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RAMANUJAN SANTHAROUBANE

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Algebraic generators of the skein algebra of a surface

RAMANUJAN SANTHAROUBANE

Let Σ be a surface with negative Euler characteristic, genus at least one and at most one boundary component. We prove that the Kauffman bracket skein algebra of Σ over the field of rational functions can be algebraically generated by a finite number of simple closed curves that are naturally associated to certain generators of the mapping class group of Σ. The action of the mapping class group on the skein algebra gives canonical relations between these generators. From this, we conjecture a presentation for a skein algebra of Σ.

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1 Introduction

1.1 Main results

This paper is focused on finding algebraic generators of the Kauffman bracket skein algebra of a surface. Throughout this paper, we will refer to the Kauffman bracket skein algebra simply as the skein algebra. Let Σ be a compact oriented connected surface of genus at least one and with at most one boundary component. Moreover, we will suppose that Σ has negative Euler characteristic. We denote by $\mathcal{S}(\Sigma, \mathbb{Q}(A))$ the skein module of $\Sigma \times [0,1]$ with coefficients in the field of rational function $\mathbb{Q}(A)$ and by $S(\Sigma)$ the skein module over $\mathbb{Z}[A^{\pm 1}]$. The module $\mathcal{S}(\Sigma, \mathbb{Q}(A))$ is equipped with a natural product given by stacking banded links. For γ is a simple closed curve on Σ, we write γ for the element $\gamma \times \left[\frac{1}{2}, \frac{2}{3}\right]$ in $\mathcal{S}(\Sigma, \mathbb{Q}(A))$ and we denote by t_γ the Dehn twist along γ.

Theorem 1.1 *Let $\{\gamma_j\}_{j\in I}$ be a finite set of nonseparating simple closed curves such that*

(1) *for any $i, j \in I$, the curves γ_i and γ_j intersect at most once;*

(2) *the set $\{t_{\gamma_j}\}_{j\in I}$ generates the mapping class group of Σ.*

Then $\{\gamma_j\}_{j\in I}$ generates $\mathcal{S}(\Sigma, \mathbb{Q}(A))$ as a $\mathbb{Q}(A)$–algebra. Moreover, $\{\gamma_j\}_{j\in I}$ generates

$$S(\Sigma) \otimes \mathbb{Z}\left[A^{\pm 1}, \frac{1}{A^2 - A^{-2}}\right]$$

as a $\mathbb{Z}[A^{\pm 1}, 1/(A^2 - A^{-2})]$–algebra.

We recall that the mapping class group of Σ is $\pi_0(\mathrm{Homeo}^+(\Sigma, \partial\Sigma))$. We will now give an interpretation of some relations that should hold for the generators in the previous theorem. Let us fix $\{\gamma_j\}_{j\in I}$ a set of simple closed curves on Σ satisfying the hypothesis of Theorem 1.1. Let $\mathbb{Q}(A)\langle I\rangle$ be the free

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noncommutative $\mathbb{Q}(A)$–algebra generated by $\{X_j\}_{j\in I}$. The theorem says that there exists a surjective algebra homomorphism

$$\Psi\colon \mathbb{Q}(A)\langle I\rangle \to \mathcal{S}(\Sigma, \mathbb{Q}(A)) \tag{1}$$

defined by

$$\Psi(X_j) = \frac{\gamma_j}{A^2 - A^{-2}} \quad \text{for all } \ j \in I.$$

Now for $j \in I$ and $\epsilon \in \{-1, 1\}$, let $T_j^{\epsilon}\colon \mathbb{Q}(A)\langle I\rangle \to \mathbb{Q}(A)\langle I\rangle$ be the homomorphism of $\mathbb{Q}(A)$–algebra defined by

$$T_j^{\epsilon}(X_k) = \begin{cases} X_k & \text{if } \iota(\gamma_j, \gamma_k) = 0, \\ \epsilon(A^{\epsilon} X_j X_k - A^{-\epsilon} X_k X_j) & \text{if } \iota(\gamma_j, \gamma_k) = 1. \end{cases} \tag{2}$$

Here ι is the geometric intersection of simple closed curves. With the convention that a Dehn twist always turn to the right with respect to the orientation of the surface, we can check that Ψ exchanges the actions of the T_j and the t_j in the sense that

$$\Psi(T_j^{\epsilon} X) = t_{\gamma_j}^{\epsilon}(\Psi(X)) \quad \text{for all } \ X \in \mathbb{Q}(A)\langle I\rangle, \ j \in I, \ \text{and } \epsilon \in \{-1, 1\}.$$

Let $\Gamma(\Sigma)$ be the mapping class group of Σ and let $\overline{\Gamma}(\Sigma)$ be the group $\Gamma(\Sigma)$ modulo its center. Suppose that $I = \{1, \dots, N\}$ and let us denote each t_{γ_j} simply by t_j. Note that the map $t_j^{\epsilon} \mapsto T_j^{\epsilon}$ does not extend to an action of $\overline{\Gamma}(\Sigma)$ on $\mathbb{Q}(A)\langle I\rangle$. Indeed $\mathbb{Q}(A)\langle I\rangle$ is a free noncommutative algebra and the relations in $\overline{\Gamma}(\Sigma)$ satisfied by the Dehn twists $\{t_j^{\pm 1}\}_{j\in J}$ have no reason to be satisfied by the operators $\{T_j^{\pm 1}\}_{j\in J}$. Hence the relations between the $\{t_j^{\pm 1}\}_{j\in J}$ give relations between the generators $\{\gamma_j\}_{j\in J}$ in $\mathcal{S}(\Sigma, \mathbb{Q}(A))$. Suppose that $\overline{\Gamma}(\Sigma)$ has the following presentation with respect to the generators $\{t_j\}_{j\in I}$:

$$\overline{\Gamma}(\Sigma) = \langle t_1, \dots, t_N \mid R_1(t_1, \dots, t_N) = \cdots = R_K(t_1, \dots, t_N) = 1\rangle$$

where K is an integer and the $R_k(t_1, \dots, t_N)$ are some words in $\{t_j^{\pm 1}\}_{j\in I}$. Let $\mathcal{R}$ be the bi-ideal of $\mathbb{Q}(A)\langle I\rangle$ generated by the elements

$$\begin{aligned} R_k(T_1, \dots, T_N) X_i - X_i \quad & \text{for } \ 1 \le i \le N \ \text{ and } \ 1 \le k \le K, \\ T_j T_j^{-1} X_i - X_i \quad & \text{for } \ 1 \le i, j \le N, \\ X_i X_j - X_j X_i \quad & \text{for } \ \iota(\gamma_i, \gamma_j) = 0. \end{aligned}$$

We define

$$\mathcal{A}(\Gamma(\Sigma)) = \frac{\mathbb{Q}(A)\langle I\rangle}{\mathcal{R}}, \tag{3}$$

which is a quotient of $\mathbb{Q}(A)\langle I\rangle$ on which the actions of the $T_j^{\pm 1}$ extend to a canonical action of $\overline{\Gamma}(\Sigma)$. A direct consequence of Theorem 1.1 is the following:

Corollary 1.2 *The canonical map*

$$\mathcal{A}(\Gamma(\Sigma)) \to \mathcal{S}(\Sigma, \mathbb{Q}(A))$$

is surjective.

Conjecture 1.3 *There exists a presentation of $\overline{\Gamma}(\Sigma)$ for which $\mathscr{A}(\Gamma(\Sigma))$ is isomorphic to $\mathscr{S}(\Sigma,\mathbb{Q}(A))$ as a noncommutative $\mathbb{Q}(A)$–algebra.*

1.2 Notes and references

Bullock [1999] was the first to find algebraic generators of the skein algebra of a surface. His generators are over $\mathbb{Z}[A^{\pm1}]$ and not over $\mathbb{Q}(A)$. The number of his generators is exponential in the genus of the surface whereas here we have a linear number (by choosing the right generators of $\Gamma(\Sigma)$).

It was shown in [Przytycki and Sikora 2000] that each $\mathscr{S}(\Sigma)$ has a generating set of cardinality which is cubic in the genus of the surface.

Finite generation was also prove by Abdiel and Frohman [2017, Theorem 3.7]. Frohman and Kania-Bartoszynska [2018] studied the skein algebra when A is evaluated at a root of unity. They proved that it is generated over its center by a pair of subalgebras from pants decomposition. Their generators have some similarities with the one in the current paper.

Presentations of skein algebras of surfaces are only known in genus zero and one. Bullock and Przytycki [2000] found such a presentation for the one-holed torus, the four-holed sphere and two-holed torus. They related some of these algebras to nonstandard deformations of lie algebras.

When A is specialized to -1, it was shown by Bullock [1997] and Przytycki and Sikora [2000] that the skein algebra of a surface is isomorphic to the ring of algebraic functions of the $\mathrm{SL}(2,\mathbb{C})$ character variety of the surface. Moreover, for $A=\sqrt{-1}$, Marché [2011] gave an homological interpretation of the skein algebra of the surface. Note that the map Ψ defined in (1) is not defined if A is specialized to a 4^{th} primitive root of unity. It is possible to see that if we specialize A at a 4^{th} root of unity in the algebra $\mathscr{A}(\Gamma(\Sigma))$ we find something different from the algebras studied by Bullock, Marché, Przytycki and Sikora.

Humphries generators [1979] and Lickorish generators [1964] are examples of generators of the mapping class groups satisfying the hypothesis of Theorem 1.1. Moreover, presentations for both of these generating sets are known; we refer to book of Farb and Margalit [2012] for more details.

We consider $\Gamma(\Sigma)$ quotiented by its center because the center of the mapping class group acts trivially on the skein algebra of the surface.

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2 Quick review of the skein algebra

For any compact oriented manifold M (maybe with boundary), we denote by $\mathscr{S}(M)$ the Kauffman bracket skein module with coefficients in $\mathbb{Z}[A^{\pm1}]$. We recall that it is the free $\mathbb{Z}[A^{\pm1}]$–module generated by the

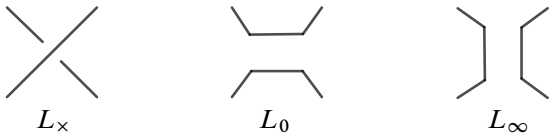

Figure 1: Kauffman triple.

set of isotopy classes of banded links in the interior of M quotiented by the following so-called skein relations. First

$$L_\times = AL_\infty + A^{-1}L_0 \tag{4}$$

where $L_\times$, L_0 and L_∞ are any three banded links in M which are the same outside a small 3–ball but differ inside as in Figure 1. In this case, the triple $L_\times, L_0, L_\infty$ is called a Kauffman triple. The second relation satisfied in $\mathcal{S}(M)$ is

$$L \cup D = -(A^2 + A^{-2})L$$

where L is any link in M and D is a trivial banded knot. We define $\mathcal{S}(M, \mathbb{Q}(A))$ to be the $\mathbb{Q}(A)$–vector space $\mathcal{S}(M) \otimes \mathbb{Q}(A)$ where the tensor product is made over $\mathbb{Z}[A^{\pm 1}]$.

Let Σ be a compact oriented connected surface; we denote by $\mathcal{S}(\Sigma, \mathbb{Q}(A))$ the space $\mathcal{S}(\Sigma \times [0,1], \mathbb{Q}(A))$. Stacking banded links on top of each other gives $\mathcal{S}(\Sigma, \mathbb{Q}(A))$ the structure of a $\mathbb{Q}(A)$–algebra.

A multiloop is a disjoint union of non-null-homotopic simple close curve inside Σ. For $\gamma \subset \Sigma$ a multiloop we write γ for the banded link $\gamma \times \left[\frac{1}{2}, \frac{2}{3}\right]$ in $\mathcal{S}(\Sigma, \mathbb{Q}(A))$ and we will still call this banded link a multiloop. A well-known theorem is the following:

Theorem 2.1 *The set of isotopy classes of multiloops on Σ is a basis of the $\mathbb{Q}(A)$–vector space $\mathcal{S}(\Sigma, \mathbb{Q}(A))$.*

In particular this theorem clearly implies that simple closed curves generate $\mathcal{S}(\Sigma, \mathbb{Q}(A))$ as an algebra. Recall that $\Gamma(\Sigma) = \pi_0(\mathrm{Homeo}^+(\Sigma, \partial\Sigma))$ acts canonically on $\mathcal{S}(\Sigma, \mathbb{Q}(A))$. If $\gamma \subset \Sigma$ is a simple closed curve, we denote by t_γ the Dehn twist along γ. We use the turn right convention for t_γ. More precisely, let $\mathcal{N} \subset \Sigma$ be an annulus neighborhood of γ, we can identify $\mathcal{N}$ with $S^1 \times [0,1]$ using an orientation preserving homeomorphism. Outside $\mathcal{N}$, the map t_γ is defined to be the identity and on $\mathcal{N}$ it is given by the map $(e^{i\theta}, t) \mapsto (e^{i(\theta - 2\pi t)}, t)$. Figure 2 helps to visualize this definition.

The following lemma can be obtained by applying the skein relation (4).

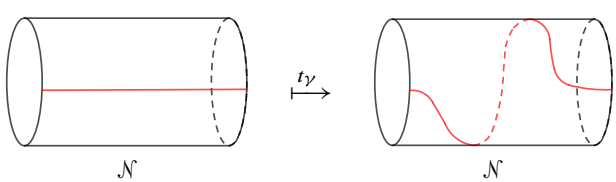

Figure 2

Lemma 2.2 *Let α and β be two simple close curves intersecting once. Then in $\mathcal{S}(\Sigma, \mathbb{Q}(A))$,*

$$t_\alpha^\epsilon(\beta) = \frac{A^\epsilon \alpha\beta - A^{-\epsilon}\beta\alpha}{\epsilon(A^2 - A^{-2})}.$$

3 Proof of Theorem 1.1

Let $\Gamma(\Sigma)$ be the mapping class group of Σ. Let $\{\gamma_j\}_{j\in I}$ be a set of simple closed curves satisfying the hypothesis of the Theorem 1.1 and let $\mathcal{B}$ be the subalgebra of $\mathcal{S}(\Sigma, \mathbb{Q}(A))$ generated by $\{\gamma_j\}_{j\in I}$.

Lemma 3.1 *$\mathcal{B}$ is stable by the action of $\Gamma(\Sigma)$.*

Proof Since $\{t_{\gamma_j}\}_{j\in I}$ generates $\Gamma(\Sigma)$, it enough to prove that for any $j, k \in I$ we have $t_{\gamma_j}^{\pm 1}(\gamma_k) \in \mathcal{B}$. If γ_j does not intersect γ_k then $t_{\gamma_j}^{\pm 1}(\gamma_k) = \gamma_k \in \mathcal{B}$. Now if γ_j intersects γ_k once then, by Lemma 2.2,

$$t_{\gamma_j}^{\pm 1}(\gamma_k) = \frac{A^{\pm 1}\gamma_j\gamma_k - A^{\mp 1}\gamma_k\gamma_j}{\pm(A^2 - A^{-2})} \in \mathcal{B}. \qquad \square$$

Lemma 3.2 *If $\mathcal{C}_0$ denotes the set of nonseparating simple closed curves then $\mathcal{C}_0 \subset \mathcal{B}$.*

Proof Let $\gamma \in \mathcal{C}_0$ and $\gamma_0 \in \{\gamma_j\}_{j\in I}$; there exists $\phi \in \Gamma(\Sigma)$ such that $\phi(\gamma_0) = \gamma$. Since γ_0 belongs to $\mathcal{B}$ which is stable by the action of $\Gamma(\Sigma)$ (see the previous lemma), we have $\gamma \in \mathcal{B}$. $\square$

Lemma 3.3 *If $\mathcal{C}_1$ denotes the set of separating simple closed curves then $\mathcal{C}_1 \subset \mathcal{B}$.*

Proof Suppose that the genus of Σ is $g \geq 1$. Let $\delta_1, \dots, \delta_g$ be the curves in Figure 3, where δ_g is trivial when Σ does not have boundary. Let $j \in \{1, \dots, g\}$ and let z_j and z_j' be the two nonseparating curves in the torus with two boundary components defined by δ_j, δ_{j-1}, as shown in Figure 4.

By applying the skein relations, we have

$$z_j' z_j = A^2 x_j x_j' + A^{-2} y_j y_j' + \delta_j + \delta_{j-1}$$

where $\delta_0 = -A^2 - A^{-2}$ and x_j, x_j', y_j and y_j' are nonseparating curves. By Lemma 3.2, z_j', z_j, x_j, x_j', y_j and y_j' are in $\mathcal{B}$, so by an induction on j we can prove that for all $1 \leq j \leq g$ we have $\delta_j \in \mathcal{B}$.

Now if γ is a separating curve, there exists $\phi \in \Gamma(\Sigma)$ and j_0 such that $\gamma = \phi(\delta_{j_0})$. Since $\mathcal{B}$ is stable by the action of $\Gamma(\Sigma)$, we have $\gamma \in \mathcal{B}$. $\square$

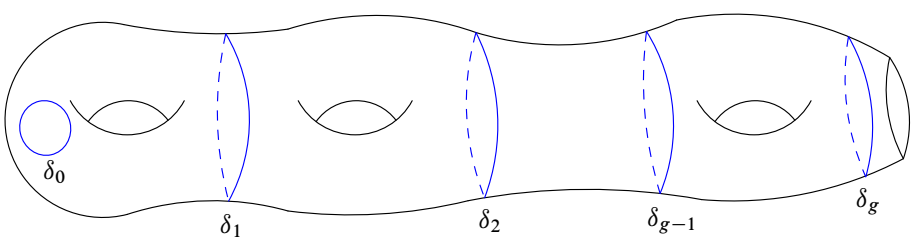

Figure 3

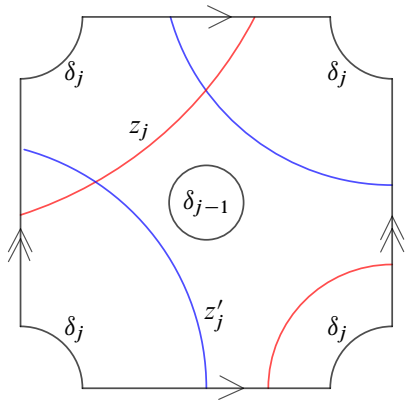

Figure 4

Proof of Theorem 1.1 $\mathcal{S}(\Sigma, \mathbb{Q}(A))$ is algebraically generated by simple closed curves. Hence, combining Lemmas 3.2 and 3.3, we can conclude the proof. Moreover, we can still conclude by Lemmas 3.1, 3.2 and 3.3 that the $\mathbb{Z}[A^{\pm 1}, 1/(A^2-A^{-2})]$–algebra generated by $\{\gamma_j\}_{j\in I}$ is

$$S(\Sigma) \otimes \mathbb{Z}\Big[A^{\pm 1}, \frac{1}{A^2-A^{-2}}\Big]. \qquad \square$$

4 Interpretation of the relations in the skein algebra

Let $\{\gamma_j\}_{j\in I}$ be a set of simple closed curves on Σ satisfying the hypothesis of Theorem 1.1. Recall that $\mathbb{Q}(A)\langle I\rangle$ is the free noncommutative $\mathbb{Q}(A)$–algebra generated by $\{X_j\}_{j\in I}$.

Definition 4.1 For $X, Y \in \mathbb{Q}(A)\langle I\rangle$ we define $[X, Y]_A := AXY - A^{-1}YX$.

Recall that the maps $\{T_j^{\epsilon}\}_{j\in I}$ are defined by (2) in the introduction. Recall also that given a presentation of $\overline{\Gamma}(\Sigma)$ with respected to the generating set $\{t_{\gamma_j}\}_{j\in I}$, we defined $\mathscr{A}(\Gamma(\Sigma))$ via (3) (see the introduction). By definition, any relation satisfied by the $\{t_{\gamma_j}\}_{j\in I}$ (which may not appear in the given presentation) gives some relation in $\mathscr{A}(\Gamma(\Sigma))$. Let us focus on the relations

$$T_j^{-1}T_i^{-1}T_j^{-1}T_iT_jT_iX_a - X_a = 0 \quad \text{for } \iota(\gamma_i,\gamma_j) = 1 \text{ and } a \in I, \tag{5}$$

$$T_jT_j^{-1}X_i - X_i = 0 \quad \text{for } i, j \in I, \tag{6}$$

$$X_iX_j - X_jX_i = 0 \quad \text{for } \iota(\gamma_i,\gamma_j) = 0. \tag{7}$$

Note that these relations hold but are not a complete set of relations in $\mathscr{A}(\Gamma(\Sigma))$. The first relation comes from the braid relations in the mapping class group.

Proposition 4.2 *In $\mathscr{A}(\Gamma(\Sigma))$, the relations (5), (6) and (7) are equivalent to*

$$[[X_j, X_i]_A, X_j]_A = X_i \quad \text{for } \iota(\gamma_i,\gamma_j) = 1, \tag{8}$$

$$X_iX_j - X_jX_i = 0 \quad \text{for } \iota(\gamma_i,\gamma_j) = 0. \tag{9}$$

Proof Let $i, j \in I$. Note that if $\iota(\gamma_i, \gamma_j) = 0$, the relation (6) is empty and if $\iota(\gamma_i, \gamma_j) = 1$, this relation gives $[[X_j, X_i]_A, X_j]_A = X_i$.

Let $i, j \in I$ such that $\iota(\gamma_i, \gamma_j) = 1$. Because of (6), the relation (5) can be rewritten as

$$T_i T_j T_i X_k = T_j T_i T_j X_k$$

for all $k \in I$. It is easy to check that this relation is implied by (8) and (9). □

Remark 4.3 We did not include $T_j^{-1} T_j X_i = X_i$ in the relations defining $\mathscr{A}(\Gamma(\Sigma))$ because they give $[X_j, [X_i, X_j]_A]_A = X_i$ for $\iota(\gamma_i, \gamma_j) = 1$ which is the same as $[[X_j, X_i]_A, X_j]_A = X_i$.

4.1 The case of the one-holed torus

Let Σ be a surface of genus one with one boundary component. Its mapping class group is the braid group B_3 whose presentation is $\langle t_1, t_2 \mid t_1 t_2 t_1 = t_2 t_1 t_2 \rangle$. Here is t_1 is Dehn twist along the canonical meridian of Σ and t_2 is the Dehn twist around the longitude of Σ. Note that these two curves satisfy the hypothesis of Theorem 1.1. The center of this group is the group generated by $(t_1 t_2 t_1)^2$ and $\overline{\Gamma}(\Sigma)$ is $\mathrm{PSL}_2(\mathbb{Z})$ with presentation

$$\overline{\Gamma}(\Sigma) = \langle t_1, t_2 \mid t_1 t_2 t_1 = t_2 t_1 t_2,\ (t_1 t_2 t_1)^2 = 1 \rangle.$$

In this case $\mathscr{A}(\Gamma(\Sigma))$ is a noncommutative algebra generated by X_1 and X_2. Because of Proposition 4.2, the only relations between X_1 and X_2 are

$$[X_1, [X_2, X_1]_A]_A = X_2, \quad [X_2, [X_1, X_2]_A]_A = X_1, \quad (T_1 T_2 T_1)^2 X_1 = X_1, \quad (T_1 T_2 T_1)^2 X_2 = X_2.$$

It is easy to check that the two last relations are implied by the two first one. Therefore,

$$\mathscr{A}(\Gamma(\Sigma)) = \langle X_1, X_2 \mid [X_1, [X_2, X_1]_A]_A = X_2,\ [X_2, [X_1, X_2]_A]_A = X_1 \rangle.$$

From [Bullock 1999, Theorem 2.1] the skein module of the one-holed torus is isomorphic to $\mathscr{A}(\Gamma(\Sigma))$. Therefore Conjecture 1.3 holds for the one-holed torus.

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Laboratoire Mathématiques d'Orsay, CNRS, Université Paris-Saclay
Orsay, France

Current address: *Department of Mathematics, University of Virginia*
Charlottesville, VA, United States

ramanujan.santharoubane@gmail.com

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