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Topological Hochschild homology of truncated Brown–Peterson spectra, I

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We compute the topological Hochschild homology of sufficiently structured forms of truncated Brown–Peterson spectra with coefficients. In particular, we compute $\mathrm{THH}_*(B\langle n\rangle; H\mathbb{Z}_{(p)})$ for all n, where $B\langle n\rangle$ is an E_3 form of $\mathrm{BP}\langle n\rangle$ for certain primes p, and $\mathrm{THH}_*(B\langle 2\rangle; M)$ for $M \in \{k(1), k(2)\}$. For example, this gives a computation of $\mathrm{THH}(\mathrm{taf}^D; M)$ for $M \in \{H\mathbb{Z}_{(3)}, k(1), k(2)\}$ where taf^D is the E_∞ form of $\mathrm{BP}\langle 2\rangle$ constructed by Hill and Lawson.

16E40, 19D55, 55N22, 55P43, 55Q51; 55P42, 55Q10, 55T99

1 Introduction

Topological Hochschild homology and cohomology are rich invariants of rings, or more generally ring spectra, with applications to such fields as string topology [Cohen and Jones 2002], deformation theory of A_∞ algebras [Angeltveit 2008], and integral p–adic Hodge theory [Bhatt et al. 2019]. Topological Hochschild homology is also a first order approximation to algebraic K–theory in a sense made precise using Goodwillie calculus by [Dundas and McCarthy 1994].

Algebraic K–theory of ring spectra that arise in chromatic stable homotopy theory are of particular interest because of the program of Ausoni and Rognes [2002], which suggests that algebraic K–theory shifts chromatic complexity up by one, a higher chromatic height analogue of conjectures of Lichtenbaum [1973] and Quillen [1975]. A higher chromatic height analogue of one of the Lichtenbaum–Quillen conjectures was recently proven for truncated Brown–Peterson spectra $\mathrm{BP}\langle n\rangle$ by [Hahn and Wilson 2018]. However, it is still desirable to have a more explicit computational understanding of algebraic K–theory of $\mathrm{BP}\langle n\rangle$ in order to understand the étale cohomology of $\mathrm{BP}\langle n\rangle$ as suggested by Rognes [2014, Sections 5 and 6].

One of the most fundamental objects in chromatic stable homotopy theory is the Brown–Peterson spectrum BP, which is a complex oriented cohomology theory that carries the universal p–typical formal group. The coefficients of BP are the symmetric algebra over $\mathbb{Z}_{(p)}$ on generators v_i for $i \geq 1$, and we may form truncated versions of BP, denoted by $\mathrm{BP}\langle n\rangle$, by coning off the regular sequence $(v_{n+1}, v_{n+2}, \dots)$. More generally, we consider forms of $\mathrm{BP}\langle n\rangle$, in the spirit of [Morava 1989], which are constructed by coning off some sequence $(v'_{n+1}, v'_{n+2}, \dots)$ of indecomposable algebra generators in BP_* where $|v'_k| = |v_k|$ (see Definition 2.1 for a precise definition). We will be most interested in working with forms of $\mathrm{BP}\langle n\rangle$ that are E_m–ring spectra for sufficiently large m. We will refer to such spectra as E_m forms of $\mathrm{BP}\langle n\rangle$. For example, the spectrum $H\mathbb{Z}_{(p)}$ is an E_∞ form of $\mathrm{BP}\langle 0\rangle$, and ℓ is an E_∞ form of $\mathrm{BP}\langle 1\rangle$ at all primes by [Baker and Richter 2008].

In the last decade E_∞ forms of $\mathrm{BP}\langle 2\rangle$ were constructed at the prime $p=2$ by [Lawson and Naumann 2012] and $p=3$ by [Hill and Lawson 2010]. Lawson and Naumann [2012] used the moduli stack of formal groups with a $\Gamma_1(3)$–structure to construct an E_∞ form of $\mathrm{BP}\langle 2\rangle$ at the prime 2 denoted by $\mathrm{tmf}_1(3)$. Hill and Lawson [2010] used a quaternion algebra D of discriminant 14 and its associated Shimura curve $\mathscr{X}^D$ to construct an E_∞ form of $\mathrm{BP}\langle 2\rangle$ at the prime $p=3$, denoted by taf^D. Even more recently, Hahn and Wilson [2022] constructed an E_3 form of $\mathrm{BP}\langle n\rangle$ at all primes and for all n, which we denote by $\mathrm{BP}\langle n\rangle'$. This is especially interesting since no $E_{2(p^2+2)}$ form of $\mathrm{BP}\langle n\rangle$ exists for $n\geq 4$ by Lawson [2018] at the prime $p=2$ and Senger [2017] at primes $p>2$. Highly structured models for truncated Brown–Peterson spectra make computations of invariants of these truncated Brown–Peterson spectra more tractable, and therefore they will be important for our calculations.

For small values of n, the calculations of $\mathrm{THH}_*(\mathrm{BP}\langle n\rangle)$ are known and of fundamental importance. The first known computations of topological Hochschild homology are those of Bökstedt [1985] for $\mathrm{THH}_*(H\mathbb{F}_p)$ and $\mathrm{THH}_*(H\mathbb{Z}_{(p)})$. To illustrate how fundamental these computations are, we point out that the computation $\mathrm{THH}_*(H\mathbb{F}_p)\cong P(\mu_0)$ where $|\mu_0|=2$ is the linchpin for a new proof of Bott periodicity [Hesselholt and Nikolaus 2020]. McClure and Staffeldt [1993] computed the Bockstein spectral sequence

$$\mathrm{THH}_*(\ell;H\mathbb{F}_p)[v_1]\Rightarrow \mathrm{THH}_*(\ell;k(1)),$$

which was extended by Angeltveit, Hill, and Lawson [Angeltveit et al. 2010] to compute the square of spectral sequences

$$\begin{array}{ccc}\mathrm{THH}_*(\mathrm{BP}\langle 1\rangle;H\mathbb{F}_p)[v_0,v_1] & \Longrightarrow & \mathrm{THH}_*(\mathrm{BP}\langle 1\rangle;H\mathbb{Z}_{(p)})_p[v_1]\\ \Downarrow & & \Downarrow\\ \mathrm{THH}_*(\mathrm{BP}\langle 1\rangle;k(1))[v_0] & \Longrightarrow & \mathrm{THH}_*(\mathrm{BP}\langle 1\rangle;\mathrm{BP}\langle 1\rangle)_p\end{array}$$

This gives a complete computation of $\mathrm{THH}_*(\mathrm{BP}\langle 1\rangle)$.

Let $B\langle n\rangle$ denote an E_3 form of $\mathrm{BP}\langle n\rangle$ (see Definition 2.1).[1] In Proposition 2.7 we compute

$$\mathrm{THH}_*(B\langle n\rangle;H\mathbb{F}_p)\cong E(\lambda_1,\ldots,\lambda_{n+1})\otimes P(\mu_{n+1}),$$

where $|\lambda_i|=2p^i-1$ and $|\mu_{n+1}|=2p^{n+1}$, as a consequence of work of [Angeltveit and Rognes 2005]. Hahn and Wilson [2018] calculated the groups $\mathrm{THH}_*(B\langle n\rangle/\mathrm{MU})$, but working over MU significantly simplifies the calculation. Ausoni and Richter [2020] computed $\mathrm{THH}_*(E(2))$ under the assumption that $E(2)=\mathrm{BP}\langle 2\rangle[v_2^{-1}]$ has an E_∞–ring structure and gave a conjectural answer for $\mathrm{THH}_*(E(n))$, which is consistent with our calculations. These are currently the only known results for $n\geq 2$.

[1] Note that there is a spectrum commonly denoted by $B(n)=v_n^{-1}P(n)$ in other references (eg [Ravenel 1986]) and our notation and meaning is distinct.

The main three results of this paper are computations of the Bockstein spectral sequences

$$\mathrm{THH}_*(B\langle n\rangle; H\mathbb{F}_p)[v_0] \Rightarrow \mathrm{THH}_*(B\langle n\rangle; H\mathbb{Z}_{(p)})_p, \tag{1-1}$$

$$\mathrm{THH}_*(B\langle 2\rangle; H\mathbb{F}_p)[v_1] \Rightarrow \mathrm{THH}_*(B\langle 2\rangle; k(1)), \tag{1-2}$$

$$\mathrm{THH}_*(B\langle 2\rangle; H\mathbb{F}_p)[v_2] \Rightarrow \mathrm{THH}_*(B\langle 2\rangle; k(2)), \tag{1-3}$$

where $B\langle n\rangle$ is an E_3 form of $BP\langle n\rangle$ and we assume $p \geq 3$ for our computation of the spectral sequence (1-2). The Bockstein spectral sequences (1-2) and (1-3) are of similar computational complexity to the main result of McClure and Staffeldt [1993] and we were inspired by their work.

We summarize our three main results as follows: First, we compute the topological Hochschild homology of an E_3 form of $\mathrm{BP}\langle n\rangle$ with $H\mathbb{Z}_{(p)}$ coefficients.

Theorem A (Theorem 3.8) *Let $B\langle n\rangle$ be an E_3 form of $\mathrm{BP}\langle n\rangle$ and at $p > 2$ assume the error term (3-7) vanishes. Then there is an isomorphism of graded $\mathbb{Z}_{(p)}$–modules*

$$\mathrm{THH}_*(B\langle n\rangle; H\mathbb{Z}_{(p)}) \cong E_{\mathbb{Z}_{(p)}}(\lambda_1, \dots, \lambda_n) \otimes (\mathbb{Z}_{(p)} \oplus T_0^n)$$

where T_0^n is an explicit torsion $\mathbb{Z}_{(p)}$–module defined in (3-11).

In particular, the error term (3-7) vanishes for any E_4 form of $\mathrm{BP}\langle n\rangle$ such as $B\langle 2\rangle = \mathrm{taf}^D$. It is possible that the error term (3-7) also vanishes for $B\langle n\rangle = \mathrm{BP}\langle n\rangle'$ where $\mathrm{BP}\langle n\rangle'$ is the E_3 form of $\mathrm{BP}\langle n\rangle$ constructed by Hahn and Wilson [2022] at odd primes, but it is not known to the authors. Theorem 3.8 also holds for $B\langle 2\rangle = \mathrm{tmf}_1(3)$ and $B\langle n\rangle = \mathrm{BP}\langle n\rangle'$, where $\mathrm{BP}\langle n\rangle'$ is the E_3 form of $\mathrm{BP}\langle n\rangle$ at the prime 2 constructed by Hahn and Wilson [2022].

Second, we compute the topological Hochschild homology of an E_3 form $B\langle 2\rangle$ of $\mathrm{BP}\langle 2\rangle$ at $p \geq 3$ with $k(1)$ coefficients.

Theorem B (Theorem 4.6) *Let $B\langle 2\rangle$ denote an E_3 form of $\mathrm{BP}\langle 2\rangle$ at an odd prime p. There is an isomorphism of $P(v_1)$–modules*

$$\mathrm{THH}_*(B\langle 2\rangle; k(1)) \cong E(\lambda_1) \otimes (P(v_1) \oplus T_1^2)$$

where T_1^2 is an explicit v_1–torsion $P(v_1)$–module defined in (4-3).

In particular, this result holds for $B\langle 2\rangle = \mathrm{taf}^D$ and $\mathrm{BP}\langle 2\rangle'$ at odd primes.

Finally, we compute topological Hochschild homology of any E_3 form of $\mathrm{BP}\langle 2\rangle$ with $k(2)$ coefficients.

Theorem C (Theorem 5.5) *Let $B\langle 2\rangle$ be an E_3 form of $\mathrm{BP}\langle 2\rangle$. There is an isomorphism of $P(v_2)$–modules*

$$\mathrm{THH}_*(B\langle 2\rangle; k(2)) \cong P(v_2) \oplus T_2^2,$$

where T_2^2 is an explicit v_2–torsion $P(v_2)$–module defined in (5-2).

In particular, this result holds for $B\langle 2\rangle = \mathrm{taf}^D$, $B\langle 2\rangle = \mathrm{tmf}_1(3)$, and $BP\langle 2\rangle'$ at any prime. We end with a conjectural answer (see Conjecture 5.6) for $\mathrm{THH}_*(B\langle n\rangle; k(m))$ for all integers $1 \le m \le n$ and any E_3 form of $B\langle n\rangle$ at a prime p.

We now outline our approach to computing $\mathrm{THH}_*(\mathrm{taf}^D)$ in the sequels to this paper. There is a cube of Bockstein spectral sequences

$$\begin{array}{ccccccc}
H\mathbb{F}_3 & & \Longrightarrow & & H\mathbb{Z}_{(3)} & & \\
\Downarrow & \searrow\!\!\searrow & & & \Downarrow & \searrow\!\!\searrow & \\
 & & k(1) & & \Longrightarrow & & B\langle 2\rangle/v_2 \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
k(2) & & \Longrightarrow & & B\langle 2\rangle/v_1 & & \\
 & \searrow\!\!\searrow & \Downarrow & & & \searrow\!\!\searrow & \Downarrow \\
 & & B\langle 2\rangle/3 & & \Longrightarrow & & B\langle 2\rangle
\end{array} \tag{1-4}$$

where we use the abbreviation $M/x \Rightarrow M$ for the Bockstein spectral sequence with signature

$$\mathrm{THH}_*(\mathrm{taf}^D; M/x)[x] \Rightarrow \mathrm{THH}_*(\mathrm{taf}^D; M),$$

where $M \in \{H\mathbb{Z}_{(3)}, k(1), k(2), \mathrm{taf}^D/3, \mathrm{taf}^D/v_1, \mathrm{taf}^D/v_2, \mathrm{taf}^D\}$. Here we write taf^D/x for the cofiber of a representative of an element $x \in \pi_{2k}\mathrm{taf}^D$ regarded as a taf^D–module map $\Sigma^{2k}\mathrm{taf}^D \to \mathrm{taf}^D$. In the sequels to this paper, we plan to compute $\mathrm{THH}_*(\mathrm{taf}^D; M)$ for $M = \mathrm{taf}^D/3$ and $M = \mathrm{taf}^D/v_1$ by comparing the edges of the cube of Bockstein spectral sequences to the Hochschild–May spectral sequence [Angelini-Knoll and Salch 2018] and the Brun spectral sequence [Höning 2020], which compute the diagonals of the faces of the cube directly. Finally, we plan to compute $\mathrm{THH}_*(\mathrm{taf}^D)$ by again comparing the Hochschild–May spectral sequence to the relevant Bockstein spectral sequences in addition to cosimplicial descent techniques.

Conventions We write F_*X for $\pi_*(F \wedge X)$ for any spectra F and X. We also use the shorthand $H_*(X)$ for $(H\mathbb{F}_p)_*X$ for any spectrum X. We write $\doteq$ to mean that an equality holds up to multiplication by a unit. The dual Steenrod algebra $H_*(H\mathbb{F}_p)$ will be denoted by $\mathscr{A}_*$ with coproduct $\Delta\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$. Given a left $\mathscr{A}_*$–comodule M, its left coaction will be denoted by $\nu\colon \mathscr{A}_* \to \mathscr{A}_* \otimes M$, where the comodule M is understood from the context. The antipode $\chi\colon \mathscr{A}_* \to \mathscr{A}_*$ will not play a role except that we will write $\bar{\xi}_i := \chi(\xi_i)$ and $\bar{\tau}_i := \chi(\tau_i)$.

When not otherwise specified, tensor products will be taken over $\mathbb{F}_p$ and $\mathrm{HH}_*(A)$ denotes the Hochschild homology of a graded $\mathbb{F}_p$–algebra relative to $\mathbb{F}_p$. We will let $P_R(x)$, $E_R(x)$ and $\Gamma_R(x)$ denote a polynomial algebra, exterior algebra, and divided power algebra over R on a generator x. When $R = \mathbb{F}_p$, we omit it from the notation. Let $P_i(x)$ denote the truncated polynomial algebra $P(x)/(x^i)$.

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2 Topological Hochschild homology mod $(p, \dots, v_n)$

We begin by giving a precise definition of an E_m form $B\langle n\rangle$ of $\mathrm{BP}\langle n\rangle$. We then compute topological Hochschild homology of an E_3 form $B\langle n\rangle$ of $\mathrm{BP}\langle n\rangle$ at an arbitrary prime p with coefficients in $H\mathbb{F}_p$. First, recall that there is an isomorphism $\mathrm{BP}_* \cong \mathbb{Z}_{(p)}[v_i \mid i \geq 1]$ and an isomorphism

$$\mathrm{BP}_*\mathrm{BP} \cong \mathbb{Z}_{(p)}[v_i \mid i \geq 1][t_i \mid i \geq 1]$$

where the degrees of the generators are $|v_i| = |t_i| = 2p^i - 2$ for $i \geq 1$. The generators t_i are determined by the canonical strict isomorphism f from the universal p–typical formal group law to itself given by the power series

$$f^{-1}(x) = \sum_{i\geq 0}^{F} t_i x^{p^i}$$

where F is the universal p–typical formal group law [Ravenel 1986, Lemma A2.1.26]. We let v_i be the Araki generators. Note that the Araki generators agree with Hazewinkel generators mod p [Ravenel 1986, Theorem A2.2.3].

2.1 Forms of $\mathrm{BP}\langle n\rangle$

We fix a precise notion of a form of the truncated Brown–Peterson spectrum in the spirit of [Morava 1989] below.

Definition 2.1 (cf [Lawson and Naumann 2014, Definition 4.1]) Fix integers $m \geq 1$ and $n \geq 0$. By an E_m form of $\mathrm{BP}\langle n\rangle$ (at the prime p), we mean a p–local E_m–ring spectrum R equipped with a complex orientation $\mathrm{MU}_{(p)} \to R$ such that the composite

$$\mathbb{Z}_{(p)}[v_1, \dots, v_n] \to \mathrm{BP}_* \to \pi_*\mathrm{MU}_{(p)} \to \pi_* R$$

is an isomorphism.

Remark 2.2 Note that we do not assume that an E_m form of $\mathrm{BP}\langle n\rangle$ at the prime p is an E_m MU–algebra, and therefore Definition 2.1 differs slightly from the definition of an E_m MU–algebra form of $\mathrm{BP}\langle n\rangle$ appearing in work of Hahn and Wilson [2022, Definition 2.0.1]. An E_m MU–algebra form of $\mathrm{BP}\langle n\rangle$ in the sense of [Hahn and Wilson 2022, Definition 2.0.1] is an E_m form of $\mathrm{BP}\langle n\rangle$ at the prime p in the sense of Definition 2.1. The distinction arises because, for example, taf^D is an E_∞ form of $BP\langle 2\rangle$; however it is not known, at least to the authors, whether the complex orientation $\mathrm{MU} \to \mathrm{taf}^D$ can be elevated to an E_∞–ring spectrum map. Nonetheless, we know that the map $\mathrm{MU} \to \mathrm{taf}^D$ is an E_2–ring spectrum map by [Chadwick and Mandell 2015, Theorem 1.2], which is sufficient for our purposes.

Notation 2.3 Throughout, we let $B\langle n\rangle$ denote an E_3 form of $\mathrm{BP}\langle n\rangle$ at the prime p in the sense of Definition 2.1 for $n \geq 0$.

We collect some consequences of Definition 2.1.

Proposition 2.4 *Since $B\langle n\rangle$ is an E_3 form of $\mathrm{BP}\langle n\rangle$ at the prime p for $m \geq 3$, the following hold:*

(1) *There are indecomposable algebra generators v_i' with $v_i' = v_i$ for $1 \leq i \leq n$ such that*

$$\mathrm{BP}_*/(v_k' \mid k \geq n+1) \cong \pi_* B\langle n\rangle.$$

(2) *The orientation $\mathrm{MU}_{(p)} \to B\langle n\rangle$ lifts to an E_2–ring spectrum map and consequently there is an E_2–ring spectrum map $\mathrm{BP} \to B\langle n\rangle$ realizing the canonical quotient map $\mathrm{BP}_* \to \mathrm{BP}_*/(v_k' \mid k \geq n+1)$ on homotopy groups.*

(3) *There is an E_3–ring spectrum map $B\langle n\rangle \to H\mathbb{Z}_{(p)}$ and the map induced by the composite*

$$B\langle n\rangle \to H\mathbb{Z}_{(p)} \to H\mathbb{F}_p \tag{2-1}$$

in mod p homology provides an isomorphism

$$H_*(B\langle n\rangle) \cong \mathcal{A}//E(n)_* \subset \mathcal{A}_*$$

of $\mathcal{A}_*$–comodule $\mathbb{F}_p$–algebras onto its image in the dual Steenrod algebra.

(4) *If $B\langle n\rangle$ is E_3 and $x_1, \dots, x_n$ is a regular sequence of elements in $B\langle n\rangle_*$, then one can construct the spectrum $B\langle n\rangle/(x_1, x_2, \dots, x_n)$ as an E_1 $B\langle n\rangle$–algebra.*

(5) *The p–completion of $B\langle n\rangle$ is weakly equivalent to the p–completion of any other E_m form of $\mathrm{BP}\langle n\rangle$ at the prime p in the category of spectra.*

Proof For part (1) set $v_i' := v_i - f_i(v_1, \dots, v_n)$ for $i \geq n+1$, where $f_i(v_1, \dots, v_n)$ is the image of v_i under $\mathrm{BP}_* \to \mathrm{BP}\langle n\rangle_* \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]$. Part (2) follows by applying [Chadwick and Mandell 2015, Theorem 1.2]. Part (3) is [Lawson and Naumann 2014, Theorem 4.4]. Part (4) follows from [Angeltveit 2008, Section 3] (cf [Hahn and Wilson 2018, Theorem A]). Part (5) is [Angeltveit and Lind 2017, Theorem A]. □

Example 2.5 The Eilenberg–Mac Lane spectrum $H\mathbb{Z}_{(p)}$ is an E_∞ form of $\mathrm{BP}\langle 0\rangle$. The Adams summand ℓ is an E_∞ form of $\mathrm{BP}\langle 1\rangle$ by [Baker and Richter 2008, Corollary 1.4].

Notation 2.6 Let $\mathrm{tmf}_1(3)$ denote the E_∞ form of $\mathrm{BP}\langle 2\rangle$ constructed by Lawson and Naumann [2012] at $p = 2$. Let taf^D denote the E_∞ form of $\mathrm{BP}\langle 2\rangle$ constructed by Hill and Lawson [2010] at $p = 3$. Let $\mathrm{BP}\langle n\rangle'$ denote the E_3 form of $\mathrm{BP}\langle n\rangle$ constructed by Hahn and Wilson [2022] at all primes.

2.2 Topological Hochschild homology mod $(p, \dots, v_n)$

The mod p homology of $\mathrm{THH}(\mathrm{BP}\langle n\rangle)$ has been calculated by Angeltveit and Rognes [2005, Theorem 5.12] assuming that $\mathrm{BP}\langle n\rangle$ is an E_3–ring spectrum. Their argument also applies to topological Hochschild homology of any E_3 form $B\langle n\rangle$ of $\mathrm{BP}\langle n\rangle$ at a prime p, as we now explain. By Proposition 2.4, the linearization map (2-1) induces an isomorphism

$$H_*(B\langle n\rangle) \cong \begin{cases} P(\bar\xi_1, \bar\xi_2, \dots) \otimes E(\bar\tau_{n+1}, \bar\tau_{n+2}, \dots) & \text{if } p \geq 3, \\ P(\bar\xi_1^2, \dots, \bar\xi_{n+1}^2, \bar\xi_{n+2}, \dots) & \text{if } p = 2, \end{cases}$$

with its image in $\mathscr{A}_*$ as an $\mathscr{A}_*$–subcomodule algebra of $\mathscr{A}_*$. By [Brun et al. 2007, Theorem 3.4], the spectrum $\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p)$ is an E_2–ring spectrum and the unit map

$$H\mathbb{F}_p \to \mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p)$$

is a map of E_2–ring spectra. Using [Brun et al. 2007, Section 3.3], the proof of [Angeltveit and Rognes 2005, Proposition 4.3] carries over mutatis mutandis and implies that the Bökstedt spectral sequence with signature

$$E^2_{*,*} = \mathrm{HH}_{*,*}(H_*(B\langle n\rangle); \mathscr{A}_*) \Rightarrow H_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p))$$

is a spectral sequence of $\mathscr{A}_*$–comodule algebras. As in [Angeltveit and Rognes 2005, Section 5.2], the spectral sequence collapses at the E^2–page if $p = 2$. If $p \geq 3$, one can use the map to the Bökstedt spectral sequence with signature

$$E^2_{*,*} = \mathrm{HH}_{*,*}(\mathscr{A}_*) \Rightarrow H_*(\mathrm{THH}(H\mathbb{F}_p))$$

to determine the differentials (cf [Angeltveit and Rognes 2005, Section 5.4]). Since $B\langle n\rangle$ is an E_3–ring spectrum, Dyer–Lashof operations are defined on $H_*(B\langle n\rangle)$ and $H_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p))$ in a range that is sufficient to resolve the multiplicative extensions (see [Angeltveit and Rognes 2005, Proof of Theorem 5.12]). We get an isomorphism of $\mathscr{A}_*$–comodule $\mathscr{A}_*$–algebras

$$H_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p)) \cong \begin{cases} \mathscr{A}_* \otimes E(\sigma\bar\xi_1, \dots, \sigma\bar\xi_{n+1}) \otimes P(\sigma\bar\tau_{n+1}) & \text{if } p \geq 3, \\ \mathscr{A}_* \otimes E(\sigma\bar\xi_1^2, \dots, \sigma\bar\xi_{n+1}^2) \otimes P(\sigma\bar\xi_{n+2}) & \text{if } p = 2. \end{cases} \tag{2-2}$$

Since $\sigma\colon H_*(B\langle n\rangle) \to H_{*+1}(\mathrm{THH}(B\langle n\rangle)) \to H_{*+1}(\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p))$ is a comodule map and a derivation, the $\mathscr{A}_*$–coaction of

$$H_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p))$$

can be deduced from that of $H_*(B\langle n\rangle) \subseteq \mathscr{A}_*$ (cf [Angeltveit and Rognes 2005, Proof of Theorem 5.12]): for $p \geq 3$ the classes $\sigma\bar\xi_i$ for $1 \leq i \leq n+1$ are $\mathscr{A}_*$–comodule primitives and we have

$$\nu(\sigma\bar\tau_{n+1}) = 1 \otimes \sigma\bar\tau_{n+1} + \bar\tau_0 \otimes \sigma\bar\xi_{n+1}. \tag{2-3}$$

For $p = 2$ the classes $\sigma\bar\xi_i^2$ for $1 \leq i \leq n+1$ are $\mathscr{A}_*$–comodule primitives and we have

$$\nu(\sigma\bar\xi_{n+2}) = 1 \otimes \sigma\bar\xi_{n+2} + \bar\xi_1 \otimes \sigma\bar\xi_{n+1}^2. \tag{2-4}$$

Proposition 2.7 *Let $B\langle n\rangle$ be an E_3 form of $\mathrm{BP}\langle n\rangle$. There is an isomorphism of graded $\mathbb{F}_p$–algebras*

$$\mathrm{THH}_*(B\langle n\rangle; H\mathbb{F}_p) \cong E(\lambda_1, \dots, \lambda_{n+1}) \otimes P(\mu_{n+1}), \tag{2-5}$$

where the degrees of the algebra generators are $|\lambda_i| = 2p^i - 1$ for $1 \le i \le n+1$ and $|\mu_{n+1}| = 2p^{n+1}$.

Proof Since $\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p)$ is an $H\mathbb{F}_p$–module, the Hurewicz homomorphism induces an isomorphism between $\mathrm{THH}_*(B\langle n\rangle; H\mathbb{F}_p)$ and the subalgebra of comodule primitives in $H_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p))$. For $1 \le i \le n+1$ we write $\lambda_i := \sigma\bar{\xi}_i$ if $p \ge 3$ and $\lambda_i := \sigma\bar{\xi}_i^2$ if $p = 2$. We also define

$$\mu_{n+1} := \begin{cases} \sigma\bar{\tau}_{n+1} - \bar{\tau}_0\sigma\bar{\xi}_{n+1} & \text{if } p \ge 3, \\ \sigma\bar{\xi}_{n+2} - \bar{\xi}_1\sigma\bar{\xi}_{n+1}^2 & \text{if } p = 2. \end{cases}$$

Then it is clear that the subalgebra of $H_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p))$ consisting of comodule primitives is as claimed. □

3 Topological Hochschild homology mod $(v_1, \dots, v_n)$

We begin by setting up the Bockstein spectral sequence. In order to ensure that this spectral sequence is multiplicative, we compare it with the Adams spectral sequence.

3.1 Bockstein and Adams spectral sequences

Let $B\langle n\rangle$ be an E_3 form of $\mathrm{BP}\langle n\rangle$ at the prime p which is equipped with a choice of generators v_i in degrees $|v_i| = 2p^i - 2$ for $0 < i \le n$ such that $B\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$. Let $v_0 = p$ by convention. Let

$$k(i) = B\langle n\rangle/(p, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$$

be the E_1 $B\langle n\rangle$–algebra constructed in Proposition 2.4 (4) where $k(0) = H\mathbb{Z}_{(p)}$. We regard $k(i)$ as a right $B\langle n\rangle \wedge B\langle n\rangle^{\mathrm{op}}$–module by restriction along the map

$$B\langle n\rangle \wedge B\langle n\rangle^{\mathrm{op}} \to B\langle n\rangle \to k(i).$$

For $0 \le i \le n$ we have cofiber sequences of right $B\langle n\rangle \wedge B\langle n\rangle^{\mathrm{op}}$–modules

$$\Sigma^{|v_i|}k(i) \xrightarrow{\cdot v_i} k(i) \to H\mathbb{F}_p.$$

Applying the functor $- \wedge_{B\langle n\rangle \wedge B\langle n\rangle^{\mathrm{op}}} B\langle n\rangle$ produces the cofiber sequence

$$\Sigma^{|v_i|}\,\mathrm{THH}(B\langle n\rangle; k(i)) \to \mathrm{THH}(B\langle n\rangle; k(i)) \to \mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p).$$

Iterating this, we produce the tower

$$\begin{array}{ccccccc} \cdots \longrightarrow & \Sigma^{2|v_i|}T(k(i)) & \xrightarrow{\cdot v_i} & \Sigma^{|v_i|}T(k(i)) & \xrightarrow{\cdot v_i} & T(k(i)) \\ & \downarrow & & \downarrow & & \downarrow \\ & \Sigma^{2|v_i|}T(H\mathbb{F}_p) & & \Sigma^{|v_i|}T(H\mathbb{F}_p) & & T(H\mathbb{F}_p) \end{array} \tag{3-1}$$

where $T(k(i)) := \mathrm{THH}(B\langle n\rangle; k(i))$ and $T(H\mathbb{F}_p) := \mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p)$.

This yields an exact couple after applying homotopy groups and it produces the v_i–Bockstein spectral sequence with E_1–page

$$E_1^{*,*} = \mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p)[v_i]. \tag{3-2}$$

Note that the fact that $B\langle n\rangle$ and $k(i)$ are connective and have homotopy groups that are degreewise finitely generated $\mathbb{Z}_{(p)}$–modules implies that the homotopy groups of $\mathrm{THH}(B\langle n\rangle; k(i))$ are degreewise finitely generated $\mathbb{Z}_{(p)}$–modules, too. It follows that $\mathrm{THH}_*(B\langle n\rangle; k(i))$ has the form

$$\mathrm{THH}_*(B\langle n\rangle; k(i)) \cong \bigoplus_l P(v_i)\{\alpha_l\} \oplus \bigoplus_k P_{r_k}(v_i)\{\beta_k\}$$

for some classes α_l and β_k. Here, for $i = 0$, $P(v_i)$ is defined to be $\mathbb{Z}_{(p)}$ and $P_r(v_i)$ is $\mathbb{Z}/p^i$. We get that

$$\mathrm{THH}_*(B\langle n\rangle; H\mathbb{F}_p) \cong \bigoplus_l \mathbb{F}_p\{a_l\} \oplus \bigoplus_k \mathbb{F}_p\{b_k\} \oplus \bigoplus_k \mathbb{F}_p\{c_k\},$$

where a_l and b_k are the images of α_l and β_k under the map $\mathrm{THH}_*(B\langle n\rangle; k(i)) \to \mathrm{THH}_*(B\langle n\rangle; H\mathbb{F}_p)$, and c_k is a preimage of $v_i^{r_k-1}\beta_k$ under the map $\mathrm{THH}_*(B\langle n\rangle; H\mathbb{F}_p) \to \Sigma^{|v_i|+1}\,\mathrm{THH}_*(B\langle 2\rangle; k(i))$. The differentials in the spectral sequence are given as follows: The classes a_l and b_k are infinite cycles. The class c_k survives to the E_{r_k}–page and we have

$$d_{r_k}(c_k) = v_i^{r_k} b_k.$$

The spectral sequence converges strongly to $\mathrm{THH}_*(B\langle n\rangle; k(i))$ for $0 < i \le n$ and $\pi_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{Z}_{(p)})_p)$ for $i = 0$. The cofibers in the tower (3-1) are $H\mathbb{F}_p$–module spectra.

We now relate the Bockstein spectral sequence to the Adams spectral sequence. In order to do this, we show that the tower (3-1) is also an Adams resolution. For the definition of an Adams resolution, the reader is referred to [Ravenel 1986, Definition 2.1.3]. In order to show that this tower is an Adams resolution, it must be shown that the vertical morphisms

$$\Sigma^{m|v_i|}\,\mathrm{THH}(B\langle n\rangle; k(i)) \to \Sigma^{m|v_i|}\,\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p) \tag{3-3}$$

induce monomorphisms in mod p homology. We have equivalences of spectra

$$\mathrm{THH}(B\langle n\rangle; M) \simeq M \wedge_{B\langle n\rangle} \mathrm{THH}(B\langle n\rangle)$$

for $M \in \{H\mathbb{F}_p, k(i) \mid 0 \le i \le n\}$ by [Hahn and Wilson 2022, Remark 6.1.4] and consequently there is an Eilenberg–Moore spectral sequence

$$\mathrm{Tor}_{*,*}^{H_*B\langle n\rangle}(H_*(M), H_*(\mathrm{THH}(B\langle n\rangle))) \Rightarrow H_*(\mathrm{THH}(B\langle n\rangle; M))$$

for each $M \in \{H\mathbb{F}_p, k(i) \mid 0 \le i \le n\}$. Since $H_*(\mathrm{THH}(B\langle n\rangle))$ is a free $H_*(B\langle n\rangle)$–module by [Angeltveit and Rognes 2005, Theorem 5.12], the Eilenberg–Moore spectral sequence collapses at the E^2–page without room for differentials. Furthermore, the morphism (3-3) induces a morphism of Eilenberg–Moore spectral sequences. Thus, we observe that the morphism (3-3) induces the map

$$H_*(k(i)) \otimes_{H_*(B\langle n\rangle)} H_*(\mathrm{THH}(B\langle n\rangle)) \to \mathscr{A}_* \otimes_{H_*(B\langle n\rangle)} H_*(\mathrm{THH}(B\langle n\rangle)) \tag{3-4}$$

in mod p homology where the map on the first factor is induced by the linearization map $k(i) \to H\mathbb{F}_p$. The map (3-4) is an injection. Since $H_*(\mathrm{THH}(B\langle n\rangle))$ is a free $H_*(B\langle n\rangle)$–module, the map (3-3) induces an injection on mod p homology. Thus, we have shown the following proposition.

Proposition 3.1 *The tower (3-1) is an Adams resolution.*

Thus, the Adams spectral sequence for $\mathrm{THH}(B\langle n\rangle; k(i))$ agrees with the Bockstein spectral sequence for $0 \le i \le n$. By [Ravenel 1986, Theorem 2.3.3], we know that the Adams spectral sequence for $\mathrm{THH}(B\langle n\rangle; k(i))$, and consequently the Bockstein spectral sequence, is multiplicative for $0 \le i \le n$ from the E_2–page onwards. To see that the Adams spectral sequence is in fact multiplicative from the E_1–page onwards, we prove explicitly in the case $i = 0$ that the d_1 differential satisfies the Leibniz rule in Lemma 3.4. In the case $i > 0$, we can apply a change of rings isomorphism and compute explicitly that the E_2–page is

$$\mathrm{Ext}^{*,*}_{E(Q_i)_*}(\mathbb{F}_p, E(\lambda_1, \dots, \lambda_{n+1}) \otimes P(\mu_{n+1})) = P(v_i) \otimes E(\lambda_1, \dots, \lambda_{n+1}) \otimes P(\mu_{n+1})$$

using the coactions discussed previously on λ_i and μ_{n+1}. Consequently, when $i > 0$ there are no nontrivial d_1 differentials. Altogether, this proves the following corollary.

Corollary 3.2 *The v_i–Bockstein spectral sequence computing $\mathrm{THH}_*(B\langle n\rangle; k(i))$ in the case $i \ge 1$ and $\pi_* \mathrm{THH}(B\langle n\rangle; H\mathbb{Z}_{(p)})_p$ in the case $i = 0$ is multiplicative from the E_1–page onwards.*

3.2 Rational topological Hochschild homology

We use the $H\mathbb{Q}$–based Bökstedt spectral sequence to compute

$$\pi_*(L_0 \mathrm{THH}(B\langle n\rangle)) = H\mathbb{Q}_* \mathrm{THH}(B\langle n\rangle) = \pi_* \mathrm{THH}(B\langle n\rangle) \otimes \mathbb{Q}$$

for $0 \le n \le \infty$ where $B\langle \infty\rangle = \mathrm{BP}$ and $L_0 = L_{H\mathbb{Q}}$ is the Bousfield localization at $H\mathbb{Q}$. Since BP and $B\langle n\rangle$ are E_3–ring spectra, the $H\mathbb{Q}$–based Bökstedt spectral sequences are strongly convergent multiplicative spectral sequence with signature

$$E^2_{**} = \mathrm{HH}^{\mathbb{Q}}_{*,*}(H\mathbb{Q}_* B\langle n\rangle) \Rightarrow H\mathbb{Q}_* \mathrm{THH}(B\langle n\rangle)$$

for $0 \le n \le \infty$. Recall that the rational homology of $B\langle n\rangle$ is

$$H\mathbb{Q}_* B\langle n\rangle \cong P_{\mathbb{Q}}(v_1, \dots, v_n)$$

with $|v_i| = 2p^i - 2$ for $1 \le i \le n \le \infty$. Thus, the E^2–term of the Bökstedt spectral sequence is

$$E^2_{*,*} = P_{\mathbb{Q}}(v_1, \dots, v_n) \otimes_{\mathbb{Q}} E_{\mathbb{Q}}(\sigma v_1, \dots, \sigma v_n)$$

where the bidegree of σv_i is $(1, 2(p^i-1))$ for $1 \le i \le n \le \infty$. Since the E^2–page is generated as a $\mathbb{Q}$–algebra by classes in Bökstedt filtration degree 0 and 1, the first quadrant spectral sequence collapses at the E^2–page and $E^2_{*,*} = E^\infty_{*,*}$. There are no multiplicative extensions, because the E^∞–pages are free graded-commutative $\mathbb{Q}$–algebras. Therefore, we produce isomorphisms of graded $\mathbb{Q}$–algebras

$$\mathrm{THH}_*(B\langle n\rangle) \otimes \mathbb{Q} \cong P_{\mathbb{Q}}(v_1, \dots, v_n) \otimes_{\mathbb{Q}} E_{\mathbb{Q}}(\sigma v_1, \dots, \sigma v_n)$$

with $|\sigma v_i| = 2p^i - 1$ for $1 \le i \le n \le \infty$. It follows that there is an equivalence

$$L_0\,\mathrm{THH}(B\langle n\rangle) \simeq \bigvee_{x \in B_n} \Sigma^{|x|} L_0 B\langle n\rangle,$$

where B_n is a graded basis for $E_{\mathbb{Q}}(\sigma v_1, \dots, \sigma v_n)$ as a graded $\mathbb{Q}$–vector space, since L_0 is a smashing localization. We may also let $n = \infty$ and in this case $B\langle\infty\rangle = \mathrm{BP}$ and B_∞ is a graded basis for $E_{\mathbb{Q}}(\sigma v_1, \sigma v_2, \dots)$ as a graded $\mathbb{Q}$–vector space.

By Proposition 2.4, the linearization map $\mathrm{BP}\langle n\rangle \to H\mathbb{Z}_{(p)}$ is an E_3–ring spectrum map. Since the localization map $H\mathbb{Z}_{(p)} \to H\mathbb{Q}$ is an E_∞–ring spectrum map, we may infer that the Bökstedt spectral sequence

$$E^2_{**} = \mathrm{HH}^{\mathbb{Q}}_{*,*}(H\mathbb{Q}_* B\langle n\rangle; \mathbb{Q}) \Rightarrow H\mathbb{Q}_*\,\mathrm{THH}(B\langle n\rangle; H\mathbb{Q})$$

is a spectral sequence of $\mathbb{Q}$–algebras by using [Brun et al. 2007, Section 3.3] to adapt the proof of [Angeltveit and Rognes 2005, Proposition 4.3]. This spectral sequence collapses without extensions by the same argument as before. All of these computations are functorial with respect to the map of E_2–ring spectra $\mathrm{BP} \to B\langle n\rangle$ from Proposition 2.4. This proves the following result.

Proposition 3.3 *There is an isomorphism of graded $\mathbb{Q}$–algebras*

(3-5) $$\mathrm{THH}_*(B\langle n\rangle; H\mathbb{Q}) \cong E_{\mathbb{Q}}(\sigma v_1, \dots, \sigma v_n)$$

for all $0 \le n \le \infty$. The map

$$\mathrm{THH}_*(\mathrm{BP}; H\mathbb{Q}) \to \mathrm{THH}_*(B\langle n\rangle; H\mathbb{Q})$$

sends σv_i to σv_i for $0 \le i \le n$.

3.3 The v_0–Bockstein spectral sequence

In this section, we compute the v_0–Bockstein spectral sequence with signature

(3-6) $$E_1^{*,*} = \mathrm{THH}_*(B\langle n\rangle; H\mathbb{F}_p)[v_0] \Rightarrow \mathrm{THH}_*(B\langle n\rangle; H\mathbb{Z}_{(p)})_p$$

where $B\langle n\rangle$ is an E_3 form of $\mathrm{BP}\langle n\rangle$. At odd primes, we must assume that a certain error term (3-7) vanishes. This error term vanishes for any E_4 form of $\mathrm{BP}\langle n\rangle$ at odd primes, for example taf^D.

Lemma 3.4 *There is a differential*

$$d_1(\mu_{n+1}) \doteq v_0 \lambda_{n+1}$$

in the v_0–Bockstein spectral sequence (3-6) and the d_1 differential satisfies the Leibniz rule.

Proof We just give the argument for $p \geq 3$ to simplify the discussion since the argument for $p = 2$ is the same up to a change of symbols. Recall that the classes μ_{n+1} and λ_{n+1} in $\mathrm{THH}_*(B\langle n\rangle; H\mathbb{F}_p)$ correspond to the comodule primitives $\sigma\bar{\tau}_{n+1} - \bar{\tau}_0\sigma\bar{\xi}_{n+1}$ and $\sigma\bar{\xi}_{n+1}$ in $H_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p))$. We therefore have to show that $\sigma\bar{\tau}_{n+1} - \bar{\tau}_0\sigma\bar{\xi}_{n+1}$ maps to $\sigma\bar{\xi}_{n+1}$ under the map β_1 that is given by applying $H_*(-)$ to

$$\Sigma^{-1}\,\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p) \to \mathrm{THH}(B\langle n\rangle; H\mathbb{Z}_{(p)}) \to \mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p).$$

As above, one sees that

$$H_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{Z}_{(p)})) \cong H_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma\bar{\xi}_1, \dots, \sigma\bar{\xi}_{n+1}) \otimes P(\sigma\bar{\tau}_{n+1}).$$

The map

$$H_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{Z}_{(p)})) \to H_*(\mathrm{THH}(B\langle n\rangle; H\mathbb{F}_p))$$

is induced by the inclusion $H_*(H\mathbb{Z}_{(p)}) \to H_*(H\mathbb{F}_p)$. Since the elements $\sigma\bar{\xi}_{n+1}$ and $\sigma\bar{\tau}_{n+1}$ are in the image of this map, they map to zero under β_1. Since $\bar{\tau}_0$ is not in the image, it maps to 1 under β_1 (up to a unit). Since β_1 is a derivation, we get $\beta_1(\sigma\bar{\tau}_{n+1} - \bar{\tau}_0\sigma\bar{\xi}_{n+1}) \doteq \sigma\bar{\xi}_{n+1}$.[2] Finally, we observe that the d_1–differential satisfies the Leibniz rule because the Hurewicz map is a ring map and the Bockstein operator β_1 is a derivation. □

To compute the differentials d_r for $r > 1$ we use [May 1970, Proposition 6.8].

Lemma 3.5 [May 1970, Proposition 6.8] *If $d_{r-1}(x) \neq 0$ in the v_0–Bockstein spectral sequence (3-6) and $|x| = 2q$, then*

$$d_r(x^p) \doteq v_0 x^{p-1} d_{r-1}(x)$$

if $r > 2$. If $r = 2$ and $p = 2$, then

$$d_r(x^p) \doteq v_0 x^{p-1} d_{r-1}(x) + Q^{|x|}(d_1(x)).$$

If $r = 2$ and $p > 2$, then

$$d_r(x^p) \doteq v_0 x^{p-1} d_{r-1}(x) + \mathsf{E},$$

where

$$\mathsf{E} = \sum_{j=1}^{(p-1)/2} j[d_1(x)x^{j-1}, d_1(x)x^{p-j-1}]_1 \tag{3-7}$$

and $[-,-]_1$ denotes the Browder bracket.

Remark 3.6 The result above also appears in [Bruner 1977] in the context of the Adams spectral sequence for an H_∞–ring spectrum (cf [Bruner et al. 1986, Chapter VI Theorems 1.1 and 1.2]).

We note that in order to apply [May 1970, Proposition 6.8], we need the $\cup_1$–product on $\mathrm{THH}(B\langle n\rangle; H\mathbb{Z}_{(p)})$ to satisfy the Hirsch formula, which states that $- \cup_1 c$ is a derivation. We observe that the $\cup_1$–product is

[2] Note that the Bockstein operator β_1 is defined for any $H\mathbb{Z}$–algebra R and it is a derivation at this level of generality by [Browder 1961; Shipley 2007].

a chain homotopy from $x \cdot y$ to $y \operatorname{cof} x$, which corresponds to a braiding in a braided monoidal category. From this perspective, the Hirsch formula corresponds to the first Hexagon axiom in the definition of a braided monoidal category [Joyal and Street 1985, Section 1, B1]. It is well documented that there is an E_2–operad in small categories with the property that algebras over this operad are braided monoidal categories [Dunn 1997]. The n^{th} category in this operad is the translation groupoid $\mathrm{Br}_n \int \Sigma_n$ of the action of the pure Artin braid group Br_n on Σ_n via the canonical inclusion $\mathrm{Br}_n \to \Sigma_n$. We consider the corresponding operad $\mathscr{B}_2$ in $H\mathbb{Z}$–modules by applying the nerve of the category $\mathrm{Br}_n \int \Sigma_n$ and then applying the functor $H\mathbb{Z}_{(p)} \wedge -$. In other words, the n^{th} chain complex in the operad in chain complexes is $\mathscr{B}_2(n) = H\mathbb{Z}_{(p)} \wedge N(\mathrm{Br}_n \int \Sigma_n)_+$. The fact that $\mathrm{THH}(B\langle n\rangle; H\mathbb{Z}_{(p)})$ satisfies the Hirsch formula now follows from two facts:

1. (1) algebras over the operad $\mathscr{B}_2$ in chain complexes satisfy the Hirsch formula (cf [Dunn 1997, Theorem 1.6]), and
2. (2) using [May 1972, Construction 9.6], we replace the E_2 $H\mathbb{Z}_{(p)}$–algebra $\mathrm{THH}(B\langle n\rangle; \mathbb{Z}_{(p)})$ with an $\mathscr{B}_2$ algebra without changing the underlying spectrum.

We therefore tacitly replace our E_2–ring spectrum $\mathrm{THH}(B\langle n\rangle; H\mathbb{Z}_{(p)})$ in $H\mathbb{Z}_{(p)}$–modules with an algebra over the operad $\mathscr{B}_2$ throughout the remainder of the section. The authors thank T Lawson for suggesting this argument.

We can consequently prove the following differential pattern.

Corollary 3.7 *In the spectral sequence (3-6), there are differentials*

$$d_{r+1}(\mu_{n+1}^{p^r}) \doteq v_0^{r+1} \mu_{n+1}^{p^r-1} \lambda_{n+1} \tag{3-8}$$

when $p = 2$ under the assumption that $B\langle n\rangle$ is an E_3 form. Consequently, there are differentials

$$d_{\nu_p(k)+1}(\mu_{n+1}^{k}) \doteq v_0^{\nu_p(k)+1} \mu_{n+1}^{k-1} \lambda_{n+1}$$

where $\nu_p(k)$ denotes the p–adic valuation of k. The same formulas hold for $p \geq 3$ when the error term (3-7) vanishes, for example when $B\langle n\rangle$ is an E_4 form of $\mathrm{BP}\langle n\rangle$.

Proof There is a differential

$$d_1(\mu_{n+1}) \doteq v_0 \lambda_{n+1}$$

by Lemma 3.4 for any prime p. We will argue that this differential implies the differentials (3-8) for $r \geq 1$ by applying Lemma 3.5 and observing that the obstructions vanish.

When $r = 1$ and $p > 2$, the formula (3-8) holds whenever the error term (3-7) vanishes by Lemma 3.5. The Browder bracket $[-,-]_1$ vanishes by [May 1970, Proposition 6.3(iii)] when $B\langle n\rangle$ is an E_4 form of $\mathrm{BP}\langle n\rangle$ since in that case $\mathrm{THH}(B\langle n\rangle; H\mathbb{Z}_{(p)})$ is an E_3–ring spectrum. This completes the base step in the induction for $p > 2$.

If $p=2$ and $r=1$, Lemma 3.4 implies that the error term for $d_2(\mu_{n+1}^2)$ is $Q^{2^{n+2}}\lambda_{n+1}$. At $p=2$,

$$Q^{2^{n+2}}\lambda_{n+1}=Q^{2^{n+2}}(\sigma\bar{\xi}_{n+1}^2)=\sigma(Q^{2^{n+2}}(\bar{\xi}_{n+1}^2))=\sigma((Q^{2^{n+1}}\bar{\xi}_{n+1})^2)=\sigma(\bar{\xi}_{n+2}^2)=0 \tag{3-9}$$

as we now explain. First, the operation $Q^{2^{n+2}}$ is defined on λ_{n+1} because $2^{n+2}=|\lambda_{n+1}|+1$ and $B\langle n\rangle$ is an E_3 form of $\mathrm{BP}\langle n\rangle$ by assumption. The first equality in (3-9) holds by definition of λ_3, the second equality holds because σ commutes with Dyer–Lashof operations by [Angeltveit and Rognes 2005, Proposition 5.9], the third equality holds by [Bruner et al. 1986, Chapter III, Theorem 2.2], and the last equality holds because σ is a derivation in mod p homology, by [Angeltveit and Rognes 2005, Proposition 5.10]. This completes the base step in the induction at $p=2$.

Now let $\alpha=v_p(k)$ and let p be any prime. We have that $k=p^\alpha j$ where p does not divide j. So, by the Leibniz rule,

$$\begin{aligned} d_{\alpha+1}(\mu_{n+1}^k)=d_{\alpha+1}((\mu_{n+1}^{p^\alpha})^j)&=j\mu_{n+1}^{p^\alpha(j-1)}d_{\alpha+1}(\mu_{n+1}^{p^\alpha})\\ &=jv_0^{\alpha+1}\mu_{n+1}^{p^\alpha(j-1)}\mu_{n+1}^{p^\alpha-1}\lambda_{n+1}=v_0^{\alpha+1}\mu_{n+1}^{k-1}\lambda_{n+1}\end{aligned}$$

since j is not divisible by p and therefore is a unit in $\mathbb{F}_p$. □

We now argue that the classes λ_i for $1\le i\le n$ are not p–torsion in $\mathrm{THH}(B\langle n\rangle;H\mathbb{Z}_{(p)})$. Recall from Proposition 3.3 that there is an isomorphism

$$\mathrm{THH}_*(B\langle n\rangle;H\mathbb{Q})\cong E_{\mathbb{Q}}(\sigma v_1,\dots,\sigma v_n).$$

We claim that the map

$$\mathrm{THH}_*(B\langle n\rangle;H\mathbb{Z}_{(p)})\to\mathrm{THH}_*(B\langle n\rangle;H\mathbb{Q})$$

sends λ_i to $p^{-1}\sigma v_i$ $1\le i\le n$. To see this, we note that there is a map of E_2–ring spectra $\mathrm{BP}\to B\langle n\rangle$ by Proposition 2.4 and this produces a commutative diagram

$$\begin{array}{ccc} \mathrm{THH}(\mathrm{BP}) & \longrightarrow & \mathrm{THH}(\mathrm{BP};H\mathbb{Q})\\ \downarrow & & \downarrow\\ \mathrm{THH}(B\langle n\rangle;H\mathbb{Z}_{(p)}) & \longrightarrow & \mathrm{THH}(B\langle n\rangle;H\mathbb{Q})\end{array}$$

of E_1–ring spectra by [Brun et al. 2007]. By Proposition 3.3, we know σv_i maps to σv_i for $1\le i\le n$ under the left vertical map. By [Rognes 2020, Theorem 1.1], we know that

$$\sigma v_i\equiv p\tilde{\lambda}_i \mod (v_i\mid i\ge 1)$$

up to a unit for some classes $\tilde{\lambda}_i=\sigma t_i$. Note that the choice of generators v_i in [Rognes 2020, Theorem 1.1] differ from ours, but they are the same up to a unit and modulo decomposables. Therefore there isn't a difference up to a unit modulo $(v_i\mid i\ge 1)$ after applying the derivation σ. There is an isomorphism

$$\mathrm{THH}_*(\mathrm{BP})\cong E_{\mathrm{BP}_*}(\tilde{\lambda}_k\mid k\ge 1)$$

and we know that $\tilde{\lambda}_i$ maps to λ_i under the map

$$\mathrm{THH}_*(\mathrm{BP})\to\mathrm{THH}_*(B\langle n\rangle;\mathbb{Z}_{(p)})$$

for $1 \le i \le n$ by Zahler [1971] and this does not depend on our choice of E_3 form of $\mathrm{BP}\langle n\rangle$. Therefore, the elements $\lambda_1, \dots, \lambda_n$ are not p–torsion and there are no further differentials in the v_0–Bockstein spectral sequence (3-6). We define

(3-10)
$$\lambda_s := \begin{cases} \lambda_s & \text{if } 1 \le s \le n+1, \\ \lambda_{s-1}\mu_{n+1}^{p^{s-(n+2)}(p-1)} & \text{if } s > n+1. \end{cases}$$

Note that $\mathrm{THH}_*(B\langle n\rangle; \mathbb{Z}_{(p)})$ is finite type so we can compute $\mathrm{THH}_*(B\langle n\rangle; \mathbb{Z}_{(p)})$ from $\mathrm{THH}_*(B\langle n\rangle; \mathbb{Q})$ and $\mathrm{THH}_*(B\langle n\rangle; \mathbb{Q}_p)$ using the arithmetic fracture square

$$\begin{array}{ccc} \mathrm{THH}_*(B\langle n\rangle; \mathbb{Z}_{(p)}) & \longrightarrow & \prod_p \mathrm{THH}_*(B\langle n\rangle; \mathbb{Z}_p) \\ \downarrow & & \downarrow \\ \mathrm{THH}_*(B\langle n\rangle; \mathbb{Q}) & \longrightarrow & \prod_p \mathrm{THH}_*(B\langle n\rangle; \mathbb{Q}_p) \end{array}$$

This proves the following theorem.

Theorem 3.8 *Let $B\langle n\rangle$ be an arbitrary E_3 form of $\mathrm{BP}\langle n\rangle$ and at $p > 2$ assume the error term (3-7) vanishes. Then there is an isomorphism of graded $\mathbb{Z}_{(p)}$–modules*

$$\mathrm{THH}_*(B\langle n\rangle; H\mathbb{Z}_{(p)}) \cong E_{\mathbb{Z}_{(p)}}(\lambda_1, \dots, \lambda_n) \otimes (\mathbb{Z}_{(p)} \oplus T_0^n),$$

where T_0^n is a torsion $\mathbb{Z}_{(p)}$–module defined by

(3-11)
$$T_0^n = \bigoplus_{s \ge 1} \mathbb{Z}/p^s \otimes P_{\mathbb{Z}_{(p)}}(\mu_{n+1}^{p^s}) \otimes \mathbb{Z}_{(p)}\{\lambda_{n+s}\mu_{n+1}^{jp^{s-1}} \mid 0 \le j \le p-2\}.$$

4 Topological Hochschild homology mod (p, v_2)

In this section, we compute topological Hochschild homology of $B\langle 2\rangle$ with coefficients in $k(1)$. First we compute topological Hochschild homology with coefficients in $K(1)$.

4.1 $K(1)$–local topological Hochschild homology

In this section we assume that $p \ge 3$ and write $B\langle 2\rangle$ for an E_3 form of $\mathrm{BP}\langle 2\rangle$. Write $k(1) = B\langle 2\rangle/(p, v_2)$ for the E_1 $B\langle 2\rangle$–algebra constructed as in Proposition 2.4 and let $K(1) = k(1)[v_1^{-1}]$. In order to determine the topological Hochschild homology of $B\langle 2\rangle$ with coefficients in $k(1)$, we first determine

$$\mathrm{THH}(B\langle 2\rangle; K(1)) = \mathrm{THH}(B\langle 2\rangle; K(1)).$$

To compute the multiplicative Bökstedt spectral sequence

$$E^2_{*,*} = \mathrm{HH}^{K(1)_*}_{*,*}(K(1)_* B\langle 2\rangle) \Rightarrow K(1)_* \mathrm{THH}(B\langle 2\rangle),$$

we first need to compute $K(1)_* B\langle 2\rangle$. To compute $K(1)_* B\langle 2\rangle$ we first relate it to $\mathrm{BP}_*\mathrm{BP}$. Recall that

$$\mathrm{BP}_*\mathrm{BP} = \mathrm{BP}_*[t_1, t_2, \dots]$$

with $|t_i| = 2p^i - 2$. By [Ravenel 1986, Theorem A2.2.6], the right unit η_R is determined by

$$\sum_{i,j\geq 0}^{F} t_i \eta_R(v_j)^{p^i} = \sum_{i,j\geq 0}^{F} v_i t_j^{p^i} \tag{4-1}$$

where $t_0 = 1$ and $v_0 = p$.

Lemma 4.1 *The composite map*

$$K(1)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} B\langle 2\rangle_* \to \pi_*(K(1) \wedge_{\mathrm{BP}} (\mathrm{BP} \wedge \mathrm{BP}) \wedge_{\mathrm{BP}} B\langle 2\rangle) \cong K(1)_* B\langle 2\rangle$$

is an isomorphism.

Proof Consider the commutative diagram

$$\begin{array}{ccc} \pi_*(K(1) \wedge B\langle 2\rangle) & \longrightarrow & \pi_*(K(1) \wedge B\langle 2\rangle[v_1^{-1}]) \\ \cong\uparrow & & \cong\uparrow \\ \pi_*(K(1) \wedge_{\mathrm{BP}} (\mathrm{BP} \wedge \mathrm{BP}) \wedge_{\mathrm{BP}} B\langle 2\rangle) & \longrightarrow & \pi_*(K(1) \wedge_{\mathrm{BP}} (\mathrm{BP} \wedge \mathrm{BP}) \wedge_{\mathrm{BP}} B\langle 2\rangle[v_1^{-1}]) \\ \uparrow & & \uparrow \\ K(1)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} B\langle 2\rangle_* & \longrightarrow & \pi_*(K(1)) \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} B\langle 2\rangle_*[v_1^{-1}] \end{array} \tag{4-2}$$

Since $B\langle 2\rangle[v_1^{-1}]$ is Landweber exact, the right-hand vertical map is an isomorphism. In (4-1) the F–summands in degree $\leq 2p-2$ are $\eta_R(v_0)$, $t_1\eta_R(v_0)^p$, $\eta_R(v_1)$, v_0, v_1 and $v_0 t_1$. Thus, $\eta_R(v_1) = v_1$ in $K(1)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} = K(1)_*[t_i \mid i \geq 1]$, because $p = 0$ in this ring. In $K(1)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} B\langle 2\rangle_*$,

$$v_1 \otimes 1 \otimes 1 = 1 \otimes v_1 \otimes 1 = 1 \otimes \eta_R(v_1) \otimes 1 = 1 \otimes 1 \otimes v_1$$

holds. This implies that the upper and lower horizontal map in the diagram are isomorphisms. It follows that the left vertical map is an isomorphism too. □

Notation 4.2 Let $f_i(v_1, v_2) \in B\langle 2\rangle_* = \mathbb{Z}_{(p)}[v_1, v_2]$ be the image of v_i under $\mathrm{BP}_* \to B\langle 2\rangle_*$. Define

$$v_i' := v_i - f_i(v_1, v_2) \in \mathrm{BP}_*.$$

Then v_i' is in the kernel of $\mathrm{BP}_* \to B\langle 2\rangle_*$ and $\mathrm{BP}_* = \mathbb{Z}_{(p)}[v_1, v_2, v_3', \dots]$.

By Lemma 4.1,

$$\begin{aligned} K(1)_* B\langle 2\rangle &= (K(1)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*[t_1, \dots]) \otimes_{\mathbb{Z}_{(p)}[v_1, v_2, v_3', \dots]} \mathbb{Z}_{(p)}[v_1, v_2] \\ &= K(1)_*[t_i \mid i \geq 1]/(\eta_R(v_3'), \dots). \end{aligned}$$

Lemma 4.3 *For $i \geq 0$ the element $\eta_R(v_{i+1}) \in K(1)_*[t_i \mid i \geq 1]$ actually lies in $K(1)_*[t_1, \dots, t_i]$. In fact,*

$$\eta_R(v_{i+1}) = v_{i+1} + v_1 t_i^p - v_1^{p^i} t_i + g_i,$$

where $g_i \in K(1)_*[t_1, \dots, t_{i-1}]$.

Proof We will prove the claim in $k(1)_*[t_i \mid i \geq 1]$; from this the result will follow. The reason we do this is because we will want to make degree arguments, and hence will want to avoid negative gradings.

In $\mathrm{BP}_*\mathrm{BP}/(p)$, we have $\eta_R(v_1) = v_1$. It also follows from (4-1) that, for $i \geq 0$,

$$\eta_R(v_{i+1}) \equiv v_{i+1} + v_1 t_i^p - v_1^{p^i} t_i \mod (t_1, t_2, \ldots, t_{i-1})$$

in $\mathrm{BP}_*\mathrm{BP}/(p)$. Thus, this congruence also holds in $k(1)_*[t_i \mid i \geq 1]$. Since $\eta_R(v_{i+1})$ lifts to $\mathrm{BP}_*\mathrm{BP}/(p)$ we may make our degree arguments in $k(1)_*[t_i \mid i \geq 1]$. In the ring $k(1)_*[t_i \mid i \geq 1]$, we therefore have

$$\eta_R(v_{i+1}) = v_{i+1} + v_1 t_i^p - v_1^{p^i} t_i + g_i,$$

where g_i is a polynomial in the ideal generated by $t_1, t_2, \ldots, t_{i-1}$. Thus far we have not excluded the possibility that a monomial divisible by t_j with $j \geq i$ occurs as a summand of g_i.

For $j > i+1$, we can exclude this possibility for degree reasons. Indeed, $\eta_R(v_{i+1})$ is homogenous of degree $2(p^{i+1}-1)$, and when $j > i+1$ the element t_j has degree greater than $2(p^{i+1}-1)$. Consider the case when $j = i+1$. To exclude this case, suppose there exists a monomial m in $k(1)_*[t_1, \ldots, t_i]$ which is a summand of g_i and is divisible by t_{i+1}. Then as the degrees of t_{i+1} and $\eta_R(v_{i+1})$ are the same, it follows that $m = a t_{i+1}$ for some $a \in \mathbb{F}_p$. If $a \neq 0$, then this contradicts the assumption that g_i is in the ideal $(t_1, \ldots, t_{i-1})$. This shows that $g_i \in k(1)_*[t_1, \ldots, t_i]$.

We now exclude the possibility that a monomial divisible by t_i occurs as a summand of g_i. Note that the summands $t_k \eta_R(v_j)^{p^k}$ and $v_k t_j^{p^k}$ in (4-1) both have degree $2(p^{k+j}-1)$. Cross terms in (4-1) from those summands with degree less than or equal than $2(p^{i+1}-1)$ could potentially produce a t_i divisible monomial as a summand of g_i. On the right-hand side of (4-1), the possible summands are those of the form $v_j t_i^{p^j}$. As this must have degree at most $2(p^{i+1}-1)$, we must have $j = 0, 1$. These correspond, respectively, to $v_0 t_i = p t_i$ and $v_1 t_i^p$. But $p = 0$ in $k(1)_*$, so the only one to consider is $v_1 t_i^p$. This has degree exactly $2(p^{i+1}-1)$, and so a monomial divisible by this element does not occur in g_i. In fact, it has already been accounted for.

For the left-hand side, we similarly find that the only summand which could potentially produce a t_i divisible monomial as a summand of $\eta_R(v_{i+1})$ is

$$t_i \eta_R(v_1)^{p^i} = t_i v_1^{p^i}.$$

As this has exactly degree $2(p^{i+1}-1)$, it does not occur in g_i because it cannot be written as an element in the ideal $(t_1, \ldots, t_{i-1})$ for degree reasons. In fact, this element has already been accounted for. Thus there are no t_i divisible monomials appearing as summands of g_i. Consequently, we have shown that $g_i \in k(1)_*[t_1, \ldots, t_{i-1}]$ as desired. □

Recall from the proof of Proposition 2.4 that

$$v_i' = v_i - f_i(v_1, v_2)$$

for some $f_i \in \mathbb{Z}_{(p)}[x, y]$. In light of the previous lemma, we conclude that the class

$$\eta_R(v_i') = \eta_R(v_i) - f_i(\eta_R(v_1), \eta_R(v_2)) \in K(1)_*[t_i \mid i \geq 1]$$

also lies in $K(1)_*[t_1, \ldots, t_{i-1}]$ for each $i \geq 3$.

Lemma 4.4 *The maps of commutative $K(1)_*$–algebras*

$$K(1)_*[t_1,\dots,t_{i-1}]/(\eta_R(v_3'),\dots,\eta_R(v_i')) \to K(1)_*[t_1,\dots,t_i]/(\eta_R(v_3'),\dots,\eta_R(v_{i+1}'))$$

induced by precomposing the canonical quotient map with the canonical inclusion map are étale for $i \geq 2$.

Proof For ease of notation, set

$$A_i := K(1)_*[t_1,\dots,t_{i-1}]/(\eta_R(v_3'),\dots,\eta_R(v_i'))$$

for $i \geq 2$. Note that Lemma 4.3 allows us to make this definition. Note also that $A_{i+1} = A_i[t_i]/(\eta_R(v_{i+1}'))$. We wish to show that the map

$$A_i \to A_{i+1}$$

is an étale morphism. To do this, it is enough to show that the partial derivative of $\eta_R(v_{i+1}')$ with respect to t_i is a unit in A_i. Write ∂_i for the partial derivative with respect to t_i. Since

$$v_{i+1}' = v_{i+1} - f_{i+1}(v_1, v_2)$$

for some $f_{i+1} \in \mathbb{Z}_{(p)}[x,y]$, we can infer that

$$\eta_R(v_{i+1}') = \eta_R(v_{i+1}) - f_{i+1}(\eta_R(v_1), \eta_R(v_2)).$$

In $K(1)_*[t_1,t_2,\dots]$, we know $\eta_R(v_1) = v_1$ since $p = 0$ in $K(1)_*$, and we have

$$\eta_R(v_2) = v_1 t_1^p - v_1^p t_1.$$

Thus,

$$\partial_i \eta_R(v_{i+1}') = \partial_i \eta_R(v_{i+1})$$

for $i \geq 2$ and it suffices to show that $\partial_i \eta_R(v_{i+1})$ is a unit in A_i.

By Lemma 4.3, we have the formula

$$\eta_R(v_{i+1}) = v_{i+1} + v_1 t_i^p - v_1^{p^i} t_i + g_i,$$

where $g_i \in K(1)_*[t_1,\dots,t_{i-1}]$. Thus, we conclude that

$$\partial_i \eta_R(v_{i+1}) = -v_1^{p^i} \in K(1)_*[t_1,\dots,t_{i-1}].$$

Since $v_1^{p^i}$ is a unit in $K(1)_*$, this shows that $\partial_i \eta_R(v_{i+1})$ is a unit in A_i. □

We continue to use the notation from the proof of the previous lemma. Since each map $A_i \to A_{i+1}$ is étale, we may apply [Weibel and Geller 1991, Theorem 0.1] to conclude that

$$\begin{aligned}
\mathrm{HH}_{*,*}^{K(1)_*}(K(1)_* B\langle 2\rangle) &= \operatorname{colim} \mathrm{HH}_{*,*}^{K(1)_*}(A_i) \\
&= \operatorname{colim} \mathrm{HH}_{*,*}^{K(1)_*}(A_2) \otimes_{A_2} A_i \\
&= \mathrm{HH}_{*,*}^{K(1)_*}(A_2) \otimes_{A_2} K(1)_* B\langle 2\rangle \\
&= E(\sigma t_1) \otimes K(1)_* B\langle 2\rangle.
\end{aligned}$$

Since this is concentrated in Bökstedt filtration 0 and 1, the Bökstedt spectral sequence collapses, yielding

$$E(\sigma t_1) \otimes K(1)_* B\langle 2\rangle \cong K(1)_* \operatorname{THH}(B\langle 2\rangle).$$

In the Hopf algebroid $(\mathrm{BP}_*, \mathrm{BP}_*\mathrm{BP})$, we have the formula

$$\sum_{i\geq 0}^{F} \Delta(t_i) = \sum_{i,j\geq 0}^{F} t_i \otimes t_j^{p^i}$$

by [Ravenel 1986, Theorem A2.1.27]. Since the $\mathrm{BP}_*\mathrm{BP}$–coaction on t_i agrees with the coproduct, it is determined by the formula

$$\Delta(t_1) = 1 \otimes t_1 + t_1 \otimes 1.$$

Note that $(K(1)_*, K(1)_*K(1))$ is a flat Hopf algebroid and $K(1)_*(X)$ is a left $K(1)_*K(1)$–comodule for every spectrum X. By naturality, we observe that $t_1 \in K(1)_*B\langle 2\rangle$ has the $K(1)_*K(1)$–coaction $1 \otimes t_1 + t_1 \otimes 1$. Let

$$\sigma\colon K(1)_*B\langle 2\rangle \to K(1)_{*+1} \operatorname{THH}(B\langle 2\rangle)$$

be the usual σ operator analogous to the one defined in [McClure and Staffeldt 1993]. By [Angeltveit and Rognes 2005, Proposition 5.10], which also applies to our setting because the Hopf element $\eta = 0 \in K(1)_*$, the operator σ is a derivation. It is also clear that σ is compatible with the $K(1)_*K(1)$–comodule action in the sense that

$$\psi(\sigma x) = (1 \otimes \sigma)(\psi(x)),$$

where

$$\psi\colon K(1)_* \operatorname{THH}(B\langle 2\rangle) \to K(1)_*K(1) \otimes K(1)_* \operatorname{THH}(B\langle 2\rangle).$$

It follows that $\sigma t_1 \in K(1)_* \operatorname{THH}(B\langle 2\rangle)$ is a comodule primitive. Since there is a weak equivalence $\operatorname{THH}(B\langle 2\rangle, K(1)) \simeq K(1) \wedge_{B\langle 2\rangle} \operatorname{THH}(B\langle 2\rangle)$ by [Hahn and Wilson 2022, Remark 6.1.4], we may infer from the Künneth isomorphism that there is an isomorphism of $K(1)_*$–modules

$$K(1)_* \operatorname{THH}(B\langle 2\rangle; K(1)) \cong K(1)_*K(1) \otimes E(\sigma t_1),$$

where σt_1 is a comodule primitive. Since $\operatorname{THH}(B\langle 2\rangle; K(1))$ is a $K(1)$–module spectrum and $K(1)_*$ is a graded field, we have that it splits as a sum of suspensions of $K(1)$ and that its homotopy is isomorphic to the comodule primitives in $K(1)_* \operatorname{THH}(B\langle 2\rangle; K(1))$. Thus, there is an isomorphism of $K(1)_*$–modules

$$\operatorname{THH}_*(B\langle 2\rangle; K(1)) = K(1)_* \otimes E(\sigma t_1).$$

Since σt_1 lifts to a class in $\tilde{\lambda}_1 \in \operatorname{THH}_*(B\langle 2\rangle; k(1))$ which projects onto λ_1 via the map

$$\operatorname{THH}_*(B\langle 2\rangle; k(1)) \to \operatorname{THH}_*(B\langle 2\rangle; H\mathbb{F}_p)$$

induced by the linearization map $k(1) \to H\mathbb{F}_p$ by [Zahler 1971], we simply rename this class λ_1.

In summary, we have proven the following theorem.

Theorem 4.5 *For $B\langle 2\rangle$ an E_3 form of* $\mathrm{BP}\langle 2\rangle$ *and $p\geq 3$, the following hold*:

(1) *There is a weak equivalence*
$$K(1)\vee\Sigma^{2p-1}K(1)\simeq \mathrm{THH}(B\langle 2\rangle;K(1)).$$

(2) *The $P(v_1)$–module $\mathrm{THH}_*(B\langle 2\rangle;k(1))$, modulo v_1–torsion, is freely generated by 1 and λ_1.*

4.2 The v_1–Bockstein spectral sequence

We compute $\mathrm{THH}_*(B\langle 2\rangle;k(1))$ using the spectral sequence (3-2) for $n=2$ and $i=1$. For $s\geq 4$, we recursively define
$$\lambda_s:=\lambda_{s-2}\mu_3^{p^{s-4}(p-1)}.$$
For $s\geq 1$, we define
$$r(s,1):=\begin{cases}p^{s+1}+p^{s-1}+\cdots+p^2 & \text{if } s\equiv 1 \bmod 2,\\ p^{s+1}+p^{s-1}+\cdots+p^3 & \text{if } s\equiv 0 \bmod 2.\end{cases}$$

Theorem 4.6 *Let $B\langle 2\rangle$ be an E_3 form of* $\mathrm{BP}\langle 2\rangle$ *and let $p\geq 3$. There is an isomorphism of $P(v_1)$–modules*
$$\mathrm{THH}_*(B\langle 2\rangle;k(1))\cong E(\lambda_1)\otimes(P(v_1)\oplus T_1^2),$$
where

(4-3)
$$T_1^2=\bigoplus_{s\geq 1}P_{r(s,1)}(v_1)\otimes E(\lambda_{s+2})\otimes P(\mu_3^{p^s})\otimes\mathbb{F}_p\{\lambda_{s+1}\mu_3^{jp^{s-1}}\mid 0\leq j\leq p-2\}.$$

Proof We prove by induction on $s\geq 1$ that
$$E^{*,*}_{r(s,1)}=E(\lambda_1)\otimes\bigl(P(v_1)\otimes E(\lambda_{s+1},\lambda_{s+2})\otimes P(\mu_3^{p^{s-1}})\oplus M_s\bigr)$$
with
$$M_s=\bigoplus_{t=1}^{s-1}P_{r(t,1)}(v_1)\otimes E(\lambda_{t+2})\otimes P(\mu_3^{p^t})\otimes\mathbb{F}_p\{\lambda_{t+1}\mu_3^{jp^{t-1}}\mid 0\leq j\leq p-2\},$$
that we have a differential $d_{r(s,1)}(\mu_3^{p^{s-1}})\doteq v_1^{r(s,1)}\lambda_{s+1}$, and that the classes λ_{s+1} and λ_{s+2} are infinite cycles. This implies the statement.

By Theorem 4.5, the elements v_1^s are permanent cycles for every s, so the classes λ_2 and λ_3 cannot support differentials and thus are infinite cycles. Note that we use $p\geq 3$ here; for $p=2$ we would have a possible differential $d_2(\lambda_3)\doteq v_1^2\lambda_1\lambda_2$. Since the classes $v_1^n\lambda_1$ survive by Theorem 4.5, the only possible differential on μ_3 is
$$d_{p^2}(\mu_3)\doteq v_1^{p^2}\lambda_2$$
for bidegree reasons. This differential must exist because otherwise the spectral sequence would collapse at the E_2–page by multiplicativity which would contradict Theorem 4.5. This proves the base step $s=1$ of the induction. Now, assume that the statement holds for some $s\geq 1$. We then get
$$E^{*,*}_{r(s,1)+1}=E(\lambda_1)\otimes(P(v_1)\otimes E(\lambda_{s+2},\lambda_{s+1}\mu_3^{p^{s-1}(p-1)})\otimes P(\mu_3^{p^s})\oplus M_{s+1},$$

and it suffices to show that $\lambda_{s+3} = \lambda_{s+1}\mu_3^{p^{s-1}(p-1)}$ is an infinite cycle and that

$$d_{r(s+1,1)}(\mu_3^{p^s}) \doteq v_1^{r(s+1,1)}\lambda_{s+2}.$$

Note that the class λ_{s+2} is an infinite cycle by the induction hypothesis. The class λ_{s+3} is an infinite cycle for bidegree reasons and because the classes v_1^s are permanent cycles. Note that we use $p \geq 3$ here; for $p = 2$ and s even we would have a possible differential $d_{r(s,1)+p}(\lambda_{s+3}) \doteq v_1^{r(s,1)+p}\lambda_1\lambda_{s+2}$. The class $\mu_3^{p^s}$ must support a differential because otherwise the spectral sequence would collapse at this stage which would contradict Theorem 4.5. Since the classes $v_1^n\lambda_1$ are permanent cycles,

$$d_{r(s+1,1)}(\mu_3^{p^s}) \doteq v_1^{r(s+1,1)}\lambda_{s+2}$$

for bidegree reasons. Here note that $v_1^{r(s,1)}\lambda_{s+3}$ has the right topological degree, but the filtration degree is too low for it to be the target of a differential on $\mu_3^{p^s}$ at the E_ℓ–page for $\ell > r(s,1)$. This completes the induction step. □

5 Topological Hochschild homology mod (p, v_1)

In this section $B\langle 2\rangle$ is again an E_3 form of $\mathrm{BP}\langle 2\rangle$, eg $\mathrm{tmf}_1(3)$ at $p = 2$, taf^D at $p = 3$, or $\mathrm{BP}\langle n\rangle'$ at an arbitrary prime p. We let $k(2) := B\langle 2\rangle/(p, v_1)$ be the E_1 $B\langle 2\rangle$–algebra constructed in Proposition 2.4 and let $K(2) = k(2)[v_2^{-1}]$. The goal of this section is to compute the homotopy groups of $\mathrm{THH}(B\langle 2\rangle; K(2))$. In Section 5.1, we first show that the unit map

$$K(2) \to \mathrm{THH}_*(B\langle 2\rangle; K(2))$$

is an equivalence. This implies that in the abutment of the v_2–Bockstein spectral sequence

$$\mathrm{THH}_*(B\langle 2\rangle; H\mathbb{F}_p)[v_2] \Rightarrow \mathrm{THH}_*(B\langle 2\rangle; k(2))$$

all classes are v_2–torsion besides the powers of v_2. This allows us to compute this spectral sequence in Section 5.2.

5.1 $K(2)$–local topological Hochschild homology

Considering a diagram analogous to (4-2), one sees that we have an isomorphism

$$K(2)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} B\langle 2\rangle_* \to \pi_*(K(2) \wedge B\langle 2\rangle).$$

For this, note that

$$\eta_R(v_1) = v_1 = 0 \in K(2)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} = K(2)_*[t_i \mid i \geq 1]$$

and therefore $\eta_R(v_2) = v_2$. This implies that the equality

$$v_2 \otimes 1 \otimes 1 = 1 \otimes 1 \otimes v_2$$

holds in the tensor product

$$K(2)_* \otimes_{\mathrm{BP}_*} \mathrm{BP}_*\mathrm{BP} \otimes_{\mathrm{BP}_*} B\langle 2\rangle_*.$$

From this, we determine that

$$K(2)_* B\langle 2\rangle = K(2)_*[t_i \mid i \geq 1]/(\eta_R(v_3'), \dots).$$

In particular, this is a graded commutative $K(2)_*$–algebra even at $p = 2$ where $K(2)$ is not homotopy commutative (cf [Angeltveit and Rognes 2005, Lemma 8.9]).

Lemma 5.1 *In* $K(2)_*[t_1 \mid i \geq 1]$,

$$\eta_R(v_{i+2}) = v_{i+2} + v_2 t_i^{p^2} - v_2^{p^i} t_i + g_i$$

where $g_i \in K(2)_*[t_1, \dots, t_{i-1}]$.

Proof We argue similarly to Lemma 4.3 and make our arguments in the ring $k(2)_*[t_i \mid i \geq 1]$. The result will follow from this. We have that

$$\eta_R(v_{i+2}) \equiv v_{i+2} + v_2 t_i^{p^2} - v_2^{p^i} t_i \mod (t_1, t_2, \dots, t_{i-1}),$$

in $\mathrm{BP}_*\mathrm{BP}/(p, v_1)$ (see [Ravenel 1986, Proof of Theorem 4.3.2]). Consequently, this formula also holds in $k(2)_*[t_i \mid i \geq 1]$. This shows that in $k(2)_*[t_i \mid i \geq 1]$,

$$\eta_R(v_{i+2}) = v_{i+2} + v_2 t_i^{p^2} - v_2^{p^i} t_i + g_i$$

for some g_i in the ideal $(t_1, t_2, \dots, t_{i-1})$. Since $\eta_R(v_{i+2})$ lifts to the graded abelian group $\mathrm{BP}_*\mathrm{BP}/(p, v_1)$, we may also make degree arguments in $k(2)_*[t_i \mid i \geq 1]$.

Note that for degree reasons, there can be no instance of a t_j with $j > i+2$ dividing a monomial summand of g_i. We can also exclude the possibility of t_{i+2} dividing a monomial in g_i. Indeed, a monomial in g_i divisible by t_{i+2} would necessarily be just t_{i+2} itself, contradicting that g_i is in the ideal $(t_1, \dots, t_{i-1})$. This shows that

$$\eta_R(v_{i+2}) \in k(2)_*[t_1, \dots, t_{i+1}].$$

for all $i \geq 0$.

We now exclude the possibility that t_{i+1} divides a monomial in $\eta_R(v_{i+2})$. To do this, we note that a t_{i+1} divisible monomial in g_i could arise from cross terms involving the universal p–typical formal group law and the formula (4-1). Note that the only terms to consider on the right-hand side are $v_0 t_{i+1}$ and $v_1 t_{i+1}^p$, which are 0 since $p = v_1 = 0 \in k(2)_*$. On the left-hand side, we only need to consider the terms $t_k \eta_R(v_{j+2})^{p^k}$ of degree less than or equal to $2(p^{i+2}-1)$. This immediately implies that $j \leq i$. For $k = i+1$, the term of smallest degree is $t_{i+1}\eta_R(v_2)^{p^{i+1}}$. The degree of this term is $2(p^{i+3}-1)$, which is too large. Thus we can exclude the possibility that $k = i+1$. Now as $j \leq i$ and since we have shown that $\eta_R(v_{j+2}) \in k(2)_*[t_1, \dots, t_{j+1}]$, we see that none of the relevant terms on the left-hand side can contribute a t_{i+1} divisible monomial summand to $\eta_R(v_{i+2})$. Thus we have that $g_i \in K(2)_*[t_1, \dots, t_i]$.

We are left to consider whether a t_i divisible monomial could occur as a summand of g_i via the cross terms coming from the formal group law F in (4-1). On the right-hand side, we only need to consider the term $v_2 t_i^{p^2}$. Here we use the fact that $v_1 = 0 \in k(2)_*$. This term has already been accounted for and is not in g_i. On the left-hand side, since we have shown that $\eta_R(v_{j+2}) \in k(2)_*[t_1, \dots, t_j]$, the only term we need to consider is $t_i v_2^{p^i}$. Again, we have already considered this term. We can therefore conclude that $g_i \in k(2)_*[t_1, \dots, t_{i-1}]$. □

Definition 5.2 We define commutative $K(2)_*$–algebras

$$C_0 := K(2)_*,$$
$$C_i := C_{i-1}[t_i]/\eta_R(v'_{i+2}), \quad i \geq 1,$$

and write $h_i : C_{i-1} \to C_i$ for the map of commutative $K(2)_*$–algebras defined as the composite of the canonical inclusion map $C_{i-1} \to C_{i-1}[t_i]$ with the canonical quotient map $C_{i-1}[t_i] \to C_{i-1}[t_i]/\eta_R(v'_{i+2})$.

Thus we have

$$C_i = K(2)_*[t_1, \dots, t_i]/(\eta_R(v'_3), \dots, \eta_R(v'_{i+2}))$$

for $i \geq 1$ and

$$K(2)_* B\langle 2\rangle = \operatorname*{colim}_i C_i.$$

We proceed in the same fashion as in Section 4.1 and argue that $h_i : C_{i-1} \to C_i$ is étale by examining the derivative of $\eta_R(v'_{i+2})$ with respect to t_i.

Lemma 5.3 *The map of commutative rings $h_i : C_{i-1} \to C_i$ from Definition 5.2 is étale.*

Proof We have that

$$v'_{i+2} = v_{i+2} - f_{i+2}(v_1, v_2) = v_{i+2} - f_{i+2}(0, v_2).$$

Hence,

$$\eta_R(v'_{i+2}) = \eta_R(v_{i+2}) - f_{i+2}(0, v_2).$$

Let ∂_i denote the partial derivative with respect to t_i. Since $C_i = C_{i-1}[t_i]/(\eta_R(v'_{i+2}))$, to show the morphism $C_{i-1} \to C_i$ is étale, it is enough to show that $\partial_i \eta_R(v'_{i+2})$ is a unit. We have

$$\partial_i \eta_R(v'_{i+2}) = \partial_i \eta_R(v_{i+2}) - \partial_i f_{i+2}(0, v_2) = \partial_i \eta_R(v_{i+2}).$$

From Lemma 5.1, we find that $\partial_i g_i = 0$, and hence

$$\partial_i \eta_R(v_{i+2}) = \partial_i (v_{i+2} + v_2 t_i^{p^2} - v_2^{p^i} t_i + g_i) = -v_2^{p^i}$$

which is a unit. □

Since each map $C_i \to C_{i+1}$ is étale, we may apply [Weibel and Geller 1991, Theorem 0.1] to conclude that the unit map

(5-1) $$K(2)_* B\langle 2\rangle \to \mathrm{HH}_{*,*}^{K(2)_*}(K(2)_* B\langle 2\rangle)$$

is an isomorphism of graded commutative $\mathbb{F}_p$–algebras (even at $p=2$). The unit map

$$K(2)_* B\langle 2\rangle \to K(2)_* \operatorname{THH}(B\langle 2\rangle)$$

is the edge homomorphism in the Bökstedt spectral sequence

$$E^2_{*,*} = \operatorname{HH}^{K(2)_*}_{*,*}(K(2)_* B\langle 2\rangle) \Rightarrow K(2)_* \operatorname{THH}(B\langle 2\rangle)$$

and the input is concentrated in Bökstedt filtration zero by (5-1), so the spectral sequence collapses without extensions yielding an isomorphism

$$K(2)_* B\langle 2\rangle \cong K(2)_* \operatorname{THH}(B\langle 2\rangle)$$

of graded commutative $\mathbb{F}_p$–algebras (even at the prime $p=2$).

By the Künneth isomorphism, the map

$$K(2)_* K(2) \to K(2)_* \operatorname{THH}(B\langle 2\rangle, K(2))$$

is an isomorphism as well. Since both $K(2)$ and $\operatorname{THH}(B\langle 2\rangle; K(2))$ are $K(2)$–local, we obtain the following result.

Corollary 5.4 *The unit map*

$$\eta\colon K(2) \to \operatorname{THH}(B\langle 2\rangle; K(2))$$

is an equivalence. Consequently, the $P(v_2)$–module $\operatorname{THH}_*(B\langle 2\rangle; k(2))$ modulo v_2–torsion is freely generated by 1.

5.2 The v_2–Bockstein spectral sequence

Recall from Section 3.1 that the tower of spectra used to build the Bockstein spectral sequence (3-1) can be identified as an Adams tower and therefore the Bockstein spectral sequence is multiplicative.

For $s \geq 4$, recursively define

$$\lambda_s := \lambda_{s-3}\mu_3^{p^{s-4}(p-1)}.$$

For $s \geq 1$, set

$$r(s,2) = \begin{cases} p^s + p^{s-3} + \cdots + p^4 + p & \text{if } s \equiv 1 \bmod 3, \\ p^s + p^{s-3} + \cdots + p^5 + p^2 & \text{if } s \equiv 2 \bmod 3, \\ p^s + p^{s-3} + \cdots + p^6 + p^3 & \text{if } s \equiv 0 \bmod 3. \end{cases}$$

Theorem 5.5 *Let $B\langle 2\rangle$ be an E_3 form of $\operatorname{BP}\langle 2\rangle$. There is an isomorphism of $P(v_2)$–modules*

$$\operatorname{THH}_*(B\langle 2\rangle; k(2)) \cong P(v_2) \oplus T_2^2,$$

where

$$T_2^2 \cong \bigoplus_{s\geq 1} P_{r(s,2)}(v_2) \otimes E(\lambda_{s+1}, \lambda_{s+2}) \otimes P(\mu_3^{p^s}) \otimes \mathbb{F}_p\{\lambda_s \mu_3^{jp^{s-1}} \mid 0 \leq j \leq p-2\}. \tag{5-2}$$

Proof We prove by induction on $s \geq 1$ that

$$E^{*,*}_{r(s,2)} = P(v_2) \otimes E(\lambda_s, \lambda_{s+1}, \lambda_{s+2}) \otimes P(\mu_3^{p^{s-1}}) \oplus M_s$$

with

$$M_s = \bigoplus_{t=1}^{s-1} P_{r(t,2)}(v_2) \otimes E(\lambda_{t+1}, \lambda_{t+2}) \otimes P(\mu_3^{p^t}) \otimes \mathbb{F}_p\{\lambda_t \mu_3^{jp^{t-1}} \mid 0 \leq j \leq p-2\},$$

that λ_s, λ_{s+1} and λ_{s+2} are infinite cycles, and that $d_{r(s,2)}(\mu_3^{p^{s-1}}) \doteq v_2^{r(s,2)}\lambda_s$. This implies the statement. Since the v_2^n survive to the E_∞–page by Corollary 5.4, the classes λ_1, λ_2 and λ_3 are infinite cycles. The class μ_3 needs to support a differential, because otherwise the spectral sequence would collapse at the E_2–page by multiplicativity, which is a contradiction to Corollary 5.4. For bidegree reasons the only possibility is

$$d_p(\mu_3) \doteq v_2^p \lambda_1.$$

This proves the base step $s = 1$ of the induction. We now assume that the statement holds for some $s \geq 1$. We then get

$$E^{*,*}_{r(s,2)+1} = P(v_2) \otimes E(\lambda_{s+1}, \lambda_{s+2}, \lambda_s \mu_3^{p^{s-1}(p-1)}) \otimes P(\mu_3^{p^s}) \oplus M_{s+1}.$$

It now suffices to show that $\lambda_{s+3} = \lambda_s \mu_3^{p^{s-1}(p-1)}$ is an infinite cycle and that we have a differential $d_{r(s+1,2)}(\mu_3^{p^s}) \doteq v_2^{r(s+1,2)}\lambda_{s+1}$. We cannot have a differential of the form

$$d_r(\lambda_{s+3}) \doteq v_2^n \lambda_{s+1}\lambda_{s+2}$$

for degree reasons, so λ_{s+3} is an infinite cycle. The class $\mu_3^{p^s}$ must support a differential, because otherwise the spectral sequence would collapse at this stage, which is a contradiction to Corollary 5.4. For bidegree reasons the only possibility is

$$d_{r(s+1,2)}(\mu_3^{p^s}) \doteq v_2^{r(s+1,2)}\lambda_{s+1}.$$

Note that $v_2^{r(s,2)}\lambda_{s+3}$ has the right topological degree, but a too small filtration degree to be the target of a differential on $\mu_3^{p^s}$. This completes the inductive step. □

We end with a conjectural answer for $\mathrm{THH}(\mathrm{BP}\langle n\rangle; k(m))$ for all $1 \leq m \leq n$.

Conjecture 5.6 Suppose $1 \leq m \leq n$. Let $B\langle n\rangle$ be an E_3 form of $\mathrm{BP}\langle n\rangle$. There is an isomorphism

$$\mathrm{THH}_*(B\langle n\rangle; k(m)) \cong E(\lambda_1, \dots, \lambda_{n-m}) \otimes (P(v_m) \oplus T^n_m),$$

where

$$T^n_m = \bigoplus_{s\geq 1} P_{r_n(s,m)}(v_m) \otimes E(\lambda_{n-m+s+1}, \dots, \lambda_{n+s}) \otimes P(\mu_{n+1}^{p^s}) \otimes \mathbb{F}_p\{\lambda_{n-m+s}\mu_{n+1}^{\ell p^{s-1}} \mid 0 \leq \ell \leq p-2\}$$

and by convention $E(\lambda_1, \dots, \lambda_{n-m}) = \mathbb{F}_p$ when $n = m$. The sequence of integers $r_n(s,m)$ is defined by

$$r_n(s,m) = p^{n-m+s} + p^{n-m+s-(m+1)} + \cdots + p^{n+j-m},$$

where j is the unique element in $\{1, \dots, m+1\}$ such that $s \equiv j \bmod m+1$.

Here the class λ_s is defined recursively by the formula

$$\lambda_s := \lambda_{s-(m+1)} \mu_{n+1}^{p^{s-(n+2)}(p-1)}$$

for $s \geq n+2$ and we name the classes in the abutment that are not divisible by v_n by their projection to $\mathrm{THH}_*(B\langle n\rangle; H\mathbb{F}_p)$.

Remark 5.7 When $m = 1$ and $n = 2$, we observe that this is consistent with Theorem 4.6 where $r_2(s,1) = r(s,1)$. When $m = 2$ and $n = 2$, we observe that this is consistent with Theorem 5.5 where $r_2(s,2) = r(s,2)$.

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