, we can take $G = \mathbb{Z}/37^2$ [Curtis and Reiner 1987, Theorem 50.56]. However, in our situation, we have the following.

Proposition 3.12 *If G has k–periodic cohomology, and $\sigma_k(G) = [P] + T_G$, then*

$$[P] = -[P^*] \in C(\mathbb{Z}G)/T_G.$$

Proof By Proposition 3.7, there exists $E \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ with $e(E) = [P]$ and, by forming the direct sum with length two extensions $\mathbb{Z}G \xrightarrow{\cong} \mathbb{Z}G$, we can assume that

$$E \cong (0 \to \mathbb{Z} \xrightarrow{i} P \xrightarrow{\partial_{k-1}} F_{k-2} \xrightarrow{\partial_{k-2}} \cdots \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0)$$

for some F_i free. Dualising then gives that

$$E^* \cong (0 \to \mathbb{Z} \xrightarrow{\varepsilon^*} F_0 \xrightarrow{\partial_1^*} \cdots \xrightarrow{\partial_{k-2}^*} F_{k-2} \xrightarrow{\partial_{k-1}^*} P^* \xrightarrow{i^*} \mathbb{Z} \to 0)$$

and, since k is necessarily even [Cartan and Eilenberg 1956, page 261], Schanuel's lemma implies that

$$\mathbb{Z} \oplus P \oplus P^* \oplus F \cong \mathbb{Z} \oplus F'$$

for some F and F' free. By [Swan 1960b, Lemma 6.2], we then get that $[P \oplus P^*] \in T_G$. $\square$

Remark 3.13 For a finite group G, the standard involution on $C(\mathbb{Z}G)$ is given by $[P] \mapsto -[P^*]$; see [Curtis and Reiner 1987, Section 50E]. This turns $C(\mathbb{Z}G)$ into a $\mathbb{Z}C_2$–module where the C_2–action is given by the involution. This additional structure has proven to be a useful for computing class groups [Curtis and Reiner 1987, page 284]. Note that T_G is fixed by this involution. This follows from the fact that $(I, r)^* \cong (N, r) \cong (I, r^{-1})$ by [Swan 1983, Lemma 17.1] and Proposition 3.4 respectively. Hence the involution induces an involution on $C(\mathbb{Z}G)/T_G$ and so endows it with a natural $\mathbb{Z}C_2$–module structure. With respect to this action, Proposition 3.12 says that $\sigma_k(G) \in (C(\mathbb{Z}G)/T_G)^{C_2}$.

4 Classification of projective chain complexes

We would now like to consider more generally the classification of projective extensions over $\mathbb{Z}G$ with only one fixed end. Throughout this section, G will denote a finite group. For $n \geq 0$, a *projective n–complex* $E = (P_*, \partial_*)$ over $\mathbb{Z}G$ is a chain complex consisting of an exact sequence

$$E = (P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} P_0)$$

where $H_0(P_*) \cong \mathbb{Z}$ and the P_i are (finitely generated) projective $\mathbb{Z}G$–modules. An *algebraic n–complex* is a projective n–complex such that the P_i are free.

Let $\mathrm{Proj}(G,n)$ denote the set of chain homotopy types of projective n–complexes over $\mathbb{Z}G$, which is a graded graph with edges between each $E = (P_*, \partial_*)$ and

$$E \oplus \mathbb{Z}G = (P_n \oplus \mathbb{Z}G \xrightarrow{(\partial_n,0)} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} P_0).$$

Similarly, let $\mathrm{Alg}(G,n)$ denote the set of chain homotopy types of algebraic n–complexes over $\mathbb{Z}G$, which is also a graded graph under stabilisation. By extending the projective n–complex by $\mathrm{Ker}(\partial_n)$, it is easy to see that there is a bijection

$$\mathrm{Proj}(G,n) \cong \coprod_{A \in \mathrm{Mod}(\mathbb{Z}G)} \mathrm{hProj}^{n+1}_{\mathbb{Z}G}(\mathbb{Z}, A).$$

By abuse of notation, we will assume they are the same, i.e. that an extension $E \in \mathrm{Proj}(G,n)$ lies in $\mathrm{hProj}^{n+1}_{\mathbb{Z}G}(\mathbb{Z}, A)$ for some A. For a class $\chi \in C(\mathbb{Z}G)$, let $\mathrm{Proj}(G,n;\chi)$ denote the subset of projective extensions E with $e(E) = \chi$. Note that $\mathrm{Alg}(G,n) \cong \mathrm{Proj}(G,n;0)$ for $n \geq 2$.

4.1 General classification of projective n–complexes

The following is well known; see [Mannan 2007, Theorem 1.1; Hambleton et al. 2013, Proof of Lemma 8.12].

Theorem 4.1 *If $n \geq 0$ and $\chi \in C(\mathbb{Z}G)$, then $\mathrm{Proj}(G,n;\chi)$ is a graded tree, i.e. if $E, E' \in \mathrm{Proj}(G,n)$ have $e(E) = e(E')$, then $E \oplus \mathbb{Z}G^i \simeq E' \oplus \mathbb{Z}G^j$ for some $i, j \geq 0$.*

We will now prove a cancellation theorem for projective n–complexes. Our proof will be modelled on Hambleton and Kreck's proof [1993, Theorem B] that, if X and Y are finite 2–complexes with finite fundamental group such that $X \simeq X_0 \vee S^2$ and $X \vee S^2 \simeq Y \vee S^2$, then $X \simeq Y$. This idea was applied to algebraic 2–complexes in [Hambleton 2019].

If A is a $\mathbb{Z}G$–module, then $x \in A$ is *unimodular* if there exists a map $f\colon A \to \mathbb{Z}G$ such that $f(x) = 1$. Let $\mathrm{Um}(A) \subseteq A$ denote the set of unimodular elements in A.

Lemma 4.2 *Let A and B be $\mathbb{Z}G$–modules. Then*:

(i) *If $\varphi\colon A \to B$ is an isomorphism, then $\varphi(\mathrm{Um}(A)) = \mathrm{Um}(B)$.*

(ii) *$(0,1) \in \mathrm{Um}(A \oplus \mathbb{Z}G)$, i.e. if $\varphi\colon A \oplus \mathbb{Z}G \to B$ is an isomorphism, then $\varphi(0,1) \in \mathrm{Um}(B)$.*

Suppose a $\mathbb{Z}G$–module A has a splitting $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$. Then a map $f\colon A_i \to A_j$ can be viewed as an endomorphism of A by extending it to vanish everywhere else. Write $\mathrm{GL}(A)$ for the group of automorphisms of A and define

$$E(A_i, A_j) = \langle 1 + f, 1 + g \mid f\colon A_i \to A_j, g\colon A_j \to A_i \rangle \le \mathrm{GL}(A)$$

to be the subgroup of *elementary automorphisms* for $i \ne j$, where $1\colon A \to A$ denotes the identity map.

The main result we will use is the following, which can be proven by combining [Hambleton and Kreck 1993, Corollary 1.12 and Lemma 1.16]. Let $\mathbb{Z}_{(p)} = \{a/b \mid a, b \in \mathbb{Z},\ p \nmid b\} \le \mathbb{Q}$ denote the localisation at a prime p and $A_{(p)} = A \otimes \mathbb{Z}_{(p)}$.

Theorem 4.3 *Suppose A is a $\mathbb{Z}G$–module for which $\mathbb{Z}_{(p)} \oplus A_{(p)}$ is a free $\mathbb{Z}_{(p)}G$–module for all but finitely many primes p. If $F_1, F_2 \cong \mathbb{Z}G$, then*

$$\mathscr{G} = \langle E(F_1, A \oplus F_2), E(F_2, A \oplus F_1) \rangle \le \mathrm{GL}(A \oplus F_1 \oplus F_2)$$

acts transitively on $\mathrm{Um}(A \oplus F_1 \oplus F_2)$.

We will now establish criteria for which the above conditions hold for a $\mathbb{Z}G$–module A. First recall that, by an extension of Maschke's theorem of representations, the group ring RG is semisimple whenever R is a commutative ring such that $|G| \in R^\times$. This is the case when $R = \mathbb{Z}_{(p)}$ for p a prime not dividing $|G|$. This has the following consequence.

Lemma 4.4 *Let $n \ge 1$ be odd, let p be a prime not dividing $|G|$ and let A be a $\mathbb{Z}G$–module for which $\mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ is nonempty. Then $\mathbb{Z}_{(p)} \oplus A_{(p)}$ is a free $\mathbb{Z}_{(p)}G$–module.*

Proof Let $E = (P_*, \partial_*) \in \mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$. Recall that localisation is an exact functor (since, for example, $\mathbb{Z}_{(p)}$ is a flat module). Hence we obtain $E_{(p)} = ((P_*)_{(p)}), \partial_*) \in \mathrm{Proj}^n_{\mathbb{Z}_{(p)}G}(\mathbb{Z}_{(p)}, A_{(p)})$ where the ∂_* are the induced maps. By the extension of Maschke's theorem mentioned above, $\mathbb{Z}_{(p)}G$ is semisimple and so the exact sequence $E_{(p)}$ splits completely. This implies that there is an isomorphism of $\mathbb{Z}_{(p)}G$–modules

$$\mathbb{Z}_{(p)} \oplus A_{(p)} \oplus \bigoplus_{i \text{ odd}} (P_i)_{(p)} \cong \bigoplus_{i \text{ even}} (P_i)_{(p)}.$$

By Proposition 3.2, the $(P_i)_{(p)}$ are all free $\mathbb{Z}_{(p)}G$–modules. It follows that $\mathbb{Z}_{(p)} \oplus A_{(p)}$ is a stably free $\mathbb{Z}_{(p)}G$–module. Since $\mathbb{Z}_{(p)}G$ is semisimple, this implies that $\mathbb{Z}_{(p)} \oplus A_{(p)}$ is a free $\mathbb{Z}_{(p)}G$–module. □

Note that the fact that $\mathrm{GL}(A \oplus \mathbb{Z}G^2)$ acts transitively on $\mathrm{Um}(A \oplus \mathbb{Z}G^2)$ already implies the following cancellation theorem for modules.

Corollary 4.5 *Suppose A is a $\mathbb{Z}G$–module, $A \cong A_0 \oplus \mathbb{Z}G$ and $\mathbb{Z}_{(p)} \oplus (A_0)_{(p)}$ is a free $\mathbb{Z}_{(p)}G$–module for all but finitely many primes p. Then $A \oplus \mathbb{Z}G \cong A' \oplus \mathbb{Z}G$ implies $A \cong A'$.*

Proof Let $\psi\colon A \oplus \mathbb{Z}G \to A' \oplus \mathbb{Z}G$ be an isomorphism and let $x = \psi^{-1}(0, 1) \in \mathrm{Um}(A \oplus \mathbb{Z}G)$. Since $A = A_0 \oplus \mathbb{Z}G$, Theorem 4.3 implies that $\mathrm{GL}(A \oplus \mathbb{Z}G)$ acts transitively on $\mathrm{Um}(A \oplus \mathbb{Z}G)$ and so there is an isomorphism $\varphi\colon A\oplus\mathbb{Z}G\to A\oplus\mathbb{Z}G$ such that $\varphi(0,1)=x$. Hence $\psi\circ\varphi\colon A\oplus\mathbb{Z}G\to A'\oplus\mathbb{Z}G$ has $(\psi\circ\varphi)(0,1)=(0,1)$ and so induces an isomorphism $(\psi\circ\varphi)|_A\colon A\to A'\oplus\mathbb{Z}G/\operatorname{Im}(0\oplus\mathbb{Z}G)\cong A'$. $\square$

We will upgrade the above argument from modules to projective n–complexes. The existence of a well-understood subgroup $\mathcal{G}\le\mathrm{GL}(A\oplus\mathbb{Z}G^2)$ which acts transitively on $\mathrm{Um}(A\oplus\mathbb{Z}G^2)$ is important since we need only show that elements in $\mathcal{G}$ can be extended to chain homotopy equivalences on the short exact sequences.

Theorem 4.6 *Let $n\ge 0$ be even and let $E,E'\in\mathrm{Proj}(G,n)$. If $E\simeq E_0\oplus\mathbb{Z}G$ and $E\oplus\mathbb{Z}G\simeq E'\oplus\mathbb{Z}G$, then $E\simeq E'$.*

Proof Let $E_0\in\mathrm{hProj}^{n+1}_{\mathbb{Z}G}(\mathbb{Z},A_0)$, $E=(P_*,\partial_*)\in\mathrm{hProj}^{n+1}_{\mathbb{Z}G}(\mathbb{Z},A)$ and $E'=(P'_*,\partial'_*)\in\mathrm{hProj}^{n+1}_{\mathbb{Z}G}(\mathbb{Z},A')$. If $\psi: E\oplus\mathbb{Z}G\to E'\oplus\mathbb{Z}G$ denotes the given chain homotopy equivalence in $\mathrm{hProj}^{n+1}_{\mathbb{Z}G}(\mathbb{Z},A_0\oplus\mathbb{Z}G^2)$ and $\psi_A\colon A_0\oplus\mathbb{Z}G^2\to A'\oplus\mathbb{Z}G$ is the induced map on the left, consider $x=\psi_A^{-1}(0,1)\in\mathrm{Um}(A_0\oplus\mathbb{Z}G^2)$.

We now claim that there exists a self chain homotopy equivalence $\varphi\colon E\oplus\mathbb{Z}G\to E\oplus\mathbb{Z}G$ such that the induced map $\varphi_A\colon A\oplus\mathbb{Z}G\to A\oplus\mathbb{Z}G$ has $\varphi_A(0,1)=x$.

Let $F_1,F_2\cong\mathbb{Z}G$ be such that $A=A_0\oplus F_1$ and $A\oplus\mathbb{Z}G=A_0\oplus F_1\oplus F_2$. Since $\mathrm{Proj}^{n+1}_{\mathbb{Z}G}(\mathbb{Z},A_0)$ is nonempty and $n+1$ is odd, we can combine Theorem 4.3 and Lemma 4.4 to get that there exists $\varphi_A\in\mathcal{G}=\langle E(F_1,A_0\oplus F_2),E(F_2,A_0\oplus F_1)\rangle\le\mathrm{GL}(A_0\oplus F_1\oplus F_2)$ such that $\varphi_A(0,0,1)=x$. We claim that φ_A can be extended to a chain homotopy equivalence $\varphi\colon E\oplus\mathbb{Z}G\to E\oplus\mathbb{Z}G$.

First recall that $E(F_2,A_0\oplus F_1)=E(F_2,A)\le\mathrm{GL}(A\oplus F_2)$ is generated by elements of the form $\left(\begin{smallmatrix}1&0\\f&1\end{smallmatrix}\right)$ for $f: A\to F_2$ and $\left(\begin{smallmatrix}1&g\\0&1\end{smallmatrix}\right)$ for $g\colon F_2\to A$.

If $i: A\hookrightarrow P$, then there exists $\tilde f\colon P\to\mathbb{Z}G$ such that $\tilde f\circ i=f$ by Lemma 2.8. It is straightforward to verify that the following diagrams commute, and so are chain homotopy equivalences:

$$\begin{tikzcd} E\oplus\mathbb{Z}G \arrow[d, "\varphi_1"] \\ E\oplus\mathbb{Z}G \end{tikzcd} = \left(\begin{tikzcd} 0 \arrow[r] & A\oplus\mathbb{Z}G \arrow[r, "\left(\begin{smallmatrix}i&0\\0&1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}1&0\\f&1\end{smallmatrix}\right)"] & P_n\oplus\mathbb{Z}G \arrow[r, "(\partial_n{,}\,0)"] \arrow[d, "\left(\begin{smallmatrix}1&0\\\tilde f&1\end{smallmatrix}\right)"] & P_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\mathrm{id}_{P_{n-1}}"] & \cdots \arrow[r, "\partial_1"] & P_0 \arrow[r] \arrow[d, "\mathrm{id}_{P_0}"] & 0 \\ 0 \arrow[r] & A\oplus\mathbb{Z}G \arrow[r, "\left(\begin{smallmatrix}i&0\\0&1\end{smallmatrix}\right)"] & P_n\oplus\mathbb{Z}G \arrow[r, "(\partial_n{,}\,0)"] & P_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_1"] & P_0 \arrow[r] & 0 \end{tikzcd}\right)$$

$$\begin{tikzcd} E\oplus\mathbb{Z}G \arrow[d, "\varphi_2"] \\ E\oplus\mathbb{Z}G \end{tikzcd} = \left(\begin{tikzcd} 0 \arrow[r] & A\oplus\mathbb{Z}G \arrow[r, "\left(\begin{smallmatrix}i&0\\0&1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}1&g\\0&1\end{smallmatrix}\right)"] & P_n\oplus\mathbb{Z}G \arrow[r, "(\partial_n{,}\,0)"] \arrow[d, "\left(\begin{smallmatrix}1&i\circ g\\0&1\end{smallmatrix}\right)"] & P_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\mathrm{id}_{P_{n-1}}"] & \cdots \arrow[r, "\partial_1"] & P_0 \arrow[r] \arrow[d, "\mathrm{id}_{P_0}"] & 0 \\ 0 \arrow[r] & A\oplus\mathbb{Z}G \arrow[r, "\left(\begin{smallmatrix}i&0\\0&1\end{smallmatrix}\right)"] & P_n\oplus\mathbb{Z}G \arrow[r, "(\partial_n{,}\,0)"] & P_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots \arrow[r, "\partial_1"] & P_0 \arrow[r] & 0 \end{tikzcd}\right)$$

Similarly, we can show that the generators of $E(F_1,A_0\oplus F_2)$ extend to chain homotopy equivalences. Hence, by writing $\varphi_A\in\mathcal{G}$ as the composition of maps of this form, we can get a chain homotopy equivalence $\varphi\colon E\oplus\mathbb{Z}G\to E\oplus\mathbb{Z}G$ by taking the composition of equivalences on each of the generators.

Now consider the map

$$\psi\circ\varphi=(\psi_A\circ\varphi_A,\psi_P\circ\varphi_P,\mathrm{id},\dots,\mathrm{id})\colon E\oplus\mathbb{Z}G\to E'\oplus\mathbb{Z}G.$$

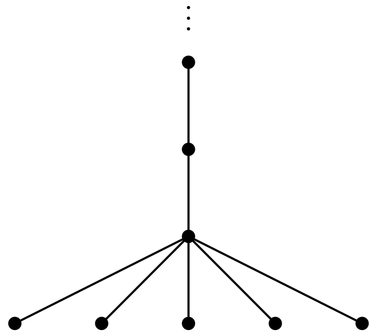


Figure 1: A graded tree which is a fork.

Since $(\psi_A \circ \varphi_A)(0,1) = (0,1)$, it must have the form $\psi_A \circ \varphi_A = \left(\begin{smallmatrix} \phi_A & 0 \\ 0 & 1 \end{smallmatrix}\right)$ since it is an isomorphism. By commutativity, $(\psi_P \circ \varphi_P)(0,1) = (0,1)$ and so similarly $\psi_P \circ \varphi_P = \left(\begin{smallmatrix} \phi_P & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for some $\phi_P \colon P \to P'$. We are now done by noting that the triple $(\phi_A, \phi_P, \mathrm{id}, \dots, \mathrm{id})$ defines a chain homotopy equivalence $E \simeq E'$. □

We say that a graded tree is a *fork* if it has a single vertex at each nonminimal grade and a finite set of vertices at the minimal grade.

Corollary 4.7 *If $n \geq 0$ is even, G is a finite group and $\chi \in C(\mathbb{Z}G)$, then* $\mathrm{Proj}(G, n; \chi)$ *is a fork. In particular,* $\mathrm{Alg}(G, n)$ *is a fork for $n \geq 2$ even.*

This recovers the even-dimensional case of a result of Browning [1978, Theorem 5.4]. This fails in odd dimensions, i.e. there are examples of finite groups G for which $\mathrm{Alg}(G, n)$ is not a fork for some n odd [Dyer 1979].

4.2 Projective 0–complexes and the unstable Euler class

We now consider the case $n = 0$. Recall that $P(\mathbb{Z}G)$ denotes the set of $\mathbb{Z}G$–module isomorphism classes of (finitely generated) nonzero projective $\mathbb{Z}G$–modules. This is a graded graph with edges between each P and $P \oplus \mathbb{Z}G$.

Note that a projective 0–complex has the form

$$E = (0 \to A \xrightarrow{i} P \xrightarrow{\varepsilon} \mathbb{Z} \to 0),$$

and so consists of a nonzero projective module $P \in P(\mathbb{Z}G)$ as well as the additional data (A, i, ε). If $\hat{e}$ is the unstable Euler class, then $\hat{e} \colon \mathrm{Proj}(G, 0) \to P(\mathbb{Z}G)$ is a map of graded graphs since

$$\hat{e}(E \oplus \mathbb{Z}G) \cong \hat{e}(E) \oplus \mathbb{Z}G.$$

We will now show the following:

Theorem 4.8 *If G is a finite group, then the unstable Euler class gives an isomorphism of graded graphs*

$$\hat{e}\colon \mathrm{Proj}(G,0) \to P(\mathbb{Z}G).$$

Remark 4.9 Such a statement is implicit in the proof of [Johnson 2003, Theorem IV, Theorem 57.4], though the argument there contains an error and can only be used to recover the statement above in the case of projective modules of rank one. This, however, suffices since one can instead rely on the cancellation theorems of Hambleton and Kreck [1993, Theorem B] or Browning [1978, Theorem 5.4] at that stage in the proof.

Proof To see that $\hat{e}$ is surjective, let $P \in P(\mathbb{Z}G)$. By Corollary 3.3, there is a surjection $\varphi\colon P \to \mathbb{Z}$ and this defines an extension $E = (P,-) \in \mathrm{hProj}^1_{\mathbb{Z}G}(\mathbb{Z}, \mathrm{Ker}(\varphi))$ which has $\hat{e}(E) = P$.

We will now show injectivity. First let $E = (P,-) \in \mathrm{hProj}^1_{\mathbb{Z}G}(\mathbb{Z}, A)$ and let $E' = (P,-) \in \mathrm{hProj}^1_{\mathbb{Z}G}(\mathbb{Z}, A')$. We will begin by considering the case where P has rank one. To show that $E \simeq E'$, it suffices to find isomorphisms $\varphi_A\colon A \to A'$ and $\varphi_{\mathbb{Z}}\colon \mathbb{Z} \to \mathbb{Z}$ such that the following diagram commutes:

$$\begin{array}{c} E \\ \downarrow{\scriptstyle\varphi} \\ E' \end{array} = \left(\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, dashed, "\varphi_A"] & P \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{id}"] & \mathbb{Z} \arrow[r] \arrow[d, dashed, "\varphi_{\mathbb{Z}}"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"] & P \arrow[r, "\varepsilon'"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}\right)$$

Consider the maps $\bar{\varepsilon} = \varepsilon \otimes \mathbb{Q}, \bar{\varepsilon}' = \varepsilon' \otimes \mathbb{Q}\colon P \otimes \mathbb{Q} \cong \mathbb{Q}G \to \mathbb{Q}$. Since $\mathbb{Q}$ has trivial G–action, each map is determined by the fact that $\bar{\varepsilon}(g) = \bar{\varepsilon}'(g) = 0$ for all $g \in G$ and $\bar{\varepsilon}(1) = \bar{\varepsilon}'(1) = x_i$ for some $x_i \in \mathbb{Q}^{\times}$. Hence $\mathrm{Ker}(\bar{\varepsilon}) = \mathrm{Ker}(\bar{\varepsilon}')$ and so $(\varepsilon' \circ i) \otimes \mathbb{Q} = 0$. Since A is a $\mathbb{Z}G$ lattice, this implies that $\varepsilon' \circ i = 0$ and so we can define maps φ_A and $\varphi_{\mathbb{Z}}$ as above. Now $\varphi_{\mathbb{Z}}$ is necessarily surjective and so an isomorphism. Hence φ_A is an isomorphism by the five lemma, and so $E \simeq E'$.

Now suppose E and E' are as above but with $\mathrm{rank}(P) \geq 2$. By Proposition 3.1, this implies that there exists P_0 of rank one such that $P \cong P_0 \oplus \mathbb{Z}G^i$ for some $i \geq 1$. Since $\hat{e}$ is surjective, there exists $E_0 = (P_0,-) \in \mathrm{hProj}^1_{\mathbb{Z}G}(\mathbb{Z}, A_0)$ for some A_0. By Theorem 4.1, there exists $j \geq 0$ for which $E_0 \oplus \mathbb{Z}G^{i+j} \simeq E \oplus \mathbb{Z}G^j \simeq E' \oplus \mathbb{Z}G^j$. Since $i \geq 1$, Theorem 4.6 then implies $E_0 \oplus \mathbb{Z}G^i \simeq E \simeq E'$. □

For use in later sections, it will be necessary to further refine the isomorphism given by $\hat{e}$. Consider following two decompositions (where $\cong$ denotes bijection):

$$\mathrm{Proj}(G,0) \cong \coprod_{\chi \in C(\mathbb{Z}G)} \mathrm{Proj}(G,0;\chi) \cong \coprod_{A \in \mathrm{Mod}(\mathbb{Z}G)} \mathrm{hProj}^1_{\mathbb{Z}G}(\mathbb{Z}, A).$$

We will begin by determining the image of $\mathrm{Proj}(G,0;\chi)$ under $\hat{e}$. This is immediate from Theorem 4.8 and the definition of $\mathrm{hProj}^1_{\mathbb{Z}G}(\mathbb{Z}, A;\chi)$. For convenience, we will write $\chi = [P]$ for some $P \in P(\mathbb{Z}G)$.

Proposition 4.10 *Let $P \in P(\mathbb{Z}G)$. Then there is an isomorphism of graded trees*

$$\hat{e}\colon \mathrm{Proj}(G,0;[P]) \to [P].$$

We will next determine the image of $\mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, A)$ under $\hat{e}$ for A a fixed $\mathbb{Z}G$–module such that $\mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, A)$ is nonempty. Recall that, if $E \in \mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, A)$, then Proposition 3.5 implies that there is a bijection

$$(m_\bullet)^* \colon (\mathbb{Z}/|G|)^\times \to \mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, A)$$

given by $r \mapsto (m_r)^*(E)$, where $m_r \colon \mathbb{Z} \to \mathbb{Z}$ denotes multiplication by r.

If M is a (left) $\mathbb{Z}G$–module and $r \in (\mathbb{Z}/|G|)^\times$, then the tensor product $(I, r) \otimes M$ can be considered as a (left) $\mathbb{Z}G$–module since (I, r) is a two-sided ideal. This allows us to find an explicit form for pullbacks of extensions. We will begin with the following special case.

Lemma 4.11 *Let A be a $\mathbb{Z}G$–module and suppose $E = (P, -) \in \mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, A)$ where $\mathrm{rank}(P) = 1$. Then, for any $r \in (\mathbb{Z}/|G|)^\times$, there are maps $\bar{i}$ and $\bar{\varepsilon}$ such that*

$$(m_r)^*(E) \cong (0 \to A \xrightarrow{\bar{i}} (I, r) \otimes P \xrightarrow{\bar{\varepsilon}} \mathbb{Z} \to 0).$$

Proof Let $E = (P, -) \in \mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, A)$ and note that we have the diagrams

$$\begin{tikzcd} (I,r) \arrow[r, "(1/r)\varepsilon"] \arrow[d, "i"] & \mathbb{Z} \arrow[d, "r"] \\ \mathbb{Z}G \arrow[r, "\varepsilon"] & \mathbb{Z} \end{tikzcd} \qquad \begin{tikzcd} (I,r)\otimes P \arrow[r, "(1/r)\varepsilon\otimes 1"] \arrow[d, "i\otimes 1"] & \mathbb{Z}\otimes P \arrow[d, "r\otimes 1"] \\ \mathbb{Z}G\otimes P \arrow[r, "\varepsilon\otimes 1"] & \mathbb{Z}\otimes P \end{tikzcd}$$

where $i \colon (I, r) \hookrightarrow \mathbb{Z}G$ is inclusion. It can be checked directly that the first diagram is a pullback, and this implies that the second diagram is a pullback since P is projective and so flat. Since $\mathrm{rank}(P) = 1$, we can choose identifications $\mathbb{Z}G \otimes P \cong P$ and $\mathbb{Z} \otimes P \cong \mathbb{Z}$ for which $\varepsilon \otimes 1$ corresponds to ε^E. We now have a map $(\mathrm{id}_A, \varphi, m_r) \colon E' \to E$ where $E' = ((I, r) \otimes P, -)$. Hence $E' \cong (m_r)^*(E)$ by uniqueness of pullbacks. □

We can now upgrade this to the general case using Theorem 4.8.

Lemma 4.12 *Let A be a $\mathbb{Z}G$–module and suppose $E = (P, -) \in \mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, A)$.*

(i) *There exists a projective $\mathbb{Z}G$–module P_0 with $\mathrm{rank}(P_0) = 1$ and $k \geq 0$ such that $P \cong P_0 \oplus \mathbb{Z}G^k$ and*

$$E \cong (0 \to A \xrightarrow{i_0} P_0 \oplus \mathbb{Z}G^k \xrightarrow{(\varepsilon_0, 0)} \mathbb{Z} \to 0)$$

for some maps i_0 and $\varepsilon_0 \colon P_0 \to \mathbb{Z}$.

(ii) *With P_0, i_0 and ε_0 as above,*

$$(m_r)^*(E) \cong (0 \to A \xrightarrow{\bar{i}_0} ((I, r) \otimes P_0) \oplus \mathbb{Z}G^k \xrightarrow{(\bar{\varepsilon}_0, 0)} \mathbb{Z} \to 0)$$

for some maps $\bar{i}_0$ and $\bar{\varepsilon}_0 \colon (I, r) \otimes P_0 \to \mathbb{Z}$.

Proof (i) Since $P \twoheadrightarrow \mathbb{Z}$, we know that P is nonzero. Hence, by Proposition 3.1, there exists a projective $\mathbb{Z}G$–module P_0 with $\mathrm{rank}(P_0) = 1$ and $k \geq 0$ such that $P \cong P_0 \oplus \mathbb{Z}G^k$. Since $\hat{e}$ is an isomorphism of graded trees, there exists $E_0 \in \mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, A_0)$ for some $\mathbb{Z}G$–module A_0 such that $E \cong E_0 \oplus \mathbb{Z}G^k$. Write

$$E_0 = (0 \to A_0 \xrightarrow{i_0'} P_0 \xrightarrow{\varepsilon_0} \mathbb{Z} \to 0)$$

for some i_0' and ε_0. The result follows by forming $E_0 \oplus \mathbb{Z}G^k$.

(ii) The result follows by noting that $(m_r)^*(E_0 \oplus \mathbb{Z}G^k) \cong (m_r)^*(E_0) \oplus \mathbb{Z}G^k$ and evaluating $(m_r)^*(E_0)$ using Lemma 4.11. $\square$

Remark 4.13 The proof of (i) also implies that $A \cong A_0 \oplus \mathbb{Z}G^k$.

This implies the following. This is the analogue of Proposition 3.7 which established the corresponding result for the stable Euler class e.

Proposition 4.14 *Let A be a $\mathbb{Z}G$–module and suppose $E = (P, -) \in \mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, A)$. Then*

$$\hat{e}(\mathrm{Proj}^1_{\mathbb{Z}G}(\mathbb{Z}, A)) = \{((I,r) \otimes P_0) \oplus \mathbb{Z}G^k \mid r \in (\mathbb{Z}/|G|)^\times\} \subseteq P(\mathbb{Z}G)$$

where P_0 is any rank one projective $\mathbb{Z}G$–module such that $P \cong P_0 \oplus \mathbb{Z}G^k$ for $k \geq 0$.

For completeness, as well as for later use, we will note the following which is a consequence of [Fröhlich et al. 1974, Remark 1.30]. This shows that Propositions 3.7 and 4.14 agree in the case $n = 1$.

Lemma 4.15 *Let P be a projective $\mathbb{Z}G$–module with $\mathrm{rank}(P) = 1$ and let $r \in (\mathbb{Z}/|G|)^\times$. Then*

$$[(I,r) \otimes P] = [(I,r)] + [P] \in C(\mathbb{Z}G).$$

5 Polarised homotopy classification of (G,n)–complexes

Recall that, for a group G, a *G–polarised* space is a pair (X, ρ_X) where X is a topological space and $\rho_X \colon \pi_1(X, *) \to G$ is a given isomorphism. We say that two G–polarised spaces (X, ρ_X) and (Y, ρ_Y) are *polarised homotopy equivalent* if there exists a homotopy equivalence $h \colon X \to Y$ such that $\rho_X = \rho_Y \circ \pi_1(h)$.

Let $\mathrm{PHT}(G,n)$ denote the set of polarised homotopy types of finite (G,n)–complexes over G. This is a graded graph with edges between each (X, ρ_X) and $(X \vee S^2, (\rho_X)^+)$ where $(\rho_X)^+$ is induced by ρ_X and the collapse map $X \vee S^2 \to X$.

If X is a finite CW–complex, then the cellular chain complex $C_*(\widetilde{X})$ can be viewed as a chain complex of $\mathbb{Z}[\pi_1(X)]$–modules under the monodromy action. We can use a G–polarisation $\rho \colon \pi_1(X) \to G$ to get a chain complex of $\mathbb{Z}G$–modules $C_*(\widetilde{X}, \rho)$ which is the same as $C_*(\widetilde{X})$ as a chain complex of abelian groups but with action $g \cdot x = \rho^{-1}(g)x$ for all $g \in G$ and $x \in C_i(\widetilde{X})$ for some $i \geq 0$.

The following is a mild generalisation of [Nicholson 2021b, Theorem 1.1]:

Proposition 5.1 *Let G be a finitely presented group and let $n \geq 2$. Then there is an injective map of graded trees*

$$\widetilde{C}_* \colon \mathrm{PHT}(G,n) \to \mathrm{Alg}(G,n)$$

induced by the map $(X,\rho) \mapsto C_*(\widetilde{X},\rho)$. Furthermore:

(i) *If $n \geq 3$, then $\widetilde{C}_*$ is bijective.*

(ii) *If $n = 2$, then $\widetilde{C}_*$ is bijective if and only if G has the D2 property.*

Remark 5.2 (a) Even if G does not satisfy the D2 property, Proposition 5.1 can be replaced with an isomorphism $\widetilde{C}_* \colon \mathrm{D2}(G) \to \mathrm{Alg}(G,2)$ where $\mathrm{D2}(G)$ denotes the polarised homotopy tree of D2–complexes over G [Nicholson 2021b, Theorems 1.1].

(b) This is often vacuous in the case $n \geq 3$ since $\mathrm{PHT}(G,n)$ and $\mathrm{Alg}(G,n)$ are often empty. More specifically, $\mathrm{PHT}(G,n)$ is nonempty if and only if G is of type F_n. $\mathrm{Alg}(G,n)$ is nonempty if and only if G has type FP_n (see [Bieri 1976]), and it is well known that $\mathrm{F}_n \iff \mathrm{FP}_n$ for finitely presented groups. This situation arises since there exist finitely presented groups which are not of type F_n for $n \geq 3$ [Stallings 1963].

(c) This fails in general for nonfinitely presented groups. In particular, for each $n \geq 2$, Bestvina and Brady [1997] constructed a nonfinitely presented group G of type FP_n. Here $\mathrm{PHT}(G,n)$ is empty and $\mathrm{Alg}(G,n)$ is nonempty and so $\widetilde{C}_*$ is not bijective.

We will now use the results from the previous section to study projective n–complexes over groups with periodic cohomology. By Proposition 5.1, this will lead to a proof of the following more detailed version of Theorem A. Note that, if X is a finite (G,n)–complex, then

$$\pi_n(X) \cong H_n(\widetilde{X}) \cong \mathrm{Ker}(\partial_n \colon C_n(\widetilde{X}) \to C_{n-1}(\widetilde{X}))$$

are isomorphisms of $\mathbb{Z}G$–modules.

Theorem 5.3 *Let G have k–periodic cohomology, let $n = ik$ or $ik-2$ for some $i \geq 1$ and let $P_{(G,n)}$ be a projective $\mathbb{Z}G$–module with $\sigma_{ik}(G) = [P_{(G,n)}] \in C(\mathbb{Z}G)/T_G$. Let $F \in \mathrm{Proj}^{ik}_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ be such that $e(F) = [P_{(G,n)}]$. Then there is an injective map of graded trees*

$$\Psi \colon \mathrm{PHT}(G,n) \to [P_{(G,n)}],$$

defined as follows:

(i) *If $n = ik-2$, then $\Psi \colon X \mapsto P$, where P is the unique projective $\mathbb{Z}G$–module for which*

$$(0 \to \mathbb{Z} \xrightarrow{\alpha} P^* \xrightarrow{\beta} \pi_n(X) \to 0) \circ C_*(\widetilde{X}) \simeq F$$

for some α and β.

(ii) *If $n = ik$, then $\Psi \colon X \mapsto P$, where P is the unique projective $\mathbb{Z}G$–module for which*
$$C_*(\widetilde{X}) \simeq (0 \to \pi_n(X) \xrightarrow{\alpha} P \xrightarrow{\beta} \mathbb{Z} \to 0) \circ F$$
for some α and β.

Furthermore, Ψ is bijective if and only if $n \geq 3$ or $n = 2$ and G has the D2 *property.*

Remark 5.4 The definition of $P_{(G,n)}$ depends on G, n and k. Note that n and k determine i except when $k = 2$ where $n = ik = (i+1)k - 2$. However, in this case there is no ambiguity since G is cyclic [Swan 1965, Lemma 5.2] and so $\sigma_{2i}(G) = 0$ for all i.

First note that, when G has periodic cohomology, we get the following two relations between projective complexes of different dimensions.

Lemma 5.5 *Suppose G has k–periodic cohomology and let $F \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$. If $n \geq 0$, then we have isomorphisms of graded graphs*
$$- \circ F \colon \mathrm{Proj}(G,n) \to \mathrm{Proj}(G, n+k), \qquad * \circ \Psi_F \colon \mathrm{Proj}(G,n) \to \mathrm{Proj}(G, k-(n+2)),$$
where $n + 2 \leq k$ in the second case.

Proof The first isomorphism is immediate from the shifting lemma. The second isomorphism consists of the compositions
$$\mathrm{hProj}^{n+1}_{\mathbb{Z}G}(\mathbb{Z}, A) \xrightarrow{\Psi_F} \mathrm{hProj}^{k-n-1}_{\mathbb{Z}G}(A, \mathbb{Z}) \xrightarrow{*} \mathrm{hProj}^{k-n-1}_{\mathbb{Z}G}(\mathbb{Z}, A^*)$$
for all $\mathbb{Z}G$–modules A. These are bijections by the duality and reflexivity lemmas.

To see that the image of the full map is $\mathrm{Proj}(G, k-(n+2))$ note that, if B is such that $\mathrm{hProj}^{k-n-2}_{\mathbb{Z}G}(\mathbb{Z}, B)$ is nonzero, then B is a $\mathbb{Z}G$–lattice since it is a submodule of a free module. By Lemma 2.5, $B^{**} \cong B$ and so there is an isomorphism $* \circ \Psi_F \colon \mathrm{hProj}^{n+1}_{\mathbb{Z}G}(\mathbb{Z}, B^*) \to \mathrm{hProj}^{k-n-1}_{\mathbb{Z}G}(\mathbb{Z}, B)$. ◻

Remark 5.6 Furthermore, if $E \in \mathrm{Proj}(G,n)$ has $\chi = e(E)$, then it is easy to see that $e(E \circ F) = e(F) + \chi$ since k is even and $e((\psi_F(E))^*) = e(F)^* - \chi^*$.

The proof of Theorem 5.3 will now consist of applying Lemma 5.5 in the case $k \mid n$ or $n+2$ and then composing with the isomorphism from Theorem 4.8.

We will need the following result of Wall [1979a, Corollary 12.6].

Proposition 5.7 *If G has k–periodic cohomology, then*
$$2\sigma_k(G) = 0 \in C(\mathbb{Z}G)/T_G.$$

By iterating extensions using the Yoneda product, it can be shown that $n\sigma_k(G) = \sigma_{nk}(G)$ and so this theorem is equivalent to showing that $\sigma_{2k}(G) = 0$, i.e. that the obstruction vanishes whenever k is not the minimal period.

Theorem 5.8 *If G has k–periodic cohomology and $\sigma_k(G) = [P_{(G,n)}] + T_G$ for some $P_{(G,n)} \in P(\mathbb{Z}G)$, then there exists $F \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ such that there are isomorphisms of graded trees*

$$\mathrm{Alg}(G,k) \xrightarrow{(-\circ F)^{-1}} \mathrm{Proj}(G,0;[P_{(G,n)}]) \xrightarrow{\hat{e}} [P_{(G,n)}].$$

Proof By Proposition 5.7, $\sigma_k(G) = [P_{(G,n)}] + T_G = -[P_{(G,n)}] + T_G$ and so there exists $F \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ with $e(F) = -[P_{(G,n)}]$ by Proposition 3.7.

If $E \in \mathrm{Alg}(G,k)$, then $e(E) = 0$ and so $e((-\circ F)^{-1}) = -(-1)^k e(F)$ by Lemma 2.10. Since k is even, this is equal to $[P_{(G,n)}]$. Hence the map $(-\circ F)^{-1}$ is as described. By Lemma 5.5, we get that $(-\circ F)^{-1}$ is an isomorphism.

That $\hat{e}$ is an isomorphism follows from Proposition 4.10. □

Theorem 5.9 *If G has k–periodic cohomology and $\sigma_k(G) = [P_{(G,n)}] + T_G$ for some $P_{(G,n)} \in P(\mathbb{Z}G)$, then there exists $F \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ such that there are isomorphisms of graded trees*

$$\mathrm{Alg}(G,k-2) \xrightarrow{*\circ\Psi_F} \mathrm{Proj}(G,0;[P_{(G,n)}]) \xrightarrow{\hat{e}} [P_{(G,n)}].$$

Proof By Proposition 3.12, we have that $\sigma_k(G) = [P_{(G,n)}] + T_G = -[P^*_{(G,n)}] + T_G$ and so there exists $F \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ with $e(F) = -[P^*_{(G,n)}]$ by Proposition 3.7.

If $E \in \mathrm{Alg}(G,k-2)$, then $e(\Psi_F(E)) = -e(F)$ by Lemma 2.10. This implies that

$$e((*\circ\Psi_F)(E)) = -e(F)^* = [P_{(G,n)}]$$

and so the map $*\circ\Psi_F$ is as described. By Lemma 5.5, $*\circ\Psi_F$ is an isomorphism.

That $\hat{e}$ is an isomorphism follows from Proposition 4.10, as in the previous theorem. □

Proof of Theorem 5.3 If G has k–periodic cohomology, then it also has ik–periodic cohomology for any $i \geq 1$. Hence, by swapping k for ik, we can assume $i = 1$. By combining Theorems 5.8 and 5.9 with Proposition 5.1, we obtain injective maps of graded trees $\Psi\colon \mathrm{PHT}(G,n) \to [P_{(G,n)}]$ for $n = k$ or $k-2$, which are bijective as required. It remains to show that, in each case, Ψ has the form given in (i) and (ii).

If $n = k-2$, then $(*\circ\Psi_F)(C_*(\tilde{X})) \simeq (0 \to A \to P \to \mathbb{Z} \to 0)$ for some A and some $P \in [P_{(G,n)}]$. By Lemma 2.7, $\Psi_F(C_*(\tilde{X})) \simeq (0 \to \mathbb{Z} \to P^* \to A^* \to 0)$. Hence $(0 \to \mathbb{Z} \to P^* \to A^* \to 0) \circ C_*(\tilde{X}) \simeq F$ and $A^* \cong \pi_n(X)$.

If $n = k$, then $(-\circ F)^{-1}(C_*(\tilde{X})) \simeq (0 \to A \to P \to \mathbb{Z} \to 0)$ for some A and some $P \in [P_{(G,n)}]$. Hence $C_*(\tilde{X}) \simeq (0 \to A \to P \to \mathbb{Z} \to 0) \circ F$ and $A \cong \pi_n(X)$. □

This completes the proof of Theorem A.

6 Homotopy classification of (G,n)–complexes

For a finitely presented group G, an automorphism $\theta \in \mathrm{Aut}(G)$ acts on $\mathrm{PHT}(G,n)$ by $(X,\rho) \mapsto (X, \theta \circ \rho)$. It is straightforward to see that

$$\mathrm{HT}(G,n) \cong \mathrm{PHT}(G,n)/\mathrm{Aut}(G)$$

and the goal of this chapter will be to determine the induced action of $\mathrm{Aut}(G)$ on $[P_{(G,n)}]$ under the isomorphism $\mathrm{PHT}(G,n) \cong [P_{(G,n)}]$ obtained in Theorem 5.3.

6.1 Preliminaries on the action of Aut(G)

We begin by defining natural actions of $\mathrm{Aut}(G)$ on $\mathbb{Z}G$–modules and chain complexes of $\mathbb{Z}G$–modules. First, for a $\mathbb{Z}G$–module A and $\theta \in \mathrm{Aut}(G)$, let A_θ denote the $\mathbb{Z}G$–module whose underlying abelian group is that of A and whose action is $g \cdot x = \theta(g)x$ where $g \in G$, $x \in A$. This action has the following basic properties:

Lemma 6.1 *Let $\theta \in \mathrm{Aut}(G)$.*

(i) *There is a $\mathbb{Z}G$–module isomorphism*

$$i_\theta \colon \mathbb{Z}G \to \mathbb{Z}G_\theta, \quad \sum_{g \in G} a_i g_i \mapsto \sum_{g \in G} a_i \theta(g_i).$$

(ii) *If $A, B \in \mathrm{Mod}(\mathbb{Z}G)$, then $(A \oplus B)_\theta \cong A_\theta \oplus B_\theta$.*

(iii) *If $P \in P(\mathbb{Z}G)$, then $P_\theta \in P(\mathbb{Z}G)$.*

We can extend the action to chain complexes as follows. If A and B are $\mathbb{Z}G$–modules and

$$E = (E_*, \partial_*) \in \mathrm{Ext}^n_{\mathbb{Z}G}(A, B),$$

then we define $E_\theta \in \mathrm{Ext}^n_{\mathbb{Z}G}(A_\theta, B_\theta)$ by

$$E_\theta = (0 \to B_\theta \xrightarrow{\partial_n} (E_{n-1})_\theta \xrightarrow{\partial_{n-1}} (E_{n-2})_\theta \to \cdots \to (E_1)_\theta \xrightarrow{\partial_1} (E_0)_\theta \xrightarrow{\partial_0} A_\theta \to 0).$$

It is easy to see that this is well defined up to chain homotopy and, by the lemma above, it preserves projective extensions and so also induces a map on $\mathrm{hProj}^n_{\mathbb{Z}G}(A, B)$. The following is immediate from the definition of $\widetilde{C}_*(X, \rho)$.

Lemma 6.2 *If $E \in \mathrm{Alg}(G,n)$, then the induced action of $\theta \in \mathrm{Aut}(G)$ on E is given by $\theta \cdot E = E_\theta$, i.e. if $E = \widetilde{C}_*(X, \rho)$, then $E_\theta = \widetilde{C}_*(X, \theta \circ \rho)$.*

We now establish a few basic properties of this action which we will use later in this section. From now on, we will specialise to the case where G is a finite group. First, we note that the action commutes with dualising.

Lemma 6.3 *If A and B are $\mathbb{Z}G$–lattices, $E \in \mathrm{Proj}^n_{\mathbb{Z}G}(A, B)$ for $n \geq 1$ and $\theta \in \mathrm{Aut}(G)$, then*

$$(E_\theta)^* \cong (E^*)_\theta.$$

Proof We begin by proving the corresponding statement for modules, i.e. that, if A is a $\mathbb{Z}G$–lattice, then $(A_\theta)^* \cong (A^*)_\theta$. Let $A \cong_{\mathrm{Ab}} \mathbb{Z}^k$, so that the $\mathbb{Z}G$–module structure is determined by an integral representation $\rho_A \colon G \to \mathrm{GL}_k(\mathbb{Z})$. As remarked earlier, $\rho_{A^*}(g) = \rho_A(g^{-1})^T$ and it is easy to see that $\rho_{A_\theta} = \rho_A \circ \theta$. Therefore $(A_\theta)^* \cong (A^*)_\theta$ follows by noting that

$$\rho_{(A_\theta)^*}(g) = \rho_{A_\theta}(g^{-1})^T = \rho_A(\theta(g^{-1}))^T = \rho_A(\theta(g)^{-1})^T$$

and

$$\rho_{(A^*)_\theta}(g) = \rho_{A^*}(\theta(g)) = \rho_A(\theta(g)^{-1})^T.$$

The result for extensions now follows immediately since θ only affects the underlying modules and not the maps between them. □

In light of this, for $\mathbb{Z}G$–lattices A and B and $E \in \mathrm{Proj}^n_{\mathbb{Z}G}(A, B)$, it now makes sense to write A^*_θ and E^*_θ. Note that the action also commutes with pushouts.

Lemma 6.4 *If $\theta \in \mathrm{Aut}(G)$, $f \colon B_1 \to B_2$ is a $\mathbb{Z}G$–module homomorphism and $E \in \mathrm{Ext}^n_{\mathbb{Z}G}(A, B_1)$, then*

$$f_*(E_\theta) \cong (f_*(E))_\theta.$$

6.2 Proof of Theorem B

In the case where $A = B = \mathbb{Z}$, we can consider this as an action on $\mathrm{Proj}^n_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ by using the identification $\mathbb{Z}_\theta \cong \mathbb{Z}$.

Lemma 6.5 *If G has k–periodic cohomology, then there exists a unique map $\psi_k \colon \mathrm{Aut}(G) \to (\mathbb{Z}/|G|)^\times$ such that, for every $F \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ and $\theta \in \mathrm{Aut}(G)$,*

$$F_\theta \cong (m_{\psi_k(\theta)})_*(F).$$

Proof Fix an extension $F_0 \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$. By dualising and then applying Proposition 3.5, it follows that every extension in $\mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ is of the form $(m_r)_*(F_0)$ for some $r \in (\mathbb{Z}/|G|)^\times$. For $\theta \in \mathrm{Aut}(G)$, define $\psi_k(\theta) = r \in (\mathbb{Z}/|G|)^\times$ for any $r \in (\mathbb{Z}/|G|)^\times$ such that $(F_0)_\theta \cong (m_r)_*(F_0)$.

If $F \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$, then $F \cong (m_r)_*(F_0)$ for a unique $r \in (\mathbb{Z}/|G|)^\times$. By Lemma 6.4, we now have that

$$\begin{aligned} F_\theta &\cong ((m_r)_*(F_0))_\theta \cong (m_r)_*((F_0)_\theta) \cong (m_r)_*((m_{\psi_n(\theta)})_*(F_0)) \\ &\cong (m_{\psi_n(\theta)})_*((m_r)_*(F_0)) \cong (m_{\psi_n(\theta)})_*(F). \end{aligned}$$ □

Lemma 6.6 *If $E, E' \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$ and $r \in \mathbb{Z}$ is coprime to $|G|$, then*

$$E \circ (m_r)_*(E') \cong (m_r)_*(E) \circ E'.$$

Proof Consider the pushout map $\nu\colon E' \to (m_r)_*(E')$. Since this induces m_r on the left copy of $\mathbb{Z}$, we can extend it to a map $\tilde{\nu}\colon E \circ E' \to E \circ (m_r)_*(E')$ which induces multiplication by $r \in \mathbb{Z} \subseteq \mathbb{Z}G$ on every module in E, i.e.

$$\begin{array}{c} E \circ E' \\ \Big\downarrow \tilde{\nu} \\ E \circ (m_r)_*(E') \end{array} = \left(\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] \arrow[d, "r"] & P_{k-1} \arrow[r, "\partial_{k-1}"] \arrow[d, "r"] & \cdots \arrow[r, "\partial_1"] & P_0 \arrow[r, "i' \circ \varepsilon"] \arrow[d, "r"] & P'_{k-1} \arrow[r, "\partial'_{k-1}"] \arrow[d, "\nu_{k-1}"] & \cdots \arrow[r, "\partial'_1"] & P'_0 \arrow[r, "\varepsilon'"] \arrow[d, "\nu_0"] & \mathbb{Z} \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "i"] & P_{k-1} \arrow[r, "\partial_{k-1}"] & \cdots \arrow[r, "\partial_1"] & P_0 \arrow[r, "i' \circ \varepsilon"] & P'_{k-1} \arrow[r, "\partial'_{k-1}"] & \cdots \arrow[r, "\partial'_1"] & P'_0 \arrow[r, "\varepsilon'"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd} \right)$$

By the uniqueness of pushouts, this implies that $E \circ (m_r)_*(E') \cong (m_r)_*(E \circ E') = (m_r)_*(E) \circ E'$ as required. $\square$

Note that, if G has k–periodic cohomology and $k \mid n$, then it also has n–periodic cohomology and so ψ_n can still be defined using Lemma 6.5. The above lemma now allows us to give the following relation between ψ_k and ψ_n for $k \mid n$.

Lemma 6.7 *If G has k–periodic cohomology, $i \geq 1$ and $\theta \in \mathrm{Aut}(G)$, then*

$$\psi_{ik}(\theta) = \psi_k(\theta)^i.$$

Proof For $F \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ and $F^i \in \mathrm{Proj}^{ik}_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$, Lemma 6.5 implies that $F_\theta \cong (m_{\psi_k(\theta)})_*(F)$ and $(F^i)_\theta \cong (m_{\psi_{ik}(\theta)})_*(F^i)$. Since $(F^i)_\theta \cong (F_\theta)^i$, this implies that $(m_{\psi_{ik}(\theta)})_*(F^i) \cong ((m_{\psi_k(\theta)})_*(F))^i$.

By repeated application of Lemma 6.6,

$$(m_{\psi_{ik}(\theta)})_*(F^i) \cong ((m_{\psi_k(\theta)})_*(F))^i \cong (m_{\psi_k(\theta)})^i_*(F^i) \cong (m_{\psi_k(\theta)^i})_*(F^i)$$

and so $\psi_{ik}(\theta) \cong \psi_k(\theta)^i \bmod |G|$ by the extension of Proposition 3.5 to arbitrary extensions via the shifting lemma. $\square$

In order to prove Theorem B, it suffices to check what the action of $\mathrm{Aut}(G)$ corresponds to under the isomorphisms described in Section 5. Similarly to Section 5, it will suffice to consider the cases where $k = n$ or $n+2$.

Theorem 6.8 *Suppose that G has k–periodic cohomology and $\sigma_k(G) = [P_{(G,n)}] + T_G$ for some $P_{(G,n)} \in P(\mathbb{Z}G)$. If $F \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ is such that $e(F) = -[P_{(G,n)}]$, then*

$$\begin{gathered} \mathrm{hProj}^{k+1}_{\mathbb{Z}G}(\mathbb{Z}, A; 0) \xrightarrow{(-\circ F)^{-1}} \mathrm{hProj}^1_{\mathbb{Z}G}(\mathbb{Z}, A; [P_{(G,n)}]) \xrightarrow{\hat{e}} [P_{(G,n)}], \\ E \mapsto E' \mapsto P \oplus \mathbb{Z}G^r, \\ E_\theta \mapsto (m_{\psi_k(\theta)})^*((E')_\theta) \mapsto ((I, \psi_k(\theta)) \otimes P_\theta) \oplus \mathbb{Z}G^r, \end{gathered}$$

where P is a rank one projective $\mathbb{Z}G$–module and $r \geq 0$.

Proof For the first map, it suffices to check that $(\psi_k(\theta))^*((E')_\theta) \circ F \simeq E_\theta$. Since $E' \circ F = E$, we have $(E')_\theta \circ F_\theta \simeq E_\theta$. By Lemma 6.5, $F_\theta \cong (m_{\psi_n(\theta)})_*(F)$ and so

$$E_\theta \simeq (E')_\theta \circ (m_{\psi_n(\theta)})_*(F) = (m_{\psi_n(\theta)})^*((E')_\theta) \circ F.$$

The form for the second map follows directly from Lemma 4.12. □

Theorem 6.9 *Suppose that G has k–periodic cohomology and $\sigma_k(G) = [P_{(G,n)}] + T_G$ for some $P_{(G,n)} \in P(\mathbb{Z}G)$. If $F \in \mathrm{Proj}^k_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ is such that $e(F) = -[P^*_{(G,n)}]$, then*

$$\mathrm{hProj}^{k-1}_{\mathbb{Z}G}(\mathbb{Z}, A; 0) \xrightarrow{\Psi_F} \mathrm{hProj}^1_{\mathbb{Z}G}(A,\mathbb{Z}; [P^*_{(G,n)}]) \xrightarrow{*} \mathrm{hProj}^1_{\mathbb{Z}G}(\mathbb{Z}, A^*; [P_{(G,n)}]) \xrightarrow{\hat{e}} [P_{(G,n)}],$$
$$E \mapsto E' \mapsto (E')^* \mapsto P \oplus \mathbb{Z}G^r,$$
$$E_\theta \mapsto (m_{\psi_k(\theta)^{-1}})_*((E')_\theta) \mapsto (m_{\psi_k(\theta)})^*((E')^*_\theta) \mapsto ((I, \psi_k(\theta)) \otimes P_\theta) \oplus \mathbb{Z}G^r,$$

where P is a rank one projective $\mathbb{Z}G$–module and $r \geq 0$.

Proof For this first map, it suffices to check that $(m_{\psi_k(\theta)^{-1}})_*((E')_\theta) \circ E_\theta \simeq F$. Since $E' \circ E \simeq F$, we have $(E')_\theta \circ E_\theta \simeq F_\theta$. By Lemma 6.5, $F_\theta \cong (m_{\psi_n(\theta)})_*(F)$ and so

$$F \simeq (m_{\psi_k(\theta)^{-1}})_*((E')_\theta \circ E_\theta) \simeq (m_{\psi_k(\theta)^{-1}})_*((E')_\theta) \circ E_\theta.$$

For the second map, it is easy to see that pushouts dualise to pullbacks in the other direction, i.e. if $E_0 = (m_{\psi_k(\theta)^{-1}})_*((E')_\theta))$, then $(m_{\psi_k(\theta)^{-1}})_*(E^*_0) \simeq (E')^*_\theta$ and so $E^*_0 \simeq (m_{\psi_k(\theta)})^*((E')^*_\theta))$. The form for the third map follows directly from Lemma 4.12. □

If G has k–periodic cohomology and $n = ik$ or $ik - 2$ for some $i \geq 1$, then the above shows that the induced action of $\theta \in \mathrm{Aut}(G)$ on $[P_{(G,n)}]$ is given by $P \oplus \mathbb{Z}G^r \mapsto ((I, \psi_{ik}(\theta)) \otimes P_\theta) \oplus \mathbb{Z}G^r$ where P has rank one and $r \geq 0$. Furthermore, $\psi_{ik}(\theta) = \psi_k(\theta)^i$ by Lemma 6.7.

This completes the proof of Theorem B except for a possible discrepancy in the case where $k = 2$ and i is not determined by the fact that $n = ik$ or $ik - 2$ (see Remark 5.4). However, in this case, G is cyclic and so $(I, r) \cong \mathbb{Z}G$ for all $r \in (\mathbb{Z}/|G|)^\times$ by [Swan 1960b, Corollary 6.1]. Hence $(I, \psi_k(\theta)^i) \cong \mathbb{Z}G$ is independent of i.

7 Stably free Swan modules and (G,n)–complexes

Before computing the action of $\mathrm{Aut}(G)$ on $[P_{(G,n)}]$, we will pause to consider the role of Swan modules in the classification of (G,n)–complexes. We begin by considering the map

$$\psi_k \colon \mathrm{Aut}(G) \to (\mathbb{Z}/|G|)^\times$$

where G has k–periodic cohomology.

If $\theta \in \mathrm{Aut}(G)$, then the action $E \mapsto E_\theta$ induces an action of $\mathrm{Aut}(G)$ on $H^k(G;\mathbb{Z}) = \mathrm{Ext}^k_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$. This agrees with the usual action coming from the alternate definition of $H^k(-;\mathbb{Z})$ as a functor on groups

[Cartan and Eilenberg 1956, Chapter XII]. This implies that $\mathrm{Im}(\psi_k) = \mathrm{Aut}_k(G)$ which is defined in [Dyer 1976, Section 8]. We will now give several examples of maps $\psi_k\colon \mathrm{Aut}(G) \to (\mathbb{Z}/|G|)^\times$.

Cyclic If $C_n = \langle x \mid x^n = 1\rangle$ is the cyclic group of order n, then

$$\mathrm{Aut}(C_n) = \{\theta_i\colon x \mapsto x^i \mid i \in (\mathbb{Z}/n)^\times\}$$

and $\psi_2\colon \mathrm{Aut}(C_n) \to (\mathbb{Z}/n)^\times$ sends $\theta_i \mapsto i$ by [Swan 1960b, Proposition 8.1]. This is surjective and so recovers the classical results $T_{C_n} = 1$.

Dihedral If $D_{4n+2} = \langle x, y \mid x^{2n+1} = y^2 = 1, yxy^{-1} = x^{-1}\rangle$ is the dihedral group of order $4n+2$, then

$$\mathrm{Aut}(D_{4n+2}) = \{\theta_{i,j}\colon x \mapsto x^i,\ y \mapsto x^j y \mid i \in (\mathbb{Z}/(2n+1))^\times,\ j \in \mathbb{Z}/(2n+1)\}$$

and $\psi_4\colon \mathrm{Aut}(D_{4n+2}) \to (\mathbb{Z}/(4n+2))^\times$ sends $\theta_{i,j} \mapsto i^2$ by the discussion in [Johnson 2002, Section 5]. Since $(\mathbb{Z}/(4n+2))^\times = \pm((\mathbb{Z}/(4n+2))^\times)^2$, this recovers the result $T_{D_{4n+2}} = 1$.

Quaternionic Let $Q_{4n} = \langle x, y \mid x^n = y^2, yxy^{-1} = x^{-1}\rangle$ is the quaternion group of order $4n$. For $n = 2$, it is shown in [Swan 1960b, Proposition 8.3] that $\psi_4\colon \mathrm{Aut}(Q_8) \to (\mathbb{Z}/8)^\times$ sends $\theta \mapsto 1$ for all $\theta \in \mathrm{Aut}(G)$. For $n \geq 3$,

$$\mathrm{Aut}(Q_{4n}) = \{\theta_{i,j}\colon x \mapsto x^i,\ y \mapsto x^j y \mid i \in (\mathbb{Z}/2n)^\times,\ j \in \mathbb{Z}/2n\}$$

and $\psi_4\colon \mathrm{Aut}(Q_{4n}) \to (\mathbb{Z}/4n)^\times$ sends $\theta_{i,j} \mapsto i^2$ by, for example, [Golasiński and Gonçalves 2004, Proposition 1.1].

The following was noted by Davis [1983] and Dyer [1976, Note (b)]. It would be interesting to know, as was asked by Davis, whether this holds in the case $\sigma_k(G) \neq 0$.

Proposition 7.1 *If G has free period k, then $S \circ \psi_k = 0$, i.e. $(I, \psi_k(\theta))$ is stably free for all $\theta \in \mathrm{Aut}(G)$.*

Proof Note that Theorems 6.8 and 6.9 each show that $[P] = [(I, \psi_k(\theta)) \otimes P_\theta]$ for all $P \in P(\mathbb{Z}G)$ of rank one such that $\sigma_k(G) = [P] + T_G$. By Lemma 4.15, the composition

$$\mathrm{Aut}(G) \xrightarrow{\psi_k} (\mathbb{Z}/|G|)^\times \xrightarrow{S} T_G \leq C(\mathbb{Z}G)$$

is given by $S \circ \psi_k\colon \theta \mapsto [P] - [P_\theta]$ which is well defined since θ gives a well-defined action on $C(\mathbb{Z}G)$. By Lemma 6.1, $(\mathbb{Z}G)_\theta \cong \mathbb{Z}G$ and so the composition is trivial in the case where $\sigma_k(G) = 0$. □

We say that a finite group G has *weak cancellation* if every stably free Swan module is free. The following was asked by Dyer [1976, page 266] and later appeared in Wall's problems list [1979b, Problem A4].

Question 7.2 *Does there exist G with periodic cohomology and $r \in (\mathbb{Z}/|G|)^\times$ such that (I, r) is stably free but not free?*

This is equivalent to asking whether every group with periodic cohomology has weak cancellation and is still open, even for arbitrary finite groups. There are two important consequences that a negative answer to Question 7.2 would have.

First, recall the following question from the introduction. Note that, if $(I, \psi_k(\theta))$ is free, then the action described in Theorem B has the simpler form $P \mapsto P_\theta$.

Question 7.3 *Does there exist G with k–periodic cohomology and $\theta \in \mathrm{Aut}(G)$ for which $(I, \psi_k(\theta))$ is not free?*

It follows from Proposition 7.1 that, if G has free period k and has weak cancellation, then $(I, \psi_k(\theta)) \cong \mathbb{Z}G$ for all $\theta \in \mathrm{Aut}(G)$. In particular, if Question 7.2 has a negative answer, then the only groups for which the action in Theorem B might not have the form $P \mapsto P_\theta$ are the groups with $\sigma_k(G) \neq 0$.

Second, consider the following:

Question 7.4 *Let $n \geq 2$, let G be finite and let X and Y be finite (G, n)–complexes with $\chi(X) = \chi(Y)$. Then $X \vee rS^n \simeq Y \vee rS^n$ for some r. Does $r = 1$ always work?*

This is equivalent to asking whether $\mathrm{HT}(G, n)$ is a fork when G is finite. The case where n is even was proven by Browning [1978], and also follows by combining Corollary 4.7 and Proposition 5.1. When n is odd, this is known to hold provided G does not have k–periodic cohomology for any $k \mid n+1$. If G has k–periodic cohomology for $k \mid n+1$, then this holds provided G has weak cancellation (see [Dyer 1976, pages 276–277]). In particular, if Question 7.2 has a negative answer, then Question 7.4 has an affirmative answer. Note that the corresponding question for infinite groups is also still open (see [Nicholson 2021c, Problem B2]).

8 Milnor squares and the classification of projective modules

Given the observations in the previous section, the primary obstacle to computing sufficiently interesting examples of $\mathrm{HT}(G, n)$ and $\mathrm{PHT}(G, n)$ for our groups is the classification of projective $\mathbb{Z}G$–modules.

One method to classify projective R–modules over a ring R is to relate this to the classification of projective modules over simpler rings using Milnor squares. In this section, we will present a refinement of the basic theory of Milnor squares which will also allow us to determine how a ring automorphism $\alpha \in \mathrm{Aut}(R)$ acts on the class of projective R–modules. We will then apply these methods in Section 9.

Suppose R and S are rings and $f : R \to S$ is a ring homomorphism. We can use this to turn S into an (S, R)–bimodule, with right-multiplication by $r \in R$ given by $x \cdot r = xf(r)$ for any $x \in S$. If M is an R–module, we can define the *extension of scalars* of M by f as the tensor product

$$f_\#(M) = S \otimes_R M$$

since S as a right R–module and M as a left R–module, and we consider this as a left S–module where left multiplication by $s \in S$ is given by $s \cdot (x \otimes m) = (sx) \otimes m$ for any $x \in S$ and $m \in M$. This comes equipped with maps of abelian groups

$$f_*\colon M \to f_\#(M)$$

sending $m \mapsto 1 \otimes m$, and defines a covariant functor from R–modules to S–modules [Curtis and Reiner 1981, page 227]. It has the following basic properties which follow from the standard properties of tensor products such as associativity [Mac Lane 1963, page 145].

Lemma 8.1 *Let $f\colon R \to S$ and $g\colon S \to T$ be ring homomorphisms and let M and N be R–modules. Then*

(i) $f_\#(M \oplus N) \cong f_\#(M) \oplus f_\#(N)$,

(ii) $f_\#(R) \cong S$,

(iii) $(g \circ f)_\#(M) \cong (g_\# \circ f_\#)(M)$.

If $P(R)$ denotes the set of isomorphism classes of projective R–modules, then the first two properties show that $f_\#$ induces a map $f_\#\colon P(R) \to P(S)$ which restricts to each stable class.

Recall that, if R, R_1, R_2 and R_0 are rings, then a pullback diagram

$$\mathcal{R} = \begin{array}{ccc} R & \xrightarrow{i_2} & R_2 \\ \Big\downarrow{\scriptstyle i_1} & & \Big\downarrow{\scriptstyle j_2} \\ R_1 & \xrightarrow{j_1} & R_0 \end{array}$$

is a Milnor square if either j_1 or j_2 are surjective. If $P_1 \in P(R_1)$ and $P_2 \in P(R_2)$ are such that there is an R_0–module isomorphism $h\colon (j_1)_\#(P_1) \to (j_2)_\#(P_2)$, then define

$$M(P_1, P_2, h) = \{(x, y) \in P_1 \times P_2 \mid h((j_1)_*(x)) = (j_2)_*(y)\} \le P_1 \times P_2,$$

which is an R–module where multiplication by $r \in R$ is given by $r \cdot (x, y) = ((i_1)_*(r)x, (i_2)_*(r)y)$. It was shown by Milnor that $M(P_1, P_2, h)$ is projective [Milnor 1971, Theorem 2.1]. Let $\mathrm{Aut}_R(P)$ denote the set of R–module automorphisms of an R–module P. The main result on Milnor squares is as follows. This is a consequence of the results in [Milnor 1971, Section 2] and the precise statement can be found in [Swan 1980, Proposition 4.1].

Theorem 8.2 *Suppose $\mathcal{R}$ is a Milnor square and $P_i \in P(R_i)$ for $i = 0, 1, 2$ are such that*

$$P_0 \cong (j_1)_\#(P_1) \cong (j_2)_\#(P_2)$$

as R_0–modules. Then there is a one-to-one correspondence

$$\mathrm{Aut}_{R_1}(P_1)\backslash\mathrm{Aut}_{R_0}(P_0)/\mathrm{Aut}_{R_2}(P_2) \leftrightarrow \{P \in P(R) \mid (i_1)_\#(P) \cong P_1,\ (i_2)_\#(P) \cong P_2\}$$

given by sending a coset $[h]$ to $M(P_1, P_2, h)$ for any representative h.

Now suppose $\alpha \in \operatorname{Aut}(R)$. If M is an R–module, define M_α as the R–module whose abelian group is that of M but whose R–action is given by $r \cdot m = \alpha(r)m$ for $r \in R$ and $m \in M$. For example, if $R = \mathbb{Z}G$, then $\theta \in \operatorname{Aut}(G)$ induces a map $\theta \in \operatorname{Aut}(\mathbb{Z}G)$ and M_θ coincides with the definition given earlier.

This is a special case of restriction of scalars, but can also be viewed as a part of extension of scalars as follows.

Lemma 8.3 *Let R be a ring and let $\alpha \in \operatorname{Aut}(R)$. If M is an R–module, then there is an isomorphism of R–modules*

$$\psi : M_\alpha \to (\alpha^{-1})_\#(M)$$

given by $m \mapsto 1 \otimes m$.

From this, it is clear that this action has basic properties which are analogous to Lemma 6.1. The following is then immediate by combining Lemmas 8.1 and 8.3.

Corollary 8.4 *Suppose $f : R \to S$ is a ring homomorphism and $\alpha \in \operatorname{Aut}(R)$ and $\beta \in \operatorname{Aut}(S)$ are such that $f \circ \alpha = \beta \circ f$. If M is an R–module, then*

$$f_\#(M_\alpha) \cong f_\#(M)_\beta.$$

We can turn the set of Milnor squares into a category with morphisms defined as follows. If $\mathscr{R}$ and $\mathscr{R}'$ are Milnor squares, then a morphism is a quadruple

$$\hat{\alpha} = (\alpha, \alpha_1, \alpha_2, \alpha_0) : \mathscr{R} \to \mathscr{R}'$$

where $\alpha : R \to R'$ and $\alpha_i : R_i \to R'_i$ such that there is a commutative diagram

$$\begin{tikzcd}
R \arrow[rr] \arrow[dd] \arrow[dr, "\alpha"] & & R_2 \arrow[dd] \arrow[dr, "\alpha_2"] & \\
& R \arrow[rr] \arrow[dd] & & R_2 \arrow[dd] \\
R_1 \arrow[rr] \arrow[dr, "\alpha_1"] & & R_0 \arrow[dr, "\alpha_0"] & \\
& R_1 \arrow[rr] & & R_0
\end{tikzcd}$$

Let $\operatorname{Aut}(\mathscr{R})$ denote the set of automorphisms of a Milnor square $\mathscr{R}$, i.e. the set of isomorphisms $\hat{\alpha} : \mathscr{R} \to \mathscr{R}$.

Lemma 8.5 *Let $\mathscr{R}$ be a Milnor square and let $P_1 \in P(R_1)$ and $P_2 \in P(R_2)$ be such that there is an R_0–module isomorphism $h : (j_1)_\#(P_1) \to (j_2)_\#(P_2)$. If $\hat{\alpha} = (\alpha, \alpha_1, \alpha_2, \alpha_0) \in \operatorname{Aut}(\mathscr{R})$, then*

$$M(P_1, P_2, h)_\alpha \cong M((P_1)_{\alpha_1}, (P_2)_{\alpha_2}, h)$$

where, on the right, we view h as a map $h : (j_1)_\#(P_1)_{\alpha_0} \to (j_2)_\#(P_2)_{\alpha_0}$.

Proof Let $P = M(P_1, P_2, h)$ so that, by Theorem 8.2, $(i_1)_\#(P) \cong P_1$ and $(i_2)_\#(P) \cong P_2$. It is easy to see directly that the natural map

$$M((i_1)_\#(P), (i_2)_\#(P), h) \to M((i_1)_\#(P_\alpha), (i_2)_\#(P_\alpha), h)$$

is an isomorphism. We are then done by applying Corollary 8.4. □

This has the following simplification when P_1 and P_2 are free of rank one. Here we will use the identification $\mathrm{Aut}_{R_0}(R_0) \cong R_0^\times$ which sends $h\colon R_0 \to R_0$ to $h(1) \in R_0^\times$.

Lemma 8.6 *Let $\mathcal{R}$ be a Milnor square and let $u \in R_0^\times$. If $\hat{\alpha} = (\alpha, \alpha_1, \alpha_2, \alpha_0) \in \mathrm{Aut}(\mathcal{R})$, then*

$$M(R_1, R_2, u)_\alpha \cong M(R_1, R_2, \alpha_0^{-1}(u)).$$

Proof Fix identifications $\psi_i\colon (j_i)_\#(R_i) \to R_0$ and let $h\colon (j_1)_\#(R_1) \to (j_1)_\#(R_1)$ be such that

$$(\psi_2 \circ h \circ \psi_1^{-1})(1) = u \in R_0^\times.$$

By Lemma 8.5,

$$M(R_1, R_2, h)_\alpha \cong M((R_1)_{\alpha_1}, (R_2)_{\alpha_2}, h)$$

where $h\colon ((j_1)_\#(R_1))_{\alpha_0} \to ((j_1)_\#(R_1))_{\alpha_0}$ coincides with h as a map of abelian groups. For $i = 0, 1, 2$, let $c_i\colon R_i \to (R_i)_{\alpha_i}$ be the isomorphism which sends $1 \mapsto 1$. Then it is easy to see that

$$\begin{tikzcd}
(j_i)_\#(R_i) \arrow[r, "1\otimes c_i"] \arrow[d, "\psi_i"] & ((j_i)_\#((R_i)_{\alpha_i}) \arrow[r, "f"] & ((j_i)_\#(R_i))_{\alpha_0} \arrow[d, "\psi_i"] \\
R_0 \arrow[rr, "c_0"] & & (R_0)_{\alpha_0}
\end{tikzcd}$$

commutes for $i = 1, 2$, where $f\colon (j_i)_\#((R_i)_{\alpha_i}) \to ((j_i)_\#(R_i))_{\alpha_0}$ is the isomorphism coming from Corollary 8.4. Using the isomorphisms c_i for $i = 1, 2$, we get that

$$M((R_1)_{\alpha_1}, (R_2)_{\alpha_2}, h) \cong M(R_1, R_2, h_0)$$

where $h_0\colon (j_1)_\#(R_1) \to (j_2)_\#(R_2)$ induces $h\colon ((j_1)_\#(R_1))_{\alpha_0} \to ((j_1)_\#(R_1))_{\alpha_0}$ via $f \circ (1 \otimes c_i)$. Let $u_0 = (\psi_2 \circ h_0 \circ \psi_1^{-1})(1) \in R_0^\times$. Then, since the above diagram commutes, we get the commutative diagram

$$\begin{tikzcd}
R_0 \arrow[r, "\psi_2\circ h\circ\psi_1^{-1}"] \arrow[d, "c_0"] & R_0 \arrow[d, "c_0"] \\
(R_0)_{\alpha_0} \arrow[r, "\psi_2\circ h_0\circ\psi_1^{-1}"] & (R_0)_{\alpha_0}
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, mapsto] \arrow[d, mapsto] & u_0 \arrow[d, mapsto] \\
1 \arrow[r, mapsto] & \alpha_0(u_0)
\end{tikzcd}$$

which implies that $u = \alpha_0(u_0)$ and so $u_0 = \alpha_0^{-1}(u)$, as required. □

If $\mathcal{R}$ is a Milnor square, we say that $\alpha \in \mathrm{Aut}(R)$ *extends across* $\mathcal{R}$ if there exists $\hat{\alpha} = (\alpha, \alpha_1, \alpha_2, \alpha_0) \in \mathrm{Aut}(\mathcal{R})$. The following gives conditions under which this induced map is unique.

Lemma 8.7 *Let $\mathcal{R}$ be a pullback square with all maps surjective. If $\alpha \in \mathrm{Aut}(R)$ extends across $\mathcal{R}$, then it does so uniquely. That is, there exist unique maps α_1, α_2 and α_0 for which $\hat{\alpha} = (\alpha, \alpha_1, \alpha_2, \alpha_0) \in \mathrm{Aut}(\mathcal{R})$.*

Proof This follows from the simple observation that, if $f: R \twoheadrightarrow S$ is a surjective ring homomorphism and $\alpha: R \to R$ and $\beta_1, \beta_2: S \to S$ are ring homomorphisms such that $f \circ \alpha = \beta_i \circ f$ for $i = 1, 2$, then $\beta_1 = \beta_2$. To see this, note that the conditions imply that $(\beta_1 - \beta_2) \circ f = 0$ and so $\beta_1 = \beta_2$ on $\mathrm{Im}(f)$. Since f is surjective, $\mathrm{Im}(f) = S$ and so $\beta_1 = \beta_2$. □

We conclude this section with the following result which is a consequence of Theorem 8.2 and Lemmas 8.6 and 8.7.

Proposition 8.8 *Let $\mathcal{R}$ be a pullback square with all maps surjective and such that every $\alpha \in \mathrm{Aut}(R)$ extends across $\mathcal{R}$. Then there is a one-to-one correspondence*

$$R_1^\times \backslash (R_0^\times / \mathrm{Aut}(R)) / R_2^\times \leftrightarrow \{P \in P(R) : (i_1)_\#(P) \cong R_1, (i_2)_\#(P) \cong R_2\} / \mathrm{Aut}(R)$$

where $\alpha \in \mathrm{Aut}(R)$ acts on $R_0^\times$ by sending $r \mapsto \alpha_0^{-1}(r)$ for $r \in R_0^\times$ and where $\alpha_0 \in \mathrm{Aut}(R_0)$ is the unique automorphism such that $\hat{\alpha} = (\alpha, \alpha_1, \alpha_2, \alpha_0) \in \mathrm{Aut}(\mathcal{R})$.

9 Example: quaternion groups

The aim of this section is to illustrate how Theorems A and B can be combined with the known techniques to classify projective $\mathbb{Z}G$–modules to obtain a detailed classification of finite (G, n)–complexes up to homotopy equivalence.

For $k \geq 2$, recall that the quaternion group of order $4k$ has presentation

$$Q_{4k} = \langle x, y \mid x^k = y^2, yxy^{-1} = x^{-1} \rangle.$$

It is a finite 3–manifold group and so has free period 4. For $n \geq 2$ even, Theorem A and Proposition 5.1 imply that $\mathrm{PHT}(Q_{4k}, n) \cong [\mathbb{Z}Q_{4k}] = \bigcup_{r \geq 1} \mathrm{SF}_r(\mathbb{Z}Q_{4k})$ where $\mathrm{SF}_r(\mathbb{Z}Q_{4k})$ is the set of stably free $\mathbb{Z}Q_{4k}$–modules of rank $r \geq 1$.

Since stably free $\mathbb{Z}G$–modules of rank ≥ 2 are free for G finite [Swan 1960a] (or since $\mathrm{PHT}(G, n)$ is a fork by Corollary 4.7), it remains to compute $\mathrm{SF}_1(\mathbb{Z}Q_{4k})$. This was completed by Swan [1983, Theorem III] for $k \leq 9$. For $k \leq 7$, he showed that $|\mathrm{SF}_1(\mathbb{Z}Q_{4k})| = 1$ for $2 \leq k \leq 5$, $|\mathrm{SF}_1(\mathbb{Z}Q_{24})| = 3$ and $|\mathrm{SF}_1(\mathbb{Z}Q_{28})| = 2$. It also follows from his classification that $\mathbb{Z}Q_{4k}$ has weak cancellation in all these cases and so the action of $\theta \in \mathrm{Aut}(Q_{4k})$ on $[\mathbb{Z}Q_{4k}]$ sends $P \mapsto P_\theta$ (see Section 7).

In the case Q_{28}, the action of $\mathrm{Aut}(Q_{28})$ on $[\mathbb{Z}Q_{28}]$ is trivial since $(\mathbb{Z}Q_{28})_\theta \cong \mathbb{Z}Q_{28}$ for all $\theta \in \mathrm{Aut}(Q_{28})$ and so this must also hold for the nonfree stably free module also. The main result of this section will be to compute the action in the case Q_{24}.

Theorem 9.1 *$\mathrm{Aut}(Q_{24})$ acts nontrivially on $[\mathbb{Z}Q_{24}]$. More specifically, we have $|\mathrm{SF}_1(\mathbb{Z}Q_{24})| = 3$ and $|\mathrm{SF}_1(\mathbb{Z}Q_{24})/\mathrm{Aut}(Q_{24})| = 2$.*

G	Q_8	Q_{12}	Q_{16}	Q_{20}	Q_{24}	Q_{28}
$\mathrm{PHT}(G,n)$	•	•	•	•	• • •	• •
$\mathrm{HT}(G,n)$	•	•	•	•	• •	• •

Table 1: Minimal complexes for any n even with $n \neq 2$.

All of this is summarised in Table 1, which gives the structure of $\mathrm{PHT}(G,n)$ and $\mathrm{HT}(G,n)$ when $n \neq 2$ is even. These graded trees are both forks by Corollary 4.7 and each dot represents a finite (G,n)–complex at the minimal level.

Remark 9.2 This also holds in the case $n=2$ provided G has the D2 property. This holds trivially in the cases Q_8, Q_{12}, Q_{16} and Q_{20}, and is otherwise only known to be true in the case Q_{28} by [Nicholson 2021b, Theorem 7.7] using the presentation of Mannan and Popiel [2021].

We will now proceed to the proof of Theorem 9.1. First let x and y be generators for Q_{24} in the presentation given above. Let $\Lambda = \mathbb{Z}Q_{24}/(x^6+1)$ and note that the quotient map $f\colon \mathbb{Z}Q_{24} \twoheadrightarrow \Lambda$ induces a map

$$f_\#\colon \mathrm{SF}_1(\mathbb{Z}Q_{24}) \to \mathrm{SF}_1(\Lambda)$$

by Lemma 8.1. This is a bijection by the proof of [Swan 1983, Theorem 11.14].

Now note that the factorisation $x^6+1 = (x^2+1)(x^4-x^2+1)$ implies that the ideals $I=(x^2+1)$ and $J=(x^4-x^2+1)$ have $I \cap J = (x^6+1)$ and $I+J=(3,x^2+1)$. It follows from [Curtis and Reiner 1987, Example 42.3] that we have a pullback diagram

$$\begin{tikzcd}
\Lambda \arrow[r] \arrow[d] & \mathbb{Z}Q_{24}/(x^4-x^2+1) \arrow[d] \\
\mathbb{Z}Q_{24}/(x^2+1) \arrow[r] & \mathbb{F}_3 Q_{24}/(x^2+1)
\end{tikzcd}$$

which is a Milnor square since all maps are surjective.

For a field $\mathbb{F}$, let $\mathbb{H}_{\mathbb{F}} = \mathbb{F}[i,j]$ denote the quaternions over $\mathbb{F}$ and we define $\mathbb{H}_{\mathbb{Z}} = \mathbb{Z}[i,j]$ and $\mathbb{Z}[\zeta_{12},j]$ to be subrings of $\mathbb{H}_{\mathbb{R}}$, where $\zeta_{12} = e^{2\pi i/12}$ is the 12^{th} root of unity in the i direction. It is straightforward to check that there are isomorphisms of rings

$$\phi_1\colon \mathbb{H}_{\mathbb{Z}} \to \mathbb{Z}Q_{24}/(x^2+1), \quad i \mapsto x,\ j \mapsto y$$
$$\phi_2\colon \mathbb{Z}[\zeta_{12},j] \to \mathbb{Z}Q_{24}/(x^4-x^2+1), \quad \zeta_{12} \mapsto x,\ j \mapsto y.$$

Using this, we can rewrite the Milnor square above as

$$\mathcal{R} = \begin{tikzcd}
\Lambda \arrow[r, "i_2"] \arrow[d, "i_1"] & \mathbb{Z}[\zeta_{12},j] \arrow[d, "j_2"] \\
\mathbb{H}_{\mathbb{Z}} \arrow[r, "j_1"] & \mathbb{H}_{\mathbb{F}_3}
\end{tikzcd} \qquad \begin{tikzcd}
x,y \arrow[r, mapsto] \arrow[d, mapsto] & \zeta_{12},j \arrow[d, mapsto] \\
i,j \arrow[r, mapsto] & i,j
\end{tikzcd}$$

By [Swan 1983, Lemma 8.14], the induced map $(i_2)_*: C(\Lambda) \to C(\mathbb{Z}[\zeta_{12}, j])$ is an isomorphism. It also follows from [Swan 1983, page 84] that the rings $\mathbb{H}_{\mathbb{Z}}$ and $\mathbb{Z}[\zeta_{12}, j]$ both have stably free cancellation, i.e. that every stably free module is free. It follows easily that

$$\mathrm{SF}_1(\Lambda) = \{P \in P(\Lambda) : (i_1)_\#(P) \cong \mathbb{H}_{\mathbb{Z}}, (i_2)_\#(P) \cong \mathbb{Z}[\zeta_{12}, j]\}.$$

In particular, by combining with Theorem 8.2, we get that there is a bijection

$$\mathrm{SF}_1(\Lambda) \leftrightarrow \mathbb{H}_{\mathbb{Z}}^\times \backslash \mathbb{H}_{\mathbb{F}_3}^\times / \mathbb{Z}[\zeta_{12}, j]^\times.$$

Lemma 9.3 $\mathbb{H}_{\mathbb{Z}}^\times \backslash \mathbb{H}_{\mathbb{F}_3}^\times / \mathbb{Z}[\zeta_{12}, j]^\times = \{[1], [1+j], [1+k]\}$.

Proof If $N : \mathbb{H}_{\mathbb{F}_3} \to \mathbb{F}_3$ is the norm, then $\mathbb{H}_{\mathbb{F}_3}^\times = N^{-1}(\pm 1)$. Now note that $\mathbb{H}_{\mathbb{Z}}^\times = \{\pm 1, \pm i, \pm j, \pm k\}$, and it is easy to check that

$$\mathbb{H}_{\mathbb{Z}}^\times \backslash \mathbb{H}_{\mathbb{F}_3}^\times = \big\{[1], [1+i], [1+j], [1+k], [1+i+j+k], [1-i-j-k]\big\}.$$

By [Magurn et al. 1983, Lemma 7.5(b)], $\mathbb{Z}[\zeta_{12}, j]^\times = \mathbb{Z}[\zeta_{12}]^\times \cdot \langle j \rangle$ and so it remains to determine

$$\mathrm{Im}(\mathbb{Z}[\zeta_{12}, j]^\times \to \mathbb{H}_{\mathbb{Z}}^\times \backslash \mathbb{H}_{\mathbb{F}_3}^\times) = \mathrm{Im}(\mathbb{Z}[\zeta_{12}]^\times \to \mathbb{H}_{\mathbb{Z}}^\times \backslash \mathbb{H}_{\mathbb{F}_3}^\times) \subseteq \{[1], [1+i]\},$$

where the last inclusion follows since $\zeta_{12} \mapsto i$ and $\mathbb{H}_{\mathbb{Z}}^\times \backslash \langle 1, i \rangle = \{[1], [1+i]\}$.

Consider the n^{th} cyclotomic polynomial

$$\Phi_n(x) = \prod_{k \in \mathbb{Z}_n^\times} (x - \zeta_n^k).$$

It is well known, and can be shown using Möbius inversion, that $\Phi_n(1) = 1$ if n is not a prime power. In particular, $\Phi_{12}(1) = 1$ and this implies that $1 - \zeta_{12} \in \mathbb{Z}[\zeta_{12}]^\times$. Hence

$$[1+i] = [1-i] \in \mathrm{Im}(\mathbb{Z}[\zeta_{12}]^\times \to \mathbb{H}_{\mathbb{Z}}^\times \backslash \mathbb{H}_{\mathbb{F}_3}^\times).$$

The result then follows since

$$j(1+i+j+k)(1+i) = 1+k, \quad -j(1-i-j-k)(1+i) = 1+j$$

implies that $[1+j] = [1-i-j-k]$ and $[1+k] = [1+i+j+k]$ in $\mathbb{H}_{\mathbb{Z}}^\times \backslash \mathbb{H}_{\mathbb{F}_3}^\times / \mathbb{Z}[\zeta_{12}, j]^\times$. □

This implies that $|\mathrm{SF}_1(\mathbb{Z}Q_{24})| = 3$, which recovers the result of Swan. In order to determine the action of $\mathrm{Aut}(Q_{24})$ on $\mathrm{SF}_1(\mathbb{Z}Q_{24})$, first recall from Section 7 that

$$\mathrm{Aut}(Q_{24}) = \{\theta_{a,b} : x \mapsto x^a,\ y \mapsto x^b y \mid a \in (\mathbb{Z}/12)^\times, b \in \mathbb{Z}/12\}.$$

If $\mathcal{R}$ denotes the Milnor square defined above, then the following is easy to check.

Lemma 9.4 *If $a \in (\mathbb{Z}/12)^\times$ and $b \in \mathbb{Z}/12$, then $\theta_{a,b} \in \mathrm{Aut}(Q_{24})$ extends to a Milnor square automorphism*

$$\hat{\theta}_{a,b} = (\theta'_{a,b}, \theta^1_{a,b}, \theta^2_{a,b}, \bar{\theta}_{a,b}) \in \mathrm{Aut}(\mathcal{R})$$

where, for $a = 2a_0 + 1$, the maps are defined as follows:

(i) $\theta'_{a,b} \in \mathrm{Aut}(\mathbb{Z}Q_{24}/(x^6+1))$ *is given by* $x \mapsto x^a$ *and* $y \mapsto x^b y$.

(ii) $\theta^1_{a,b} \in \mathrm{Aut}(\mathbb{H}_{\mathbb{Z}})$ *and* $\bar{\theta}_{a,b} \in \mathrm{Aut}(\mathbb{H}_{\mathbb{F}_3})$ *are each given by*

$$i \mapsto i^a = (-1)^{a_0} i, \qquad j \mapsto j^b = \begin{cases} (-1)^{b_0} j & \text{if } b = 2b_0 + 1, \\ (-1)^{b_0} k & \text{if } b = 2b_0. \end{cases}$$

(iii) $\theta^2_{a,b} \in \mathrm{Aut}(\mathbb{Z}[\zeta_{12}, j])$ *is given by* $\zeta_{12} \mapsto \zeta_{12}^a$ *and* $j \mapsto \zeta_{12}^b j$.

Since $\mathcal{R}$ is a pullback square with all maps surjective, we can now apply Proposition 8.8. By combining with Lemma 9.3, this implies that there is a bijection

$$\mathrm{SF}_1(\mathbb{Z}Q_{24})/\mathrm{Aut}(Q_{24}) \leftrightarrow \{[1],[1+j],[1+k]\}/\mathrm{Aut}(Q_{24})$$

where $\theta_{a,b} \in \mathrm{Aut}(Q_{24})$ acts on the double cosets via the action described in Lemma 9.4. In particular,

$$\bar{\theta}_{a,b}([1+j]) = \begin{cases} [1+(-1)^{b_0} j] = [1+j] & \text{if } b = 2b_0+1, \\ [1+(-1)^{b_0} k] = [1+k] & \text{if } b = 2b_0, \end{cases}$$

and so $\bar{\theta}_{a,b}$ acts nontrivially when b is even. Hence $|\mathrm{SF}_1(\mathbb{Z}Q_{24})/\mathrm{Aut}(Q_{24})| = 2$. This completes the proof of Theorem 9.1.

References

[Bestvina and Brady 1997] **M Bestvina**, **N Brady**, *Morse theory and finiteness properties of groups*, Invent. Math. 129 (1997) 445–470 MR Zbl

[Bieri 1976] **R Bieri**, *Homological dimension of discrete groups*, Queen Mary College, London (1976) MR Zbl

[Bokor et al. 2021] **I Bokor**, **D Crowley**, **S Friedl**, **F Hebestreit**, **D Kasproswki**, **M Land**, **J Nicholson**, *Connected sum decompositions of high-dimensional manifolds*, from "2019–20 MATRIX annals" (D R Wood, J de Gier, C E Praeger, T Tao, editors), MATRIX Book Ser. 4, Springer (2021) 5–30 MR Zbl

[Brown 1982] **K S Brown**, *Cohomology of groups*, Grad. Texts in Math. 87, Springer (1982) MR Zbl

[Browning 1978] **W J Browning**, *Homotopy types of certain finite CW–complexes with finite fundamental group*, PhD thesis, Cornell University (1978) Available at `https://www.proquest.com/docview/302945517`

[Browning 1979] **W J Browning**, *Finite CW–complexes of cohomological dimension* 2 *with finite abelian* π_1, unpublished (1979)

[Cartan and Eilenberg 1956] **H Cartan**, **S Eilenberg**, *Homological algebra*, Princeton Landmarks Math. Phys. 19, Princeton Univ. Press (1956) MR Zbl

[Cockcroft and Swan 1961] **W H Cockcroft**, **R G Swan**, *On the homotopy type of certain two-dimensional complexes*, Proc. Lond. Math. Soc. 11 (1961) 194–202 MR Zbl

[Cohen 1977] **J M Cohen**, *Complexes dominated by a 2–complex*, Topology 16 (1977) 409–415 MR Zbl

[Conway et al. 2021] **A Conway**, **D Crowley**, **M Powell**, **J Sixt**, *Stably diffeomorphic manifolds and modified surgery obstructions*, preprint (2021) arXiv 2109.05632

[Conway et al. 2023] **A Conway**, **D Crowley**, **M Powell**, **J Sixt**, *Simply connected manifolds with large homotopy stable classes*, J. Aust. Math. Soc. 115 (2023) 172–203 MR Zbl

[Curtis and Reiner 1981] **C W Curtis**, **I Reiner**, *Methods of representation theory, I: With applications to finite groups and orders*, Wiley, New York (1981) MR Zbl

[Curtis and Reiner 1987] **C W Curtis**, **I Reiner**, *Methods of representation theory, II: With applications to finite groups and orders*, Wiley, New York (1987) MR Zbl

[Davis 1983] **J F Davis**, *A remark on: 'Homotopy equivalences and free modules' [Topology* **21** *(1982) 91–99] by S Plotnick*, Topology 22 (1983) 487–488 MR Zbl

[Dyer 1976] **M N Dyer**, *Homotopy classification of (π, m)–complexes*, J. Pure Appl. Algebra 7 (1976) 249–282 MR Zbl

[Dyer 1979] **M N Dyer**, *Nonminimal roots in homotopy trees*, Pacific J. Math. 80 (1979) 371–380 MR Zbl

[Dyer and Sieradski 1973] **M N Dyer**, **A J Sieradski**, *A homotopy classification of 2–complexes with finite cyclic fundamental group*, Bull. Amer. Math. Soc. 79 (1973) 75–77 MR Zbl

[Dyer and Sieradski 1975] **M N Dyer**, **A J Sieradski**, *Trees of homotopy types of 2–dimensional CW complexes, II*, Trans. Amer. Math. Soc. 205 (1975) 115–125 MR Zbl

[Fröhlich et al. 1974] **A Fröhlich**, **I Reiner**, **S Ullom**, *Class groups and Picard groups of orders*, Proc. Lond. Math. Soc. 29 (1974) 405–434 MR Zbl

[Golasiński and Gonçalves 2004] **M Golasiński**, **D L Gonçalves**, *Spherical space forms: homotopy types and self-equivalences*, from "Categorical decomposition techniques in algebraic topology" (G Arone, J Hubbuck, R Levi, M Weiss, editors), Progr. Math. 215, Birkhäuser, Basel (2004) 153–165 MR Zbl

[Hambleton 2019] **I Hambleton**, *Two remarks on Wall's $D2$ problem*, Math. Proc. Cambridge Philos. Soc. 167 (2019) 361–368 MR Zbl

[Hambleton and Kreck 1993] **I Hambleton**, **M Kreck**, *Cancellation of lattices and finite two-complexes*, J. Reine Angew. Math. 442 (1993) 91–109 MR Zbl

[Hambleton et al. 2013] **I Hambleton**, **S Pamuk**, **E Yalçın**, *Equivariant CW–complexes and the orbit category*, Comment. Math. Helv. 88 (2013) 369–425 MR Zbl

[Jacobinski 1968] **H Jacobinski**, *Genera and decompositions of lattices over orders*, Acta Math. 121 (1968) 1–29 MR Zbl

[Johnson 2002] **F E A Johnson**, *Explicit homotopy equivalences in dimension two*, Math. Proc. Cambridge Philos. Soc. 133 (2002) 411–430 MR Zbl

[Johnson 2003] **F E A Johnson**, *Stable modules and the $D(2)$–problem*, Lond. Math. Soc. Lect. Note Ser. 301, Cambridge Univ. Press (2003) MR Zbl

[Kreck and Schafer 1984] **M Kreck**, **J A Schafer**, *Classification and stable classification of manifolds: some examples*, Comment. Math. Helv. 59 (1984) 12–38 MR Zbl

[Linnell 1993] **P A Linnell**, *Minimal free resolutions and (G, n)–complexes for finite abelian groups*, Proc. Lond. Math. Soc. 66 (1993) 303–326 MR Zbl

[Mac Lane 1963] **S Mac Lane**, *Homology*, Grundl. Math. Wissen. 114, Springer (1963) MR Zbl

[Magurn et al. 1983] **B Magurn**, **R Oliver**, **L Vaserstein**, *Units in Whitehead groups of finite groups*, J. Algebra 84 (1983) 324–360 MR Zbl

[Mannan 2007] **W H Mannan**, *The* $D(2)$ *property for* D_8, Algebr. Geom. Topol. 7 (2007) 517–528 MR Zbl

[Mannan and Popiel 2021] **W H Mannan**, **T Popiel**, *An exotic presentation of* Q_{28}, Algebr. Geom. Topol. 21 (2021) 2065–2084 MR Zbl

[Metzler 1976] **W Metzler**, *Über den Homotopietyp zweidimensionaler CW–Komplexe und Elementartransformationen bei Darstellungen von Gruppen durch Erzeugende und definierende Relationen*, J. Reine Angew. Math. 285 (1976) 7–23 MR Zbl

[Milgram 1985] **R J Milgram**, *Evaluating the Swan finiteness obstruction for periodic groups*, from "Algebraic and geometric topology" (A Ranicki, N Levitt, F Quinn, editors), Lecture Notes in Math. 1126, Springer (1985) 127–158 MR Zbl

[Milnor 1971] **J Milnor**, *Introduction to algebraic* K*–theory*, Ann. of Math. Ser. 72, Princeton Univ. Press (1971) MR Zbl

[Nicholson 2020] **J Nicholson**, *Cancellation for* (G, n)*–complexes and the Swan finiteness obstruction*, preprint (2020) arXiv 2005.01664 To appear in Int. Math. Res. Not.

[Nicholson 2021a] **J Nicholson**, *A cancellation theorem for modules over integral group rings*, Math. Proc. Cambridge Philos. Soc. 171 (2021) 317–327 MR Zbl

[Nicholson 2021b] **J Nicholson**, *On CW–complexes over groups with periodic cohomology*, Trans. Amer. Math. Soc. 374 (2021) 6531–6557 MR Zbl

[Nicholson 2021c] **J Nicholson**, *Stably free modules and the unstable classification of* 2*–complexes*, preprint (2021) arXiv 2108.02220

[Sieradski 1977] **A J Sieradski**, *A semigroup of simple homotopy types*, Math. Z. 153 (1977) 135–148 MR Zbl

[Sieradski and Dyer 1979] **A J Sieradski**, **M N Dyer**, *Distinguishing arithmetic for certain stably isomorphic modules*, J. Pure Appl. Algebra 15 (1979) 199–217 MR Zbl

[Stallings 1963] **J Stallings**, *A finitely presented group whose* 3*–dimensional integral homology is not finitely generated*, Amer. J. Math. 85 (1963) 541–543 MR Zbl

[Swan 1960a] **R G Swan**, *Induced representations and projective modules*, Ann. of Math. 71 (1960) 552–578 MR Zbl

[Swan 1960b] **R G Swan**, *Periodic resolutions for finite groups*, Ann. of Math. 72 (1960) 267–291 MR Zbl

[Swan 1965] **R G Swan**, *Minimal resolutions for finite groups*, Topology 4 (1965) 193–208 MR Zbl

[Swan 1970] **R G Swan**, K*–theory of finite groups and orders*, Lect. Notes in Math. 149, Springer (1970) MR Zbl

[Swan 1980] **R G Swan**, *Strong approximation and locally free modules*, from "Ring theory and algebra, III" (B R McDonald, editor), Lect. Notes Pure Appl. Math. 55, Dekker, New York (1980) 153–223 MR Zbl

[Swan 1983] **R G Swan**, *Projective modules over binary polyhedral groups*, J. Reine Angew. Math. 342 (1983) 66–172 MR Zbl

[Wall 1965] **C T C Wall**, *Finiteness conditions for CW–complexes*, Ann. of Math. 81 (1965) 56–69 MR Zbl

[Wall 1979a] **C T C Wall**, *Periodic projective resolutions*, Proc. Lond. Math. Soc. 39 (1979) 509–553 MR Zbl

[Wall 1979b] *List of problems*, from "Homological group theory" (C T C Wall, editor), Lond. Math. Soc. Lect. Note Ser. 36, Cambridge Univ. Press (1979) 369–394 MR Zbl

[Weibel 1994] **C A Weibel**, *An introduction to homological algebra*, Cambridge Stud. Adv. Math. 38, Cambridge Univ. Press (1994) MR Zbl

[Whitehead 1939] **J H C Whitehead**, *Simplicial spaces, nuclei and m–groups*, Proc. Lond. Math. Soc. 45 (1939) 243–327 MR Zbl

School of Mathematics and Statistics, University of Glasgow
Glasgow, United Kingdom

john.nicholson@glasgow.ac.uk

Received: 19 July 2022 Revised: 4 April 2023

Algebraic & Geometric Topology **24**:4 (2024) 2285–2305
DOI: 10.2140/agt.2024.24.2285
Published: 16 July 2024

Realization of Lie algebras of derivations and moduli spaces of some rational homotopy types

YVES FÉLIX
MARIO FUENTES
ANICETO MURILLO

We construct Lie algebras of derivations (and identify their geometric realizations) whose Maurer–Cartan sets provide moduli spaces describing the classes of homotopy types of rational spaces having the same homotopy Lie algebra, homology or cohomology.

55P62

Introduction

Derivations of a Lie algebra are ubiquitous objects in topology. A particular instance is the following classical result (see M Schlessinger and J Stasheff [16] and D Tanré [17]): when L is a differential graded Lie algebra (dgl henceforth) characterizing the rational homotopy type of a finite simply connected CW–complex X, the dgl of positive derivations of L characterizes in the same fashion the rational homotopy type of the universal covering of $B\operatorname{aut}^*(X)$, the classifying space of pointed self-homotopy equivalences of X. With the recent extension of the Quillen approach to rational homotopy theory (see U Buijs, Y Félix, A Murillo and Tanré [6]) we were able to extend this result to connected dgls of derivations as long as the degree-zero derivations characterize a $\mathbb{Q}$–complete (in the sense of Maltsev) subgroup of $\operatorname{aut}^*(X)$; see Félix, M Fuentes and Murillo [8] and also the recent approach of A Berglund and T Zeman [2] to the description of the rational homotopy type of the classifying spaces of self-homotopy equivalences.

At this stage is convenient to remark that, under the mentioned extension of Quillen theory, which is the one we consider, only dgls that are complete are susceptible to being topologically realized (see Section 1 for a brief compendium on this theory). Nevertheless, the reader may find other classes of dgls whose topological realizations have been considered. See for instance the integration procedure of the class of *absolute* dgls recently developed by V Roca i Lucio in [12].

Complete dgls contain much more geometric data than their connected covers. For instance, the Maurer–Cartan set of a dgl modulo the gauge relation ($\widetilde{\mathrm{MC}}$ set from now on) corresponds to the set of path-connected components in which the realization of the given dgl decomposes. We try to collect this extra data for some sub-Lie algebras of derivations of a given dgl, which are complete and still provide important geometric information. In all these cases, their $\widetilde{\mathrm{MC}}$ sets, or the space of orbit of a certain action on them, turn out to be a moduli space governing classes of rational homotopy types sharing certain structures.

To begin with, we consider in Section 3 an extended dual "Lie version" of the deep result of Schlessinger and Stasheff [16, Main Theorem 4.1] which will also be considered later. Let π be a complete connected graded Lie algebra and let $\operatorname{Ho}\mathbf{sset}_\pi$ be the class of homotopy types of rational simplicial sets whose homotopy Lie algebra is isomorphic to π. If L denotes the bigraded minimal Lie model of π, which is properly introduced in Theorem 3.1, we prove the following (see Theorem 3.5 and Corollary 3.6 for precise and detailed statements):

Theorem *There exists a complete sub-dgl* $\mathfrak{Der}\, L$ *of* $\operatorname{Der} L$ *such that*

$$\operatorname{Ho}\mathbf{sset}_\pi \cong \widetilde{\mathrm{MC}}(\mathfrak{Der}\, L).$$

Via this bijection, the quotient stack $\widetilde{\mathrm{MC}}(\mathfrak{Der}\, L) = \mathrm{MC}(\mathfrak{Der}\, L)/\exp(\mathfrak{Der}_0\, L)$ can be seen as the moduli space of $\operatorname{Ho}\mathbf{sset}_\pi$.

It is important to remark that, in the simply connected case, this result was already sketched by D Blanc [3, Section 3] and explicitly developed by M Zawodniak in his thesis [18].

Then in Section 4 we construct a complete dgl of derivations, which provides a moduli space governing the class $\operatorname{Ho}\mathbf{sset}^1_H$ of homotopy types of rational finite-dimensional simply connected complexes sharing the same reduced homology with no additional structure. For it, let $\mathbb{L}(V)$ be the free Lie algebra generated by $V = s^{-1}H$ and consider the dgl $L = (\mathbb{L}(V), 0)$ with trivial differential. With this notation, Corollary 4.4 can be summarized as follows:

Theorem *There exists a complete sub-dgl* $\mathcal{D}\mathrm{er}\, L$ *of* $\operatorname{Der} L$ *and a natural action of* $\operatorname{aut}(V)$ *on* $\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)$ *for which*

$$\operatorname{Ho}\mathbf{sset}^1_H \cong \widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)/\operatorname{aut}(V). \tag{1}$$

Moreover,

$$\langle \mathcal{D}\mathrm{er}\, L\rangle = \coprod_{X\in \operatorname{Ho}\mathbf{sset}^1_H}\ \coprod_{\mathcal{O}_X} B\operatorname{aut}^*_{\mathcal{H}}(X).$$

Here $\langle\,\cdot\,\rangle$ denotes the realization functor on complete dgls (see Section 1), $\mathcal{O}_X$ denotes the (cardinality of the) orbit by the action of $\operatorname{aut}(V)$ of any element in $\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)$ representing X by the bijection (1), and finally $\operatorname{aut}^*_{\mathcal{H}}(X)$ is the subgroup of pointed homotopy equivalences of X which induces the identity on homology.

In other words, the realization of $\mathcal{D}\mathrm{er}\, L$ is the disjoint union of simplicial sets, one for each $X \in \operatorname{Ho}\mathbf{sset}^1_H$. Moreover, each of these pieces also decomposes in as many path components as points in the orbit $\mathcal{O}_X$, each of which is of the homotopy type of the classifying space $B\operatorname{aut}^*_{\mathcal{H}}(X)$.

Thus in this case $\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)$ is too big to describe $\operatorname{Ho}\mathbf{sset}^1_H$. Nevertheless, there is an action of $\operatorname{aut}(L)$ on $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)$ which provides the quotient stack $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\operatorname{aut}(L)$ responsible for $\operatorname{Ho}\mathbf{sset}^1_H$.

We remark that this result is a particular instance of the extended version in Theorem 4.2.

Finally, in Section 5 we consider the augmentation ideal A of a given simply connected finite-dimensional commutative graded algebra and denote by $\mathrm{Ho}\,\mathbf{sset}^1_A$ the class of homotopy types of rational simply connected spaces sharing A as rational (reduced) cohomology algebra. We then present a different description of $\mathrm{Ho}\,\mathbf{sset}^1_A$ than the one given by Schlessinger and Stasheff in [16, Main Theorem 4.1]. For it (see Section 5 for details), denote by $L = \mathscr{L}(A^\sharp)$ the classical Quillen functor on the coalgebra given by the dual of A. This is a dgl with a purely quadratic differential for which we prove (see Theorem 5.3 for a precise statement):

Theorem *There exists a complete sub-dgl* $\mathscr{D}\mathrm{er}\,L$ *of* $\mathrm{Der}\,L$ *and a natural action of* $\mathrm{aut}(A)$ *on* $\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\,L)$ *such that*

$$\mathrm{Ho}\,\mathbf{sset}^1_A \cong \widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\,L)/\mathrm{aut}(A).$$

Moreover,

$$\langle \mathscr{D}\mathrm{er}\,L\rangle \simeq \coprod_{X\in \mathrm{Ho}\,\mathbf{sset}^1_A}\ \coprod_{\mathcal{O}_X} B\mathrm{aut}^*_{\mathcal{H}}(X).$$

Here $\mathcal{O}_X$ again denotes the orbit by the action of $\mathrm{aut}(A)$ of any element in $\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\,L)$ representing X by the bijection in (i). On the other hand, as before, $\mathrm{aut}^*_{\mathcal{H}}(X)$ stands for the subgroup of pointed self-homotopy equivalences of X which induce the identity on homology. As X is rational, this trivially coincides with the group of self-homotopy equivalences inducing the identity on cohomology.

As a consequence we can also exhibit a particular quotient stack over $\mathrm{MC}(\mathscr{D}\mathrm{er}\,L)$ as a moduli space of $\mathrm{Ho}\,\mathbf{sset}^1_A$.

To prove the above results we need some technical statements, which are contained in Section 2. This section extends and reformulates some results of Félix, Fuentes and Murillo [8, Section 6] to obtain certain complete sub-Lie algebras of a general $\mathrm{Der}\,L$ containing the whole connected cover.

Acknowledgments We thank the referee for helpful suggestions and corrections which have considerably improved the content and presentation of this paper. The authors have been partially supported by the MICINN grant PID2020-118753GB-I00 of the Spanish government and the EXCEL-00827 grant of the Junta de Andalucía. Murillo also thanks the *Instituto de Matemáticas de la UNAM en Oaxaca* for its hospitality during a short stay in which part of this work was developed.

1 Preliminaries

This section is devoted to recalling the basic facts that we use from the homotopy theory of complete differential graded Lie algebras. We refer to the monograph [6], or the original references [5; 7], for details.

All considered differential graded vector spaces, possibly endowed with additional structures, are rational and graded over $\mathbb{Z}$. The *suspension* and *desuspension* of such a graded vector space V are denoted by sV and $s^{-1}V$, respectively. That is $(sV)_n = V_{n-1}$ and $(s^{-1}V)_n = V_{n+1}$ for any $n\in\mathbb{Z}$.

We often do not distinguish objects of the category **sset** of simplicial sets from the topological spaces given by their realization, which are therefore of the homotopy type of CW–complexes.

We denote by **dgl** the category of differential graded Lie algebras (dgls henceforth). A dgl L, or (L,d) if we want to specify the differential, is *connected* if $L = L_{\geq 0}$.

A *Maurer–Cartan* element, or simply MC element, of a given dgl L is an element $a \in L_{-1}$ satisfying the Maurer–Cartan equation $da = -\frac{1}{2}[a,a]$. We denote by $\mathrm{MC}(L)$ the set of MC elements in L. Given $a \in \mathrm{MC}(L)$, we denote by $d_a = d + \mathrm{ad}_a$ the *perturbed differential* on L where d is the original one and ad is the usual adjoint operator. The *component* of L at a is the connected sub-dgl L^a of (L, d_a) given by

$$L_p^a = \begin{cases} \ker d_a & \text{if } p = 0, \\ L_p & \text{if } p > 0. \end{cases}$$

The *derivations* $\mathrm{Der}\, L$ of a given dgl L is a dgl with the usual Lie bracket and differential $D = [d, -]$:

$$[\theta,\eta] = \theta\circ\eta - (-1)^{|\theta||\eta|}\eta\circ\theta, \quad D\theta = d\circ\theta - (-1)^{|\theta|}\theta\circ d.$$

A *filtration* of a dgl L is a decreasing sequence of differential Lie ideals

$$L = F^1 \supset \cdots \supset F^n \supset F^{n+1} \supset \cdots$$

such that $[F^p, F^q] \subset F^{p+q}$ for $p, q \geq 1$. In particular, the lower central series of L,

$$L^1 \supset \cdots \supset L^n \supset L^{n+1} \supset \cdots,$$

where $L^1 = L$ and $L^n = [L, L^{n-1}]$ for $n > 1$, is a filtration for any dgl which satisfies $L^n \subset F^n$ for any $n \geq 1$ and any other filtration $\{F^n\}_{n\geq 1}$ of L.

A *complete differential graded Lie algebra*, cdgl henceforth, is a dgl L equipped with a filtration $\{F^n\}_{n\geq 1}$ for which the natural map

$$L \xrightarrow{\cong} \varprojlim_n L/F^n$$

is a dgl isomorphism. A cdgl *morphism* between cdgls is a dgl morphism which preserves the filtrations. We denote by **cdgl** the corresponding category. By a *complete graded Lie algebra*, cgl hereafter, we mean a cdgl endowed with the trivial differential.

If L is a dgl filtered by $\{F^n\}_{n\geq 1}$, its *completion* is the dgl

$$\widehat{L} = \varprojlim_n L/F^n,$$

which is always complete with respect to the filtration

$$\widehat{F}^n = \ker(\widehat{L} \to L/F^n).$$

If no specific filtration is given, the completion of a generic dgl is always taken over the lower central series. In particular, if $\mathbb{L}(V)$ denotes the free Lie algebra generated by the graded vector space V, the completion of a dgl of the form $(\mathbb{L}(V), d)$ is the cdgl

$$\widehat{\mathbb{L}}(V) = \varprojlim_n \mathbb{L}(V)/\mathbb{L}(V)^n.$$

This is an important object in this theory, whose main properties are detailed in [6, Section 3.2]. Note that if $V = V_{>0}$ then $\widehat{\mathbb{L}}(V) = \mathbb{L}(V)$.

Given a cdgl L, the group L_0, endowed with the Baker–Campbell–Hausdorff product (BCH product henceforth), acts on the set $\mathrm{MC}(L)$ by

$$x \mathcal{G} a = e^{\mathrm{ad}_x}(a) - \frac{e^{\mathrm{ad}_x} - 1}{\mathrm{ad}_x}(dx) = \sum_{i \geq 0} \frac{\mathrm{ad}_x^i(a)}{i!} - \sum_{i \geq 0} \frac{\mathrm{ad}_x^i(dx)}{(i+1)!} \quad \text{for } x \in L_0 \text{ and } a \in \mathrm{MC}(L).$$

This is the *gauge* action and we denote by $\widetilde{\mathrm{MC}}(L) = \mathrm{MC}(L)/\mathcal{G}$ the corresponding orbit set. A homotopical description of the gauge action is given in [6, Section 5.3].

The homotopy theory of cdgls lies in the existence of a pair of adjoint functors [6, Chapter 7]: (*global*) *model* and *realization*,

$$\textbf{sset} \underset{\langle \cdot \rangle}{\overset{\mathfrak{L}}{\rightleftarrows}} \textbf{cdgl} \tag{2}$$

The set of 0–simplices of $\langle L \rangle$ coincides with $\mathrm{MC}(L)$. Moreover, if $\langle L \rangle^a$ denotes the path component of $\langle L \rangle$ containing the MC element a, then

$$\langle L \rangle^a \simeq \langle L^a \rangle \quad \text{and} \quad \langle L \rangle \simeq \amalg_{a \in \widetilde{\mathrm{MC}}(L)} \langle L^a \rangle. \tag{3}$$

If L is connected, for any $n \geq 1$ we have group isomorphisms

$$\pi_n \langle L \rangle \cong H_{n-1}(L),$$

where the group structure in $H_0(L)$ is considered with the BCH product. Under the homotopy equivalence $\langle L \rangle \simeq \mathrm{MC}_\bullet(L)$ between the realization of L and the *Deligne–Getzler–Hinich groupoid* of L (see [6, Section 11.4]), this is the original explicit isomorphism of Berglund $\pi_n \mathrm{MC}_\bullet(L) \cong H_{n-1}(L)$ in [1, Theorem 1.1].

We will also use the fact that the realization of a cdgl is invariant under perturbations. That is, for any cdgl L and any $a \in \mathrm{MC}(L)$,

$$\langle L \rangle \cong \langle (L, d_a) \rangle. \tag{4}$$

Finally, the realization functor coincides with any other known geometric realization of cdgls. In particular, if L is a 1–connected dgl of finite type, then (see [6, Corollary 11.17]) $\langle L \rangle$ has the homotopy type of the classical Quillen realization of L [15].

On the other hand (see again [6, Chapter 7] for details), the global model $\mathfrak{L}_X$ of a simplicial set X completely reflects its simplicial structure. In particular, the 0–simplices of X are the Maurer–Cartan elements of $\mathfrak{L}_X$.

If X is a simply connected simplicial set of finite type and a is any of its vertices, then [6, Theorem 10.2] $\mathfrak{L}_X^a$ is quasi-isomorphic to $\lambda(X)$ where λ is the classical Quillen dgl model functor [15]. Moreover (see [6, Theorem 11.14]), for any connected simplicial set X of finite type, $\langle \mathfrak{L}_X^a \rangle$ is weakly homotopy equivalent to $\mathbb{Q}_\infty X$, the Bousfield–Kan $\mathbb{Q}$–completion of X [4]. Recall that, whenever X is nilpotent, $\mathbb{Q}_\infty X$ and has the homotopy type of $X_{\mathbb{Q}}$, the rationalization of X.

The category **cdgl** has a *cofibrantly generated model structure* (see [6, Chapter 8]), for which the functors in (2) become a Quillen pair. With this structure the induced functors in the respective homotopy categories extend the classical Quillen equivalence between rational homotopy types of simply connected simplicial sets and homotopy types of simply connected dgls.

A *model* of a connected cdgl L is a connected cdgl of the form $(\widehat{\mathbb{L}}(V), d)$ together with a quasi-isomorphism (and hence a weak equivalence)

$$(\widehat{\mathbb{L}}(V), d) \xrightarrow{\simeq} L.$$

If d is decomposable we say $(\widehat{\mathbb{L}}(V), d)$ is the *minimal model* of L and is unique up to cdgl isomorphism.

Definition 1.1 Let X be a connected simplicial set and a any of its vertices. The *minimal model of* X is the minimal model of $\mathfrak{L}_X^a$.

If $(\widehat{\mathbb{L}}(V), d)$ is the minimal model of X, then (see [6, Proposition 8.35]) $sV \cong \widetilde{H}_*(X;\mathbb{Q})$ and, provided X is of finite type, $sH_*(\widehat{\mathbb{L}}(V), d) \cong \pi_*(\mathbb{Q}_\infty X)$. Again, the group $H_0(\widehat{\mathbb{L}}(V), d)$ is considered with the BCH product. If X is simply connected, the minimal model of X is isomorphic to its classical Quillen minimal model, for which we refer to [13] or [15].

2 Complete Lie algebras of derivations

Derivations of a cdgl are essential objects in this paper. However, even if L is 1–connected, $\operatorname{Der} L$ may fail to be complete, and thus their MC set are not defined and they are unable to be topologically realized as described in the previous section. For instance, let $L = (\mathbb{L}(x, y), 0)$ with $|x| = |y| = 2$, and consider $\theta_1, \theta_2, \theta_3 \in \operatorname{Der}_0 L$ defined by

$$\theta_1(x) = x, \quad \theta_1(y) = -y, \quad \theta_2(x) = y, \quad \theta_2(y) = 0 \quad \theta_3(x) = 0 \quad \text{and} \quad \theta_3(y) = x.$$

Note that $[\theta_1, \theta_2] = -2\theta_2$, $[\theta_1, \theta_3] = 2\theta_3$ and $[\theta_2, \theta_3] = -\theta_1$. Hence for any given filtration $\{F^n\}_{n\geq 1}$ of $\operatorname{Der} L$, $\theta_i \in F^n$ for any n and any i. That is, these derivations live in the kernel of the natural map $\operatorname{Der} L \to \varprojlim_{n\geq 1} \operatorname{Der} L/F^n$ and thus $\operatorname{Der} L$ is not complete.

Nevertheless, for any complete sub-dgl M of $(\operatorname{Der} L, D)$ we shall use the following general fact:

$$\operatorname{MC}(M) = \{\delta \in M_{-1} \text{ such that } d + \delta \text{ is a differential in } L\}. \tag{5}$$

Moreover, the gauge relation is characterized by the following result:

Proposition 2.1 *Two Maurer–Cartan elements $\delta, \eta \in \operatorname{MC}(M)$ are gauge related if and only if there exists an isomorphism of the form*

$$e^\theta \colon (L, d + \delta) \xrightarrow{\cong} (L, d + \eta)$$

with $\theta \in M_0$. Moreover, the gauge action is given by $\theta \mathcal{G} \delta = \eta$.

The proof is an obvious extension of [6, Theorem 4.31] to any complete sub-dgl of derivations.

Proof Suppose first that δ and η are gauge related. Thus, there exists $\theta \in M_0$ such that

$$\eta = e^{\mathrm{ad}_\theta}(\delta) - \frac{e^{\mathrm{ad}_\theta} - 1}{\mathrm{ad}_\theta}(D\theta).$$

As $D\theta = [d, \theta]$,

$$\frac{e^{\mathrm{ad}_\theta} - 1}{\mathrm{ad}_\theta}(D\theta) = \sum_{i \geq 0} \frac{\mathrm{ad}_\theta^i}{(i+1)!}[d, \theta] = -\sum_{i \geq 1} \frac{\mathrm{ad}_\theta^i}{i!} d.$$

Therefore,

$$d + \eta = d + e^{\mathrm{ad}_\theta}(\delta) + \sum_{i \geq 1} \frac{\mathrm{ad}_\theta^i}{i!}(d) = e^{\mathrm{ad}_\theta}(d + \delta).$$

We then use the general formula $e^{\mathrm{ad}_\theta}(d+\delta) = e^\theta(d+\delta)e^{-\theta}$ (see for instance [6, Proposition 4.13]) to conclude that

$$d + \eta = e^\theta (d+\delta) e^{-\theta}, \quad \text{that is,} \quad (d+\eta)e^\theta = e^\theta(d+\delta),$$

and e^θ is the required isomorphism.

For the other implication simply reverse the above argument. □

Due to this fact, we often identify M_0 with

$$\exp(M_0) = \{e^\theta, \theta \in M_0\},$$

and write $\widetilde{\mathrm{MC}}(M) = \mathrm{MC}(M)/\exp(M_0)$.

If M is of finite type, choose bases $\{\partial_i\}_{i=1}^s$ and $\{\sigma_\ell\}_{\ell=1}^r$ of M_{-1} and M_{-2}, respectively, and write

$$[\partial_i, \partial_j] = \sum_\ell \lambda_{ij}^\ell \sigma_\ell \quad \text{for } \lambda_{ij}^\ell \in \mathbb{Q}.$$

Then, given $\delta \in M_{-1}$, the derivation $d + \delta = \sum_i \alpha_i \partial_i$ is a differential if and only if

$$\sum_{i,j} \lambda_{ij}^\ell \alpha_i \alpha_j = 0 \quad \text{for } \ell = 1, \dots, r.$$

In other words, if we denote by $V_L \subset \mathbb{C}^s$ the affine algebraic variety defined by the polynomials $\sum_{i,j} \lambda_{ij}^\ell \alpha_i \alpha_j$, with $\ell = 1, \dots, r$, we conclude that

$$\mathrm{MC}(M) = \{\text{rational points of } V_L\}. \tag{6}$$

So $\widetilde{\mathrm{MC}}(M) = \mathrm{MC}(M)/\exp(M_0)$ can be considered as a quotient stack.

Next, consider $L = (\widehat{\mathbb{L}}(V), d)$ a connected minimal cdgl in which V is bounded above, that is $V_{>m} = 0$ for some m. We then identify some complete sub-dgls of $\operatorname{Der} L$ which conserve its "connected cover". For it, choose an arbitrary finite filtration of V by graded vector subspaces:

$$V = V^0 \supset V^1 \supset \cdots \supset V^{q-1} \supset V^q = 0. \tag{7}$$

As in [8, Section 6], for $n \geq 1$ and $p \geq 0$, write

$$\widehat{\mathbb{L}}^{n,p}(V) = \operatorname{Span}\left\{ \big[v_1, \big[v_2, [\dots, [v_{n-1}, v_n]\dots\big] \in \widehat{\mathbb{L}}^{n}(V) \;\Big|\; v_i \in V^{\alpha_i} \text{ and } \sum_{i=1}^{n} \alpha_i = p \right\},$$

and define

$$F^{n,p} = \widehat{\mathbb{L}}^{n,p}(V) \oplus \widehat{\mathbb{L}}^{\geq n+1}(V),$$

so that

$$\widehat{\mathbb{L}}(V) = F^{1,0} \supset \cdots \supset F^{1,q-1} \supset F^{2,0} \supset \cdots \supset F^{2,2q-1} \supset \cdots \supset F^{n,0} \supset \cdots \supset F^{n,nq-1} \supset \cdots.$$

In the order given by this sequence, $F^{n,p}$ takes the position

$$t = q + \cdots + (n-1)q + p + 1 = \tfrac{1}{2}(n-1)nq + p + 1$$

and we define $F^t = F^{n,p}$ for n, p and t as above. In [8, Proposition 6.3] it is proved that $\{F^t\}_{t\geq 1}$ is a filtration of L for which it is complete.

This filtration of L naturally determines a decreasing sequence of sub-dgls of $\operatorname{Der} L$

$$\mathcal{F}^1 \supset \cdots \supset \mathcal{F}^n \supset \mathcal{F}^{n+1} \supset \cdots, \tag{8}$$

where, for any $n \geq 1$,

$$\mathcal{F}^n = \{\theta \in \operatorname{Der} L \mid \theta(F^r) \subset F^{n+r} \text{ for all } r \geq 0\}.$$

Note that $\{\mathcal{F}^n\}_{n\geq 1}$ is a filtration of the dgl $\mathcal{F}^1$. Moreover, a simple inspection shows that

$$\mathcal{F}^1 = \{\theta \in \operatorname{Der} L \mid \theta_*(V^i) \subset V^{i+1} \text{ for all } i\}, \tag{9}$$

where $\theta_*\colon V \to V$ denotes the linear part of θ. Then:

Proposition 2.2 *$\mathcal{F}^1$ is a complete dgl.*

Proof As $\bigcap_n \mathcal{F}^n = 0$, the map $\mathcal{F}^1 \to \varprojlim_n \mathcal{F}^1/\mathcal{F}^n$ is injective. On the other hand, write a given element of $\varprojlim_n \mathcal{F}^1/\mathcal{F}^n$ as a series $\sum_n \theta_n$ with $\theta_n \in \mathcal{F}^n$. Note that, for each $v \in V$ and any integer $m \geq 1$, the series $\sum_n \theta_n(v)$ contains a finite sum of elements in $\mathbb{L}^m(V)$, and thus $\sum_n \theta_n(v)$ is a well-defined element in $\widehat{\mathbb{L}}(V)$. Hence $\sum_n \theta_n \in \mathcal{F}^1$, and the above map is also surjective. □

We now "enlarge" the cdgl $\mathcal{F}^1$ as much as possible in positive degrees: starting from the original filtration (7) we define a new filtration of V as follows:

$$V \supset V_1^1 \oplus V_{\geq 2} \supset V_1^2 \oplus V_{\geq 2} \supset \cdots \supset V_1^{q-1} \oplus V_{\geq 2} \supset V_{\geq 2} \supset V_2^1 \oplus V_{\geq 3} \supset V_2^2 \oplus V_{\geq 3} \supset \cdots \supset V_2^{q-1} \oplus V_{\geq 3} \supset V_{\geq 3}$$
$$\supset \cdots \supset V_m^1 \supset V_m^2 \supset \cdots \supset V_m^{q-1} \supset 0,$$

where m is such that $V_{>m} = 0$. If we rename this filtration of subspaces of V by

$$V = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \mathcal{V}^2 \supset \cdots \supset \mathcal{V}^{m(q-1)} \supset 0,$$

it clearly satisfies the following property:

$$\mathcal{V}_\ell^i \neq 0 \;\text{ implies }\; V_{>\ell} \subset \mathcal{V}^{i+1}. \tag{10}$$

Definition 2.3 For this new filtration of V, the procedure above determines again a decreasing sequence of sub-dgls of $\operatorname{Der} L$ as in (8), whose first term we denote by $\mathcal{D}\mathrm{er}\, L$.

By Proposition 2.2, $\mathcal{D}\mathrm{er}\, L$ is complete and, in view of (9), it can be written as

$$\mathcal{D}\mathrm{er}\, L = \{\theta \in \operatorname{Der} L \mid \theta_*(\mathcal{V}^i) \subset \mathcal{V}^{i+1} \text{ for all } i\}.$$

Furthermore, from the characterization of $\mathcal{F}^1$ in (9) one easily observes that

$$\mathcal{F}^1_{>0} \subset \mathcal{D}\mathrm{er}_{>0}\, L, \quad \mathcal{F}^1_0 = \mathcal{D}\mathrm{er}_0\, L \quad \text{and} \quad \mathcal{F}^1_{<0} \supset \mathcal{D}\mathrm{er}_{<0}\, L. \tag{11}$$

Moreover:

Proposition 2.4
$$\mathcal{D}\mathrm{er}_k\, L = \begin{cases} \operatorname{Der}_k L & \text{if } k>0, \\ \theta \in \operatorname{Der}_0 L \text{ such that } \theta(V^i) \subset V^{i+1} \oplus \widehat{\mathbb{L}}^{\geq 2}(V) & \text{if } k=0, \\ \theta \in \operatorname{Der}_k L \text{ such that } \theta(V) \subset \widehat{\mathbb{L}}^{\geq 2}(V) & \text{if } k<0. \end{cases}$$

That is, $\mathcal{D}\mathrm{er}\, L$ is a cdgl consisting of all derivations in positive degrees, those derivations of degree 0 which increase the original filtration degree on V modulo decomposables, and all derivations of negative degrees which increase the word length.

Proof Let $k>0$ and $\theta \in \operatorname{Der}_k L$. Then, for any i and any nonzero element of degree $v \in \mathcal{V}^i_\ell$, it follows by (10) that $\theta_*(v) \in V_{k+\ell} \subset \mathcal{V}^{i+1}$. By (9), $\theta \in \mathcal{D}\mathrm{er}_k\, L$.

Let $k<0$ and $\theta \in \mathcal{D}\mathrm{er}_k\, L$ such that $\theta(V) \subset \widehat{\mathbb{L}}^{\geq 2}(V)$. By definition $\theta \in \mathcal{D}\mathrm{er}_k\, L$. Conversely, let $\theta \in \mathcal{D}\mathrm{er}_k\, L$ and let $v \in V$ be a nonzero element. Assume $v \in V_\ell$ and let i be the maximal filtration index such that $v \in \mathcal{V}^i$ but $v \notin \mathcal{V}^{i+1}$. Then $\theta_*(v)=0$. Otherwise $\theta_*(v) \in \mathcal{V}^{i+1}_{<\ell}$. Hence by (10) $V_\ell \subset \mathcal{V}^{i+2}$, which contradicts the fact that $v \notin V^{i+1}$.

Finally, for $k=0$, the obvious fact $\mathcal{F}^1_0 = \mathcal{D}\mathrm{er}_0\, L$ in (11) amounts to the required equality. □

Remark 2.5 Of special interest in what follows is the particular instance of choosing the trivial filtration $V = V^0 \supset V^1 = 0$ on V. In this case,

$$\mathcal{D}\mathrm{er}_k\, L = \begin{cases} \operatorname{Der}_k L & \text{if } k>0, \\ \theta \in \operatorname{Der}_k L \text{ such that } \theta(V) \subset \widehat{\mathbb{L}}^{\geq 2}(V) & \text{if } k \leq 0. \end{cases}$$

3 Rational homotopy types with prescribed homotopy Lie algebras and their moduli space

In this section we check that the method for building the moduli space of rational simply connected homotopy types with prescribed homotopy Lie algebra, already sketched in [3, Section 3] and explicitly developed in [18], also works in the nonsimply connected case by means of the homotopy theory of cdgls.

First, a simple inspection shows that the procedure to obtain the bigraded model of a simply connected graded Lie algebra (see [11, théorème 1] or [14, Chapter I]), dual of the classical commutative context [10, Section 3], extends mutatis mutandis to any connected complete cdgl:

Theorem 3.1 (complete bigraded Lie model) *Let π be a connected cgl. Then the cdgl $(\pi, 0)$ admits a Lie minimal model*

$$\rho\colon (\widehat{\mathbb{L}}(V), d) \xrightarrow{\simeq} (\pi, 0)$$

satisfying:

- *$V = \bigoplus_{p,q\geq 0} V_p^q$ is bigraded, being the lower grading the usual homological one. This bigradation extends bracketwise to $\widehat{\mathbb{L}}(V)$.*
- *$dV^0 = 0$ and $d(V^{n+1}) \subset \widehat{\mathbb{L}}(V^{\leq n})^n$ for $n \geq 0$. In particular d decreases by one the upper degree so that $H(\widehat{\mathbb{L}}(V), d) = \bigoplus_{p,q\geq 0} H_p^q(\widehat{\mathbb{L}}(V, d)$ is also bigraded.*
- *$\rho\colon \widehat{\mathbb{L}}(V^0) \twoheadrightarrow \pi$ is surjective, $\rho(V^n) = 0$ for $n \geq 1$, $H^0(\rho)\colon H^0(\widehat{\mathbb{L}}(V), d) \xrightarrow{\cong} \pi$ is an isomorphism and $H^+(\widehat{\mathbb{L}}(V), d) = 0$.*

For completeness we include here a sketch of the proof:

Proof Let π be filtered by $\{F^n\}_{n\geq 1}$ so that $\pi \cong \varprojlim_n \pi/F^n$, and consider the projection $q\colon \pi \to \pi/[\pi,\pi]$ onto the indecomposables of π. Define V^0 to be a space of generators of π by $V^0 = \pi/[\pi,\pi]$ and choose $\rho\colon V^0 \to \pi$ a section of q. Set $dV^0 = 0$ and extend ρ first to $\mathbb{L}(V^0) \to \pi$ and then, by completion, to

$$\rho\colon \widehat{\mathbb{L}}(V^0) = \varprojlim_n \mathbb{L}(V^0)/\mathbb{L}^n(V^0) \to \varprojlim_n \pi/\pi^n \to \varprojlim_n \pi/F^n = \pi.$$

Next, define V^1 to be a space of relations of π by $V^1 = \ker\rho/[\widehat{\mathbb{L}}(V^0), \ker\rho]$, set $\rho(V^1) = 0$ and extend d to V^1 as a section of the projection $\ker\rho \twoheadrightarrow V^1$.

For $n \geq 1$ define $V^{n+1} = H^n(\widehat{\mathbb{L}}(V^{\leq n})/[H^n(\widehat{\mathbb{L}}(V^{\leq n}), H^0(\widehat{\mathbb{L}}(V^{\leq n})]$, set $\rho(V^{n+1}) = 0$ and define $d\colon V^{n+1} \to \widehat{\mathbb{L}}(V^{\leq n})^n \cap \ker d$ to be a section of $\widehat{\mathbb{L}}(V^{\leq n})^n \cap \ker d \twoheadrightarrow V^{n+1}$. □

Definition 3.2 The cdgl $(\widehat{\mathbb{L}}(V), d)$ is the (complete) *bigraded model of* π. We say that the elements of $\widehat{\mathbb{L}}(V)_p^n$ have *weight* $p-n$. Note that the differential d preserves weight as $dV_p^n \subset \widehat{\mathbb{L}}(V)_{p-1}^{n-1}$

We now show that any cdgl whose homology is isomorphic to the cgl π has a Lie model (not minimal in general) obtained by perturbing in a particular way the bigraded Lie model of π. The following is again a straightforward extension to the complete connected setting of [11, théorème 2] or [14, Chapter II], which is in turn the dual of [10, Theorem 4.4] in the commutative context.

Theorem 3.3 (complete filtered Lie model) *Let $\rho\colon (\widehat{\mathbb{L}}(V), d) \xrightarrow{\simeq} \pi$ be the bigraded model for the cgl π and let L be a cdgl whose homology is isomorphic to π. Then there is a Lie model of L of the form*

$$\phi\colon (\widehat{\mathbb{L}}(V), d + \varphi) \xrightarrow{\simeq} L$$

such that φ increases the weight and $[\phi(v)] = \rho(v)$ for each $v \in V^0$.

Moreover, if $\gamma\colon (\widehat{\mathbb{L}}(V), d + \psi) \xrightarrow{\simeq} L$ is another Lie model under the same conditions, there exists an isomorphism

$$f\colon (\widehat{\mathbb{L}}(V), d + \varphi) \xrightarrow{\cong} (\widehat{\mathbb{L}}(V), d + \psi)$$

such that $f - \mathrm{id}_{\widehat{\mathbb{L}}(V)}$ increases the weight and γf is homotopic to ϕ. □

Definition 3.4 Let π be a connected cgl and $(\widehat{\mathbb{L}}(V),d)$ be its bigraded model. Define the sub-Lie algebra

$$\mathfrak{Der}\,\widehat{\mathbb{L}}(V)\subset \operatorname{Der}\widehat{\mathbb{L}}(V)$$

of derivations which raise the weight. That is, if $W^m\subset\widehat{\mathbb{L}}(V)$ denotes the subspace of elements of weight m, then $\theta\in\mathfrak{Der}\,\widehat{\mathbb{L}}(V)$ if $\theta(W^m)\subset W^{\geq m+1}$ for all $m\in\mathbb{Z}$.

We can now easily prove the dual of [16, Theorem 4.1]:

Theorem 3.5 *We have that $(\mathfrak{Der}\,\widehat{\mathbb{L}}(V),D)$ is a cdgl whose $\widetilde{\mathrm{MC}}$ set is in bijective correspondence with the set* $\mathrm{Ho}\,\mathbf{cdgl}_\pi$ *of homotopy types of cdgls whose homology is isomorphic to π.*

Proof Filter $\mathfrak{Der}\,\widehat{\mathbb{L}}(V)$ by $\{F^n\}_{n\geq 1}$, where

$$F^n=\{\theta\in\mathfrak{Der}\,\widehat{\mathbb{L}}(V)\mid \theta(W^m)\subset W^{m+n}\text{ for all }m\}.$$

A simple inspection shows that $\{F^n\}_{n\geq1}$ is indeed a filtration of the dgl $(\mathfrak{Der}\,\widehat{\mathbb{L}}(V),D)$. Moreover, $\bigcap_{n\geq1}F^n=0$ so the natural map $\zeta\colon\mathfrak{Der}\,\widehat{\mathbb{L}}(V)\to\varprojlim_n\mathfrak{Der}\,\widehat{\mathbb{L}}(V)/F^n$ is injective.

On the other hand, write any $\theta\in\varprojlim_n\mathfrak{Der}\,\widehat{\mathbb{L}}(V)/F^n$ of degree q as

$$\theta=\sum_{n\geq1}\theta_n\quad\text{for }\ \theta_n\in F^n,$$

and observe that, for any $p,m\geq0$, $\theta_n(V_p^m)=0$ as long as $n>q+m$. Hence, for any $v\in V$, $\sum_{n\geq1}\theta_n(v)$ is always a finite sum. That is, θ is a well-defined element in $\mathfrak{Der}\,\widehat{\mathbb{L}}(V)$ and thus ζ is also surjective. This shows that $(\mathfrak{Der}\,\widehat{\mathbb{L}}(V),D)$ is a complete dgl. Note that $d\notin\mathfrak{Der}\,\widehat{\mathbb{L}}(V)$ as it does not raise the weight.

We next see that

$$\exp(\mathfrak{Der}_0\widehat{\mathbb{L}}(V))=\{f\in\operatorname{aut}\widehat{\mathbb{L}}(V)\text{ such that }f-\mathrm{id}_{\widehat{\mathbb{L}}(V)}\text{ raises the weight}\}.$$

Indeed, given $\theta\in\mathcal{D}\mathrm{er}_0\,\widehat{\mathbb{L}}(V)$ we have $e^\theta-\mathrm{id}_{\widehat{\mathbb{L}}(V)}=\sum_{n\geq1}\theta^n/n!$, which clearly raises the weight. Conversely, given $f\in\operatorname{aut}\widehat{\mathbb{L}}(V)$ such that $f-\mathrm{id}_{\widehat{\mathbb{L}}(V)}$ raises the weight, the linear map

$$\theta\colon V\to\widehat{\mathbb{L}}(V)\quad\text{given by }\ \theta(v)=\sum_{n\geq1}(-1)^{n+1}\frac{(f-\mathrm{id})^n}{n}$$

is well defined and clearly raises the degree. In fact, the same argument used above shows that for any $p,m\geq0$ and any $v\in V_p^m$ we have $(f-\mathrm{id})^n(v)=0$ for n big enough. To conclude, extend θ as a derivation in $\mathfrak{Der}_0\widehat{\mathbb{L}}(V)$ so that $\theta=\log f$, or equivalently, $f=e^\theta$.

Finally, regard the MC set as in (5) and consider the map

$$\mathrm{MC}(\mathfrak{Der}\,\widehat{\mathbb{L}}(V),D)\to\mathrm{Ho}\,\mathbf{cdgl}_\pi\qquad\text{given by }\ \bar\varphi\mapsto\text{homotopy type of }(\widehat{\mathbb{L}}(V),d+\varphi).$$

It clearly factors through the orbit set

$$\widetilde{\mathrm{MC}}(\mathfrak{Der}\,\widehat{\mathbb{L}}(V),D)=\mathrm{MC}(\mathfrak{Der}\,\widehat{\mathbb{L}}(V),D)/\exp(\mathfrak{Der}_0\,\widehat{\mathbb{L}}(V))\to\mathrm{Ho}\,\mathbf{cdgl}_\pi,$$

and, by a direct application of Theorem 3.3, this is a bijection. □

Corollary 3.6 *Let π be a finite type connected cgl and let $(\widehat{\mathbb{L}}(V), d)$ be its bigraded model. Then the set* $\operatorname{Ho}\mathbf{sset}_\pi$ *of homotopy types of rational simplicial sets whose homotopy Lie algebra is isomorphic to π is in bijective correspondence with $\widetilde{\mathrm{MC}}(\mathfrak{Der}\,\widehat{\mathbb{L}}(V), D)$.*

Proof We first note that any rational simplicial set whose homotopy Lie algebra is isomorphic to π is a nilpotent finite type simplicial set. Indeed, every complete finite type Lie algebra π is degreewise nilpotent [1, Proposition 5.2]. That is, for each degree n there is an integer $k \geq 1$ such that any bracket of length k and degree n vanishes. Moreover, if π is connected, being degreewise nilpotent is equivalent to π_0 being nilpotent and acting nilpotently on π_n for all $n \geq 1$. Hence, any simplicial set having π as homotopy Lie algebra is necessarily rational nilpotent and of finite type.

On the other hand, the pair of adjoint functors in (2) restrict to equivalences between the homotopy categories of rational nilpotent finite type simplicial sets and that of connected cdgls whose homology is complete and of finite type, ie degreewise nilpotent [6, Chapter 10]. To finish, apply Theorem 3.5 □

Remark 3.7 Identifying, as in (6), $\mathrm{MC}(\mathfrak{Der}\,\widehat{\mathbb{L}}(V), D)$ with the rational points of the variety V_L with $L = (\widehat{\mathbb{L}}(V), d)$, the above corollary exhibits the quotient stack $V_L/\exp(\mathfrak{Der}_0\,\widehat{\mathbb{L}}(V))$ as a moduli space of $\operatorname{Ho}\mathbf{sset}_\pi$.

We are aware that, as we work over the rationals, this topological space is not a quotient of a variety. Nevertheless, following [16, Section 7], where the authors study the commutative dual context, one could properly define and study $\mathfrak{Der}\,L$ as a scheme and $\exp(\mathfrak{Der}_0\,\widehat{\mathbb{L}}(V))$ as an algebraic group acting on $\mathfrak{Der}\,L$. In this way $\operatorname{Ho}\mathbf{sset}_\pi$ would become a quotient stack. This remark applies to the subsequent sections

4 Rational homotopy types with prescribed homology and their moduli space

We describe the geometric realization of the cdgls of derivations provided in the previous section and interpret their $\widetilde{\mathrm{MC}}$ sets from the topological point of view.

Definition 4.1 Let H be a simply connected graded vector space bounded above. Denote by $\operatorname{Ho}\mathbf{sset}^1_H$ the class of homotopy types of rational simply connected simplicial sets with reduced homology isomorphic to H. To avoid excessive notation, we will not distinguish a simplicial set from the homotopy type it represents.

We fix such a graded vector space H and a finite filtration of it,

$$H = H^0 \supset H^1 \supset \cdots \supset H^{q-1} \supset H^q = 0.$$

This induces a filtration on $V = s^{-1}H$ as in (7). Let $L = (\mathbb{L}(V), 0)$ and consider the cdgl $\mathcal{D}\mathrm{er}\,L$ given in Proposition 2.4 corresponding to this filtration.

For each $X \in \mathbf{sset}^1_H$, denote by G the subgroup of homotopy classes of self-homotopy equivalences of X which raise the degree of the homology filtration:

$$G = \{[f] \in \mathcal{E}(X) \mid H(f)(H^i) \subset H^{i+1} \text{ for all } i\}.$$

Consider also the subgroup $\operatorname{aut}^*_G(X) \subset \operatorname{aut}^*(X)$ of pointed homotopy automorphisms whose homotopy classes (free or pointed, as X is simply connected) live in G:

$$\operatorname{aut}^*_G(X) = \{f \in \operatorname{aut}^*(X) \mid [f] \in G\}.$$

In $\mathbf{sset}^1_H$ there is a particular element that we denote by X_0, whose minimal model is L. This is the (co)formal space with free rational homotopy Lie algebra generated by H consisting of a wedge of rational spheres, one for each generator of H.

Theorem 4.2 (i) *There are actions of* $\operatorname{aut}(L)$ *on* $\operatorname{MC}(\mathcal{D}\mathrm{er}\, L)$ *and* $\operatorname{aut}(L)/\exp(\mathcal{D}\mathrm{er}_0\, L)$ *on* $\widetilde{\operatorname{MC}}(\mathcal{D}\mathrm{er}\, L)$ *which induce bijections*

$$\widetilde{\operatorname{MC}}(\mathcal{D}\mathrm{er}\, L)/(\operatorname{aut}(L)/\exp(\mathcal{D}\mathrm{er}_0\, L)) \cong \operatorname{MC}(\mathcal{D}\mathrm{er}\, L)/\operatorname{aut}(L) \cong \operatorname{Ho}\mathbf{sset}^1_H.$$

(ii) *Moreover*,

$$\langle \mathcal{D}\mathrm{er}\, L\rangle = \coprod_{X \in \operatorname{Ho}\mathbf{sset}^1_H} \coprod_{\mathcal{O}_X} B\operatorname{aut}^*_G(X).$$

Here $\mathcal{O}_X$ denotes the (cardinality of the) orbit by the action of $\operatorname{aut}(L)/\exp(\mathcal{D}\mathrm{er}_0\, L)$ of any element in $\widetilde{\operatorname{MC}}(\mathcal{D}\mathrm{er}\, L)$ providing X via the bijection in (i). In other words, when H is finite dimensional, the realization of $\mathcal{D}\mathrm{er}\, L$ is the disjoint union of simplicial sets, one for each $X \in \operatorname{Ho}\mathbf{sset}^1_H$, and each of which with as many path components as points in the orbit $\mathcal{O}_X$. Finally, each of these path components has the homotopy type of the classifying space $B\operatorname{aut}^*_G(X)$, which is nilpotent but clearly not simply connected.

Remark 4.3 In (ii) we may replace the zero differential on L by any other decomposable differential d. Indeed, in view of (5), any such differential is an MC element in $\mathcal{D}\mathrm{er}\, L$, and by (4),

$$\langle \mathcal{D}\mathrm{er}\, L\rangle = \langle(\mathcal{D}\mathrm{er}\, L, 0)\rangle \simeq \langle(\mathcal{D}\mathrm{er}\, L, 0_d)\rangle = \langle(\mathcal{D}\mathrm{er}\, L, D)\rangle,$$

where $D = [d, -]$, the differential induced by d.

Proof (i) In view of Proposition 2.4, the MC elements of $\mathcal{D}\mathrm{er}\, L$ are simply decomposable differentials on $\mathbb{L}(V)$. Therefore, the group $\operatorname{aut}(L)$ acts on $\operatorname{MC}(\mathcal{D}\mathrm{er}\, L)$ by

$$\varphi \cdot \delta = \varphi\delta\varphi^{-1} \quad \text{for } \varphi \in \operatorname{aut}(L) \text{ and } \delta \in \operatorname{MC}(\mathcal{D}\mathrm{er}\, L). \tag{12}$$

That is, $\varphi \cdot \delta = \delta'$ if

$$\varphi\colon (\mathbb{L}(V), \delta) \xrightarrow{\cong} (\mathbb{L}(V), \delta')$$

is a dgl isomorphism. Note also that the map

$$\operatorname{MC}(\mathcal{D}\mathrm{er}\, L) \to \operatorname{Ho}\mathbf{sset}^1_H \quad \text{given by } \delta \mapsto \langle(\mathbb{L}(V), \delta)\rangle$$

induces a map on the orbit set,

$$\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(L) \xrightarrow{\cong} \mathrm{Ho}\,\mathbf{sset}^1_H, \tag{13}$$

which is clearly a bijection.

On the other hand, and although $\exp(\mathcal{D}\mathrm{er}_0\, L)$ is not in general a normal subgroup of $\mathrm{aut}(L)$, we still can consider the short exact sequence of pointed sets

$$\exp(\mathcal{D}\mathrm{er}_0\, L) \to \mathrm{aut}(L) \to \mathrm{aut}(L)/\exp(\mathcal{D}\mathrm{er}_0\, L), \tag{14}$$

and observe that the action of $\mathrm{aut}(L)$ on $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)$ restricts to the gauge action of $\mathcal{D}\mathrm{er}_0\, L$ on $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)$: $\theta \mathscr{G} \delta = \delta'$ if, again,

$$(\mathbb{L}(V), \delta) \xrightarrow{e^\theta} (\mathbb{L}(V), \delta')$$

is a dgl isomorphism.

Hence, $\mathrm{aut}(L)/\exp(\mathcal{D}\mathrm{er}_0\, L)$ acts on the orbit set $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\exp(\mathcal{D}\mathrm{er}_0\, L) = \widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)$ and

$$\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)/(\mathrm{aut}(L)/\exp(\mathcal{D}\mathrm{er}_0\, L)) \cong \mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(L).$$

This and (13) proves (i).

(ii) By (3), the number of connected components of $\pi_0\langle \mathcal{D}\mathrm{er}\, L\rangle$ is in bijective correspondence with $\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)$. But, in view of (i), each homotopy type of $\mathrm{Ho}\,\mathbf{sset}^1_H$ contains as many $\widetilde{\mathrm{MC}}$ elements of $\mathcal{D}\mathrm{er}\, L$ as points in $\mathcal{O}_X$. Hence, the number of connected components of $\langle \mathcal{D}\mathrm{er}\, L\rangle$ is as asserted.

Next choose $d \in \widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)$, which again corresponds to a decomposable differential d in $L = \widehat{\mathbb{L}}(V)$. Then the (algebraic) component $(\mathcal{D}\mathrm{er}\, L)^d$ is the connected cdgl

$$(\mathcal{D}\mathrm{er}\, L)^d_k = \begin{cases} \mathrm{Der}_k\, L & \text{if } k > 0, \\ \mathcal{D}\mathrm{er}_0\, L \cap \ker D & \text{if } k = 0, \end{cases}$$

whose differential is $D = [d, -]$, induced by d. By [8, Theorem 7.13], if we denote by $X \in \mathbf{sset}^1_H$ the (rational homotopy type of the) simplicial set whose minimal model is $(\mathbb{L}(V), d)$, we deduce that

$$\langle (\mathcal{D}\mathrm{er}\, L)^d\rangle \simeq B\mathrm{aut}^*_G(X),$$

and (ii) follows. $\square$

The following instance is of special interest. If we choose in H the trivial filtration $H = H^0 \supset H^1 = 0$, Theorem 4.2 reads:

Corollary 4.4 (i) *There are actions of* $\mathrm{aut}(L)$ *on* $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)$ *and* $\mathrm{aut}(V)$ *on* $\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)$ *which induce bijections*

$$\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(V) \cong \mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(L) \cong \mathrm{Ho}\,\mathbf{sset}^1_H.$$

(ii) *Moreover,*

$$\langle \mathcal{D}\mathrm{er}\, L\rangle = \coprod_{X \in \mathrm{Ho}\,\mathbf{sset}^1_H} \coprod_{\mathcal{O}_X} B\mathrm{aut}^*_{\mathcal{H}}(X).$$

Here, for each $X \in \mathrm{Ho}\,\mathbf{sset}^1_H$, $\mathcal{H}$ denotes the subgroup of homotopy classes of self-homotopy equivalences that induce the identity on homology. Again, $\mathcal{O}_X$ denotes the (cardinality of the) orbit by the action of $\mathrm{aut}(V)$ of any element in $\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)$ representing X by the bijection in (i).

Proof Recall from Remark 2.5 that in this case

$$\mathcal{D}\mathrm{er}_0\, L = \{\theta \in \mathrm{Der}_0\, L \text{ such that } \theta(V) \subset \widehat{\mathbb{L}}^{\geq 2}(V)\}.$$

Then

$$\exp(\mathcal{D}\mathrm{er}_0\, L) = \{\varphi \in \mathrm{aut}(L) \mid \varphi_* = \mathrm{id}_V\},$$

where $\varphi_* \colon V \to V$ denotes the induced map on the indecomposables. Hence (14) becomes

$$\exp(\mathcal{D}\mathrm{er}_0\, L) \to \mathrm{aut}(L) \to \mathrm{aut}(V), \tag{15}$$

and Theorem 4.2(i) translates to (i). With this, and the fact that $G = \mathcal{H}$ in this case, (ii) is obvious. □

In view of the isomorphism in (13),

$$\mathrm{Ho}\,\mathbf{sset}^1_H \cong \mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(L),$$

we can identify the quotient stack $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(L)$ as a moduli space of the set of simply connected rational homotopy types with prescribed reduced homology H. Moreover, two proportional (nontrivial) differentials in $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)$ are in the same orbit. Hence, as the polynomials defining $\boldsymbol{V_L}$ are homogeneous, we can think of $\mathrm{Ho}\,\mathbf{sset}^1_H - \{X_0\}$ as a quotient stack of a subvariety of a projective space.

Example 4.5 Let H be the vector space with two generators of degrees 2 and 4 and another two generators of degree 6. We compute the moduli space of $\mathrm{Ho}\,\mathbf{sset}^1_H$.

Let $L = (\mathbb{L}(V), 0)$ where V is the vector space with generators x, y, z and w of degrees 1, 3, 5 and 5, respectively. We endow V with the trivial filtration. Then $\mathcal{D}\mathrm{er}_{-1}\, L$ is a 3–dimensional vector space, generated by the derivations δ_y, δ_z and δ_w defined by

$$\delta_y(y) = [x, x], \quad \delta_z(z) = [x, y] \quad \text{and} \quad \delta_w(w) = [x, y],$$

and are zero otherwise. In this particular case, $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L) = \mathcal{D}\mathrm{er}_{-1}\, L$. Moreover, one easily checks that the gauge action is trivial and thus, in view of Corollary 4.4(i),

$$\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(L) \cong \mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(V) \cong \mathrm{Ho}\,\mathbf{sset}^1_H.$$

Hence if we use $\{\delta_y, \delta_z, \delta_w\}$ as basis, we identify four different orbits in $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(V)$ represented by the derivations $(0,0,0)$, $(\alpha, 0, 0)$ with $\alpha \neq 0$, $(0, \beta, \gamma)$ with either β or γ not zero, and (α, β, γ) with $\alpha \neq 0$ and either β or γ not zero. By considering which spaces correspond to these differentials, we obtain

$$\mathrm{Ho}\,\mathbf{sset}^1_H = \{S^2 \vee S^4 \vee S^6 \vee S^6 \mid (S^2 \times S^4) \vee S^6, \mathbb{C}P^2 \vee S^6 \vee S^6, \mathbb{C}P^3 \vee S^6\}.$$

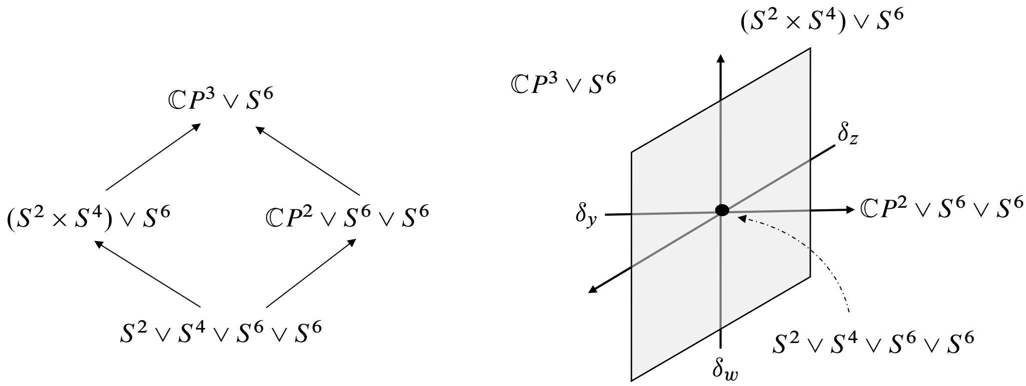


Figure 1

In other words, the moduli space of $\operatorname{Ho}\mathbf{sset}^1_H$ consists of four points, where $S^2 \vee S^4 \vee S^6 \vee S^6$ is a closed point, $\mathbb{C}P^2 \vee S^6$ is an open point, and any neighborhood of $(S^2 \times S^4) \vee S^6$ or $\mathbb{C}P^2 \vee S^6 \vee S^6$ contains the point $\mathbb{C}P^3 \vee S^6$. As a finite topological space, it is characterized by its corresponding poset in which $x \leq y$ if and only if x belongs to the closure of y.

In Figure 1 we depict this poset and the algebraic variety V_L, which in this case is all $\mathbb{C}^3$, or $\mathbb{C}P^2$ if we consider the corresponding projective variety by removing the origin. There, we identify the rational points belonging to each orbit of the moduli space: the origin is the only point in its orbit and corresponds to $S^2 \vee S^4 \vee S^6 \vee S^6$, all rational points of the δ_y axis are in the orbit of $\mathbb{C}P^2 \vee S^6 \vee S^6$, rational points of the plane generated by $\{\delta_z, \delta_w\}$ comprise the orbit of $(S^2 \times S^4) \vee S^6$, and the rest of the rational points are in the orbit of $\mathbb{C}P^3 \vee S^6$.

Note that in general, for any L, the zero differential is always alone in its orbit space $\operatorname{MC}(\mathcal{D}\mathrm{er}\, L)/\operatorname{aut}(L)$; it is a closed set and it corresponds to X_0, a wedge of spheres determined by a set of generators of H.

Also, $\operatorname{Ho}\mathbf{sset}^1_H \cong \operatorname{MC}(\mathcal{D}\mathrm{er}\, L)/\operatorname{aut}(L)$ is not always a finite space as one can check by, for instance, computing the example in which $H = \operatorname{Span}\{x, y, z\}$ with $|x| = 5$, $|y| = 6$ and $|z| = 22$.

An interesting property of the set of elliptic homotopy types sharing the same homology is:

Proposition 4.6 *The set* $\mathbf{Ell}_H \subset \operatorname{Ho}\mathbf{sset}^1_H$ *of homotopy types of elliptic spaces is always an open subset of the moduli space.*

Proof Fix $\delta \in \operatorname{MC}(\mathcal{D}\mathrm{er}_{-1}\, L)$ such that (L, δ) is elliptic. Since L is of finite type, the function

$$\dim_k : \operatorname{MC}(\mathcal{D}\mathrm{er}\, L) \to \mathbb{Z} \quad \text{given by} \quad \dim_k(\delta') = \dim H_k(L, \delta')$$

is well defined for each $k \geq 1$. Note that if $V = V_{\leq N}$ then $\dim_k(\delta) = 0$ for $k \geq 2N$; see for instance [9, Corollary 1, Section 32]. Moreover, by elementary linear algebra, regarding $\dim_k(\delta')$ as $\dim\ker \delta'|_{L_k} - \dim\operatorname{Im}\delta'|_{L_{k+1}}$, the map $\dim_k$ is semicontinuous for all k. In particular, for each $k \geq 1$ there is a neighborhood θ_k of δ such that $\dim_k(\delta') \leq \dim(H_k(L, \delta))$ for any $\delta' \in \theta_k$.

Consider the open set $\theta = \bigcap_{i=2N}^{3N} \theta_i$ in which $\dim_k = 0$ for all $k = 2N, \dots, 3N$. This implies that for any $\delta' \in \theta$, (L, δ') is elliptic. Otherwise, by [9, Theorem 33.3], if (L, δ') is hyperbolic, there must be an integer k_0 with $2N \le k_0 \le 3N$ such that $\dim_{k_0}(\delta') \neq 0$.

Finally, if we denote by $p\colon \mathrm{MC}(\mathcal{D}\mathrm{er}\, L) \to \mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(L)$ the projection, then $p(\theta) \subset \mathbf{Ell}_H$ is clearly an open set of the moduli space $\mathrm{MC}(\mathcal{D}\mathrm{er}\, L)/\mathrm{aut}(L)$ containing the orbit of δ. □

5 Rational homotopy types with prescribed cohomology algebra and their moduli space

Let $L = (\widehat{\mathbb{L}}(V), d)$ be a connected free cdgl and consider in V, which is supposed to be bounded above, the trivial filtration, so that Remark 2.5 applies.

Definition 5.1 Define $\mathscr{D}\mathrm{er}(L)$ as the complete sub-dgl of $\mathcal{D}\mathrm{er}\, L$ given by

$$\mathscr{D}\mathrm{er}_k\, L = \begin{cases} \mathcal{D}\mathrm{er}_k\, L & \text{if } k \ge 0, \\ \{\eta \in \mathrm{Der}_k\, L \text{ such that } \eta(V) \subset \widehat{\mathbb{L}}^{\ge 3}(V)\} & \text{if } k < 0. \end{cases}$$

This cdgl will be essential in what follows.

Definition 5.2 Consider a simply connected commutative graded algebra of finite dimension whose augmentation ideal we denote by A. Define $\mathrm{Ho}\,\mathbf{sset}^1_A$ as the class of homotopy types of rational simply connected simplicial sets with reduced cohomology algebra isomorphic to A. Again, we will not distinguish a simplicial set from the homotopy type that it represents.

Recall that given $X \in \mathrm{Ho}\,\mathbf{sset}^1_A$, a classical fact (see for instance [17, III.3.(9)]) states that the differential d in $\mathscr{L}(A^\sharp)$, necessarily quadratic, is naturally identified with the cup product of X. Here $\mathscr{L}$ denotes the classical Quillen functor from coalgebras to Lie algebras. We then fix A, rename $L = \mathscr{L}(A^\sharp)$ and prove:

Theorem 5.3 (i) *There is an action of* $\mathrm{aut}(A)$ *on* $\widetilde{\mathrm{MC}}(\mathrm{Der}\, L)$ *which induces a bijection*

$$\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\, L)/\mathrm{aut}(A) \cong \mathrm{Ho}\,\mathbf{sset}^1_A.$$

(ii) *Moreover,*

$$\langle \mathscr{D}\mathrm{er}\, L\rangle \simeq \coprod_{X \in \mathrm{Ho}\,\mathbf{sset}^1_A} \coprod_{\mathcal{O}_X} B\mathrm{aut}^*_{\mathcal{H}}(X).$$

Once again, $\mathcal{O}_X$ denotes the (cardinality of the) orbit by the action of $\mathrm{aut}(A)$ of any element in $\widetilde{\mathrm{MC}}(\mathcal{D}\mathrm{er}\, L)$ representing X by the bijection in (i).

Proof Write $L = (\mathbb{L}(V), d)$ where $V = s^{-1}A^\sharp$ and d is quadratic. Recall that the differential in $\mathscr{D}\mathrm{er}\, L$ is $[d, -]$. Hence, an MC element of $\mathscr{D}\mathrm{er}\, L$ is, by definition, a derivation δ of L such that $\delta(V) \subset \mathbb{L}^{\ge 3}(V)$ and $d + \delta$ is a differential on L. In what follows we use the trivial identification

$$\mathrm{MC}(\mathscr{D}\mathrm{er}\, L) \cong \{\text{differentials } d + \delta \text{ on } L \text{ such that } \delta(V) \subset \mathbb{L}^{\ge 3}(V)\},$$

so that

$$\mathrm{MC}(\mathscr{D}\mathrm{er}\,L) \subset \mathrm{MC}(\mathcal{D}\mathrm{er}\,L, 0) = \{\text{decomposable differentials on } L\}.$$

We then consider the stabilizer $\mathrm{aut}_d(L)$ of $\mathrm{MC}(\mathscr{D}\mathrm{er}\,L)$ of the action (12) of the (nondifferential automorphisms) $\mathrm{aut}(L)$ on $\mathrm{MC}(\mathcal{D}\mathrm{er}\,L, 0)$. That is,

$$\mathrm{aut}_d(L) = \{\varphi \in \mathrm{aut}(L) \text{ such that the quadratic part of } \varphi^{-1} \circ d \circ \varphi \text{ is } d\}.$$

On the other hand, the surjective map

$$\mathrm{MC}(\mathscr{D}\mathrm{er}\,L) \to \mathrm{Ho}\,\mathbf{sset}^1_A \quad \text{given by } d + \delta \mapsto \langle (L, d+\delta) \rangle$$

clearly induces a map on the set of orbits

$$\mathrm{MC}(\mathscr{D}\mathrm{er}\,L)/\mathrm{aut}_d(L) \to \mathrm{Ho}\,\mathbf{sset}^1_A.$$

Now, $\langle (L, d+\delta)\rangle \simeq \langle (L, d+\delta')\rangle$ if and only if there is dgl isomorphism

$$\varphi\colon (L, d+\delta) \xrightarrow{\cong} (L, d+\delta').$$

Thus $\varphi \in \mathrm{aut}_d(L)$ and $\varphi \cdot (d+\delta) = d+\delta'$. So the above map is also injective, and we have a bijection

$$\mathrm{MC}(\mathscr{D}\mathrm{er}\,L)/\mathrm{aut}_d(L) \cong \mathrm{Ho}\,\mathbf{sset}^1_A.$$

Next, observe that $\exp(\mathscr{D}\mathrm{er}_0\,L) \subset \mathrm{aut}_d(L)$ and the quotient $\mathrm{aut}_d(L)/\exp(\mathscr{D}\mathrm{er}_0\,L)$ is trivially identified to the group of automorphisms of V which respect the quadratic differential d. That is,

$$\mathrm{aut}_d(L)/\exp(\mathscr{D}\mathrm{er}_0\,L) \cong \{\phi\colon V \xrightarrow{\cong} V \text{ such that } \phi^{-1} \circ d \circ \phi = d\}.$$

But this group is in bijective correspondence with the algebra automorphisms $\mathrm{aut}(A)$ and we have the following short exact sequence, analogous to (15),

$$\exp(\mathscr{D}\mathrm{er}_0\,L) \to \mathrm{aut}_d(L) \to \mathrm{aut}(A).$$

Next observe that the action of $\mathrm{aut}_d(L)$ on $\mathrm{MC}(\mathscr{D}\mathrm{er}\,L)$ restricts to the gauge action of $\mathscr{D}\mathrm{er}_0\,L$ on $\mathrm{MC}(\mathscr{D}\mathrm{er}\,L)$. Hence, as in the proof of Theorem 4.2(i), we deduce that $\mathrm{aut}(A)$ acts on $\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\,L)$ and

$$\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\,L)/\mathrm{aut}(A) \cong \mathrm{MC}(\mathscr{D}\mathrm{er}\,L)/\mathrm{aut}_d(L) \cong \mathrm{Ho}\,\mathbf{sset}^1_A. \tag{16}$$

On the other hand, via this bijection, each homotopy type X of $\mathrm{Ho}\,\mathbf{sset}^1_H$ contains as many $\widetilde{\mathrm{MC}}$ elements of $\mathscr{D}\mathrm{er}\,L$ as points in the orbit $\mathcal{O}_X$, and thus the number of path components of $\langle \mathscr{D}\mathrm{er}\,L\rangle$ is as asserted in (ii) for a general A.

Finally, since A is finite dimensional and $\mathscr{D}\mathrm{er}_{\geq 0}\,L = \mathcal{D}\mathrm{er}_{\geq 0}\,L$, every connected component of $\langle \mathscr{D}\mathrm{er}\,L\rangle$ is necessarily of the homotopy type of $B\mathrm{aut}^*_{\mathcal{H}}(X)$ for the corresponding $X \in \mathrm{Ho}\,\mathbf{sset}^1_A$, just as in Corollary 4.4(ii). □

Remark 5.4 We can also exhibit the set of simply connected homotopy types sharing the same cohomology algebra A as a quotient stack. Indeed, in view of (16),

$$\mathrm{Ho}\,\mathbf{sset}^1_A \cong \mathrm{MC}(\mathscr{D}\mathrm{er}\,L)/\mathrm{aut}_d(L),$$

which by (6) is a quotient of rational points in a variety. Moreover, observe that

$$\mathcal{D}\mathrm{er}_{-1}\,L = \mathcal{D}\mathrm{er}^2_{-1}\,L \oplus \mathscr{D}\mathrm{er}_{-1}\,L,$$

where $\mathcal{D}\mathrm{er}^2_{-1}\,L = \{\eta \in \mathrm{Der}_{-1}\,L \text{ such that } \eta(V) \subset \mathbb{L}^2(V)\}$ is the set of quadratic derivations. Therefore we can identify $\mathrm{MC}(\mathscr{D}\mathrm{er}\,L)$ with the intersection of the algebraic variety $\mathrm{MC}(\mathcal{D}\mathrm{er}\,L, 0)$ and the affine linear subspace $d + \mathscr{D}\mathrm{er}_{-1}\,L$:

$$\mathrm{MC}(\mathscr{D}\mathrm{er}\,L) = \mathrm{MC}(\mathcal{D}\mathrm{er}\,L, 0) \cap (d + \mathscr{D}\mathrm{er}_{-1}\,L).$$

Example 5.5 Consider the commutative graded algebra A generated by the elements a, b, c, p and q of degrees 4, 6, 13, 15 and 19, respectively, and whose only nontrivial products are

$$ap = q = bc.$$

We determine the moduli space of $\mathrm{Ho}\,\mathbf{sset}^1_A$. Note that $L = \mathscr{L}(A) = (\mathbb{L}(V), d)$, where V is generated by elements x, y, z, u and v of degrees 3, 5, 12, 14 and 18, respectively. The differential is given by

$$dv = [x,u] + [y,z]$$

and zero on any other generator.

We now compute $\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\,L)/\mathrm{aut}(A)$. First, we check that $\mathscr{D}\mathrm{er}_{-1}\,L$ is generated by three derivations δ_z, δ_u and δ_v defined by

$$\delta_z(z) = \mathrm{ad}^2_x(y), \quad \delta_u(u) = \mathrm{ad}^2_y(x) \quad \text{and} \quad \delta_v(v) = \mathrm{ad}^4_x(y),$$

and zero otherwise. Direct computation shows that a general element $\alpha\delta_z + \beta\delta_u + \gamma\delta_v$ is in $\mathrm{MC}(\mathscr{D}\mathrm{er}\,L, D)$ if and only if $\alpha = \beta$.

To compute the gauge action, we first check that $\mathscr{D}\mathrm{er}_0\,L$ is generated by three derivations defined by

$$\theta(u) = \mathrm{ad}^3_x(y), \quad \theta'(v) = \mathrm{ad}^2_x(y) \quad \text{and} \quad \theta''(v) = -\mathrm{ad}^3_y(x),$$

and zero otherwise. Another straightforward computation shows that

$$D\theta = -\delta_v, \quad D\theta' = 0, \quad D\theta'' = 0 \quad \text{and} \quad [\mathscr{D}\mathrm{er}_0\,L, \mathscr{D}\mathrm{er}_{-1}\,L] = 0.$$

Therefore the only nontrivial gauge action is

$$(t\theta)\,\mathscr{G}(\alpha\delta_z + \alpha\delta_u + \gamma\delta_v) = \alpha\delta_z + \alpha\delta_u + (\gamma + t)\delta_v$$

for any $t \in \mathbb{Q}$. Hence in $\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\,L)$, we can take representatives with $\gamma = 0$, so that

$$\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\,L) = \{\alpha(\delta_z + \delta_u) \text{ with } \alpha \in \mathbb{Q}\}.$$

Finally, any automorphism $\phi \in \operatorname{aut}(A)$ is given by

$$\phi(a) = \lambda_a a, \quad \phi(b) = \lambda_b b, \quad \phi(c) = \lambda_c c, \quad \phi(p) = \lambda_p p \quad \text{and} \quad \phi(q) = \lambda_q q,$$

where the scalars are nonzero and satisfy

$$\lambda_a \lambda_p = \lambda_q = \lambda_b \lambda_c.$$

For $\alpha \neq 0$ choose ϕ with $\lambda_a = \lambda_c = \lambda_q = 1/\alpha$ and $\lambda_b = \lambda_p = 1$. Then one checks that the action of ϕ on the MC element $\delta_z + \delta_u$ gives $\alpha(\delta_z + \delta_u)$.

Therefore in $\widetilde{\mathrm{MC}}(\mathscr{D}\mathrm{er}\, L)/\operatorname{aut}(A)$ there are only two orbits corresponding to $\alpha \neq 0$ and $\alpha = 0$. By Theorem 5.3(i) we conclude that

$$\mathrm{Ho}\, \mathbf{sset}^1_A = \{X_0, X_1\},$$

where $X_0 = \langle (L, d) \rangle$ is the formal space of cohomology algebra A and $X_1 = \langle (L, d + \delta_z + \delta_u) \rangle$ is the rationalization of $\mathrm{SU}(6)/\mathrm{SU}(3) \times \mathrm{SU}(3)$.

Moreover, as a moduli space $\mathrm{Ho}\, \mathbf{sset}^1_A$ has the Sierpinski topology, in which X_1 is open.

References

[1] **A Berglund**, *Rational homotopy theory of mapping spaces via Lie theory for L_∞–algebras*, Homology Homotopy Appl. 17 (2015) 343–369 MR Zbl

[2] **A Berglund**, **T Zeman**, *Algebraic models for classifying spaces of fibrations*, preprint (2022) arXiv 2203.02462

[3] **D Blanc**, *Moduli spaces of homotopy theory*, from "Geometry, spectral theory, groups, and dynamics" (M Entov, Y Pinchover, M Sageev, editors), Contemp. Math. 387, Amer. Math. Soc., Providence, RI (2005) 37–63 MR Zbl

[4] **A K Bousfield**, **D M Kan**, *Homotopy limits, completions and localizations*, Lecture Notes in Math. 304, Springer (1972) MR Zbl

[5] **U Buijs**, **Y Félix**, **A Murillo**, **D Tanré**, *Homotopy theory of complete Lie algebras and Lie models of simplicial sets*, J. Topol. 11 (2018) 799–825 MR Zbl

[6] **U Buijs**, **Y Félix**, **A Murillo**, **D Tanré**, *Lie models in topology*, Progr. Math. 335, Birkhäuser, Cham (2020) MR Zbl

[7] **U Buijs**, **Y Félix**, **A Murillo**, **D Tanré**, *Lie models of simplicial sets and representability of the Quillen functor*, Israel J. Math. 238 (2020) 313–358 MR Zbl

[8] **Y Félix**, **M Fuentes**, **A Murillo**, *Lie models of homotopy automorphism monoids and classifying fibrations*, Adv. Math. 402 (2022) art. id. 108359 MR Zbl

[9] **Y Félix**, **S Halperin**, **J-C Thomas**, *Rational homotopy theory*, Graduate Texts in Math. 205, Springer (2001) MR Zbl

[10] **S Halperin**, **J Stasheff**, *Obstructions to homotopy equivalences*, Adv. Math. 32 (1979) 233–279 MR Zbl

[11] **Y Haralambous**, *Coformalité modérée et formalité des CW-complexes de dimension finie*, C. R. Acad. Sci. Paris Sér. I Math. 311 (1990) 365–368 MR Zbl

[12] **V Roca i Lucio**, *The integration theory of curved absolute homotopy Lie algebras*, preprint (2022) arXiv 2207.11115

[13] **J Neisendorfer**, *Lie algebras, coalgebras and rational homotopy theory for nilpotent spaces*, Pacific J. Math. 74 (1978) 429–460 MR Zbl

[14] **A Oukili**, *Sur l'homologie d'une algébre différentielle* (*de Lie*), PhD thesis, Université de Nice (1978)

[15] **D Quillen**, *Rational homotopy theory*, Ann. of Math. 90 (1969) 205–295 MR Zbl

[16] **M Schlessinger**, **J Stasheff**, *Deformation theory and rational homotopy type*, preprint (2012) arXiv 1211.1647

[17] **D Tanré**, *Homotopie rationnelle: modèles de Chen, Quillen, Sullivan*, Lecture Notes in Math. 1025, Springer (1983) MR Zbl

[18] **M D Zawodniak**, *A moduli space for rational homotopy types with the same homotopy Lie algebra*, PhD thesis, University of Georgia (2016) Available at `https://esploro.libs.uga.edu/esploro/outputs/9949334836202959`

Département de Mathématiques, Université Catholique de Louvain
Louvain-la-Neuve, Belgium

Departamento de Algebra, Geometria y Topología, Universidad de Málaga
Málaga, Spain

Current address: *CIMI, Insitut de Mathématiques, Université Paul Sabatier*
Toulouse, France

Departamento de Algebra, Geometria y Topología, Universidad de Málaga
Málaga, Spain

`yves.felix@uclouvain.be`, `m_fuentes@uma.es`, `aniceto@uma.es`

Received: 22 September 2022 Revised: 11 February 2023

Algebraic & Geometric Topology **24**:4 (2024) 2307–2329
DOI: 10.2140/agt.2024.24.2307
Published: 16 July 2024

On the positivity of twisted L^2–torsion for 3–manifolds

JIANRU DUAN

For any compact orientable irreducible 3–manifold N with empty or incompressible toral boundary, the twisted L^2–torsion is a nonnegative function defined on the representation variety $\mathrm{Hom}(\pi_1(N), \mathrm{SL}(n, \mathbb{C}))$. We show that if N has infinite fundamental group, then the L^2–torsion function is strictly positive. Moreover, this torsion function is continuous when restricted to the subvariety of upper triangular representations.

57K31

1 Introduction

Let N be a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. The L^2–torsion of N is a numerical topological invariant of N that equals $\exp(\mathrm{Vol}(N)/6\pi)$, where $\mathrm{Vol}(N)$ is the simplicial volume of N; see [Lück 2002, Theorem 4.3]. The idea of twisting is to use a linear representation of $\pi_1(N)$ to define more L^2–torsion invariants. The first attempt was made by Li and Zhang [2006a; 2006b] in which they defined the L^2–Alexander invariants for knot complements, making use of the one-dimensional representations of the knot group. Later Dubois, Friedl and Lück [Dubois et al. 2015a] introduced the L^2–Alexander torsion for 3–manifolds which recovers the L^2–Alexander invariants. A recent breakthrough was made independently by Liu [2017] and Lück [2018] who proved that the L^2–Alexander torsion is always positive. More interesting properties of the L^2–Alexander torsion are revealed in [Liu 2017; Friedl and Lück 2019]; for example, we now know that the L^2–Alexander torsion is continuous and its limiting behavior recovers the Thurston norm of N.

Generally, let $\mathcal{R}_n(\pi_1(N)) := \mathrm{Hom}(\pi_1(N), \mathrm{SL}(n, \mathbb{C}))$ be the representation variety. One wishes to define L^2–torsion twisted by any representation $\rho \in \mathcal{R}_n(\pi_1(N))$, and we have this *twisted L^2–torsion function* abstractly defined on the representation variety of $\pi_1(N)$:

$$\rho \mapsto \tau^{(2)}(N, \rho) \in [0, +\infty), \quad \rho \in \mathcal{R}_n(\pi_1(N)).$$

A technical obstruction to defining a reasonable L^2–torsion is that the corresponding L^2–chain complex must be weakly L^2–acyclic and of determinant class (see Definition 2.3). If either condition is not satisfied, we define the L^2–torsion to be 0 by convention.

It is natural to question the positivity and continuity of this function. The first result of this paper is the following:

Theorem 1.1 *Let N be a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. Suppose N has infinite fundamental group; then the twisted L^2–torsion $\tau^{(2)}(N,\rho)$ is positive for any group homomorphism $\rho\colon \pi_1(N)\to \mathrm{SL}(n,\mathbb{C})$.*

When N is a graph manifold, the twisted L^2–torsion function is explicitly computed in Theorem 4.1. Other cases are dealt with in Theorem 4.5 where we only need to consider fibered 3–manifolds thanks to the virtual fibering arguments. We carefully construct a CW structure for N as in [Dubois et al. 2015a] and observe that the matrices in the corresponding twisted L^2–chain complex are in a special form so that we can apply Liu's result [2017, Theorem 5.1] to guarantee the positivity of the Fuglede–Kadison determinant.

We have the following partial result regarding continuity of the twisted L^2–torsion function, We say $\rho\in\mathcal{R}_n(\pi_1(N))$ is an upper triangular representation if $\rho(g)$ is an upper triangular matrix for every $g\in\pi_1(N)$.

Theorem 1.2 *Let N be a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. Suppose N has infinite fundamental group. Define $\mathcal{R}_n^{\mathrm{t}}(\pi_1(N))$ to be the subvariety of $\mathcal{R}_n(\pi_1(N))$ consisting of upper triangular representations. Then the twisted L^2–torsion function*

$$\rho\mapsto\tau^{(2)}(N,\rho)$$

is continuous with respect to $\rho\in\mathcal{R}_n^{\mathrm{t}}(\pi_1(N))$.

The continuity of the twisted L^2–torsion function in general is open. It is mainly because the Fuglede–Kadison determinant of an arbitrary matrix over $\mathbb{C}[\pi_1(N)]$ is very difficult to compute. However, the L^2–torsion twisted by upper triangular representations is simpler because we can reduce many problems to the one-dimensional case, which is well studied under the name of the L^2–Alexander torsion (see Section 5). We remark that the work of Bénard and Raimbault [2022] based on the strong acyclicity property by Bergeron and Venkatesh [2013] shows that the twisted L^2–torsion function is positive and real analytic near any holonomy representation $\rho_0\colon\pi_1(N)\to\mathrm{SL}(2,\mathbb{C})$ of a hyperbolic 3–manifold N.

The proof relies on the continuity of L^2–Alexander torsion with respect to the cohomology classes, which is conjectured by [Lück 2018, Chapter 10]. This is done by introducing the concept of Alexander multitwists (see Section 5). One can similarly define the "multivariable L^2–Alexander torsion" and our argument essentially shows that the multivariable function is multiplicatively convex (compare Theorem 5.7), generalizing [Liu 2017, Theorem 5.1]. This then applies to show the continuity as desired.

The organization of this paper is as follows. In Section 2, we introduce the terminology of this paper and some algebraic facts. In Section 3, we define the twisted L^2–torsion for CW complexes and state some basic properties. In Section 4, we prove Theorem 1.1 in two steps: first for graph manifolds, then for hyperbolic or mixed manifolds. In Section 5, we begin with the L^2–Alexander torsion and then prove Theorem 1.2.

Acknowledgement The author wishes to thank his advisor Yi Liu for guidance and many conversations.

2 Notation and some algebraic facts

In this section we define the twisting functor and introduce L^2–torsion theory. The reader can refer to [Lück 2018] where discussions are taken on in a more general setting.

2.1 Twisting $\mathbb{C}G$–modules via $\mathrm{SL}(n,\mathbb{C})$ representations

Let G be a finitely generated group and let $\mathbb{C}G$ be its group ring. In this paper our main objects are finitely generated free left $\mathbb{C}G$–modules with a preferred ordered basis. We will abbreviate it as *based $\mathbb{C}G$–modules* unless otherwise stated. A natural example of a based $\mathbb{C}G$–module is $\mathbb{C}G^m$ as a free left $\mathbb{C}G$–module of rank m, with the natural ordered basis $\{\sigma_1,\dots,\sigma_m\}$ where σ_i is the unit element of the i^{th} direct summand. Any based $\mathbb{C}G$–module is canonically isomorphic to $\mathbb{C}G^m$ for some nonnegative integer m and this identification is used throughout.

We fix V to be an n–dimensional complex vector space with a fixed choice of basis $\{e_i\}_{i=1}^n$. Let $\rho\colon G\to\mathrm{SL}(n,\mathbb{C})$ be a group homomorphism. Then V can be viewed as a left $\mathbb{C}G$–module via ρ,

$$\gamma\cdot e_i=\sum_{j=1}^n\rho(\gamma^{-1})_{i,j}\cdot e_j,\quad \gamma\in G,$$

where $\rho(\gamma^{-1})\in\mathrm{SL}(n,\mathbb{C})$ is a square matrix. We extend this action $\mathbb{C}$–linearly so that V is a left $\mathbb{C}G$–module. In other words, left action of γ corresponds to right multiplication to the row coordinate vector of the matrix $\rho(\gamma^{-1})$.

We are interested in twisting a based $\mathbb{C}G$–module via ρ. In literature, there are two different ways to twist a based $\mathbb{C}G$–module, namely the "diagonal twisting" and the "partial twisting" (compare [Lück 2018]). They are naturally isomorphic. We only consider the diagonal twisting.

Definition 2.1 Recall that $\mathbb{C}G^m$ is a based $\mathbb{C}G$–module with a natural basis $\{\sigma_i\}$ for $i=1,\dots,m$. We define $(\mathbb{C}G^m\otimes_{\mathbb{C}}V)_{\mathrm{d}}$ to be the $\mathbb{C}G$–module with diagonal $\mathbb{C}G$–action, ie

$$(\mathbb{C}G^m\otimes_{\mathbb{C}}V)_{\mathrm{d}}:=\mathbb{C}G^m\otimes_{\mathbb{C}}V,\quad g\cdot(u\otimes v)=gu\otimes gv$$

for any $g\in G$, $u\in\mathbb{C}G^m$ and $v\in V$, and then extend $\mathbb{C}$–linearly to define a $\mathbb{C}G$–module structure.

With the definition above, we can see that

$$(\mathbb{C}G^m\otimes_{\mathbb{C}}V)_{\mathrm{d}}=\bigoplus_{i=1}^m(\mathbb{C}G\otimes_{\mathbb{C}}V)_{\mathrm{d}}$$

is a based $\mathbb{C}G$–module with a basis

$$\{\sigma_1\otimes e_1,\sigma_1\otimes e_2,\dots,\sigma_1\otimes e_n,\sigma_2\otimes e_1,\dots,\sigma_m\otimes e_n\}.$$

Let $\mathscr{A}$ be the category whose objects are finitely generated free left $\mathbb{C}G$–modules with a preferred ordered basis and whose morphisms are $\mathbb{C}G$–linear homomorphisms. We consider the following "diagonal twisting" functor

$$\mathscr{D}(\rho)\colon \mathscr{A} \to \mathscr{A}$$

which sends any object M to the based $\mathbb{C}G$–module $(M \otimes_{\mathbb{C}} V)_{\mathrm{d}}$ and sends any morphism f to

$$\mathscr{D}(\rho) f := f \otimes_{\mathbb{C}} \mathrm{id}_V.$$

The following proposition describes how matrices behave under the twisting functor.

Proposition 2.2 *Let $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ be any group homomorphism. Suppose that a homomorphism between based $\mathbb{C}G$–modules*

$$f\colon \mathbb{C}G^r \to \mathbb{C}G^s$$

is presented by a matrix $(\Lambda_{i,j})$ over $\mathbb{C}G$ of size $r \times s$; ie if

$$\{\sigma_1, \dots, \sigma_r\}, \quad \{\tau_1, \dots, \tau_s\}$$

are the natural bases of $\mathbb{C}G^r$ and $\mathbb{C}G^s$, respectively, then

$$f(\sigma_i) = \sum_{j=1}^{s} \Lambda_{i,j} \tau_j, \quad i = 1, \dots, r.$$

We form a new matrix Ω of size $nr \times ns$ by replacing each entry $\Lambda_{i,j}$ with an $n \times n$ square matrix $\Lambda_{i,j} \cdot \rho(\Lambda_{i,j})$. Then Ω is a matrix presenting the diagonal twisting morphism $\mathscr{D}(\rho) f$, under the natural bases

$$\{\sigma_1 \otimes e_1, \dots, \sigma_1 \otimes e_n, \sigma_2 \otimes e_1, \dots, \sigma_r \otimes e_n\},$$

$$\{\tau_1 \otimes e_1, \dots, \tau_1 \otimes e_n, \tau_2 \otimes e_1, \dots, \tau_s \otimes e_n\}$$

of the diagonal twisting based $\mathbb{C}G$–modules $\mathscr{D}(\rho)(\mathbb{C}G^r)$ and $\mathscr{D}(\rho)(\mathbb{C}G^s)$, respectively.

Proof Let $\Phi = (\Phi_{i,j})$ for $i = 1, \dots, r$ and $j = 1, \dots, s$ be a block matrix of size $nr \times ns$, with each entry $\Phi_{i,j}$ an $n \times n$ matrix, such that Φ is the matrix presenting $\mathscr{D}(\rho) f$ under the natural basis. We only need to verify that $\Phi_{i,j} = \Lambda_{i,j} \cdot \rho(\Lambda_{i,j})$. The submatrix $\Phi_{i,j}$ can be characterized as follows. Let $\pi_j\colon \mathscr{D}(\rho)(\mathbb{C}G^r) \to \mathscr{D}(\rho)(\mathbb{C}G)$ be the projection to the j^{th} direct component which is spanned by $\{(\sigma_j \otimes e_1)_{\mathrm{d}}, \dots, (\sigma_j \otimes e_n)_{\mathrm{d}}\}$. Then

$$\pi_j \circ \mathscr{D}(\rho) f \begin{pmatrix} (\sigma_i \otimes e_1)_{\mathrm{d}} \\ \vdots \\ (\sigma_i \otimes e_n)_{\mathrm{d}} \end{pmatrix} = \Phi_{i,j} \begin{pmatrix} (\tau_j \otimes e_1)_{\mathrm{d}} \\ \vdots \\ (\tau_j \otimes e_n)_{\mathrm{d}} \end{pmatrix}.$$

On the other hand, for any $k = 1, \dots, n$,

$$\begin{aligned} \pi_j \circ \mathscr{D}(\rho) f((\sigma_i \otimes e_k)_{\mathrm{d}}) &= \pi_j \biggl(\sum_{l=1}^{s} (\Lambda_{i,l} \tau_l \otimes e_k)_{\mathrm{d}} \biggr) = \pi_j \biggl(\sum_{l=1}^{s} \Lambda_{i,l} \cdot (\tau_l \otimes \Lambda_{i,l}^{-1} e_k)_{\mathrm{d}} \biggr) \\ &= \Lambda_{i,j} \cdot (\tau_j \otimes \Lambda_{i,j}^{-1} e_k)_{\mathrm{d}} = \Lambda_{i,j} \cdot \sum_{l=1}^{n} \rho(\Lambda_{i,j})_{k,l} (\tau_j \otimes e_l)_{\mathrm{d}}. \end{aligned}$$

This shows that $\Phi_{i,j} = \Lambda_{i,j} \cdot \rho(\Lambda_{i,j})$, and hence $\Phi = \Omega$. □

We now mention that the twisting functor can be naturally generalized to the category of *based $\mathbb{C}G$–chain complexes*. More explicitly, let C_* be a based $\mathbb{C}G$–chain complex, ie

$$C_* = (\cdots \to C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \to \cdots)$$

is a chain of based $\mathbb{C}G$–modules with $\mathbb{C}G$–linear connecting morphisms $\{\partial_p\}$ such that $\partial_{p-1} \circ \partial_p = 0$. We can apply the functor $\mathfrak{D}(\rho)$ to obtain a new $\mathbb{C}G$–chain complex

$$\mathfrak{D}(\rho)C_* = (\cdots \to \mathfrak{D}(\rho)C_{p+1} \xrightarrow{\mathfrak{D}(\rho)\partial_{p+1}} \mathfrak{D}(\rho)C_p \xrightarrow{\mathfrak{D}(\rho)\partial_p} \mathfrak{D}(\rho)C_{p-1} \to \cdots)$$

with connecting homomorphisms $\{\mathfrak{D}(\rho)\partial_p\}$. If f_* is a chain map between based $\mathbb{C}G$–chain complexes, the twisting chain map $\mathfrak{D}(\rho) f_*$ is a $\mathbb{C}G$–chain map between the corresponding twisted chain complexes. So $\mathfrak{D}(\rho)$ generalizes to be a functor of the category of based $\mathbb{C}G$–chain complexes.

2.2 L^2–torsion theory

Let

$$l^2(G) = \left\{ \sum_{g \in G} c_g \cdot g \;\middle|\; c_g \in \mathbb{C}, \sum_{g \in G} |c_g|^2 < \infty \right\}$$

be the Hilbert space orthonormally spanned by all elements in G. Since G is finitely generated, $l^2(G)$ is a separable Hilbert space with isometric left and right $\mathbb{C}G$–module structure. We denote by $\mathcal{N}(G)$ the *group von Neumann algebra* of G which consists of all bounded Hilbert operators of $l^2(G)$ that commute with the right $\mathbb{C}G$–action. We will treat $l^2(G)$ as a left $\mathcal{N}(G)$–module and a right $\mathbb{C}G$–module. The *l^2–completion* of a based $\mathbb{C}G$–chain complex C_* is then a *Hilbert $\mathcal{N}(G)$–chain complex* defined as

$$l^2(G) \otimes_{\mathbb{C}G} C_*,$$

and the l^2–completions of the connecting homomorphism ∂ and chain map f are $\mathrm{id} \otimes_{\mathbb{C}G} \partial$ and $\mathrm{id} \otimes_{\mathbb{C}G} f$, respectively. Note that each chain module of $l^2(G) \otimes_{\mathbb{C}G} C_*$ is simply a direct sum of $l^2(G)$,

$$l^2(G) \otimes_{\mathbb{C}G} C_p = l^2(G) \otimes_{\mathbb{C}G} \mathbb{C}G^{r_p} = l^2(G)^{r_p},$$

where r_p is the rank of C_p.

The l^2–completion process converts a based $\mathbb{C}G$–chain complex into a finitely generated, free Hilbert $\mathcal{N}(G)$–chain complex.

Definition 2.3 A finitely generated, free Hilbert $\mathcal{N}(G)$–chain complex is called *weakly acyclic* if the l^2–Betti numbers are all trivial. A finitely generated, free Hilbert $\mathcal{N}(G)$–chain complex is *of determinant class* if all the Fuglede–Kadison determinants of the connecting homomorphisms are positive real numbers.

Definition 2.4 Let C_* be a finitely generated, free Hilbert $\mathcal{N}(G)$–chain complex. Suppose C_* is of finite length, ie there exists an integer $N > 0$ such that $C_p = 0$ for $|p| > N$. Furthermore, if C_* is weakly acyclic and of determinant class, we define the L^2*–torsion* of C_* to be the alternating product of the Fuglede–Kadison determinants of the connecting homomorphisms:

$$\tau^{(2)}(C_*) = \prod_{p\in\mathbb{Z}} (\det_{\mathcal{N}(G)} \partial_p)^{(-1)^p}.$$

Otherwise, we artificially set $\tau^{(2)}(C_*) = 0$.

We recommend [Lück 2002] for the definition of the L^2–Betti number and the Fuglede–Kadison determinant. We remark that our notational convention follows [Dubois et al. 2015a; 2015b; Liu 2017], and the exponential of the torsion in [Lück 2002; 2018] is the multiplicative inverse of our torsion.

Let A be a $p\times p$ matrix over $\mathcal{N}(G)$. The *regular Fuglede–Kadison determinant* of A is defined to be

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(A) = \begin{cases} \det_{\mathcal{N}(G)}(A) & \text{if } A \text{ is full rank of determinant class,} \\ 0 & \text{otherwise.} \end{cases}$$

We will need the following two lemmas in order to do explicit calculations; the proof can be found in [Dubois et al. 2015b, Lemmas 2.6 and 3.2] combining with the basic properties of the Fuglede–Kadison determinant (see [Lück 2002, Theorem 3.14]).

Lemma 2.5 *Let $\mathbb{Z}^k$ be a free abelian subgroup of G generated by $z_1,\dots,z_k$. Let A be a $p\times p$ matrix over $\mathbb{C}\mathbb{Z}^k$. Identify $\mathbb{C}\mathbb{Z}^k$ with the k–variable Laurent polynomial ring $\mathbb{C}[z_1^{\pm},\dots,z_k^{\pm}]$, and denote by $p(z_1,\dots,z_k)$ the ordinary determinant of A. Then*

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(A) = \mathrm{Mah}(p(z_1,\dots,z_k)),$$

where $\mathrm{Mah}(p(z_1,\dots,z_k))$ is the Mahler measure of the polynomial $p(z_1,\dots,z_k)$.

Lemma 2.6 *Let*

$$D_* = (0 \to \mathbb{C}G^j \xrightarrow{C} \mathbb{C}G^k \xrightarrow{B} \mathbb{C}G^{k+l-j} \xrightarrow{A} \mathbb{C}G^l \to 0)$$

be a complex, $L\subset\{1,\dots,k+l-j\}$ be a subset of size l and $J\subset\{1,\dots,k\}$ a subset of size j. Define

- *$A(J)$ to be the rows in A corresponding to J;*
- *$B(J,L)$ to be the result of deleting the columns of B corresponding to J and deleting the rows corresponding to L;*
- *$C(L)$ to be the columns of C corresponding to L.*

View A, B and C as matrices over $\mathcal{N}(G)$. If $\det^{\mathrm{r}}_{\mathcal{N}(G)}(A(J)) \neq 0$ and $\det^{\mathrm{r}}_{\mathcal{N}(G)}(C(L)) \neq 0$, then

$$\tau^{(2)}(l^2(G)\otimes_{\mathbb{C}G} D_*) = \det^{\mathrm{r}}_{\mathcal{N}(G)}(B(J,L))\cdot \det^{\mathrm{r}}_{\mathcal{N}(G)}(A(J))^{-1}\cdot \det^{\mathrm{r}}_{\mathcal{N}(G)}(C(L))^{-1}.$$

3 Twisted L^2–torsion for CW complexes

Let X be a finite CW complex with fundamental group G. Denote by $\widehat{X}$ the universal cover of $|X|$ with the natural CW complex structure coming from X. Choose a lifting $\hat{\sigma}_i$ for each cell σ_i in the CW structure of X. The deck group G acts freely on the cellular chain complex of $\widehat{X}$ on the left, which makes the $\mathbb{C}$–coefficient cellular chain complex $C_*(\widehat{X})$ a based $\mathbb{C}G$–chain complex with basis $\{\hat{\sigma}_i\}$. Recall that $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ is any group homomorphism.

For future convenience, we introduce the concept of *admissible triple* for higher-dimensional linear representations, generalizing the admissibility condition in [Dubois et al. 2015b].

Definition 3.1 (admissible triple) Let $\gamma\colon G \to H$ be a homomorphism to a countable group H. We say that $(G,\rho;\gamma)$ forms an *admissible triple* if $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ factors through γ, ie for some homomorphism $\psi\colon H \to \mathrm{SL}(n,\mathbb{C})$, the following diagram commutes:

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[dr, "\rho"'] & H \arrow[d, "\psi"] \\ & \mathrm{SL}(n,\mathbb{C}) \end{tikzcd}$$

Definition 3.2 Let $(G,\rho;\gamma)$ be an admissible triple. Consider $l^2(H)$ as a left Hilbert $\mathcal{N}(H)$–module, and a right $\mathbb{C}G$–module induced by γ. Define the L^2–chain complex of X twisted by $(G,\rho;\gamma)$ to be the Hilbert $\mathcal{N}(H)$–chain complex

$$C_*^{(2)}(X,\rho;\gamma) := l^2(H) \otimes_{\mathbb{C}G} \mathfrak{D}(\rho) C_*(\widehat{X}).$$

We define the L^2–torsion of X twisted by $(G,\rho;\gamma)$ as

$$\tau^{(2)}(X,\rho;\gamma) := \tau^{(2)}(C_*^{(2)}(X,\rho;\gamma)).$$

Proposition 3.3 *The definition of $\tau^{(2)}(X,\rho;\gamma)$ with respect to any admissible triple $(G,\rho;\gamma)$ does not depend on the order or orientation of the basis $\{\sigma_i\}$, nor the choice of lifting $\{\hat{\sigma}_i\}$. Moreover, let $\rho'\colon G \to \mathrm{SL}(n,\mathbb{C})$ be conjugate to ρ, ie there exists a matrix $T \in \mathrm{SL}(n,\mathbb{C})$, such that $\rho' = T \cdot \rho \cdot T^{-1}$. Then $(G,\rho';\gamma)$ is also an admissible triple and $\tau^{(2)}(X,\rho;\gamma) = \tau^{(2)}(X,\rho';\gamma)$.*

Proof The property of being weakly L^2–acyclic does not depend on the choices in the statement. We only need to analyze how these choices change the Fuglede–Kadison determinant of the connecting morphisms.

Abbreviate by $C_*(\widehat{X},\rho) := \mathfrak{D}(\rho) C_*(\widehat{X};\mathbb{C})$ the diagonal twisting chain complex. Suppose the based cellular chain complex of $\widehat{X}$ has the form

$$C_*(\widehat{X}) = (\cdots \to \mathbb{C}G^{r_{i+1}} \xrightarrow{\partial_{i+1}} \mathbb{C}G^{r_i} \xrightarrow{\partial_i} \mathbb{C}G^{r_{i-1}} \to \cdots),$$

where ∂_i is an $r_i \times r_{i-1}$ matrix over $\mathbb{C}G$ for all i. Then the diagonal twisting chain complex $C_*(\widehat{X},\rho)$ has the form

$$C_*(\widehat{X},\rho) = (\cdots \to \mathbb{C}G^{nr_{i+1}} \xrightarrow{\partial^{\rho}_{i+1}} \mathbb{C}G^{nr_i} \xrightarrow{\partial^{\rho}_i} \mathbb{C}G^{nr_{i-1}} \to \cdots),$$

where $\partial_i^\rho = \mathfrak{D}(\rho)\partial_i$ is an $nr_i \times nr_{i-1}$ matrix over $\mathbb{C}G$ for all i. An explicit formula for ∂_i^ρ is presented in Proposition 2.2. Then the L^2–chain complex of X twisted by $(G, \rho; \gamma)$ has the form

$$C_*^{(2)}(X, \rho; \gamma) = (\cdots \to l^2(H)^{nr_{i+1}} \xrightarrow{\gamma(\partial_{i+1}^\rho)} l^2(H)^{nr_i} \xrightarrow{\gamma(\partial_i^\rho)} l^2(H)^{nr_{i-1}} \to \cdots),$$

where $\gamma(\partial_i^\rho)$ means applying the group homomorphism γ to each monomial of any entry of the matrix ∂_i^ρ, resulting in a matrix over $\mathbb{C}H \subset \mathcal{N}(H)$.

We now analyze how the choices affect the value of $\tau^{(2)}(X, \rho; \gamma)$. If the basis of $C_i(X)$ is permuted, and the orientations are changed, then $\gamma(\partial_i^\rho)$ and $\gamma(\partial_{i+1}^\rho)$ change by multiplying a permutation matrix, with entries ± 1.

If one choose another lifting $g\hat{\sigma}$ instead of $\hat{\sigma}$ for some $g \in G$, then $\gamma(\partial_i^\rho)$ and $\gamma(\partial_{i+1}^\rho)$ change by multiplying a block matrix of the form

$$\begin{pmatrix} I^{n\times n} & & & & \\ & \ddots & & & \\ & & \rho(g)^{\pm 1} \cdot I^{n\times n} & & \\ & & & \ddots & \\ & & & & I^{n\times n} \end{pmatrix}.$$

If one replaces ρ by $\rho' = T \cdot \rho \cdot T^{-1}$ for a matrix $T \in \mathrm{SL}(n, \mathbb{C})$, the corresponding connecting homomorphism is of the form

$$\gamma(\partial_i^{\rho'}) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix} \gamma(\partial_i^\rho) \begin{pmatrix} T^{-1} & & \\ & \ddots & \\ & & T^{-1} \end{pmatrix}.$$

In all cases, the regular Fuglede–Kadison determinant of $\gamma(\partial_i^\rho)$ and $\gamma(\partial_{i+1}^\rho)$ are unchanged by basic properties of Fuglede–Kadison determinant; see [Lück 2002, Theorem 3.14]. □

Note that the "moreover" part of the previous lemma tells us that we don't need to worry about the different choices of the base point when identifying the fundamental group $\pi_1(X)$ with G.

Lemma 3.4 *Let T be a two-dimensional torus. For any admissible triple*

$$(T, \rho\colon \pi_1(T) \to \mathrm{SL}(n, \mathbb{C}); \gamma\colon \pi_1(T) \to H),$$

if $\operatorname{im}\gamma$ *is infinite, then*

$$\tau^{(2)}(T, \rho; \gamma) = 1.$$

Proof We consider the standard CW structure for T constructed by identifying pairs of sides of a square. Let P be the 0–cell, let E_1 and E_2 be the 1–cells, and let

$$e_1 = [E_1] \in \pi_1(T), \quad e_2 = [E_2] \in \pi_1(T).$$

Then $\pi_1(T)$ is the free abelian group generated by e_1 and e_2. There is a 2–cell σ whose boundary is the loop $E_1E_2E_1^{-1}E_2^{-1}$. Let $\widehat{T}$ be the universal covering of T with the induced CW structure. It is easy to see that the L^2–chain complex of T twisted by $(\pi_1(T),\rho;\gamma)$ is

$$C_*^{(2)}(T,\rho;\gamma) = (0 \to l^2(H)\langle\sigma\rangle\otimes_{\mathbb{C}} V \xrightarrow{\gamma(\partial_2^\rho)} l^2(H)\langle E_1,E_2\rangle\otimes_{\mathbb{C}} V \xrightarrow{\gamma(\partial_1^\rho)} l^2(H)\langle P\rangle\otimes_{\mathbb{C}} V \to 0)$$

in which

$$\gamma(\partial_2^\rho) = \begin{pmatrix} I^{n\times n}-\gamma(e_2)\rho(e_2) & -I^{n\times n}+\gamma(e_1)\rho(e_1)\end{pmatrix}, \quad \gamma(\partial_1^\rho) = \begin{pmatrix}\gamma(e_1)\rho(e_1)-I^{n\times n}\\ \gamma(e_2)\rho(e_2)-I^{n\times n}\end{pmatrix}.$$

We assume without loss of generality that $\gamma(e_1)$ has infinite order. Set $p(z) := \det(z\rho(e_1)-I^{n\times n})$ as a polynomial of indeterminant z. Then by Lemma 2.5,

$$\det{}^{\mathrm{r}}_{\mathcal{N}(H)}(\gamma(e_1)\rho(e_1)-I^{n\times n}) = \mathrm{Mah}(p(z)) \neq 0.$$

The conclusion follows from [Dubois et al. 2015b, Lemma 3.1] which is a formula analogous to Lemma 2.6 but applies to shorter chain complexes. □

There is another way to define twisted L^2–torsion, following Lück [2018]. Let H be a finitely generated group. Recall that $\widetilde{X}$ is called a *finite free H–CW complex* if $\widetilde{X}$ is a regular covering space of a finite CW complex X, with deck transformation group H acting on $\widetilde{X}$ on the left. Choose an H–equivariant CW structure for $\widetilde{X}$, and choose one representative cell for each H–orbit. Then the cellular chain complex $C_*(\widetilde{X})$ becomes a based $\mathbb{C}H$–chain complex. For any group homomorphism $\phi\colon H\to \mathrm{SL}(n,\mathbb{C})$, we form the diagonal twisting chain complex $\mathfrak{D}(\phi)C_*(\widetilde{X})$ (recall the definition of the twisting functor $\mathfrak{D}$ in Section 2). The *ϕ–twisted L^2–torsion* of the H–CW complex $\widetilde{X}$ is defined to be

$$\rho_H^{(2)}(\widetilde{X},\phi) := \log\tau^{(2)}(l^2(H)\otimes_{\mathbb{C}H}\mathfrak{D}(\phi)C_*(\widetilde{X})).$$

Note that ϕ is a unimodular representation in our setting; this torsion does not depend on a specific $\mathbb{C}H$–basis for $C_*(\widetilde{X})$ (compare Proposition 3.3). We point out in the following proposition that both definitions of twisted L^2–torsion are essentially the same.

Proposition 3.5 *Following the notation above, let G be the fundamental group of $X = H\backslash\widetilde{X}$. There is a natural quotient map $\gamma\colon G\to H$ by covering space theory, and it is obvious that $(G,\phi\circ\gamma;\gamma)$ is an admissible triple. Then*

$$\tau^{(2)}(X,\phi\circ\gamma;\gamma) = \exp\rho_H^{(2)}(\widetilde{X},\phi).$$

Proof Let $\widehat{X}$ be the universal covering space of X, with the natural CW structure coming from X. Choose a lifting for each cell in X and then $C_*(\widehat{X})$ becomes a based $\mathbb{C}G$–chain complex. It is a pure algebraic fact that the two based $\mathbb{C}H$–chain complexes are $\mathbb{C}H$–isomorphic:

$$(*)\qquad \mathfrak{D}(\phi)C_*(\widetilde{X}) \cong \mathbb{C}H\otimes_{\mathbb{C}G}\mathfrak{D}(\phi\circ\gamma)C_*(\widehat{X}).$$

Indeed, the $\mathbb{C}H$–chain complex $\mathbb{C}H \otimes_{\mathbb{C}G} \mathfrak{D}(\phi\circ\gamma)C_*(\widehat{X})$ is obtained from

$$C_*(\widehat{X}) = (\cdots \to \mathbb{C}G^{r_{i+1}} \xrightarrow{\partial_{i+1}} \mathbb{C}G^{r_i} \xrightarrow{\partial_i} \mathbb{C}G^{r_{i-1}} \to \cdots)$$

by the following two operations:

(1) **The diagonal twist** First, replace every direct summand $\mathbb{C}G$ by its n^{th} power $\mathbb{C}G^n$, and replace any entry $\Lambda_{i,j}$ of the matrix ∂_* by a block matrix $\Lambda_{i,j}\phi\circ\gamma(\Lambda_{i,j})$, as in Proposition 2.2, resulting in a new matrix $\partial_*^{\phi\circ\gamma}$.

(2) **Tensoring with $\mathbb{C}H$** Then replace every direct summand $\mathbb{C}G$ of the chain module by $\mathbb{C}H$, and apply γ to every entry of $\partial_*^{\phi\circ\gamma}$, resulting in a block matrix whose i,j–submatrix is $\gamma(\Lambda_{i,j})\phi\circ\gamma(\Lambda_{i,j})$.

The resulting chain complex is exactly the chain complex $\mathfrak{D}(\phi)(\mathbb{C}H \otimes_{\mathbb{C}G} C_*(\widehat{X}))$ (this can be seen by doing the above operations in the reversed order, thanks to the admissible condition). Combined with the well-known $\mathbb{C}H$–isomorphism $C_*(\widetilde{X}) \cong \mathbb{C}H \otimes_{\mathbb{C}G} C_*(\widehat{X})$, the isomorphism $(*)$ follows.

Finally, we tensor $l^2(H)$ on the left of both $\mathbb{C}H$–chain complexes and the conclusion follows from taking L^2–torsion of each. □

The following useful properties are obtained by translating the statements of [Lück 2018, Theorem 6.7] into our terminology.

Lemma 3.6 *Some basic properties of twisted L^2–torsions*:

(1) ***G–homotopy equivalence*** *Let X and Y be two finite CW complexes with fundamental group G. For any admissible triple $(G,\rho;\gamma)$, suppose there is a simple homotopy equivalence $f\colon X \to Y$ such that the induced homomorphism $f_*\colon G \to G$ preserves $\ker\gamma$. Then*

$$\tau^{(2)}(X,\rho;\gamma) = \tau^{(2)}(Y,\rho;\gamma).$$

(2) ***Restriction*** *Let X be a finite CW complex with fundamental group G. Let $\widetilde{X}$ be a finite regular cover of X with the induced CW structure. Suppose $\pi_1(\widetilde{X}) = \widetilde{G} \lhd G$ is a normal subgroup of index d. Let $\tilde{\rho}\colon \widetilde{G} \to \mathrm{SL}(n,\mathbb{C})$ be the restriction of $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$. Then*

$$\tau^{(2)}(\widetilde{X},\tilde{\rho}) = \tau^{(2)}(X,\rho)^d.$$

(3) ***Sum formula*** *Let X be a finite CW complex with fundamental group G and $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ be a homomorphism. Let*

$$i_1\colon X_1 \hookrightarrow X, \quad i_2\colon X_2 \hookrightarrow X, \quad i_0\colon X_1\cap X_2 \hookrightarrow X$$

be subcomplex of X with $X_1 \cup X_2 = X$. Let

$$\rho_1 = \rho|_{\pi_1(X_1)}, \quad \rho_2 = \rho|_{\pi_1(X_2)}, \quad \rho_0 = \rho|_{\pi_1(X_1\cap X_2)}$$

be the restriction of ρ. If $\tau^{(2)}(X_1\cap X_2,\rho_0;i_{0*}) \neq 0$, then

$$\tau^{(2)}(X,\rho) = \tau^{(2)}(X_1,\rho_1;i_{1*})\cdot\tau^{(2)}(X_2,\rho_2;i_{2*})/\tau^{(2)}(X_1\cap X_2,\rho_0;i_{0*}).$$

4 Twisted L^2–torsion for 3–manifolds

For the remainder of this paper, we will assume that N is a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. We denote by G the fundamental group of N and assume G is infinite. It is well known that G is finitely generated and residually finite [Hempel 1987]. For any group homomorphism $\rho\colon G \to \mathrm{SL}(n, \mathbb{C})$ and $\gamma\colon G \to H$, we say $(N, \rho; \gamma)$ is an admissible triple if $(G, \rho; \gamma)$ is. In this case, we define the *twisted L^2–torsion of* $(N, \rho; \gamma)$ by

$$\tau^{(2)}(N, \rho; \gamma) := \tau^{(2)}(X, \rho; \gamma),$$

where X is any CW structure for N. This definition does not depend on the choice of X, thanks to Lemma 3.6. Indeed, if X and Y are two CW structures for N, and $f: X \to Y$ is the corresponding homeomorphism, then f is a simple homotopy equivalence by Chapman [1974, Theorem 1] and certainly preserves $\ker\gamma$. So $\tau^{(2)}(X, \rho; \gamma) = \tau^{(2)}(Y, \rho; \gamma)$.

The remaining part of this section is devoted to the proof of Theorem 1.1.

4.1 Twisted L^2–torsion for graph manifolds

We prove Theorem 1.1 for a graph manifold N with infinite fundamental group G.

Theorem 4.1 *Suppose M is a Seifert-fibered piece of the graph manifold N. Let $h \in \pi_1(M)$ be represented by the regular fiber of M. Consider the product of all eigenvalues (with multiplicity) of $\rho(h)$ whose modulus is not greater than 1, and denote by Λ the modulus of this product. Suppose the orbit space M/S^1 has orbifold Euler characteristic χ_{orb}. Then*

$$\tau^{(2)}(N, \rho) = \prod_{M \subset N \text{ is a Seifert piece}} \Lambda^{\chi_{\mathrm{orb}}}.$$

The proof is a direct generalization of [Bénard and Raimbault 2022, Proposition 4.3], though the technique in both proofs essentially goes back to T Kitano [1994], where he computed the $\mathrm{SL}(2, \mathbb{C})$–twisted Reidemeister torsion of graph manifolds.

Proof Fix any Seifert-fibered piece M of the JSJ decomposition of N. Then $\pi_1(M)$ is infinite as well. Suppose that M is isomorphic to a model

$$M(g, b; q_1/p_1, \dots, q_k/p_k), \quad k \geqslant 1,\ p_1, \dots, p_k > 0,$$

following Hatcher [2007]. More explicitly, take a surface of genus g with b boundary components, namely $E_1, \dots, E_b$, then drill out k–disjoint disks from it to form a new surface Σ with k additional boundary circles $F_1, \dots, F_k$. These k boundary circles correspond to k boundary tori of $\Sigma \times S^1$, namely $T_1, \dots, T_k$. Then M is obtained by a Dehn filling of slope $(q_1/p_1, \dots, q_k/p_k)$ along $(T_1, \dots, T_k)$, respectively. So

$$M = (\Sigma \times S^1) \cup_{T_1} D_1 \cup_{T_2} \cdots \cup_{T_k} D_k,$$

in which D_i is a solid torus whose meridian $(0,1)$–curve is attached to the (q_i, p_i)–curve of T_i. The orbit space can be viewed as a 2–dimensional orbifold, whose underlying topological space is a surface $\Sigma_{g,b}$ with k singularities of indices $p_1, \dots, p_k$, respectively. The orbifold Euler characteristic is

$$\chi_{\mathrm{orb}} = 2 - 2g - b - \sum_{i=1}^{k}\Big(1 - \frac{1}{p_i}\Big).$$

More details can be found in [Scott 1983].

Retract Σ along the boundary circle F_k to an 1–dimensional complex X; it is a bunch of circles with one common vertex P, and edges

$$A_1, B_1, \dots, A_g, B_g, E_1, \dots, E_b, F_1, \dots, F_{k-1}$$

where $A_1, B_1, \dots, A_g, B_g$ come from the standard polygon representation of a closed surface Σ_g. Suppose that A_i, B_i, E_i and F_i represent a_i, b_i, e_i and f_i, respectively, in $\pi_1(M)$. Let H be the 1–cell of S^1 representing $h \in \pi_1(M)$. Then $\Sigma \times S^1$ is given the product CW structure, with the cells in each dimension being

$$\{A_1 \times H, B_1 \times H, \dots, A_g \times H, B_g \times H, E_1 \times H, \dots, E_b \times H, F_1 \times H, \dots, F_{k-1} \times H\},$$
$$\{A_1, B_1, \dots, A_g, B_g, E_1, \dots, E_b, F_1, \dots, F_{k-1}, H\}, \quad \{P\}.$$

We have $f_i^{p_i} h^{q_i} = 1$ for $i = 1, \dots, k-1$ by the Dehn filling.

Denote by

$$\kappa\colon \Sigma \times S^1 \hookrightarrow N, \quad \iota_i\colon T_i \hookrightarrow N, \quad \zeta_i\colon D_i \hookrightarrow N, \quad i = 1, \dots, k,$$

the inclusion maps to the ambient manifold N. Our strategy is as follows: cut N along all JSJ tori and all tori $\{T_1, \dots, T_k\}$ that appear in each Seifert piece of the JSJ decomposition of N as above. By Lemma 3.4, the JSJ tori do not contribute to the L^2–torsion. Then, by the sum formula of Lemma 3.6,

$$\tau^{(2)}(N,\rho) = \prod_{M \subset N \text{ is a Seifert piece}} \frac{\tau^{(2)}(\Sigma \times S^1, \rho \circ \kappa_*; \kappa_*) \prod_{i=1}^{k} \tau^{(2)}(D_i, \rho \circ \zeta_{i*}; \zeta_{i*})}{\prod_{i=1}^{k} \tau^{(2)}(T_i, \rho \circ \iota_{i*}; \iota_{i*})}. \tag{1}$$

It remains to calculate the terms appearing in (1).

First, the easiest part. Since $\iota_{i*}(\pi_1(T_i))$ has infinite order in G, the twisted L^2–torsion of the admissible triple $(T_i, \rho \circ \iota_{i*}; \iota_{i*})$ is trivially 1 by Lemma 3.4.

We now compute $\tau^{(2)}(\Sigma \times S^1, \rho \circ \kappa_*; \kappa_*)$. Set $\pi := \pi_1(\Sigma \times S^1)$. The CW chain complex of the universal cover $\widehat{\Sigma \times S^1}$ is

$$C_*(\widehat{\Sigma \times S^1}) = (0 \to \mathbb{C}\pi^{2g+b+k-1} \xrightarrow{\partial_2} \mathbb{C}\pi^{2g+b+k} \xrightarrow{\partial_1} \mathbb{C}\pi \xrightarrow{\partial_0} 0)$$

in which

$$\partial_2 = \begin{pmatrix} 1-h & 0 & \cdots & 0 & * \\ 0 & 1-h & & \vdots & \vdots \\ \vdots & & \ddots & 0 & * \\ 0 & \cdots & 0 & 1-h & * \end{pmatrix}, \quad \partial_1 = \begin{pmatrix} * \\ \vdots \\ * \\ 1-h \end{pmatrix}.$$

Then the L^2–chain complex of $\Sigma \times S^1$ twisted by $(\pi, \rho\circ\kappa_*; \kappa_*)$ is

$$C_*^{(2)}(\Sigma \times S^1, \rho\circ\kappa_*; \kappa_*) = (0 \to l^2(G)^{2g+b+k-1} \xrightarrow{\partial_2^\rho} l^2(G)^{2g+b+k} \xrightarrow{\partial_1^\rho} l^2(G) \to 0)$$

in which

$$\partial_2^\rho = \begin{pmatrix} I^{n\times n} - h\rho(h) & 0 & \cdots & 0 & * \\ 0 & I^{n\times n} - h\rho(h) & & \vdots & \vdots \\ \vdots & & \ddots & 0 & * \\ 0 & \cdots & 0 & I^{n\times n} - h\rho(h) & * \end{pmatrix}, \quad \partial_1^\rho = \begin{pmatrix} * \\ \vdots \\ * \\ I^{n\times n} - h\rho(h) \end{pmatrix}.$$

We have identified h with its image under κ_* in $\pi_1(N) = G$ for notational convenience. If the modulus of all eigenvalues of $\rho(h)$ are $\lambda_1, \dots, \lambda_n$, by properties of the regular Fuglede–Kadison determinant and Lemmas 2.5 and 2.6, we know that

$$\begin{aligned}\tau^{(2)}(\Sigma \times S^1, \rho\circ\kappa_*; \kappa_*) &= \det{}^{\mathrm{r}}_{\mathcal{N}(G)}(I^{n\times n} - h\rho(h))^{2g+b+k-2} \\ &= \operatorname{Mah}\Big(\prod_{r=1}^{n}(1 - z\lambda_r)\Big)^{2g+b+k-2} = \Lambda^{-(2g+b+k-2)}.\end{aligned}$$

Then we compute $\tau^{(2)}(D_i, \rho\circ\zeta_{i*}; \zeta_{i*})$. It is easy to see that the generator of $\pi_1(D_i)$ is represented by $h^{m_i} f_i^{n_i}$, where (m_i, n_i) is a pair of integers such that $m_i p_i - n_i q_i = 1$. Then

$$\tau^{(2)}(D_i, \rho\circ\zeta_{i*}; \zeta_{i*}) = \det{}^{\mathrm{r}}_{\mathcal{N}(G)}(I^{n\times n} - h^{m_i} f_i^{n_i} \cdot \rho(h^{m_i} f_i^{n_i}))^{-1},$$

where h and f_i are again viewed as elements in G. Since h and f_i commute and are simultaneously upper triangularizable, the modulus of all eigenvalues of $\rho(h^{m_i} f_i^{n_i})$ are $\lambda_1^{1/p_i}, \dots, \lambda_n^{1/p_i}$. Note that $h^{m_i} f_i^{n_i}$ is an infinite order element. By Lemma 2.5,

$$\det{}^{\mathrm{r}}_{\mathcal{N}(G)}(I^{n\times n} - h^{m_i} f_i^{n_i} \cdot \rho(h^{m_i} f_i^{n_i})) = \operatorname{Mah}\Big(\prod_{r=1}^{n}(1 - z\lambda_r^{1/p_i})\Big) = \Lambda^{-1/p_i},$$

and then $\tau^{(2)}(D_i, \rho\circ\zeta_{i*}; \zeta_{i*}) = \Lambda^{1/p_i}$.

Finally, combining the calculations above,

$$\begin{aligned}\frac{\tau^{(2)}(\Sigma \times S^1, \rho\circ\kappa_*; \kappa_*) \prod_{i=1}^{k} \tau^{(2)}(D_i, \rho\circ\zeta_{i*}; \zeta_{i*})}{\prod_{i=1}^{k} \tau^{(2)}(T_i, \rho\circ\iota_{i*}; \iota_{i*})} &= \Lambda^{-(2g+b+k-2)+\sum_{i=1}^{k} 1/p_i} \\ &= \Lambda^{2-2g-b-\sum_{i=1}^{k}(1-1/p_i)} = \Lambda^{\chi_{\mathrm{orb}}},\end{aligned}$$

and the conclusion follows from (1). □

4.2 Twisted L^2–torsion for hyperbolic or mixed manifolds

In this part, we assume that N is not a graph manifold, or equivalently, N contains at least one hyperbolic piece in its geometrization decomposition. Then N is either hyperbolic or so-called mixed. By Agol's

RFRS criterion [2008] for virtual fibering and the virtual specialness of 3–manifolds having at least one hyperbolic piece [Agol 2013; Przytycki and Wise 2018], we can assume that N has a regular finite cover that fibers over the circle.

For future convenience, we introduce the following notions.

Definition 4.2 Let G be a finitely generated, residually finite group. For any cohomology class $\psi \in H^1(G;\mathbb{R})$, and any real number $t > 0$, there is an 1–dimensional representation

$$\psi_t \colon G \to \mathbb{C}^\times, \quad g \mapsto t^{\psi(g)}.$$

This representation can be used to twist $\mathbb{C}G$, determining a $\mathbb{C}G$–homomorphism

$$\kappa(\psi,t)\colon \mathbb{C}G \to \mathbb{C}G, \qquad g \mapsto t^{\psi(g)} g, \quad g \in G,$$

and extend $\mathbb{C}$–linearly. The $\mathbb{C}G$–homomorphism $\kappa(\psi,t)$ is called the *Alexander twist of* $\mathbb{C}G$ *associated to* (ψ,t).

Definition 4.3 A positive function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is multiplicatively convex if the function

$$F\colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto \log f(e^t),$$

is a convex function. In particular, a multiplicatively convex function is continuous and everywhere positive.

Our main technical tool is the following theorem due to Liu [2017, Theorem 5.1].

Theorem 4.4 *Let G be a finitely generated, residually finite group. For any square matrix A over $\mathbb{C}G$ and any 1–cohomology class $\psi \in H^1(G;\mathbb{R})$, the function*

$$t \mapsto \det^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\psi,t)A), \quad t > 0,$$

is either constantly zero or multiplicatively convex (and in particular everywhere positive).

With the above preparations, we are now ready to prove Theorem 1.1 for hyperbolic or mixed 3–manifolds.

Theorem 4.5 *Suppose N is a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. Assume that N is hyperbolic or mixed. Then $\tau^{(2)}(N,\rho) > 0$.*

Proof Since twisted L^2–torsion behaves multiplicatively with respect to finite covers by Lemma 3.6, we may assume without loss of generality that N itself fibers over the circle.

The following procedure is analogous to [Dubois et al. 2015b, Theorem 8.5]. Denote by Σ a fiber of N, and $f\colon \Sigma \to \Sigma$ the monodromy such that N is homeomorphic to the mapping torus

$$T_f(N) = \Sigma \times [-1,1]/(x,-1) \sim (f(x),1).$$

We can assume by isotopy that f has a fixed point P. Construct a CW structure X modeled on Σ with a single 0–cell P, k 1–cells $E_1,\dots,E_n$, and a 2–cell σ. By CW approximation, there is a cellular map $g\colon \Sigma\to\Sigma$ homotopic to f. Then the mapping torus $T_g(\Sigma)$ is homotopy equivalent to N, which is a simple homotopy equivalence since the Whitehead group of a fibered 3–manifold is trivial; see [Waldhausen 1978, Theorems 19.4 and 19.5]. Hence, by Lemma 3.6,

$$\tau^{(2)}(N,\rho)=\tau^{(2)}(T_g(\Sigma),\rho).$$

We proceed to describe a CW complex for the mapping torus $T_g(\Sigma)$. Suppose $\pi_1(N)=\pi_1(T_g(\Sigma))=:G$. The cells in each dimensions are

$$\{\sigma\times I\},\quad \{\sigma, E_1\times I,\dots,E_k\times I\},\quad \{E_1,\dots,E_k,P\times I\},\quad \{P\},$$

where $I=[-1,1]$. Let $e_i:=[E_i]\in G$ and $h:=[P\times I]\in G$ be the fundamental group elements represented by the corresponding loops. Denote by $\psi\in H^1(G;\mathbb{R})$ the 1–cohomology class dual to the fiber Σ. Then

$$\psi(h)=1,\quad \psi(e_1)=\cdots=\psi(e_k)=0.$$

The CW chain complex of $\widehat{T_g(\Sigma)}$ has the form

$$C_*(\widehat{T_g(\Sigma)})=(0\to\mathbb{C}G\xrightarrow{\partial_3}\mathbb{C}G^{k+1}\xrightarrow{\partial_2}\mathbb{C}G^{k+1}\xrightarrow{\partial_1}\mathbb{C}G\xrightarrow{\partial_0}0)$$

in which

$$\partial_3=(1-h,*,\dots,*),\quad \partial_2=\begin{pmatrix} * & * \\ I^{k\times k}-h\cdot A & * \end{pmatrix},\quad \partial_1=\begin{pmatrix} * \\ 1-h\end{pmatrix}$$

where "$*$" stands for matrices of appropriate size, and A is a matrix over $\mathbb{C}[\ker\psi]$ of size $k\times k$. Denote by A_ρ the matrix A twisted by ρ, as in Proposition 2.2. Then the L^2–chain complex of $T_g(\Sigma)$ twisted by $(G,\rho;\mathrm{id}_G)$ is

$$C_*^{(2)}(T_g(\Sigma),\rho)=(0\to l^2(G)^n\xrightarrow{\partial_3^\rho} l^2(G)^{n(k+1)}\xrightarrow{\partial_2^\rho} l^2(G)^{n(k+1)}\xrightarrow{\partial_1^\rho} l^2(G)^n\to 0)$$

in which

$$\partial_3^\rho=(I^{n\times n}-h\rho(h),*,\dots,*),\quad \partial_2^\rho=\begin{pmatrix} * & * \\ I^{nk\times nk}-h\cdot\rho(h)A_\rho & *\end{pmatrix},\quad \partial_1^\rho=\begin{pmatrix} * \\ I^{n\times n}-h\rho(h)\end{pmatrix}.$$

Consider the matrices

$$S:=I^{n\times n}-h\rho(h),\quad T:=I^{nk\times nk}-h\rho(h)A_\rho,$$

and the matrices under the Alexander twist associated to (ψ,t),

$$S(t):=\kappa(\psi,t)S=I^{n\times n}-t\cdot h\rho(h),\quad T(t):=\kappa(\psi,t)T=I^{nk\times nk}-t\cdot h\rho(h)A_\rho.$$

For any real number $t>0$ sufficiently small, the two matrices $S(t)$ and $T(t)$ are both invertible with regular Fuglede–Kadison determinant equal to 1; see [Dubois et al. 2015b, Proposition 8.8]. Then Liu's Theorem 4.4 applies to show that these two Fuglede–Kadison determinants are positive when $t=1$. It follows from Lemma 2.6 that $\tau^{(2)}(N,\rho)=\det^{\mathrm{r}}_{\mathcal{N}(G)}T(1)\cdot\det^{\mathrm{r}}_{\mathcal{N}(G)}S(1)^{-2}$ is positive. □

Theorem 1.1 then follows from Theorems 4.1 and 4.5.

5 Continuity of twisted L^2–torsion on representation varieties

Let N be any compact orientable irreducible 3–manifold with empty or incompressible toral boundary, and set $G := \pi_1(N)$. Suppose that G is infinite, and denote by $\mathscr{R}_n(G) := \mathrm{Hom}(G, \mathrm{SL}(n, \mathbb{C}))$ the representation variety. Then Theorem 1.1 implies that the twisted L^2–torsion can be viewed as a positive function

$$\rho \mapsto \tau^{(2)}(N, \rho), \quad \rho \in \mathscr{R}_n(G).$$

The continuity of this torsion function is an interesting but rather hard question. Work of Liu [2017, Theorem 1.2] has shown that the torsion function is continuous in $\mathrm{Hom}(G, \mathbb{R})$ along the Alexander twists. We remark that in his article the twist is not unimodular, and an equivalence class for torsion functions is introduced to guarantee well-definedness. If N is hyperbolic, $\rho_0\colon G \to \mathrm{PSL}(2, \mathbb{C})$ is a holonomy representation associated to the hyperbolic structure, and $\rho \in \mathscr{R}_2(G)$ is a lifting of ρ_0 (such lifting always exists, see [Culler 1986, Corollary 2.2]), then Bénard and Raimbault [2022] proved that the torsion function is analytic near ρ. The continuity of the torsion function in general is wide open. In this section we present a partial result on the continuity of the twisted L^2–torsion function, namely Theorem 1.2. We start with a brief discussion of the L^2–Alexander torsions since it is closely related to the proof of Theorem 1.2.

5.1 L^2–Alexander torsion

The L^2–torsion twisted by 1–dimensional representations is called *L^2–Alexander torsion*. To be precise, for any 1–cohomology class $\psi \in H^1(G; \mathbb{R})$ and any real number $t > 0$, the *L^2–Alexander torsion* of N associated to (ψ, t) is defined to be

$$A^{(2)}(N, \psi, t) := \tau^{(2)}(C_*^{(2)}(N, \psi_t)).$$

Recall $\psi_t\colon G \to \mathbb{C}^\times$ that maps $g \in G$ to $t^{\psi(g)}$ is the representation associated to (ψ, t). Since ψ_t is not a unimodular representation, the L^2–Alexander torsion depends on the based $\mathbb{C}G$–chain complex $C_*(\widehat{N})$. Indeed, altering the $\mathbb{C}G$–basis of $C_*(\widehat{N})$, the base change matrix for $C_*^{(2)}(N, \psi_t)$ will be a permutation matrix with entries $\pm t^{\pm\psi(g_i)} g_i$ (compare Proposition 3.3), whose regular Fuglede–Kadison determinant is $t^{\sum_i \pm\psi(g_i)}$. Since $g_i \in G$ are independent of ψ and t, the continuity of $A^{(2)}(N, \psi, t)$ as a function of $(\psi, t) \in H^1(G; \mathbb{R}) \times \mathbb{R}_+$ is independent of the choice of cellular basis; here $H^1(N; \mathbb{R})$ is given the usual real vector space topology.

In [Dubois et al. 2015a; 2015b], one considers $A^{(2)}(N, \psi, t)$ as a function of t, and introduces an equivalence relation between functions. Namely, two functions $f_1, f_2\colon \mathbb{R}_+ \to [0, +\infty)$ are equivalent if and only if there exists a real number r such that

$$f_1(t) = t^r \cdot f_2(t)$$

holds for all $t > 0$. In this case we denote by $f_1 \doteq f_2$. So the equivalence class of $A^{(2)}(N, \psi, t)$ as a function of t does not depend on the choice of cellular basis.

Another way to cure the ambiguity is to modify ψ_t to be a unimodular 2–dimensional representation. Set

$$\psi_t \oplus \psi_{t^{-1}} \colon G \to \mathrm{SL}(2,\mathbb{C}), \quad g \mapsto \begin{pmatrix} t^{\psi(g)} & 0 \\ 0 & t^{-\psi(g)} \end{pmatrix}.$$

Then it is easy to observe that $C_*^{(2)}(N, \psi_t \oplus \psi_{t^{-1}}) = C_*^{(2)}(N, \psi_t) \oplus C_*^{(2)}(N, \psi_{t^{-1}})$, and hence by Lück [2002, Theorem 3.35],

$$A^{(2)}(N,\psi,t) \cdot A^{(2)}(N,\psi,t^{-1}) = \tau^{(2)}(N, \psi_t \oplus \psi_{t^{-1}}),$$

which does not depend on the choice of cellular basis. This fact motivates the following definition.

Definition 5.1 For any $\psi \in H^1(G;\mathbb{R})$ and $t > 0$, we define the *symmetric L^2–Alexander torsion of N associated to* (ψ, t) to be

$$A^{(2)}_{\mathrm{sym}}(N,\psi,t) := \tau^{(2)}(N, \psi_t \oplus \psi_{t^{-1}})^{1/2}.$$

It is shown in [Dubois et al. 2015a, Chapter 6] that the L^2–Alexander torsion satisfies

$$A^{(2)}(N,\psi,t) = t^{-\psi(c_1(e))} \cdot A^{(2)}(N,\psi,t^{-1})$$

where $c_1(e) \in H_1(N;\mathbb{Z})$ is independent of (ψ, t). This shows that

$$A^{(2)}_{\mathrm{sym}}(N,\psi,t) = t^r \cdot A^{(2)}(N,\psi,t)$$

for some real number r. We remark that, as a function of (ψ, t), the continuity of $A^{(2)}(N,\psi,t)$ defined by any CW structure is equivalent to the continuity of $A^{(2)}_{\mathrm{sym}}(N,\psi,t)$.

As an illustration of the various definitions, we rediscover the L^2–Alexander torsion $A^{(2)}(N,\psi,t)$ for a graph manifold N using Theorem 4.1. The calculation is first carried out by Herrmann [2017] for Seifert fibering space and by [Dubois et al. 2015a] for graph manifolds.

Theorem 5.2 *Let N be a graph manifold with infinite fundamental group. Suppose that $N \neq S^1 \times D^2$ and $N \neq S^1 \times S^2$. Then a representative of the L^2–torsion twisted by (ψ, t) is*

$$A^{(2)}(N,\psi,t) = \max\{1, t^{x_N(\psi)}\},$$

where x_N is the Thurston norm for $H^1(N;\mathbb{R})$.

Proof For $t \geqslant 1$, set $\rho := \psi_t \oplus \psi_{t^{-1}}$. Then, by Theorem 4.1,

$$A^{(2)}_{\mathrm{sym}}(N,\psi,t)^2 = \tau^{(2)}(N, \psi_t \oplus \psi_{t^{-1}}) = \prod_{M \subset N \text{ is a Seifert piece}} t^{-|\psi(h)| \cdot \chi_{\mathrm{orb}}},$$

where $h \in H^1(M;\mathbb{R})$ is represented by the regular fiber of M and χ_{orb} is the orbifold Euler characteristic of M/S^1. By our assumption on N, we know that $\chi_{\mathrm{orb}} \leqslant 0$, so $-|\psi(h)| \cdot \chi_{\mathrm{orb}} = x_M(\psi)$ by [Herrmann

2017, Lemma A], where x_M is the Thurston norm for $H^1(M;\mathbb{R})$. Then by [Eisenbud and Neumann 1985, Proposition 3.5],

$$\sum_{M\subset N \text{ is a Seifert piece}} x_M(\psi) = x_N(\psi)$$

and so

$$A^{(2)}_{\mathrm{sym}}(N,\psi,t)^2 = t^{x_N(\psi)}, \quad t \geqslant 1.$$

Since the symmetric L^2–Alexander torsion is by definition symmetric,

$$A^{(2)}_{\mathrm{sym}}(N,\psi,t) = \max\{t^{\frac{1}{2}x_N(\psi)}, t^{-\frac{1}{2}x_N(\psi)}\} \doteq \max\{1, t^{x_N(\psi)}\}. \qquad \square$$

It follows that the L^2–Alexander torsion of graph manifolds is continuous in $(\psi,t)\in H^1(G;\mathbb{R})\times\mathbb{R}^+$. For a general 3–manifold N, the continuity of the L^2–Alexander torsion is a hard question. Liu [2017] and Lück [2018] independently proved that the L^2–Alexander torsion function is always positive. Moreover Liu proved in the same article that $A^{(2)}(N,\psi,t)$ is continuous with respect to t. Lück [2018, Chapter 10] conjectured that this function is continuous with respect to $(\psi,t)\in H^1(N;\mathbb{R})\times\mathbb{R}^+$. We will see that this statement is true.

Theorem 5.3 *Let N be a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. Suppose $\pi_1(N)=G$ is infinite. Then any representative of the L^2–Alexander torsion function $A^{(2)}(N,\psi,t)$ is continuous with respect to $(\psi,t)\in H^1(N;\mathbb{R})\times\mathbb{R}^+$.*

Theorem 1.2 is now a corollary of Theorem 5.3, as we restate here.

Theorem 5.4 *Let N be a compact orientable irreducible 3–manifold with empty or incompressible toral boundary. Suppose $\pi_1(N)=G$ is infinite. Define $\mathcal{R}^{\mathrm{t}}_n(G)$ to be the subvariety of $\mathcal{R}_n(G)$ consisting of upper triangular representations. Then the twisted L^2–torsion function*

$$\rho \mapsto \tau^{(2)}(N,\rho)$$

is continuous with respect to $\rho\in\mathcal{R}^{\mathrm{t}}_n(G)$.

Proof Fix a CW structure for N and fix a choice of cell-lifting to $\widehat{N}$, so we can talk about the L^2–Alexander torsion unambiguously. For any $\rho\in\mathcal{R}^{\mathrm{t}}_n(G)$, we can assume that

$$\rho(g) = \begin{pmatrix} \chi_1(g) & \cdots & * \\ & \ddots & \vdots \\ & & \chi_n(g) \end{pmatrix},$$

where $\chi_k : G\to\mathbb{C}^\times$ are characters. The modulus of those characters can be written as

$$|\chi_k| = e^{\phi_k}, \quad g\mapsto e^{\phi_k(g)},$$

for some real 1–cohomology classes $\phi_k\in H^1(G;\mathbb{R})$. The classes $\phi_1,\ldots,\phi_n$ are continuous with respect to $\rho\in\mathcal{R}^{\mathrm{t}}_n(G)$.

Let V_n be the G–invariant subspace of V corresponding to χ_n, and let $V' := V/V_n$, then there is an exact sequence of G–representations

$$0 \to V_n \to V \to V' \to 0,$$

where the G–actions are given by

$$\rho_n(g) = \chi_n(g), \quad \rho(g) = \begin{pmatrix} \chi_1(g) & \cdots & * \\ & \ddots & \vdots \\ & & \chi_n(g) \end{pmatrix}, \quad \rho'(g) = \begin{pmatrix} \chi_1(g) & \cdots & * \\ & \ddots & \vdots \\ & & \chi_{n-1}(g) \end{pmatrix},$$

respectively. Then, by Lück [2018, Lemma 3.3],

$$\tau^{(2)}(N,\rho) = \tau^{(2)}(N,\rho_n)\tau^{(2)}(N,\rho').$$

Since unitary twists have no effect on L^2–torsions by Lück [2018, Theorem 4.1], we have

$$\tau^{(2)}(N,\rho_n) = \tau^{(2)}(N, e^{\phi_n}) = A^{(2)}(N,\phi_n,e).$$

The above process can then be applied to ρ' and finally we have the formula

$$\tau^{(2)}(N,\rho) = A^{(2)}(N,\phi_1,e)\cdots A^{(2)}(N,\phi_n,e).$$

Since the cohomology classes $\phi_1,\ldots,\phi_n$ vary continuously with respect to $\rho \in \mathfrak{R}_n^{\mathrm{t}}(G)$, the conclusion follows from Theorem 5.3. □

The remaining part of this section is devoted to the proof of Theorem 5.3. We will need the notion of Alexander multitwists.

5.2 Alexander multitwists of matrices

Recall that G is any finitely generated, residually finite group. For any collection of 1–cohomology classes $\Phi = (\phi_1,\ldots,\phi_n) \in \prod_{i=1}^n H^1(G;\mathbb{R})$ and any collection of positive real numbers $T = (t_1,\ldots,t_n) \in \mathbb{R}_+^n$, we define a $\mathbb{C}G$–homomorphism

$$\kappa(\Phi,T)\colon \mathbb{C}G \to \mathbb{C}G, \quad g \to t_1^{\phi_1(g)} \cdots t_n^{\phi_n(g)} \cdot g, \quad g \in G.$$

This is called the *Alexander multitwist of $\mathbb{C}G$ associated to* (Φ,T).

Proposition 5.5 *Basic properties of the Alexander multitwist*:

(1) ***Associativity*** *Suppose* $\Phi = (\phi_1,\ldots,\phi_n)$ *and* $T = (t_1,\ldots,t_n)$. *Then*

$$\kappa(\Phi,T) = \kappa(\phi_1,t_1) \circ \cdots \circ \kappa(\phi_n,t_n).$$

(2) ***Commutativity*** $\kappa(\phi_1,t_1) \circ \kappa(\phi_2,t_2) = \kappa(\phi_2,t_2) \circ \kappa(\phi_1,t_1)$.

(3) ***Change of coordinate*** *Let* $r_1,r_2 \in \mathbb{R}$; *then*

$$\kappa(r_1\phi_1 + r_2\phi_2, t) = \kappa(\phi_1, t^{r_1}) \circ \kappa(\phi_2, t^{r_2}),$$
$$\kappa(\phi, t_1^{r_1} t_2^{r_2}) = \kappa(r_1\phi, t_1) \circ \kappa(r_2\phi, t_2).$$

The Alexander multitwist extends to an endomorphism of the matrix algebra with entries in $\mathbb{C}G$.

In the following part of this section, we shall fix a square matrix Ω over $\mathbb{C}G$, and suppose that $\det^{\mathrm{r}}_{\mathcal{N}(G)}(\Omega)$ is not zero. For any collection of 1–cohomology classes $\Phi = (\phi_1, \dots, \phi_n)$ and positive real numbers $T = (t_1, \dots, t_n)$, we introduce the notation

$$V_\Phi(T) := \det^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\Phi, T)\Omega).$$

Proposition 5.6 *For any fixed choice of Φ, the multivariable function $V_\Phi(T)$ is everywhere positive and is multiplicatively convex in each coordinate with respect to $T = (t_1, \dots, t_n) \in \mathbb{R}^n_+$.*

Proof By associativity and commutativity of the Alexander multitwist,

$$\kappa(\Phi, T)\Omega = \kappa(\phi_i, t_i) \circ \kappa(\Phi', T')\Omega$$

where (Φ', T') are variables other than (ϕ_i, t_i). The conclusion then follows from applying Theorem 4.4 to each i. $\square$

Theorem 5.7 *For any fixed choice of Φ, the multivariable real function $V_\Phi(T)$ is multiplicatively convex with respect to $T = (t_1, \dots, t_n) \in \mathbb{R}^n_+$.*

Proof We will prove that for any fixed choice of Φ and every positive integer $k \leqslant n$, the function $V_\Phi(T)$ is multiplicatively convex with respect to the first k coordinates.

The case $k = 1$ is proved by Proposition 5.6. Assume the claim holds for $(k-1)$ and consider

$$V_{\phi_1,\dots,\phi_k}(t_1, \dots, t_k) = V_\Phi(T)$$

as a function of the first k variables of Φ and T. It suffices to prove that for any $\theta \in (0, 1)$ and any collection of positive numbers $r_1, \dots, r_k > 0$ and $s_1, \dots, s_k > 0$,

$$\big(V_{\phi_1,\dots,\phi_k}(r_1, \dots, r_k)\big)^\theta \cdot \big(V_{\phi_1,\dots,\phi_k}(s_1, \dots, s_k)\big)^{1-\theta} \geqslant V_{\phi_1,\dots,\phi_k}(r_1^\theta s_1^{1-\theta}, \dots, r_k^\theta s_k^{1-\theta}).$$

We can assume that $r_1 \neq s_1$, otherwise this inequality degenerates to the $(k-1)$ case after permuting the coordinates. Consider $\psi_1 = \phi_1 + \lambda\phi_k$ for a real number λ which will be determined later. We have the identity that for all $t_1, \dots, t_k > 0$,

$$V_{\psi_1,\phi_2,\dots,\phi_k}(t_1, \dots, t_{k-1}, t_k) = V_{\phi_1,\phi_2,\dots,\phi_k}(t_1, \dots, t_{k-1}, t_1^\lambda t_k).$$

By the induction hypothesis, for all $r > 0$,

$$\big(V_{\psi_1,\phi_2,\dots,\phi_k}(r_1, \dots, r_{k-1}, r)\big)^\theta \cdot \big(V_{\psi_1,\phi_2,\dots,\phi_k}(s_1, \dots, s_{k-1}, r)\big)^{1-\theta} \\ \geqslant V_{\psi_1,\phi_2,\dots,\phi_k}(r_1^\theta s_1^{1-\theta}, \dots, r_{k-1}^\theta s_{k-1}^{1-\theta}, r),$$

which is equivalent to

$$\big(V_{\phi_1,\dots,\phi_k}(r_1, \dots, r_{k-1}, r_1^\lambda r)\big)^\theta \cdot \big(V_{\phi_1,\dots,\phi_k}(s_1, \dots, s_{k-1}, s_1^\lambda r)\big)^{1-\theta} \\ \geqslant V_{\phi_1,\dots,\phi_k}\big(r_1^\theta s_1^{1-\theta}, \dots, r_{k-1}^\theta s_{k-1}^{1-\theta}, (r_1^\lambda r)^\theta \cdot (s_1^\lambda r)^{1-\theta}\big).$$

Since $r_1 \neq s_1$, we can prescribe $\lambda \in \mathbb{R}$ and $r > 0$ by solving the equations

$$r_1^{\lambda} r = r_k, \quad s_2^{\lambda} r = s_k.$$

This finishes the induction. □

Corollary 5.8 *For any fixed $(\Phi, T) \in \prod_{i=1}^{n} H^1(G;\mathbb{R}) \times \mathbb{R}_+^n$, the function $W_{\Phi,T}\colon \mathbb{R}^n \to \mathbb{R}$,*

$$W_{\Phi,T}(s_1, \dots, s_n) := \log(V_{s_1\phi_1,\dots,s_n\phi_s}(T)),$$

is convex. In particular, it is continuous.

Proof This follows from the identity

$$W_{\Phi,T}(s_1, \dots, s_n) := \log(V_{s_1\phi_1,\dots,s_n\phi_s}(T)) = \log(V_{\Phi}(t_1^{s_1}, \dots, t_n^{s_n}))$$

and the multiplicative convexity of $V_\Phi(T)$. □

Theorem 5.9 *The regular Fuglede–Kadison determinant map $\det^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\phi,t)\Omega)$ is continuous with respect to $(\phi, t) \in H^1(G;\mathbb{R}) \times \mathbb{R}_+$.*

Proof Let $\Psi = (\psi_1, \dots, \psi_k)$ be a basis for the real vector space $H^1(G;\mathbb{R})$. Suppose

$$\phi = \sum_{i=1}^{k} c_j \psi_j, \quad 1 \leqslant i \leqslant n,$$

where the coefficients c_j are continuous with respect to $\phi \in H^1(G;\mathbb{R})$. Then

$$\begin{aligned}\kappa(\phi,t)\Omega &= \kappa(c_1\psi_1, t) \circ \cdots \circ \kappa(c_k\psi_k, t)\Omega \\ &= \kappa(c_1 \log t \cdot \psi_1, e) \circ \cdots \circ \kappa(c_k \log t \cdot \psi_k, e)\Omega \\ &= \kappa\big((c_1 \log t \cdot \psi_1, \dots, c_k \log t \cdot \psi_k), (e, \dots, e)\big)\Omega.\end{aligned}$$

By definition,

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\phi,t)\Omega) = \exp W_{\Psi,(e,\dots,e)}(c_1 \log t, \dots, c_k \log t).$$

The continuity follows from Corollary 5.8. □

5.3 Applications to 3–manifolds

Proof of Theorem 5.3 If N is a graph manifold, then Theorem 5.2 offers an explicit formula for the L^2–Alexander torsion; the theorem holds since the Thurston norm is continuous in $H^1(N;\mathbb{R})$.

If N is a compact connected orientable irreducible 3–manifold which is hyperbolic or mixed, then as in the proof of Theorem 4.5, we can find a regular finite covering $p\colon \widetilde{N} \to N$ of degree d such that $\widetilde{N}$ fibers over the circle. Since by Lemma 3.6 we have

$$\tau^{(2)}(N, \psi_t \oplus \psi_{t^{-1}})^d = \tau^{(2)}(\widetilde{N}, p^*\psi_t \oplus p^*\psi_{t^{-1}}),$$

it follows that $A^{(2)}_{\mathrm{sym}}(N,\psi,t)^d = A^{(2)}_{\mathrm{sym}}(\widetilde{N},\, p^*\psi,t)$. Note that the pullback map $p^*\colon H^1(N;\mathbb{R})\to H^1(\widetilde{N};\mathbb{R})$ is a continuous embedding, so we only need to prove the theorem for $\widetilde{N}$. We can assume without loss of generality that our manifold N fibers over circle. From the proof of Theorem 4.5, we see that

$$A^{(2)}(N,\psi,t) = \det{}^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\psi,t)T)\cdot \det{}^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\psi,t)S)^{-2},$$

where $T = I^{k\times k} - hA_\rho$ and $S = 1-h$ are square matrices over $\mathbb{C}G$ with positive regular Fuglede–Kadison determinant. The conclusion follows immediately from Theorem 5.9. □

The continuity result can be used to improve previous calculations of the L^2–Alexander torsion associated to fibered classes. In [Dubois et al. 2015b, Theorem 8.2], the calculation is carried out for rational homology classes only. Liu's result [2017, Theorem 1.2] shows that the asymptotic degree of the L^2–Alexander torsion associated to any class equals its Thurston norm, but does not offer an explicit formula.

Theorem 5.10 *Let N be any compact, connected, irreducible, orientable 3–manifold with empty or incompressible toral boundary. Suppose $\pi_1(N)$ is infinite, $N \neq S^1\times D^2$ and $N\neq S^1\times S^2$. Let $\phi\in H^1(N;\mathbb{R})$ be in the interior of a fibered cone. Then there exists a representative of the L^2–Alexander torsion associated to (ϕ,t) such that*

$$A^{(2)}(N,\phi,t) = \begin{cases} 1 & \text{if } t < 1/h(\phi), \\ t^{x_N(\phi)} & \text{if } t > h(\phi), \end{cases}$$

where $h(\phi)$ is the entropy function defined on the fibered cone of $H^1(N;\mathbb{R})$ (compare [Dubois et al. 2015b, Section 8]).

Proof Let $\phi_n\in H^1(N;\mathbb{Q})$ be a sequence in the fibered cone that converge to ϕ. By [Dubois et al. 2015b, Theorem 8.5], for any n,

$$A^{(2)}(N,\phi_n,t) = \begin{cases} 1 & \text{if } t < 1/h(\phi_n), \\ t^{x_N(\phi_n)} & \text{if } t > h(\phi_n). \end{cases}$$

By Theorem 5.3,

$$A^{(2)}(N,\phi_n,t)\to A^{(2)}(N,\phi,t),\quad n\to\infty,$$

for any $t\in\mathbb{R}$. Since the entropy and the Thurston norm are continuous functions of $H^1(N;\mathbb{R})$,

$$h(\phi_n)\to h(\phi),\quad x_N(\phi_n)\to x_N(\phi),\quad n\to\infty.$$ □

References

[Agol 2008] **I Agol**, *Criteria for virtual fibering*, J. Topol. 1 (2008) 269–284 MR Zbl

[Agol 2013] **I Agol**, *The virtual Haken conjecture*, Doc. Math. 18 (2013) 1045–1087 MR Zbl

[Bénard and Raimbault 2022] **L Bénard**, **J Raimbault**, *Twisted L^2–torsion on the character variety*, Publ. Mat. 66 (2022) 857–881 MR Zbl

[Bergeron and Venkatesh 2013] **N Bergeron**, **A Venkatesh**, *The asymptotic growth of torsion homology for arithmetic groups*, J. Inst. Math. Jussieu 12 (2013) 391–447 MR Zbl

[Chapman 1974] **T A Chapman**, *Topological invariance of Whitehead torsion*, Amer. J. Math. 96 (1974) 488–497 MR Zbl

[Culler 1986] **M Culler**, *Lifting representations to covering groups*, Adv. Math. 59 (1986) 64–70 MR Zbl

[Dubois et al. 2015a] **J Dubois**, **S Friedl**, **W Lück**, *The L^2–Alexander torsion is symmetric*, Algebr. Geom. Topol. 15 (2015) 3599–3612 MR Zbl

[Dubois et al. 2015b] **J Dubois**, **S Friedl**, **W Lück**, *The L^2–Alexander torsions of 3–manifolds*, C. R. Math. Acad. Sci. Paris 353 (2015) 69–73 MR Zbl

[Eisenbud and Neumann 1985] **D Eisenbud**, **W Neumann**, *Three-dimensional link theory and invariants of plane curve singularities*, Ann. of Math. Stud. 110, Princeton Univ. Press (1985) MR Zbl

[Friedl and Lück 2019] **S Friedl**, **W Lück**, *The L^2–torsion function and the Thurston norm of 3–manifolds*, Comment. Math. Helv. 94 (2019) 21–52 MR Zbl

[Hatcher 2007] **A Hatcher**, *Notes on basic 3–manifold topology*, preprint (2007) Available at `https://pi.math.cornell.edu/~hatcher/3M/3Mdownloads.html`

[Hempel 1987] **J Hempel**, *Residual finiteness for 3–manifolds*, from "Combinatorial group theory and topology" (S M Gersten, J R Stallings, editors), Ann. of Math. Stud. 111, Princeton Univ. Press (1987) 379–396 MR Zbl

[Herrmann 2017] **G Herrmann**, *The L^2–Alexander torsion for Seifert fiber spaces*, Arch. Math. (Basel) 109 (2017) 273–283 MR Zbl

[Kitano 1994] **T Kitano**, *Reidemeister torsion of Seifert fibered spaces for $\mathrm{SL}(2;\mathbb{C})$–representations*, Tokyo J. Math. 17 (1994) 59–75 MR Zbl

[Li and Zhang 2006a] **W Li**, **W Zhang**, *An L^2–Alexander–Conway invariant for knots and the volume conjecture*, from "Differential geometry and physics" (M-L Ge, W Zhang, editors), Nankai Tracts Math. 10, World Sci., Hackensack, NJ (2006) 303–312 MR Zbl

[Li and Zhang 2006b] **W Li**, **W Zhang**, *An L^2–Alexander invariant for knots*, Commun. Contemp. Math. 8 (2006) 167–187 MR Zbl

[Liu 2017] **Y Liu**, *Degree of L^2–Alexander torsion for 3–manifolds*, Invent. Math. 207 (2017) 981–1030 MR Zbl

[Lück 2002] **W Lück**, *L^2–invariants: theory and applications to geometry and K–theory*, Ergebnisse der Math. 44, Springer (2002) MR Zbl

[Lück 2018] **W Lück**, *Twisting L^2–invariants with finite-dimensional representations*, J. Topol. Anal. 10 (2018) 723–816 MR Zbl

[Przytycki and Wise 2018] **P Przytycki**, **D T Wise**, *Mixed 3–manifolds are virtually special*, J. Amer. Math. Soc. 31 (2018) 319–347 MR Zbl

[Scott 1983] **P Scott**, *The geometries of 3–manifolds*, Bull. Lond. Math. Soc. 15 (1983) 401–487 MR Zbl

[Waldhausen 1978] **F Waldhausen**, *Algebraic K–theory of generalized free products, II*, Ann. of Math. 108 (1978) 205–256 MR Zbl

Beijing International Center for Mathematical Research, Peking University
Beijing, China

`duanjr@stu.pku.edu.cn`

Received: 27 October 2022 Revised: 3 January 2023

Algebraic & Geometric Topology **24**:4 (2024) 2331–2350
DOI: 10.2140/agt.2024.24.2331
Published: 16 July 2024

An algebraic C_2–equivariant Bézout theorem

STEVEN R COSTENOBLE
THOMAS HUDSON
SEAN TILSON

One interpretation of Bézout's theorem, nonequivariantly, is as a calculation of the Euler class of a sum of line bundles over complex projective space, expressing it in terms of the rank of the bundle and its degree. We generalize this calculation to the C_2–equivariant context, using the calculation of the cohomology of C_2–complex projective spaces from an earlier paper, which used ordinary C_2–cohomology with Burnside ring coefficients and an extended grading necessary to define the Euler class. We express the Euler class in terms of the equivariant rank of the bundle and the degrees of the bundle and its fixed subbundles. We do similar calculations using constant $\mathbb{Z}$ coefficients and Borel cohomology and compare the results.

55N91; 14N10, 14N15, 55R40, 55R91

Introduction

Suppose that we have n nonzero homogeneous polynomials f_i for $1 \le i \le n$ in N variables where $n < N$, let d_i be the degree of f_i, and let $\Delta = d_1 d_2 \cdots d_n$. If $\mathbb{P}^{N-1}$ is the complex projective space, we can consider each f_i as giving a section of the complex line bundle $O(d_i)$, the d_i–fold tensor power of the dual of the tautological line bundle over $\mathbb{P}^{N-1}$. Each f_i determines a hypersurface $H_i \subset \mathbb{P}^{N-1}$, its zero locus. In this context, the (nonequivariant) Bézout theorem, as given by Fulton [5], for example, can be stated in several ways. Geometrically, it says that the intersection of the hypersurfaces H_i, counted with multiplicities, is generically rationally equivalent to Δ copies of $\mathbb{P}^{N-n-1}$. In the classical case, when $n = N-1$, the hypersurfaces intersect in Δ points.

We can restate Bézout's theorem as a purely algebraic statement: in the cohomology ring

$$H^*(\mathbb{P}^{N-1}; \mathbb{Z}) \cong \mathbb{Z}[\hat{c}]/\langle \hat{c}^N \rangle,$$

the Euler class of $F = O(d_1) \oplus O(d_2) \oplus \cdots \oplus O(d_n)$ is

$$e(F) = \Delta \hat{c}^n,$$

where $\hat{c} = e(O(1))$. As a consequence, $e(F)$ determines and is completely determined by the rank n of F (that is, the complex dimension of each of its fibers) and its degree Δ. (The connection to the geometric statement is via the Chow ring, isomorphic to cohomology in this case, in which $\hat{c}^n$ is represented by $\mathbb{P}^{N-n-1}$.) Here we want to generalize the algebraic calculation, including giving a generalization of the notions of rank and degree, and discussing how they determine and are determined by the Euler class; in a followup paper we will pursue a geometric interpretation.

In [3] we began to examine how this generalizes in the presence of an action of the two-element group C_2. Let $\mathbb{C}$ denote the trivial complex representation of C_2 and let $\mathbb{C}^\sigma$ denote the nontrivial representation. If $p \geq 0$ and $q \geq 0$ are integers, let $\mathbb{C}^{p+q\sigma}$ be the sum of p copies of $\mathbb{C}$ and q copies of $\mathbb{C}^\sigma$, and let $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ be its (complex) projective space, a C_2–space. Using the equivariant ordinary cohomology with extended grading defined in [4], we computed the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ in [3] with Burnside ring coefficients. We also gave the zero-dimensional version of an equivariant Bézout theorem, showing that the equivariant Euler class in equivariant ordinary cohomology with Burnside ring coefficients allows us to determine the finite C_2–set in $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ given by the intersection of $p+q-1$ equivariant hypersurfaces.

Let us set up the context for a generalization to higher dimensions. As mentioned above, if F is a nonequivariant vector bundle over $\mathbb{P}^{N-1}$, its Euler class has the form $e(F) = \Delta\hat{c}^n$, where n is the rank of F, Δ is its degree, and we set $\Delta = 0$ if $n \geq N$.

Now suppose that we have $(n<p+q)$–many C_2–line bundles over $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ with direct sum F. We let Δ be the nonequivariant degree of F. We can also consider the fixed-set bundle F^{C_2} over $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2} = \mathbb{P}(\mathbb{C}^p) \sqcup \mathbb{P}(\mathbb{C}^{q\sigma})$. Let n_0 denote the rank of the restriction of F^{C_2} to $\mathbb{P}(\mathbb{C}^p)$ and let Δ_0 be its degree. We know that $n_0 \leq n$, and, to keep the situation geometrically meaningful, we would like the generic intersection of the corresponding hypersurfaces in $\mathbb{P}(\mathbb{C}^p)$ to have dimension no more than the dimension of the intersection of all the hypersurfaces in $\mathbb{P}(\mathbb{C}^{p+q\sigma})$. For that, we require that $p-n_0-1 \leq p+q-n-1$, that is, $n_0 \geq n-q$. Similarly, let n_1 denote the rank of F^{C_2} over $\mathbb{P}(\mathbb{C}^{q\sigma})$ and let Δ_1 be its degree; we require that $n_1 \geq n-p$. We record these notations and conditions for later reference.

Bézout context 0.1 F is the sum of n–many C_2–line bundles over $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ and Δ is its nonequivariant degree. The restriction of F^{C_2} to $\mathbb{P}(\mathbb{C}^p)$ has rank n_0 and degree Δ_0, while its restriction to $\mathbb{P}(\mathbb{C}^{q\sigma})$ has rank n_1 and degree Δ_1. We assume that

$$n < p+q, \quad n-q \leq n_0 \leq n \quad \text{and} \quad n-p \leq n_1 \leq n.$$

We call the triple (n, n_0, n_1) the C_2*–ranks* of F and the triple $(\Delta, \Delta_0, \Delta_1)$ the C_2*–degrees* of F.

In this context we will calculate the Euler class $e(F)$ as an element of the equivariant cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$, as computed in [3].

Bézout theorem, part I *In the context above, the Euler class $e(F)$ is completely determined by the ranks (n, n_0, n_1) and the degrees $(\Delta, \Delta_0, \Delta_1)$. Moreover, these ranks and degrees can be recovered from $e(F)$. The ranks are additive and the degrees are multiplicative.*

This will be proved as Theorem 2.11. When we say that the degrees are multiplicative, we really mean the following: Suppose that we have two such bundles F and F' with ranks (n, n_0, n_1) and (n', n'_0, n'_1), respectively, and corresponding degrees. We assume that $F \oplus F'$ still satisfies the conditions of the Bézout context above. This allows the possibility that $n_0 + n'_0 \geq p$, in which case the corresponding degree is not $\Delta_0\Delta'_0$ but 0, and similarly if $n_1 + n'_1 \geq q$.

Nonequivariantly, the cohomology of $\mathbb{P}^{N-1}$ is a free $\mathbb{Z}$–module with a basis given by the powers of $\hat{c}$. Explicitly,

$$H^*(\mathbb{P}^{N-1}) \cong \bigoplus_{i=0}^{N-1} \hat{c}^i \mathbb{Z}. \tag{0.2}$$

The nonequivariant Bézout theorem can be viewed as expressing $e(F)$ in terms of this basis. In any given grading, there is at most one basis element, so there is only one coefficient to specify, which turns out to be the degree Δ. Equivariantly, the result is more complicated. In [3] we showed that the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ is free over the $\mathrm{RO}(C_2)$–graded equivariant cohomology of a point and gave an explicit basis that maps to the nonequivariant one. That is, we have a decomposition similar to (0.2), with $\mathbb{Z}$ replaced by the $\mathrm{RO}(C_2)$–graded cohomology of a point and the powers of $\hat{c}$ replaced by our preferred basis. Because the cohomology of a point is not concentrated in grading 0 outside of the $\mathbb{Z}$–graded part, in any given grading of the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ there are up to $p+q$ basis elements that can contribute, so an element potentially requires a $(p+q)$–tuple of coefficients (from the cohomology of a point) to specify. Our second main result is summarized as follows:

Bézout theorem, part II *In the context above, the Euler class $e(F)$ is the linear combination of at most three basis elements.*

This is proved as Theorem 2.12, which also gives the details as to which three basis elements are involved and what their coefficients are. The three basis elements are determined by (p and q and) the ranks (n, n_0, n_1). The coefficients are determined by the degrees $(\Delta, \Delta_0, \Delta_1)$, but are not simply equal to them.

This paper is structured as follows. In Section 1 we review the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ as computed in [3], including our preferred basis. In Section 2 we give the main results, proving the two theorems above. There are two other equivariant ordinary cohomology theories in common use: cohomology with constant $\mathbb{Z}$ coefficients and Borel cohomology. In Section 3 we discuss how the computation changes if we use constant $\mathbb{Z}$ coefficients rather than Burnside ring coefficients, and in Section 4 we discuss the similar computation in Borel cohomology. There are maps from cohomology with Burnside ring coefficients to cohomology with constant $\mathbb{Z}$ coefficients, and from that theory to Borel cohomology, both respecting Euler classes, and we will see that the Euler classes in the last two theories carry less information than the Euler class in cohomology with Burnside ring coefficients. In particular, we cannot recover the degrees Δ_0 and Δ_1 from the Euler class in cohomology with constant $\mathbb{Z}$ coefficients or the class in Borel cohomology.

Acknowledgements Hudson and Tilson were partially supported by the DFG through the SPP 1786: *Homotopy theory and algebraic geometry*, project number 405468058: *C_2–equivariant Schubert calculus of homogeneous spaces.* While Hudson and Tilson were affiliated to the Bergische Universität Wuppertal, the research was conducted in the framework of the research training group GRK 2240: *Algebro-geometric methods in algebra, arithmetic and topology*, which is funded by the DFG,

1 The cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$

1.1 Ordinary cohomology

We will use C_2–equivariant ordinary cohomology with the extended grading developed in [4]. This is an extension of Bredon's ordinary cohomology to be graded on representations of the fundamental groupoids of C_2–spaces. We review here some of the notation and computations we will be using. A more detailed description of this theory can be found in [3].

For an ex-C_2–space Y over X, we write $H_{C_2}^{RO(\Pi X)}(Y;\underline{T})$ for the ordinary cohomology of Y with coefficients in a Mackey functor $\underline{T}$, graded on $RO(\Pi X)$, the representation ring of the fundamental groupoid of X. Through most of this paper we will use the Burnside ring Mackey functor $\underline{A}$ as the coefficients, and write simply $H_{C_2}^{RO(\Pi X)}(Y)$.

In [4; 3] we considered cohomology to be Mackey functor-valued, which is useful for many computations, and wrote $\underline{H}_{C_2}^{RO(\Pi X)}(Y)$ for the resulting theory. Here we concentrate on the values at level C_2/C_2, and write

$$H_{C_2}^{RO(\Pi X)}(Y) = \underline{H}_{C_2}^{RO(\Pi X)}(Y)(C_2/C_2).$$

However, we will still refer to the restriction functor ρ from equivariant cohomology to nonequivariant cohomology, and the transfer map τ going in the other direction.

For all X and Y, $H_{C_2}^{RO(\Pi X)}(Y)$ is a graded module over

$$\mathbb{H} = \mathbb{H}^{RO(C_2)} = H_{C_2}^{RO(C_2)}(S^0),$$

the cohomology of a point. The grading on the latter is just $RO(C_2)$, the real representation ring of C_2, which is free abelian on 1, the class of the trivial representation $\mathbb{R}$, and σ, the class of the sign representation $\mathbb{R}^\sigma$. The cohomology of a point was calculated by Stong in an unpublished manuscript, and first published by Lewis in [6]. We can picture the calculation as in Figure 1, in which a group in grading $a + b\sigma$ is plotted at the point (a, b), and the spacing of the grid lines is 2 (which is more convenient for other graphs we will give). The box at the origin is a copy of $A(C_2)$, the Burnside ring of C_2, closed circles are copies of $\mathbb{Z}$, and open circles are copies of $\mathbb{Z}/2$.

Recall that $A(C_2)$ is the Grothendieck group of finite C_2–sets, with multiplication given by products of sets. Additively, it is free abelian on the classes of the orbits of C_2, for which we will write $1 = [C_2/C_2]$ and $g = [C_2/e]$. The multiplication is given by $g^2 = 2g$. We will also write $\kappa = 2 - g$. Other important elements are shown in the figure: The group in degree σ is generated by an element e, while the group in degree $-2 + 2\sigma$ is generated by an element ξ. The groups in the second quadrant are generated by the products $e^m\xi^n$, with $2e\xi = 0$. We have $g\xi = 2\xi$ and $ge = 0$. The groups in gradings $-m\sigma$, for $m \ge 1$, are generated by elements $e^{-m}\kappa$, so named because $e^m \cdot e^{-m}\kappa = \kappa$. We also have $ge^{-m}\kappa = 0$.

To explain $\tau(\iota^{-2})$, we think for a moment about the nonequivariant cohomology of a point. If we grade it on $RO(C_2)$, we get $H^{RO(C_2)}(S^0;\mathbb{Z}) \cong \mathbb{Z}[\iota^{\pm 1}]$, where $\deg\iota = -1 + \sigma$. (Nonequivariantly, we cannot tell

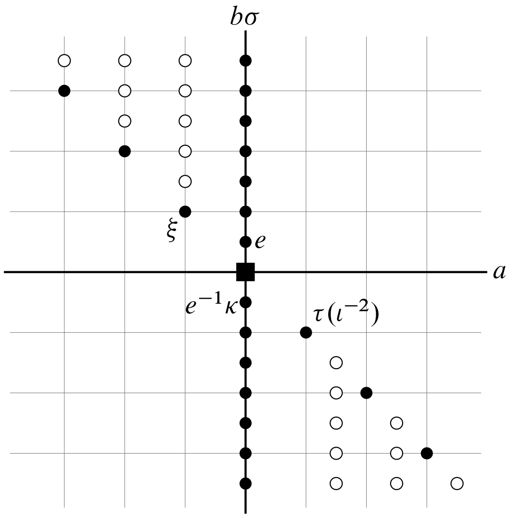


Figure 1: $\mathbb{H}^{\mathrm{RO}(C_2)}$.

the difference between $\mathbb{R}$ and $\mathbb{R}^\sigma$.) We have $\rho(\xi)=\iota^2$ and $\tau(\iota^2)=g\xi=2\xi$. Note also that $\tau(1)=g$. In the fourth quadrant the group in grading $n(1-\sigma)$ for $n\geq 2$ is generated by $\tau(\iota^{-n})$. The remaining groups in the fourth quadrant will not concern us here. For more details, see [2; 3].

1.2 The cohomology of projective space

As described in the introduction, the form of Bézout's theorem we shall give expresses the Euler class of a bundle over $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ in terms of a basis of its cohomology. We now review the structure of that cohomology as calculated in [3].

Write $B=\mathbb{P}(\mathbb{C}^{\infty+\infty\sigma})$. Its fixed set is

$$B^{C_2}=\mathbb{P}(\mathbb{C}^\infty)\sqcup\mathbb{P}(\mathbb{C}^{\infty\sigma})=B^0\sqcup B^1,$$

where we use the indices 0 and 1 to evoke the trivial and nontrivial representations of C_2, respectively. (We will use this convention, that a subscript 0 refers to something related to B^0 and subscript 1 refers to something related to B^1, throughout.) Representations of ΠB are determined by their restrictions to B^0 and B^1, which are elements of $\mathrm{RO}(C_2)$ that must have the same nonequivariant rank and the same parity for the ranks of their fixed-point representations. As shown in [3, Section 2.2, page 13] this leads to the calculation

$$\mathrm{RO}(\Pi B)=\mathbb{Z}\{1,\sigma,\Omega_0,\Omega_1\}/\langle\Omega_0+\Omega_1=2\sigma-2\rangle,$$

where Ω_0 is the representation whose value on B^0 is $2\sigma-2$ and on B^1 is 0, while Ω_1 is the representation whose value on B^0 is 0 and on B^1 is $2\sigma-2$. For any $\alpha\in\mathrm{RO}(\Pi B)$, write $|\alpha|\in\mathbb{Z}$ for its underlying nonequivariant rank, and α_0 and $\alpha_1\in\mathrm{RO}(C_2)$ for its restrictions to B^0 and B^1, respectively. What we said above can be phrased as: α is completely determined by the triple of ranks $(|\alpha|,|\alpha_0^{C_2}|,|\alpha_1^{C_2}|)$, where the last two ranks have the same parity.

We think of the finite projective spaces as spaces over B by the evident inclusions $\mathbb{P}(\mathbb{C}^{p+q\sigma}) \to \mathbb{P}(\mathbb{C}^{\infty+\infty\sigma})$, and so will grade their cohomologies on $RO(\Pi B)$. Let ω denote the tautological line bundle over B, let $\omega^\vee$ be its dual bundle, let $\chi\omega = \omega \otimes_{\mathbb{C}} \mathbb{C}^\sigma$, and let $\chi\omega^\vee$ be the dual of $\chi\omega$. We will also use the notation from algebraic geometry in which $\omega = O(-1)$ and $\omega^\vee = O(1)$; we write $\chi O(-1) = \chi\omega$ and $\chi O(1) = \chi\omega^\vee$.

Associated to any bundle over B is a representation in $RO(\Pi B)$ that we think of as the equivariant rank of the bundle; this representation is given by the fiber representations over B^0 and B^1. In the case of ω and $\chi\omega$, we have

$$\omega = 2 + \Omega_1 \quad \text{and} \quad \chi\omega = 2 + \Omega_0,$$

where we write ω and $\chi\omega$ again for the associated elements of $RO(\Pi B)$.

Let $\hat{c}_\omega$ and $\hat{c}_{\chi\omega}$ denote the Euler classes of $\omega^\vee$ and $\chi\omega^\vee$, respectively. The cohomology of $\mathbb{P}(\mathbb{C}^{\infty+\infty\sigma})$ was calculated in [2] as follows:

Theorem 1.1 *$H_{C_2}^{RO(\Pi B)}(B_+)$ is an algebra over $\mathbb{H}$ generated by the Euler classes $\hat{c}_\omega$ and $\hat{c}_{\chi\omega}$ together with classes ζ_0 and ζ_1. These elements live in gradings*

$$\operatorname{grad} \hat{c}_\omega = \omega, \quad \operatorname{grad} \hat{c}_{\chi\omega} = \chi\omega, \quad \operatorname{grad} \zeta_1 = \omega - 2 \quad \textit{and} \quad \operatorname{grad} \zeta_0 = \chi\omega - 2.$$

They satisfy the relations

$$\zeta_0\zeta_1 = \xi \quad \textit{and} \quad \zeta_1\hat{c}_{\chi\omega} = (1-\kappa)\zeta_0\hat{c}_\omega + e^2,$$

which completely determine the algebra. Moreover, $H_{C_2}^{RO(\Pi B)}(B_+)$ is free as a module over $\mathbb{H}$. □

There are two restriction maps we will use,

$$\rho\colon H^\alpha_{C_2}(B_+) \to H^{|\alpha|}(B_+),$$

restriction to nonequivariant cohomology, and

$$(-)^{C_2}\colon H^\alpha_{C_2}(B_+) \to H^{\alpha_0^{C_2}}(B^0_+) \oplus H^{\alpha_1^{C_2}}(B^1_+),$$

the fixed-point map. These are ring maps and their values on the multiplicative generators are given by the following:

$$\begin{aligned}
&\rho(\zeta_0) = 1, && \rho(\zeta_1) = 1, && \rho(\hat{c}_\omega) = \hat{c}, && \rho(\hat{c}_{\chi\omega}) = \hat{c},\\
&\zeta_0^{C_2} = (0,1), && \zeta_1^{C_2} = (1,0), && \hat{c}_\omega^{C_2} = (\hat{c},1), && \hat{c}_{\chi\omega}^{C_2} = (1,\hat{c}).
\end{aligned}$$

Here $\hat{c}$ denotes the first nonequivariant Chern class of $O(1)$. We also need the values of the similar restriction maps

$$\rho\colon \mathbb{H}^\alpha \to H^{|\alpha|}(S^0) \quad \text{and} \quad (-)^{C_2}\colon \mathbb{H}^\alpha \to H^{\alpha^{C_2}}(S^0).$$

The particular values we will need are

$$\rho(\tau(\iota^{2k})) = 1, \quad \rho(e^{-k}\kappa) = 0, \quad \rho(e^k) = 0, \quad \tau(\iota^{2k})^{C_2} = 0, \quad (e^{-k}\kappa)^{C_2} = 2 \quad \text{and} \quad (e^k)^{C_2} = 1.$$

Moving now to finite projective spaces, on pulling back along the inclusion $\mathbb{P}(\mathbb{C}^{p+q\sigma}) \hookrightarrow \mathbb{P}(\mathbb{C}^{\infty+\infty\sigma})$, the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ contains elements we will also call $\hat{c}_\omega$, $\hat{c}_{\chi\omega}$, ζ_0, and ζ_1.

Theorem 1.2 [3, Theorem A] *Let $0 \le p, q < \infty$ with $p+q > 0$. Then $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ is a free module over $\mathbb{H}$, and as a (graded) commutative algebra over $\mathbb{H}$ the ring $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ is generated by $\hat{c}_\omega$, $\hat{c}_{\chi\omega}$, ζ_0, and ζ_1, together with the following classes: $\hat{c}_\omega^p$ is infinitely divisible by ζ_0, meaning that, for $k \ge 1$, there are unique elements $\zeta_0^{-k}\hat{c}_\omega^p$ such that*

$$\zeta_0^k \cdot \zeta_0^{-k}\hat{c}_\omega^p = \hat{c}_\omega^p.$$

Similarly, $\hat{c}_{\chi\omega}^q$ is infinitely divisible by ζ_1, so for $k \ge 1$ there are unique elements $\zeta_1^{-k}\hat{c}_{\chi\omega}^q$ such that

$$\zeta_1^k \cdot \zeta_1^{-k}\hat{c}_{\chi\omega}^q = \hat{c}_{\chi\omega}^q.$$

The generators satisfy the following further relations:

$$\zeta_0\zeta_1 = \xi, \quad \zeta_1\hat{c}_{\chi\omega} = (1-\kappa)\zeta_0\hat{c}_\omega + e^2 \quad \text{and} \quad \hat{c}_\omega^p\hat{c}_{\chi\omega}^q = 0. \qquad \square$$

We also gave an explicit basis for $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ over $\mathbb{H}$, which we can describe as follows. We define sets $F_{p,q}(m)$, recursively on p and q, that give bases for $H_{C_2}^{m\omega+\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$. For $m \in \mathbb{Z}$, let

$$F_{p,0}(m) := \{\zeta_1^m, \zeta_1^{m-1}\hat{c}_\omega, \zeta_1^{m-2}\hat{c}_\omega^2, \dots, \zeta_1^{m-p+1}\hat{c}_\omega^{p-1}\}$$

and

$$F_{0,q}(m) := \{\zeta_0^m, \zeta_0^{m-1}\hat{c}_{\chi\omega}, \zeta_0^{m-2}\hat{c}_{\chi\omega}^2, \dots, \zeta_0^{m-q+1}\hat{c}_{\chi\omega}^{q-1}\}.$$

(Note that ζ_1 is invertible in the first case and ζ_0 is invertible in the second.) For $p, q > 0$ we then define

$$F_{p,q}(m) := \begin{cases} \{\zeta_1^m\} \cup i_! F_{p-1,q}(m-1) & \text{if } m \ge 0, \\ \{\zeta_0^{|m|}\} \cup j_! F_{p,q-1}(m+1) & \text{if } m < 0, \end{cases}$$

where $i\colon \mathbb{P}(\mathbb{C}^{p-1+q\sigma}) \to \mathbb{P}(\mathbb{C}^{p+q\sigma})$ and $j\colon \mathbb{P}(\mathbb{C}^{p+(q-1)\sigma}) \to \mathbb{P}(\mathbb{C}^{p+q\sigma})$ are the inclusions. The pushforward $i_!$ is given algebraically by multiplication by $\hat{c}_\omega$, and $j_!$ is multiplication by $\hat{c}_{\chi\omega}$.

It is possible from this description to write down the bases explicitly, but the results are messy, having to be broken down by cases depending on where m falls in relation to p and q; this is done in [3, Proposition 4.7]. However, we can make the following general statements.

(1) For fixed p, q, and m, there are exactly $p+q$ basis elements lying in $H_{C_2}^{m\omega+\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$.

(2) Those basis elements have gradings of the form $m(\omega-2) + 2a_i + 2b_i\sigma$ for $0 \le i \le p+q-1$, where $a_i + b_i = i$.

(3) The basis element with grading $m(\omega-2) + 2a + 2b\sigma$ restricts to the nonequivariant class $\hat{c}^{a+b}$, where again $\hat{c}$ is the first nonequivariant Chern class of $O(1)$.

(4) For a given integer k, there are at most two indices i such that $a_i = k$.

Figure 2 illustrates, in the case of $\mathbb{P}(\mathbb{C}^{4+5\sigma})$, how the basis elements can be arranged for various values of m. In each case, the basis element with grading $m(\omega-2) + 2a + 2b\sigma$ is marked by a dot at (a, b).

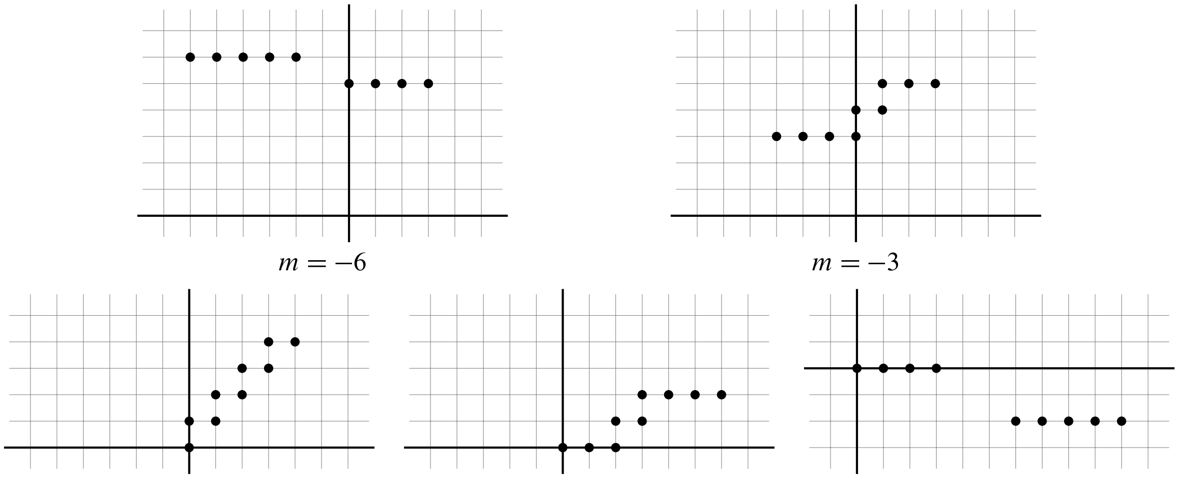


Figure 2: Bases for $H_{C_2}^{m\omega+\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{4+5\sigma})_+)$.

For ease of reference, we will write the bases as

$$F_{p,q}(m) = \{P_0^{(m)}, P_1^{(m)}, \ldots, P_{p+q-1}^{(m)}\},$$

where $P_i^{(m)}$ is the basis element in $H_{C_2}^{m\omega+\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ restricting to the element $\hat{c}^i$ nonequivariantly. When m is understood, we will simply write P_i for $P_i^{(m)}$. We can also say that P_i is the basis element in grading $m(\omega-2)+2a+2b\sigma$ with $a+b=i$, as illustrated for $m=0$ in Figure 3.

Definition 1.3 Given any element $x \in H_{C_2}^{m\omega+\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$, we can write x uniquely as

$$x = \sum_{i=0}^{p+q-1} \alpha_i P_i^{(m)}$$

with each coefficient $\alpha_i \in \mathbb{H}$. We call the $(p+q)$–tuple (α_i) the *coefficient vector* of x.

Because elements of $\mathbb{H}$ lie in a restricted set of gradings, the number of nonzero coefficients possible for a given x may be limited, depending on the grading of x, though there are elements x for which all coefficients are nonzero.

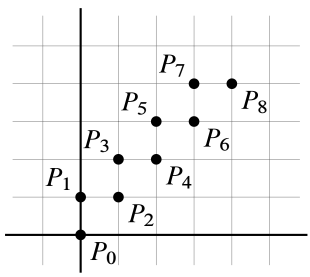


Figure 3: Basis for $H_{C_2}^{\mathrm{RO}(C_2)}(\mathbb{P}(\mathbb{C}^{4+5\sigma})_+)$.

2 The algebraic equivariant Bézout theorem

It is possible to take the calculation of Euler classes in [3] and, by brute force, work out their expression in terms of the basis for the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ discussed in the preceding section. Instead, we will take advantage of some features of the cohomology of a point to give a more conceptual approach that shows better why the calculation works the way it does.

Definition 2.1

- Let $T \subset \mathbb{H}$ consist of the elements $a\tau(\iota^{2\ell})$ for $a \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, the elements $ae^{-m}\kappa$ for $a \in \mathbb{Z}$ and $m \geq 1$, the elements ae^m for $a \in \mathbb{Z}$ and $m \geq 1$, and all of $A(C_2) = \mathbb{H}^0$.
- Let $I_e \subset T$ consist of the elements $a\tau(\iota^{2\ell})$ for $a \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, $ae^m\kappa$ for $a \in \mathbb{Z}$ and $m \in \mathbb{Z}$, and $a + bg \in A(C_2)$ such that a is even.

Note that $e^m\kappa = 2e^m$ if $m > 0$.

Proposition 2.2 *I_e is an ideal of $\mathbb{H}$.*

Proof This is a straightforward check from the known structure of $\mathbb{H}$, as given in [3]. □

On the other hand, T is not an ideal, because $e\xi \notin T$ while $e \in T$. But T is an additive subgroup.

An important fact about T is that, as shown in Figure 4, all of its elements lie in gradings of the form $n\sigma$ or $2n(1-\sigma)$, that is, on the vertical line through the origin or the diagonal through the origin with slope -1. Closed circles indicate copies of $\mathbb{Z}$, while the box at the origin is $A(C_2)$. T is a free $\mathbb{Z}$–module.

Another fact that follows from the known structure of $\mathbb{H}$ is that the quotient ring $\mathbb{H}/I_e$ is all 2–torsion.

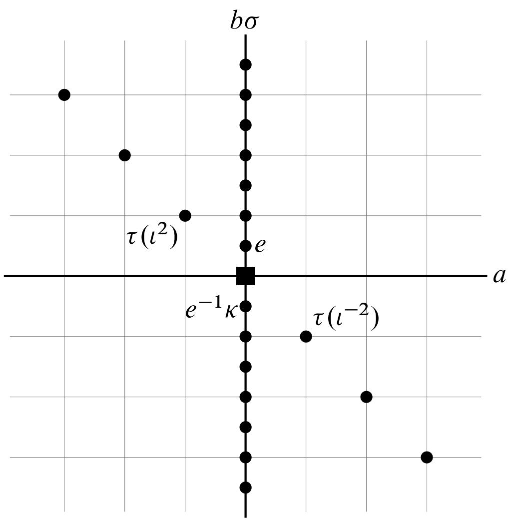


Figure 4: The subset T of $\mathbb{H}$.

Remark 2.3 The ideal I_e is almost, but not quite, the kernel of the restriction map

$$\mathbb{H} = H_{C_2}^{\mathrm{RO}(C_2)}(S^0; \underline{A}) \to H_{C_2}^{\mathrm{RO}(C_2)}(S^0; \underline{\mathbb{Z}/2}).$$

That kernel would not contain all the elements $a\tau(\iota^{-2m})$ for $m \geq 1$, but only those of the form $2a\tau(\iota^{-2m})$. Either ideal would serve our purpose here, but we chose to use the one that is slightly simpler to describe.

Definition 2.4
- Denote the set of linear combinations of elements of our preferred basis of $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ with coefficients in T by $\widetilde{T} \subset H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$.
- Let J_e be the ideal defined by
$$J_e = I_e H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+) \subset H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+).$$

Every element of J_e is a linear combination of elements from our preferred basis with coefficients in I_e (and this would be true for any basis we used). Because $J_e \subset \widetilde{T}$, the following facts about $\widetilde{T}$ apply to J_e as well.

Lemma 2.5 *Every element $x \in \widetilde{T}$ is a linear combination of at most three basis elements: if x lies in grading $m(\omega - 2) + a + b\sigma$, the only basis elements that can contribute to x are the one (if any) lying on the same diagonal as x, that is, in a grading $m(\omega - 2) + a' + b'\sigma$ with $a' + b' = a + b$, and the two (at most) lying in the same vertical line as x, that is, in gradings $m(\omega - 2) + a + b'\sigma$.*

Proof This follows from the description of the locations of the basis elements given in the preceding section together with the locations of the elements of T. □

See the example in Remark 2.15 below for an illustration of this lemma.

Proposition 2.6 *If $x \in \widetilde{T}$, then x is determined by its restrictions $\rho(x)$ and x^{C_2}.*

Proof By the preceding lemma, x can be written as a linear combination of at most three elements from our standard basis. There are various cases that should be considered. Suppose, for example, that x lies on the same diagonal as a basis element P_n and lies above two basis elements P_k and P_{k-1}. Then we can write

$$x = \alpha\tau(\iota^{2\ell})P_n + \beta e^m P_k + \gamma e^{m+2} P_{k-1}$$

for some integers α, β, γ, ℓ, and m. We now appeal to [3, Proposition 4.6], where we showed that our standard basis restricts to a nonequivariant basis for $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ and a nonequivariant basis for $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}$. We have $\rho(x) = 2\alpha\rho(P_n)$, so α is determined by $\rho(x)$. On the other hand, $x^{C_2} = \beta P_k^{C_2} + \gamma P_{k-1}^{C_2}$, so β and γ are determined by x^{C_2}.

There are other cases, for example, where x lies below two basis elements rather than above, or where it lies in the same grading as a basis element. Each of these can be handled in the same way as the case above. □

Note that this is not true for general elements of $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ because there are elements of $\mathbb{H}$ that vanish under both ρ and $(-)^{C_2}$.

For any $x \in H_{C_2}^{RO(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ we have

$$\rho(x) \in H^{\mathbb{Z}}(\mathbb{P}(\mathbb{C}^{p+q})_+),$$

so $\rho(x) = \Delta\hat{c}^k$ for some integers Δ and k, or it is 0, in which case we set $\Delta = 0$. We also have

$$x^{C_2} \in H^{\mathbb{Z}}(\mathbb{P}(\mathbb{C}^p)_+) \oplus H^{\mathbb{Z}}(\mathbb{P}(\mathbb{C}^q)_+),$$

so $x^{C_2} = (\Delta_0\hat{c}^i, \Delta_1\hat{c}^j)$ for some integers Δ_0, Δ_1, i, and j. (Again, we set $\Delta_0 = 0$ if $\Delta_0\hat{c}^i = 0$ and $\Delta_1 = 0$ if $\Delta_1\hat{c}^j = 0$.)

Definition 2.7 We call the triple of integers $(\Delta, \Delta_0, \Delta_1)$ determined as above the C_2*–degrees* of x.

Corollary 2.8 *If $x \in \widetilde{T}$, then x is determined by its grading and its C_2–degrees.*

Proof Suppose that x lies in grading $m(\omega - 2) + a + b\sigma$ and that the degrees of x are $(\Delta, \Delta_0, \Delta_1)$. By the structure of $\widetilde{T}$ and the locations of the basis elements, we can assume that a is even. Then

$$\rho(x) = \begin{cases} \Delta\hat{c}^{(a+b)/2} & \text{if } b \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad x^{C_2} = (\Delta_0\hat{c}^{a/2}, \Delta_1\hat{c}^{a/2-m}).$$

Thus the grading of x and its degrees determine $\rho(x)$ and x^{C_2}, so the result follows from the preceding proposition. □

In order to apply these results to derive the two parts of our Bézout theorem, we need to know a little more about the line bundles that are the summands of F as in Bézout context 0.1. In [3] we showed that the line bundles over $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ all have the form $O(d)$ or $\chi O(d)$. It is useful to further break these down into four types:

type I	bundles of the form $O(2d+1)$,
type II	bundles of the form $O(2d)$,
type III	bundles of the form $\chi O(2d+1)$,
type IV	bundles of the form $\chi O(2d)$.

The fixed points $O(2d+1)^{C_2}$ of a bundle of type I have fiber $\mathbb{C}$ over $\mathbb{P}(\mathbb{C}^p)$ and 0 over $\mathbb{P}(\mathbb{C}^{q\sigma})$, while the reverse is true for a bundle of type III. The fixed points $O(2d)^{C_2}$ of a bundle of type II have fiber $\mathbb{C}$ over both components of $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}$, while the fixed points of a bundle of type IV have fiber 0 over both components.

In [3], for $\dagger \in \{\mathrm{I,II,III,IV}\}$ we wrote $n_\dagger$ for the number of summands of type $\dagger$ and $d_\dagger$ for the products of their degrees. These are related to the ranks and C_2–degrees of F by

$$n = n_{\mathrm{I}} + n_{\mathrm{II}} + n_{\mathrm{III}} + n_{\mathrm{IV}}, \quad n_0 = n_{\mathrm{I}} + n_{\mathrm{II}}, \quad n_1 = n_{\mathrm{II}} + n_{\mathrm{III}}, \quad \Delta = d_{\mathrm{I}}d_{\mathrm{II}}d_{\mathrm{III}}d_{\mathrm{IV}},$$

(2.9)
$$\Delta_0 = \begin{cases} d_{\mathrm{I}}d_{\mathrm{II}} & \text{if } n_0 < p, \\ 0 & \text{if } n_0 \geq p, \end{cases}$$

(2.10)
$$\Delta_1 = \begin{cases} d_{\mathrm{II}}d_{\mathrm{III}} & \text{if } n_1 < q, \\ 0 & \text{if } n_1 \geq q. \end{cases}$$

Now, d_{I} and d_{III} are always odd, and d_{II} and d_{IV} are even if and only if there is a summand of type II or IV, respectively. Notice that, when $n_{\mathrm{II}} > 0$, the quantities Δ, Δ_0, and Δ_1 will all be even. If $n_{\mathrm{II}} = 0$, then $n_0 + n_1 \le n$, which implies that

$$n_0 \le n - n_1 \le n - (n-p) = p$$

and $n_1 \le q$, similarly, with equality possible only if $n_{\mathrm{IV}} = 0$. So, if $n_{\mathrm{II}} = 0$ but $n_{\mathrm{IV}} > 0$, we will have Δ even and both Δ_0 and Δ_1 odd. When $n_{\mathrm{II}} = 0$ and $n_{\mathrm{IV}} = 0$, we will have Δ odd while Δ_0 and Δ_1 will be odd if nonzero.

Theorem 2.11 (Bézout theorem, part I) *Let F be as in Bézout context 0.1. Then $e(F)$ lies in $\widetilde{T}$, and hence is determined by its grading, which is*

$$(n_0 - n_1)(\omega - 2) + 2n_0 + 2(n - n_0)\sigma,$$

and its C_2–degrees, which are $(\Delta, \Delta_0, \Delta_1)$. Moreover, the grading and degrees can be recovered from $e(F)$. The ranks (n, n_0, n_1) are additive while the degrees are multiplicative.

Proof The additivity of the grading and the multiplicativity of the degrees are clear (but see the caveat about multiplicativity given in the introduction).

Given that n is the nonequivariant (complex) rank of F and n_0 and n_1 are the ranks of the restriction of F^{C_2} to $\mathbb{P}(\mathbb{C}^p)$ and $\mathbb{P}(\mathbb{C}^{q\sigma})$, respectively, $e(F)$ must lie in the grading given, which is the grading α with $|\alpha| = n$, $\alpha_0 = 2n_0 + 2(n - n_0)\sigma$, and $\alpha_1 = n_1 + 2(n - n_1)\sigma$.

Conversely, if $e(F)$ lies in grading $m(\omega - 2) + 2a + 2b\sigma$, then we can recover $n = a + b$, $n_0 = a$, and $n_1 = a - m$.

The degrees $(\Delta, \Delta_0, \Delta_1)$ are, by the nonequivariant Bézout theorem, given by

$$\rho(e(F)) = \Delta \hat{c}^n \quad \text{and} \quad e(F)^{C_2} = (\Delta_0 \hat{c}^{n_0}, \Delta_1 \hat{c}^{n_1}),$$

using the fact that ρ and $(-)^{C_2}$ preserve Euler classes. Thus, we can recover the degrees from $e(F)$.

It remains to show that $e(F)$ is determined by its grading and C_2–degrees.

Recall the discussion above of the four types of line bundles over $\mathbb{P}(\mathbb{C}^{p+q\sigma})$. In [3, Proposition 6.5] we computed their Euler classes, which are

$$\begin{aligned}
e(O(2d+1)) &= \hat{c}_\omega + d(\tau(1)\hat{c}_\omega + e^{-2}\kappa\zeta_1\hat{c}_\omega\hat{c}_{\chi\omega}) \equiv \hat{c}_\omega \pmod{J_e},\\
e(O(2d)) &= d(\tau(\iota^{-2})\zeta_0\hat{c}_\omega + e^{-2}\kappa\hat{c}_\omega\hat{c}_{\chi\omega}) \equiv 0 \pmod{J_e},\\
e(\chi O(2d+1)) &= \hat{c}_{\chi\omega} + d(\tau(1)\hat{c}_{\chi\omega} + e^{-2}\kappa\zeta_0\hat{c}_\omega\hat{c}_{\chi\omega}) \equiv \hat{c}_{\chi\omega} \pmod{J_e},\\
e(\chi O(2d)) &= e^2 + d\tau(1)\zeta_0\hat{c}_\omega \equiv e^2 \pmod{J_e}.
\end{aligned}$$

From (2.9) and (2.10), we see that Δ_0 and Δ_1 are both even if and only if F contains at least one summand of the form $O(2d)$ (type II). If F does not contain such a summand, then n_0 is the number of summands of the form $O(2d+1)$ and n_1 is the number of summands of the form $\chi O(2d+1)$, and we will have $n_0+n_1 \le n$. From the congruences above, we have, modulo J_e, that

$$e(F) \equiv \begin{cases} 0 & \text{if } \Delta_0 \text{ and } \Delta_1 \text{ are even,} \\ e^{2(n-n_0-n_1)}\hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{n_1} & \text{if } \Delta_0 \text{ or } \Delta_1 \text{ is odd.} \end{cases}$$

When Δ_0 or Δ_1 is odd, $n_0 \le p$ and $n_1 \le q$, with at least one of the inequalities being strict, so $\hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{n_1}$ is a basis element and $e^{2(n-n_0-n_1)}\hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{n_1} \in \widetilde{T}$. It follows that $e(F) \in \widetilde{T}$, and then the fact that $e(F)$ is determined by its grading and C_2–degrees follows from Corollary 2.8. □

By Lemma 2.5, the Euler class $e(F)$ can be written as a linear combination of just three basis elements. We next work out the explicit expression, which, by Theorem 2.11, is determined by the grading of $e(F)$ and its C_2–degrees.

Theorem 2.12 (Bézout theorem, part II) *Let F be as in Bézout context 0.1. Then we can write*

$$e(F) = \alpha P_n^{(m)} + \beta P_k^{(m)} + \gamma P_{k-1}^{(m)}$$

for some $1 \le k < p+q$ and some coefficients α, β, and γ in $\mathbb{H}$, so the coefficient vector of $e(F)$ has at most three nonzero components. Allowing for the possibility that $n=k$ or $n=k-1$, we can arrange that the coefficient α is always an integer multiple of $\tau(\iota^{2i})$ for some $i \in \mathbb{Z}$, and the coefficients β and γ are always integer multiples of e^{2i} or $e^{-2i}\kappa$ for some $i \ge 0$.

Use the briefer notation P_n and write $\epsilon = 0$ or 1 for the remainder on dividing $n+n_0+n_1$ by 2. We have

$$P_n = \begin{cases} \zeta_0^{-(n+n_0-n_1-2p)}\hat{c}_\omega^{p}\hat{c}_{\chi\omega}^{n-p} & \text{if } n+n_0-n_1 > 2p, \\ \zeta_1^{-(n-n_0+n_1-2q)}\hat{c}_\omega^{n-q}\hat{c}_{\chi\omega}^{q} & \text{if } n-n_0+n_1 > 2q, \\ \zeta_0^{\epsilon}\hat{c}_\omega^{(n+n_0-n_1+\epsilon)/2}\hat{c}_{\chi\omega}^{(n-n_0+n_1-\epsilon)/2} & \text{otherwise,} \end{cases}$$

$$P_k = \begin{cases} \zeta_0\hat{c}_\omega^{n_0+1}\hat{c}_{\chi\omega}^{n_1} & \text{if } n_0 < p, \\ \zeta_0^{-(n_0-p)}\hat{c}_\omega^{p}\hat{c}_{\chi\omega}^{n_1} & \text{if } n_0 \ge p, \end{cases} \quad \text{and} \quad P_{k-1} = \begin{cases} \hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{n_1} & \text{if } n_1 < q, \\ \zeta_1^{-(n_1-q)}\hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{q} & \text{if } n_1 \ge q. \end{cases}$$

The coefficient α will be an integer multiple of

$$\tau_n = \begin{cases} \tau(\iota^{2(n-n_1-p)}) & \text{if } n+n_0-n_1 > 2p, \\ \tau(\iota^{2(n-n_0-q)}) & \text{if } n-n_0+n_1 > 2q, \\ \tau(\iota^{n-n_0-n_1-\epsilon}) & \text{otherwise.} \end{cases}$$

Finally, write $\bar{n}_0 = \min\{n_0, p-1\}$ and $\bar{n}_1 = \min\{n_1, q\}$. Then we break the result into the following cases:

(1) *If Δ is even, then*

$$\alpha = \tfrac{1}{2}\Delta\tau_n, \quad \beta = \tfrac{1}{2}(\Delta_1-\Delta_0)e^{-2(\bar{n}_0+\bar{n}_1-n+1)}\kappa, \quad \gamma = \tfrac{1}{2}(\Delta_0)e^{-2(\bar{n}_0+\bar{n}_1-n)}\kappa, \quad k = \bar{n}_0+\bar{n}_1+1.$$

(2) *If Δ is odd and $\Delta_0 \neq 0$, then*

$$\alpha = \tfrac{1}{2}(\Delta-\Delta_0)\tau(1), \quad \beta = \tfrac{1}{2}(\Delta_1-\Delta_0)e^{-2}\kappa, \quad \gamma = \Delta_0, \quad k = n+1.$$

(3) *If Δ is odd and $\Delta_0 = 0$, then*

$$\alpha = \tfrac{1}{2}(\Delta-\Delta_1)\tau(1), \quad \beta = 0, \quad \gamma = \Delta_1, \quad k = n+1.$$

Remark 2.13 We should point out some abuses of notation we are indulging in. The formulas for P_k and P_{k-1} evaluate to 0, not basis elements, when both $n_0 \geq p$ and $n_1 \geq q$. In the case $n_0 < p-1$ and $n_1 \geq q$, the formula for P_k is not a basis element, but we know that its coefficient will be a multiple of $e^m\kappa$ for some integer m, and the product $e^m\kappa P_k = 0$ in that case because of the relations in the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$. A similar vanishing happens in the case of P_{k-1} when $n_0 > p$ and $n_1 < q$. Finally, the formulas for P_k and P_{k-1} coincide when $n_0 = p$ and $n_1 < q$, but in that case $\Delta_0 = 0$ so only one copy of this basis element appears in the formula for $e(F)$.

Proof Theorem 2.11 and Lemma 2.5 imply the first claim, that we can write $e(F)$ in terms of just three basis elements.

To determine P_n, P_k, and P_{k-1}, we recall from [3, Proposition 4.7] that the basis elements take one of the six possible forms

$$\begin{array}{llllll} \zeta_1^m \hat{c}_\omega^a & \text{for } m>1, a<p, & \zeta_0^m \hat{c}_{\chi\omega}^b & \text{for } m>1, b<q, & \hat{c}_\omega^a \hat{c}_{\chi\omega}^b & \text{for } a\leq p, b\leq q, \\ \zeta_0 \hat{c}_\omega^a \hat{c}_{\chi\omega}^b & \text{for } a\leq p, b<q, & \zeta_0^{-m} \hat{c}_\omega^p \hat{c}_{\chi\omega}^b & \text{for } m>0, b<q, & \zeta_1^{-m} \hat{c}_\omega^a \hat{c}_{\chi\omega}^q & \text{for } m>0, a<p, \end{array}$$

where we recall that $\hat{c}_\omega^p \hat{c}_{\chi\omega}^q = 0$, so we do not have $a = p$ and $b = q$ above.

We noted earlier that $e(F)$ lies in grading

$$\operatorname{grad} e(F) = (n_0 - n_1)(\omega - 2) + 2n_0 + 2(n - n_0)\sigma.$$

P_n is the unique basis element having grading in $(n_0 - n_1)(\omega - 2) + RO(C_2)$ restricting to $\hat{c}^n$, and we can check that the formula given in the statement of the theorem has those properties. Similarly, P_k and P_{k-1} are the (at most) two basis elements having gradings of the form $(n_0 - n_1)(\omega - 2) + 2n_0 + 2b\sigma$, and we can check that the formulas given have that property. The coefficient τ_n is the element of the form $\tau(\iota^{2i})$ such that $\tau_n P_n$ lies in the same grading as $e(F)$. The terms of the form $e^m\kappa$ multiplying P_k and P_{k-1} in the formulas for $e(F)$ are determined similarly.

To verify the coefficients of P_n, P_k, and P_{k-1}, we use the fact that $e(F)$ is determined by the nonequivariant elements

$$\rho(e(F)) = \Delta \hat{c}^n \quad \text{and} \quad e(F)^{C_2} = (\Delta_0 \hat{c}^{n_0}, \Delta_1 \hat{c}^{n_1}),$$

so we simply need to check that the formulas of the theorem have the correct values on applying these restriction maps.

First note that, regardless of which case we fall in, we will always have

$$\rho(\tau_n P_n) = 2\hat{c}^n \quad \text{and} \quad (\tau_n P_n)^{C_2} = (0, 0).$$

For P_k and P_{k-1} we have

$$\rho(P_k) = \hat{c}^k, \quad \rho(P_{k-1}) = \hat{c}^{k-1}, \quad P_k^{C_2} = (0, \hat{c}^{n_1}) \quad \text{and} \quad P_{k-1}^{C_2} = (\hat{c}^{n_0}, \hat{c}^{n_1}),$$

which includes the possibility that $P_{k-1}^{C_2} = (\hat{c}^{n_0}, 0)$ if $n_1 \geq q$.

Now, when Δ is even, in the formulas given, β and γ each have a factor of the form $e^m\kappa$, and $\rho(e^m\kappa)=0$ and $(e^m\kappa)^{C_2}=2$. Combined with the formulas above, this verifies case (1) of the theorem, except that we should say something about the parities of Δ_0 and Δ_1. From the discussion before Theorem 2.11, because Δ is even, Δ_0 and Δ_1 have the same parity. There is a possibility that Δ_0 is odd, but this can happen only when $n_{\mathrm{II}}=0$ and $n_{\mathrm{IV}}>0$, in which case $n_0<p$, $n_1<q$, and $n_0+n_1<n$. The coefficient γ in that case is

$$\gamma=\tfrac{1}{2}\Delta_0 e^{-2(n_0+n_1-n)}\kappa=\tfrac{1}{2}(\Delta_0)2e^{2(n-n_0-n_1)},$$

which we interpret as $\Delta_0 e^{2(n-n_0-n_1)}$ by another abuse of notation. (The abuse is that division by 2 is not well defined in $\mathbb{H}$.) We then use that $\rho(e^m)=0$ and $(e^m)^{C_2}=1$ for $m>0$.

If Δ is odd, then $n=n_0+n_1$, $n_0\le p$, and $n_1\le q$. If Δ_0 and Δ_1 are both nonzero, then $n_0<p$ and $n_1<q$, $P_n=P_{k-1}$, and the formula in case (2) of the theorem is easily verified.

If $\Delta_0\neq 0$ but $\Delta_1=0$, then $n_0<p$ and $n_1=q$. In this case,

$$e^{-2}\kappa P_k=e^{-2}\kappa\zeta_0\hat{c}_\omega^{n_0+1}\hat{c}_{\chi\omega}^q=0,$$

so we allow the abuse of notation that $\Delta_1-\Delta_0$ is odd in the formula for β. With that caveat, the verification of case (2) can be completed.

In case (3), since $\Delta_0=0$ we must have $\Delta_1\neq 0$ and odd. The verification is then just as for the previous cases.

The asymmetry in these formulas comes from an asymmetry in our preferred basis regarding $\hat{c}_\omega$ vs $\hat{c}_{\chi\omega}$. □

Remark 2.14 Theorems 2.11 and 2.12 give us two related ways of determining $e(F)$: by the ranks (n,n_0,n_1) and the C_2–degrees $(\Delta,\Delta_0,\Delta_1)$, and also by its triple of nonzero coefficients. The advantage of using the degrees is that they are multiplicative. This is simpler to calculate with, and also parallels the result of the nonequivariant Bézout theorem that degrees are multiplicative under intersection of projective varieties.

Remark 2.15 The summary of Theorem 2.12 is that $e(F)$ can be expressed in terms of at most three basis elements. This is not a restriction imposed by the locations of the basis elements. As an example, consider $\mathbb{P}(\mathbb{C}^{5+5\sigma})$ and the bundle $F=4\chi O(2)$, the sum of four copies of $\chi O(2)$, so $n=4$ and $n_0=n_1=0$. This Euler class lives in grading

$$(n_0-n_1)(\omega-2)+2n_0+2(n-n_0)\sigma=8\sigma.$$

Figure 5 shows the location of $e(F)$, the "×" at 8σ, and the locations of the basis elements in the $RO(C_2)$–grading. The five basis elements within the shaded area have nonzero multiples in degree 8σ, so could conceivably contribute to $e(F)$, but the theorem says that it can be written in terms of just three of them: P_4, the one on the same diagonal as $e(F)$, and the two below it, P_0 and P_1.

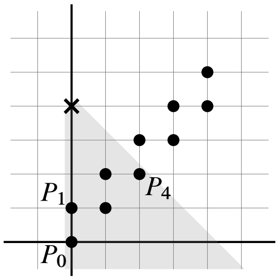


Figure 5: Location of $e(4\chi O(2))$.

In fact, we are in Theorem 2.12(1), with $\Delta = 8$ and $\Delta_0 = \Delta_1 = 1$, so

$$e(4\chi O(2)) = 8\tau(\iota^4)P_4 + 0P_1 + e^8 P_0 = 8\tau(\iota^4)\hat{c}_\omega^2\hat{c}_{\chi\omega}^2 + e^8.$$

As it happens, P_1 does not actually contribute in this example.

Remark 2.16 In [3], we looked in detail at the case $n = p+q-1$, where the hypersurfaces associated with the line bundle summands of F intersect generically in a C_2–set of points in $\mathbb{P}(\mathbb{C}^{p+q\sigma})$. In that case, we showed that the explicit formula for $e(F)$ can be read as telling us how that collection of points breaks down as free orbits versus fixed points in each of the components of $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}$. In a followup to this paper, we will show how the Euler class more generally gives us geometric information about the intersection of hypersurfaces.

3 Comparison with constant $\mathbb{Z}$ coefficients

Another equivariant cohomology theory commonly used is ordinary cohomology with coefficients in $\underline{\mathbb{Z}}$, the constant-$\mathbb{Z}$ Mackey functor. We calculate the Euler class $e(F)$ with $\underline{\mathbb{Z}}$ coefficients and compare it to the class obtained with Burnside ring coefficients.

As shown in [2], $H_{C_2}^{\mathrm{RO}(C_2)}(S^0;\underline{\mathbb{Z}})$ is obtained from $\mathbb{H}$ by setting $\kappa = 0$. This has the effect of removing the elements $e^{-n}\kappa$ and making $2e = 0$. Since $\kappa = 2-g$, it also has the effect of setting $g = 2$. Put another way, this theory cannot distinguish between a free orbit and two fixed points.

Because the cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ with $\underline{A}$ coefficients is free over the cohomology of a point, we obtain the cohomology with $\underline{\mathbb{Z}}$ coefficients by setting $\kappa = 0$. The result is the following:

Theorem 3.1 [3, Corollary 5.4] *Let $0 \le p, q < \infty$ with $p+q > 0$. Then $H_{C_2}^{\mathrm{RO}(\Pi B)}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+;\underline{\mathbb{Z}})$ is a free module over $H_{C_2}^{\mathrm{RO}(C_2)}(S^0;\underline{\mathbb{Z}})$. Its structure as a graded commutative algebra over $H_{C_2}^{\mathrm{RO}(C_2)}(S^0;\underline{\mathbb{Z}})$ is described as in Theorem 1.2, except that the relation $\zeta_1\hat{c}_{\chi\omega} = (1-\kappa)\zeta_0\hat{c}_\omega + e^2$ is replaced by the relation*

$$\zeta_1\hat{c}_{\chi\omega} = \zeta_0\hat{c}_\omega + e^2. \qquad \square$$

Setting $\kappa = 0$ in Theorem 2.12, remembering that e^m is 2–torsion, and paying attention to the abuses of notation mentioned in the proof of that theorem, we get the following:

Theorem 3.2 (Bézout's theorem for constant $\mathbb{Z}$ coefficients) *Let F be as in Bézout context 0.1. Then the Euler class $e_{\mathbb{Z}}(F) \in H^{\mathrm{RO}(\Pi B)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+;\underline{\mathbb{Z}})$ is given by*

$$e_{\mathbb{Z}}(F) = \begin{cases} \frac{1}{2}\Delta\tau_n P_n^{(m)} & \text{if } \Delta,\ \Delta_0 \text{ and } \Delta_1 \text{ are even,} \\ \frac{1}{2}\Delta\tau_n P_n^{(m)} + e^{2(n-n_0-n_1)} P_{k-1}^{(m)} & \text{if } \Delta \text{ is even and } \Delta_0 \text{ or } \Delta_1 \text{ is odd,} \\ \Delta P_n^{(m)} & \text{if } \Delta \text{ is odd,} \end{cases}$$

where, writing $\epsilon = 0$ or 1 for the remainder on dividing $n+n_0+n_1$ by 2, we set

$$P_n^{(m)} = \begin{cases} \zeta_0^{-(n+n_0-n_1-2p)}\hat{c}_\omega^{p}\hat{c}_{\chi\omega}^{n-p} & \text{if } n+n_0-n_1 > 2p, \\ \zeta_1^{-(n-n_0+n_1-2q)}\hat{c}_\omega^{n-q}\hat{c}_{\chi\omega}^{q} & \text{if } n-n_0+n_1 > 2q, \\ \zeta_0^{\epsilon}\hat{c}_\omega^{(n+n_0-n_1+\epsilon)/2}\hat{c}_{\chi\omega}^{(n-n_0+n_1-\epsilon)/2} & \text{otherwise,} \end{cases}$$

$$\tau_n = \begin{cases} \tau(\iota^{2(n-n_1-p)}) & \text{if } n+n_0-n_1 > 2p, \\ \tau(\iota^{2(n-n_0-q)}) & \text{if } n-n_0+n_1 > 2q, \\ \tau(\iota^{n-n_0-n_1-\epsilon}) & \text{otherwise,} \end{cases}$$

and, when Δ is even and Δ_0 or Δ_1 is odd,

$$P_{k-1}^{(m)} = \hat{c}_\omega^{n_0}\hat{c}_{\chi\omega}^{n_1}. \qquad \square$$

While this result has the benefit of relative simplicity, it carries significantly less information than Theorem 2.12. In particular, we cannot reconstruct Δ_0 and Δ_1 from $e_{\mathbb{Z}}(F)$. This follows from the formula in the theorem, but we can also look again at the fixed-point map $(-)^{C_2}$ to see why this must happen. As defined in [4], the fixed-point map takes G–equivariant cohomology with coefficients in a Mackey functor $\underline{T}$ to nonequivariant cohomology with coefficients in $\underline{T}^G$. In the case of the group C_2, we have

$$\underline{T}^{C_2} = \underline{T}(C_2/C_2)/\tau(\underline{T}(C_2/e)).$$

This gives $\underline{A}^{C_2} = \mathbb{Z}$, but $\underline{\mathbb{Z}}^{C_2} = \mathbb{Z}/2$. We then get the following:

Corollary 3.3 *With F as in Bézout context 0.1, we have*

$$e_{\mathbb{Z}}(F)^{C_2} = (\Delta_0\hat{c}^{n_0}, \Delta_1\hat{c}^{n_1}) \in H^{2a}(\mathbb{P}(\mathbb{C}^p)_+;\mathbb{Z}/2) \oplus H^{2(a-m)}(\mathbb{P}(\mathbb{C}^{q\sigma})_+;\mathbb{Z}/2),$$

so

$$e_{\mathbb{Z}}(F)^{C_2} = \begin{cases} (0,0) & \text{if } \Delta_0 \text{ and } \Delta_1 \text{ are even,} \\ (\hat{c}^{n_0}, \hat{c}^{n_1}) & \text{if } \Delta_0 \text{ or } \Delta_1 \text{ is odd.} \end{cases}$$

Proof From the commutativity of the diagram

$$\begin{tikzcd} H^{\mathrm{RO}(\Pi B)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+;\underline{A}) \arrow[r] \arrow[d, "(-)^{C_2}"'] & H^{\mathrm{RO}(\Pi B)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+;\underline{\mathbb{Z}}) \arrow[d, "(-)^{C_2}"] \\ H^{\mathbb{Z}}(\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}_+;\mathbb{Z}) \arrow[r] & H^{\mathbb{Z}}(\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}_+;\mathbb{Z}/2) \end{tikzcd}$$

where the horizontal arrows are given by change of coefficients, $e_{\mathbb{Z}}(F)^{C_2}$ is just the reduction of $e(F)^{C_2}$ modulo 2. $\square$

Thus, from this Euler class we cannot recover Δ_0 and Δ_1, only their parities. This goes back to the fact that, because $g = 2$, cohomology with $\underline{\mathbb{Z}}$ coefficients cannot distinguish between a free orbit and two fixed points, and hence retains only parity information about fixed points.

For example, in the case $n = p + q - 1$ discussed in detail in [3], we can think of the Euler class in terms of the finite C_2–set given by the zero locus of a section of F, or the intersection of the hypersurfaces given by the zero loci of sections of the line bundles making up F. The Euler class with Burnside ring coefficients completely determines this C_2–set, including how many fixed points lie in each component of $\mathbb{P}(\mathbb{C}^{p+q\sigma})^{C_2}$. The Euler class with constant $\mathbb{Z}$ coefficients can tell us only the parity of the number of fixed points in each component.

4 Comparison with Borel cohomology

Borel cohomology was the first theory thought of as equivariant ordinary cohomology, but is a considerably weaker theory than Bredon cohomology. (See, for example, May's discussion in [7].) There is a map from ordinary cohomology with $\underline{\mathbb{Z}}$ coefficients to Borel cohomology, so the latter is also weaker than cohomology with $\underline{\mathbb{Z}}$ coefficients. To see how much weaker, let us look at the calculation of $e(F)$ in Borel cohomology.

We take Borel cohomology to be the $\mathrm{RO}(C_2)$–graded theory defined on based C_2–spaces by

$$BH_{C_2}^{\mathrm{RO}(C_2)}(X) = H_{C_2}^{\mathrm{RO}(C_2)}((EC_2)_+ \wedge X),$$

where, as usual, we use Burnside ring coefficients on the right, but suppress them from the notation. (Because EC_2 is free, and $\underline{A} \to \underline{\mathbb{Z}}$ is an isomorphism at the C_2/e level, we could instead use $\underline{\mathbb{Z}}$ coefficients and get naturally isomorphic results.) This is the usual Borel cohomology with $\mathbb{Z}$ coefficients, but we have expanded the grading from the common $\mathbb{Z}$ to $\mathrm{RO}(C_2)$. As shown in [2], the Borel cohomology of a point is $\mathbb{H}$ with ξ inverted:

$$BH_{C_2}^{\mathrm{RO}(C_2)}(S^0) \cong \mathbb{Z}[e, \xi, \xi^{-1}]/\langle 2e \rangle.$$

Here $\deg e = \sigma$ and $\deg \xi = 2\sigma - 2$, as before. In the map $\mathbb{H} \to BH_{C_2}^{\mathrm{RO}(C_2)}(S^0)$, κ goes to 0. As with cohomology with $\underline{\mathbb{Z}}$ coefficients, Borel cohomology cannot tell the difference between g and 2.

Note that, if we restrict to the $\mathbb{Z}$ grading, as is usually done, we get a polynomial algebra in $e^2\xi^{-1}$ modulo $2e^2\xi^{-1} = 0$, a copy of the group cohomology of C_2 with $\mathbb{Z}$ coefficients. If we restrict the grading to $\sigma + \mathbb{Z}$, we see the group cohomology of C_2 with twisted $\mathbb{Z}$ coefficients. That the twisted and untwisted cohomologies can be combined in a single algebra like this seems to have been first observed by Čadek [1].

Because the ordinary C_2–cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ is free over the cohomology of a point, we obtain its Borel cohomology also by inverting ξ. On doing so, the elements ζ_0 and ζ_1 become invertible, with the result that, if we continued to grade on $\mathrm{RO}(\Pi B)$, the groups outside the $\mathrm{RO}(C_2)$ grading would all be isomorphic to groups in the $\mathrm{RO}(C_2)$ grading via multiplication by an appropriate power of, say, ζ_0.

So we lose nothing by considering the $RO(C_2)$–graded part only. To give the explicit result, let $\hat{c}$ be the image of $\zeta_0\hat{c}_\omega$ in $BH^{2\sigma}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$. The following is then a corollary of Theorem 1.2:

Corollary 4.1 *Let $0 \le p, q < \infty$ with $p+q > 0$. Then $BH^{RO(C_2)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ is a free module over $BH^{RO(C_2)}_{C_2}(S^0)$, and as a (graded) commutative algebra over $BH^{RO(C_2)}_{C_2}(S^0)$, $BH^{RO(\Pi B)}_{C_2}(\mathbb{P}(\mathbb{C}^{p+q\sigma})_+)$ is generated by $\hat{c}$ in degree 2σ, which satisfies the single relation*

$$\hat{c}^p(\hat{c}+e^2)^q = 0. \qquad \square$$

Of course, we could also use as a generator the element $c' = \xi^{-1}\hat{c}$ in degree 2, but the relation is then

$$(c')^p(c'+e^2\xi^{-1})^q = 0.$$

For the simplicity of the relation, and to keep the generator more closely related to an element from ordinary cohomology, we prefer to use $\hat{c}$.

We view $\hat{c}$ as the Euler class of $\omega^\vee$. The Euler class of $\chi\omega^\vee$ is then $\hat{c}+e^2$, the image of $\zeta_1\hat{c}_{\chi\omega}$. In doing this, we are choosing to say that ω is a rank-2σ bundle over $EC_2 \times \mathbb{P}(\mathbb{C}^{p+q\sigma})$. Because EC_2 is free, we are as free to say ω has rank 2σ as to say it has rank 2.

Another way of seeing that $e(\chi\omega) = \hat{c}+e^2$ is to recall that $\chi\omega = \omega \otimes_{\mathbb{C}} \mathbb{C}^\sigma$, then use the additive formal group law of nonequivariant ordinary cohomology and the fact that $e(\mathbb{C}^\sigma) = e^2$.

Now consider the Euler classes of the bundles $O(d)$ and $\chi O(d)$, all of which we will think of as having rank 2σ. As a corollary of [3, Proposition 6.5], or as a consequence of the formal group law for nonequivariant cohomology, we have the following:

Proposition 4.2 *In the Borel cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ we have*

$$e(O(d)) = d\hat{c} \quad \text{and} \quad e(\chi O(d)) = d\hat{c}+e^2$$

for every $d \in \mathbb{Z}$. $\square$

Theorem 4.3 (Bézout's theorem for Borel cohomology) *Let F be as in Bézout context 0.1. The Euler class of F in the Borel cohomology of $\mathbb{P}(\mathbb{C}^{p+q\sigma})$ is*

$$e_{BH}(F) = \begin{cases} \Delta\hat{c}^n & \text{if } \Delta, \Delta_0 \text{ and } \Delta_1 \text{ are even,} \\ \Delta\hat{c}^n + e^{2(n-n_0-n_1)}\hat{c}^{n_0}(\hat{c}+e^2)^{n_1} & \text{if } \Delta \text{ is even and } \Delta_0 \text{ or } \Delta_1 \text{ is odd,} \\ \Delta\hat{c}^{n_0}(\hat{c}+e^2)^{n-n_0} & \text{if } \Delta \text{ is odd.} \end{cases}$$

Proof These formulas can be derived from the preceding proposition or from Theorem 3.2, using the fact that $\tau(1) = 2$ in Borel cohomology. $\square$

As we saw with ordinary cohomology with $\underline{\mathbb{Z}}$ coefficients, the Euler class in Borel cohomology contains significantly less information than the one in ordinary cohomology with Burnside ring coefficients. The fixed-point map would be

$$(-)^{C_2}\colon H^{RO(C_2)}_{C_2}((EC_2)_+ \wedge \mathbb{P}(\mathbb{C}^{p+q\sigma})_+) \to H^{\mathbb{Z}}(((EC_2)_+ \wedge \mathbb{P}(\mathbb{C}^{p+q\sigma})_+)^{C_2};\mathbb{Z}) = H^{\mathbb{Z}}(*;\mathbb{Z}) = 0.$$

Thus, Borel cohomology contains no information at all about fixed points.

References

[1] **M Čadek**, *The cohomology of* $\mathrm{BO}(n)$ *with twisted integer coefficients*, J. Math. Kyoto Univ. 39 (1999) 277–286 MR Zbl

[2] **S R Costenoble**, *The* $RO(\Pi B)$*–graded* C_2*–equivariant ordinary cohomology of* $B_{C_2}U(1)$, Topology Appl. 338 (2023) art. id. 108660 MR Zbl

[3] **S R Costenoble**, **T Hudson**, **S Tilson**, *The* C_2*–equivariant cohomology of complex projective spaces*, Adv. Math. 398 (2022) art. id. 108245 MR Zbl

[4] **S R Costenoble**, **S Waner**, *Equivariant ordinary homology and cohomology*, Lecture Notes in Math. 2178, Springer (2016) MR Zbl

[5] **W Fulton**, *Intersection theory*, Ergebnisse der Math. 2, Springer (1984) MR Zbl

[6] **L G Lewis, Jr**, *The* $\mathrm{RO}(G)$*–graded equivariant ordinary cohomology of complex projective spaces with linear* $\mathbb{Z}/p$ *actions*, from "Algebraic topology and transformation groups" (T tom Dieck, editor), Lecture Notes in Math. 1361, Springer (1988) 53–122 MR Zbl

[7] **J P May**, *Characteristic classes in Borel cohomology*, J. Pure Appl. Algebra 44 (1987) 287–289 MR Zbl

Department of Mathematics, Hofstra University
Hempstead, NY, United States

College of Transdisciplinary Studies, DGIST
Daegu, South Korea

Hörstel, Germany

steven.r.costenoble@hofstra.edu, hudson@dgist.ac.kr, sean.tilson@gmail.com

Received: 9 November 2022 Revised: 21 March 2023

Algebraic & Geometric Topology **24**:4 (2024) 2351–2365
DOI: 10.2140/agt.2024.24.2351
Published: 16 July 2024

Topologically isotopic and smoothly inequivalent 2–spheres in simply connected 4–manifolds whose complement has a prescribed fundamental group

RAFAEL TORRES

We describe a procedure to construct infinite sets of pairwise smoothly inequivalent 2–spheres in simply connected 4–manifolds, which are topologically isotopic and whose complement has a prescribed fundamental group that satisfies some conditions. This class of groups include cyclic groups and the binary icosahedral group. These are the first known examples of such exotic embeddings of 2–spheres in 4–manifolds. Examples of locally flat embedded 2–spheres in a nonsmoothable 4–manifold whose complements are homotopy equivalent to smoothly embedded ones are also given.

57K45, 57R55; 57R40, 57R52

1 Main results

The first main result of this note is the following theorem.

Theorem A *Fix $p \geq 2$. There is an infinite set*

$$\{S_{n,p} : n \in \mathbb{Z}\}$$

of smoothly embedded 2–spheres in $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ that satisfies the following properties:

- *Any two elements are topologically isotopic.*
- *There is a diffeomorphism of pairs*

$$(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_1,p}) \to (2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_2,p})$$

  if and only if $n_1 = n_2$.
- *The fundamental group of the complement is*

$$\pi_1(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 \setminus \nu(S_{n,p})) = \mathbb{Z}/p$$

  for every $n \in \mathbb{Z}$.
- *$[S_{n,p}] \neq 0 \in H_2(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2; \mathbb{Z})$ for every $n \in \mathbb{Z}$.*
- *Surgery along each of these 2–spheres yields an infinite set of pairwise homeomorphic and pairwise nondiffeomorphic closed smooth 4–manifolds with fundamental group $\mathbb{Z}/p$.*

Theorem A provides the first known example of an infinite set of 2–spheres smoothly embedded in a simply connected 4–manifold that are pairwise topologically isotopic, pairwise smoothly inequivalent and having a complement with finite cyclic fundamental group. Schwartz [2019, Theorem 2] pointed out the existence of closed simply connected 4–manifolds containing pairs of smoothly embedded 2–spheres that are both smoothly equivalent and topologically isotopic, but not smoothly isotopic. Examples of these exotic embeddings of 2–spheres in closed 4–manifolds with simply connected complement have been constructed by Akbulut [2015] and Auckly, Kim, Melvin and Ruberman [Auckly et al. 2015]. Exotic embeddings of surfaces with positive genus in simply connected 4–manifolds and complement having nontrivial fundamental group were found by Kim [2006] and Kim and Ruberman [2008]. An ingredient in the proof of Theorem A is of independent interest: we point out in Theorem 1 that constructions of inequivalent smooth structures on simply connected 4–manifolds of Fintushel and Stern [2011; 2012] can be extended to produce such structures on 4–manifolds with nontrivial fundamental group too.

The second main result provides a construction procedure for topologically equivalent yet smoothly inequivalent homologically essential 2–spheres whose complement can be chosen to have the same fundamental group as a wide range of $\mathbb{Q}$–homology 4–spheres. We work with the modified Seiberg–Witten SW'_X invariant of a closed 4–manifold X as defined, for example, in [Fintushel et al. 2007, Section 2], and denote by $\mathcal{B}_X$ the set of basic classes.

Theorem B *Let $\{Z_n : n \in \mathbb{Z}\}$ be an infinite set of closed smooth simply connected 4–manifolds with pairwise different integer invariants*

$$S_n = \max\{|\mathrm{SW}'_{Z_n}(k_{Z_n})| : k_{Z_n} \in \mathcal{B}_{Z_n}\}, \tag{1-1}$$

which are pairwise homeomorphic to a given closed 4–manifold Z and such that the connected sum $Z_n \# S^2 \times S^2$ is diffeomorphic to $Z \# S^2 \times S^2$ for every $n \in \mathbb{Z}$. Let M be a closed smooth 4–manifold with $H_*(M;\mathbb{Q}) \cong H_*(S^4;\mathbb{Q})$ and set $\pi := \pi_1 M$. Suppose that there is a loop $\alpha \subset M$ and a choice of framing such that

$$S^2 \times S^2 = M \setminus \nu(\alpha) \cup D^2 \times S^2. \tag{1-2}$$

There is an infinite set

$$\{S_{n,\pi} : n \in \mathbb{Z}\}$$

of smoothly embedded 2–spheres in $Z \# S^2 \times S^2$ that satisfies the following properties.

- *There is a homeomorphism of pairs*
$$(Z \# S^2 \times S^2, S_{n_1,\pi}) \to (Z \# S^2 \times S^2, S_{n_2,\pi})$$
for every $n_i \in \mathbb{Z}$.
- *There is a diffeomorphism of pairs*
$$(Z \# S^2 \times S^2, S_{n_1,\pi}) \to (Z \# S^2 \times S^2, S_{n_2,\pi})$$
if and only if $n_1 = n_2$.

- *The fundamental group of the complement is*
$$\pi_1(Z \# S^2 \times S^2 \setminus \nu(S_{n,\pi})) = \pi$$
and its homology class satisfies
$$[S_{n,\pi}] \neq 0 \in H_2(Z \# S^2 \times S^2; \mathbb{Z})$$
for every $n \in \mathbb{Z}$.
- *Surgery along each of these 2–spheres yields an infinite set* $\{Z_n \# M : n \in \mathbb{Z}\}$ *of pairwise nondiffeomorphic closed smooth 4–manifolds with fundamental group* π *that are pairwise homeomorphic to the connected sum* $Z \# M$.

See [Fintushel et al. 2007, Proof of Theorem 1] for details on the definition of the invariant (1-1). Fintushel and Stern [2011; 2012] constructed infinite sets as in the hypothesis of Theorem B for $Z = \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ for $2 \leq k \leq 7$. Baykur and Sunukian [2013] showed that Fintushel and Stern's examples become diffeomorphic after a connected sum with a single copy of $S^2 \times S^2$. Examples of $\mathbb{Q}$–homology 4–spheres M that satisfy the hypothesis are spun 4–manifolds with the fundamental group of any lens space and the Poincaré homology 3–sphere. A similar result holds if (1-2) is substituted for the nontrivial bundle $S^2 \tilde{\times} S^2$. It is possible to strengthen the conclusion of Theorem B to topologically isotopic 2–spheres, although we do not pursue this endeavor here; see Sunukjian [2015].

A contribution of this note is to point out the simplicity of the proofs of Theorems A and B. The reader will notice that the 4–manifolds in the last clause of Theorem B are smoothly reducible (see [Gompf and Stipsicz 1999, Definition 10.1.17]), while those in the last clause of Theorem A are not. We explain in Remark 10 how an instance of Theorem B implies the claims on the existence of the homeomorphism of pairs and the nonexistence of the diffeomorphism of pairs of Theorem A. An independent proof of Theorem A is given in Section 2.7 as well. The following consequence of Theorem B is another contribution.

Corollary C *Let* G *be a finite cyclic group or the icosahedral group*
$$G = \langle g_1, g_2 : {g_1}^5 = (g_1 g_2)^2 = {g_2}^3 \rangle.$$
There is an infinite set of smoothly embedded 2–spheres in $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ *that are pairwise topologically equivalent, yet pairwise smoothly inequivalent, and the fundamental group of the complement is* G.

These are the first examples of exotic embeddings of 2–spheres in simply connected 4–manifolds whose complement has a fundamental group isomorphic to the binary icosahedral group among several other choices of groups. We exhibit interesting smooth embeddings of nullhomotopic 2–spheres in the fourth main result of this note.

Theorem D *There is an infinite set*

$$\{S_n : n \in \mathbb{Z}\} \tag{1-3}$$

of 2–spheres smoothly embedded in $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ that satisfies the following properties.

- *The fundamental group of the complement of an element in (1-3) is*
$$\pi_1(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 \setminus \nu(S_n)) = \mathbb{Z}$$
and $[S_n] = 0 \in \pi_2(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2)$ for every $n \in \mathbb{Z}$.
- *There is a diffeomorphism of pairs*
$$(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_1}) \to (2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_2})$$
if and only if $n_1 = n_2$.

Notice that elements in (1-3) do not bound a smoothly embedded 3–ball in $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$. The smoothly inequivalent embeddings of homotopically trivial 2–spheres of Theorem D are related to a construction of an infinite set of closed smooth 4–manifolds with infinite cyclic fundamental group and the homology of the connected sum $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 \# S^1 \times S^3$, which is given in Theorem 2.

While any 2–sphere in a closed simply connected 4–manifold can be assumed to be regularly immersed, Hambleton and Kreck [1993b] and Lee and Wilczyński [1990; 1997] completely characterized when a homology class of nonzero divisibility can be represented by a locally flat embedded 2–sphere. The fifth and last result to be mentioned in this introduction records the existence of a myriad of explicit examples of locally flat embedded 2–spheres in closed simply connected 4–manifolds whose exteriors are homotopy equivalent but not homeomorphic.

Theorem E *For every $p \geq 2$, there is a locally flat embedded 2–sphere*

$$S_p \subset *\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \tag{1-4}$$

whose complement has finite cyclic group $\mathbb{Z}/p$, and it is homotopy equivalent to the complement of a smoothly embedded 2–sphere

$$S'_p \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2. \tag{1-5}$$

Theorem E is essentially derived from an existence result of nonsmoothable $\mathbb{Q}$–homology 4–spheres due to Hambleton and Kreck [1993a]. Other interesting examples were found by Kasprowski, Lambert-Cole, Land and Lecuona [Kasprowski et al. 2021].

Acknowledgements

Email exchanges with Aru Ray, Bob Gompf, Danny Ruberman and Ian Hambleton have increased our understanding on the subject. We are grateful to them for patiently sharing their knowledge. We thank the referees for their careful reading of the manuscript and for their input, which helped us improve the manuscript. Del tega dela je nastal na Rogu in Metelkovi.

2 Proofs

2.1 Infinitely many inequivalent smooth structures

Fintushel and Stern [2012, Theorem 1] showed that there is a nullhomologous 2–torus T smoothly embedded in $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ such that performing surgeries on T results in infinitely many inequivalent smooth structures on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$. We point out that changing the coefficients of the torus surgery on T introduces homotopically nontrivial loops to the resulting 4–manifold, and their procedure also yields infinitely many smooth structures on 4–manifolds with prescribed cyclic fundamental group. The latter will serve as raw material to construct the knotted 2–spheres.

We introduce terminology to state the result and follow the notation in [Fintushel and Stern 2012, Section 3]. Let $T \subset X$ be a smoothly embedded 2–torus with trivial tubular neighborhood $\nu(T) = T^2 \times D^2$. Let $\{a, b\}$ be loops in T that form a symplectic basis of $\pi_1 T = \mathbb{Z}^2$, and let $\{S^1_a, S^1_b\}$ be loops in $\partial\nu(T) = T^2 \times \partial D^2 = T^2 \times S^1$ that are homologous to a and b, respectively. The meridian of T is denoted by μ_T and it is any curve in the same isotopy class of the curve $\{x\} \times \partial D^2 \subset \partial\nu(T)$. The smooth 4–manifold

$$X_{T,S^1_b}(p/n) := (X \setminus \nu(T)) \cup_\varphi (T^2 \times D^2), \tag{2-1}$$

where the gluing diffeomorphism satisfies $\varphi_*([\partial D^2]) = n[S^1_b] + p[\mu_T]$, is said to be obtained by performing a p/n–torus surgery to X on T along the curve b.

We first consider the case of finite cyclic fundamental group and postpone the infinite cyclic case to the end of the section.

Theorem 1 *Fix $p \geq 2$. There is a smoothly embedded nullhomologous 2–torus $T \subset \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ and a nullhomologous curve in its complement $S^1_b \subset \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \setminus \nu(T)$ such that performing a p/n–torus surgery to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ on T along S^1_b yields an infinite set*

$$\{X_{T,S^1_b}(p/n) : n \in \mathbb{Z}\} \tag{2-2}$$

of pairwise nondiffeomorphic 4–manifolds such that every element is homeomorphic to the connected sum

$$\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# \Sigma_p, \tag{2-3}$$

where Σ_p is a $\mathbb{Q}$–homology 4–sphere with fundamental group $\pi_1\Sigma_p = \mathbb{Z}/p$.

Proof The only contribution in this note to the work of Fintushel and Stern [2011; 2012] that provides a proof of Theorem 1 is the change in a coefficient of the torus surgery. We then employ a homeomorphism criteria of Hambleton and Kreck to pin down the homeomorphism class of the closed 4–manifolds that are constructed this way. Set $X := \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ so to not overload the notation. Fintushel and Stern [2012, Theorem 1.1] showed the existence of a nullhomologous torus $T \subset X$ and the curve $b \subset T$

with framing $S^1_b \subset X \setminus \nu(T)$ as in the statement of Theorem 1. Build $X_{T,S^1}(p/n)$ as in (2-1). Since $[T] = 0 \in H_2(X;\mathbb{Z})$, we have that $H_1(X_{T,S^1_b}(p/n);\mathbb{Z}) = \mathbb{Z}/p$ for every $n \in \mathbb{Z}$; in this notation, $p = 0$ corresponds to $\mathbb{Z}$. We now fix $p \geq 2$.

To see that the fundamental group of $X_{T,S^1_b}(p/n)$ is $\mathbb{Z}/p$, we take a closer look at the constructions of Fintushel and Stern [2011; 2012], where six torus surgeries along six nullhomologous 2–tori $\{T_{1,i}, T_{2,i} : i = 1,2,3\}$ are performed to X to produce a symplectic 4–manifold Q with fundamental group $\pi_1(Q) = \mathbb{Z}^6$ and that contains six Lagrangian 2–tori $\{L_{1,i}, L_{2,i} : i = 1,2,3\}$ [Fintushel and Stern 2011, Proposition 7; 2012, page 77]. The complements of these 2–tori are the same, ie

$$X \setminus \bigcup_{i=1}^{3}(\nu(T_{1,i}) \sqcup \nu(T_{2,i})) = Q \setminus \bigcup_{i=1}^{3}(\nu(L_{1,i}) \sqcup \nu(L_{2,i})). \tag{2-4}$$

By applying six surgeries to the symplectic 4–manifold Q along the Lagrangian 2–tori with a given choice of surgery curves [Fintushel and Stern 2011, Theorem 2], one obtains an infinite set of inequivalent smooth structures on X. The first five surgeries are $|1/1|$–torus surgeries, while the last one is a $1/n$–torus surgery [Fintushel and Stern 2011, page 1685]. In particular, this infinite set can be obtained by performing torus surgeries to X on six nullhomologous 2–tori. For our purposes, we perform the first five surgeries verbatim as in the proof of [Fintushel and Stern 2011, Theorem 2], but change the surgery coefficients of the sixth surgery to perform a p/n–torus surgery in order to obtain an infinite set

$$\{X_{T,S^1_b}(p/n) : n \in \mathbb{Z}\} \tag{2-5}$$

for a fixed $p \geq 2$. It follows from the Seifert–van Kampen theorem that the fundamental group is $\pi_1(X_{T,S^1_b}(p/n)) = \mathbb{Z}/p$ [Baldridge and Kirk 2009, page 321] for every $n \in \mathbb{Z}$; a detailed account on the computation of the fundamental group of the 4–manifolds obtained with such a change in the surgery coefficient can be found in several places in the literature, for example [Akhmedov and Park 2010, page 595; Baldridge and Kirk 2009, Section 5]. We have explained so far that six surgeries on six nullhomologous 2–tori in X as in [Fintushel and Stern 2011, Theorem 2] produce an infinite set (2-5) of 4–manifolds with fundamental group $\mathbb{Z}/p$.

We now appeal to the main result of Fintushel and Stern [2012, Section 8], which is that the first five surgeries on X do not change the diffeomorphism type of X and, thus, there is a single nullhomologous 2–torus $T \subset X$ along with a nullhomologous curve $S^1_b \subset X \setminus \nu(T)$ such that a $1/n$–torus surgery produces an infinite set of smooth structures on X, as we had mentioned before [Fintushel and Stern 2012, Theorem 1.1]. Thus, we conclude that each element in the set (2-5) is obtained by performing a p/n–torus surgery on $T \subset X$ along S^1_b.

We now argue that these 4–manifolds are homeomorphic to (2-3). An inclusion-exclusion argument indicates that the Euler characteristic is unchanged under torus surgeries, ie

$$\chi(X_{T,S^1_b}(p/n)) = \chi(X) = 6. \tag{2-6}$$

Novikov additivity [Gompf and Stipsicz 1999, Remark 9.1.7] implies

$$\sigma(X_{T,S^1_b}(p/n)) = \sigma(X) = -2, \tag{2-7}$$

and we conclude that the second Stiefel–Whitney class of $X_{T,S^1_b}(p/n)$ does not vanish employing a result of Rohklin; see [Gompf and Stipsicz 1999, Theorem 1.2.29]. A classification result of Hambleton and Kreck [1993a, Theorem C] allows us to conclude that the 4–manifold $X_{T,S^1_b}(p/n)$ is homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# \Sigma_p$, where Σ_p is a closed smooth 4–manifold with Euler characteristic two and signature zero for every $n \in \mathbb{Z}$ and $p \geq 2$.

To argue that we have constructed infinitely many 4–manifolds that are pairwise nondiffeomorphic, we compute their Seiberg–Witten invariants using an argument well documented in the literature [Akhmedov et al. 2008; Baldridge and Kirk 2009; Fintushel et al. 2007; Fintushel and Stern 2011; 2012]. We reproduce the argument here for the sake of completeness, which requires us to describe the relation between the Seiberg–Witten invariants of the 4–manifolds $X_{T,S^1_b}(p/n)$, X and $X_{T,S^1_b}(0/1)$. Given a characteristic element $k_0 \in H_2(X_{T,S^1_b}(0/1);\mathbb{Z})$, there are unique characteristic elements $k_X \in H_2(X_{T,S^1_b}(1/0);\mathbb{Z}) = H_2(X;\mathbb{Z})$ and $k_{p/n} \in H_2(X_{T,S^1_b}(p/n);\mathbb{Z})$ [Akhmedov et al. 2008, Remark 4; Fintushel and Stern 2011, page 64]. The 4–manifolds $X_{T,S^1_b}(1/0) = X$ and $X_{T,S^1_b}(0/1)$ will have at most one basic class up to sign in our setting; cf [Akhmedov et al. 2008; Fintushel and Stern 2011; 2012]. As described in [Fintushel and Stern 2012, Section 3], the 4–manifold $X_{T,S^1_b}(0/1)$ has infinite cyclic fundamental group and it admits a symplectic structure [Fintushel and Stern 2012, Section 4]; cf [Fintushel et al. 2007, Section 3]. A result of Taubes [1994] says that the canonical class $k_0 = -c_1(X_{T,S^1_b}(0/1))$ is a basic class of $X_{T,S^1_b}(0/1)$ and $\mathrm{SW}_{X_{T,S^1_b}(0/1)}(\pm k_0) = \pm 1$. Moreover, the adjunction inequality — see [Akhmedov et al. 2008, Section 2.1] — implies that $k_0 \in \mathcal{B}$ is the only basic class up to sign.

It follows that there is a unique $k_{p/n} \in \mathcal{B}_{X_{T,S^1_b}(p/n)}$ up to sign for every $n \geq 1$, and the product formula of Morgan, Mrowka and Szabó [Morgan et al. 1997, Theorem 1.1] yields

$$\mathrm{SW}_{X_{T,S^1_b}(p/n)}(k_{p/n}) = p \cdot \mathrm{SW}_X(k_X) + n \cdot \sum_i \mathrm{SW}_{X_{T,S^1_b}(0/1)}(k_0 + i[T_0]). \tag{2-8}$$

There is a 2–torus $T_d \subset X_{T,S^1_b}(0/1)$ that is geometrically dual to the core 2–torus $T_0 \subset X_{T,S^1_b}(0/1)$ of the surgery. Along with this fact, an adjunction inequality argument implies that the sum on the right-hand side of (2-8) contains at most one nonvanishing term; see [Akhmedov et al. 2008, Section 4.1] for the argument. We have the equality

$$\mathrm{SW}_{X_{T,S^1_b}(p/n)}(k_{p/n}) = p \cdot \mathrm{SW}_X(k_X) + n \cdot \mathrm{SW}_{X_{T,S^1_b}(0/1)}(k_0) \tag{2-9}$$

and we conclude that there is an infinite set of pairwise nondiffeomorphic closed 4–manifolds (2-2). □

What is obtained when we set $p = 0$ in the statement of Theorem 1 and the previous proof, is an infinite set $\{X_{T,S^1_b}(0/n) : n \in \mathbb{Z} - \{0\}\}$ of pairwise nondiffeomorphic closed 4–manifolds with infinite cyclic fundamental group and the same homology of the connected sum $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 \# S^1 \times S^3$; see Fintushel and Stern [2012, Theorem 1.1].

Theorem 2 *There is a smoothly embedded nullhomologous 2–torus $T \subset \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ and a nullhomologous curve in its complement $S_b^1 \subset \mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \setminus \nu(T)$ such that performing a $0/n$–torus surgery to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ on T along S_b^1 yields an infinite set*

$$\{X_{T,S_b^1}(0/n) : n \in \mathbb{Z} - \{0\}\} \tag{2-10}$$

of pairwise nondiffeomorphic 4–manifolds with infinite cyclic fundamental group and such that every element has the homology of the connected sum

$$2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 \# S^1 \times S^3. \tag{2-11}$$

Similar statements to Theorems 1 and 2 for further choices of homeomorphism types of 4–manifolds with cyclic fundamental group are produced by employing other results of Fintushel and Stern [2012].

2.2 2–spheres whose complement has a prescribed fundamental group

Let X be a closed smooth 4–manifold whose fundamental group has a presentation

$$\pi_1 X = \langle g_1, \dots, g_j : r_1, \dots, r_k \rangle \tag{2-12}$$

such that adding the relation $g_1 = 1$ to it for a given generator g_1, one obtains the trivial group. Cyclic groups and the group $\langle g_1, g_2 : {g_1}^5 = (g_1 g_2)^2 = {g_2}^3 \rangle$ are examples of such groups.

Let $\alpha_1 \subset X$ be a based loop whose homotopy class is $[\alpha_1] = g_1 \in \pi_1 X$. Build the closed smooth simply connected 4–manifold

$$Z := X \setminus \nu(\alpha_1) \cup (D^2 \times S^2) \tag{2-13}$$

and consider the belt 2–sphere

$$S := \{0\} \times S^2 \subset D^2 \times S^2 \subset Z. \tag{2-14}$$

Further information is needed on the framing of the loop $\alpha_1 \subset X$ to pin down the diffeomorphism or homeomorphism type of Z. Once this is taken care of, this process provides a 2–sphere (2-14) smoothly embedded in Z and whose complement has fundamental group G. A topological construction of locally flat 2–surfaces in topological 4–manifolds is obtained by using locally flat embedded submanifolds in the surgery (2-13); see [Freedman and Quinn 1990, Section 9.3] for existence and uniqueness results on tubular neighborhoods of locally flat embedded submanifolds.

We set some notation and construct the 2–spheres of Theorem A using this procedure in the following example. It includes the choice of framing on the loop whose homotopy class generates the fundamental group.

Example 3 Fix $p \geq 2$ and an integer $n \in \mathbb{Z}$. Consider the 4–manifold $X_{T,S_b^1}(p/n)$ in the set (2-2) and let $\widehat{T} \subset X_{T,S_b^1}(p/n)$ be the core 2–torus of the surgery. Let $\alpha \subset X_{T,S_b^1}(p/n)$ be a loop such that the 4–manifold

$$Z_{n,p} := (X_{T,S_b^1}(p/n) \setminus \nu(\alpha)) \cup (D^2 \times S^2) \tag{2-15}$$

is simply connected and consider the belt 2–sphere

$$S_{n,p} := \{0\} \times S^2 \subset D^2 \times S^2 \subset Z_{n,p}. \tag{2-16}$$

Notice that the loop α lies on the boundary of $\partial\nu(\widehat{T})$. The framing on the loop α is induced by the product framing of core torus of the p/n–torus surgery. The complement of the 2–sphere (2-16) has fundamental group

$$\pi_1(Z_{n,p} \setminus \nu(S_n)) = \mathbb{Z}/p, \tag{2-17}$$

and the homology class of (2-16) satisfies $[S_{n,p}] \neq 0 \in H_2(Z_{n,p}; \mathbb{Z})$. Moreover, the 4–manifold $X_{n,p}$ is recovered by applying surgery to $Z_{n,p}$ along $S_{n,p}$.

2.3 The ambient 4–manifold of Theorems A and D

We prove in this section that the 2–spheres (2-16) of Example 3 are all smoothly embedded in $2\mathbb{CP}^2\#4\overline{\mathbb{CP}}^2$, and postpone to Section 2.7 the proof that they are pairwise smoothly inequivalent.

Proposition 4 *The 4–manifold $Z_{n,p}$ from (2-15) is diffeomorphic to the connected sum $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ for every $n \in \mathbb{Z}$ and a fixed $p \geq 2$. In particular, there is an infinite set*

$$\{S_{n,p} : n \in \mathbb{Z}\} \tag{2-18}$$

of 2–spheres smoothly embedded in $Z = 2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ such that the complement $Z \setminus \nu(S_{n,p})$ has fundamental group $\mathbb{Z}/p$ for every $n \in \mathbb{Z}$.

Proof We use an argument due to Moishezon [1977, Lemma 13] (see also Gompf [1991, Lemma 3]) and work of Baykur and Sunukjian [2013] to establish the diffeomorphism type of our 4–manifolds. We follow the notation in [Gompf 1991, Lemma 3], fix an $n \in \mathbb{Z}$ and a $p \geq 2$, and consider the 4–manifold $X_{T,S^1_b}(p/n)$ in (2-2) that is constructed from $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ using torus surgeries and the 4–manifold $Z_{n,p}$ built in (2-15). Perform a torus surgery to $X_{T,b}(p/n)$ which identifies the loop that generates its fundamental group with the normal disk to the 2–torus to obtain a simply connected 4–manifold $\widehat{N}$; this gluing map is described on [Gompf 1991, page 101]. The latter 4–manifold can also be obtained by applying a torus surgery to $\mathbb{CP}^2\#3\overline{\mathbb{CP}}^2$. Moishezon's argument implies that $Z_{n,p}$ is obtained from $\widehat{N}$ by doing surgery along a loop [Gompf and Stipsicz 1999, Section 5.2], ie $Z_{n,p} = N^* = \widehat{N} \# S^2 \times S^2$ [Gompf and Stipsicz 1999, Propositions 5.2.3 and 5.2.4]. Results of Baykur and Sunukjian [2013, Section 3] imply that $\widehat{N} \# S^2 \times S^2$ is diffeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# S^2 \times S^2 = 2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$. Since the choice of n and p was arbitrary, we conclude that $Z_{n,p}$ is diffeomorphic to $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ for every $n \in \mathbb{Z}$ and $p \geq 2$. □

A tweak to the proof of Proposition 4 pins down the diffeomorphism type of the 4–manifolds constructed in the proof of Theorem D.

2.4 Topological isotopy

Locally flat embeddings of 2–spheres in 4–manifolds whose complement has finite cyclic fundamental group have been studied by Lee and Wilczyński [1990] and Hambleton and Kreck [1993b, Theorem 4.5]. The next result from their work is of particular importance for our purposes.

Theorem 5 (Lee–Wilczyński, Hambleton–Kreck) *Let X be a closed simply connected topological 4–manifold such that $b_2(X) > |\sigma(X)| + 2$ and let $h \in H_2(X;\mathbb{Z})$ be a homology class of nonzero divisibility $p \neq 0$. Let $S_1, S_2 \subset X$ be locally flat embedded 2–spheres with homology classes*

$$[S_1] = [S_2] = h \in H_2(X;\mathbb{Z}),$$

and whose complement has fundamental group $\pi_1(X \setminus \nu(S_1)) = \mathbb{Z}/p = \pi_1(X \setminus \nu(S_2))$ for $p \geq 2$. If

$$b_2(X) > \max_{0 \leq j < p} |\sigma(X) - 2j(p-j)(1/p^2)h \cdot h|, \tag{2-19}$$

then there is a topological isotopy between S_1 and S_2.

Notice that our ambient 4–manifold $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ is within the range of the hypothesis of Theorem 5. Moreover, the homology class of the belt 2–sphere (2-16) of Example 3 has nonzero divisibility and self-intersection zero by construction. We conclude that the 2–spheres that were constructed in the previous sections are all topologically isotopic to each other by Theorem 5.

Corollary 6 *The infinite set $\{S_{n,p} : n \in \mathbb{Z}\}$ of Proposition 4 is made of smoothly embedded 2–spheres in $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ that are pairwise topologically isotopic.*

2.5 Some examples of $\mathbb{F}$–homology 4–spheres

Constructions of 4–manifolds that have the same $\mathbb{F}$–homology as S^4 are not scarce in the literature. For example, a surgery theory construction of $\mathbb{Q}$–homology 4–spheres with finite cyclic fundamental group is given by Hambleton and Kreck [1993a, Proposition 4.1]. Their examples include 4–manifolds with nonvanishing Kirby–Siebermann invariant and they admit no smooth structure. We describe two constructions of such objects in this section.

The first involves doing surgery on the product of a 3–manifold with a circle. Spun closed smooth 4–manifolds form a classical set of examples of 4–manifolds that share the homology of S^4 with $\mathbb{F}$–coefficients and whose fundamental group is a 3–manifold group. We briefly recall their construction and suppose that N is a closed orientable 3–manifold. A homology 4–sphere Σ_N with fundamental group $\pi_1 \Sigma_N = \pi_1 N$ is constructed as

$$\Sigma_N := (N \times S^1) \setminus \nu(\{\mathrm{pt}\} \times S^1) \cup_{\mathrm{id}} (D^2 \times S^2), \tag{2-20}$$

where we use the identity map to identify the common boundary. There is another choice of framing, yet results of Plotnick [1986] state that there is a unique diffeomorphism class of (2-20) for the 3–manifolds employed in this paper. If N is an $\mathbb{F}$–homology 3–sphere, then Σ_N is an $\mathbb{F}$–homology 4–sphere. There are two principal choices of 3–manifold used in the proofs of our results:

- For $N = L(p,1)$, we obtain a $\mathbb{Q}$–homology 4–sphere $\Sigma_{L(p,1)}$ with fundamental group
$$\pi_1 \Sigma_{L(p,1)} = \mathbb{Z}/p.$$
- For $N = \Sigma(2,3,5)$, we obtain a $\mathbb{Z}$–homology 4–sphere $\Sigma_{\Sigma(2,3,5)}$ with fundamental group
$$\pi_1 \Sigma_{\Sigma(2,3,5)} = \langle a, b : a^5 = (ab)^2 = b^3 \rangle.$$

A second construction of smooth $\mathbb{Q}$–homology 4–spheres with finite cyclic fundamental group is through handlebodies. Gompf and Stipsicz's [1999, Figure 5.46] depiction of a pair of orientable S^2–bundles over $\mathbb{RP}^2$ describes a handlebody of a pair of $\mathbb{Q}$–homology 4–spheres with fundamental group of order two whose second Stiefel–Whitney class can be chosen to vanish or not depending on the n–framing of one of the two 2–handles. Handlebodies of pairs of $\mathbb{Q}$–homology 4–spheres $\{\Sigma_{p,n} : n \in \{0,1\}\}$ with fundamental group
$$\pi_1 \Sigma_{p,n} = \mathbb{Z}/p$$
for every $p \geq 2$ and second Stiefel–Whitney class
$$w_2 \Sigma_{p,n} = n$$
consisting of one 0–handle, one 1–handle, one 0–framed 2–handle, one n–framed 2–handle, one 3–handle, and one 4–handle are drawn as a straight-forward extension of the $p = 2$ case [Gompf and Stipsicz 1999, Figure 5.46].

2.6 2–spheres in simply connected 4–manifolds via $\mathbb{F}$–homology 4–spheres

The 4–manifolds of the previous section and the procedure of Section 2.2 yields knotted 2–spheres smoothly embedded in the total space of an S^2–bundle over S^2. The case of most interest for us is summarized in the following lemma.

Lemma 7 [Sato 1991, Section 3] *There is a smoothly embedded 2–sphere $S_p \hookrightarrow S^2 \times S^2$ whose complement has fundamental group $\mathbb{Z}/p$ for every $p \geq 2$.*

There is a smoothly embedded 2–sphere $S_G \hookrightarrow S^2 \times S^2$ whose complement has fundamental group $G = \langle a, b : a^5 = (ab)^2 = b^3 \rangle$ or $\mathbb{Z}/p$.

A variation of the proof of Proposition 4 yields a proof of Lemma 7 by using Moishezon's argument [1977], a lemma of Gompf [1991, Lemma 1.6] and a result of Akbulut [1999, Theorem]; cf [Tange 2014]. Another proof of Lemma 7 is obtained by using handlebodies [Akbulut 1999; 2016; Gompf and Stipsicz 1999].

2.7 Proof of Theorem A

We collect the results of previous sections into a proof of the following theorem, which is equivalent to Theorem A.

Theorem 8 *Fix $p \geq 2$. There is an infinite set*

$$\{S_{n,p} \subset 2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2 : n \in \mathbb{Z}\} \tag{2-21}$$

made of topologically isotopic 2–spheres whose complement has fundamental group $\mathbb{Z}/p$, and for which doing surgery on each element yields the infinite set (2-2) of pairwise nondiffeomorphic smooth 4–manifolds in the homeomorphism class of $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# \Sigma_p$.

In particular, there is a diffeomorphism of pairs

$$(2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_1,p}) \to (2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2, S_{n_2,p}) \tag{2-22}$$

if and only if $n_1 = n_2$, and the infinite set (2-21) consists of pairwise smoothly inequivalent 2–spheres.

Proof The infinite set (2-21) was constructed in Section 2.2. The fundamental group of the complement of any 2–sphere is a prescribed finite cyclic group; see Example 3. Corollary 6 says that elements in (2-21) are pairwise topologically isotopic. As indicated in Example 3, the 4–manifold $X_{T,S^1_b}(p/n)$ in the infinite set (2-2) is obtained by carving out a tubular neighborhood $\nu(S_{n,p})$ of a 2–sphere in (2-21) from $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$, and capping off the boundary with $S^1 \times D^3$. Given that the infinite set (2-2) is made of pairwise nondiffeomorphic 4–manifolds, we conclude that the infinite set (2-21) is made of pairwise smoothly inequivalent 2–spheres. □

Remark 9 A minor modification to the previous argument yields a proof of Theorem D.

2.8 Proof of Theorem B

Let $\{Z_n : n \in \mathbb{Z}\}$ be an infinite set of pairwise nondiffeomorphic 4–manifolds in the homeomorphism class of Z. Taking a connected sum with any $\mathbb{Q}$–homology 4–sphere M yields an infinite set

$$\{Z_n \# M : n \in \mathbb{Z}\} \tag{2-23}$$

of reducible pairwise nondiffeomorphic 4–manifolds that are pairwise homeomorphic to $Z \# M$. The smooth structures are distinguished with the Seiberg–Witten invariant of the connected sums using the fact that $b_1(M) = 0 = b_2^+(M)$ and results of Kotschick, Morgan and Taubes [Kotschick et al. 1995]. By hypothesis, there is a $\mathrm{Spin}^{\mathbb{C}}$–structure on Z_n for which the Seiberg–Witten invariant SW_{Z_n} is nonzero. As explained in [Kotschick et al. 1995, Proof of Proposition 2], the $\mathrm{Spin}^{\mathbb{C}}$–structure can be extended to the connected sum $Z_n \# M$ and conclude that there is a $\mathrm{Spin}^{\mathbb{C}}$–structure for which $\mathrm{SW}_{Z_n \# M} = \mathrm{SW}_{Z_n}$. This implies that the infinite set $\{Z_n \# M : n \in \mathbb{Z}\}$ consists of pairwise nondiffeomorphic 4–manifolds that are pairwise homeomorphic to $Z \# M$.

We do surgery along the loop $\alpha \subset Z_n \# M$ as in the hypothesis of Theorem B verbatim to the procedure described in Example 3 to construct an infinite set

$$\{S_{n,\pi} : n \in \mathbb{Z}, \pi = \pi_1 M\} \tag{2-24}$$

of smoothly embedded 2–spheres in $Z\#S^2\times S^2$ whose complement has fundamental group $\pi = \pi_1 M$. By construction we obtain a homeomorphism of pairs between $(Z\#S^2\times S^2, S_{n_1,\pi})$ and $(Z\#S^2\times S^2, S_{n_2,\pi})$ for every $n_i \in \mathbb{Z}$. Surgery on the belt 2–sphere $S_{n,\pi} \subset Z \# S^2 \times S^2$ gives us $Z_n \# M$ back. Since the infinite set (2-23) is made of pairwise nondiffeomorphic 4–manifolds, we conclude that there is no diffeomorphism of pairs

$$(Z \# S^2 \times S^2, S_{n_1,\pi}) \to (Z \# S^2 \times S^2, S_{n_2,\pi}) \tag{2-25}$$

if $n_1 \neq n_2$. □

Remark 10 We elaborate on an argument to prove Theorem A by using the construction procedure of Theorem B. The ingredients that satisfy the hypothesis of the latter are the following. Take the infinite set $\{Z_n : n \in \mathbb{Z}\}$ of pairwise nondiffeomorphic 4–manifolds that are homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ that was constructed by Fintushel and Stern [2012]. These 4–manifolds have different Seiberg–Witten invariant. A result of Baykur and Sunukjian [2013] implies that $Z_n \# S^2 \times S^2$ is diffeomorphic to $2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ for every $n \in \mathbb{Z}$. As the 4–manifold M in the statement of Theorem B, use the $\mathbb{Q}$–homology 4–sphere $\Sigma_{p,0}$ that was discussed in Section 2.6 with $\pi_1 \Sigma_{p,0} = \mathbb{Z}/p$. Build the infinite set

$$\{Z_n \# \Sigma_{p,0} : n \in \mathbb{Z}\} \tag{2-26}$$

of closed reducible 4–manifolds that are homeomorphic to $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2 \# \Sigma_{p,0}$. The set (2-26) consists of pairwise nondiffeomorphic 4–manifolds, where the diffeomorphism classes are distinguished by their Seiberg–Witten invariants [Kotschick et al. 1995, Proposition 2]. Proceed as in the proof of Theorem B and build an infinite set (2-24) of pairwise smoothly inequivalent 2–spheres. These submanifolds have the required properties by construction and they are pairwise topologically isotopic by Theorem 5.

2.9 Proof of Corollary C

We check that the hypothesis of Theorem B are met in these cases. As the infinite set $\{Z_n : n \in \mathbb{Z}\}$ we can take the infinite inequivalent smooth structures on $\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$ constructed by Fintushel and Stern [2012]. Work of Baykur and Sunukjian [2013, Theorem] implies that $Z_n \# S^2 \times S^2 = 2\mathbb{CP}^2 \# 4\overline{\mathbb{CP}}^2$ for every $n \in \mathbb{Z}$; this connected sum is the simply connected 4–manifold in the statement of Corollary C. The $\mathbb{Q}$–homology 4–spheres with the desired fundamental group were constructed in Section 2.5; see Lemma 7. □

2.10 Proof of Theorem E

Hambleton and Kreck [1993a, Proposition 4.1] used surgery to prove the existence of a $\mathbb{Q}$–homology 4–sphere M_p with nonzero second Stiefel–Whitney class $w_2(M_p) \neq 0$, nonvanishing Kirby–Siebenmann invariant $\mathrm{KS}(M_p) \neq 0$, and fundamental group $\pi_1 M_p = \mathbb{Z}/p$ for every $p \geq 2$. Carve out the loop in M_p whose homotopy class generates the group $\pi_1 M_p = \mathbb{Z}/p$, and glue back a locally flat copy of

$D^2 \times S^2$ to obtain a simply connected 4–manifold $\widehat{M}$ with Euler characteristic $\chi(\widehat{M}) = \chi(M_p) + 2 = 4$, signature $\sigma(\widehat{M}) = \sigma(M_p)$, second Stiefel–Whitney class $w_2(\widehat{M}) \neq 0$ and Kirby–Siebenmann invariant $\mathrm{KS}(\widehat{M}) \neq 0$. A result of Freedman and Quinn [1990, Section 10.1] states that $\widehat{M}$ is homeomorphic to the connected sum $*\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ of the Chern manifold and the complex projective space with the opposite orientation for every $p \geq 2$; cf [Gompf and Stipsicz 1999, Theorem 1.2.27]. The fundamental group of the complement of the belt 2–sphere S_p of the surgery is isomorphic to $\pi_1 M_p = \mathbb{Z}/p$.

To produce the smoothly embedded 2–sphere $S'_p \subset \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ as in (1-5) and prove the last clause of Theorem E, perform surgery to the smooth $\mathbb{Q}$–homology 4–sphere $\Sigma_{p,1}$ described in Section 2.6. □

References

[Akbulut 1999] **S Akbulut**, *Scharlemann's manifold is standard*, Ann. of Math. 149 (1999) 497–510 MR Zbl

[Akbulut 2015] **S Akbulut**, *Isotoping 2–spheres in 4–manifolds*, from "Proceedings of the Gökova Geometry/Topology Conference 2014" (S Akbulut, D Auroux, T Önder, editors), GGT, Gökova (2015) 264–266 MR Zbl

[Akbulut 2016] **S Akbulut**, *4–manifolds*, Oxford Grad. Texts in Math. 25, Oxford Univ. Press (2016) MR Zbl

[Akhmedov and Park 2010] **A Akhmedov**, **B D Park**, *Exotic smooth structures on small 4–manifolds with odd signatures*, Invent. Math. 181 (2010) 577–603 MR Zbl

[Akhmedov et al. 2008] **A Akhmedov**, **R İ Baykur**, **B D Park**, *Constructing infinitely many smooth structures on small 4–manifolds*, J. Topol. 1 (2008) 409–428 MR Zbl

[Auckly et al. 2015] **D Auckly**, **H J Kim**, **P Melvin**, **D Ruberman**, *Stable isotopy in four dimensions*, J. Lond. Math. Soc. 91 (2015) 439–463 MR Zbl

[Baldridge and Kirk 2009] **S Baldridge**, **P Kirk**, *Constructions of small symplectic 4–manifolds using Luttinger surgery*, J. Differential Geom. 82 (2009) 317–361 MR Zbl

[Baykur and Sunukjian 2013] **R İ Baykur**, **N Sunukjian**, *Round handles, logarithmic transforms and smooth 4–manifolds*, J. Topol. 6 (2013) 49–63 MR Zbl

[Fintushel and Stern 2011] **R Fintushel**, **R J Stern**, *Pinwheels and nullhomologous surgery on 4–manifolds with $b^+ = 1$*, Algebr. Geom. Topol. 11 (2011) 1649–1699 MR Zbl

[Fintushel and Stern 2012] **R Fintushel**, **R Stern**, *Surgery on nullhomologous tori*, from "Proceedings of the Freedman Fest" (R Kirby, V Krushkal, Z Wang, editors), Geom. Topol. Monogr. 18, Geom. Topol. Publ., Coventry (2012) 61–81 MR Zbl

[Fintushel et al. 2007] **R Fintushel**, **B D Park**, **R J Stern**, *Reverse engineering small 4–manifolds*, Algebr. Geom. Topol. 7 (2007) 2103–2116 MR Zbl

[Freedman and Quinn 1990] **M H Freedman**, **F Quinn**, *Topology of 4–manifolds*, Princeton Math. Ser. 39, Princeton Univ. Press (1990) MR Zbl

[Gompf 1991] **R E Gompf**, *Sums of elliptic surfaces*, J. Differential Geom. 34 (1991) 93–114 MR Zbl

[Gompf and Stipsicz 1999] **R E Gompf**, **A I Stipsicz**, *4–manifolds and Kirby calculus*, Grad. Stud. Math. 20, Amer. Math. Soc., Providence, RI (1999) MR Zbl

[Hambleton and Kreck 1993a] **I Hambleton**, **M Kreck**, *Cancellation, elliptic surfaces and the topology of certain four-manifolds*, J. Reine Angew. Math. 444 (1993) 79–100 MR Zbl

[Hambleton and Kreck 1993b] **I Hambleton**, **M Kreck**, *Cancellation of hyperbolic forms and topological four-manifolds*, J. Reine Angew. Math. 443 (1993) 21–47 MR Zbl

[Kasprowski et al. 2021] **D Kasprowski**, **P Lambert-Cole**, **M Land**, **A G Lecuona**, *Topologically flat embedded 2–spheres in specific simply connected 4–manifolds*, from "2019–20 MATRIX annals" (D R Wood, J de Gier, C E Praeger, T Tao, editors), MATRIX Book Ser. 4, Springer (2021) 111–116 MR Zbl

[Kim 2006] **H J Kim**, *Modifying surfaces in 4–manifolds by twist spinning*, Geom. Topol. 10 (2006) 27–56 MR Zbl

[Kim and Ruberman 2008] **H J Kim**, **D Ruberman**, *Smooth surfaces with non-simply-connected complements*, Algebr. Geom. Topol. 8 (2008) 2263–2287 MR Zbl

[Kotschick et al. 1995] **D Kotschick**, **J W Morgan**, **C H Taubes**, *Four-manifolds without symplectic structures but with nontrivial Seiberg–Witten invariants*, Math. Res. Lett. 2 (1995) 119–124 MR Zbl

[Lee and Wilczyński 1990] **R Lee**, **D M Wilczyński**, *Locally flat 2–spheres in simply connected 4–manifolds*, Comment. Math. Helv. 65 (1990) 388–412 MR Zbl Correction in 67 (1992) 334–335

[Lee and Wilczyński 1997] **R Lee**, **D M Wilczyński**, *Representing homology classes by locally flat surfaces of minimum genus*, Amer. J. Math. 119 (1997) 1119–1137 MR Zbl

[Moishezon 1977] **B Moishezon**, *Complex surfaces and connected sums of complex projective planes*, Lecture Notes in Math. 603, Springer (1977) MR Zbl

[Morgan et al. 1997] **J W Morgan**, **T S Mrowka**, **Z Szabó**, *Product formulas along T^3 for Seiberg–Witten invariants*, Math. Res. Lett. 4 (1997) 915–929 MR Zbl

[Plotnick 1986] **S P Plotnick**, *Equivariant intersection forms, knots in S^4, and rotations in 2–spheres*, Trans. Amer. Math. Soc. 296 (1986) 543–575 MR Zbl

[Sato 1991] **Y Sato**, *Locally flat 2–knots in $S^2 \times S^2$ with the same fundamental group*, Trans. Amer. Math. Soc. 323 (1991) 911–920 MR Zbl

[Schwartz 2019] **H R Schwartz**, *Equivalent non-isotopic spheres in 4–manifolds*, J. Topol. 12 (2019) 1396–1412 MR Zbl

[Sunukjian 2015] **N S Sunukjian**, *Surfaces in 4–manifolds: concordance, isotopy, and surgery*, Int. Math. Res. Not. 2015 (2015) 7950–7978 MR Zbl

[Tange 2014] **M Tange**, *The link surgery of $S^2 \times S^2$ and Scharlemann's manifolds*, Hiroshima Math. J. 44 (2014) 35–62 MR Zbl

[Taubes 1994] **C H Taubes**, *The Seiberg–Witten invariants and symplectic forms*, Math. Res. Lett. 1 (1994) 809–822 MR Zbl

Scuola Internazionale Superiori di Studi Avanzati (SISSA)
Trieste, Italy

rtorres@sissa.it

Received: 1 December 2022 Revised: 27 March 2023

Algebraic & Geometric Topology **24**:4 (2024) 2367–2384
DOI: 10.2140/agt.2024.24.2367
Published: 16 July 2024

Remarks on symplectic circle actions, torsion and loops

MARCELO S ATALLAH

We study loops of symplectic diffeomorphisms of closed symplectic manifolds. Our main result, which is valid for a large class of symplectic manifolds, shows that the flux of a symplectic loop vanishes whenever its orbits are contractible. As a consequence, we obtain a new vanishing result for the flux group and new instances where the presence of a fixed point of a symplectic circle action is a sufficient condition for it to be Hamiltonian. We also obtain applications to symplectic torsion; more precisely, nontrivial elements of $\mathrm{Symp}_0(M,\omega)$ that have finite order.

53D22, 53D40, 57R17; 57S15, 57S17

1 Introduction

1.1 The flux group and the e–homomorphism

Let (M,ω) be a closed symplectic manifold and denote by $\widetilde{\mathrm{Symp}}_0(M,\omega)$ the universal cover of the identity component $\mathrm{Symp}_0(M,\omega)$ of the group of symplectic diffeomorphisms. The *flux homomorphism*

$$\widetilde{\mathrm{Flux}}\colon \widetilde{\mathrm{Symp}}_0(M,\omega)\to \mathrm{H}^1(M;\mathbb{R})$$

is defined by assigning to each class $\tilde{\psi}\in\widetilde{\mathrm{Symp}}_0(M,\omega)$ a cohomology class

$$\widetilde{\mathrm{Flux}}(\tilde{\psi})=\int_0^1[\iota_{X^t}\omega]\,dt,$$

where X^t is the time-dependent vector field induced by a symplectic path $\{\psi_t\}$ representing $\tilde{\psi}$. In particular, if γ is a 1–cycle in M, we have that

$$\langle\widetilde{\mathrm{Flux}}(\{\psi_t\}),[\gamma]\rangle=\int_{[0,1]\times\mathbb{R}/\mathbb{Z}}\alpha^*\omega, \tag{1}$$

where $\alpha\colon[0,1]\times\mathbb{R}/\mathbb{Z}\to M$ is given by setting $\alpha(t,s)=\psi_t(\gamma(s))$. We shall often denote the flux of $\tilde{\psi}$ by $\widetilde{\mathrm{Flux}}(\{\psi_t\})$. The *flux group* Γ is the image of the restriction of the flux homomorphism to $\pi_1(\mathrm{Symp}_0(M,\omega))\subset\widetilde{\mathrm{Symp}}_0(M,\omega)$. The $\widetilde{\mathrm{Flux}}$ map descends to a homomorphism

$$\mathrm{Flux}\colon \mathrm{Symp}_0(M,\omega)\to \mathrm{H}^1(M;\mathbb{R})/\,\Gamma,$$

whose kernel was shown by Banyaga [2] to be equal to $\mathrm{Ham}(M,\omega)$. In particular, we have the exact sequence

$$1\to\mathrm{Ham}(M,\omega)\to\mathrm{Symp}_0(M,\omega)\to\mathrm{H}^1(M;\mathbb{R})/\,\Gamma\to 1,$$

which implies that $\mathrm{Ham}(M,\omega)$ coincides with $\mathrm{Symp}_0(M,\omega)$ if and only if $\mathrm{H}^1(M;\mathbb{R})$ vanishes. For a basepoint $x_0 \in M$, denote by

$$\mathrm{ev}\colon \pi_1(\mathrm{Symp}_0(M,\omega),\mathrm{id}) \to \pi_1(M,x_0)$$

the *evaluation homomorphism* given by setting $\mathrm{ev}([\{\psi_t\}]) = [\{\psi_t(x_0)\}]$. The image of ev lies in the center of $\pi_1(M)$; see Polterovich [25]. Therefore, since M is connected, the evaluation maps for different choices of basepoints $x_0 \in M$ are identified. Hence, we write ev without making specific reference to x_0. Observe that the evaluation map factors through the flux group Γ yielding the following commutative diagram:[1]

(2)
$$\begin{tikzcd} \pi_1(\mathrm{Symp}_0(M,\omega)) \arrow[d, "\widetilde{\mathrm{Flux}}"'] \arrow[dr, "\mathrm{ev}"] & \\ \Gamma \arrow[r, "e"] & \pi_1(M) \end{tikzcd}$$

To see that the *e–homomorphism* is well-defined note that if

$$\widetilde{\mathrm{Flux}}(\{\phi_t\}) = \widetilde{\mathrm{Flux}}(\{\psi_t\}),$$

then the loop $\{\phi_t \circ \psi_t^{-1}\}$ of symplectic diffeomorphisms can be homotoped to a Hamiltonian loop, which is known to have contractible orbits; see McDuff [16]. An important consequence of diagram (2) is that whenever e is injective, a symplectic loop with contractible orbits can be homotoped to a Hamiltonian loop. It is not hard to see that this is the case for (M,ω) symplectically aspherical: $[\omega]$ vanishes on the image $\mathrm{H}_2^S(M;\mathbb{Z})$ of the Hurewicz map $\pi_2(M) \to \mathrm{H}_2(M;\mathbb{Z})$. Indeed, if a symplectic loop $\{\psi_t\}$ has trivial evaluation and γ is any 1–cycle in M, then the torus $\alpha(\mathrm{T}^2)$, where $\alpha(t,s) = \psi_t(\gamma(s))$, will have the symplectic area of a sphere. Equation (1) then implies that the flux of the loop $\{\psi_t\}$ vanishes.

The injectivity of the e–homomorphism has important consequences in the theory of symplectic circle actions discussed in Section 1.2. Furthermore, it allows one to deduce the vanishing of the Flux group in cases where the evaluation map is trivial. In particular, we have the following:

Proposition 1.1 *Let (M,ω) be a closed symplectic manifold with injective e–homomorphism, and suppose further that either $\chi(M) \neq 0$ or that $\pi_1(M)$ has finite center. Then the flux group Γ is trivial.*

Proof When $\chi(M) \neq 0$ we have that ev is trivial, a fact that is true even in the more general setting of loops of diffeomorphisms on a closed manifold; see Lê and Ono [12]. Therefore, Γ is trivial by the injectivity of e. Now, suppose that $\pi_1(M)$ has finite center and let $\{\psi_t\}$ be a symplectic loop. Then, there exists some positive integer k such that $\mathrm{ev}(\{\psi_t\})^k = 1$. Thus, we have

$$e(\widetilde{\mathrm{Flux}}(\{\psi_t^k\})) = \mathrm{ev}(\{\psi_t^k\}) = \mathrm{ev}(\{\psi_t\})^k = 1.$$

We conclude, by the injectivity of e and the fact that $\widetilde{\mathrm{Flux}}$ is a homomorphism to a torsion-free group, that the loop $\{\psi_t\}$ has no flux. □

[1] The fact that ev factors through the flux group was pointed out to me by Egor Shelukhin.

Remark In the case where (M,ω) is symplectically aspherical and the Euler characteristic is nonzero, Proposition 1.1 can be obtained from [12, Proposition 1.7, Corollary 4.2]. Furthermore, the fact that the flux group vanishes was established in Lalonde, McDuff and Polterovich [11] and Polterovich [24]. We also note that in Kędra, Kotschick and Morita [9], the implication of Proposition 1.1 is proven under the weak Lefschetz assumption.

Our main result provides more sufficient conditions for the injectivity of the e–homomorphism.

Definition 1.2 Let (M,ω) be a closed symplectic manifold of dimension $2n$. We say that it satisfies condition $(\star)$ if at least one of the following is true:

- **Symplectically aspherical** The cohomology class $[\omega]$ vanishes on the image $\mathrm{H}_2^S(M;\mathbb{Z})$ of the Hurewicz map $\pi_2(M) \to \mathrm{H}_2(M;\mathbb{Z})$.
- **Spherically monotone** There exists a constant $\lambda \in \mathbb{R} \setminus \{0\}$ such that
$$[\omega]|_{\pi_2(M)} = \lambda \cdot c_1(M)|_{\pi_2(M)},$$
where $c_1(M)$ denotes the first Chern class associated with (M,ω). We say *positive* (resp. *negative*) *spherically monotone* when $\lambda > 0$ (resp. $\lambda < 0$).
- **Weak Lefschetz property** The multiplication map
$$[\omega]^{n-1}\colon \mathrm{H}^1(M;\mathbb{R}) \to \mathrm{H}^{2n-1}(M;\mathbb{R})$$
is injective (hence an isomorphism).

Remark (examples satisfying $(\star)$) The standard symplectic torus $(T^{2n}, dp\wedge dq)$ and any closed oriented surface Σ_g of genus $g \ge 1$ with the standard area form are symplectically aspherical. Symplectic products of the form $(M\times N, \omega_M \oplus \omega_N)$, where (M,ω_M) is positive (resp. negative) homologically monotone and (N,ω_N) is aspherical, are positive (resp. negative) spherically monotone but not homologically monotone. In particular, $\mathbb{C}P^n \times \mathrm{T}^{2m}$ with symplectic form $\omega_{\mathrm{FS}} \oplus dp\wedge dq$ is positive spherically monotone. Hypersurfaces of $\mathbb{C}P^n$ defined by setting $z_0^m + \cdots + z_n^m = 0$, with $m > n+1$, are negative homologically monotone; see McDuff and Salamon [18]. Finally, all closed Kähler manifolds satisfy the weak Lefschetz property.

Theorem 1.3 *Let (M,ω) be a closed symplectic manifold satisfying $(\star)$. Then the e–homomorphism is injective.*

When the weak Lefschetz property is satisfied the injectivity of the e–homomorphism follows from classical arguments that were known at least as early as the work of Ono in [20]; see also McDuff and Salamon [19] and Lalonde, McDuff and Polterovich [11]. Indeed, if $\{\psi_t\}$ is a symplectic loop inducing a symplectic vector field X_t, then the homology classes of its orbits are Poincaré dual to the class
$$\frac{1}{\mathrm{Vol}(M)}\left[\widetilde{\mathrm{Flux}}(\{\psi_t\}) \wedge \frac{\omega^{n-1}}{(n-1)!}\right] \in \mathrm{H}^{2n-1}(M;\mathbb{R}).$$

Therefore, by the injectivity of $[\omega]^{n-1}$, we have that $\widetilde{\mathrm{Flux}}(\{\psi_t\}) = 0$ whenever $\mathrm{ev}(\{\psi_t\}) = 1$, since every orbit of $\{\psi_t\}$ is homologically trivial.

1.1.1 Outline of the proof of Theorem 1.3 When (M,ω) is spherically monotone we provide two proofs of the injectivity of the e–homomorphism, which are detailed in Section 3. The first proof relies on looking at this problem from the perspective of Floer–Novikov theory developed by Lê and Ono [30]; see also Ono [22; 23]. Floer–Novikov cohomology is a natural generalization of Hamiltonian Floer cohomology in the sense that to a symplectic path $\{\psi_t\}$ based at identity with nondegenerate endpoint $\psi = \psi_1$, we can associate a Floer-type cohomology group $\mathrm{HFN}^*(\{\psi_t\}; J)$ that, up to a natural isomorphism, depends only on the flux of $\{\psi_t\}$. The proof has two key steps. First, using ideas in [23] we obtain an isomorphism

$$\mathrm{HFN}(\{\psi_t\}; J) \cong \mathrm{H}^*(M;\mathbb{Q}) \otimes \Lambda_\omega$$

for a symplectic path $\{\psi_t\}$ with $\widetilde{\mathrm{Flux}}(\{\psi_t\}) = [\theta] \in \ker e$. Next, we show that when (M,ω) is spherically monotone, the Floer–Novikov cohomology of a symplectic path $\{\psi_t\}$ is isomorphic to the Morse–Novikov cohomology $\mathrm{HN}^*(M,\theta)$ of its flux. A simple rank comparison then shows that this is only possible when $\widetilde{\mathrm{Flux}}(\{\psi_t\}) = 0$, which concludes the argument.

The second proof follows from a result of McDuff [14, Theorem 1], from which the triviality of Γ in the homologically monotone setting follows. Let $\{\psi_t\}$ be a symplectic loop. For a loop γ, set $\alpha(t,s) = \psi_t(\gamma(s))$ as before. McDuff's result implies that the 2–cycle A_γ represented by the torus $\mathrm{im}(\alpha)$ satisfies

$$\langle c_1(M), A_\gamma \rangle = 0. \tag{3}$$

If $\{\psi_t\}$ has trivial evaluation, then $\mathrm{im}(\alpha)$ can be represented by a sphere. Thus, we obtain

$$\langle \widetilde{\mathrm{Flux}}(\{\psi_t\}), [\gamma] \rangle = \langle [\omega], A_\gamma \rangle = \lambda \cdot \langle c_1(M), A_\gamma \rangle = 0,$$

where $\lambda \neq 0$ is the monotonicity constant. Since γ is arbitrary, we conclude that $\widetilde{\mathrm{Flux}}(\{\psi_t\}) = 0$. While this proof is easier, it heavily relies on McDuff's result, which was proven using highly nontrivial topological arguments. The first proof, on the other hand, is symplectic in nature.

1.2 Symplectic circle actions

Let $\mathrm{S}^1 = \mathbb{R}/\mathbb{Z}$ be the standard circle group. Let (M,ω) be a symplectic manifold equipped with a smooth S^1–action generated by a vector field X. The contraction of the symplectic form with the vector field X defines a 1–form $\iota_X\omega$ on M. The circle action is called *symplectic* whenever $\iota_X\omega$ is closed, and *Hamiltonian* if it is also exact. Knowing that the action is Hamiltonian has several advantages. For example, one can use a primitive H of $\iota_X\omega$, referred to as a *moment map*, to obtain a symplectic quotient of M at a regular value of H. This procedure is used, in particular, to reduce the dimension of the phase-space associated to problems arising in classical mechanics that have a circular symmetry.

If $\mathrm{H}^1(M;\mathbb{R}) = 0$, it is clear that every symplectic circle action is Hamiltonian. Otherwise, it becomes substantially more difficult to determine whether $\iota_X\omega$ is an exact 1–form. Finding sufficient and necessary conditions assuring that a symplectic circle action is Hamiltonian has been a subject of interest at least as early as the work of T Frankel in the late 1950s showing the following:

Theorem 1.4 (Frankel [6]) *A symplectic circle action on a closed Kähler manifold is Hamiltonian if and only if it has fixed points.*

This result was later generalized by Ono [20] to closed Lefschetz manifolds. The condition of having fixed points was shown to be sufficient to guarantee the exactness of the circle action in a few other instances. McDuff [15] proved it when M has dimension four, while Ono [21] and Ginzburg [7] showed it in the symplectically aspherical case. Tolman and Weitsman [29] proved that it remains true for semifree circle actions with isolated fixed points. Finally, Lupton and Oprea [13] and McDuff [14] showed that every symplectic circle action is Hamiltonian when (M,ω) is homologically monotone, ie the cohomology class $[\omega]$ is a nonzero multiple of the first Chern class $c_1(M)$ — a fact which is not true in the more general spherically monotone setting. Indeed, one can consider the symplectic circle action on $\mathrm{T}^2\times\mathrm{S}^2$ (with the product symplectic form of the standard symplectic structures) given by rotation in the first factor and identity on the second.

On the other hand, McDuff constructed in [15] a non-Hamiltonian circle action with fixed tori on a closed 6–dimensional Calabi–Yau symplectic manifold (see Cho and Kim [3]), showing that the condition $M^{\mathrm{S}^1}\neq\varnothing$ alone is not sufficient to guarantee the exactness of $\iota_X\omega$ for general closed symplectic manifolds. McDuff and Salamon then asked if every symplectic circle action with isolated fixed points on a closed symplectic manifold is Hamiltonian. This question was answered in the negative by Tolman in [28]; see also Jang and Tolman [8]. It remains unclear when such examples can exist, or from another viewpoint, how large the class is of closed symplectic manifolds for which the presence of fixed points is equivalent to the exactness of the circle action. Nonetheless, the e–homomorphism gives a partial characterization of this class. In particular, we have the following:

Proposition 1.5 *Let (M,ω) be a closed symplectic manifold such that the e–homomorphism is injective. Then a symplectic circle action is Hamiltonian if and only if it has fixed points.*

Proof If a symplectic circle action has a fixed point, then the symplectic loop $\{\psi_t\}$ induced by it has trivial evaluation. The injectivity of the e–homomorphism implies that $[\iota_X\omega] = \widetilde{\mathrm{Flux}}(\{\psi_t\}) = 0$. Here, X is the time-independent vector field generating the action. □

In view of Theorem 1.3, we obtain the following.

Theorem 1.6 *Let (M,ω) be a closed spherically monotone symplectic manifold. Then, a symplectic circle action is Hamiltonian if and only if it has a fixed point.*

A result of McDuff [17] showed that every closed symplectic manifold that admits a Hamiltonian circle action is uniruled in the Gromov–Witten sense. In particular, they are geometrically uniruled: for each ω–compatible almost complex structure J and each point $x \in M$, there is a nonconstant J–holomorphic sphere u such that $x \in \operatorname{im}(u)$. Symplectic manifolds that are symplectically Calabi–Yau are not geometrically uniruled, neither are (spherically) negative monotone ones. By considering this fact in addition to Theorem 1.6 we obtain the following corollary.

Corollary 1.7 *Let the circle act symplectically and nontrivially on a closed symplectic manifold (M, ω) such that*

$$c_1(M)|_{\pi_2(M)} = \lambda \cdot [\omega]|_{\pi_2(M)}$$

for $\lambda \in \mathbb{R}$. Then:

(i) *If $\lambda > 0$, the action is Hamiltonian if and only if it has fixed points.*

(ii) *If $\lambda < 0$, the action is non-Hamiltonian, and has no fixed points.*

(iii) *If $\lambda = 0$ the action is non-Hamiltonian.*

Remark When $\lambda \neq 0$ there are examples of non-Hamiltonian symplectic circle actions. In particular, consider the symplectic product $\mathbb{T}^2 \times M$ of the standard symplectic torus with any closed spherically monotone symplectic manifold, and the symplectic circle action given by $t \cdot (x, y, p) = (x + t, y, p)$ for $(x, y) \in T^2$ and $p \in M$.

Another closely related question was raised by McDuff and Salamon [19]. They asked if there exists a symplectic free circle action whose orbits are contractible. Kotschick [10] proved this to be the case for all symplectic manifolds of dimension four, even if the action is only assumed to be smooth. Furthermore, Kotschick produced examples of symplectic free circle actions with contractible orbits in every even dimension greater than or equal to six. As a corollary of Theorem 1.3 and the argument in the proof of Proposition 1.5, we obtain the following.

Theorem 1.8 *Let (M, ω) be a closed symplectic manifold satisfying $(\star)$. Then every free symplectic circle action must have noncontractible orbits.*

1.3 Applications to symplectic torsion

The injectivity of the e–homomorphism, when combined with results in Atallah and Shelukhin [1], provides applications to questions about the existence of finite subgroups of $\operatorname{Symp}_0(M, \omega)$. In [1], it was shown that if (M, ω) is positive spherically monotone, then the existence of a nontrivial finite subgroup of $\operatorname{Ham}(M, \omega)$ implies that (M, ω) is geometrically uniruled. Furthermore, it was shown that if (M, ω) is negative spherically monotone, then there are no nontrivial finite subgroups of $\operatorname{Ham}(M, \omega)$. Therefore, as a corollary of Proposition 1.1 and Theorem 1.3 we obtain the following.

Theorem 1.9 *Let (M,ω) be a closed symplectic manifold such that*

$$c_1(M)|_{\pi_2(M)} = \lambda \cdot [\omega]|_{\pi_2(M)}$$

for $\lambda \in \mathbb{R} \setminus \{0\}$. Further, suppose that either $\chi(M) \neq 0$ or $\pi_1(M)$ has finite center. Then:

(i) *If $\lambda > 0$, then the existence of a nontrivial finite subgroup of $\mathrm{Symp}_0(M,\omega)$ implies that (M,ω) is geometrically uniruled.*

(ii) *If $\lambda < 0$, then all finite subgroups of $\mathrm{Symp}_0(M,\omega)$ are trivial.*

Remark The symplectic product of $(\mathbb{C}P^n, \omega_{\mathrm{FS}})$ with any closed oriented surface Σ_g of genus $g \geq 2$ with the standard area form satisfies condition (i) of Theorem 1.9. More generally, any symplectic product of the form $(M \times N, \omega_M \oplus \omega_N)$, where (M,ω_M) is positive spherically monotone with $\chi(M) \neq 0$ and (N,ω_N) is symplectically aspherical with $\chi(N) \neq 0$. Furthermore, following Dimca [4], the Euler characteristic of the degree m hypersurface $X_m \subset \mathbb{C}P^n$ defined by setting $z_0^m + \cdots + z_n^m = 0$ is given by

$$\chi(X_m) = \frac{1}{m}((1-m)^{n+1} - 1) + n + 1.$$

Therefore, $\chi(X_m) \neq 0$ when $m > n+1$. Hence, X_m, which is negative monotone, satisfies condition (ii) of Theorem 1.9.

The injectivity of the e–homomorphism also gives information about the presence of fixed points of a nontrivial symplectic diffeomorphism $\psi \in \mathrm{Symp}_0(M,\omega)$ of finite order. The following definition given by Polterovich in [24] naturally fits into this context.

Definition 1.10 A fixed point x of a symplectic diffeomorphism $\psi \in \mathrm{Symp}_0(M,\omega)$ is of *contractible type* if there exists a symplectic path $\{\psi_t\}$ based at the identity with $\psi_1 = \psi$ such that the loop $\{\psi_t(x)\}$ is contractible in M.

The presence of a fixed point of contractible type of a nontrivial $\psi \in \mathrm{Symp}_0(M,\omega)$ of finite order implies that it must be Hamiltonian. Indeed, we have the following:

Proposition 1.11 *Let (M,ω) be a closed symplectic manifold such that the e–homomorphism is injective. Further suppose that $\psi \in \mathrm{Symp}_0(M,\omega)$ is nontrivial of finite order, ie $\psi^d = \mathrm{id}$ for some integer $d > 1$. Then ψ is Hamiltonian if and only if it admits a fixed point of contractible type.*

Proof Suppose $x \in \mathrm{Fix}(x)$ is of contractible type. Let $\{\psi_t\}$ be a symplectic path with $\psi_1 = \psi$ such that $\{\psi_t(x)\}$ is contractible in M. Note that $\{\psi_t^d\}$ is a symplectic loop. Then,

$$e(\widetilde{\mathrm{Flux}}(\{\psi_t^d\})) = [\{\psi_t^d(x)\}] = [\{\psi_t(x)\}]^d = 1.$$

Hence, by the injectivity of e and the fact that $\widetilde{\mathrm{Flux}}$ is a homomorphism to a torsion-free group, we have that $\widetilde{\mathrm{Flux}}(\{\psi_t\}) = 0$. In particular, ψ is Hamiltonian. □

Corollary 1.12 *Let (M,ω) be a closed symplectic manifold satisfying $(\bigstar)$ such that $\chi(M) \neq 0$. Then all nontrivial symplectic diffeomorphisms $\psi \in \mathrm{Symp}_0(M,\omega)$ of finite order are Hamiltonian.*

We note that any orientation-preserving diffeomorphism of a closed manifold, which is of finite order and has finitely many fixed points, must have the number of fixed points equal to the Lefschetz number of the diffeomorphism. Therefore, when the Euler characteristic of (M,ω) is zero and $\psi \in \mathrm{Symp}_0(M,\omega)$ is torsion, it either has no fixed points or nonisolated ones. In the latter case, there is no a priori reason for the existence of a fixed point of contractible type. However, in the case of the standard symplectic torus, Polterovich [24] showed that any fixed point of a symplectic diffeomorphism $\psi \in \mathrm{Symp}_0(\mathrm{T}^{2n}, dp \wedge dq)$ is of contractible type. We can adapt that argument to prove the following:

Corollary 1.13 *Let (M,ω) be a simply connected closed symplectic manifold satisfying $(\bigstar)$. Then every non-Hamiltonian $\psi \in \mathrm{Symp}_0(\mathrm{T}^{2n} \times M, dp \wedge dq \oplus \omega)$ of finite order has no fixed points.*

Proof Let $(x,p) \in \mathrm{T}^{2n} \times M$ be a fixed point of ψ, and $\{\psi_t\}$ a symplectic path such that $\psi_0 = \mathrm{id}$ and $\psi_1 = \psi$. Consider the lift

$$\widetilde{\psi}_t \colon \mathbb{R}^{2n} \times M \to \mathbb{R}^{2n} \times M$$

of ψ_t to the universal cover of $\mathrm{T}^{2n} \times M$ and pick $(\tilde{x}, p) \in \pi^{-1}(\{(x,p)\})$. Then, $\widetilde{\psi}(\tilde{x}, p) = (\tilde{x} + a, p)$ for some $a \in \mathbb{Z}^{2n}$. We can then define a symplectic flow $f_t \times \mathrm{id}_M$ on $\mathrm{T}^{2n} \times M$ by setting

$$f_t(y) = y - t \cdot a \mod 1.$$

Note that its lift to the universal cover is given by

$$(\widetilde{f}_t \times \mathrm{id}_M)(y,q) = (y - t \cdot a, q).$$

Therefore, by setting $\varphi_t = f_t \circ \psi_t$, we obtain a symplectic path based at the identity with $\varphi_1 = \psi$, and whose lift $\widetilde{\varphi}_t$ satisfies $\widetilde{\varphi}(\tilde{x}, p) = (\tilde{x}, p)$. Consequently, the loop $\{\varphi_t(x,p)\}$ is contractible in $\mathrm{T}^{2n} \times M$. The corollary then follows by noting that $T^{2n} \times M$ satisfies condition $(\bigstar)$ and by Proposition 1.11. $\square$

2 Preliminaries

2.1 Floer–Novikov cohomology

In Section 2.1.2 we review the construction of Floer–Novikov cohomology after Ono [23]. This is a cohomological version of the construction introduced by Lê and Ono [30] with a slightly smaller coefficient ring. In Section 2.1.3 we review a variant of Floer–Novikov cohomology introduced in Ono [22] which enables the comparison between symplectic paths with different flux. We refer to Lê and Ono [12] for an in-depth discussion of the variants of Floer–Novikov cohomology and the relations among them. We also briefly recall the definition of classical Morse–Novikov cohomology and outline a few important properties it satisfies. For further details on Morse–Novikov cohomology, see [30; 26; 5].

2.1.1 Morse–Novikov cohomology Let M be a closed smooth manifold and let θ be a closed 1–form on M. Fix a ground field $\mathbb{K}$. For our purposes it will be sufficient to consider the case $\mathbb{K} = \mathbb{Q}$. Denote by $I_\theta \colon \pi_1(M) \to \mathbb{R}$ the homomorphism given by integrating θ over a representative loop; that is,

$$I_\theta([\gamma]) = \int_\gamma \theta.$$

In particular, I_θ depends only on the cohomology class $[\theta]$. Denote by $\pi \colon \overline{M}^\theta \to M$ the covering space of M corresponding to $\ker I_\theta \subset \pi_1(M)$. It is the minimal abelian covering on which $\pi^*\theta$ is exact. The covering transformation group is given by $G_\theta = \pi_1(M)/\ker I_\theta$. We denote by Λ_θ the completion of the group ring of G_θ with respect to the filtration induced by I_θ; that is,

$$\Lambda_\theta = \left\{ \sum_i a_i g_i \;\middle|\; a_i \in \mathbb{K} \text{ and } g_i \in G_\theta \text{ satisfy condition (A)} \right\},$$

where condition (A) is:

(A) For each $c \in \mathbb{R}$, the set $\{i \mid a_i \neq 0,\ I_\theta(g_i) < c\}$ is finite.

The fact that $\mathbb{K}$ is a field implies that so is Λ_θ. Let $\bar{f}$ be a choice of primitive for $\pi^*\theta$. Then, to each $\tilde{x} \in \operatorname{Crit}(\bar{f})$ there corresponds a zero $x = \pi(\tilde{x})$ of θ. A 1–form θ is said to be Morse if all the critical points of $\bar{f}$ are nondegenerate. The Morse–Novikov cochain complex in degree k with coefficients in $\mathbb{K}$ is defined as

$$\mathrm{CN}^k(M,\theta) = \left\{ \sum_i a_i \tilde{x}_i \;\middle|\; a_i \in \mathbb{K} \text{ and } \tilde{x}_i \in \operatorname{Crit}(\bar{f}),\ \text{where } \operatorname{index}(\tilde{x}_i) = k,\ \text{satisfy condition (B)} \right\},$$

where condition (B) is:

(B) For each $c \in \mathbb{R}$ the set $\{i \mid a_i \neq 0,\ \bar{f}(\tilde{x}_i) < c\}$ is finite.

Note that $\mathrm{CN}^*(M,\theta)$ is finitely generated over Λ_θ. For a choice of Riemannian metric g on M, the coboundary operator δ is defined by counting bounded gradient trajectories of $\bar{f}$ with respect to the pullback metric π^*g that are emerging from a critical point $\tilde{x} \in \operatorname{Crit}(\bar{f})$ and converging to critical points $\tilde{y} \in \operatorname{Crit}(\bar{f})$ such that $\operatorname{index}(\tilde{y}) - \operatorname{index}(\tilde{x}) = 1$. We may assume that the gradient flow is of Morse–Smale type. The Morse–Novikov cohomology of θ is defined as $\mathrm{HN}^*(M,\theta) = H^*((\mathrm{CN}(M,\theta),\delta))$, and is a finitely generated Λ_θ–module. The resulting cohomology is independent of the choice of Riemannian metric for which the flow is of Morse–Smale type. Furthermore, cohomologous 1–forms have canonically isomorphic Morse–Novikov cohomologies. That is, if $[\theta_1] = [\theta_2]$, then $\mathrm{HN}^*(M,\theta_1)$ is isomorphic to $\mathrm{HN}^*(M,\theta_2)$ as graded vector spaces over $\Lambda_{\theta_1} = \Lambda_{\theta_2}$. More generally, we have the following.

Proposition 2.1 (Lê and Ono [30, Theorem C.2]) *Suppose that θ_1 and θ_2 are closed 1–forms such that $\ker I_{\theta_1} \subset \ker I_{\theta_2}$. Then, for each degree k,*

$$\dim_{\Lambda_{\theta_1}} \mathrm{HN}^k(M,\theta_1) \le \dim_{\Lambda_{\theta_2}} \mathrm{HN}^k(M,\theta_2).$$

In addition, we have the following useful proposition, which distinguishes Morse–Novikov cohomology from usual Morse cohomology.

Proposition 2.2 (Ono [23, Proposition 4.12]) *Let θ be a closed 1–form such that $[\theta] \neq 0$. Then*

$$\dim_{\Lambda_\theta} \mathrm{HN}^k(M,\theta) = 0 \quad \textit{for } k = 0 \textit{ or } k = \dim(M).$$

In particular, when θ is not exact, we have that

$$\dim_{\Lambda_\theta} \mathrm{HN}(M,\theta) < \dim_{\mathbb{Q}} \mathrm{H}(M). \tag{4}$$

2.1.2 Floer–Novikov cohomology Let (M,ω) be a closed symplectic manifold, and $\{\psi_t\}$ a symplectic path based at the identity with endpoint $\psi_1 = \psi$. The deformation lemma in [30] implies that we can suppose, without loss of generality, that there exists a 1–periodic smooth family of smooth functions $F_t \in C^\infty(M)$, with $F_{t+1} = F_t$, such that

$$\iota_{X^t}\omega = \theta + dF_t, \tag{5}$$

where X^t is the time-dependent vector field induced by the symplectic isotopy $\{\psi_t\}$ and θ is a closed 1–form representing $\widetilde{\mathrm{Flux}}(\{\psi_t\})$. When θ is exact, equation (5) becomes the usual Hamiltonian equation for $H_t = F_t + f$, where $df = \theta$. Similarly, we have a formal closed 1–form on the contractible component $\mathcal{L}M$ of the loop space of M. Indeed, for a loop $x \in \mathcal{L}M$ and $v \in T_x\mathcal{L}M = \Gamma(x^*TM)$, we define

$$a_{\{\psi_t\}}(v) = \int_0^1 \omega(v(t), x'(t) - X^t(x(t)))\,dt = \int_0^1 \omega(v(t), x'(t))\,dt + \int_0^1 (\theta + dF_t)(v(t))\,dt.$$

The idea is to find a suitable cover on which $a_{\{\psi_t\}}$ is exact, and then to define Floer–Novikov cohomology as an analog of Morse theory for a primitive $\mathcal{A}_{\psi_t}$. Consider the homomorphisms $\mathcal{I}_\theta, \mathcal{I}_\omega, \mathcal{I}_{c_1} : \pi_1(\mathcal{L}M) \to \mathbb{R}$ defined as

$$\mathcal{I}_\theta = I_\theta \circ \mathrm{Ev}_*, \quad \mathcal{I}_\omega(\{x_s\}) = \int_{C(\{x_s\})} \omega, \quad \mathcal{I}_{c_1}(\{x_s\}) = \langle c_1(M), [C(\{x_s\})]\rangle,$$

for a loop $\{x_s\}_{s\in[0,1]}$ in $\mathcal{L}M$. Here, $\mathrm{Ev}\colon \mathcal{L}M \to M$ is given by evaluation at $t=0$, and $C(\{x_s\})$ is the 2–cycle represented by

$$\mathrm{S}^1 \times \mathrm{S}^1 \ni (s,t) \mapsto x_s(t) \in M.$$

We denote by $\widetilde{\mathcal{L}}M \to \mathcal{L}M$ the covering space of $\mathcal{L}M$ corresponding to

$$\ker(\mathcal{I}_\omega + \mathcal{I}_\theta) \cap \ker \mathcal{I}_{c_1} \subset \pi_1(\mathcal{L}M).$$

A description of $\widetilde{\mathcal{L}}M$ can be given in the following manner. Choose a primitive $\bar{f}$ of $\pi^*\theta$ on $\overline{M}^\theta$, and consider pairs $(\tilde{x}, u)$ composed of a loop $\tilde{x} \in \mathcal{L}\overline{M}^\theta$ and a capping $u\colon \mathbb{D} \to M$ such that $u|_{\partial\mathbb{D}} = \pi \circ \tilde{x}$. Define the equivalence relation $\sim$ given by $(\tilde{x}, u) \sim (\tilde{y}, w)$ if and only if

$$\pi \circ \tilde{x} = \pi \circ \tilde{y}, \quad \int_u \omega + \bar{f}(\tilde{x}(0)) = \int_w \omega + \bar{f}(\tilde{y}(0)), \quad \langle c_1(M), u \# (-w)\rangle = 0.$$

Here, $u\#(-w)$ corresponds to the sphere obtained by gluing the two disks along their common boundaries with the orientation of w reversed. Each element in $\widetilde{\mathscr{L}M}$ corresponds to an equivalence class $[\tilde{x},u]$ of such a pair. With this description, the covering map $\Pi\colon \widetilde{\mathscr{L}M} \to \mathscr{L}M$ is given by $[\tilde{x},u] \mapsto \pi \circ \tilde{x} = x$. A choice of primitive for $\Pi^* a_{\{\psi_t\}}$ is given by

$$\mathscr{A}_{\{\psi_t\}}([\tilde{x},u]) = \int_0^1 (\bar{f} + F_t \circ \pi)(\tilde{x}(t))\,dt + \int_u \omega,$$

where F_t is as in equation (5). The critical points $\mathscr{P}(\{\psi_t\})$ of $\mathscr{A}_{\{\psi_t\}}$ are the lifts to $\widetilde{\mathscr{L}M}$ of the fixed points $x \in \mathrm{Fix}(\psi)$ of ψ such that $[\{\psi_t(x)\}] = 1$ in $\pi_1(M)$. To a critical point $[\tilde{x},u]$ we assign an index $\mathrm{CZ}([\tilde{x},u])$ given by the Conley–Zehnder index of $x = \pi \circ \tilde{x}$ with respect to the trivialization x^*TM, which extends to u^*TM. Note that the covering transformation group of $\mathscr{L}M$ is given by

$$G_{\theta,\omega} = \frac{\pi_1(\mathscr{L}M)}{\ker(\mathscr{I}_\omega + \mathscr{I}_\theta) \cap \ker \mathscr{I}_{c_1}}.$$

Let $\Lambda_{\theta,\omega}$ be Novikov ring given by the completion of the group ring of $G_{\theta,\omega}$ with respect to the filtration induced by $\mathscr{I}_\omega + \mathscr{I}_\theta$; that is,

$$\Lambda_{\theta,\omega} = \left\{ \sum_i a_i g_i \;\middle|\; a_i \in \mathbb{Q} \text{ and } g_i \in G_{\theta,\omega} \text{ satisfy condition (A}')\right\},$$

where condition (A′) is:

(A′) For each $c \in \mathbb{R}$ the set $\{i \mid a_i \neq 0, (\mathscr{I}_\omega + \mathscr{I}_\theta)(g_i) < c\}$ is finite.

Suppose that ψ is nondegenerate and let $J = \{J_t\}$ be a family of ω–compatible almost complex structures. The Floer–Novikov cochain complex is defined as

$$\mathrm{CFN}^k(\{\psi_t\}; J) = \left\{ \sum_i a_i[\tilde{x}_i,u_i] \;\middle|\; a_i \in \mathbb{Q} \text{ and } [\tilde{x}_i,u_i] \in \mathscr{P}(\{\psi_t\}) \text{ satisfy condition (B}')\right\},$$

where condition (B′) is:

(B′) For each $c \in \mathbb{R}$ the set $\{i \mid a_i \neq 0, \mathscr{A}_{\{\psi_t\}}([\tilde{x},u]) < c\}$ is finite and $\mathrm{CZ}([\tilde{x}_i,u_i]) = k$ for all i.

The graded $\mathbb{Q}$–vector space $\mathrm{CFN}^*(\{\psi_t\}; J)$ is endowed with the Floer–Novikov coboundary operator δ_{FN}, which is defined as the signed count of isolated solutions (modulo the $\mathbb{R}$–action) of the asymptotic boundary value problem on maps $u\colon \mathbb{R} \times \mathrm{S}^1 \to M$ defined by the gradient of $\mathscr{A}_{\{\psi_t\}}$; see [30; 23]. In other words, the coboundary operator counts the finite-energy solutions to the Floer equation

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right) = 0, \qquad \lim_{s\to\pm\infty} \tilde{u}(s,t) = \tilde{x}^{\pm}(t),$$

for some lift $\tilde{u}\colon \mathbb{R} \times \mathrm{S}^1 \to M$ such that $\mathrm{CZ}([\tilde{x}^+,u^+]) - \mathrm{CZ}([\tilde{x}^-,u^-]) = 1$ and $[\tilde{x}^+, u^- \# u] = [\tilde{x}^+,u^+]$. The Floer–Novikov cohomology is defined as

$$\mathrm{HFN}^*(\{\psi_t\}; J) = H^*(\mathrm{CFN}(\{\psi_t\}; J), \delta_{\mathrm{FN}}).$$

It can be shown to be independent of the choice of J. In addition, it only depends on the flux of the symplectic path.

Theorem 2.3 (Lê and Ono [30, Theorem 4.3]) *Suppose that $\{\psi_t^{(1)}\}$ and $\{\psi_t^{(2)}\}$ are symplectic paths with nondegenerate endpoints. Suppose that $\{\psi_t^{(1)} \circ (\psi_t^{(2)})^{-1}\}$ has zero flux. Then,*

$$\mathrm{HFN}^*(\{\psi_t^{(1)}\}; J^{(1)}) \cong \mathrm{HFN}^*(\{\psi_t^{(2)}\}; J^{(2)})$$

as $\Lambda_{\theta,\omega}$–modules. Here, $[\theta]$ is the flux of both paths.

Remark When $\{\varphi_H^t\}$ is a Hamiltonian isotopy, $\overline{M}^\theta = M$ and $(\mathrm{CFN}^*(\{\psi\}; J), \delta)$ reduces to the usual Floer cochain complex $(\mathrm{CF}(H; J), \delta)$ associated with the Hamiltonian function H. Therefore, the PSS isomorphism yields

$$\mathrm{HFN}^*(\{\varphi_H^t\}; J) = \mathrm{HF}^*(H; J) \cong \mathrm{H}^{*+n}(M; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_\omega. \tag{6}$$

The main reason for the choice of Novikov ring in this construction is so that we have the following two statements.

Theorem 2.4 (Ono [23, Theorem 4.10]) *Let $\{\psi_t^{(1)}\}$ and $\{\psi_t^{(2)}\}$ be symplectic paths with nondegenerate endpoints $\psi_1^{(1)} = \psi_1^{(2)} = \psi$. Suppose that the symplectic loop $\{\psi_t^{(1)} \circ (\psi_t^{(2)})^{-1}\}$ has trivial evaluation, ie $\mathrm{ev}(\{\psi_t^{(1)} \circ (\psi_t^{(2)})^{-1}\}) = 1$. Then, we have a ring isomorphism $\Psi \colon \Lambda_{\theta_1,\omega} \xrightarrow{\cong} \Lambda_{\theta_2,\omega}$, and*

$$\mathrm{HFN}^*(\{\psi_t^{(1)}\}; J^{(1)}) \cong \mathrm{HFN}^*(\{\psi_t^{(2)}\}; J^{(2)})$$

as $\Lambda_{\theta_1,\omega}$–modules, where $[\theta_i]$ corresponds to the flux of $\{\psi_t^{(i)}\}$ for $i = 1, 2$. The module action of $\Lambda_{\theta_1,\omega}$ on $\mathrm{HFN}^*(\{\psi_t^{(2)}\}, J^{(2)})$ is the one induced by Ψ.

Proof Let $\xi_t = \psi_t^{(1)} \circ (\psi_t^{(2)})^{-1}$. In [23, Theorem 4.10] the conclusion of Theorem 2.4 was proven under the assumption that the map $\Psi \colon \{\gamma(t)\} \mapsto \{\xi_t(\gamma(t))\}$ preserves the component $\mathscr{L}M$ of the free loop space consisting of contractible loops. Therefore, Theorem 2.4 follows from the observation that Ψ preserves $\mathscr{L}M$ whenever $\mathrm{ev}(\{\xi_t\}) = 1$. As detailed in [23, Section 4], it is the map $\Psi \colon \mathscr{L}M \to \mathscr{L}M$ that induces the isomorphism $\mathrm{HFN}^*(\{\psi_t^{(1)}\}; J^{(1)}) \cong \mathrm{HFN}^*(\{\psi_t^{(2)}\}; J^{(2)})$. □

Theorem 2.5 (Ono [23, Theorem 3.12]) *Let $\{\psi_t\}$ be a symplectic path based at identity with sufficiently small flux $[\theta]$. Then,*

$$\mathrm{HFN}^*(\{\psi_t\}; J) \cong \mathrm{HN}^{*+n}(M, \theta) \otimes_{\Lambda_\theta} \Lambda_{\theta,\omega}.$$

Remark The Floer–Novikov cohomology $\mathrm{HFN}^*(\{\psi_t\}; J)$ of a symplectic path $\{\psi_t\}$ is not always isomorphic to the Morse–Novikov cohomology $\mathrm{HN}(M, \theta)$ of its flux $[\theta]$; this has also been observed in Seidel [27]. This can be seen by studying the examples in Jang and Tolman [8], Tolman [28] and McDuff [15] of non-Hamiltonian symplectic circle actions with fixed points. Indeed, in these cases we have symplectic paths with nontrivial flux and trivial evaluation. In particular, if we suppose that such an isomorphism exists, Theorem 2.4 together with Theorem 2.5 would imply that $\dim_{\Lambda_\theta} \mathrm{HN}(M,\theta) = \dim_{\mathbb{Q}} \mathrm{H}(M;\mathbb{Q})$, which is in contradiction to Proposition 2.2. This also shows that the e–homomorphism is not injective in general.

2.1.3 A variant of Floer–Novikov cohomology for changing flux In this section we recall a variant, introduced in [22], of the Floer–Novikov cohomology construction presented in Section 2.1.2, which allows the comparison between symplectic paths that have different flux.

Let (M,ω) be a closed symplectic manifold. Suppose $\{\psi_t\}$ is a symplectic path with endpoint $\psi_1=\psi$ and flux $[\theta]$. Let $p\colon \widetilde{M}\to M$ be an abelian covering space of M such that $p^*\theta$ is exact; $\overline{M}^\theta$ is the smallest choice of such a covering space. Let $\{\widetilde{F}_t\}$ be a smooth family of smooth functions on $\widetilde{M}$ such that $d\widetilde{F}_t = p^*\theta$. Just as before, we would like to make a choice of covering space of $\mathcal{L}M$ on which the pullback of $a_{\{\psi_t\}}$ is exact. We denote by $P\colon \widetilde{\mathcal{L}}\widetilde{M}\to\mathcal{L}M$ the covering space of $\mathcal{L}M$ associated with

$$\ker \mathcal{I}_\omega \cap \ker \mathcal{I}_{c_1} \cap \mathrm{Ev}_*^{-1}(p_*(\pi_1(\widetilde{M}))) \subset \pi_1(\mathcal{L}M).$$

This covering can be seen as the space of pairs $(\tilde{x},u)$, $u|_{\partial\mathbb{D}} = \pi\circ\tilde{x}$, under the equivalence relation defined by $(\tilde{x},u)\sim(\tilde{y},w)$ if and only if

$$\tilde{x}=\tilde{y},\quad \int_u \omega = \int_w \omega,\quad \langle c_1(M), u\,\#\,(-w)\rangle = 0.$$

On $\widetilde{\mathcal{L}}M$, the pullback $P^*a_{\psi_t}$ is exact and a choice of primitive is given by

$$\widetilde{\mathcal{A}}_{\{\psi_t\}}([\tilde{x},u]) = \int_0^1 \widetilde{F}_t(\tilde{x}(t))\,dt + \int_u \omega.$$

Similarly, the critical points $\widetilde{\mathcal{P}}(\{\psi_t\})$ of $\widetilde{\mathcal{A}}_{\{\psi_t\}}$ are lifts to $\widetilde{\mathcal{L}}\widetilde{M}$ of the fixed points $x\in\mathrm{Fix}(\psi)$ of ψ that satisfy $[\{\psi_t(x)\}] = 1\in\pi_1(M)$. Just as in Section 2.1.2, to each critical point $[\tilde{x},u]$ we assign a Conley–Zehnder type index. The covering transformation group of $\widetilde{\mathcal{L}}\widetilde{M}$ is given by

$$\widetilde{G}_{\theta,\omega} = \frac{\pi_1(\mathcal{L}M)}{\ker \mathcal{I}_\omega \cap \ker \mathcal{I}_{c_1} \cap \mathrm{Ev}_*^{-1}(p_*(\pi_1(\widetilde{M})))},$$

and we denote by $\widetilde{\Lambda}_{\theta,\omega}$ the Novikov ring given by the completion of its group ring with respect to the filtration induced by $\mathcal{I}_\omega + \mathcal{I}_\theta$; that is,

$$\widetilde{\Lambda}_{\theta,\omega} = \left\{ \sum_i a_i g_i \;\middle|\; a_i\in\mathbb{Q} \text{ and } g_i\in\widetilde{G}_{\theta,\omega} \text{ satisfy condition (A}'\text{)} \right\}.$$

Suppose that ψ is nondegenerate and let $J=\{J_t\}$ be a family of ω–compatible almost complex structures. The Floer–Novikov cochain complex is defined as

$$\mathrm{CFN}^k(\{\psi_t\},\widetilde{M};J) = \left\{ \sum_i a_i[\tilde{x}_i,u_i] \;\middle|\; a_i\in\mathbb{Q} \text{ and } [\tilde{x}_i,u_i]\in\widetilde{\mathcal{P}}(\{\psi_t\}) \text{ satisfy condition (B}'\text{)} \right\}.$$

The coboundary operator $\widetilde{\delta}_{FN}$ is defined by the same formula as in Section 2.1.2. The Floer–Novikov homology $\mathrm{HFN}_*(\{\psi_t\}, \widetilde{M}; J)$ associated with the covering space $\widetilde{M}$ is defined as the homology of $(\mathrm{CFN}(\{\psi_t\}, \widetilde{M}; J), \widetilde{\delta}_{FN})$. It is independent of the choice of almost complex structure J and depends only on the flux of the symplectic path. The following theorem allows the comparison between the ranks of the Floer–Novikov cohomology of symplectic paths with flux lying in the kernel of p^*.

Proposition 2.6 (Ono [22, Proposition 4.8]) *Let $\{\psi_t^{(1)}\}$ and $\{\psi_t^{(2)}\}$ be symplectic paths such that $\widetilde{\mathrm{Flux}}(\{\psi_t^{(i)}\}) \in \ker\{p^*\colon \mathrm{H}^1(M;\mathbb{R}) \to \mathrm{H}^1(\widetilde{M};\mathbb{R})\}$ for $i = 1, 2$. Then,*

$$\mathrm{rank}_{\widetilde{\Lambda}_{\theta_1,\omega}} \mathrm{HFN}^*(\{\psi_t^{(1)}\}, \widetilde{M}; J^{(1)}) = \mathrm{rank}_{\widetilde{\Lambda}_{\theta_2,\omega}} \mathrm{HFN}^*(\{\psi_t^{(2)}\}, \widetilde{M}; J^{(2)}),$$

where $[\theta_i]$ corresponds to the flux of $\{\psi_t^{(i)}\}$ for $i = 1, 2$.

3 Proof of Theorem 1.3

3.1 Proof using Floer–Novikov theory

When (M, ω) is either symplectically aspherical or satisfies the weak Lefschetz property, we have presented proofs based on classical arguments in Section 1.1. We shall, therefore, consider the spherically monotone case. Let $\{\psi_t\}$ be a symplectic loop with

$$\widetilde{\mathrm{Flux}}(\{\psi_t\}) = [\theta] \in \Gamma.$$

Observe that spherical monotonicity implies that

$$\ker(\mathcal{I}_\omega + \mathcal{I}_\theta) \cap \ker \mathcal{I}_{c_1} = \ker \mathcal{I}_\omega \cap \ker \mathcal{I}_\theta \cap \ker \mathcal{I}_{c_1}. \tag{7}$$

Set $\widetilde{M} = \overline{M}^\theta$ in the construction of the variant of Floer–Novikov homology defined in Section 2.1.3. In this case, note that $\mathrm{Ev}_*^{-1}\bigl(p_*(\pi_1(\overline{M}^\theta))\bigr) = \ker \mathcal{I}_\theta$. Indeed, this is a consequence of $\pi_1(\overline{M}^\theta) = \ker I_\theta$, which follows from the defining property of $\overline{M}^\theta$, and of the equality $\mathcal{I}_\theta = I_\theta \circ \mathrm{Ev}_*$. These observations allow one to deduce that for any symplectic path $\{\xi_t\}$ with flux $[\theta]$, we have $\widetilde{\Lambda}_{\theta,\omega} = \Lambda_{\theta,\omega}$, and

$$(\mathrm{CFN}^*(\{\xi_t\}, \overline{M}^\theta; J), \widetilde{\delta}_{FN}) = (\mathrm{CFN}^*(\{\xi_t\}; J), \delta_{FN}) \tag{8}$$

by definition (we reiterate that monotonicity is used in an important way; see the remark below).

Remark In general, $\mathrm{HFN}_*(\{\psi_t\}, \widetilde{M}; J)$ is different from $\mathrm{HFN}_*(\{\psi_t\}; J)$ even in the case $\widetilde{M} = \overline{M}^\theta$. Nonetheless, whenever $\ker \mathcal{I}_{c_1} \subset \ker \mathcal{I}_\omega$ we have that equation (8) holds and, arguing as in the preceding paragraph, that

$$\mathrm{HFN}_*(\{\psi_t\}, \overline{M}^\theta; J) = \mathrm{HFN}_*(\{\psi_t\}; J).$$

This holds for (M, ω) spherically (positive or negative) monotone or symplectically aspherical. We refer to [12] for more details on the relationship between these cohomology theories.

Now, further suppose that $[\{\psi_t\}] \in \ker \mathrm{ev}$. Let $\{\varphi_t\}$ be a Hamiltonian path with endpoint φ, which is generated by a nondegenerate Hamiltonian H, and set $\psi'_t = \psi_t \circ \varphi_t$. Then $\{\psi'_t\}$ and $\{\varphi_t\}$ are two symplectic paths with nondegenerate endpoints $\psi' = \varphi$. Since $\mathrm{ev}(\{\psi_t\}) = 1$, Theorem 2.4 implies that $\Lambda_{\theta,\omega} \cong \Lambda_\omega$ and that

$$\mathrm{HFN}^*(\{\psi'_t\}, J') \cong \mathrm{HFN}^*(\{\varphi_t\}; J) \tag{9}$$

as $\Lambda_{\theta,\omega}$–modules. Since φ_t is Hamiltonian, we have that

$$\mathrm{HFN}^*(\{\varphi_t\}; J) \cong \mathrm{HF}^*(H; J) \cong \mathrm{H}^*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_\omega. \tag{10}$$

For $\varepsilon > 0$, let $\{\psi_t^{\varepsilon\theta}\}_{t\in[0,1]}$ be the symplectic path induced by the symplectic vector field $X_{\varepsilon\theta}$ defined by $\iota_{X_{\varepsilon\theta}}\omega = \varepsilon\theta$. Then, for all $\varepsilon > 0$, we have that

$$\widetilde{\mathrm{Flux}}(\{\psi_t^{\varepsilon\theta}\}) = \varepsilon[\theta] \in \ker\{\pi^* : \mathrm{H}^1(M; \mathbb{R}) \to \mathrm{H}^1(\overline{M}^\theta; \mathbb{R})\}.$$

Therefore, Proposition 2.6 implies that

$$\mathrm{rank}_{\widetilde{\Lambda}_{\theta,\omega}} \mathrm{HFN}^*(\{\psi'_t\}, \overline{M}^\theta; J) = \mathrm{rank}_{\widetilde{\Lambda}_{\varepsilon\theta,\omega}} \mathrm{HFN}^*(\{\psi_t^{\varepsilon\theta}\}, \overline{M}^\theta; J). \tag{11}$$

Finally, equations (8)–(11), Theorem 2.5 and Proposition 2.1 imply that for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}
\mathrm{rank}_{\Lambda_\omega} \mathrm{H}^*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_\omega &= \mathrm{rank}_{\Lambda_\omega} \mathrm{HFN}^*(\{\varphi_t\}; J) = \mathrm{rank}_{\Lambda_{\theta,\omega}} \mathrm{HFN}^*(\{\psi'_t\}, J') \\
&= \mathrm{rank}_{\widetilde{\Lambda}_{\theta,\omega}} \mathrm{HFN}^*(\{\psi'_t\}, \overline{M}^\theta; J) = \mathrm{rank}_{\widetilde{\Lambda}_{\varepsilon\theta,\omega}} \mathrm{HFN}^*(\{\psi_t^{\varepsilon\theta}\}, \overline{M}^\theta; J) \\
&= \mathrm{rank}_{\Lambda_{\varepsilon\theta,\omega}} \mathrm{HFN}^*(\{\psi_t^{\varepsilon\theta}\}; J) = \mathrm{rank}_{\Lambda_{\varepsilon\theta,\omega}} \mathrm{HN}^*(M, \varepsilon\theta) \otimes_{\Lambda_{\varepsilon\theta}} \Lambda_{\varepsilon\theta,\omega} \\
&= \mathrm{rank}_{\Lambda_{\theta,\omega}} \mathrm{HN}^*(M, \theta) \otimes_{\Lambda_\theta} \Lambda_{\theta,\omega}.
\end{aligned}$$

Hence, θ must be exact. Indeed, if $[\theta] \neq 0$, then by Proposition 2.2

$$\begin{aligned}
\mathrm{rank}_{\Lambda_\omega} \mathrm{H}^*(M; \mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_\omega &= \dim_{\mathbb{Q}} \mathrm{H}^*(M; \mathbb{Q}) \\
&> \dim_{\Lambda_\theta} \mathrm{HN}^{*+n}(M, \theta) = \mathrm{rank}_{\Lambda_{\theta,\omega}} \mathrm{HN}^{*+n}(M, \theta) \otimes_{\Lambda_\theta} \Lambda_{\theta,\omega},
\end{aligned}$$

in contradiction with the above equalities. This concludes the proof of Theorem 1.3.

3.2 A proof using a result of McDuff

It follows from McDuff [14, Theorem 1] that if $\{\psi_t\}$ is a symplectic loop, then $\mathscr{I}_{c_1}$ vanishes on the elements of $\pi_1(\mathscr{L}M)$ that are represented by $\{\psi_t(\gamma(s))\}_{s,t\in[0,1]}$ for a loop $\gamma \in \mathscr{L}M$. If $\{\psi_t\}$ has trivial evaluation, then $\{\psi_t(\gamma(s))\}_{s,t\in[0,1]}$ determines a homotopy class $A_\gamma \in \pi_2(M)$ with $\langle c_1(M), A_\gamma\rangle = 0$. If (M, ω) is spherically monotone, for every 1–cycle in M given by a loop γ we have that

$$\langle \widetilde{\mathrm{Flux}}(\{\psi_t\}), [\gamma]\rangle = \langle [\omega], A_\gamma\rangle = \lambda \langle c_1(M), A_\gamma\rangle = 0.$$

Therefore, $\widetilde{\mathrm{Flux}}(\{\psi_t\}) = 0$, which yields once again the conclusion of Theorem 1.3. In addition, note that if $\{\psi_t\}$ has trivial evaluation and flux $[\theta] \neq 0$ we can produce a homotopy class $\alpha \in \pi_1(\mathscr{L}M)$ such that $\alpha \in \ker(\mathscr{I}_\omega + \mathscr{I}_\theta) \cap \ker \mathscr{I}_{c_1}$ while $\mathscr{I}_\theta(\alpha) = -\mathscr{I}_\omega(\alpha) \neq 0$. Indeed the class represented by the loop-of-loops $\alpha(s,t) = \psi_t(\gamma(s))$ satisfies these properties. We are then able to conclude the following.

Corollary 3.1 *If the e–homomorphism is not injective, then*

$$\ker \mathcal{I}_\omega \cap \ker \mathcal{I}_\theta \cap \ker \mathcal{I}_{c_1} \subsetneq \ker(\mathcal{I}_\omega + \mathcal{I}_\theta) \cap \ker \mathcal{I}_{c_1}$$

for all $[\theta] \in \ker e$.

This shows, in hindsight, why the equality of Novikov rings in the spherically monotone setting yielded a proof of injectivity of the e–homomorphism.

Acknowledgements

This work is part of the author's PhD thesis carried out at the Université de Montréal under the supervision of Egor Shelukhin, whom we thank for the inspiring interactions and countless suggestions. We thank Dustin Connery-Grigg for the suggestion to re-examine McDuff's result [14], which yielded an alternative proof of the main theorem. This research was partially supported by Fondation Courtois, by the ISM's excellence scholarship, and by the J Armand Bombardier excellence scholarship. This material is based upon work supported by the National Science Foundation under grant DMS-1928930, while the author was in residence at the Simons Laufer Mathematical Sciences Institute (previously known as MSRI), Berkeley, California during the Fall 2022 semester.

References

[1] **M S Atallah**, **E Shelukhin**, *Hamiltonian no-torsion*, Geom. Topol. 27 (2023) 2833–2897 MR Zbl

[2] **A Banyaga**, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique*, Comment. Math. Helv. 53 (1978) 174–227 MR Zbl

[3] **Y Cho**, **M K Kim**, *An example of circle actions on symplectic Calabi–Yau manifolds with non-empty fixed points*, preprint (2013) arXiv 1304.0540

[4] **A Dimca**, *Singularities and topology of hypersurfaces*, Springer (1992) MR Zbl

[5] **M Farber**, *Topology of closed one-forms*, Math. Surv. Monogr. 108, Amer. Math. Soc., Providence, RI (2004) MR Zbl

[6] **T Frankel**, *Fixed points and torsion on Kähler manifolds*, Ann. of Math. 70 (1959) 1–8 MR Zbl

[7] **V L Ginzburg**, *Some remarks on symplectic actions of compact groups*, Math. Z. 210 (1992) 625–640 MR Zbl

[8] **D Jang**, **S Tolman**, *Non-Hamiltonian actions with fewer isolated fixed points*, Int. Math. Res. Not. 2023 (2023) 6045–6077 MR Zbl

[9] **J Kędra**, **D Kotschick**, **S Morita**, *Crossed flux homomorphisms and vanishing theorems for flux groups*, Geom. Funct. Anal. 16 (2006) 1246–1273 MR Zbl

[10] **D Kotschick**, *Free circle actions with contractible orbits on symplectic manifolds*, Math. Z. 252 (2006) 19–25 MR Zbl

[11] **F Lalonde**, **D McDuff**, **L Polterovich**, *On the flux conjectures*, from "Geometry, topology, and dynamics" (F Lalonde, editor), CRM Proc. Lecture Notes 15, Amer. Math. Soc., Providence, RI (1998) 69–85 MR Zbl

[12] **H V Lê**, **K Ono**, *Floer–Novikov cohomology and symplectic fixed points, revisited*, Proc. Int. Geom. Cent. 13 (2020) 89–114 MR Zbl

[13] **G Lupton**, **J Oprea**, *Cohomologically symplectic spaces: toral actions and the Gottlieb group*, Trans. Amer. Math. Soc. 347 (1995) 261–288 MR Zbl

[14] **D McDuff**, *Symplectic diffeomorphisms and the flux homomorphism*, Invent. Math. 77 (1984) 353–366 MR Zbl

[15] **D McDuff**, *The moment map for circle actions on symplectic manifolds*, J. Geom. Phys. 5 (1988) 149–160 MR Zbl

[16] **D McDuff**, *A survey of the topological properties of symplectomorphism groups*, from "Topology, geometry and quantum field theory" (U Tillmann, editor), Lond. Math. Soc. Lect. Note Ser. 308, Cambridge Univ. Press (2004) 173–193 MR Zbl

[17] **D McDuff**, *Hamiltonian S^1–manifolds are uniruled*, Duke Math. J. 146 (2009) 449–507 MR Zbl

[18] **D McDuff**, **D Salamon**, *J–holomorphic curves and symplectic topology*, Amer. Math. Soc. Colloq. Publ. 52, Amer. Math. Soc., Providence, RI (2004) MR Zbl

[19] **D McDuff**, **D Salamon**, *Introduction to symplectic topology*, 3rd edition, Oxford Univ. Press (2017) MR Zbl

[20] **K Ono**, *Equivariant projective imbedding theorem for symplectic manifolds*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 35 (1988) 381–392 MR Zbl

[21] **K Ono**, *Obstruction to circle group actions preserving symplectic structure*, Hokkaido Math. J. 21 (1992) 99–102 MR Zbl

[22] **K Ono**, *Floer–Novikov cohomology and symplectic fixed points*, J. Symplectic Geom. 3 (2005) 545–563 MR Zbl

[23] **K Ono**, *Floer–Novikov cohomology and the flux conjecture*, Geom. Funct. Anal. 16 (2006) 981–1020 MR Zbl

[24] **L Polterovich**, *Growth of maps, distortion in groups and symplectic geometry*, Invent. Math. 150 (2002) 655–686 MR Zbl

[25] **L Polterovich**, *Floer homology, dynamics and groups*, from "Morse theoretic methods in nonlinear analysis and in symplectic topology" (P Biran, O Cornea, F Lalonde, editors), NATO Sci. Ser. II Math. Phys. Chem. 217, Springer (2006) 417–438 MR Zbl

[26] **L Polterovich**, **D Rosen**, *Function theory on symplectic manifolds*, CRM Monogr. Ser. 34, Amer. Math. Soc., Providence, RI (2014) MR Zbl

[27] **P Seidel**, *Braids and symplectic four-manifolds with abelian fundamental group*, Turkish J. Math. 26 (2002) 93–100 MR Zbl

[28] **S Tolman**, *Non-Hamiltonian actions with isolated fixed points*, Invent. Math. 210 (2017) 877–910 MR Zbl

[29] **S Tolman**, **J Weitsman**, *On semifree symplectic circle actions with isolated fixed points*, Topology 39 (2000) 299–309 MR Zbl

[30] **L H Vân**, **K Ono**, *Symplectic fixed points, the Calabi invariant and Novikov homology*, Topology 34 (1995) 155–176 MR Zbl

Department of Mathematics and Statistics, University of Montreal
Montreal QC, Canada

marcelo.sarkis.atallah@umontreal.ca

Received: 20 December 2022 Revised: 14 May 2023

Algebraic & Geometric Topology **24**:4 (2024) 2385–2387
DOI: 10.2140/agt.2024.24.2385
Published: 16 July 2024

Correction to the article
Hopf ring structure on the mod p cohomology of symmetric groups

LORENZO GUERRA

We correct Propositions 5.4, 5.5 and 5.6 of the author's previous article (Algebr. Geom. Topol. 17 (2017) 957–982).

20J06

In our previous paper [1], the formulas describing the action of the mod p Steenrod algebra on Hopf ring generators are incorrect. This is not caused by a flaw in the general argument, but by our misstatement of Hung and Minh's theorem on mod p modular invariants. In this note, we refer to our original article for notation and conventions.

We begin by providing the accurate statement of Hung and Minh's theorem. Although these two authors decided to compute coefficients explicitly, we believe that, for our purpose, their result is best restated using the total Steenrod class $\mathcal{P}^* = \sum_{r\geq 0} \mathcal{P}^r$.

Hung and Minh's theorem [2, Theorem 4.1, page 42] *In* $[\mathbb{Z}_p[y_1, \dots, y_n] \otimes \Lambda(x_1, \dots, x_n)]^{\mathrm{Gl}_n(\mathbb{Z}_p)}$, *the algebra of modular invariants for* $\mathrm{Gl}_n(\mathbb{Z}_p)$, *for all* $0 \leq s < n$ *and* $0 \leq s_1 < \cdots < s_k < n$,

$$\mathcal{P}^*(d_{s,n-s}) = (d_{0,n}+\cdots+d_{s,n-s})(d_{0,n}+\cdots+d_{n-1,1}+1)^{p-1} - (d_{0,n}+\cdots+d_{s-1,n-s+1})^p,$$

$$\mathcal{P}^*(R_{n,s_1,\dots,s_k}) = \sum_{\substack{0\leq t_1<t_2<\cdots<t_{k+1}\leq n \\ s_{i-1}<t_i\leq s_i}} \sum_{i=1}^{k+1} [(-1)^{k+1-i} R_{n,t_1,\dots,\hat{t}_i,\dots,t_{k+1}} d_{t_i,n-t_i}](d_{0,n}+\cdots+d_{n-1,1}+1)^{p-2}.$$

In the expression above, we let, by convention, $s_{-1} = -1$, $s_{k+1} = n$, $R_{n,t_1,\dots,t_k,n} = 0$, *and* $d_{n,0} = 1$.

We now provide the corrected formulas for the Steenrod powers of generators.

Corrected version of Proposition 5.4 *Let* $0 \leq k < n$. *Let* $\mathrm{Outgrowth}(\gamma_{n-k,p^k})$ *be the set of full-width Hopf monomials* $x \in H^*(\Sigma_{p^n}; \mathbb{Z}_p)$ *of type* C *with* $\mathrm{effsc}(x) \geq n-k$ *and* $\mathrm{ht}(x) \leq p$. *Then, for all* $0 \leq r \leq p^n - p^k$,

$$\mathcal{P}^r(\gamma_{n-k,p^k}) = \sum_{\substack{x\in \mathrm{Outgrowth}(\gamma_{n-k,p^k}) \\ \deg(x)=2(p^n-p^k+r(p-1))}} c_{n,k,x} x,$$

where $c_{n,k,x}$ is a scalar coefficient calculated as follows.

If $x=\prod_{i=1}^{n}\gamma_{i,p^{n-i}}^{e_i}$ $(e_i\geq 0)$ is a gathered block, then

$$c_{n,k,x}=(-1)^{n-k+\sum_{i=1}^{n} ie_i}\frac{1}{p}\frac{p!}{\prod_{i=0}^{n}e_i!}\biggl(\sum_{i=n-k}^{n}e_i\biggr),$$

where we let by convention $e_0=p-\sum_{i=1}^{n}e_i$ and we reduce $c_{n,k,x}$ mod p after calculating its expression in $\mathbb{Z}$ first. If x is not a gathered block, we can write x as a nonzero scalar multiple of a transfer product $b_1\odot\cdots\odot b_r$, where each b_j is a column of width equal to a power p^{l_j} of p. Then, we put $c_{n,k,x}=\prod_{j=1}^{r}c_{l_j,l_j-n+k,b_j}$.

Corrected version of Proposition 5.5 • *Let $1\leq j\leq k$. Let $\mathrm{Outgrowth}(\alpha_{j,k})$ be the set of full-width Hopf monomials $x\in H^*(\Sigma_{p^k};\mathbb{Z}_p)$ of type A with $\mathrm{effsc}(x)\geq k$ and $\mathrm{ht}(x)\leq p$. Then, for all $0\leq r<p^k-p^{k-j}$,*

$$\mathcal{P}^r(\alpha_{j,k})=\sum_{\substack{x\in\mathrm{Outgrowth}(\alpha_{j,k})\\ \deg(x)=2p^k-2p^{k-j}-1+2r(p-1)}}c'_{k,j,x}x,$$

where $c'_{k,j,x}$ is a scalar coefficient calculated as follows.

If $x=\alpha_{t,k}\prod_{i=1}^{k}\gamma_{i,p^{k-i}}^{e_i}$ with $e_i\geq 0$ and $1\leq t\leq k$ is a gathered block, then

$$c'_{k,j,x}=\begin{cases}(-1)^{j+t+\sum_{i=1}^{k} ie_i}\bigl((p-1)!/\prod_{i=0}^{k}e_i!\bigr)\bigl(\sum_{l=j}^{k}e_l\bigr) & \text{if } 1\leq t<j,\\ -(-1)^{j+t+\sum_{i=1}^{k} ie_i}\bigl((p-1)!/\prod_{i=0}^{k}e_i!\bigr)\bigl(\sum_{l=0}^{j-1}e_l\bigr) & \text{if } j\leq t\leq k,\end{cases}$$

where we let by convention $e_0=p-1-\sum_{i=1}^{n}e_i$. All Hopf monomials $x\in\mathrm{Outgrowth}(\alpha_{j,k})$ have this form.

• *Let $1\leq i<j\leq k$. Let $\mathrm{Outgrowth}(\beta_{i,j,p^{k-j}})$ be the set of full-width Hopf monomials $x\in H^*(\Sigma_{p^k};\mathbb{Z}_p)$ of type B with $\mathrm{effsc}(x)\geq j$ and $\mathrm{ht}(x)\leq p$. Then, for all $0\leq r\leq p^k-p^{k-j}-p^{k-i}$,*

$$\mathcal{P}^r(\beta_{i,j,p^{k-j}})=\sum_{\substack{x\in\mathrm{Outgrowth}(\beta_{i,j,p^{k-j}})\\ \deg(x)=2(p^k-p^{k-j}-p^{k-i}+r(p-1))}}c''_{k,i,j,x}x,$$

where $c''_{k,i,j,x}$ is a scalar coefficient calculated as follows.

If $x=\beta_{t,u,p^{k-u}}\prod_{i=1}^{k}\gamma_{i,p^{k-i}}^{e_i}$ with $e_i\geq 0$ and $1\leq t<u\leq k$ is a gathered block, then

$$c''_{k,i,j,x}=\begin{cases}-(-1)^{i+j+t+u+\sum_{m=1}^{k} me_m}\bigl((p-1)!/\prod_{m=0}^{k}e_m!\bigr)\bigl(\sum_{l=j}^{k}e_l\bigr) & \text{if } 1\leq t<i\leq u<j\leq k,\\ (-1)^{i+j+t+u+\sum_{m=1}^{k} me_m}\bigl((p-1)!/\prod_{m=0}^{k}e_m!\bigr)\bigl(\sum_{l=i}^{j-1}e_l\bigr) & \text{if } 1\leq t<i<j\leq u\leq k,\\ -(-1)^{i+j+t+u+\sum_{m=1}^{k} me_m}\bigl((p-1)!/\prod_{m=0}^{k}e_m!\bigr)\bigl(\sum_{l=0}^{i-1}e_l\bigr) & \text{if } i\leq t<j\leq u\leq k,\\ 0 & \text{otherwise,}\end{cases}$$

where we let by convention $e_0 = p - 1 - \sum_{m=1}^{k} e_m$. If x is not a gathered block, we can write x as a nonzero scalar multiple of a transfer product $b_1 \odot \cdots \odot b_r$, where each b_m is a column of width equal to a power p^{l_m} of p. Then, we put $c''_{k,i,j,x} = \prod_{m=1}^{r} c''_{l_m,i,j,b_m}$.

Corrected version of Proposition 5.6 *The following formulas hold*:

- $\beta(\alpha_{j,k}) = \gamma_{k,1}$ *if* $j = k$ *and is equal to* 0 *otherwise.*
- $\beta(\beta_{i,j,p^k}) = \alpha_{i,j} \odot \beta_{i,j,p^k-1}$.
- $\beta(\gamma_{j,p^k}) = 0$.

The proof of these propositions is essentially unchanged from the author's original article, except that we use the correct statement of Hung and Mihn's theorem to compute coefficients.

References

[1] **L Guerra**, *Hopf ring structure on the* mod *p cohomology of symmetric groups*, Algebr. Geom. Topol. 17 (2017) 957–982 MR Zbl

[2] **N H V Hung**, **P A Minh**, *The action of the mod p Steenrod operations on the modular invariants of linear groups*, Vietnam J. Math. 23 (1995) 39–56 MR Zbl

Dipartimento di Matematica, Università di Roma Tor Vergata
Rome, Italy

guerra@axp.mat.uniroma2.it

Received: 21 December 2023

Guidelines for Authors

Submitting a paper to Algebraic & Geometric Topology

Papers must be submitted using the upload page at the AGT website. You will need to choose a suitable editor from the list of editors' interests and to supply MSC codes.

The normal language used by the journal is English. Articles written in other languages are acceptable, provided your chosen editor is comfortable with the language and you supply an additional English version of the abstract.

Preparing your article for Algebraic & Geometric Topology

At the time of submission you need only supply a PDF file. Once accepted for publication, the paper must be supplied in LaTeX, preferably using the journal's class file. More information on preparing articles in LaTeX for publication in AGT is available on the AGT website.

`arXiv` papers

If your paper has previously been deposited on the `arXiv`, we will need its `arXiv` number at acceptance time. This allows us to deposit the DOI of the published version on the paper's `arXiv` page.

References

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited at least once in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used, but will be converted to the house style (see a current issue for examples).

Figures

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Fuzzy or sloppily drawn figures will not be accepted. For labeling figure elements consider the pinlabel LaTeX package, but other methods are fine if the result is editable. If you're not sure whether your figures are acceptable, check with production by sending an email to graphics@msp.org.

Proofs

Page proofs will be made available to authors (or to the designated corresponding author) in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 4 (pages 1809–2387) 2024