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To each oriented surface Σ, we associate a differential graded category $\mathcal{K}o(\Sigma)$. The homotopy category $\mathrm{Ho}(\mathcal{K}o(\Sigma))$ is a triangulated category which satisfies properties akin to those of the contact categories studied by K Honda. These categories are also related to the algebraic contact categories of Y Tian and to the bordered sutured categories of R Zarev.

53D10; 18G55

1 Introduction

Our purpose here is to associate a differential graded category $\mathcal{K}o(\Sigma)$ to each oriented surface Σ. This category is used to study comparison problems between the categories associated to surfaces by Seiberg–Witten-type manifold invariants. For example, we prove that the categories associated to the disk $(D^2, 2n)$ with $2n$ marked points by each theory are equivalent, and there is a functorial relationship between the categories associated to a surfaces with boundary when they can be defined.

1.1 The unicity of Floer-type invariants of 3–manifolds

In [42; 43] P Ozsváth and Z Szabó introduced invariants of 3–manifolds known as the Heegaard–Floer homologies. Depending upon the setting of a parameter U, there are homology groups: $HF^-_*(M)$, $HF^+_*(M)$ and $HF^\infty_*(M)$ which fit into a long exact sequence:

$$\cdots \to HF^-_*(M) \to HF^\infty_*(M) \to HF^+_*(M) \to \cdots. \tag{1-1}$$

When $U=0$, there are simpler invariants $\widehat{HF}_*(M)$. The Heegaard–Floer theory has had a profound effect on the study of 3–manifolds and 4–manifolds; see Juhász [23]. This is in part because it was originally conceived of as a way to obtain information in the Seiberg–Witten invariants; see Donaldson [7], Kronheimer and Mrowka [28] and Witten [61]. The relationship between the Heegaard–Floer homology theory and the Seiberg–Witten Floer homology was recently articulated by two independent groups of researchers: Ç Kutluhan, Y-J Lee and C H Taubes [29; 30; 31; 32; 33] and V Colin, P Ghiggini and K Honda [3; 4; 5]. Both teams built upon the earlier work of Taubes [49; 50; 51; 52; 53], which identified the Seiberg–Witten Floer homologies $\widehat{HM}_*(M)$ with the embedded contact homology $ECH_*(M)$ due to M Hutchings [20] and Hutchings and Taubes [21; 22]:

$$\Omega\colon ECH_*(M) \xrightarrow{\sim} \widehat{HM}_*(M).$$

Using the embedded contact homology as an intermediary, both groups completed the diagram

$$\begin{array}{ccc} & & HF^-_*(M) \\ & \nearrow & \downarrow \\ ECH_*(M) & & \\ & \searrow_{\Omega} & \\ & & \widehat{HM}_*(M) \end{array}$$

in a fashion which preserved essential properties of the three homology theories. In particular, the maps defined respect decompositions with respect to $\mathrm{Spin}^{\mathbb{C}}$ structures, carry invariants of contact structures to invariants of contact structures, preserve the long exact sequence (1-1) and support reductions to the simpler $U=0$ theory:

$$\widehat{ECH}_*(M) \cong \widehat{HF}_*(M) \cong \widetilde{HM}_*(M). \tag{1-2}$$

Intuitively, each component in the equation above corresponds to a codimension-1 piece of a 4–dimensional topological field theory. It is evident that such a theory satisfies the following properties. In codimension 1, a topological field theory associates a chain complex $C(M)$ to each oriented 3–manifold M. The homology of this chain complex $H_*(C(M))$ is an invariant of the diffeomorphism type of the 3–manifold. In codimension 2, a topological field theory associates a differential graded category $\mathcal{C}(\Sigma)$ to each oriented surface Σ. The derived category $D(\mathcal{C}(\Sigma))$ of right $\mathcal{C}(\Sigma)$–modules (see Keller [25; 26]) is an invariant of the diffeomorphism type of the surface, and reversing the orientation of the surface produces the opposite dg category:

$$\mathcal{C}(\overline{\Sigma}) \cong \mathcal{C}(\Sigma)^{\mathrm{op}}.$$

For each 3–manifold X with boundary $\partial X = \Sigma$, there is a right $\mathcal{C}(\Sigma)$–module X_*. When a 3–manifold M is split along a surface $M = X \cup_\Sigma Y$, the invariant $C(M)$ corresponding to M is quasi-isomorphic to the tensor product,

$$C(M) \xrightarrow{\sim} X_* \otimes^{\mathbb{L}}_{\mathcal{C}(\Sigma)} (Y_*)^{\mathrm{op}},$$

of the modules associated to each piece. If the identifications made by (1-2) result from an equivalence between topological field theories, then the codimension-2 extensions of these topological field theories must be equivalent as well.

Question 1.1 *Is there an equivalence between codimension-*2 *extensions of Seiberg–Witten Floer*, *Heegaard–Floer and embedded contact homology*?

We study the simpler question of establishing a relationship between the categories associated to oriented surfaces Σ by Heegaard–Floer theory and contact topology.

The Heegaard–Floer homology $\widehat{HF}^*(M)$ was extended to surfaces and 3–manifolds with boundary, in the manner described above, by Ozsváth, R Lipshitz and D Thurston [36]. The theory was further developed by R Zarev [62; 63]. In particular, when an oriented surface Σ sports a handle decomposition, determined by combinatorial data $\mathcal{Z}$ called an *arc parametrization*, there is a dg category $\mathcal{A}(-\mathcal{Z})$ which is associated to the surface Σ. The Morita homotopy class of the corresponding categories of dg modules are independent of the handle decomposition $\mathcal{Z}$.

On the contact side, Honda has conjectured the existence of a family of triangulated categories $\mathcal{C}o(\Sigma)$ associated to oriented surfaces Σ called *contact categories* [15]. These categories might function as part of a codimension-2 component of the embedded contact homology. The morphisms of contact categories are isotopy classes of tight contact structures on a thickened surface $\Sigma \times [0, 1]$. Maps in $\mathcal{C}o(\Sigma)$ are composed by gluing $\Sigma \times [0, 1]$ to $\Sigma \times [0, 1]$ and rescaling. The contact categories $\mathcal{C}o(\Sigma)$ are conjectured to contain distinguished triangles associated to special contact structures called bypass moves. Unfortunately, this construction is not yet available in its full generality. For disks and annuli, algebraic analogues of these categories were introduced and studied by Y Tian [54; 55].

1.2 Summary of main results

We associate a $\mathbb{Z}/2$–linear dg category $\mathcal{K}o(\Sigma)$ to each oriented surface Σ. This category satisfies a universal property which guarantees the existence of a unique map to a dg enhancement of any contact category $\mathcal{C}o(\Sigma)$, when it exists.

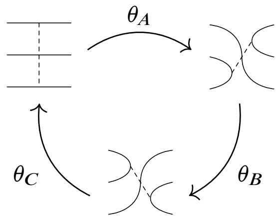
Universal property 1.2 *Suppose $\mathcal{X}$ is a pretriangulated dg category for which there are choices of maps $\theta\colon \gamma \to \gamma'$ corresponding to bypass moves between dividing sets $\gamma, \gamma' \subset \Sigma$, and these maps satisfy four properties*:

(1) *Bypass moves are cycles.*

(2) *Trivial bypass moves are equal to the identity.*

(3) *Disjoint bypass moves commute.*

(4) *Associated to each bypass move is an exact triangle of the form*



Then there is a unique map $\mathcal{K}o(\Sigma) \to \mathfrak{X}$ in the homotopy category of differential graded categories. See Section 3 for details.

Section 2 contains algebraic background necessary to produce and study $\mathcal{K}o(\Sigma)$. The definition of pretriangulated hull and a review of Drinfeld–Toën localization construction for dg categories is included. A variation of this localization construction is introduced and related to the standard localization.

Section 3 contains a discussion of surface topology needed for the main construction. The construction of the formal contact categories $\mathcal{K}o(\Sigma)$ follows immediately by combining these topological considerations and the localization construction introduced in Section 2. The remainder of the paper is dedicated to the study of formal contact categories.

In Section 4, we check that the categories satisfy several elementary properties which were outlined by Honda. In particular, Corollary 4.10 shows that nontrivial boundary conditions are necessary for Giroux's tightness criterion to be satisfied. Theorem 4.14 shows that when such boundary conditions are present, the triangulated structure allows one to simplify the category by writing dividing sets which do not interact with the boundary in terms of those which do, up to homotopy equivalence. In Section 4.5, formal contact categories $\mathcal{K}o(\Sigma)$ are split into a product of two isomorphic copies of a subcategory $\mathcal{K}o_+(\Sigma)$, called the positive half of the formal contact category.

In Section 5, Theorem 5.2 shows that the mapping class group $\Gamma(\Sigma)$ of Σ acts naturally on the category $\mathcal{K}o(\Sigma)$. Theorem 5.11 shows that when the surface Σ supports a handle decomposition, determined by an arc parametrization $\mathcal{Z}$, this produces a collection of generators $\mathfrak{Z}(\mathcal{Z})$ for the category $\mathcal{K}o(\Sigma)$. After proving the second statement above, in Section 5.4 we study additive invariants of $\mathcal{K}o_+(\Sigma_{g,1}, 2)$.

The remainder of the paper is dedicated to an investigation of the comparison problem between two codimension-2 extensions: contact categories and Heegaard–Floer categories. The strategy pursued is illustrated by the diagram

$$\begin{array}{ccc} \mathcal{K}o(\Sigma) & \longleftrightarrow & \mathcal{A}(-\mathcal{Z})\text{–mod} \\ & \searrow \quad \nearrow & \\ & \mathcal{C}o(\Sigma) & \end{array}$$

When a reasonable candidate for the geometric contact category $\mathcal{C}o(\Sigma)$ exists, the dashed lines should be taken to be solid.

In Section 6, we study the relationship between three categories associated to the disk $(D^2, 2n)$ with $2n$ points fixed along its boundary. In [54], Tian constructed a candidate $\mathcal{Y}_n$ for $\mathcal{C}\mathrm{o}(D^2, 2n)$, and we introduce an arc parametrization $\mathcal{M}_n$ of the disk $(D^2, 2n)$ which gives a dg category $\mathcal{A}(-\mathcal{M}_n)$ associated to the Heegaard–Floer package; see Zarev [62]. The main result of this section is to prove that the three dg categories are Morita equivalent:

$$\mathcal{C}\mathrm{o}(D^2, 2n) \cong \mathcal{K}\mathrm{o}_+(D^2, 2n) \cong \mathcal{A}(-\mathcal{M}_n). \tag{1-3}$$

The category $\mathcal{A}(-\mathcal{M}_n)$ is a k–linear category because the differential d is always equal to zero. There are several other instances in which categories with this property can be associated to surfaces. In Section 7, we show that functors from these categories to the homotopy categories of the appropriate formal contact categories can be defined.

Section 8 applies the universal property, discussed above, in a much broader context. The section begins with a discussion of the relationship between the formal contact categories $\mathcal{K}\mathrm{o}(\Sigma)$ and the contact categories $\mathcal{C}\mathrm{o}(\Sigma)$. The main theorem leverages the universal property to construct a map

$$\mathcal{K}\mathrm{o}_+(\Sigma) \to \mathcal{A}(-\mathcal{Z})\text{–mod}$$

in the homotopy category of dg categories from the formal contact category associated to Σ to the Heegaard–Floer category associated to Σ, when Σ is parametrized by $\mathcal{Z}$, for any oriented surface Σ with sufficient boundary conditions.

Acknowledgments

The construction of contact categories was inspired by the ideas of K Honda, Y Tian and K Walker [15; 54; 55; 59]. I would especially like to thank Tian for his helpful correspondence and the Simons Center for facilitating our discussion, and also Y Huang, R Lipshitz, A Manion and I Petkova for several helpful emails, and my colleagues C Frohman, A Kaloti, K Kawamuro and R Kinser for their cordiality and their conversation.

In more detail, the author's involvement in this subject began because of the mention of a contact TQFT in Walker's 2006 notes [59, Section 9.4]. These were discussed at length with J Roberts between 2006 and 2009. Several years later, the author spoke to Walker about the possibility of fitting Heegaard–Floer theory into his framework in [59]. In 2014 the author found Tian's papers [54; 55] and a recording of Honda at MSRI discussing his ideas [15]. This project began after several conversations with Kaloti.

The author would like to thank the referees for their readings and very helpful feedback.

The author would also like to thank the organizers and participants of the *Categorifications of quantum groups, representations and knot invariants* session at the AMS–EMS–SPM meeting in June 2015, where some of these results were announced.

After this paper was posted, a few papers with complementary results have appeared; see [18; 39].

2 Algebraic constructions

In this section, a discussion of localizations follows a review of pretriangulated hulls. Section 2.2 reviews the standard localization procedure for dg categories. Section 2.3 introduces a form of localization which creates formal extensions among objects in a dg category: rather than creating homotopy equivalences amongst objects, this *Postnikov* localization introduces distinguished triangles. In Section 2.4, properties of Postnikov localizations are discussed.

Most of the materials in this section are standard. Some review is found in the appendix. A review of differential graded categories can be found in [26; 57] or [10, Section 1]; consult [46; 47; 56] for technical details. The language of model categories is reviewed in [37, Section A.2]; more details can be found in [19; 45].

2.1 Pretriangulated hull

This section contains a brief discussion of pretriangulated hulls of dg categories. The key ideas were introduced in [2, Section 4]; see also [1; 8].

Definition 2.1 [8, Section 2.4] If $\mathcal{C}$ is a dg category then there exists a dg category $\mathcal{C}^{\mathrm{pretr}}$, called the *pretriangulated hull* of $\mathcal{C}$. The objects of $\mathcal{C}^{\mathrm{pretr}}$ are one-sided twisted complexes, ie formal expressions

$$x = \Big(\bigoplus_{i=1}^{n} x_i[r_i], p\Big) \quad \text{such that } dp + p^2 = 0$$

where $n \geq 0$, $x_i \in \mathrm{Ob}(\mathcal{C}) \cup \{0\}$ and $r_i \in \mathbb{Z}$. The map $p = (p_{i,j})$ is a matrix such that $|p_{i,j}| = 1$ and

$$p_{i,j} = \begin{cases} x_i[r_i] \to x_j[r_j] & \text{if } j > i, \\ 0 & \text{if } j \leq i. \end{cases}$$

If $x, x' \in \mathrm{Ob}(\mathcal{C}^{\mathrm{pretr}})$ with $x = \big(\bigoplus_{i=1}^{n} x_i[r_i], p\big)$ and $x' = \big(\bigoplus_{i=1}^{n} x'_i[r'_i], p'\big)$, then $\mathrm{Hom}(x, x')$ consists of matrices $f = (f_{i,j})$ for $f_{i,j} \in \mathrm{Hom}^{r'_j - r_i}(x_i, x'_j)$, the composition is given by matrix multiplication and the differential $d : \mathrm{Hom}(x, x') \to \mathrm{Hom}(x, x')$ is determined by the formula

$$(df)_{i,j} = (df)_{i,j} + (p' f)_{i,j} - (-1)^{|f_{i,j}|}(fp)_{i,j}.$$

Remark 2.2 [8, Section 2.4] If $x, y \in \mathrm{Ob}(\mathcal{C})$ and $f : x \to y$ is a closed map of degree zero, then the cone of f exists in $\mathcal{C}^{\mathrm{pretr}}$ by construction: $C(f) = (x \oplus y[-1], p) \in \mathrm{Ob}(\mathcal{C})$ where $p_{1,2} = f$ and $p_{1,1} = p_{2,1} = p_{2,2} = 0$. The objects in $\mathcal{C}^{\mathrm{pretr}}$ can be obtained by iterated applications of the cone construction.

A referee notes that the construction in Remark 2.2 is sometimes called a "cocone."

By construction, the pretriangulated dg category $\mathcal{C}^{\mathrm{pretr}}$ associated to a k–linear category $\mathcal{C}$ factors through its additive closure $\mathrm{Mat}(\mathcal{C})$:

$$\mathrm{Mat}(\mathcal{C})^{\mathrm{pretr}} \cong \mathcal{C}^{\mathrm{pretr}}.$$

(Or set $p = 0$ in Definition 2.1.) The canonical inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\text{pretr}}$ is fully faithful. A dg category $\mathcal{C}$ is *pretriangulated* when the functor $\text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{C}^{\text{pretr}})$ induced by inclusion between the associated homotopy categories is an equivalence of categories. The *category of pretriangulated dg categories* will be denoted by $\text{dgcat}_k^{\text{pretr}}$.

Unfamiliar readers may wish to recall that $\text{Ob}(\mathcal{C} \amalg \mathcal{D}) := \text{Ob}(\mathcal{C}) \sqcup \text{Ob}(\mathcal{D})$ and

$$\text{Hom}_{\mathcal{C} \amalg \mathcal{D}}(x, y) := \begin{cases} \text{Hom}_{\mathcal{C}}(x, y) & \text{if } x, y \in \text{Ob}(\mathcal{C}), \\ \text{Hom}_{\mathcal{D}}(x, y) & \text{if } x, y \in \text{Ob}(\mathcal{D}), \\ 0 & \text{otherwise.} \end{cases}$$

The proposition below shows how the pretriangulated hull operation distributes over coproducts of dg categories. This is a $p \neq 0$ generalization of the analogous statement about additive closures. It will be used in Theorem 4.4.

Proposition 2.3 *If $\mathcal{C}$ and $\mathcal{D}$ are k–linear then* $(\mathcal{C} \amalg \mathcal{D})^{\text{pretr}} \cong \mathcal{C}^{\text{pretr}} \prod \mathcal{D}^{\text{pretr}}$.

Proof Since there are no nonzero maps between $\mathcal{C}$ and $\mathcal{D}$, thought of as subcategories of $\mathcal{C} \amalg \mathcal{D}$, a twisted complex $\left(\bigoplus_{i=1}^n x_i[r_i], p\right) \in (\mathcal{C} \amalg \mathcal{D})^{\text{pretr}}$ splits into a direct sum of twisted complexes in $\mathcal{C}^{\text{pretr}}$ and $\mathcal{D}^{\text{pretr}}$. Likewise, matrices $(f_{i,j})$ of maps between twisted complexes in $(\mathcal{C} \amalg \mathcal{D})^{\text{pretr}}$ consist of blocks. It follows that there are functors $\pi_{\mathcal{C}} \colon (\mathcal{C} \amalg \mathcal{D})^{\text{pretr}} \to \mathcal{C}^{\text{pretr}}$ and $\pi_{\mathcal{D}} \colon (\mathcal{C} \amalg \mathcal{D})^{\text{pretr}} \to \mathcal{D}^{\text{pretr}}$ which satisfy the universal property of the product. □

The following proposition is well known; see [1, Section 1.5].

Many of the constructions to follow in this section use ideas which are touched on in the appendix.

Proposition 2.4 *The pretriangulated hull* $-^{\text{pretr}} \colon \text{dgcat}_k \to \text{dgcat}_k^{\text{pretr}}$ *is left adjoint to the forgetful functor*:

$$\text{Hom}_{\text{dgcat}_k^{\text{pretr}}}(\mathcal{C}^{\text{pretr}}, \mathcal{D}) \cong \text{Hom}_{\text{dgcat}_k}(\mathcal{C}, \text{Forget}(\mathcal{D})).$$

If $f \colon \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ is a quasiequivalence then $f^{\text{pretr}} \colon \mathcal{C}^{\text{pretr}} \to \mathcal{D}^{\text{pretr}}$ is a quasiequivalence of dg categories.

The category Hqe is a localization of dgcat_k in which quasiequivalences between dg categories are isomorphisms. The Morita homotopy category Hmo is a localization of the homotopy category Hqe of dg categories in which derived equivalences are isomorphisms. In Hmo, the homotopy idempotent completion $\mathcal{C}^{\text{perf}}$ of the pretriangulated hull $\mathcal{C}^{\text{pretr}}$ is fibrant replacement; see [46].

2.2 Inverting maps in dg categories

This section contains a brief review of the localization construction for dg categories. Many authors have studied this problem; see [8; 25; 26; 48; 56, Section 8.2].

Definition 2.5 The symbol I will be used to denote the dg category freely generated by a cycle $f \colon 1 \to 2$ of degree 0, and I' will be used to denote the dg category freely generated by cycles $f \colon 1 \to 2$ and $g \colon 2 \to 1$ of degree 0:

$$I = 1 \xrightarrow{f} 2 \quad \text{and} \quad I' = 1 \rightleftarrows 2.$$

The symbol $\bar{I}$ denotes the dg category with a unique degree 0 isomorphism $f: 1 \xrightarrow{\sim} 2$ with $df = 0$. There are canonical inclusions

$$\kappa: I \hookrightarrow \bar{I} \quad \text{and} \quad \kappa': I' \hookrightarrow \bar{I}.$$

These maps are determined by the assignments $\kappa(f) = f$, $\kappa'(f) = f$ and $\kappa'(g) = f^{-1}$.

Definition 2.6 Suppose that $\mathcal{C}$ is a dg category and $R: \coprod_{r \in \mathcal{R}} I \to \mathcal{C}$ is a dg functor. Then the *localization of* $\mathcal{C}$ *with respect to* R is a dg functor

$$P: \mathcal{C} \to L_R\mathcal{C}$$

which satisfies:

(1) The pullback map $P^*: \mathrm{Hom}_{\mathrm{Hqe}}(L_R\mathcal{C}, \mathcal{X}) \to \mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{C}, \mathcal{X})$ is injective.

(2) The image of P^* consists of maps $f: \mathcal{C} \to \mathcal{X}$ for which there is a map α making the diagram below commute:

$$\begin{array}{ccc} \coprod_{r \in \mathcal{R}} \mathrm{Ho}(I) & \xrightarrow{\mathrm{Ho}(R^* f)} & \mathrm{Ho}(\mathcal{X}) \\ \Big\downarrow{\scriptstyle \mathrm{Ho}(\kappa)} & \overset{\alpha}{\nearrow} & \\ \coprod_{r \in \mathcal{R}} \mathrm{Ho}(\bar{I}) & & \end{array}$$

The image $\mathrm{im}(P^*)$ may be denoted by $\mathrm{Hom}^{I}_{\mathrm{Hqe}}(\mathcal{C}, \mathcal{X})$.

Corollary 8.8 in [56] shows that for any dg category $\mathcal{C}$ and any functor $R: \coprod_{r \in \mathcal{R}} I \to \mathcal{C}$, there exists a functor $P: \mathcal{C} \to L_R\mathcal{C}$ in the homotopy category Hqe of dg categories which satisfies the two properties in Definition 2.6. The functor $P: \mathcal{C} \to L_R\mathcal{C}$ is defined to be the homotopy pushout

$$\begin{array}{ccc} \coprod_{r \in \mathcal{R}} I & \xrightarrow{R} & \mathcal{C} \\ {\scriptstyle \kappa}\Big\downarrow & & \Big\downarrow{\scriptstyle P} \\ \coprod_{r \in \mathcal{R}} \bar{I} & \longrightarrow & L_R\mathcal{C} \end{array}$$

When the category $\mathcal{C}$ is cofibrant, this homotopy pushout

$$L_R\mathcal{C} = \left(\coprod_{r \in \mathcal{R}} \bar{I} \right) \amalg^{\mathbb{L}}_{R} \mathcal{C}$$

can be computed by replacing the inclusion $\kappa: I \hookrightarrow \bar{I}$ by a well-known cofibration $I \hookrightarrow \tilde{I}$. The dg category $\tilde{I}$ appears in Drinfeld, where it is denoted by $\mathcal{K}$ [8, Section 3.7.1].

Definition 2.7 The category $\tilde{I}$ has two objects: 1 and 2. Its maps are generated by the elements $f \in \mathrm{Hom}^0_{\tilde{I}}(1,2)$, $g \in \mathrm{Hom}^0_{\tilde{I}}(2,1)$, $h_{1,1} \in \mathrm{Hom}^{-1}_{\tilde{I}}(1,1)$, $h_{2,2} \in \mathrm{Hom}^{-1}_{\tilde{I}}(2,2)$ and $h_{1,2} \in \mathrm{Hom}^{-2}_{\tilde{I}}(1,2)$:

$$h_{1,1} \circlearrowright 1 \underset{g}{\overset{f, h_{1,2}}{\rightleftarrows}} 2 \circlearrowleft h_{2,2}$$

The differential is determined by the Leibniz rule together with the equations

$$df = 0, \quad dg = 0, \quad dh_{1,1} = gf - 1_1, \quad dh_{2,2} = fg - 1_2 \quad \text{and} \quad dh_{1,2} = h_{2,2}f - fh_{1,1},$$

and the maps are subject to no relations.

Remark 2.8 In Definition 2.6, the category I and the map $\kappa\colon I \hookrightarrow \bar{I}$ can be replaced by the category I' and the map $\kappa'\colon I' \hookrightarrow \bar{I}$. Suppose that $R\colon I \to \mathcal{C}$ and a candidate $R(f)^{-1}$ for the inverse of the map $R(f)$ already exists in the category $\mathcal{C}$. Then R can be extended to a functor $R'\colon I' \to \mathcal{C}$ such that $R'(f) = R(f)$ and $R'(g) = R(f)^{-1}$ and there is an analogous localization

$$P : \mathcal{C} \to L_{R'}\mathcal{C} \quad \text{where} \quad L_{R'}\mathcal{C} = \tilde{I} \amalg^{\mathbb{L}}_{R'} \mathcal{C}.$$

2.3 Postnikov localization

A variation of the localization procedure discussed in the previous section is introduced. This *Postnikov localization* introduces distinguished triangles rather than homotopy equivalences. In particular, given a sequence

$$1 \to 2 \to 3 \to 1$$

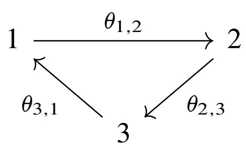
of maps S in a dg category $\mathcal{C}$, there is a dg category $L_S\mathcal{C}$ in which this sequence forms a distinguished triangle.

The dg categories considered in this section are $\mathbb{Z}/2$–graded for simplicity. The equivalences discussed below commute with the forgetful functor to the ungraded setting introduced in Section 2.5. On the other hand, $\mathbb{Z}$–graded lifts determined by grading conventions for distinguished triangles can be found in [10, Section 2.4.1]. See also [57, Section 4.3].

Historically, Postnikov systems appear in the study of triangulated categories [12]. The name Postnikov may be attached to that construction because it is a generalization of the Postnikov decomposition of topological spaces to algebraic triangulated categories.

First we introduce a dg category D' which corepresents triangles; see (2-1) and Proposition 2.11. Then Definition 2.13 introduces dg categories $\bar{D}$ and $\widetilde{D}$ which corepresent distinguished triangles. A dg functor $\kappa\colon D' \hookrightarrow \widetilde{D}$ will be used to construct the Postnikov localization in Definition 2.15.

Definition 2.9 The symbol D' will be used to denote the dg category freely generated by cycles $\theta_{1,2}\colon 1 \to 2$, $\theta_{2,3}\colon 2 \to 3$ and $\theta_{3,1}\colon 3 \to 1$:



The degrees are determined by $|\theta_{1,2}| = 1$, $|\theta_{2,3}| = 1$ and $|\theta_{3,1}| = 1$.

Since a dg functor $f: D' \to \mathcal{C}$ is uniquely determined by where it maps the generators in the definition above, there is a bijection between the set of such functors and (symmetric) triangles in $\mathcal{C}$:

$$\mathrm{Hom}_{\mathrm{dgcat}_k}(D', \mathcal{C}) \xrightarrow{\sim} \{\text{symmetric triangles in } \mathcal{C}\}. \tag{2-1}$$

Definition 2.10 If $f, g: D' \to \mathcal{C}$ are two triangles in $\mathcal{C}$, then f *is isomorphic to* g when $\mathrm{Ho}(f) \cong \mathrm{Ho}(g)$ as objects in the functor category $\mathrm{Hom}(\mathrm{Ho}(D'), \mathrm{Ho}(\mathcal{C}))$.

The proposition below states that in the homotopy category Hqe of dg categories, the left-hand side of (2-1) is in canonical bijection with isomorphism classes of triangles.

Proposition 2.11 [10, Proposition 2.4.7] *For any dg category* $\mathcal{C}$, *there is a one-to-one correspondence between homotopy classes of functors* $f: D' \to \mathcal{C}$ *and isomorphism classes of triangles in* $\mathcal{C}$:

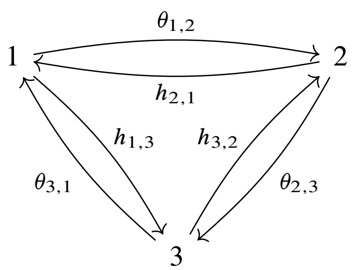
$$\mathrm{Hom}_{\mathrm{Hqe}}(D', \mathcal{C}) \leftrightarrow \{\textit{symmetric triangles in } \mathcal{C}\}/\mathrm{iso}.$$

Just as isomorphisms are distinguished types of maps, distinguished triangles are distinguished types of triangles. A distinguished triangle is a recipe for constructing one of its objects in terms of the other two.

Definition 2.12 If S is a symmetric triangle $1 \xrightarrow{\theta_{1,2}} 2 \xrightarrow{\theta_{2,3}} 3 \xrightarrow{\theta_{3,1}} 1$ in a dg category $\mathcal{C}$, then S is a *distinguished triangle* if and only if S is isomorphic to the distinguished triangle S' given by $1 \xrightarrow{\theta_{1,2}} 2 \to C(\theta_{1,2}) \to 1$ in the homotopy category of $\mathcal{C}^{\mathrm{pretr}}$.

In keeping with Section 2.2, the distinguished property of triangles is formulated as a lifting problem. An innocuous-looking dg category $\bar{D}$ which corepresents distinguished triangles and a quasiequivalent cofibrant replacement $\tilde{D} \xrightarrow{\sim} \bar{D}$ are introduced below.

Definition 2.13 [10, Section 2.4.1] The dg category $\bar{D}$ consists of objects $\mathrm{Ob}(\bar{D}) = \{1, 2, 3\}$. The maps are generated by cycles: $\theta_{1,2}: 1 \to 2$, $\theta_{2,3}: 2 \to 3$ and $\theta_{3,1}: 3 \to 1$ of degree 1, and homotopies $h_{2,1}: 2 \to 1$, $h_{3,2}: 3 \to 2$ and $h_{1,3}: 1 \to 3$ of degree 1,



with $dh_{2,1} = \theta_{3,1}\theta_{2,3}$, $dh_{3,2} = \theta_{1,2}\theta_{3,1}$ and $dh_{1,3} = \theta_{2,3}\theta_{1,2}$ and the relations

$$\theta_{2,3}h_{3,2} + h_{1,3}\theta_{3,1} = 1_3, \quad \theta_{1,2}h_{2,1} + h_{3,2}\theta_{2,3} = 1_2 \quad \text{and} \quad \theta_{3,1}h_{1,3} + h_{2,1}\theta_{1,2} = 1_1.$$

The dg category $\widetilde{D}$ consists of objects $\mathrm{Ob}(\widetilde{D}) = \{1, 2, 3\}$. The maps $\theta_{i,j}: i \to j$ in this category are clockwise-oriented paths between vertices, from i to j, in the triangular graph featured in Definition 2.9. The differential is zero on paths of length zero or one; when $\theta_{i,i}$ is a cycle, a path of topological degree one (a loop),

$$d\theta_{i,i} = 1_i - \sum_k \theta_{k,i}\theta_{i,k},$$

otherwise $d\theta_{i,j}$ is the sum over compositions of all possible factorizations of the path:

$$d\theta_{i,j} = \sum_k \theta_{k,i}\theta_{j,k}.$$

The projection $p: \widetilde{D} \to \overline{D}$ given by mapping cycles of length 1 to their respective θ–maps is a quasiequivalence [10, Proposition 2.4.13]. In the other direction, there is an inclusion $\kappa': D' \hookrightarrow \widetilde{D}$ given by sending the θ–maps to their respective length-1 cycles. There is also an inclusion $\kappa': D' \hookrightarrow \overline{D}$ given by the same formula. A $\mathbb{Z}$–graded analogue of $\widetilde{D}$ is discussed in [27]. This dg category is the cobar–bar construction on the partially wrapped Fukaya category of the disk with three stops [41].

The proposition below states that the dg category $\widetilde{D}$ corepresents distinguished triangles and satisfies the key properties necessary for the localization construction.

Proposition 2.14 [10, Proposition 2.4.14] (1) *For any dg category $\mathcal{C}$, the set of homotopy classes of dg functors from $\widetilde{D}$ to $\mathcal{C}$ is in bijection with the set of isomorphism classes of distinguished triangles in $\mathcal{C}$*:

$$\mathrm{Hom}_{\mathrm{Hqe}}(\widetilde{D}, \mathcal{C}) = \{1 \xrightarrow{\theta_{1,2}} 2 \xrightarrow{\theta_{2,3}} C(\theta_{1,2}) \to 1\}/\mathrm{iso}.$$

(2) *The image of the pullback induced by the map κ' appearing in Definition 2.13 coincides with the subset of triangles which are distinguished*:

$$(\kappa')^*: \mathrm{Hom}_{\mathrm{Hqe}}(\widetilde{D}, \mathcal{C}) \to \mathrm{Hom}_{\mathrm{Hqe}}(D', \mathcal{C}).$$

(3) *The set $\mathrm{Hom}_{\mathrm{Hqe}}(\widetilde{D}, \mathcal{C})$ is equal to the set of maps $f \in \mathrm{Hom}_{\mathrm{Hqe}}(D', \mathcal{C})$ for which there is a map $\alpha: \mathrm{Ho}(\widetilde{D}) \to \mathrm{Ho}(\mathcal{C})$ such that $\mathrm{Ho}(f) = \alpha \circ \mathrm{Ho}(\kappa')$.*

We are now ready to discuss a generalization of the localization procedure presented earlier in Section 2.2. Instead of inverting maps in the associated homotopy category, this new operation creates distinguished triangles in the associated homotopy category.

Definition 2.15 Suppose that $\mathcal{C}$ is a dg category and $S: \coprod_{s \in \mathcal{S}} D' \to \mathcal{C}$ is a dg functor. Then the *Postnikov localization of $\mathcal{C}$ with respect to S* is a dg functor

$$Q: \mathcal{C} \to L_S\mathcal{C}$$

such that for any dg category $\mathcal{X}$ the following properties are satisfied:

(1) The pullback map $Q^*: \mathrm{Hom}_{\mathrm{Hqe}}(L_S\mathcal{C}, \mathcal{X}) \to \mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{C}, \mathcal{X})$ is injective.

(2) The set of maps $\mathrm{Hom}_{\mathrm{Hqe}}(L_S\mathcal{C}, \mathcal{X})$ in the image of Q^* is equal to the set of maps $f \in \mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{C}, \mathcal{X})$ such that there is a map α making the diagram below commute:

$$\begin{array}{ccc} \coprod_{s\in\mathcal{S}} \mathrm{Ho}(D') & \xrightarrow{\mathrm{Ho}(f\circ S)} & \mathrm{Ho}(\mathcal{X}) \\ \kappa' \big\downarrow & \overset{\alpha}{\nearrow} & \\ \coprod_{s\in\mathcal{S}} \mathrm{Ho}(\widetilde{D}) & & \end{array}$$

The image $\mathrm{im}(Q^*)$ may also be denoted by $\mathrm{Hom}^T_{\mathrm{Hqe}}(\mathcal{C}, \mathcal{X})$.

Recall from above that a functor from $S\colon D' \to \mathcal{C}$ is determined by the choice of cycles $f\colon 1 \to 2$, $g\colon 2 \to 3$ and $h\colon 3 \to 1$. The Postnikov localization $L_S\mathcal{C}$ associated to the functor S requires that the sequence

$$1 \xrightarrow{f} 2 \xrightarrow{g} 3 \xrightarrow{h} 1$$

is a distinguished triangle in the sense of Definition 2.12. The category $L_S\mathcal{C}$ is uniquely determined up to homotopy by the property that a functor $f\colon \mathcal{C} \to \mathcal{X}$ factors through $Q\colon \mathcal{C} \to L_S\mathcal{C}$ in Hqe when it maps triangles in the image of S to distinguished triangles in the homotopy category $\mathrm{Ho}(\mathcal{X})$ of $\mathcal{X}$.

When $\mathcal{C}$ is a cofibrant dg category, the category $L_S\mathcal{C}$ is a pushout, obtained by gluing a copy of $\widetilde{D}$ along the subcategory determined by the image of a functor S. If $\mathcal{C}$ is not cofibrant then $L_S\mathcal{C}$ is a homotopy pushout: the pushout of a cofibrant replacement $\widetilde{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}$ of $\mathcal{C}$ [37, Section A.2.4.4].

The next proposition states that Postnikov localizations always exist.

Proposition 2.16 *For any dg category $\mathcal{C}$ and any collection $S\colon \coprod_{s\in\mathcal{S}} D' \to \mathcal{C}$, there is a Postnikov localization $Q\colon \mathcal{C} \to L_S\mathcal{C}$ in* Hqe.

Proof It follows from Proposition 2.14 that the functor $\kappa'\colon D' \to \widetilde{D}$ is a Postnikov localization in the sense of Definition 2.15. Therefore, any coproduct of inclusions $\coprod_{s\in\mathcal{S}} D' \to \coprod_{s\in\mathcal{S}} \widetilde{D}$ is a Postnikov localization. For any dg category $\mathcal{C}$, the localization $Q\colon \mathcal{C} \to L_S\mathcal{C}$ is given by the homotopy pushout:

$$\begin{array}{ccc} \coprod_{s\in\mathcal{S}} D' & \xrightarrow{S} & \mathcal{C} \\ \big\downarrow & & \big\downarrow Q \\ \coprod_{s\in\mathcal{S}} \widetilde{D} & \longrightarrow & L_S\mathcal{C} \end{array}$$

That $L_S\mathcal{C}$ is a Postnikov localization follows Definition 2.15 and properties of homotopy pushouts [19]. □

2.4 Properties of Postnikov localization

In this section, we explore properties of the Postnikov localization procedure, establish a relationship between it and the ordinary localization of dg categories, and introduce an analogue of Heller's lemma which facilitates the computation of additive invariants such as the Grothendieck group.

Triangle insertion The appendix reviews relevant concepts such quasifully faithful embedding.

The proposition below assures us that, after having added a triangle, it persists in the pretriangulated hull.

Proposition 2.17 *Suppose that* $S\colon D' \to \mathcal{C}$, $Q\colon \mathcal{C} \to L_S\mathcal{C}$ *and* $R\colon L_S\mathcal{C} \to \mathcal{X}$ *is a quasifully faithful embedding of the Postnikov localization of* $\mathcal{C}$ *into a pretriangulated category* $\mathcal{X}$. *If* $f = RQS(1 \to 2)$ *and* $c = RQS(3)$ *then* c *is isomorphic to the cone* $C(f)$ *of* f *in the homotopy category of* $\mathcal{X}$:

$$c \cong C(f) \quad \text{in } \mathrm{Ho}(\mathcal{X}).$$

Proof For the sake of notation, everything to follow takes place inside of the category $\mathrm{Ho}(\mathcal{X})$. By the triangulated category axiom TR3, there is a map h in $\mathcal{X}$ which yields a map $(1, 1, h, 1)$ from the triangle $S(1) \to S(2) \to S(3) \to S(1)$ to the triangle $S(1) \to S(2) \to C(f) \to S(1)$. For all $x \in \mathcal{X}$, both triangles determine long exact sequences after applying the functor $\mathrm{Hom}(x, -)$. By the five lemma, $h_*\colon \mathrm{Hom}^*(x, c) \to \mathrm{Hom}^*(x, C(f))$ is an isomorphism. Therefore Yoneda's lemma implies the result. □

Decategorification of localizations For references concerning short exact sequences of dg categories see [26, Section 4.6].

Lemma 2.18 *Suppose* $S\colon D' \to \mathcal{C}$ *is a triangle,* $\theta_{1,2} = S(1 \to 2)$ *and* $c = S(3)$ *in a dg category* $\mathcal{C}$. *Then* S *is isomorphic to a distinguished triangle if and only if the double cone complex* $K = C(C(\theta_{1,2}) \xrightarrow{\tilde{\theta}_{2,3}} c)$ *is contractible, where* $\tilde{\theta}_{2,3}$ *is the extension of the map* $\theta_{2,3}\colon S(2) \to c$ *to the cone* $C(\theta_{1,2})$.

Proof If S is distinguished, then the triangle $S(1) \to S(2) \to S(3) \to S(1)$ is isomorphic to $1 \to 2 \to C(\theta_{1,2}) \to 1$ in the homotopy category via the map $(1, 1, \tilde{\theta}_{2,3}, 1[1])$, so $\tilde{\theta}_{2,3}$ is a homotopy equivalence and $C(\tilde{\theta}_{2,3})$ is contractible. Conversely, $C(\tilde{\theta}_{2,3}) \simeq 0$ implies $\tilde{\theta}_{2,3}$ is a homotopy equivalence and the map above determines an equivalence of triangles. □

Recall that if $a \in \mathrm{Ob}(\mathcal{C})$, then Drinfeld's dg quotient $\mathcal{C}/\langle a\rangle$ can be formed by adding a homotopy h which satisfies $dh = 1_a$ to a cofibrant replacement of $\mathcal{C}$; see [8]. This makes the object contractible in the homotopy category of the Drinfeld quotient. (This can be reformulated as a homotopy pushout [48, Theorem 4.0.1].)

The proposition below constructs a short exact sequence of dg categories by relating the Postnikov localization $L_S\mathcal{C}$ of a dg category $\mathcal{C}$ to a Drinfeld quotient $\mathcal{C}/\langle K\rangle$. The subcategory $\langle K\rangle$ is generated by the object K in Lemma 2.18.

Proposition 2.19 *Suppose that* $S\colon D' \to \mathcal{C}$ *is a triangle,* $f = S(1 \to 2)$ *and* $c = S(3)$ *in a dg category* $\mathcal{C}$. *Then there is a short exact sequence of dg categories*

$$\langle K\rangle \to \mathcal{C} \to L_S(\mathcal{C})$$

in the Morita homotopy category Hmo, *where* $\langle K\rangle$ *is the dg category determined by the cone* $K = C(C(f) \to c)$ *of the natural map from the cone on* f *to* c *in* $\mathcal{C}^{\mathrm{pretr}}$.

Proof First assume that K is represented by an object in $\mathcal{C}$. By Definition 2.15, the Postnikov localization $L_S\mathcal{C}$ satisfies the universal property

$$\mathrm{Hom}_{\mathrm{Hqe}}(L_S\mathcal{C},\mathcal{X}) \xrightarrow{\sim} \mathrm{Hom}^T_{\mathrm{Hqe}}(\mathcal{C},\mathcal{X}), \tag{2-2}$$

the set of homotopy classes of functors from $L_S\mathcal{C}$ to any dg category $\mathcal{X}$ is in bijection with the set of homotopy classes of functors $f\colon \mathcal{C}\to\mathcal{X}$ which map $\mathrm{im}(S)$ to distinguished triangles in the homotopy categories: $\mathrm{Ho}(f)\colon \mathrm{Ho}(\mathcal{C})\to\mathrm{Ho}(\mathcal{X})$. By the lemma above, the condition that $\mathrm{Ho}(fS)\colon D'\to\mathrm{Ho}(\mathcal{X})$ maps to a distinguished triangle is equivalent to the condition that a certain double cone complex K is contractible. If $\tilde\theta_{2,3}\colon C(\theta_{1,2})\to 3$ is given by $\tilde\theta_{2,3}=(0,\theta_{2,3})$, then set $K=C(\tilde\theta_{2,3})$ so that

$$K=C(\tilde\theta_{2,3})=(1[2]\oplus 2[1]\oplus 3, d_K)\quad\text{where } d_K=\begin{pmatrix} d_1 & \theta_{1,2} & \\ & -d_2 & \theta_{2,3} \\ & & d_3\end{pmatrix}$$

is contractible in $\mathcal{X}$. So there is a bijection of sets

$$\mathrm{Hom}^T_{\mathrm{Hqe}}(\mathcal{C},\mathcal{X}) \xrightarrow{\sim} \mathrm{Hom}^{\langle K\rangle}_{\mathrm{Hqe}}(\mathcal{C},\mathcal{X}), \tag{2-3}$$

where $\mathrm{Hom}^{\langle K\rangle}(\mathcal{C},\mathcal{X})$ is the set of maps $f\colon\mathcal{C}\to\mathcal{X}$ which send K to a contractible object in $\mathcal{X}$. Then

$$\mathrm{Hom}_{\mathrm{Hqe}}(\mathcal{C}/\langle K\rangle,\mathcal{X}) \xrightarrow{\sim} \mathrm{Hom}^{\langle K\rangle}_{\mathrm{Hqe}}(\mathcal{C},\mathcal{X}). \tag{2-4}$$

See [48, Theorem 4.0.1]. The maps in (2-2), (2-3) and (2-4) combine to show that the Postnikov localization satisfies the same universal property as the Drinfeld quotient. Therefore, $\mathcal{C}/\langle K\rangle$ and $L_S\mathcal{C}$ are isomorphic in Hqe. Associated to any such Drinfeld quotient, there is a short exact sequence

$$\langle K\rangle \hookrightarrow \mathcal{C}\to\mathcal{C}/\langle K\rangle$$

in the Morita homotopy category Hmo [48, Remark 4.0.2]. Since Hmo is a quotient of Hqe, the isomorphism $\mathcal{C}/\langle K\rangle\cong L_S\mathcal{C}$ in Hqe implies the isomorphism $\mathcal{C}/\langle K\rangle\cong L_S\mathcal{C}$ in Hmo, and there is a short exact sequence of dg categories

$$\langle K\rangle\hookrightarrow\mathcal{C}\to L_S\mathcal{C}.$$

Now suppose that K is *not* representable by an object in $\mathcal{C}$. In the Morita homotopy category Hmo, the fibrant replacement $\mathcal{C}^{\mathrm{perf}}$ of $\mathcal{C}$ is the category of perfect modules over $\mathcal{C}$ — an idempotent completion of the pretriangulated hull. The object K is representable in $\mathcal{C}^{\mathrm{perf}}$ (see Remark 2.2), and so, by the argument above, there is a short exact sequence

$$\langle K\rangle\to\mathcal{C}^{\mathrm{perf}}\to L_S(\mathcal{C}^{\mathrm{perf}}).$$

In the homotopy category of any model category, every object $\mathcal{C}$ is isomorphic to its fibrant replacement $\beta\colon\mathcal{C}\xrightarrow{\sim}\mathcal{C}^{\mathrm{perf}}$. Since cofibrations in Hmo and Hqe are identical, a homotopy pushout in Hqe is a homotopy pushout in Hmo. The map β determines an equivalence of pushout diagrams from $\widetilde{D}\leftarrow\coprod_s D'\to\mathcal{C}$ to $\widetilde{D}\leftarrow\coprod_s D'\to\mathcal{C}^{\mathrm{perf}}$, from which it follows that the map $L_S\beta\colon L_S\mathcal{C}\to L_S(\mathcal{C}^{\mathrm{perf}})$ is an isomorphism in Hmo.

There is a commuting diagram extending the right-hand side of the short exact sequence in which both of the vertical maps are isomorphisms in Hmo:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & L_S\mathcal{C} \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle L_S\beta} \\ \mathcal{C}^{\text{perf}} & \longrightarrow & L_S(\mathcal{C}^{\text{perf}}) \end{array}$$

So there is a short exact sequence: $E \to \mathcal{C} \to L_S\mathcal{C}$ where E is a dg category Morita equivalent to $\langle K\rangle$. □

A short exact sequence of dg categories in Hmo induces a long exact sequence among additive invariants of dg categories [26; 46]. The corollary below is the first part of the long exact sequence associated to Hochschild homology.

Corollary 2.20 *Suppose that S, $\langle K\rangle$ and $\mathcal{C}$ are as in the proposition above. Then there is an exact sequence of abelian groups*

$$HH_0(\langle K\rangle) \to HH_0(\mathcal{C}) \to HH_0(L_S(\mathcal{C})) \to 0$$

A Postnikov localization as a module In this section, we explain how Postnikov localizations inherit the structure of a module category over $\text{End}(\widetilde{D})$ in Hmo.

If $\mathcal{C} \cong L_S\mathcal{X}$ is a Postnikov localization of a dg category $\mathcal{X}$, then the map $\iota\colon \coprod_{s\in\mathcal{S}} \widetilde{D} \to \mathcal{C}$ from the proof of Proposition 2.16 yields a map $\iota^{\text{pretr}}\colon \big(\coprod_{s\in\mathcal{S}} \widetilde{D}\big)^{\text{pretr}} \to \mathcal{C}^{\text{pretr}}$. Therefore, by Proposition 2.3 there is a map $\hat{\iota}^{\text{pretr}}\colon \prod_{s\in\mathcal{S}} \widetilde{D}^{\text{pretr}} \to \mathcal{C}^{\text{pretr}}$. The pullback of the map $\hat{\iota}^{\text{pretr}}$ along the diagonal map $\Delta_{\mathcal{S}}\colon \widetilde{D}^{\text{pretr}} \to \prod_{s\in\mathcal{S}} \widetilde{D}^{\text{pretr}}$ is a functor $j\colon \widetilde{D}^{\text{pretr}} \to \mathcal{C}^{\text{pretr}}$. The map j determines an action of $\text{End}(\widetilde{D}^{\text{pretr}})$ on $\mathcal{C}^{\text{pretr}}$:

$$\begin{array}{ccc} \widetilde{D}^{\text{pretr}} & \xrightarrow{\;j\;} & \mathcal{C}^{\text{pretr}} \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \bar{g}} \\ \widetilde{D}^{\text{pretr}} & \xrightarrow{\;j\;} & \mathcal{C}^{\text{pretr}} \end{array}$$

The universal property in Definition 2.15 gives us a lift $\bar{g}$ of $j \circ g$ for each $g \in \text{End}(\widetilde{D}^{\text{pretr}})$, and uniqueness of lifts implies that lifts commute with compositions.

2.5 Ungraded dg categories

The main body of the paper will use the trivial grading; a more sophisticated G–grading will be introduced at a later time [6]. Here we require k to be a field of characteristic 2.

There is a category Kom_k^{un} of ungraded chain complexes. In more detail, An *ungraded chain complex* is a k–vector space C and a differential $d_C\colon C \to C$ which satisfies $d_C^2 = 0$. A map $f\colon C \to D$ of ungraded chain complexes is a map of vector spaces. If $\text{Hom}(C, D)$ denotes the vector space of such maps from C to D, then there is an associative composition and for each C there is an identity map $1_C\colon C \to C$. This determines the category Kom_k^{un}.

The monoidal structure in $\mathrm{Kom}_k^{\mathrm{un}}$ is the tensor product; the differential is defined by

$$d_{C\otimes D}(x\otimes y)=d_C x\otimes y+x\otimes d_D y.$$

If $f\in\mathrm{Hom}(C,D)$ then the formula $df=fd_C-d_D f$ defines a differential which makes $(\mathrm{Hom}(C,D),d)$ an ungraded chain complex, and $\mathrm{Kom}_k^{\mathrm{un}}$ is a category which is enriched over itself.

If $\mathrm{Kom}_k^{\mathbb{Z}/2}$ denotes the dg category of $\mathbb{Z}/2$–graded chain complexes then there is an adjunction

$$\iota\colon \mathrm{Kom}_k^{\mathrm{un}}\leftrightarrow \mathrm{Kom}_k^{\mathbb{Z}/2}:\rho$$

in which ι maps (C,d) to the chain complex $(C_n,d_n)_{n\in\mathbb{Z}/2}$ where $C_n=C$ and $d_n=d$ for each $n\in\mathbb{Z}/2$. If $(C_n,d_n)_{n\in\mathbb{Z}/2}$ is a chain complex then $C=\bigoplus_n C_n$ and $d=\sum_n d_n$ determine a forgetful functor $\rho\colon \mathrm{Kom}_k^{\mathbb{Z}/2}\to\mathrm{Kom}_k^{\mathrm{un}}$.

An *ungraded dg category* $\mathcal{C}$ is a category which is enriched over $\mathrm{Kom}_k^{\mathrm{un}}$. The adjunction above induces an adjunction between the category $\mathrm{dgcat}_k^{\mathrm{un}}$ of ungraded dg categories and the category $\mathrm{dgcat}_k^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$–graded categories. This extends to a Quillen adjunction which induces model structures corresponding to Hqe and Hmo on $\mathrm{dgcat}_k^{\mathrm{un}}$; for analogous details see [9, Section 5.1].

3 Formal contact categories

In this section, a contact category $\mathcal{K}\mathrm{o}(\Sigma)$ is associated to each oriented surface Σ. The remainder of the paper will assume that k is a field of characteristic 2 and use the trivial grading.

3.1 Bypass moves

In what follows, surfaces will always be pointed in the sense defined below.

Definition 3.1 A *pointed surface* Σ is a compact connected surface Σ in which the connected components of the boundary have been ordered and each boundary component $\partial_i\Sigma$ contains a marked point $z_i\in\partial_i\Sigma$:

$$\partial\Sigma=\partial_1\Sigma\cup\cdots\cup\partial_n\Sigma,\quad z=\{z_1,\dots,z_n\}\quad\text{and}\quad z_i\in\partial_i\Sigma.$$

Every closed surface is canonically pointed.

A pointed oriented surface Σ in which a collection of points $m\subset\partial\Sigma$ satisfies the conditions

$$m\cap z=\varnothing\quad\text{and}\quad |m|\in 2\mathbb{Z}_+$$

will be denoted by (Σ,m). We write $m=\bigcup_i m_i$ where $m_i\subset\partial_i\Sigma$. Often notation will be abused and m will be used to denote both the set m and the cardinality $|m|$.

An orientation on a pointed surface Σ induces an orientation of each boundary component. The points $m_i\subset\partial_i\Sigma$ inherit an ordering by starting from the basepoint $z_i\in\partial_i\Sigma$ and traversing the boundary circle in this direction. Combining the order on each $m_i\subset\partial_i\Sigma$ with the ordering of the boundary components $\{\partial_1\Sigma,\partial_2\Sigma,\dots,\partial_n\Sigma\}$ produces a total ordering on the set m.

Recall that an arc γ is properly embedded in a pointed surface when $\partial\gamma \subset \partial\Sigma \setminus z$ and $\mathrm{int}(\gamma) \cap \partial\Sigma = \varnothing$. Arcs γ are required to intersect the boundary transversely.

Definition 3.2 Let Σ be a pointed orientable surface possibly with boundary. Then a properly embedded collection of smooth curves and arcs γ on Σ is a *multicurve*.

If γ is a multicurve on (Σ, m) then we require that the set $\gamma \cap \partial\Sigma$ coincides with the points m chosen on the boundary $\partial\Sigma$.

Definition 3.3 A nonempty multicurve γ is said to be a *dividing set on the surface* Σ when there are disjoint subsurfaces R_+ and R_- of Σ such that

$$\Sigma \setminus \gamma = R_+ \cup R_- \quad \text{and as sets} \quad \gamma = \partial R_+ \setminus \partial\Sigma = \partial R_- \setminus \partial\Sigma.$$

If Σ is a surface with boundary, then we require that the intersection number $i(\gamma, \partial\Sigma)$ is a positive even integer. In particular, when Σ has boundary we *require* that $m \geq 2$.

The subsets R_+ and R_- of Σ are the *positive region* and the *negative region* of γ on Σ, respectively. These regions may be labeled by $+$ and $-$ signs in illustrations.

If a multicurve γ is a dividing set, then for each boundary component $\partial_i\Sigma$, the number of points $\gamma \cap \partial_i\Sigma$ must be even.

Definition 3.4 For any dividing set γ on Σ, there is a *dual dividing set* $\gamma^\vee$ on Σ that is obtained by exchanging the positive and negative regions.

The *equator* $\ell = \{(x, y) : y = 0\} \subset D^2 = \{x \in \mathbb{R}^2 : |x| < 1\}$ of a disk is the line formed by the x–axis in the standard embedding $D^2 \subset \mathbb{R}^2$. The equator ℓ divides the disk D^2 into two *half-disks*, a bottom B and a top T:

$$D^2 = B \cup T \quad \text{and} \quad B \cap T = \ell.$$

The boundary ∂T of the top half-disk T consists of the equator ℓ and the northern hemisphere $\nu \subset \partial D^2$ of the boundary circle:

$$\partial T = \ell \cup \nu.$$

Definition 3.5 Suppose that γ is a dividing set on an oriented surface Σ. Then a *bypass disk on* γ is a smoothly embedded oriented half-disk $(T, \ell) \subset (\Sigma \times [0, 1], \Sigma \times 0)$ which satisfies the following properties:

(1) The equatorial arc ℓ intersects γ at exactly three points, a, b and c, such that

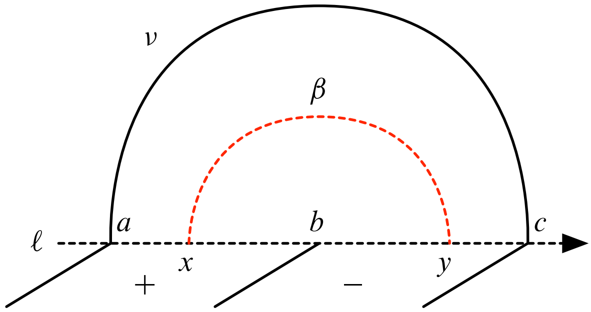
$$\ell = [a, b] \cup [b, c] \quad \text{and} \quad a < b < c,$$

where the order of the points is induced by the orientation.

(2) The boundary points of the arcs ℓ and ν are the points a and c.

A *dividing set* β of a bypass disk T is a properly embedded arc starting at a point x between a and b and ending at a point y between b and c.

Definition 3.5 is illustrated below:



This picture shows a bypass disk T embedded in a thickened surface $\Sigma \times [0,1]$. The boundary of the half-disk consists of the dashed equatorial arc ℓ and the boundary of the northern hemisphere ν. The dashed curve β is the dividing set for the bypass disk. The three straight lines at the bottom are part of a dividing set γ on the surface Σ. The labels a, b and c indicate the intersection points of the arc ℓ with the dividing set γ. The orientation of T is determined by fixing the direction of the equator ℓ and using the standard orientation along the normal axis. The equator ℓ is drawn beyond the boundary of T for aesthetic reasons.

Remark 3.6 If $\Sigma \subset (M, \xi)$ is a convex surface in a contact 3–manifold, then ξ determines a dividing set γ on Σ. A bypass disk T, embedded into a regular neighborhood of Σ, determines an operation on the dividing set called *bypass attachment* that changes the dividing set and the contact structure in a well-understood way [16]. These operations generate the contact structures on $M = \Sigma \times [0,1]$ in a sense which has been made precise by Honda [17, Lemma 3.10 (isotopy discretization)].

If Σ is an oriented surface then the space $\Sigma \times [0,1]$ will be always be oriented by appending the vertical direction to the orientation of Σ.

Definition 3.7 A bypass disk (T, ℓ) in $\Sigma \times [0,1]$ determines the product orientation on $\Sigma \times [0,1]$. In more detail, if ℓ represents the direction of the equator and n is the direction of the disk normal to the surface, then the three vectors $(\ell, \ell \times n, n)$ determine this orientation of $\Sigma \times [0,1]$. If the orientation induced by T agrees with that of $\Sigma \times [0,1]$ then the bypass disk is said to be *orientation preserving*; otherwise it is *orientation reversing*.

Definition 3.8 (bypass move) Suppose that γ is a dividing set on an oriented surface Σ, T is a bypass disk on γ and $N(T)$ is a regular neighborhood of the half-disk $T \subset \Sigma \times [0,1]$. The boundary $\partial N(T)$ contains two copies of the half-disk T, which we will call *faces*. Each face, being a parallel copy of the half-disk T, contains a collection of points

$$a < x < b < y < c$$

ordered along an equator ℓ, a dividing set β and a northern hemisphere ν. Moreover, there are three line segments γ_a, γ_b and γ_c from γ, on either side, meeting the points a, b and c, respectively. The face in the $\ell \times n$ direction of $T \times \{\frac{1}{2}\} \subset T \times [0,1]$ is called the *positive face*; the other face is the *negative face*.

There is a dividing set η on the surface $\Sigma' = \partial(\Sigma \cup N(T))$, which is constructed by regluing the segments of γ according to the prescription below:

(1) If T is orientation preserving, then on the positive face attach γ_b to the point x of β and attach γ_c to the point y of β, and on the negative face attach γ_a to the point x and attach γ_b to the point y.

(2) If T is not orientation preserving, then on the positive face attach γ_a to the point x of β and attach γ_b to the point y of β, and on the negative face attach γ_b to the point x and attach γ_c to the point y.

(3) Attach the curve γ_a on the latter face to the curve γ_c on the former face by an interval that crosses over the $\nu \times [0,1] \subset \partial N(T)$ boundary component along the diagonal.

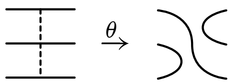
After smoothing the corners, the surface Σ' is diffeomorphic to Σ by a diffeomorphism ψ which is isotopic to the identity. If $\gamma' = \psi(\eta)$ then the *bypass move* $\theta\colon \gamma \to \gamma'$ is the tuple

$$\gamma \xrightarrow{\theta} \gamma' = (T, \gamma, \gamma')$$

given by the bypass disk T, the dividing set γ and the curve γ' determined by the operation described above.

Remark 3.9 The definition of the bypass move requires a choice of smoothing. We fix one choice and use it consistently. Any two such choices will produce equivalent categories.

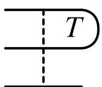
The picture below shows the orientation-preserving bypass move defined above. On the left-hand side, the dividing set γ consists of three horizontal lines and the equator ℓ of the bypass disk T is indicated by the vertical line. The rest of the bypass disk T is assumed to come out of the page. The positive and negative regions on the right are determined by the positive and negative regions on the left.



In the contact category, bypass moves are required to be orientation preserving. Since the orientation of a bypass disk T is determined by the direction of the equator, we will always choose orientations which are compatible with the ambient orientation of the surface. So it is not necessary to denote the orientation in most illustrations.

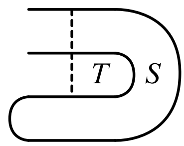
Special types of bypass moves The two special types of bypass moves isolated below correspond precisely to the relations (1) and (2) in Definition 3.15.

Definition 3.10 A bypass move $\theta\colon \gamma \to \gamma'$ is *capped* when either the subset $[a,b]$ or the subset $[b,c]$ of the associated equator ℓ is the equator ρ of an embedded half-disk $(T, T \setminus \rho) \to (\Sigma, \gamma)$ which does not intersect the equator at any other point.



Intercardinal directions will be used to locate caps. For instance, a bypass featuring a cap T in its northeastern corner is pictured above.

Example 3.11 The picture below contains one cap T in the southeastern corner. The half-disk labeled S is not a cap because it intersects the equator twice.



Capped bypass moves are the least interesting bypass moves since, depending upon where the cap is found, a capped bypass must be either nullhomotopic or equal to the identity map in the formal contact category.

Definition 3.12 Two distinct bypass moves $\theta\colon \gamma \to \gamma'$ and $\theta'\colon \gamma \to \gamma''$ are *disjoint*, up to isotopy with endpoints fixed in the dividing set, when the equators of their bypass disks have geometric intersection number zero.

If a collection of bypass moves $\{\theta_i\}_{1\le i\le n}$ on a dividing set γ is pairwise disjoint, then performing the moves in any order produces the same result: γ'. So the union

$$\coprod_{i=1}^{n} \theta_i : \gamma \to \gamma'$$

may be viewed as a kind of bypass combo move.

Isotopy of curves and disks

Definition 3.13 If γ and γ' are dividing sets on a surface Σ then they are *isotopic*, $\gamma \simeq \gamma'$, when they are isotopic as multicurves on Σ. If Σ is a pointed surface then the isotopy is required to fix the basepoints $z \subset \partial\Sigma$. If (Σ, m) is a surface with points m on each boundary component then the isotopy is required to fix the points at which the dividing sets attach to each boundary component.

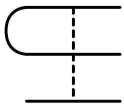
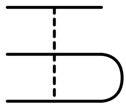
Two bypass moves $\theta = (T, \gamma, \gamma')$ and $\theta' = (S, \delta, \delta')$ are *isotopic*, $\theta \simeq \theta'$, when the graph $\gamma \cup \ell$ is isotopic to $\delta \cup \rho$ where ℓ and ρ are equators of T and S, respectively.

Remark 3.14 If Σ is realized as a convex surface in the 3–manifold $M = \Sigma\times[0, 1]$ and the dividing sets γ and γ' corresponding to two contact structures ξ and ξ' are isotopic, then ξ and ξ' are contactomorphic [16]. Since our motivation is to produce a category in which morphisms behave like contact structures up to contactomorphism, isotopic dividing sets are identified in Definition 3.15.

3.2 The contact category

Definition 3.15 The *preformal contact category* Pre–$\mathcal{K}$o(Σ) is the ungraded k–linear category with objects corresponding to isotopy classes of dividing sets on Σ and maps generated by isotopy classes of orientation-preserving bypass moves subject to the following relations:

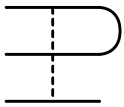
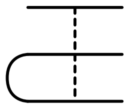
(1) If θ is a capped bypass move then $\theta = 1$ when the cap can be found in the northwest or southeast:

 $= 1$ and  $= 1.$

(2) If θ and θ' are disjoint bypass moves then the maps that they determine commute:

$$\theta\theta' = \theta \amalg \theta' = \theta'\theta.$$

These relations are required for the formal contact category, defined below, to have any bearing on contact geometry; see Remark 3.14. In Section 4.3, we will show that the first relation implies that $\theta = 0$ in the associated homotopy category when the corresponding bypass is capped in the northeast or the southwest:

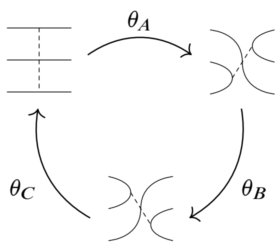
 $= 0$ and  $= 0.$

The next proposition shows that every bypass move determines a triple of composable morphisms. This determines a functor from the category D' in Definition 2.9 to the category Pre–$\mathcal{K}$o(Σ). This proposition is due to Honda and K Walker; see [15; 59].

Proposition 3.16 *For each oriented surface Σ and each dividing set γ on Σ, each bypass move θ on γ determines a functor $\tilde{\theta}\colon D' \to$ Pre–$\mathcal{K}$o(Σ).*

Proof Set $\gamma_A = \gamma$ and $\theta_A = \theta$. By definition, a bypass move $\theta_A = (T_A, \gamma_A, \gamma_B)$ is locally modeled on a bypass disk T_A in $\Sigma \times [0,1]$ which intersects γ_A in three points. There is a bypass disk T_B on the dividing set γ_B which results from the bypass move θ_A. The disk T_B determines a bypass move $\theta_B = (T_B, \gamma_B, \gamma_C)$, and there is a bypass disk T_C on the dividing set γ_C. The disk T_C determines a bypass move $\theta_C = (T_C, \gamma_C, \gamma_A)$; the result of the bypass T_C is the original dividing set $\gamma = \gamma_A$. These choices are unique up to isotopy. □

The construction above is illustrated below. Each of the arrows in the diagram is a bypass move. The solid lines represent dividing sets on the surface Σ and the dashed lines represent the equators of bypass disks:



The icon at the source of a given arrow represents a dividing set γ on the surface Σ. The icon at the target of the arrow represents the dividing set obtained by performing the bypass move with equator given by the dashed line in the source.

The proposition above allows us to associate a functor $\tilde{\theta}\colon D' \to \text{Pre–}\mathcal{K}\text{o}(\Sigma)$ to each bypass move $\theta\colon \gamma \to \gamma'$ between dividing sets on Σ. Composing the coproduct $\coprod_\theta \tilde{\theta}\colon \coprod_\theta D' \to \coprod_\theta \text{Pre–}\mathcal{K}\text{o}(\Sigma)$ of all such functors with the fold map $\coprod_\theta \text{Pre–}\mathcal{K}\text{o}(\Sigma) \to \text{Pre–}\mathcal{K}\text{o}(\Sigma)$ yields the functor

$$\Xi\colon \coprod_\theta D' \to \text{Pre–}\mathcal{K}\text{o}(\Sigma). \tag{3-1}$$

Definition 3.17 The *formal contact category* $\mathcal{K}\text{o}(\Sigma)$ is the pretriangulated hull of the Postnikov localization of the preformal contact category $\text{Pre–}\mathcal{K}\text{o}(\Sigma)$ along the functor Ξ above:

$$\mathcal{K}\text{o}(\Sigma) = L_\Xi \text{Pre–}\mathcal{K}\text{o}(\Sigma)^{\text{pretr}}$$

By Proposition 2.17, the bypass triangles introduced by the Postnikov localization remain distinguished triangles in the homotopy category of the hull. The formal contact category $\mathcal{K}\text{o}(\Sigma)$ is the universal pretriangulated category generated by bypass moves, containing bypass triangles and satisfying the relations (1) and (2).

Conjecture 3.18 *A cofibrant–fibrant replacement for* $\mathcal{K}\text{o}(\Sigma)$ *can be constructed without homotopy pushouts. Note that, before relations* (1) *and* (2) *are applied to the preformal contact category*

$$\text{Pre–}\mathcal{K}\text{o}(\Sigma) = \text{Pre–Pre-}\mathcal{K}\text{o}(\Sigma)/\langle(1),(2)\rangle,$$

the "prepreformal contact category" is freely generated by bypass moves. Any freely generated category is cofibrant as it can be obtained by a series of pushouts along generating cofibrations in Hqe*. One can then adjoin copies of Drinfeld's category* $\tilde{I}$ *via pushout and copies of a resolution for the symmetric algebra for each instance of relations* (1) *and* (2)*, respectively. The result is cofibrant in* Hqe*, so the homotopy pushout which underlies the Postnikov localization in Definition 3.17 is now an ordinary pushout and the result of this pushout is both cofibrant and fibrant in* Hqe*. The idempotent completion* $L_\Xi \text{Pre–}\mathcal{K}\text{o}(\Sigma)^{\text{perf}}$ *of* $\mathcal{K}\text{o}(\Sigma)$ *is cofibrant and fibrant in the Morita category* Hmo.

4 Elementary properties of contact categories

In this section, many of the properties which should hold for the contact categories [15] are shown to hold for the formal contact categories. The formal contact category associated to a surface decomposes into a product of formal contact categories with fixed Euler invariant. The category with Euler invariant n is equivalent to the category with Euler invariant $-n$. Reversing the orientation of the surface is equivalent to forming the opposite category. A dividing set featuring a homotopically trivial curve is contractible and dividing sets featuring regions which are disconnected from the boundary are shown to be homotopy equivalent to convolutions of dividing sets which are connected to the boundary.

4.1 Decompositions of contact categories

The contact categories $\mathcal{K}\mathrm{o}(\Sigma)$ consist of noninteracting subcategories $\mathcal{K}\mathrm{o}^n(\Sigma, m)$. Each subcategory is determined by fixing some points m on each boundary component and the Euler number $n = \mathfrak{e}(\gamma)$ of the dividing sets γ on Σ.

Euler decomposition If $(\Sigma \times [0,1), \xi)$ is a contact 3–manifold and $e(\xi)$ is the Euler class of ξ, then the Euler number of ξ is $\mathfrak{e}(\xi) = \langle e(\xi), [\Sigma]\rangle$. This number can be computed from the dividing set $\gamma \subset \Sigma$.

Definition 4.1 If γ is a dividing set on an orientable surface Σ then the *Euler number* $\mathfrak{e}(\gamma)$ of γ is the Euler characteristic of the positive region minus the Euler characteristic of the negative region:

$$\mathfrak{e}(\gamma) = \chi(R_+) - \chi(R_-).$$

The proposition below shows that this is a reasonable thing to consider.

Proposition 4.2 *The Euler number satisfies the following properties*:

(1) *If two dividing sets are isotopic then the corresponding Euler numbers are equal*:

$$\gamma \simeq \gamma' \text{ implies that } \mathfrak{e}(\gamma) = \mathfrak{e}(\gamma').$$

(2) *If $\theta\colon \gamma \to \gamma'$ is a bypass move then the Euler numbers of γ and γ' must be equal.*

Proof The first statement follows from the observation that $\gamma \simeq \gamma'$ implies that $R_+ \simeq R'_+$ and $R_- \simeq R'_-$. The second statement follows from computing each Euler characteristic as a union of the region in which the bypass move is performed and its complement. Suppose that $B \subset \Sigma$ is a small ball containing the bypass moves. If $X_\pm = R_\pm \setminus B$ and $Y_\pm = R_\pm \cap B$, then $Y_\pm$ is homeomorphic to the disjoint union of two disks and $X_\pm \cap Y_\pm$ is homeomorphic to the disjoint union of three intervals. See the illustration following Definition 3.8. □

Remark 4.3 If γ is a dividing set on a surface $(\Sigma_{g,1}, 2)$ of genus g with one boundary component and two points on the boundary, then $\chi(R_+ \cap R_-) = 1$ because γ consists of a disjoint union of circles and one interval connecting the two points which are fixed on the boundary. So $2 - 2g = \chi(R_+) + \chi(R_-)$. If $\mathfrak{e}(\gamma) = 2(g-k)$ then $\chi(R_+) = 1-k$ and $\chi(R_-) = 1-l$, where $k + l = 2g$ for $0 \le k \le 2g$.

Since the preformal contact category Pre–$\mathcal{K}\mathrm{o}(\Sigma, m)$ in Definition 3.15 is generated by bypass moves, the proposition above is equivalent to the statement that the Euler number yields a well-defined map $\mathfrak{e}\colon \mathrm{Ob}(\text{Pre–}\mathcal{K}\mathrm{o}(\Sigma, m)) \to \mathbb{Z}$ which determines a decomposition

$$\text{Pre–}\mathcal{K}\mathrm{o}(\Sigma, m) \cong \coprod_{n \in \mathbb{Z}} \text{Pre–}\mathcal{K}\mathrm{o}^n(\Sigma, m)$$

in which Pre–$\mathcal{K}\mathrm{o}^n(\Sigma, m)$ is the full subcategory of Pre–$\mathcal{K}\mathrm{o}(\Sigma, m)$ such that $\mathfrak{e}(\gamma) = n$ for all $\gamma \in \mathrm{Ob}(\text{Pre–}\mathcal{K}\mathrm{o}^n(\Sigma, m))$. The theorem below shows that this decomposition extends to the formal contact category $\mathcal{K}\mathrm{o}(\Sigma, m)$.

Theorem 4.4 *The formal contact category $\mathcal{K}\mathrm{o}(\Sigma, m)$ splits into a product of categories $\mathcal{K}\mathrm{o}^n(\Sigma, m)$:*

$$\mathcal{K}\mathrm{o}(\Sigma, m) \cong \prod_{n\in\mathbb{Z}} \mathcal{K}\mathrm{o}^n(\Sigma, m).$$

Here $\mathcal{K}\mathrm{o}^n(\Sigma, m)$ is the full subcategory of $\mathcal{K}\mathrm{o}(\Sigma, m)$ with objects that satisfy $\mathfrak{e}(\gamma) = n$.

Proof By the proposition above, $\Xi\colon \coprod D' \to \text{Pre–}\mathcal{K}\mathrm{o}(\Sigma, m)$ splits into a union $\Xi = \coprod_n \Xi_n$ where $\Xi_n\colon \coprod D' \to \text{Pre–}\mathcal{K}\mathrm{o}^n(\Sigma, m)$ corresponds to the bypass triangles contained in $\text{Pre–}\mathcal{K}\mathrm{o}^n(\Sigma, m)$. The localization functor $Q\colon \text{Pre–}\mathcal{K}\mathrm{o}(\Sigma, m) \to L_\Xi\text{Pre–}\mathcal{K}\mathrm{o}(\Sigma, m)$ splits into a union of localizations:

$$\text{Pre–}\mathcal{K}\mathrm{o}(\Sigma, m) \cong \coprod_n \text{Pre–}\mathcal{K}\mathrm{o}^n(\Sigma, m) \to L_\Xi \coprod_n \text{Pre–}\mathcal{K}\mathrm{o}^n(\Sigma, m) \cong \coprod_n L_{\Xi_n}\text{Pre–}\mathcal{K}\mathrm{o}^n(\Sigma, m).$$

The theorem follows from Proposition 2.3. □

4.2 Dualities of contact categories

Two forms of duality are introduced, corresponding to switching the labelings of the regions and the ambient orientation of the surface.

Euler duality Definition 3.4 introduced an operation $\gamma \mapsto \gamma^\vee$ on dividing sets which exchanged the positive and negative regions: $R_+ \leftrightarrow R_-$. This reverses the sign of the Euler number: $\mathfrak{e}(\gamma^\vee) = -\mathfrak{e}(\gamma)$. Here this operation is extended to an involution

$$-^\vee\colon \mathcal{K}\mathrm{o}(\Sigma, m) \to \mathcal{K}\mathrm{o}(\Sigma, m)$$

of the formal contact category which exchanges $\mathcal{K}\mathrm{o}^n(\Sigma, m)$ and $\mathcal{K}\mathrm{o}^{-n}(\Sigma, m)$ from Theorem 4.4.

Proposition 4.5 *The Euler duality map on dividing sets: $-^\vee\colon \mathrm{Ob}(\text{Pre–}\mathcal{K}\mathrm{o}^n(\Sigma, m)) \to \mathrm{Ob}(\text{Pre–}\mathcal{K}\mathrm{o}(\Sigma, m))$ extends to an involution of dg categories:*

$$-^\vee\colon \mathcal{K}\mathrm{o}^n(\Sigma, m) \to \mathcal{K}\mathrm{o}^{-n}(\Sigma, m) \quad \textit{and} \quad (-^\vee)^\vee \cong 1.$$

Proof If γ is a dividing set on Σ, then for any bypass move $\theta\colon \gamma \to \gamma'$ the positive and negative regions of γ determine positive and negative regions of γ'; see the illustration after Definition 3.8. Therefore, on the generators θ of $\text{Pre–}\mathcal{K}\mathrm{o}^n(\Sigma, m)$:

$$\theta\colon \gamma \to \gamma' \quad\mapsto\quad \theta^\vee\colon \gamma^\vee \to \gamma'^\vee.$$

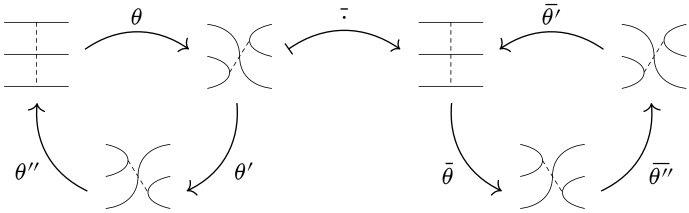
This extends to an involution of $\text{Pre–}\mathcal{K}\mathrm{o}(\Sigma, m)$ which takes triangles to triangles and so descends to a functor $-^\vee\colon \mathcal{K}\mathrm{o}^n(\Sigma, m) \to \mathcal{K}\mathrm{o}^{-n}(\Sigma, m)$. The uniqueness of this extension implies the relation $(-^\vee)^\vee \cong 1$. The map $-^\vee$ is an equivalence as it is its own inverse. □

Orientation reversal The formal contact category $\mathcal{K}\mathrm{o}(\bar{\Sigma})$ of a surface with reversed orientation is identified with the opposite formal contact category $\mathcal{K}\mathrm{o}(\Sigma)^{\mathrm{op}}$ of the surface.

Proposition 4.6 *There is an equivalence of formal contact categories*

$$\mathcal{K}\mathrm{o}^n(\Sigma, m)^{\mathrm{op}} \xrightarrow{\sim} \mathcal{K}\mathrm{o}^n(\bar{\Sigma}, m).$$

Proof It is a consequence Definition 3.8 that reversing the orientation of the surface is equivalent to reversing the orientation of each bypass half-disk or equator. It suffices to analyze the correspondence between bypass triangles. In the eyeglass-shaped diagram below, reversing the orientation of each bypass disk $\theta \mapsto \bar{\theta}$ in a triangle fixes the source and changes the sink of each map:



Reversing the arrows on the left-hand side of the diagram produces the bypass triangle for $\mathcal{K}o^n(\Sigma, m)^{\mathrm{op}}$. The assignment $\gamma \mapsto \gamma$ on objects and $\theta^{\mathrm{op}} \mapsto \bar{\theta}'$ on maps determines a functor

$$\bar{\cdot}\colon \text{Pre–}\mathcal{K}o^n(\Sigma, m)^{\mathrm{op}} \to \text{Pre–}\mathcal{K}o^n(\bar{\Sigma}, m)$$

because it preserves the cap relations and disjoint unions. Moreover, the relation $\theta^{\mathrm{op}} \mapsto \bar{\theta}'$ implies that $(\theta')^{\mathrm{op}} \mapsto \bar{\theta}''$ and $(\theta'')^{\mathrm{op}} \mapsto \bar{\theta}$, so that triangles are mapped to triangles and the functor $\bar{\cdot}$ descends to a map between formal contact categories. By applying the same construction to the surface after reversing its orientation again, one obtains an inverse functor, and so the functor $\bar{\cdot}$, introduced above, is an isomorphism of formal contact categories. □

4.3 Relations for overtwisted contact structures

A theorem of E Giroux [13] states that a contact structure on $\Sigma \times [0, 1]$, when $\Sigma \neq S^2$, is overtwisted if and only if its dividing set contains no homotopically trivial closed curves. When $\Sigma = S^2$, a contact structure is overtwisted if and only if the dividing set contains any two such curves. Corollary 4.10 states that Giroux's criterion is satisfied for surfaces with boundary. The surface Σ is assumed to be connected in this section.

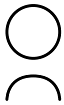
The lemma below shows that the local relations can be applied to parts of more complicated dividing sets.

Lemma 4.7 (local relations) *Suppose that R and Σ are orientable surfaces and $R \subset \Sigma$. Then a distinguished triangle in* $\mathrm{Ho}(\mathcal{K}o(R))$ *yields a distinguished triangle in* $\mathrm{Ho}(\mathcal{K}o(\Sigma))$.

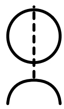
Proof The embedding $R \subset \Sigma$ determines a functor $i\colon \text{Pre–}\mathcal{K}o(R) \hookrightarrow \text{Pre–}\mathcal{K}o(\Sigma)$. A bypass triangle $\tilde{\theta}\colon D' \to \text{Pre–}\mathcal{K}o(R)$ determines a bypass triangle $D' \to \text{Pre–}\mathcal{K}o(\Sigma)$ after composing with i. □

Definition 4.8 If γ is a dividing set then we write $S^1 \subset \gamma$ when γ contains a homotopically trivial closed curve. All such curves are isotopic when Σ is connected. If γ contains any collection of $n \in \mathbb{Z}_+$ such curves then we write $nS^1 \subset \gamma$.

Proposition 4.9 *The object represented by the dividing set pictured below is contractible*:

 $\cong 0$

Proof The formal contact category $\mathrm{Ho}(\mathcal{K}\mathrm{o}(D^2, 2))$ associated to the disk D^2 with two boundary points contains a bypass move with equator indicated by the dashed line below:



All of the objects in the distinguished triangle associated to the bypass move are isotopic, and the first relation in Definition 3.15 implies two out of three of the maps are the identity. □

Corollary 4.10 (1) *If Σ is a surface with boundary*, *then for all dividing sets γ on Σ*,

$$S^1 \subset \gamma \textit{ implies } \gamma \cong 0 \textit{ in } \mathrm{Ho}(\mathcal{K}\mathrm{o}(\Sigma)).$$

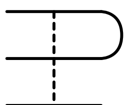
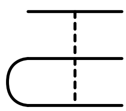
(2) *If Σ is a closed surface then for all dividing sets γ on Σ*,

$$S^1 \subset \gamma \textit{ and } \gamma \neq S^1 \textit{ implies } \gamma \cong 0 \textit{ in } \mathrm{Ho}(\mathcal{K}\mathrm{o}(\Sigma)).$$

Proof The proposition above applies to surfaces with boundary as they are required to contain properly embedded arcs. □

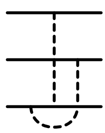
Without further complicating the main construction, this corollary appears to be optimal: bypass moves do not imply that $S^1 \cong 0$ in the disk category $\mathrm{Ho}(\mathcal{K}\mathrm{o}(D^2, 0))$, and any such proof would contradict Giroux's theorem for $\Sigma = S^2$.

Corollary 4.11 *The relation in Proposition 4.9 implies that a bypass move is zero in the homotopy category when it is capped in either the northeast or southwest*:

 $= 0$ *and*  $= 0.$

Proof The dividing set γ' resulting from either bypass move $\theta \colon \gamma \to \gamma'$ must contain a homotopically trivial curve. So the isomorphism $\gamma' \cong 0$ is obtained by applying Lemma 4.7 and Proposition 4.9. This implies the relation $\theta = 0$ in the homotopy category of the formal contact category. □

Remark 4.12 Two consecutive bypass moves occurring in a bypass triangle are disjoint:



The second bypass is capped when it is performed before the first, so the commutativity of disjoint bypasses and the corollary above suffice to imply that compositions of consecutive bypass moves must be zero in the homotopy category.

4.4 Dividing sets containing disconnected regions are convolutions

Suppose γ is a dividing set on a surface Σ with boundary and $\Sigma \setminus \gamma$ contains a connected component B which is disjoint from the boundary of Σ. We will show that γ is homotopy equivalent to an iterated cone construction on dividing sets which do not contain a region such as B.

Definition 4.13 A multicurve γ on a surface Σ with boundary is *boundary disconnected* when there is a connected component B of $\Sigma \setminus \gamma$ which does not touch the boundary:

$$B \subset \Sigma \setminus \gamma \quad \text{and} \quad B \cap \partial\Sigma = \varnothing$$

A dividing set γ is *boundary connected* when it is not boundary disconnected.

Theorem 4.14 *In the homotopy category of the formal contact category $\mathcal{K}\mathrm{o}(\Sigma, m)$ associated to a surface (Σ, m) with boundary, every boundary-disconnected dividing set γ is isomorphic to an iterated extension of dividing sets γ_i which are boundary connected.*

Proof Observe that boundary-disconnected regions can be nested. For example, an annulus can be placed within the annulus illustrated below. For the purpose of this argument, the amount of nesting $n(\gamma)$ is defined to be

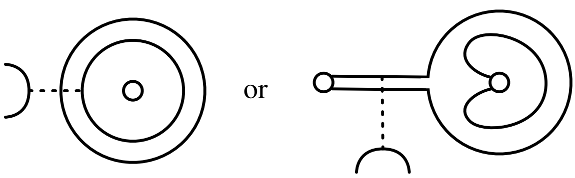
$$n(\gamma) := \max_B \min_a |a \cap \gamma|,$$

where $a\colon (I, \{0\}, \{1\}) \to (B, \partial\Sigma, \mathrm{int}(B))$ is an arc from the boundary $\partial\Sigma$ to an interior point of a connected component $B \subset \Sigma \setminus \gamma$.

The proof is by induction on the amount of nesting in boundary-disconnected regions. Fix a dividing set γ. If $n(\gamma) = 0$ and there are no boundary-disconnected regions then there is nothing to show. So assume that the statement of the theorem holds for all γ with $n(\gamma) = N$ and suppose $n(\gamma) = N + 1$.

There are innermost disconnected regions B and arcs $a\colon I \to B$ in Σ which satisfy $|a \cap \gamma| = N + 1$. Fix such a disconnected region B.

If this disconnected region is a disk then γ is isomorphic to zero because $|m| \geq 2$ by Proposition 4.9. If γ is a dividing set on a surface with boundary and $\Sigma \setminus \gamma$ contains an annulus or a punctured torus component, then there are bypass moves



respectively. The first picture above shows two concentric homotopically nontrivial circles in the annulus $(S^1 \times [0,1], 2)$. In the second picture above, the two small circles are identified by folding the page to form a torus with one boundary component $(T^2 \setminus D^2, 2)$. In either case, the triangle associated to the indicated bypass move results in two dividing sets which connect B to either the boundary, when $n(\gamma) = 1$, or a region outside of B, when $n(\gamma) > 1$. In either case this lowers $n(\gamma)$ by 1.

In general, the innermost region B is an orientable surface with boundary. Any such surface is obtained by attaching 1–handles to the boundary components of a disjoint union of punctured tori $\Sigma_{1,1}$ and annuli $\Sigma_{0,2}$. If B has genus g and $n+1$ boundary components, then B is abstractly homeomorphic to g copies of $\Sigma_{1,1}$ and n copies of $\Sigma_{0,2}$ glued together in this fashion. In particular, there is a 1–handle H which, when cut along its cocore I, produces a disjoint union of surfaces with lower genus or number of boundary components. There is an interval ℓ in Σ which is obtained by connecting I to a point on the boundary of the region outside of B (which is not in ∂B itself). By construction, this interval ℓ intersects γ at three points. The bypass move θ determined by ℓ determines a distinguished triangle

$$\gamma \xrightarrow{\theta} \gamma' \to \gamma'' \to \gamma[1]$$

with objects γ' and γ'' that must contain disconnected regions B and B'' with lower genus or number of boundary components. This procedure can be iterated until the result contains only annuli and tori, to which one applies the bypasses in the previous paragraph.

Applying the procedure in the previous two paragraphs to each innermost disconnected region expresses the result as an iterated extension of dividing sets for which $n(\gamma) < N+1$. It follows by induction that γ can be further expressed as an iterated extension of dividing sets, for which $n(\gamma) = 0$, which are boundary connected. □

4.5 The positive half of the contact category

The decomposition of the formal contact category introduced by the proposition below will clarify our discussion later.

Proposition 4.15 *The formal contact category $\mathcal{K}\mathrm{o}(\Sigma, m)$ associated to a surface with boundary splits into a product of two pieces,*

$$\mathcal{K}\mathrm{o}(\Sigma, m) \cong \mathcal{K}\mathrm{o}_+(\Sigma, m) \times \mathcal{K}\mathrm{o}_-(\Sigma, m),$$

supported on the dividing sets $\gamma \in \mathcal{K}\mathrm{o}(\Sigma, m)$, in which the basepoint $z_1 \in \partial_1 \Sigma$ is contained in a positive or negative region, respectively.

Proof If two dividing sets γ and γ' are isotopic, then the signs of the regions containing the basepoint must be equal. If $\theta \colon \gamma \to \gamma'$ is a bypass move then it cannot change the sign of the region containing the basepoint z_1. The rest of the proof follows along the same lines of the proof of Theorem 4.4. □

By Proposition 4.5, the two pieces found in the decomposition above are equivalent:

$$-^{\vee}\colon \mathcal{K}o^{n}_{+}(\Sigma, m) \xrightarrow{\sim} \mathcal{K}o^{-n}_{-}(\Sigma, m).$$

In Corollary 5.3, moving the basepoint z_1 to an adjacent region is shown to yield an equivalence $r\colon \mathcal{K}o^{n}_{+}(\Sigma, m) \xrightarrow{\sim} \mathcal{K}o^{n}_{-}(\Sigma, m)$. By composing the two maps we obtain an equivalence

$$\mathcal{K}o^{n}_{+}(\Sigma, m) \xrightarrow{\sim} \mathcal{K}o^{-n}_{+}(\Sigma, m).$$

See also Proposition 6.15.

5 Symmetries and generators of contact categories

The mapping class group of the surface Σ is shown to act naturally on the formal contact category $\mathcal{K}o(\Sigma)$. After introducing arc diagrams and parametrizations of surfaces by arc diagrams, each parametrization of Σ by an arc diagram is shown to yield a system of generators for the formal contact category. Section 5.4 contains a discussion of decategorification.

5.1 The mapping class group action

In this section, we show that the mapping class group $\Gamma(\Sigma)$ acts naturally on $\mathcal{K}o(\Sigma)$.

Definition 5.1 Suppose that Σ is an oriented surface. Then the mapping class group $\Gamma(\Sigma)$ is the group of connected components of the group of orientation-preserving and boundary-fixing diffeomorphisms:

$$\Gamma(\Sigma) = \pi_0 \operatorname{Diff}^{+}(\Sigma, \partial\Sigma).$$

Recall that an action of a group G on a dg category $\mathcal{C}$ is a homomorphism from G to the group $\operatorname{Aut}(\mathcal{C}) \subset \operatorname{End}_{\mathrm{Hmo}}(\mathcal{C})$ of derived equivalences.

Theorem 5.2 *The mapping class group $\Gamma(\Sigma)$ acts naturally on the formal contact category $\mathcal{K}o(\Sigma)$.*

Proof The proof occurs in two steps: first we construct a natural $\Gamma(\Sigma)$–action on the preformal contact category Pre–$\mathcal{K}o(\Sigma)$, and second this group action is extended to the formal contact category $\mathcal{K}o(\Sigma)$.

A diffeomorphism class $g \in \Gamma(\Sigma)$ determines a functor $f_g\colon \text{Pre–}\mathcal{K}o(\Sigma) \to \text{Pre–}\mathcal{K}o(\Sigma)$ that is defined by its action on dividing sets and bypass moves. If γ is an isotopy class of dividing set on Σ then there is a unique isotopy class of dividing set $g\gamma$, and if $\theta = (T, \gamma, \gamma')$ is a bypass move then there is a unique bypass disk gT and associated bypass move $g\theta = (gT, g\gamma, g\gamma')$. Since the category Pre–$\mathcal{K}o(\Sigma)$ is generated by bypass moves and the assignment $\theta \mapsto g\theta$ preserves disjointness of bypass moves and caps of bypass moves, there is a functor

$$f_g\colon \text{Pre–}\mathcal{K}o(\Sigma) \to \text{Pre–}\mathcal{K}o(\Sigma) \quad \text{such that } f_g(\gamma) = g\gamma \text{ and } f_g(\theta) = g\theta.$$

Both the composition law $f_{gg'} = f_g \circ f_{g'}$ and naturality follow directly from the definition. In particular, since the identity diffeomorphism $1 \in \Gamma(\Sigma)$ fixes both dividing sets and bypass moves, the functor f_1 is the identity functor $1_{\text{Pre–}\mathcal{K}o(\Sigma)}$.

Suppose that $f_g : \text{Pre–}\mathcal{K}\text{o}(\Sigma) \to \text{Pre–}\mathcal{K}\text{o}(\Sigma)$ is a functor occurring in the construction above. Composing with the localization functor $Q : \text{Pre–}\mathcal{K}\text{o}(\Sigma) \to L_{\Xi}\text{Pre–}\mathcal{K}\text{o}(\Sigma)$ from (3-1) yields a functor $\text{Pre–}\mathcal{K}\text{o}(\Sigma) \to L_{\Xi}\text{Pre–}\mathcal{K}\text{o}(\Sigma)$. By Definition 2.15, the image of

$$Q^* : \text{Hom}_{\text{Hqe}}(L_{\Xi}\text{Pre–}\mathcal{K}\text{o}(\Sigma), L_{\Xi}\text{Pre–}\mathcal{K}\text{o}(\Sigma)) \to \text{Hom}_{\text{Hqe}}(\text{Pre–}\mathcal{K}\text{o}(\Sigma), L_{\Xi}\text{Pre–}\mathcal{K}\text{o}(\Sigma))$$

is the subset of functors $f : \text{Pre–}\mathcal{K}\text{o}(\Sigma) \to L_{\Xi}\text{Pre–}\mathcal{K}\text{o}(\Sigma)$ whose restriction to a bypass triangle extends to a distinguished triangle in the localization $L_{\Xi}\text{Pre–}\mathcal{K}\text{o}(\Sigma)$.

If $\tilde{\theta} : D' \to \text{Pre–}\mathcal{K}\text{o}(\Sigma)$ is the bypass triangle

$$\gamma \xrightarrow{\theta} \gamma' \xrightarrow{\theta'} \gamma'' \xrightarrow{\theta''} \gamma[1]$$

associated to a bypass move $\theta = (T, \gamma, \gamma')$ on Σ by Proposition 3.16, then $f_g(\theta) = (gT, g\gamma, g\gamma')$ and $f_g(\tilde{\theta})$ corresponds to the bypass triangle

$$g\gamma \xrightarrow{g\theta} g\gamma' \xrightarrow{g\theta'} g\gamma'' \xrightarrow{g\theta''} g\gamma[1].$$

Since the criteria of Definition 2.15 are satisfied, there is a unique lift of the functor $Q \circ f_g$ to a functor $\tilde{f}_g : L_{\Xi}\text{Pre–}\mathcal{K}\text{o}(\Sigma) \to L_{\Xi}\text{Pre–}\mathcal{K}\text{o}(\Sigma)$. By Proposition 2.4, there is an induced functor between the associated pretriangulated hulls:

$$h_g : \mathcal{K}\text{o}(\Sigma) \to \mathcal{K}\text{o}(\Sigma) \quad \text{where } h_g = \tilde{f}_g^{\text{pretr}}.$$

Uniqueness of the lift and functoriality of $-^{\text{pretr}}$ imply that the stated group action is obtained. □

The same argument as above allows us to define an automorphism r which moves the first basepoint across the first adjacent boundary point. The corollary below records the existence of this map.

Corollary 5.3 *There is a distinguished automorphism r of $\mathcal{K}\text{o}(\Sigma, m)$ which moves the first basepoint $z_1 \in \partial_1\Sigma$ on the first boundary component over the nearest boundary point in the direction of the orientation.*

The functor r induces functors $r : \mathcal{K}\text{o}^n_{\pm}(\Sigma, m) \to \mathcal{K}\text{o}^n_{\mp}(\Sigma, m)$ with respect to the decomposition of $\mathcal{K}\text{o}^n(\Sigma, m)$ found in Proposition 4.15. See also Proposition 6.15.

5.2 Arc diagrams

An arc diagram is a combinatorial way to record a handle decomposition of a surface. The definitions below are due to Zarev [62] and constitute generalizations of ideas which were used by Lipshitz, Ozsváth and Thurston [36, Section 3.2].

Definition 5.4 An *arc diagram* $\mathcal{Z}$ consists of three things:

(1) an ordered collection $Z = \{\mathcal{Z}_1, \dots, \mathcal{Z}_l\}$ of l oriented line segments,

(2) a set $\boldsymbol{a} = \{a_1, \dots, a_{2k}\}$ of distinct points in the line segments Z, and

(3) a two-to-one function $M : \boldsymbol{a} \to \{1, \dots, k\}$ called the *matching*.

In order to apply to any version of the bordered Heegaard–Floer package, this data is required to be *nondegenerate*: after performing surgery on Z at each 0–sphere $M^{-1}(j)$ for $1 \le j \le k$, the resulting 1–manifold must have no closed components.

The set of points $\boldsymbol{a}$ receives a total ordering from the order on the set Z and the orientations of the line segments. The numbers l and k are allowed to be zero. Each arc diagram $\mathcal{Z}$ determines a surface $F(\mathcal{Z})$.

Definition 5.5 The *surface* $F(\mathcal{Z})$ associated to an arc diagram $\mathcal{Z}$ is given by thickening each line segment $\mathcal{Z}_i$ to $\mathcal{Z}_i \times [0,1]$ for $1 \le i \le l$ and attaching oriented 1–handles $D^1 \times D^1$ along the normal bundles of the 0–spheres $M^{-1}(j) \times \{0\}$ for $1 \le j \le k$. The surface $F(\mathcal{Z})$ is oriented by extending the orientation of the line segment $\mathcal{Z}_1$ and its positive normal.

Remark 5.6 One can regard $\mathcal{Z}_i$ as part of the boundary of $\mathcal{Z}_i \times [0,1]$. In Definition 5.10, an arc parametrization will be used to construct dividing sets $\mathfrak{z}_C \in \mathcal{K}o_+(F(\mathcal{Z}))$ in which the positive regions correspond to the handles of $\mathcal{Z}$. In particular, $\mathcal{Z}_i$, when regarded as part of the boundary, will always be contained in a positive region of $\mathfrak{z}_C \in \mathcal{K}o_+(F(\mathcal{Z}))$ (and a negative region of $z_C \in \mathcal{K}o_-(F(\mathcal{Z}))$.

Recall that the points m on a pointed oriented surface (Σ, m) are also ordered by the ordering of the boundary components, and the order on each boundary component is obtained by starting from each basepoint and traveling in the direction of the orientation induced on the boundary.

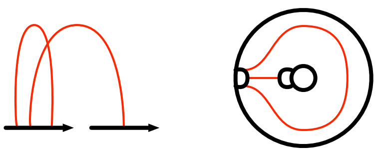
Definition 5.7 Suppose that $m \subset \partial\Sigma$ is the set of *sutures* or points fixed along the boundary of Σ. An *arc parametrization* $(\mathcal{Z}, \varphi_{\mathcal{Z}})$ of a pointed oriented surface (Σ, m) is an arc diagram $\mathcal{Z}$ and a proper orientation-preserving diffeomorphism

$$\varphi_{\mathcal{Z}} \colon \left(F(\mathcal{Z}), \bigcup_{i=1}^{l} \partial \mathcal{Z}_i \right) \to (\Sigma, m)$$

which preserves the total order on the points $\boldsymbol{a}$ and m.

Remark 5.8 An arc parametrization identifies $\bigcup_{i=1}^{l} \partial \mathcal{Z}_i$ with m. The sets m and $\boldsymbol{a}$ play different roles, but under this identification pairs in m partition the points of $\boldsymbol{a}$.

Example 5.9 The annulus $(S^1 \times [0,1], (2,2))$ with two points fixed on each boundary component is parametrized by the arc diagram $\mathcal{Z}$ pictured on the left:



This picture contains two oriented lines $Z = \{\mathcal{Z}_1, \mathcal{Z}_2\}$ and four points $\boldsymbol{a} = \{x, x', y, y'\}$ with $\mathcal{Z}_1 = xyx'$ and $\mathcal{Z}_2 = y'$. The matching function $M \colon \boldsymbol{a} \to \{1, 2\}$ is determined by the assignments $M(x) = 1 = M(x')$ and $M(y) = 2 = M(y')$. The picture on the right shows the surface $F(\mathcal{Z})$ associated to $\mathcal{Z}$.

5.3 Generators from arc diagrams

In this section, we show that a parametrization $\mathcal{P} = (\mathcal{Z}, \varphi_{\mathcal{Z}})$ of a pointed oriented surface (Σ, m) determines a canonical collection $\mathfrak{Z}(\mathcal{Z})$ of generators for the associated contact category $\mathcal{K}o(\Sigma, m)$. This material is motivated by a reading of Zarev [63].

Definition 5.10 Suppose that a pointed oriented surface (Σ, m) is parametrized by an arc diagram $\mathcal{Z}$. Then for each subset $C \subset \{1, \dots, k\}$ of matched pairs, there is an *elementary dividing set*

$$\mathfrak{z}_C = \partial R_C \quad \text{on } \Sigma,$$

where $R_C \subset \Sigma$ is the union of a thickening of the core of each 1–handle indexed by C with the collection of thickened oriented arcs $\mathcal{Z}_i \times [0,1]$. The region R_C is the positive region of $\mathfrak{z}_C$ and its complement $\Sigma \setminus R_C$ is the negative region of $\mathfrak{z}_C$.

An elementary dividing set may be also be called a *positive elementary dividing set*. The *set of positive elementary dividing sets* will be denoted by $\mathfrak{Z}_+(\mathcal{Z})$. The *set of negative elementary dividing sets* $\mathfrak{Z}_-(\mathcal{Z}) = \mathfrak{Z}_+(\mathcal{Z})^\vee$ is obtained by reversing the positive and negative regions. The *set of elementary dividing sets* is the union

$$\mathfrak{Z}(\mathcal{Z}) = \mathfrak{Z}_+(\mathcal{Z}) \cup \mathfrak{Z}_-(\mathcal{Z}).$$

Theorem 5.11 *Suppose (Σ, m) is a pointed oriented surface with boundary and (Σ, m) is parametrized by an arc diagram $\mathcal{Z}$. Then the elementary dividing sets $\mathfrak{Z}(\mathcal{Z})$ classically generate the contact category $\mathcal{K}o(\Sigma, m)$: any dividing set γ is homotopy equivalent to an iterated extension of dividing sets $\mathfrak{z} \in \mathfrak{Z}(\mathcal{Z})$.*

Proof Suppose that γ is a dividing set on Σ. We will show that γ can be expressed in terms of elementary dividing sets. The proof will be divided into a number of steps.

First By Theorem 4.14 we can assume that γ is boundary connected.

Second Here we simplify γ within the 1–handles of $F(\mathcal{Z})$.

Let $\{c_1, \dots, c_k\}$ be the set of cocores of 1–handles of $F(\mathcal{Z})$. If c_i is a cocore of a 1–handle in $F(\mathcal{Z})$ and the intersection number $|\gamma \cap c_i|$ is greater than 2, then there is a bypass disk with equator parallel to c_i with associated bypass triangle $\gamma \to \gamma' \xrightarrow{\theta_B} \gamma'' \to \gamma[1]$ with $|\gamma' \cap c_i|, |\gamma'' \cap c_i| < |\gamma \cap c_i|$. So γ is isomorphic to a cone

$$\gamma \cong C(\theta_B) \quad \text{such that } |\gamma' \cap c_i|, |\gamma'' \cap c_i| < |\gamma \cap c_i|.$$

Since γ bounds an orientable surface contained within the 1–handle, $|\gamma \cap c_i|$ is even. In more detail, γ bounds $R \subset \Sigma \setminus \gamma$ so $R \cap c_i$ is a disjoint union of intervals. Since the cardinality of the boundary of an interval is 2, $\gamma \cap c_i = \partial(R \cap c_i)$ is even.

Therefore, after iterating this procedure some number of times, we can assume that

(5-1) $$|\gamma \cap c_i| = 0 \quad \text{or} \quad |\gamma \cap c_i| = 2 \quad \text{for } 1 \le i \le k.$$

If the intersection number is 0 then the i^{th} 1–handle is *unoccupied*, and if the number is 2 then the i^{th} 1–handle is *occupied*.

Third Here we simplify γ within the 0–handles of $F(\mathcal{Z})$.

After removing the cocores from the surface, one obtains a disjoint union of disks

$$F(\mathcal{Z}) \setminus \{c_1, \dots, c_k\} = \coprod_{i=1}^{l} D_i^2. \tag{5-2}$$

The positive regions of a dividing set γ produced by the second step intersects the boundary of each such disk along intervals where occupied 1–handles are attached and the endpoints of the oriented line segment $\mathcal{Z}_i \times [0,1] \subset \partial D_i^2$.

Let us formalize the situation which we will simplify in the remainder of the proof. Suppose R is a positive region bounded by γ, and D_i is a disk from (5-2). Then R is *disconnected in* D_i if $R \cap \partial D_i \neq \varnothing$ and $(R \cap D_i) \cap \mathcal{Z}_i \times [0,1] = \varnothing$. A region R is *disconnected* if R is disconnected in D_i for some disk D_i in (5-2).

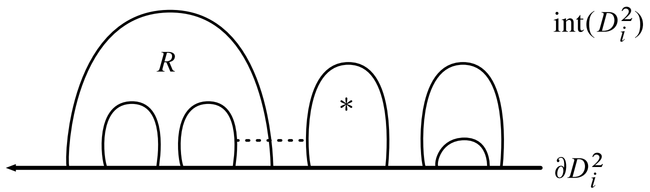
A dividing set γ is elementary if and only if there is one positive region in each disk. So in order to express γ produced by step two in terms of elementary dividing sets, we must reduce the number of disconnected regions. (This is just a version of Theorem 4.14 with the boundary components $\mathcal{Z}_i \subset \partial\Sigma$ treated separately.)

Let $R_1, \dots, R_N$ be the positive regions of γ which are disconnected. Our complexity function is

$$n(\gamma) := \sum_{i=1}^{N} \sum_{j=1}^{l} |\pi_0(R_i \cap \partial D_j)| \in \mathbb{Z}_{\geq 0},$$

the total number of 1–handles occupied by the disconnected regions. Notice that if $N > 0$ then there exists an R such that $R \cap \partial D_i \neq \varnothing$ and so $n(\gamma) > 0$. On the other hand, if $n(\gamma) = 0$ then there are no disconnected regions and $N = 0$.

We claim that any γ which satisfies (5-1) with $n(\gamma) > 0$ can be expressed as a twisted complex in dividing sets γ' which satisfy $n(\gamma') = 0$. Suppose $n(\gamma) > 0$. Then there is a disk D_i which contains a disconnected region. Let $*$ be the positive region which contains $\mathcal{Z}_i \times [0,1] \subset D_i$. Now follow the orientation around D_i to the region R disconnected in D_i which is adjacent to $*$ and consider the bypass move illustrated below:



This results in a triangle $\gamma \to \gamma' \to \gamma''$ for which $n(\gamma'), n(\gamma'') < n(\gamma)$.

Lastly, our dividing sets may still contain some positive regions which do not intersect the boundary of any disk. Such regions can be removed with Theorem 4.14. □

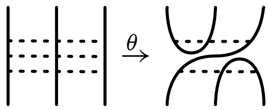
Corollary 5.12 *When a pointed oriented surface Σ is parametrized by an arc diagram $\mathcal{Z}$, the positive half of the formal contact category $\mathcal{K}o_+(\Sigma)$ is generated by the positive elementary dividing sets $\mathfrak{Z}_+(\mathcal{Z})$.*

5.4 Decategorification

In this section, we prove a variety of structural properties and conjecture a decategorification statement for the formal contact category.

Proposition 5.13 *A single bypass $\theta\colon \gamma \to \gamma$ which takes γ to γ is capped.*

Proof One can make a small perturbation a (or b) above (or below) the equator ℓ of the bypass θ, as pictured on the left-hand side below. The bypasses associated to a (or b) are isotopic to θ.



Now by assumption the right-hand side, or the result of performing θ, is isotopic to the left-hand side. This isotopy takes the caps pictured on the right-hand side to caps of the bypasses on the left-hand side, so a and b are capped. But a and b arose as perturbations of θ, so θ is capped. □

Proposition 5.14 *Let Σ be a surface with boundary together with a parametrization $(\mathcal{Z}, \varphi_{\mathcal{Z}})$. There is a surjective map*

$$\epsilon\colon \mathbb{F}_2\langle \mathrm{Ob}(\mathcal{K}o_+(\Sigma))\rangle \to \Lambda^* H_1(F(\mathcal{Z}), F(\partial\mathcal{Z}); \mathbb{F}_2),$$

where $F(\partial\mathcal{Z}) := \bigcup_i \mathcal{Z}_i \subset \partial F(\mathcal{Z})$. This map satisfies the following property: if

$$\gamma \to \gamma' \to \gamma''$$

is a bypass triangle then $\epsilon(\gamma'') = \epsilon(\gamma) + \epsilon(\gamma')$.

Proof A dividing set $\gamma \subset \Sigma$ determines a collection of positive regions: if $\Sigma \setminus \gamma = \bigsqcup_{i\in I} R_i$ then the set of positive regions is given by $\mathcal{R} := \{i \in I : R_i \text{ is positive}\}$. For each such region $R \in \mathcal{R}$, let $\partial_+ R := \partial R \cap F(\partial\mathcal{Z})$; the pair $(R, \partial_+ R)$ gives an inclusion

$$i_R\colon (R, \partial_+ R) \to (F(\mathcal{Z}), F(\partial\mathcal{Z})).$$

Let $n_R := \dim H_1(R, \partial_+ R; \mathbb{F}_2)$, so that $\Lambda^{n_R} H_1(R, \partial_+ R; \mathbb{F}_2)$ is 1–dimensional and there is a unique choice of nonzero vector $v_R \in \Lambda^{n_R} H_1(R, \partial_+ R; \mathbb{F}_2)$. Now tensoring gives a map

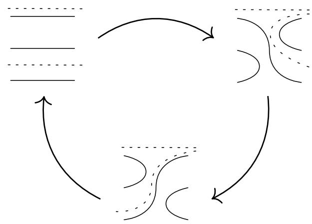
$$\hat{\imath}\colon \bigotimes_{R\in\mathcal{R}} \Lambda^{n_R} H_1(R, \partial_+ R; \mathbb{F}_2) \xrightarrow{\bar{\imath}} \bigotimes_{R\in\mathcal{R}} \Lambda^{n_R} H_1(F(\mathcal{Z}), F(\partial\mathcal{Z}); \mathbb{F}_2) \hookrightarrow \Lambda^* H_1(F(\mathcal{Z}), F(\partial\mathcal{Z}); \mathbb{F}_2),$$

where $\bar{\imath} := \bigotimes_{R\in\mathcal{R}} \wedge^{n_R}(i_R)_*$ and the last map is a composition of wedge products. The map ϵ is defined as

$$\epsilon(\gamma) := \hat{\imath}\Big(\bigwedge_{R\in\mathcal{R}} v_R\Big).$$

The 1–handles in $F(\mathcal{Z})$ span $H_1(F(\mathcal{Z}), F(\partial\mathcal{Z}); \mathbb{F}_2)$. If C corresponds to a subset of 1–handles, then by construction $\epsilon(\mathfrak{z}_C)$ is the wedge product of these classes in $\Lambda^* H_1(F(\mathcal{Z}), F(\partial\mathcal{Z}); \mathbb{F}_2)$. Since wedge products of 1–handles span the exterior algebra, ϵ is onto.

Additivity of ϵ can be observed by examining how the bypass moves affect elements in the first homology.



In the picture above, the dashed arcs represent (local) choices of generators in a positive region. If the $\epsilon(\gamma) = A \wedge C$ and $\epsilon(\gamma') = B \wedge C$ are the wedge products of arcs depicted on the left and right, respectively, then $\epsilon(\gamma'') = (A + B) \wedge C$. The other possible cases are handled similarly. □

Corollary 5.15 *For any bypass $\theta\colon \mathfrak{z}_C \to \mathfrak{z}_{C'}$ between elementary dividing sets, the third dividing set γ'' in the associated bypass triangle,*

(5-3) $$\mathfrak{z}_C \xrightarrow{\theta} \mathfrak{z}_{C'} \to \gamma'',$$

is not an elementary dividing set.

Proof As above, elementary dividing sets $\mathfrak{z}_C$ determine basis vectors for $\Lambda^* H_1(\Sigma, \partial\Sigma; \mathbb{F}_2)$ in a canonical way. Since $\epsilon(\gamma'')$ in (5-3) must be a sum of the vectors determined by $\mathfrak{z}_C$ and $\mathfrak{z}_{C'}$ in this correspondence, it cannot be an elementary generator. □

Conjecture 5.16 *For any parametrization $\mathcal{Z}$ of Σ, there is a map $\bar{\epsilon}$, induced by ϵ, which is an isomorphism, as in the following diagram:*

$$\begin{array}{ccc} \mathbb{F}_2\langle \mathrm{Ob}(\mathcal{K}o_+(\Sigma))\rangle & & \\ \big\downarrow \pi & \searrow^{\epsilon} & \\ K_0(\mathcal{K}o_+(\Sigma)) & \xrightarrow{\bar{\epsilon}} & \Lambda^* H_1(F(\mathcal{Z}), F(\partial\mathcal{Z}); \mathbb{F}_2) \end{array}$$

Here π is the quotient map found in the definition of K_0.

Relation to work of J Murakami and O Viro The representation theory of the quantum group $U_q(\mathfrak{sl}_2)$ at $q^4 = 1$ determines a degenerate instance of the Chern–Simons topological field theory that has been related to the Alexander polynomial [40; 58]. The Jones–Wenzl projector $p_3 \in \mathrm{End}_{U_q(\mathfrak{sl}_2)}(V^{\otimes 3})$ takes the form

$$p_3 = |\,|\,| - \frac{d}{d^2-1}\Big(\genfrac{}{}{0pt}{}{\cup}{\cap}| + |\genfrac{}{}{0pt}{}{\cup}{\cap} \Big) + \frac{1}{d^2-1}\Big(\genfrac{}{}{0pt}{}{\cup\,|}{|\,\cap} + \genfrac{}{}{0pt}{}{|\,\cup}{\cap\,|} \Big),$$

where $d = q + q^{-1}$. Taking $q = \sqrt{-1}$ gives $d = 0$ and $d^2 - 1 = -1$. This eliminates the middle term above, leaving the bypass triangle

$$p_3 =$$ 

Since the right-hand side should be zero, there is only a relationship between the contact geometry and representation theory after reducing by the Goodman–Wenzl ideal $\langle p_3 \rangle$ [14].

6 Comparison between categories associated to disks

In this section, we show that the categories associated to the disk $(D^2, 2n)$ with $2n$ points by the Heegaard–Floer theory $\mathcal{A}(D^2, 2n)$, the contact topology $\mathcal{C}\mathrm{o}(D^2, 2n)$ and the formal contact construction, are Morita equivalent:

$$\mathcal{A}(D^2, 2n) \cong \mathcal{C}\mathrm{o}(D^2, 2n) \cong \mathcal{K}\mathrm{o}_+(D^2, 2n).$$

This is accomplished by choosing an arc parametrization $\mathcal{M}_n$ of the disk $(D^2, 2n)$ so that the associated Heegaard–Floer category $\mathcal{A}(D^2, 2n) \cong \mathcal{A}(-\mathcal{M}_n)$ has the same quiver presentation as the algebraic contact category $\mathcal{C}\mathrm{o}(D^2, 2n) \cong \mathcal{Y}_n$ studied by Tian. This equivalence is combined with Theorem 5.11 to show that both categories are Morita equivalent to the positive half of the formal contact category $\mathcal{K}\mathrm{o}_+(D^2, 2n)$. In this section $n \geq 2$.

6.1 The Heegaard–Floer categories associated to a disk

In this section, an arc diagram $\mathcal{M}_n$ and an arc parametrization of the disk $(D^2, 2n)$ with $2n$ marked points by $\mathcal{M}_n$ are introduced. The bordered sutured Floer theory developed by Zarev associates a dg category $\mathcal{A}(\mathcal{M}_n)$ to this parametrization. In Section 6.3, we will find that this category is the same as Tian's quiver algebra $\mathcal{R}_n$.

The disk will be oriented in the opposite direction of later sections. In this way the boundary of the disk is oriented clockwise. When viewed from above, as in the illustration below, each interval $\mathcal{Z}_i \subset \partial D$ has a well-defined left direction (counterclockwise) and right direction (clockwise). This terminology is used by the definition below.

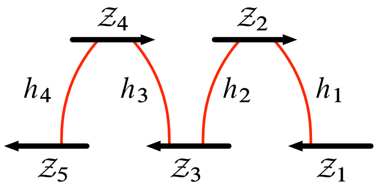
Definition 6.1 The *zigzag arc diagram* $\mathcal{M}_n$ is defined inductively as follows:

(1) The arc diagram $\mathcal{M}_2$ consists of two lines $Z = \{\mathcal{Z}_1, \mathcal{Z}_2\}$ and two points $\boldsymbol{a} = \{a_1, a_1'\}$, where $a_1 \in \mathcal{Z}_1$, $a_1' \in \mathcal{Z}_2$ and $M(a_1) = M(a_1')$.

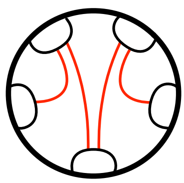
(2) If n is odd then $\mathcal{M}_n$ is obtained from $\mathcal{M}_{n-1}$ by adding a new line $\mathcal{Z}_n$, containing the point a_{n-1}, to the right of the line $\mathcal{Z}_{n-2}$ and adding the point a_{n-1}' to the line $\mathcal{Z}_{n-1}$ immediately to the left of a_{n-2}'.

(3) If n is even then $\mathcal{M}_n$ is obtained from $\mathcal{M}_{n-1}$ by adding a new line $\mathcal{Z}_n$, containing the point a_{n-1}', to the left of $\mathcal{Z}_{n-2}$ and then adding the point a_{n-1} to $\mathcal{Z}_{n-1}$ to the right of the point a_{n-2}.

If we imagine the line segments $\{\mathcal{Z}_i\}_{i=1}^n$ to be embedded sequentially along the real line $\mathbb{R}$, then an orientation on each line segment is induced by choosing an orientation of $\mathbb{R}$; they all point either to the left or to the right. The name zigzag becomes clear after rearranging the line segments into a zigzag pattern:



The arc diagram for $\mathcal{M}_5$ is pictured above. The line labeled $\mathcal{Z}_i$ is the i^{th} line segment in the construction from Definition 6.1. The lines h_i connect the matched pairs $\{a_i, a_i'\}$. If the illustration above is understood to specify an embedding of the arc diagram into the plane, then thickening each of the components produces the parametrization of the disk $(D^2, 2\cdot 5)$ with 10 points, pictured below:



Giving the plane the standard $\langle x, y\rangle$ orientation induces an orientation on $(D^2, 2n)$ in which the boundary is oriented clockwise.

The proposition below may be clear to readers who are more familiar with the algebras involved.

Proposition 6.2 *The dg category $\mathcal{A}(-\mathcal{M}_n)$ has trivial differential $d = 0$.*

Proof This follows from the definition of the differential. In more detail, by construction, as an algebra with idempotents, the dg category $\mathcal{A}(\mathcal{M}_n)$ is a subalgebra of a tensor product of copies of strands algebras $\mathcal{A}(1)$ and $\mathcal{A}(2)$. Neither of these algebras have differentials. Any tensor product of algebras without differentials does not have a differential. Any subalgebra of an algebra without differential does not have a differential; see also [62, Proposition 9.2]. □

Without a differential, the dg category $\mathcal{A}(-\mathcal{M}_n)$ is a category. The definition below comes from [62, Section 2.3]. It is summarized in Definition 6.3.

First note that the idempotents in this construction correspond to the objects of the category $\mathcal{A}(-\mathcal{M}_n)$. The idempotents are indexed by a choice of a subset

$$S \subset \{h_1, \ldots, h_{n-1}\}$$

of the 1–handles which identify matched pairs in the arc diagram $\mathcal{M}_n$ [62, Definition 2.5].

In the definition of $\mathcal{M}_n$ above, there are n line segments $\{\mathcal{Z}_1, \dots, \mathcal{Z}_n\}$. On the segment $\mathcal{Z}_1$, there is only one point a_1. If n is even then $\mathcal{Z}_n$ contains only one point a'_{n-1}. If n is odd then $\mathcal{Z}_n$ contains only the point a_{n-1}. The line segment $\mathcal{Z}_k \in \{\mathcal{Z}_2, \dots, \mathcal{Z}_{n-1}\}$ contains the two points

$$a'_k a'_{k-1} \text{ for } k \text{ even} \quad \text{or} \quad a_k a_{k+1} \text{ for } k \text{ odd.} \tag{6-1}$$

Since the algebra $\mathcal{A}(1)$ only contains the identity element, the nonidentity elements in the parts of $\mathcal{A}(-\mathcal{M}_n) \subset \mathcal{A}(1) \otimes \mathcal{A}(2)^{\otimes n-2} \otimes \mathcal{A}(1)$ correspond to the $\mathcal{A}(2)$–tensor factors. Each such factor contains a Reeb chord $\rho_{k,k+1}$ or $\rho_{k+1,k}$. If the line segment contains the points $a'_{k+1}a'_k$ then the Reeb chord $\rho_{k,k+1}$ connects $\rho^-_{k,k+1} = a'_k$ to $\rho^+_{k+1,k} = a'_{k+1}$. If the line segment contains the points $a_k a_{k+1}$ then the Reeb chord $\rho_{k+1,k}$ connects $\rho^-_{k+1,k} = a_{k+1}$ to $\rho^+_{k+1,k} = a_k$. Since the k^{th} 1–handle h_k corresponds to the matching of the pair a_k and a'_k, the Reeb chords $\rho_{k,k+1}$ and $\rho_{k+1,k}$ correspond to maps

$$\rho_{k,k+1} \colon h_k \to h_{k+1} \quad \text{and} \quad \rho_{k+1,k} \colon h_{k+1} \to h_k \tag{6-2}$$

Translating (6-1) into the language of (6-2) tells us when such maps can be found in the category $\mathcal{A}(-\mathcal{M}_n)$. If n is even then there are maps

$$h_{n-1} \xrightarrow{\rho_{n-1,n-2}} h_{n-2} \xleftarrow{\rho_{n-3,n-2}} h_{n-3} \to \cdots \leftarrow h_3 \xrightarrow{\rho_{3,2}} h_2 \xleftarrow{\rho_{1,2}} h_1$$

and if n is odd then there are maps

$$h_{n-1} \xleftarrow{\rho_{n-2,n-1}} h_{n-2} \xrightarrow{\rho_{n-2,n-3}} h_{n-3} \leftarrow \cdots \leftarrow h_3 \xrightarrow{\rho_{3,2}} h_2 \xleftarrow{\rho_{1,2}} h_1.$$

Increasing the number n by one has the effect of adding one new Reeb chord.

The generators of the full category $\mathcal{A}(-\mathcal{M}_n)$ are obtained by extending each Reeb chord by identity in all possible ways [62, Definition 2.9]. In more detail, if $S = h_{i_1} h_{i_2} \cdots h_{i_j} \cdots h_{i_{k-1}} h_{i_k}$ is a subset of 1–handles which have been ordered so that $i_j < i_{j+1}$, then there is a generator

$$h_{i_1} h_{i_2} \cdots h_{i_j} \cdots h_{i_{k-1}} h_{i_k} \to h_{i_1} h_{i_2} \cdots h_{i_j \pm 1} \cdots h_{i_{k-1}} h_{i_k} \tag{6-3}$$

in $\mathcal{A}(-\mathcal{M}_n)$ when there is a Reeb chord $\rho_{i_j, i_j \pm 1} \colon h_{i_j} \to h_{i_j \pm 1}$ as above and the 1–handle $h_{i_{j \pm 1}}$ isn't contained in set S:

$$i_{j \pm 1} \notin \{i_1, i_2, \dots, i_k\}.$$

None of the relations satisfied by the strands algebras apply in our context because the Reeb chords are contained in independent strands algebras $\mathcal{A}(2)$ of order two. There is only one relevant family of relations, stemming from the observation that maps applied to independent tensor factors commute:

$$\begin{array}{ccccc} & & \cdots h_{i_j \pm 1} \cdots h_{i_l} \cdots & & \\ & \nearrow & & \searrow & \\ \cdots h_{i_j} \cdots h_{i_l} \cdots & & & & \cdots h_{i_j \pm 1} \cdots h_{i_l \pm 1} \cdots \\ & \searrow & & \nearrow & \\ & & \cdots h_{i_j} \cdots h_{i_l \pm 1} \cdots & & \end{array} \tag{6-4}$$

Said differently, whenever generators can be applied out-of-order to form a square, as pictured above, this square must commute.

The definition below summarizes the discussion above.

Definition 6.3 $\mathcal{A}(-\mathcal{M}_n)$ is the dg category with $d = 0$. The objects

$$\mathrm{Ob}(\mathcal{A}(-\mathcal{M}_n) = \{S : S \subset \{h_1, \dots, h_{n-1}\}\}$$

are subsets of the set of arcs in Definition 6.1. We write $S = \prod_{h_{i_k} \in S} h_{i_k}$ for any $S \in \mathrm{Ob}(\mathcal{A}(-\mathcal{M}_n))$. The category $\mathcal{A}(-\mathcal{M}_n)$ is generated by maps of the form (6-3) subject to relations in (6-4).

The examples below will be compared to Examples 6.9 and 6.10 in Section 6.3.

Example 6.4 The structure of $\mathcal{A}(-\mathcal{M}_3)$ can be pictured in the following way:

$$\varnothing \qquad h_1 \xrightarrow{\rho_{1,2}} h_2 \qquad h_1 h_2$$

Example 6.5 The structure of $\mathcal{A}(-\mathcal{M}_4)$ is illustrated by the diagram below:

$$\varnothing \qquad \begin{array}{ccc} & \overset{\rho_{1,2}}{\nearrow} & h_2h_3 \\ h_1h_3 & & \\ & \underset{\rho_{3,2}}{\searrow} & h_1h_2 \end{array} \qquad \begin{array}{ccc} h_1 & \overset{\rho_{1,2}}{\searrow} & \\ & & h_2 \\ h_3 & \underset{\rho_{3,2}}{\nearrow} & \end{array} \qquad h_1h_2h_3$$

Remark 6.6 Bordered sutured theory usually associates different algebras to different parametrizations of a surface. The categories of modules associated to these algebras are equivalent. In this sense, the algebras associated to surfaces are Morita equivalent; see the appendix. In order to understand why this is the case, consider that the mapping cylinder 3–manifolds associated to a diffeomorphism between parametrizations and its inverse determine a pair of bimodules [62, Section 8]. Product with a bimodule determines a functor between modules over algebras. The composition of functors gives the bimodule associated to the identity, which is algebraically the identity [62, Section 8.6]. See also [63].

In particular, there is an arc parametrization $\mathcal{W}_n$ [62, Example 9.1] for which there is an isomorphism of dg categories $\mathcal{A}(\mathcal{W}_n) \cong \mathcal{A}(n-1)^{\mathrm{op}}$ [62, Proposition 9.1], where $\mathcal{A}(n-1)$ is the strands algebra [36, Section 3.1]. Therefore $\mathcal{A}(-\mathcal{M}_n) \cong \mathcal{A}(n-1)^{\mathrm{op}}$ in Hmo.

6.2 The contact category associated to a disk

Here we introduce the category $\mathcal{Y}_n$ that Tian associates to the disk with $2n$ boundary points [54]. We will not discuss gradings.

6.2.1 Indexing multicurves with nil-Temperley–Lieb notation Monomials in the nil-Temperley–Lieb algebra will be used to denote multicurves $\gamma \subset (D^2, 2n)$ in the disk. In particular, multicurves determined by monomials $e_{i_1} e_{i_2} \cdots e_{i_k}$, which have been ordered so as to satisfy $i_1 < i_2 < \cdots < i_k$, correspond to the objects in Tian's construction; see Definition 6.8.

Definition 6.7 The *nil-Temperley–Lieb algebra* $\mathcal{N}_n$ is the k–algebra on generators e_i for $1 \leq i < n$, subject to the relations

(1) $e_i^2 = 0$ for $1 \leq i < n$,

(2) $e_i e_j = e_j e_i$ for $|i - j| > 2$, and

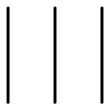
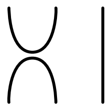
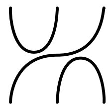
(3) $e_i e_{i\pm1} e_i = e_i$.

If the ground ring k is changed to $\mathbb{Z}[q, q^{-1}]$ and the first relation is changed from $e_i^2 = 0$ to $e_i^2 = q + q^{-1}$, then the algebra $\mathcal{N}_n$ introduced above becomes the well-known Temperley–Lieb algebra $\mathcal{TL}_n$; see [24].

The relationship between the Temperley–Lieb algebra and the planar algebra of multicurves extends to the nil-variant $\mathcal{N}_n$ introduced above. There is a basis for the algebra $\mathcal{N}_n$ consisting of monomials which is in one-to-one correspondence with isotopy classes of boundary-connected multicurves in a pointed oriented disk $(D^2, 2n)$. This can be seen after each generator e_i is identified with a multicurve $\gamma(e_i)$,

$$e_i \mapsto \gamma(e_i).$$

If the disk is pictured so that the first n points are situated on the top of the disk and the last n points are situated on the bottom of the disk, then all of the strands of $\gamma(e_i)$ are vertical except for two which connect the i^{th} and $(i+1)^{\text{st}}$ points in each collection. The products, $\gamma(e_i e_j) = \gamma(e_i)\gamma(e_j)$, of generators correspond to vertically stacking the multicurves. For instance, when $n = 3$ we have the following pictures:

$\gamma(1) =$  , $\gamma(e_1) =$  or $\gamma(e_1 e_2) =$  .

In the image of the map γ, the second and third relations in Definition 6.7 correspond to isotopy, and the first relation implies that any multicurve containing a homotopically trivial component is zero.

This observation can be used to construct a set map γ from the monomials of the nil-Temperley–Lieb algebra $\mathcal{N}_n$ to positive dividing sets on $(D^2, 2n)$. Since all of the defining relations for $\mathcal{N}_n$ preserve monomiality, the product of monomials is a monomial and each monomial $x \in \mathcal{N}_n$ corresponds to a multicurve $\gamma(x)$. After signing the regions of $D^2 \setminus \gamma(x)$, this determines a dividing set on the disk. Knowledge of the map γ is assumed throughout the next section.

6.2.2 Tian's disk category Tian's category $\mathcal{Y}_n$ is introduced by the sequence of definitions below. The construction presented here is equivalent to the original [54]. However, we will use the algebra $\mathcal{N}_n$ to express the presentation in more familiar notation.

Definition 6.8 The quiver $\mathcal{Q}_n$ has vertices $V := \{S : S = \{i_1 < i_2 < \cdots < i_k : 1 \leq i_j < n \text{ for } j = 1, \ldots, k\}\}$ and edges

$$E(S,T) := \begin{cases} \{\theta_p\} & \text{if } |T| = |S| + 2 \text{ and } T = S \cup \{p, p+1\}, \\ \varnothing & \text{otherwise.} \end{cases}$$

In more detail, the vertices S of the quiver $\mathcal{Q}_n$ are the ordered monomials

$$e_S = e_{i_1} e_{i_2} \cdots e_{i_k} \in \mathcal{N}_n \quad \text{where } S = \{i_1 < i_2 < \cdots < i_k\}$$

and $1 \le i_j < n$ for $j = 1, \dots, k$ in the nil-Temperley–Lieb algebra. There is an edge $\theta_p \colon e_S \to e_T$ from e_S to e_T when the set T can be obtained from the set S by adjoining the disjoint subset $\{p, p+1\}$.

Before introducing the category $\mathcal{Y}_n$ we illustrate this definition:

Example 6.9 When $n = 3$, the quiver $\mathcal{Q}_3$ assumes a rather unassuming form:

$$e_1 \qquad 1 \xrightarrow{\theta_1} e_1 e_2 \qquad e_2$$

Example 6.10 When $n = 4$, the quiver $\mathcal{Q}_4$ is more complicated:

$$\begin{array}{ccccccc} & & & e_1 e_2 & e_1 \xrightarrow{\theta_2} & & \\ e_1 e_3 & & 1 \nearrow^{\theta_1} \searrow_{\theta_2} & & & e_1 e_2 e_3 & e_2 \\ & & & e_2 e_3 & e_3 \xrightarrow{\theta_1} & & \end{array}$$

Each arrow $\theta_p \colon e_S \to e_T$ corresponds to a bypass move $\gamma(e_S) \to \gamma(e_T)$ between the multicurves $\gamma(e_S)$ and $\gamma(e_T)$, involving the p^{th} and $(p+1)^{\text{st}}$ regions in the disk; see (6-5).

The disk category $\mathcal{R}_n$ is the category generated by the graph $\mathcal{Q}_n$, modulo the relation that compositions of disjoint bypass moves commute.

Definition 6.11 The *disk category* $\mathcal{R}_n$ is the k–linear category generated by the graph $\mathcal{Q}_n$ subject to the relations

$$\theta_p \theta_q = \theta_q \theta_p \quad \text{for each pair of arrows } \theta_p \theta_q, \theta_q \theta_p \colon e_S \to e_T \text{ in } \mathcal{Q}_n.$$

The disk category $\mathcal{R}_n$ can be viewed as a dg category with $d = 0$. Recall the notion of a pretriangulated hull from Section 2.1.

Definition 6.12 The category $\mathcal{Y}_n$ associated to the disk $(D^2, 2n)$ is the pretriangulated hull of the disk category $\mathcal{R}_n$:

$$\mathcal{Y}_n = \mathcal{R}_n^{\text{pretr}}.$$

6.3 Relationship between the contact category and the Heegaard–Floer category

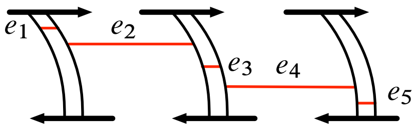
Here we show that the category $\mathcal{A}(-\mathcal{M}_n)$ found in Section 6.1 is isomorphic to Tian's disk category $\mathcal{R}_n$ from Section 6.2.2.

Theorem 6.13
$$\mathcal{R}_n \xrightarrow{\sim} \mathcal{A}(-\mathcal{M}_n).$$

Proof The similarities between Examples 6.4 and 6.9 and Examples 6.5 and 6.10 are suggestive. We will discuss the case when n is even; the case when n is odd is similar. We first give a bijective correspondence between the objects in either category. After this, the generators in either category are related to one another by representing each by geometric bypass moves.

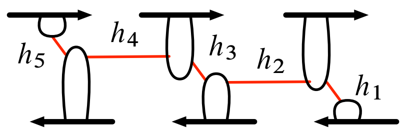
There is a one-to-one correspondence between the objects in each category. Recall that for $\mathcal{R}_n$, the objects are $\text{Ob}(\mathcal{R}_n) = V(\mathcal{Q}_n) = \{e_S : S = \{i_1 < i_2 < \cdots < i_k\}\}$, which correspond to multicurves in the disk determined by the product $e_S = e_{i_1} \cdots e_{i_k}$ in the nil-Temperley–Lieb algebra. For $\mathcal{A}(-\mathcal{M}_n)$, the objects are $\mathrm{Ob}(\mathcal{A}(-\mathcal{M}_n)) = \{S : S \subset \{h_1, h_2, \ldots, h_{n-1}\}\}$, which correspond to a selection of 1–handles in the zigzag diagram. The maps in the next two paragraphs are constructed using these two topological interpretations for S.

First we construct a map $\Phi\colon \mathrm{Ob}(\mathcal{R}_n) \to \mathrm{Ob}(\mathcal{A}(-\mathcal{M}_n))$. In this correspondence, the identity diagram $1 \in \mathcal{N}_n$ corresponds to selecting all of the odd 1–handles, $\Phi(1) = h_1 h_3 \cdots h_{n-1}$. Suppose that $e_S = e_{i_1} e_{i_2} \cdots e_{i_k} \in \mathcal{N}_n$ is an ordered monomial. Then to construct the selection of 1–handles in $\mathrm{Ob}(\mathcal{A}(-\mathcal{M}_n))$ associated to e_S we perform surgery on this identity surface $h_1 h_3 \cdots h_{n-1}$ along the arcs pictured below for each e_{i_k} appearing in e_S:



After performing this surgery, there is a uniquely determined set $S \subset \{h_1, \ldots, h_{n-1}\}$ of 1–handles in the arc diagram $\mathcal{M}_n$ corresponding to this surface; this is the map from ordered monomials to subsets S of the set of 1–handles.

Now we construct an inverse map $\Psi\colon \mathrm{Ob}(\mathcal{A}(-\mathcal{M}_n)) \to \mathrm{Ob}(\mathcal{R}_n)$. The empty set of 1–handles $\varnothing$ corresponds to the product of the odd generators, $\Psi(\varnothing) = e_1 e_3 \cdots e_{n-1}$. If $h_{i_1} h_{i_2} \cdots h_{i_k}$ is an arbitrary selection of 1–handles, then gluing each 1–handle h_{i_j} into the picture below, in the indicated fashion, uniquely determines a multicurve associated to a positive monomial:



The maps introduced above are inverse. There is a bijection between the objects of either category. Observe that performing the odd e_i surgeries in the first illustration above produces the picture below it. From this observation the following two rules can be deduced:

(1) If i is odd then the effect of choosing or not choosing e_i corresponds to removing or adding h_{n-i}.

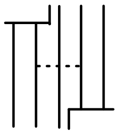
(2) If i is even then the effect of choosing or not choosing e_i corresponds to adding or removing h_{n-i}.

Here it is in algebraic notation:

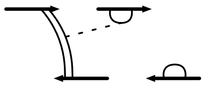
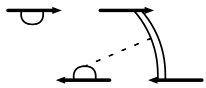
$$\Phi(e_{i_1} e_{i_2} \cdots e_{i_k}) = \{h_{n-s} : \exists r, s = i_r \text{ and } s \text{ even}\} \cup \{h_{n-s} : \forall r, s \neq i_r \text{ and } s \text{ odd}\},$$
$$\Psi(\{h_{i_1}, h_{i_2}, \ldots, h_{i_k}\}) = \{e_{n-s} : \exists r, s = i_r \text{ and } s \text{ even}\} \cup \{e_{n-s} : \forall r, s \neq i_r \text{ and } s \text{ odd}\}.$$

The variable r is restricted to the relevant subset of indices and the subscripts of a word $e_{i_1} e_{i_2} \cdots e_{i_k}$ are placed in order so as to coincide with conventions. These rules determine a bijection.

If $w, w' \in \mathcal{N}_n$ are ordered monomials, then an arrow $\theta_p\colon ww' \to we_pe_{p+1}w'$ in the graph $\mathcal{Q}_n$ corresponds to the bypass move $\theta_p\colon \gamma(ww') \to \gamma(we_pe_{p+1}w')$ pictured below:

(6-5) $\theta_p =$ 

For example, after a rotation the only arrow in the quiver $\mathcal{Q}_3$ corresponds to the bypass illustrated before Definition 3.12. Conversely, the basic Reeb chords $\rho_{k,k+1}\colon h_k \to h_{k+1}$ and $\rho_{k+2,k+1}\colon h_{k+2} \to h_{k+1}$ from Section 6.1 correspond to the pictures

(6-6)  and 

so that the two combinatorial notions perform the same function between multicurves in the correspondence between the objects.

There are no relations in either category besides the commutativity of (6-4) and Definition 6.11. □

6.4 Relationship between the disk category and the formal contact category

In this section, we will construct a Morita equivalence between the Heegaard–Floer category $\mathcal{A}(-\mathcal{M}_n)$ considered in Section 6.1 and the formal contact category $\mathcal{K}o_+(D^2, 2n)$.

The discussion in prior sections suffices to define a functor

$$\mu\colon \mathcal{A}(-\mathcal{M}_n) \to \mathcal{K}o_+(D^2, 2n).$$

To each collection of 1–handles $C = h_{i_1}h_{i_2}\cdots h_{i_k}$ we associate the elementary generator

$$\mathfrak{z}_C \in \mathrm{Ob}(\text{Pre–}\mathcal{K}o_+(D^2, 2n)).$$

The basic Reeb chords correspond to the bypass moves pictured in (6-6). Composing this functor with the quotient map $Q\colon \text{Pre–}\mathcal{K}o_+(D^2, 2n) \to \mathcal{K}o_+(D^2, 2n)$ yields μ above.

Theorem 6.14 *The functor $\mu\colon \mathcal{A}(-\mathcal{M}_n) \to \mathcal{K}o_+(D^2, 2n)$ determines a Morita equivalence.*

The proof of the theorem will use the fact that if $\mathcal{A}$ and $\mathcal{C}$ are small dg categories then $\mathcal{A}$ is Morita equivalent to $\mathcal{C}$ when $\mathcal{C}$ is quasiequivalent to a full dg subcategory $\mathcal{B}$ of the category of $\mathcal{A}$ whose objects form a set of small generators. This is a special case of a more general statement [25, Theorem 8.2].

Proof Using Theorem 5.11, it suffices to check that for each pair of collections of 1–handles C and C',

$$\mu_{C,C'}\colon \mathrm{Hom}_{\mathcal{A}(-\mathcal{M}_n)}(C, C') \to \mathrm{Hom}_{\mathcal{K}o_+(D^2,2n)}(\mathfrak{z}_C, \mathfrak{z}_{C'})$$

is a quasi-isomorphism Since the trivial bypasses must bound caps and are removed by Definition 3.15(1), the only bypasses $\mathfrak{z}_C \to \mathfrak{z}'_C$ between elementary generators are those that appear in (6-6). These bypasses and their compositions are the cycles in Pre–$\mathcal{K}o_+(D^2, 2n)$. It suffices to show that they remain cycles in the quotient.

The remainder follows from the commutativity of pushouts,

$$L_S L_{S'}\mathcal{C} \cong L_{S\amalg S'}\mathcal{C} \cong L_{S'} L_S \mathcal{C},$$

and the observation that the maps $Q_{C,C'}\colon \mathrm{Hom}_{\mathcal{C}}(C,C') \to \mathrm{Hom}_{L_S\mathcal{C}}(C,C')$ are quasi-isomorphisms for any single Postnikov localization. This can be seen by identifying a single Postnikov localization as an instance of Drinfeld localization under the Yoneda embedding; see Proposition 2.19. The Drinfeld localization modifies the homological structure of the morphisms by adding a single map h which is a boundary $dh = 1_K$, where K is as in the proof of Proposition 2.19. This makes any cycle to or from K into a boundary, but does not create any other boundaries. Since K is not an elementary generator $\mathfrak{z}_C$ for some C, the result follows. □

6.5 Dualities

Our discussion concludes with some mention of dualities. In Examples 6.9 and 6.10, duality is found in the lateral symmetry of the graph $\mathcal{Q}_n$. If $[n]$ denotes an ordered set $\{1 < 2 < \cdots < n\}$, then the assignment

$$e_S^y = e_{[n]\setminus S}$$

determines a contravariant involution:

$$-^y : \mathcal{Y}_n^{\mathrm{op}} \to \mathcal{Y}_n.$$

In $\mathcal{Y}_n$ there are no signed regions and the lateral symmetry is contravariant. So the functor $-^y$ cannot directly correspond to a functor, such as $-^\vee$, between formal contact categories. The proposition records the correct formulation. The proof is left to the reader.

Proposition 6.15 *The diagram*

$$\begin{array}{ccc} \mathcal{Y}_n^{\mathrm{op}} & \xrightarrow{\mathrm{i}_+^{\mathrm{op}}} & \mathcal{K}o_+(D^2,2n)^{\mathrm{op}} \\ {\scriptstyle -^y}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathcal{Y}_n & \xrightarrow{\mathrm{i}_+} & \mathcal{K}o_+(D^2,2n) \end{array}$$

commutes, where the functor $\alpha = (-)^\vee \circ (\overline{-}) \circ (r^{-1})^{\mathrm{op}}$ is the composition of three equivalences: r is the element of the mapping class group which rotates the basepoint z by one region clockwise (Corollary 5.3), $(\overline{-})$ reverses the orientation of the disk (Proposition 4.6) and $(-)^\vee$ changes the signs of the regions (Proposition 4.5).

7 Linear bordered Heegaard–Floer categories

Within the framework of the bordered Heegaard–Floer theory, a differential graded category $\mathcal{A}(\mathcal{Z})$ is associated to each arc diagram $\mathcal{Z}$. For some choices this category satisfies $d = 0$ and it is possible to write down a quiver presentation. In this section, these categories are related to the corresponding formal contact categories. We define functors

$$\mathcal{A}(-\mathcal{Z}_{0,n}, 1-n) \xrightarrow{\sigma_n} \mathrm{Ho}(\mathcal{K}o_+^{2n-4}(\Sigma_{0,n}, n\cdot 2)) \quad \text{and} \quad \mathcal{A}(-\mathcal{Z}_{g,1}, 2g-1) \xrightarrow{\tau_g} \mathrm{Ho}(\mathcal{K}o_+^{2g-2}(\Sigma_{g,1}, 2)),$$

where $\mathcal{Z}_{0,n}$ and $\mathcal{Z}_{g,1}$ are arc diagrams which parametrize surfaces $\Sigma_{0,n}$ and $\Sigma_{g,1}$ of genus zero with n boundary components, and of genus g with one boundary component, respectively. We fix two points on every boundary component and require that $n > 1$ and $g > 0$.

The bordered algebras studied in this section are the "one moving strand" algebras corresponding to the second-largest weight; see [62, Section 2], [34, Section 2] or [36, Section 3].

7.1 A surface $\Sigma_{0,n}$ of genus 0 with several boundary components

When n disks are removed from the 2–sphere,

$$\Sigma_{0,n} = S^2 \setminus \coprod_{i=1}^{n} D^2 \quad \text{for } n > 1,$$

and two points are fixed on each of its boundary components, the resulting surface can be parametrized by the arc diagram $\mathcal{Z}_{0,n}$ found in the definition below.

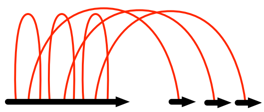
Definition 7.1 The arc diagram $\mathcal{Z}_{0,n}$ consists of n oriented line segments $Z = \{\mathcal{Z}_1, \mathcal{Z}_2, \dots, \mathcal{Z}_n\}$. On the first line segment $\mathcal{Z}_1$ there are $3n-3$ points, and there is one point on each of the remaining line segments $\{\mathcal{Z}_2 \dots \mathcal{Z}_n\}$:

$$\mathcal{Z}_1 = a_1 b_1 a_1' a_2 b_2 a_2' \cdots a_{n-1} b_{n-1} a_{n-1}' \qquad \text{and} \qquad \mathcal{Z}_i = b_{i-1}' \quad \text{for } 2 \le i \le n.$$

The set of points is given by $\boldsymbol{a} = \{a_i, a_i', b_i, b_i' : 1 \le i < n\}$. The line $\mathcal{Z}_1$ is oriented so that the subscripts of the points increase in value. The matching function is determined by the rules $M(a_i) = M(a_i')$ and $M(b_i) = M(b_i')$.

The annulus $\Sigma_{0,2}$ and its parametrization by $\mathcal{Z}_{0,2}$ are pictured in Example 5.9.

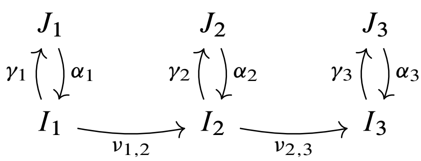
Example 7.2 When $n = 4$, the definition above is illustrated by the picture below:



Definition 7.3 The category $\mathcal{A}(-\mathcal{Z}_{0,n}, 1-n)$ associated to the arc diagram $\mathcal{Z}_{0,n}$ is the k–linear category determined by a quiver with vertices I_i and J_i corresponding to the pairs $\{a_i, a_i'\}$ and $\{b_i, b_i'\}$ for $1 \le i < n$, respectively. There are arrows $\alpha_i : I_i \to J_i$, $\gamma_i : J_i \to I_i$ and $\nu_{i,i+1} : I_i \to I_{i+1}$ subject to the relations

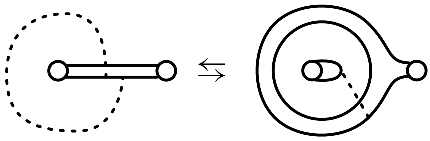
(1) $\alpha_i \gamma_i = 0 \colon J_i \to J_i$, and
(2) $\nu_{i+1,i+2} \nu_{i,i+1} = 0 \colon I_i \to I_{i+2}$.

Example 7.4 We illustrate quiver underlying the category $\mathcal{A}(-\mathcal{Z}_{0,4}, -2)$ in the definition above:



The construction of the functor $\sigma_n\colon \mathcal{A}(\mathcal{Z}_{0,n}, 1-n) \to \mathrm{Ho}(\mathcal{K}\mathrm{o}_+^{2n-4}(\Sigma_{0,n}, n\cdot 2))$ will occur in two stages. First note that the parametrization of $\Sigma_{0,n}$ by the arc diagram allows us to associate to each object I_i or J_i for $1 \le i < n$ a dividing set contained in an annulus. In fact, Theorem 5.11 states that these dividing sets generate the contact category. In each annulus we will describe bypass moves corresponding to the arrows $\alpha_i\colon I_i \to J_i$ and $\gamma_i\colon J_i \to I_i$. We will check that these bypass moves satisfy the first collection of relations in the definition above. After this has been done, bypass moves corresponding to the lateral arrows $\nu_{i,i+1}\colon I_i \to I_{i+1}$ will be introduced and shown to satisfy the second collection of relations.

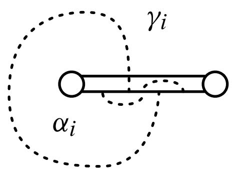
Step 1 For each annulus, the dividing sets J_i and I_i, and the bypass moves corresponding to the maps $\gamma_i\colon J_i \to I_i$ and $\alpha_i\colon I_i \to J_i$ can be depicted by the curves



The dividing set associated to J_i is featured on the left-hand side and the dividing set associated to I_i is shown on the right-hand side. The map γ_i runs from left to right and the map α_i runs from right to left. The equators of γ_i and α_i are determined by the dashed lines in the dividing sets corresponding to J_i and I_i, respectively.

Proposition 7.5 *The relation* $\alpha_i\gamma_i = 0$ *holds in the formal contact category* $\mathrm{Ho}(\mathcal{K}\mathrm{o}_+(S^1\times[0,1],(2,2)))$.

Proof The map $\alpha_i\gamma_i\colon J_i \to J_i$ is a composition of two disjoint bypass moves. This is illustrated below:



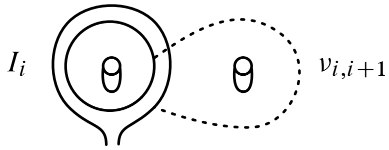
Relation (2) in the definition of the formal contact category (Definition 3.17) implies that applying the two bypass moves in either order must commute:

$$\begin{array}{ccc} J_i & \xrightarrow{\gamma_i} & I_i \\ \downarrow & & \downarrow{\scriptstyle \alpha_i} \\ 0 & \longrightarrow & J_i \end{array}$$

But performing the bypass move α_i before the bypass move γ_i must be zero since α_i is capped. □

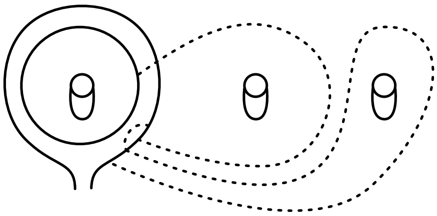
The same argument shows that one of the terms in the commutative diagram associated to the other composition $\gamma_i\alpha_i$ is a capped bypass equivalent to the identity.

Step 2 As pictured above, the idempotents I_i correspond to the boundaries of regular neighborhoods of loops about each boundary component of $\Sigma_{0,n}$. Here we think of $\Sigma_{0,n}$ as a subset of the plane $D^2 \setminus \coprod_{i=1}^{n-1} D^2 \subset \mathbb{R}^2$ with $n-1$ disks removed from its interior. The arc parametrization orders the boundary components and the associated idempotents. When two of them are adjacent, I_i and I_{i+1}, there is a bypass move $\nu_{i,i+1}\colon I_i \to I_{i+1}$ determined by the equator of the bypass disk in the illustration below:



Proposition 7.6 *In the formal contact category* $\mathrm{Ho}(\mathcal{K}o_+(\Sigma_{0,n}, n\cdot 2))$, *the relation* $\nu_{i+1,i+2}\nu_{i,i+1} = 0$ *holds.*

Proof The proof is analogous to the proof of Proposition 7.5. The bypass moves representing $\nu_{i,i+1}$ and $\nu_{i+1,i+2}$ are disjoint. Considering them simultaneously produces the visual aid below:



The curve on the far right represents the equator of the bypass $\nu_{i+1,i+2}$. Since this bypass move is capped, the composition factors through zero. □

Tian's annulus As in Section 6, in Tian's work [55, Section 2.2] the category associated to an annulus with two points on each boundary component is the pretriangulated hull on the free k–linear category associated to a quiver with five vertices: I, E, F and EF. The dividing sets associated to E and F are Euler dual and are neither the source nor the target of any nontrivial edges. There are two dividing sets I and EF generating the subcategory with Euler number zero via maps $\gamma\colon I \to EF$ and $\alpha\colon EF \to I$ which are required to satisfy the relation

$$\alpha\gamma = 0.$$

This description is summarized by the illustration below:

$$F \qquad \gamma : I \rightleftarrows EF : \alpha \qquad E$$

The quiver in the center is precisely $\mathcal{A}(-\mathcal{Z}_{0,2}, 0)$ above.

Remark 7.7 It is natural to ask about surfaces $\Sigma_{0,n}$ with $n > 2$. There are presently two constructions in the literature. In [55], the category associated to $\Sigma_{0,n}$ is a bordered Heegaard–Floer category by definition. Precisely the same can be said for the categories considered by Petkova and V Vértesi [44]. While the former chooses an arc parametrization which yields a heart encoding contact geometry, the latter chooses an arc parametrization which yields an extension [60] of the strands algebra [36]. In both cases the arc parametrizations are *degenerate*, so the bordered Heegaard–Floer construction does not suffice to imply an equivalence between the two, and the materials here do not necessarily apply.

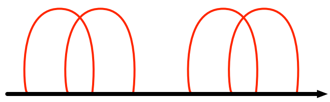
7.2 A surface $\Sigma_{g,1}$ of genus g with one boundary component

Definition 7.8 The arc diagram $\mathcal{Z}_{g,1}$ consists of $4g$ points $\boldsymbol{a} = \{a_i, a'_i, b_i, b'_i : 1 \leq i \leq g\}$ on one line segment $Z = \{\mathcal{Z}_1\}$,

$$\mathcal{Z}_1 = a_1 b_1 a'_1 b'_1 a_2 b_2 a'_2 b'_2 \cdots a_g b_g a'_g b'_g,$$

which is oriented so that the indices above are increasing. The matching function is determined by the rules $M(a_i) = M(a'_i)$ and $M(b_i) = M(b'_i)$ for $1 \leq i \leq g$.

Example 7.9 The arc diagram $\mathcal{Z}_{2,1}$ is illustrated below:



Definition 7.10 The category $\mathcal{A}(-\mathcal{Z}_{g,1}, 2g-1)$ associated to the arc diagram $\mathcal{Z}_{g,1}$ is the k–linear category determined by a quiver with vertices: I_i and J_i corresponding to the pairs $\{a_i, a'_i\}$ and $\{b_i, b'_i\}$ for $1 \leq i \leq g$, respectively. There are arrows

$$\alpha_i, \beta_i : I_i \to J_i, \quad \gamma_i : J_i \to I_i \quad \text{and} \quad \eta_{i,i+1} : I_i \to J_{i+1},$$

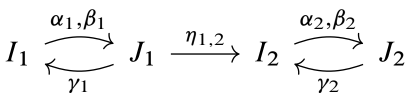
the compositions of which satisfy the relations

(1) $\alpha_i \gamma_i = 0 : J_i \to J_i$ and $\gamma_i \beta_i = 0 : I_i \to I_i$, and

(2) $\eta_{i,i+1} \alpha_i = 0 : I_i \to I_{i+1}$ and $\beta_{i+1} \eta_{i,i+1} = 0 : J_i \to J_{i+1}$.

Note that $\eta_{i,i+1} : I_i \to J_{i+1}$ is not the same as $\nu_{i,i+1} : I_i \to I_{i+1}$ in the previous section.

Example 7.11 The quiver underlying the construction of the category $\mathcal{A}(-\mathcal{Z}_{2,1}, 3)$ is illustrated below:

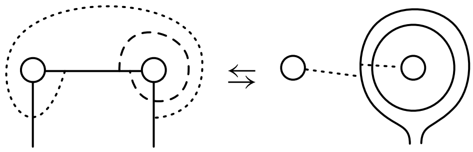


The construction of the functor $\tau_g : \mathcal{A}(-\mathcal{Z}_{g,1}, 2g-1) \to \mathcal{K}o^{2g-2}(\Sigma_{g,1}, 2)$ will occur in two stages.

First note that the parametrization of $\Sigma_{g,1}$ by the arc diagram allows us to associate to each i, for $1 \leq i \leq g$, a pair of dividing sets I_i and J_i contained in a torus $\Sigma_{1,1} \subset \Sigma_{g,1}$ with one boundary component. In fact, Theorem 5.11 states that these dividing sets generate the category. In each torus, we will describe bypass moves corresponding to the arrows $\alpha_i, \beta_i : I_i \to J_i$ and $\gamma_i : J_i \to I_i$, and check that these bypass moves satisfy the first collection of relations in the definition above.

After this has been done, bypass moves corresponding to the lateral arrows $\eta_{i,i+1}$ will be introduced and shown to satisfy the second collection of relations.

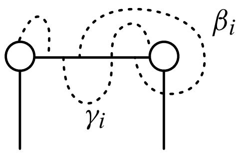
Step 1 For each torus, the dividing sets I_i and J_i, and the bypass moves corresponding to the maps $\alpha_i, \beta_i : I_i \to J_i$ and $\gamma_i : J_i \to I_i$, can be depicted by the following curves:



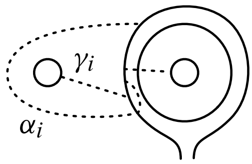
On either side of the arrows in the picture above, the two small circles are identified by folding the page to form the surface $(T^2 \setminus D^2, 2)$. The dividing set associated to I_i is featured on the left-hand side and the dividing set associated to J_i is featured on the right-hand side. The maps α_i and β_i run from left to right and the map γ_i runs from right to left. The equator of the map α_i is dotted and the equator of β_i is dashed.

Proposition 7.12 *In the formal contact category* $\mathrm{Ho}(\mathcal{K}\mathrm{o}_+(\Sigma_{1,1}, 2))$, *the relations* $\beta_i\gamma_i = 0$ *and* $\gamma_i\alpha_i = 0$ *hold.*

Proof The logic is analogous to the proof of Proposition 7.5. The map $\beta_i\gamma_i$ is a composition of two disjoint bypass moves. When performed in the opposite order the bypass γ_i is capped, implying that the composition $\beta_i\gamma_i$ factors through zero. This is illustrated below:

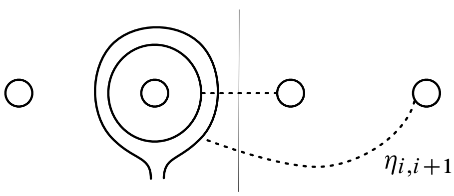


The map $\gamma_i\alpha_i$ is a composition of two disjoint bypass moves. When performed in the opposite order the bypass α_i is capped, implying that the composition $\beta_i\gamma_i$ factors through zero. This illustrated below:



□

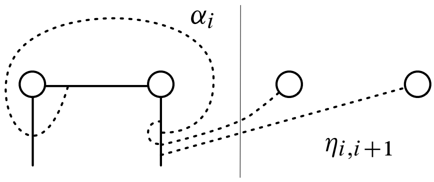
Step 2 As pictured above, the idempotents I_i correspond to the boundaries of regular neighborhoods of loops about the first 1–handle and the idempotents J_i to the boundaries of regular neighborhoods of loops about the second 1–handle in the i^{th} torus $\Sigma_{1,1} \subset \Sigma_{g,1}$. The tori $\Sigma_{1,1}$ are ordered by the arc parametrization and, when two tori are adjacent, there is a bypass move $\eta_{i,i+1} : J_i \to I_{i+1}$ from the dividing set about the second 1–handle of the first torus to the dividing set about the first 1–handle of the second torus. The map $\eta_{i,i+1}$ is pictured below:



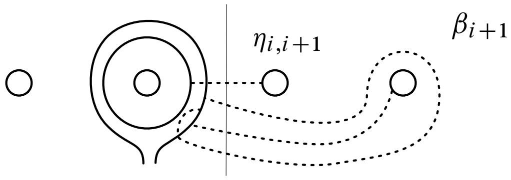
Here the first two and the second two smaller circles are connected by annuli $S^1 \times [0,1]$ to form the k^{th} and $(k+1)^{\text{st}}$ tori $\Sigma_{1,1} \subset \Sigma_{g,1}$.

Proposition 7.13 *The relations*: $\eta_{i,i+1}\alpha_i = 0\colon I_i \to I_{i+1}$ *and* $\beta_{i+1}\eta_{i,i+1} = 0\colon J_i \to J_{i+1}$ *hold in the formal contact category* $\mathrm{Ho}(\mathcal{K}\mathrm{o}_+(\Sigma_{g,1},2))$.

Proof The logic is analogous to the proof of Proposition 7.12. The map $\eta_{i,i+1}\alpha_i$ is a composition of two disjoint bypass moves. When performed in the opposite order the bypass $\eta_{i,i+1}$ is capped, implying that the composition factors through zero. This is illustrated below:



The map $\beta_{i+1}\eta_{i,i+1}$ is a composition of two disjoint bypass moves. When performed in the opposite order the bypass β_{i+1} is capped, implying that the composition factors through zero. This is illustrated below:



□

8 Comparison to geometric categories

One of the appealing qualities of the formal contact category $\mathcal{K}\mathrm{o}(\Sigma)$ is that it has a universal property with respect to other dg categories, by construction. Although there is no underlying Floer theory or contact geometry, this property allows us compare $\mathcal{K}\mathrm{o}(\Sigma)$ to other constructions which stem from observations involving either. In this section, we will discuss why the universal property of $\mathcal{K}\mathrm{o}(\Sigma)$ implies the existence of maps

$$\mathcal{C}\mathrm{o}(\Sigma) \dashleftarrow \mathcal{K}\mathrm{o}(\Sigma) \longrightarrow \mathcal{A}(-\mathcal{Z})\text{–mod}$$

in the homotopy category of dg categories which relate contact categories $\mathcal{C}\mathrm{o}(\Sigma)$ with the corresponding component of the bordered Heegaard–Floer theory. See Sections 8.1 and 8.2 for precise statements.

8.1 Relation to the contact category

Much of the material in this paper was inspired by Honda's proposed *contact category* $\mathcal{C}\mathrm{o}(\Sigma)$ [15]. Although a full account of this construction is in preparation, in this section a modest comparison is drawn between the formal and geometric contact categories.

The morphisms in the contact category $\mathcal{C}o(\Sigma)$ are tight contact structures on $\Sigma \times [0,1]$. More precisely, $\mathcal{C}o(\Sigma)$ is the additivization [38, Section 1.1.2.1] of a category with objects given by dividing sets γ on the surface Σ and morphisms $\theta\colon \gamma \to \gamma'$ given by contactomorphism classes of contact structures on $\Sigma \times [0,1]$, which induce γ and γ' on $\partial\Sigma \times [0,1]$, subject to the relation that an overtwisted contact structure is zero. The composition is induced by the pullback of contact plane fields along the rescaling diffeomorphism: $\Sigma \times [0,1] \xrightarrow{\sim} \Sigma \times [0,1] \cup_\Sigma \Sigma \times [0,1]$.

The contact category $\mathcal{C}o(\Sigma)$ plainly exists. The maps in the contact category $\mathcal{C}o(\Sigma)$ are generated by bypass moves between dividing sets [17, Lemma 3.10 (isotopy discretization)]. Since the bypass moves satisfy the elementary relations (1) and (2) in Definition 3.15, there is a functor: $\sigma\colon \text{Pre–}\mathcal{K}o(\Sigma) \to \mathcal{C}o(\Sigma)$. When (Σ, m) is a surface with boundary then the discussion in Section 4.3 suggests that these categories are very closely related.

For the purposes of comparison, we must make the nontrivial assumption below:

Assumption 8.1 *The contact category $\mathcal{C}o(\Sigma)$ has pretriangulated dg enhancement $\mathcal{C}o^{\mathrm{dg}}(\Sigma)$ in which bypass triangles are distinguished triangles.*

If this assumption is correct then there is a canonical lift

$$\tilde{\sigma}\colon \mathcal{K}o(\Sigma) \to \mathcal{C}o^{\mathrm{dg}}(\Sigma)$$

of the dg functor σ to a functor from the formal contact category to the dg category $\mathcal{C}o^{\mathrm{dg}}(\Sigma)$.

Remark 8.2 In the formal contact category $\mathcal{K}o(\Sigma)$, the bypass "an" involving the annulus in the proof of Theorem 4.14 determines a distinguished triangle

$$\gamma \xrightarrow{\mathrm{an}} \gamma' \xrightarrow{\mathrm{an}'} \gamma'' \xrightarrow{\mathrm{an}''} \gamma[1].$$

The map an' is not necessarily zero. However, in the geometric setting $\mathrm{an}' = 0$, making the convolution $\gamma \simeq C(\mathrm{an}')$ isomorphic to a direct sum (Tian, personal communication, 2015). As $\sigma(\mathrm{an}') = 0$, it is possible to view $\mathrm{Ho}(\mathcal{K}o(\Sigma))$ as a deformation.

8.2 Relation to the bordered sutured Floer categories

In this section, we construct a functor $\widetilde{\mathcal{K}o}_+(\Sigma, m) \to \mathcal{A}(-\mathcal{Z})$–mod from a cofibrant replacement of the positive part of the formal contact category to the category of left dg modules over an arc algebra of an arc diagram $\mathcal{Z}$ that parametrizes Σ. Assume that (Σ, m) has at least one boundary component, and every boundary component $\partial_i\Sigma$ contains a positive even number of points m_i. The ground ring k of $\mathcal{K}o_+(\Sigma, m)$ is fixed to be the field $\mathbb{F}_2$. We will not discuss gradings here. The cofibrant replacement is a slightly larger but quasiequivalent category; see Conjecture 3.18. In particular, there is a functor $\mathcal{K}o_+(\Sigma, m) \to \mathcal{A}(-\mathcal{Z})$–mod in Hqe.

If γ is a dividing set on Σ then Zarev associates a bordered sutured manifold [62, Section 3.2] called the *cap* W_γ to γ. The cap W_γ is the 3–manifold $\Sigma \times [0,1]$ in which the surface $\Sigma \times \{0\}$ is parametrized by the arc diagram $\mathcal{Z}$, the *sutures* m are the m boundary points, the dividing set γ appears on $\Sigma \times \{1\}$ and the two sides are connected by straight lines segments in $\partial\Sigma \times [0,1]$:

$$W_\gamma = (\Sigma \times [0,1], \gamma \times \{1\} \cup \Lambda \times [0,1], (-\Sigma \times \{0\}, -\Lambda \times \{0\})).$$

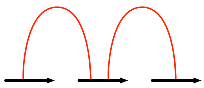
For details concerning this definition consult [63, Definition 2.5].

Associated to each bordered sutured manifold Y, there is a Heegaard diagram $H(Y)$ [62, Section 4]. Associated to each Heegaard diagram $H(Y)$, there is a left dg $\mathcal{A}(-\mathcal{Z})$–module $\widehat{\mathrm{BSD}}(Y)$ [62, Section 7.3]. Notation for the module does not include the intermediate Heegaard diagram because the homotopy type of the module is independent of this choice.

If γ is a dividing set on Σ such that the basepoint z_1 is contained in the positive region $R_+ \subset \Sigma \setminus \gamma$, then γ determines an object $\gamma \in \mathrm{Ob}(\mathcal{K}\mathrm{o}_+(\Sigma, m))$. To each such γ we associate the left dg module $\widehat{\mathrm{BSD}}(\gamma) = \widehat{\mathrm{BSD}}(W_\gamma)$ associated to the cap for some choice of Heegaard diagram:

$$\gamma \mapsto \widehat{\mathrm{BSD}}(\gamma) \quad \text{where} \quad \widehat{\mathrm{BSD}}(\gamma) = \widehat{\mathrm{BSD}}(W_\gamma). \tag{8-1}$$

The disk $(D^2, 6)$ can be parametrized by an arc diagram $\mathcal{W}_3$ pictured below:



The diagram $\mathcal{W}_3$ consists of three oriented line segments $Z = \{\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3\}$ containing the points $\{a\}$, $\{a' < b\}$ and $\{b'\}$, respectively. The matching function M is determined by $M(a) = M(a')$ and $M(b) = M(b')$.

As discussed in Proposition 3.16, the three important dividing sets γ_A, γ_B and γ_C in $(D^2, 6)$ can be connected by three bypass moves

$$\gamma_A \xrightarrow{\theta_A} \gamma_B \xrightarrow{\theta_B} \gamma_C \xrightarrow{\theta_C} \gamma_C[1]$$ or  $\xrightarrow{\theta_A}$  $\xrightarrow{\theta_B}$  $\xrightarrow{\theta_C}$  $[1]$.

(The signs of the regions are fixed by requiring that the region containing the basepoint is positive.) Associated to these three dividing sets, there are three left $\mathcal{A}(-\mathcal{W}_3)$–modules $\widehat{\mathrm{BSD}}(\gamma_A)$, $\widehat{\mathrm{BSD}}(\gamma_B)$ and $\widehat{\mathrm{BSD}}(\gamma_C)$ corresponding to the bordered sutured diagrams given by the caps W_{γ_A}, W_{γ_B} and W_{γ_A}.

In [11, Section 6.2], the authors J B Etnyre, D S Vela-Vick and Zarev made a fundamental computation: after choosing Heegaard diagrams for the caps W_{γ_A}, W_{γ_B} and W_{γ_C}, they found that there are chain maps $\phi_A : \widehat{\mathrm{BSD}}(\gamma_A) \to \widehat{\mathrm{BSD}}(\gamma_B)$, $\phi_B : \widehat{\mathrm{BSD}}(\gamma_B) \to \widehat{\mathrm{BSD}}(\gamma_C)$ and $\phi_C : \widehat{\mathrm{BSD}}(\gamma_C) \to \widehat{\mathrm{BSD}}(\gamma_A)$ such that

$$\widehat{\mathrm{BSD}}(\gamma_A) \xrightarrow{\phi_A} \widehat{\mathrm{BSD}}(\gamma_B) \xrightarrow{\phi_B} \widehat{\mathrm{BSD}}(\gamma_C) \xrightarrow{\phi_C} \widehat{\mathrm{BSD}}(\gamma_A)[1]$$

is a distinguished triangle. They show explicitly that

1. $\widehat{\mathrm{BSD}}(\gamma_A) = C(\phi_B)$,
2. ϕ_A is projection, and
3. ϕ_C is inclusion.

(Alternatively, this follows from Section 6 and the Morita invariance of the category associated to the disk by bordered sutured Floer theory.) Our functor is defined using the pairing theorem to extend the assignments $\theta_A \mapsto \phi_A$, $\theta_B \mapsto \phi_B$ and $\theta_C \mapsto \phi_C$ to all of the other bypass moves between dividing sets.

Throughout the remainder of this section we will make repeated use of the pairing theorem. Suppose that γ is a dividing set on Σ and the first basepoint z_1 is contained in a positive region. Then if $D = (D^2, 2m) \subset \Sigma$ is an embedded disk with $2m$ points on the boundary such that $\gamma^\circ = \gamma \setminus (D \cap \gamma)$ is a dividing set on $\Sigma \setminus D$, then the pairing theorem [62, Theorem 8.7] gives a homotopy equivalence

$$\widehat{\mathrm{BSD}}(\gamma) \xrightarrow{\sim} \widehat{\mathrm{BSDA}}(\gamma^\circ) \boxtimes \widehat{\mathrm{BSD}}(\gamma \cap D) \quad \text{where } \gamma = \gamma^\circ \cup_{\gamma \cap \partial D} (\gamma \cap D),$$

and where $\gamma \cap \partial D = 2m$, $\widehat{\mathrm{BSD}}(\gamma) = \widehat{\mathrm{BSD}}(W_\gamma)$ is the left dg $\mathcal{A}(-\mathcal{Z})$–module assigned to the dividing set γ, $\widehat{\mathrm{BSDA}}(\gamma^\circ) = \widehat{\mathrm{BSDA}}(W_{\gamma^\circ})$ is a left $\mathcal{A}(-\mathcal{Z})$–module and right A_∞ $\mathcal{A}(-\mathcal{W}_3)$–module, $\widehat{\mathrm{BSD}}(\gamma \cap D) = \widehat{\mathrm{BSD}}(W_{\gamma \cap D})$ is the left $\mathcal{A}(-\mathcal{W}_3)$–module determined by γ in the interior of the disk D, and the box product $\boxtimes$ is an analogue of the derived tensor product; see [36, Section 2.4].

Definition 8.3 If $\theta\colon \gamma \to \eta$ is a bypass move then the map $\theta_*\colon \widehat{\mathrm{BSD}}(\gamma) \to \widehat{\mathrm{BSD}}(\eta)$ of dg modules associated to θ is determined by the commutative diagram

$$\begin{array}{ccc} \widehat{\mathrm{BSD}}(\gamma) & \xrightarrow{\sim} & \widehat{\mathrm{BSDA}}(\gamma^\circ) \boxtimes \widehat{\mathrm{BSD}}(\gamma_A) \\ {\scriptstyle \theta_*}\big\downarrow & & \big\downarrow{\scriptstyle 1 \boxtimes \phi_A} \\ \widehat{\mathrm{BSD}}(\eta) & \xrightarrow{\sim} & \widehat{\mathrm{BSDA}}(\gamma^\circ) \boxtimes \widehat{\mathrm{BSD}}(\gamma_B) \end{array}$$

where $\gamma^\circ = \gamma \setminus D$, introduced above, denotes the dividing set minus the region containing the equator of the bypass disk associated to θ.

In order for the maps chosen above to yield a functor from the preformal contact category, we must check that relations (1) and (2) in Definition 3.15 above are satisfied. Since these relations hold up to homotopy in the category $\mathcal{A}(-\mathcal{Z})$–mod, this determines a functor from the cofibrant replacement of the preformal contact category. Lastly we will show that this functor factors through the Postnikov localization introduced by Proposition 3.16.

Relation (1) If θ is capped in the northwest or southeast then relation, (1) must hold up to homotopy by the invariance of the bordered sutured theory [62, Section 7].

In more detail, suppose that $\theta\colon \gamma \to \eta$ is a bypass move and D is a neighborhood of the equator of the underlying bypass disk. Then when there is a cap, the region D can be enlarged to a region $\widetilde{D}$ which contains the cap disk in Σ. Two applications of the pairing theorem give

$$\begin{array}{ccccc} \widehat{\mathrm{BSD}}(\gamma) & \xrightarrow{\sim} & \widehat{\mathrm{BSDA}}(\gamma^\circ) \boxtimes \widehat{\mathrm{BSD}}(\gamma_A) & \xrightarrow{\sim} & \widehat{\mathrm{BSDA}}(\tilde{\gamma}^\circ) \boxtimes \widehat{\mathrm{BSD}}(\tilde{\gamma}_A) \\ \big\downarrow{\scriptstyle \theta_*} & & \big\downarrow{\scriptstyle 1 \boxtimes \phi_A} & & \big\downarrow{\scriptstyle 1 \boxtimes \tilde{\phi}_A} \\ \widehat{\mathrm{BSD}}(\eta) & \xrightarrow{\sim} & \widehat{\mathrm{BSDA}}(\gamma^\circ) \boxtimes \widehat{\mathrm{BSD}}(\gamma_B) & \xrightarrow{\sim} & \widehat{\mathrm{BSDA}}(\tilde{\gamma}^\circ) \boxtimes \widehat{\mathrm{BSD}}(\tilde{\gamma}_B) \end{array}$$

where $\gamma^\circ = \gamma \setminus D$ and $\tilde{\gamma}^\circ = \gamma \setminus \tilde{D}$. The dividing sets $\tilde{\gamma}_A$ and $\tilde{\gamma}_B$, on the right-hand side above, are identical when the cap is either northwestern or southeastern. They are both represented by the same Heegaard diagram and the map $\tilde{\phi}_A$ is the identity. It follows that θ_* is homotopic to the identity.

Relation (2) In order to see that disjoint bypass moves $\theta \amalg \theta' \colon \gamma \to \eta$ commute, we must cut the dividing set γ along the two disjointly embedded disks corresponding to neighborhoods of the equators of our bypass moves to form $\gamma^{\circ\circ} = \gamma \setminus (D \amalg D')$. The arc algebra associated to a disjoint union splits, $\varphi \colon \mathcal{A}(-(\mathcal{W}_3 \amalg \mathcal{W}_3)) \xrightarrow{\sim} \mathcal{A}(-\mathcal{W}_3) \otimes_k \mathcal{A}(-\mathcal{W}_3)$, the module $\widehat{\mathrm{BSD}}(\gamma_A) \otimes_k \widehat{\mathrm{BSD}}(\gamma_A)$ appears in the pairing theorem,

$$\widehat{\mathrm{BSD}}(\gamma) \xrightarrow{\sim} \widehat{\mathrm{BSDA}}(\gamma^{\circ\circ}) \boxtimes [\widehat{\mathrm{BSD}}(\gamma_A) \otimes_k \widehat{\mathrm{BSD}}(\gamma_A)],$$

and the disjoint union of Heegaard diagrams splits as a tensor product compatible with the isomorphism φ above. Under this identification, the maps θ_* and θ'_* induced by θ and θ' correspond to different tensor factors and must commute by the standard algebraic fact that

$$(1_{\gamma^{\circ\circ}} \boxtimes [1_A \otimes \theta'_*])(1_{\gamma^{\circ\circ}} \boxtimes [\theta_* \otimes 1_A]) = (1_{\gamma^{\circ\circ}} \boxtimes [\theta_* \otimes 1_A])(1_{\gamma^{\circ\circ}} \boxtimes [1_A \otimes \theta'_*]),$$

where $1_{\gamma^{\circ\circ}}$ and 1_A are used to denote the identity maps $1_{\widehat{\mathrm{BSDA}}(\gamma^{\circ\circ})}$ and $1_{\widehat{\mathrm{BSD}}(\gamma_A)}$, respectively.

Triangles Finally, it is necessary to see that the objects and the maps assigned by (8-1) and Definition 8.3 factor through the Postnikov localization constructed in Proposition 3.16.

These choices form distinguished triangles because

$$\widehat{\mathrm{BSD}}(\gamma) = \widehat{\mathrm{BSD}}(\gamma \cup \gamma_A) \simeq \widehat{\mathrm{BSDA}}(\gamma^\circ) \boxtimes \widehat{\mathrm{BSD}}(\gamma_A) \simeq \widehat{\mathrm{BSDA}}(\gamma^\circ) \boxtimes C(\phi_B) \simeq C(1_{\widehat{\mathrm{BSDA}}(\gamma^\circ)} \boxtimes \phi_B) \simeq C(\theta'_*),$$

where the last equivalence corresponds to the commutative diagram in Definition 8.3 after rotating the triangle. An analogue of this argument appears in [35, Theorem 4.1].

Appendix Dg categories

This section contains some materials about dg categories and the model structures. All of the definitions are from the literature. More information about differential graded categories can be found in [26; 57] or [10, Section 1]; consult [46; 47; 56] for technical details. The language of model categories is reviewed in [37, Section A.2]; more details can be found in [19; 45].

Definition A.1 A *dg category* $\mathcal{C}$ *over* $\mathcal{A}$ is a category enriched in the monoidal category of chain complexes,

$$\mathrm{Hom}_{\mathcal{C}}(x, y) \in \mathrm{Kom}_k(\mathcal{A}) \quad \text{for all } x, y \in \mathrm{Ob}(\mathcal{C}),$$

such that composition in $\mathcal{C}$ is a map in $\mathrm{Kom}_k(\mathcal{A})$. A *functor* $f : \mathcal{C} \to \mathcal{D}$ between two such dg categories is required to consist of maps in $\mathrm{Kom}_k(\mathcal{A})$:

$$\text{(A-2)} \qquad f_{x,y} \colon \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(f(x), f(y)) \in \mathrm{Kom}_k(\mathcal{A}).$$

A dg functor $f:\mathcal{C}\to\mathcal{D}$ is *fully faithful* when, for any pair $x,y\in \mathrm{Ob}(\mathcal{C})$, the map $f_{x,y}$ in (A-2) is an isomorphism of chain complexes. If the homology $H^*(f_{x,y})$ induces an isomorphism for all pairs, then $f_{x,y}$ is called *quasifully faithful*. A functor $f:\mathcal{C}\to\mathcal{D}$ is a *quasi-isomorphism* of dg categories when $H^*(f)\colon H^*(\mathcal{C})\to H^*(\mathcal{D})$ induces an equivalence of graded k–linear categories.

Example A.2 The category of chain complexes $\mathrm{Kom}_k(\mathcal{A})$ is a subcategory $\mathrm{Kom}_k(\mathcal{A})\subset \mathrm{Kom}_k^*(\mathcal{A})$ of a dg category. The objects of $\mathrm{Kom}_k^*(\mathcal{A})$ are the chain complexes $(C,\partial_C)\in \mathrm{Kom}_k(\mathcal{A})$. The maps are now given by the chain complex $(\mathrm{Hom}^*((C,\partial_C),(D,\partial_D)),\delta)$, where

$$\mathrm{Hom}^n((C,\partial_C),(D,\partial_D)) := \prod_{m\in\mathbb{Z}} \mathrm{Hom}(C^m,D^{n+m}),$$

with differential $\delta(f):=d_Df+(-1)^{n+1}fd_C$ for f of degree n.

When $\mathcal{A}$ is Vect_k, the category of dg categories will be denoted by dgcat_k. Important for this paper is a sequence of localizations obtained by different model category structures on dgcat_k:

(A-3) $$\mathrm{dgcat}_k \xrightarrow{(1)} \mathrm{Hqe} \xrightarrow{(2)} \mathrm{Hmo}.$$

Hqe The first category $\mathrm{Hqe}:=\mathrm{dgcat}_k[W^{-1}]$ is obtained by requiring quasi-isomorphisms $f\in W$ to be isomorphisms. In this model structure, cofibrations are determined by the left lifting property with respect to fibrations, and fibrations are dg functors $f:\mathcal{C}\to\mathcal{D}$ for which $f_{x,y}$ in (A-2) are surjective and

- for $x\in\mathrm{Ob}(\mathcal{C})$ and any homotopy equivalence $\beta\colon f(x)\to y$ in $\mathcal{D}$ there is a homotopy equivalence $\alpha\colon x\to z$ in $\mathcal{C}$ such that $f(\alpha)=\beta$.

The initial object is the empty category $\varnothing$ with no objects and the final object 0 is the zero dg category consisting of one object with no endomorphisms. In Hqe nontrivial dg categories are fibrant, and cofibrant resolutions are can be obtained from cobar–bar construction.

Modules For any dg category there are associated categories of modules over that dg category.

A right dg module M over a dg category $\mathcal{C}$ is a dg functor $\mathcal{C}^{\mathrm{op}}\to \mathrm{Kom}_k^*(\mathrm{Vect}_k)$. The dg category of such functors will be denoted by $\mathrm{Mod}_{\mathcal{C}}$. The homology $H^*(M)\colon \mathcal{C}^{\mathrm{op}}\to \mathrm{Vect}_k^{\mathbb{Z}}$ of a dg module M is the functor $c\mapsto H^*(M(c))$ taking values in graded vector spaces. A *quasi-isomorphism* $g\colon M\to N$ of dg modules is a map inducing an isomorphism between their respective homologies. The derived category $D(\mathcal{C})$ of dg modules over a dg category $\mathcal{C}$ is obtained by inverting the quasi-isomorphisms Q:

$$D(\mathcal{C}):=\mathrm{Mod}_{\mathcal{C}}[Q^{-1}].$$

This is a triangulated category [25]. If $f:\mathcal{C}\to\mathcal{D}$ is a dg functor then there is a pushforward functor $f_!\colon \mathrm{Mod}_{\mathcal{C}}\to\mathrm{Mod}_{\mathcal{D}}$ which is left adjoint to the pullback $f^*\colon \mathrm{Mod}_{\mathcal{D}}\to\mathrm{Mod}_{\mathcal{C}}$. These functors induce functors between derived categories

$$f_!\colon D(\mathcal{C})\leftrightarrow D(\mathcal{D}):f^*.$$

A dg functor $f:\mathcal{C}\to\mathcal{D}$ is a *Morita equivalence* when $f^*:D(\mathcal{D})\to D(\mathcal{C})$ is an equivalence of triangulated categories.

Hmo The category Hmo is obtained by inverting Morita equivalences M:

$$\mathrm{Hmo} := \mathrm{Hqe}[M^{-1}].$$

The category Hmo is pointed: the dg category 1 consisting of a single object and a single morphism is both initial and terminal. The cofibrant objects of Hmo and Hqe remain the same. Fibrant objects become pretriangulated dg categories, as discussed in the next paragraph.

There is a full subcategory $\mathcal{C}^{\mathrm{perf}} \subset \mathrm{Mod}_{\mathcal{C}}$ consisting of modules M which are compact in $D(\mathcal{C})$. Since representable modules are compact, the Yoneda embedding factors through the subcategory of perfect modules, giving a dg functor

$$\gamma\colon \mathcal{C} \to \mathcal{C}^{\mathrm{perf}}.$$

A dg category $\mathcal{C}$ is called *perfect* when γ is a quasiequivalence. A dg category $\mathcal{C}$ in Hmo is fibrant if and only if it is perfect. So γ is fibrant replacement. An explicit model for $\mathcal{C}^{\mathrm{perf}}$ is given by the idempotent completion of the category of one-sided twisted complexes over $\mathcal{C}$ [8, Section 2.4].

Maps Toën's theorem shows that maps $\mathcal{C} \to \mathcal{D}$ in Hqe and are given by bimodules $\mathcal{C} \otimes \mathcal{D}^{\mathrm{op}} \to \mathrm{Kom}_k^*(\mathrm{Vect}_k)$ satisfying certain cofibrancy and representability conditions [56]. If $\mathcal{D}$ is fibrant then these are also the maps in Hmo. Dg functors described above define maps in each of these settings.

Constructions in Hqe vs Hmo If $\mathcal{C} \to \mathcal{D}$ and $\mathcal{C} \to \mathcal{E}$ are in Hqe, then the homotopy pushout $\mathcal{D} \sqcup_{\mathcal{C}}^h \mathcal{E}$ can be constructed by using the coproduct of dg categories on the associated pushout of cofibrant replacements. Since cofibrant objects in Hqe and Hmo agree, the quotient Hqe $\to$ Hmo commutes with homotopy pushout.

Since all of our localization constructions are homotopy pushouts, they are indifferent to the distinction between Hqe and Hmo in the manner described above.

List of symbols

After Section 2, dg categories are ungraded over a field of characteristic 2. The homotopy category of dg categories $\mathrm{Ho}(\mathrm{dgcat}_k)$ over k will be denoted by Hqe or Hmo when the equivalence relation is quasiequivalence or Morita equivalence, respectively. All surfaces denoted by Σ are connected unless otherwise mentioned. $\Sigma_{g,n}$ is the orientable surface of genus g with n boundary components.

$-^{\vee}$	Proposition 4.5
$-^{\mathrm{op}}$	opposite category
$\boldsymbol{a}$	points $\{a_1, \dots, a_{2k}\}$ in arc diagram, Definition 5.4
a_k, a'_k	points in an arc diagram, Definition 5.4
$\mathcal{A}(\mathcal{Z})$	arc algebra [36; 62]
B	bottom of D^2

B	$B \subset \Sigma$, Definition 4.13
$\widehat{\mathrm{BSD}}(\gamma)$	(8-1)
$\mathcal{C}$	dg category, after Section 2 ungraded; see Section 2.5
c_i	cocore of 1–handle
$\mathcal{C}o(\Sigma)$	geometric contact category or Tian algebraic contact category
d	differential, $d^2 = 0$
dgcat_k	category of dg categories [8; 57]
$D', \overline{D}, \widetilde{D}$	Definitions 2.9 and 2.13
D^2	unit disk
e_k	generator of $\mathcal{N}_n$, Definition 6.7
$\mathfrak{e}(\gamma)$	Definition 4.1
$F(\mathcal{Z})$	surface of arc diagram, Definition 5.5
$F(\partial\mathcal{Z})$	Proposition 5.14
γ	dividing set, Definition 3.3
$\gamma(\epsilon_i)$	dividing set associated to e_i, Section 6.2.1
$\gamma_A, \gamma_B, \gamma_C$	bypass triangle, Proposition 3.16 and Section 8.2
$\gamma^\vee$	dual dividing set, Definition 3.4 and Proposition 4.5
$\left(\bigoplus_{i=1}^n \gamma_i, p\right)$	convolution of dividing sets, Definition 2.1
$\Gamma(\Sigma)$	mapping class group, Section 5.1
h_k	1–handle in $F(\mathcal{M}_n)$
$\mathrm{Ho}(\mathcal{C})$	$[\mathcal{C}]$ or $H^0(\mathcal{C})$ [57]
Hom^I	Definition 2.6
Hom^T	Definition 2.15
$\mathrm{Hom}^{\langle K\rangle}$	Proposition 2.19
Hmo	Morita homotopy category [46]
Hqe	homotopy category [47]
$i(x, y)$	geometric intersection number
$\mathrm{int}(X)$	interior of X
$I, I', \overline{I}$	Definitions 2.5 and 2.7
k	ground field; after Section 2 $\mathrm{char}(k) = 2$
κ, κ'	Definitions 2.5 and 2.13
$\langle K\rangle$	Proposition 2.19
$K_0(\mathcal{C})$	Grothendieck group [46]
$\mathcal{K}o(\Sigma)$	Definition 3.17
$\mathcal{K}o^n(\Sigma, m)$	Theorem 4.4
$\mathcal{K}o^n_\pm(\Sigma, m)$	$\pm$–halves of $\mathcal{K}o^n(\Sigma, m)$, Section 4.5
$L_R\mathcal{C}$	Definition 2.5
$L_S\mathcal{C}$	Proposition 2.16

m	boundary points $m \subset \partial\Sigma$, Section 3.1
M	a matching $M\colon \boldsymbol{a} \to \{1,\dots,k\}$ in arc diagram, Definition 5.4
μ	Section 6.3
$\mathcal{M}_n$	zigzag diagram for $(D^2, 2n)$, Definition 6.1
$\mathrm{Mat}(\mathcal{C})$	the additive closure, Section 2.1
$\mathcal{N}_n$	nil-Temperley–Lieb algebra, Definition 6.7
$\mathbb{N}$	$\mathbb{N} = \{0\} \cup \mathbb{Z}_+$
nS^1	Definition 4.8
$N(T)$	neighborhood of disk, Definition 3.8
$\text{Pre–}\mathcal{K}\mathrm{o}(\Sigma)$	Definition 3.15
$\text{Pre–Pre-}\mathcal{K}\mathrm{o}(\Sigma)$	Conjecture 3.18
$\mathcal{Q}_n$	Tian quiver, Definition 6.8
r	basepoint automorphism, Corollary 5.3
$\rho_{k,k\pm1}$	(6-2)
$\mathcal{R}_n$	Tian disk category, Definition 6.11
$R_\pm$	positive and negative regions, Definition 3.3
S^2	the 2–sphere
$\Sigma_{g,n}$	connected surface of genus g with n boundary components
(Σ, m)	pointed oriented surface, Definition 3.1
$\bar{\Sigma}$	orientation reversal, Proposition 4.6
$\partial_i\Sigma$	i^{th} boundary component of Σ, Definition 3.1
$\theta\colon \gamma \to \gamma'$	bypass move, Definition 3.8
$\theta_{i,j}$	Definitions 2.9 and 2.13
T	top of D^2
(T, γ, γ')	bypass attachment, Definition 3.8
W_γ	cap associated to γ [63, Definition 2.5], Section 8.2
Ξ	(3-1) and Definition 3.17
$\mathcal{Y}_n$	$\mathcal{R}_n^{\text{pretr}}$, Definition 6.12
z	basepoints $z = \{z_1,\dots,z_n\}$ for $z_i \in \partial_i\Sigma$, Definition 3.1
$\mathfrak{z}_C$	$z_C \in \mathfrak{Z}(\mathcal{Z})$, Definition 5.10
$\mathbb{Z}_+$	$\{1,2,3,\dots\} \subset \mathbb{Z}$
$\mathbb{Z}/2$	$\mathbb{Z}/2\mathbb{Z}$
Z	ordered set of lines, Definition 5.4
$\mathcal{Z}$	arc diagram, Definition 5.4
$\mathcal{Z}_i$	arc in arc diagram, Definition 5.4
$\mathcal{Z}_{0,n}$	arc diagram for $\Sigma_{0,n}$, Definition 7.1
$\mathcal{Z}_{g,1}$	arc diagram for $\Sigma_{g,1}$, Definition 7.8
$\mathfrak{Z}(\mathcal{Z})$	set of elementary dividing sets, Definition 5.10

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