Algebraic & Geometric Topology

Volume 24 (2024)

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Algebraic & Geometric Topology **24**:5 (2024) 2579–2618
DOI: 10.2140/agt.2024.24.2579
Published: 19 August 2024

Bundle transfer of L–homology orientation classes for singular spaces

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We consider transfer maps on ordinary homology, bordism of singular spaces and homology with coefficients in Ranicki's symmetric L–spectrum, associated to block bundles with closed oriented PL manifold fiber and compact polyhedral base. We prove that if the base polyhedron is a Witt space, for example a pure-dimensional compact complex algebraic variety, then the symmetric L–homology orientation of the base, constructed by Laures, McClure and the author, transfers to the L–homology orientation of the total space. We deduce from this that the Cheeger–Goresky–MacPherson L–class of the base transfers to the product of the L–class of the total space with the cohomological L–class of the stable vertical normal microbundle.

55N33, 55R12, 57N80, 57Q50, 57R20; 57Q20

1 Introduction

To a fiber bundle $p\colon X \to B$ whose structure group is a compact Lie group acting smoothly on the compact smooth d–dimensional manifold fiber F, and whose base space B is a finite complex, Becker and Gottlieb associate in [8] a transfer homomorphism $p^!\colon H_n(B) \to H_{n+d}(X)$. Boardman discusses this transfer and several closely related constructions, such as the Umkehr map and pullback transfers, in [9]. Let $L^*(\alpha)$ denote the cohomological Hirzebruch L–class of a vector bundle α, and for a smooth closed oriented manifold M with tangent bundle TM, let $L_*(M) \in H_*(M;\mathbb{Q})$ denote the Poincaré dual of $L^*(TM)$. Suppose that F is oriented and the structure group of p acts in an orientation-preserving manner. If the base B of the fiber bundle is a smooth closed oriented manifold M, then

$$p^! L_*(M) = L^*(T_p)^{-1} \cap L_*(X), \tag{1}$$

where T_p is the vertical tangent bundle of p. This is a straightforward consequence of the bundle isomorphism $TX \cong p^*TM \oplus T_p$, naturality and the Whitney sum formula for the cohomological L–class, multiplicative properties of the transfer, and the fact that $p^!$ maps the fundamental class of the base to the fundamental class of the total space.

If the base B is a singular pseudomanifold then the above argument does not apply. On the other hand, intersection homology methods still allow for the construction of a homological L–class $L_*(B) \in H_*(B;\mathbb{Q})$ for many types of compact pseudomanifolds B: When B allows for a stratification with only even-codimensional strata, for example a pure-dimensional compact complex algebraic variety, $L_*(B)$ has been defined by Goresky and MacPherson in [21]. This construction has been extended by P Siegel [44] to Witt spaces, ie oriented polyhedral pseudomanifolds that may have strata of odd codimension such that the middle-dimensional middle-perversity rational intersection homology of the corresponding links vanishes. In [1; 2], the author has yet more generally defined $L_*(B)$ for topologically stratified spaces B that allow for Lagrangian structures along strata of odd codimension. A local definition of L–classes for triangulated pseudomanifolds with piecewise-flat metric was given by Cheeger [16] in terms of η–invariants of links. As for manifolds, the L–class of singular spaces plays an important role in the topological classification of such spaces, as shown by Cappell and Weinberger in [13] and by Weinberger in [49].

Let F be a closed oriented d–dimensional PL manifold, B a compact polyhedron and ξ an oriented PL F–block bundle over B; see Casson [15]. Oriented PL F–fiber bundles $p\colon X \to B$ are a special case. Block bundles, and hence our results here, do not require a locally trivial projection map p. Then ξ still admits a transfer homomorphism

$$\xi^!\colon H_n(B) \to H_{n+d}(X).$$

See Ebert and Randal-Williams [19] and Section 5. Furthermore, ξ possesses a stable vertical normal PL microbundle ν_ξ; see Hebestreit, Land, Lück and Randal-Williams [25] and Section 2. Here we develop methods that yield, among other results, formula (1) for F–block bundles over Witt spaces B:

Theorem 8.1 *Let B be a closed Witt space (eg a pure-dimensional compact complex algebraic variety) and let F be a closed oriented* PL *manifold. Let ξ be an oriented* PL *F–block bundle over B with total space X and oriented stable vertical normal microbundle ν_ξ over X. Then X is a Witt space, and the associated block bundle transfer $\xi^!$ sends the Cheeger–Goresky–MacPherson–Siegel L–class of B to the product*

$$\xi^! L_*(B) = L^*(\nu_\xi) \cap L_*(X). \tag{2}$$

Note that since the cohomological class $L^*(\nu_\xi)$ is invertible, this formula yields a method for computing the Cheeger–Goresky–MacPherson L–class of the total space.

Our method of proof rests on the geometric description of PL cobordism provided by Buoncristiano, Rourke and Sanderson [12] in terms of mock bundles. We construct a transfer $\xi^!\colon E_n(B) \to E_{n+d}(X)$ for any module spectrum E over the Thom spectrum MSPL of oriented PL bundle theory. In addition to the transfer on ordinary homology, this yields transfer homomorphisms on Ranicki's homology with coefficients in the symmetric $\mathbb{L}^\bullet$–spectrum and on Witt bordism theory, Ω_*^{Witt}. We describe the latter transfer geometrically as a pullback transfer and use this, together with mock bundle theory, to show that the Witt bordism transfer sends the fundamental class $[B]_{\mathrm{Witt}} \in \Omega_*^{\mathrm{Witt}}(B)$ to the fundamental class $[X]_{\mathrm{Witt}} \in \Omega_*^{\mathrm{Witt}}(X)$; see Proposition 6.8. Using work of Laures, McClure and the author [7], which provides a map of ring spectra $\mathrm{MWITT} \to \mathbb{L}^\bullet(\mathbb{Q})$, where MWITT represents Witt-bordism, as well as a fundamental class $[B]_{\mathbb{L}} \in \mathbb{L}^\bullet(\mathbb{Q})_*(B)$, we then show:

Theorem 7.1 *Let B be a closed Witt space of dimension n and let F be a closed oriented PL manifold of dimension d. Let ξ be an oriented PL F–block bundle over B with total space X. Then the $\mathbb{L}^\bullet$–homology block bundle transfer*

$$\xi^!\colon \mathbb{L}^\bullet(\mathbb{Q})_n(B) \to \mathbb{L}^\bullet(\mathbb{Q})_{n+d}(X)$$

maps the $\mathbb{L}^\bullet(\mathbb{Q})$–homology fundamental class of B to the $\mathbb{L}^\bullet(\mathbb{Q})$–homology fundamental class of X,

$$\xi^![B]_{\mathbb{L}} = [X]_{\mathbb{L}}.$$

The result on Cheeger–Goresky–MacPherson L–classes is then deduced from an explicit formula for the transfer by tensoring with the rationals. For a PL F–fiber bundle $p\colon X \to B$ over a PL manifold base B, the formula

$$p^![B]_{\mathbb{L}} = [X]_{\mathbb{L}} \in \mathbb{L}^\bullet(\mathbb{Z})_{n+d}(X)$$

was stated by Lück and Ranicki in [32]. The behavior of the L–class for singular spaces under transfers associated to finite-degree covering projections has already been clarified in [4], where we showed that for a closed oriented Whitney stratified pseudomanifold B admitting Lagrangian structures along strata of odd codimension (eg B Witt) and $p\colon X \to B$ an orientation-preserving covering map of finite degree, the L–class of B transfers to the L–class of the cover, ie

$$p^! L_*(B) = L_*(X).$$

For the Witt case, from our perspective this is a special case of (2).

An inclusion $g\colon Y \hookrightarrow X$ of stratified spaces is called normally nonsingular if Y possesses a tubular neighborhood in X that can be equipped with the structure of a real vector bundle; see eg work of Goresky and MacPherson [23] and the author [5]. An oriented normally nonsingular inclusion g of real codimension c has a Gysin map

$$g^!\colon H_*(X;\mathbb{Q}) \to H_{*-c}(Y;\mathbb{Q})$$

on ordinary homology,

$$g^!\colon \mathbb{L}^\bullet(\mathbb{Q})_*(X) \to \mathbb{L}^\bullet_{*-c}(\mathbb{Q})(Y)$$

on $\mathbb{L}^\bullet(\mathbb{Q})$–homology, and

$$g^!\colon \Omega_*^{\mathrm{Witt}}(X) \to \Omega_{*-c}^{\mathrm{Witt}}(Y)$$

on Witt bordism. In [5], we showed that if g is a normally nonsingular inclusion of closed oriented even-dimensional piecewise-linear Witt pseudomanifolds, for example pure-dimensional compact complex algebraic varieties, then

$$g^! L_*(X) = L^*(\nu_g) \cap L_*(Y), \tag{3}$$
$$g^![X]_{\mathbb{L}} = [Y]_{\mathbb{L}}, \quad g^![X]_{\mathrm{Witt}} = [Y]_{\mathrm{Witt}},$$

where ν_g is the normal bundle of g. These formulae have been applied in [5] to compute the Cheeger–Goresky–MacPherson L–class of certain singular Schubert varieties. No previous computations of such classes seem to be available in the literature. Together with the bundle transfer formula (2), this makes it possible to compute the transfer of the Cheeger–Goresky–MacPherson L–class associated to a normally nonsingular map, that is, a map which can be factored as a composition of a normally nonsingular inclusion, followed by the projection of an oriented PL F–fiber bundle ξ with closed PL manifold fiber F; see Section 9.

For complex algebraic, possibly singular, varieties X, Brasselet, Schürmann and Yokura [11] introduced Hodge-theoretic intersection Hirzebruch characteristic classes $\mathrm{IT}_{y,*}(X)$, which agree with $L_*(X)$ for $y = 1$ and X nonsingular or, more generally, a rational homology manifold; see de Bobadilla and Pallarés [10]. Using results of Schürmann [43] and Maxim and Schürmann [34], we established an algebraic version of (3) for $\mathrm{IT}_{1,*}$ in a context of appropriately normally nonsingular regular algebraic embeddings [5, Theorem 6.30]. Similarly, we expect IT_{1*} to satisfy a relation analogous to (2) for smooth algebraic morphisms $p\colon X \to B$, where ν_p would now be inverse to the algebraic relative tangent bundle $T_{X/B}$. Such a relation, together with our results here, then enable further comparison between the Hodge-theoretic class IT_{1*} and the topological class L_*. The aforementioned normally nonsingular maps form a topological parallel to the algebraic concept of a local complete intersection morphism, ie a morphism of varieties that can be factored into a closed regular embedding and a smooth morphism. Hence our results impact the behavior of topological characteristic classes under transfers associated to local complete intersection morphisms.

Acknowledgments This work is funded by the Deutsche Forschungsgemeinschaft (DFG, German research foundation) research grant 495696766.

2 Stable vertical normal block bundles

Block bundles with manifold fiber over compact polyhedra admit stable vertical normal closed disc block bundles; see eg [19; 25], as well as [12, page 83] for the more general mock bundle situation. We will use the vertical normal block bundle later in the description of the Umkehr map, and thus recall the construction in the form we need.

Let F be a closed oriented PL manifold of dimension d and let K be a finite ball complex with associated polyhedron $B = |K|$. (The polyhedron B is not required to be a manifold.) Let ξ be an oriented PL

F–block bundle over K (see Casson [15]) with total space $X = E(\xi)$. Let b denote the dimension of B so that $\dim X = d + b$. The block of ξ over a cell $\sigma \in K$ will be denoted by $\xi(\sigma)$. For every σ, there is a block-preserving PL homeomorphism $\xi(\sigma) \cong F \times \sigma$. Thus the blocks of ξ are compact PL manifolds with boundary

$$\xi(\partial\sigma) := \bigcup_{\tau\in\partial\sigma} \xi(\tau).$$

Over the interiors $\mathring{\sigma}$ of cells, we set $\xi(\mathring{\sigma}) := \xi(\sigma) - \xi(\partial\sigma)$.

In order to construct a stable vertical normal PL block bundle (and hence a stable PL microbundle, since $\mathrm{BSPL} \simeq \mathrm{B}\widetilde{\mathrm{SPL}}$) for ξ, choose a block-preserving PL embedding

$$\theta\colon X \hookrightarrow \mathbb{R}^s \times B$$

for sufficiently large $s > 2d + b + 1$, ie a PL embedding such that

$$\theta(\xi(\mathring{\sigma})) \subset \mathbb{R}^s \times \mathring{\sigma}$$

and

$$\theta|\colon (\xi(\sigma), \xi(\partial\sigma)) \to (\mathbb{R}^s \times \sigma, \mathbb{R}^s \times \partial\sigma)$$

is a locally flat PL embedding of manifolds for every simplex $\sigma \subset K$. One way to obtain such an embedding is to choose first a PL embedding $e\colon X \hookrightarrow \mathbb{R}^s$. By Casson [15, Lemma 6, page 43], ξ can be equipped with a choice of block fibration $p\colon X \to B$. This is a PL map such that $\xi(\sigma) = p^{-1}(\sigma)$ for every cell $\sigma \in K$. Then $\theta := (e, p)\colon X \hookrightarrow \mathbb{R}^s \times B$ is a block-preserving PL embedding. (The local flatness is ensured by requiring the codimension to be at least 3.) Another method to construct a block-preserving embedding θ is by induction over the cells $\sigma \in K$, starting with the 0–cells σ^0 and embeddings $\theta\colon \xi(\sigma^0) \cong F \subset \mathbb{R}^s \times \sigma^0 \cong \mathbb{R}^s$. These are then extended to proper embeddings of manifolds-with-boundary $\theta\colon \xi(\sigma^1) \subset \mathbb{R}^s \times \sigma^1$ for every 1–cell σ^1 in K, etc. As in [41], an embedding $j\colon M \to Q$ of manifolds is *proper* if $j^{-1}(\partial Q) = \partial M$.

Recall that one says that a PL embedding $j\colon A \to P$ of polyhedra possesses a *normal* PL *closed disc block bundle* if there exists a regular neighborhood N of $j(A)$ in P such that N is the total space of a PL closed disc block bundle over $j(A)$ whose zero section embedding agrees with the inclusion $j(A) \subset N$.

Proposition 2.1 *Let ξ be an F–block bundle over a finite cell complex K with polyhedron $B = |K|$, where F is a closed* PL *manifold. A block-preserving* PL *embedding $\theta\colon X \to \mathbb{R}^s \times B$ of the total space X of ξ possesses a normal* PL *closed $(s-d)$–disc block bundle ν_θ over $\theta(X)$. If ξ is oriented, then ν_θ is canonically oriented.*

Proof We begin by constructing a particular regular neighborhood N of $\theta(X)$ in $\mathbb{R}^s \times B$ such that N is compatible with the blocks $\theta(\xi(\sigma))$ and $\mathbb{R}^s \times \sigma$ for all cells $\sigma \in K$. There exists a locally finite simplicial complex L with subcomplexes $T, L_\sigma \subset L$ $(\sigma \in K)$ such that

(i) $|L| = \mathbb{R}^s \times B$,

(ii) $\theta(X) = |T|$,

(iii) for every cell σ in K, $\mathbb{R}^s \times \sigma = |L_\sigma|$, and

(iv) each simplex of L meets T in a single face or not at all.

It follows from (ii) and $\theta(\xi(\sigma)) \subset \mathbb{R}^s \times \sigma$ that the compact manifold $M_\sigma := \theta(\xi(\sigma))$ is triangulated by $L_\sigma \cap T$. The boundary of M_σ is triangulated by the subcomplex $L_{\partial\sigma} \cap T$ of L, where $L_{\partial\sigma}$ is the subcomplex of L given by

$$L_{\partial\sigma} = \bigcup_{\tau\in\partial\sigma} L_\tau.$$

Furthermore, (iv) implies that each simplex of L_σ meets $L_\sigma \cap T$ in a single face or not at all. Let $f\colon L \to [0,1] = \Delta^1$ be the unique simplicial map such that $f^{-1}(0) = |T|$. Then the preimage

$$N := f^{-1}\bigl[0, \tfrac{1}{2}\bigr] \subset \mathbb{R}^s \times B$$

is a regular neighborhood of $\theta(X)$ in $\mathbb{R}^s \times B$. The intersection $Q_\sigma := N \cap (\mathbb{R}^s \times \sigma)$ is a regular neighborhood of the manifold M_σ in the manifold $\mathbb{R}^s \times \sigma$. This regular neighborhood meets the boundary $\mathbb{R}^s \times \partial\sigma$ transversely, ie $N \cap (\mathbb{R}^s \times \partial\sigma)$ is a regular neighborhood of $\theta(\xi(\sigma)) \cap (\mathbb{R}^s \times \partial\sigma) = \theta(\xi(\partial\sigma))$ in $\mathbb{R}^s \times \partial\sigma$. The boundary of the compact manifold Q_σ is described by

$$\partial Q_\sigma = \bigl(f^{-1}\bigl(\tfrac{1}{2}\bigr) \cap (\mathbb{R}^s \times \sigma)\bigr) \cup \bigcup_{\tau\in\partial\sigma} Q_\tau \tag{4}$$

and M_σ is a proper submanifold of Q_σ.

We will construct a PL closed disc block bundle ν_θ over $\theta(X)$ by induction on the cells σ of K. The total space $E(\nu_\theta)$ of ν_θ is given by $E(\nu_\theta) := N$. Given a nonnegative integer n, we set

$$L_n := \bigcup_\sigma L_\sigma,$$

where the union is taken over all cells $\sigma \in K$ of dimension at most n. The corresponding polyhedron is $|L_n| = \mathbb{R}^s \times B^n$, where B^n denotes the n–skeleton of B. Set

$$Q_n := \bigcup_\sigma Q_\sigma \subset \mathbb{R}^s \times B^n,$$

where the union is taken over all cells $\sigma \in K$ of dimension at most n, so that

$$Q_n = N \cap (\mathbb{R}^s \times B^n).$$

For $n = b = \dim B$ we have $B^n = B$ and thus $Q_b = N$.

Let σ be a 0–cell of K. By [41, Theorem 4.3, page 16], there is a disc block bundle ν_σ over the complex $L_\sigma \cap T$ with total space $E(\nu_\sigma) = Q_\sigma$. Then the collection of blocks $\nu_\sigma(\beta)$, for $\beta \in L_\sigma \cap T$, of the bundles ν_σ endow Q_0 with the structure of a disc block bundle ν_0 over $L_0 \cap T$. Assume inductively that a block bundle ν_{n-1} over $L_{n-1} \cap T$ with total space

$$E(\nu_{n-1}) = Q_{n-1}$$

has been constructed such that for all cells $\sigma \in K$ with $\dim\sigma < n$, the restriction ν_σ of ν_{n-1} to the subcomplex $L_\sigma \cap T \subset L_{n-1} \cap T$ has total space $E(\nu_\sigma) = Q_\sigma$. Let $\sigma \in K$ be an n–cell. The pair $(M_\sigma, \partial M_\sigma)$ is triangulated by $(L_\sigma \cap T, L_{\partial\sigma} \cap T)$. Using the description (4) of ∂Q_σ, we have

$$E(\nu_{n-1}|_{L_{\partial\sigma}\cap T}) = \bigcup_{\tau\in\partial\sigma} E(\nu_{n-1}|_{L_\tau\cap T}) = \bigcup_{\tau} Q_\tau \subset \partial Q_\sigma.$$

Since Q_σ is a regular neighborhood of the compact manifold M_σ, there exists, again by [41, Theorem 4.3], a disc block bundle ν_σ over $L_\sigma \cap T$ with total space $E(\nu_\sigma) = Q_\sigma$ such that

$$\nu_\sigma|_{L_{\partial\sigma}\cap T} = \nu_{n-1}|_{L_{\partial\sigma}\cap T}.$$

Then the collection of blocks $\nu_\sigma(\beta)$, for $\beta \in L_\sigma \cap T$ and $\dim\sigma \le n$, of the bundles ν_σ endow $Q_n = \bigcup_{\dim\sigma\le n} Q_\sigma$ with the structure of a disc block bundle ν_n over $L_n \cap T$. By construction,

$$E(\nu_n|_{L_\sigma\cap T}) = E(\nu_\sigma) = Q_\sigma$$

for all $\sigma \in K$ and $\dim\sigma \le n$. This concludes the inductive step. For $n = b$, $\nu_\theta := \nu_b$ is a PL closed disc block bundle over $L_b \cap T = T$ with total space $E(\nu) = N$.

If P is an oriented codimension-0 submanifold of the boundary ∂M of an oriented manifold M, then the incidence number $\epsilon(P, M)$ is defined to be $+1$ if the orientation on P induced by the orientation of M agrees with the given orientation of P, and -1 otherwise. Suppose that ξ is oriented as an F–block bundle. Then K is an oriented cell complex and each block $\xi(\sigma)$ is oriented (as a manifold) so that $\epsilon(\xi(\tau), \xi(\sigma)) = \epsilon(\tau, \sigma)$ whenever τ is a codimension-1 face of a cell $\sigma \in K$. Requiring θ to be orientation preserving, we obtain orientations of all M_σ such that $\epsilon(M_\tau, M_\sigma) = \epsilon(\tau, \sigma)$. Give every $\mathbb{R}^s \times \sigma$ the product orientation determined by the orientation of the cell σ and the standard orientation of $\mathbb{R}^s$. Then the inclusion embeddings of oriented manifolds $M_\sigma \subset \mathbb{R}^s \times \sigma$ induce unique orientations of the normal bundles ν_σ. The above incidence number relation implies that these orientations fit together to give an orientation of ν_θ. □

Using the PL homeomorphism $\theta\colon X \to \theta(X)$, we may think of ν_θ as a bundle over X.

Proposition 2.2 *For s sufficiently large (compared to the dimension of X), the equivalence class of the disc block bundle ν_θ as constructed in Proposition 2.1 is independent of the choice of blockwise embedding $\theta\colon X \hookrightarrow \mathbb{R}^s \times B$, and thus only depends on the F–block bundle ξ.*

Proof Let $\theta, \theta'\colon X \hookrightarrow \mathbb{R}^s \times B$ be ξ–block-preserving PL embeddings. These give rise to vertical normal disc block bundles ν_θ and $\nu_{\theta'}$. The idea is to construct a $(\xi\times I)$–block-preserving concordance $\bar\theta\colon X \times I \hookrightarrow \mathbb{R}^s \times B \times I$ between θ and θ' and then apply Proposition 2.1 to endow inductively a suitable regular neighborhood of the image of the concordance with the structure of a disc block bundle, extending the disc block bundles ν_θ and $\nu_{\theta'}$. This implies that ν_θ and $\nu_{\theta'}$ are equivalent.

Observe that the equivalence class of the block bundle ν_θ does not change under passage to a simplicial subdivision L_0 of the complex L used in the proof of Proposition 2.1. This is a consequence of M Cohen's uniqueness theorem for regular neighborhoods in general polyhedra [17, Theorem 3.1, page 196] and Rourke and Sanderson's uniqueness theorem for disc block bundle structures [41, Theorem 4.4, page 16].

The cylinder $\overline{B} = B \times I$ is the polyhedron of the product cell complex $\overline{K} = K \times I$, where I carries the minimal cell structure. Let $\overline{X} = X \times I$. The product block bundle $\bar{\xi} := \xi \times I$ is an F–block bundle over the cell complex $\overline{K}$ with total space $E(\xi \times I) = \overline{X}$ and blocks $(\xi \times I)(\sigma \times \tau) = \xi(\sigma) \times \tau$, where σ is a cell in K and τ a cell of I. For sufficiently large s, by induction over the finitely many cells σ in $\overline{K}$, there exists a PL embedding

$$\bar{\theta}\colon \overline{X} \to \mathbb{R}^s \times \overline{B} = (\mathbb{R}^s \times B) \times I$$

such that $\bar{\theta}_0 = \theta \times 0$, $\bar{\theta}_1 = \theta' \times 1$, $\bar{\theta}(\bar{\xi}(\mathring{\sigma})) \subset \mathbb{R}^s \times \mathring{\sigma}$ and

$$\bar{\theta}|\colon (\bar{\xi}(\sigma), \bar{\xi}(\partial\sigma)) \to (\mathbb{R}^s \times \sigma, \mathbb{R}^s \times \partial\sigma)$$

is a locally flat PL embedding of manifolds for every cell $\sigma \subset \overline{K}$. Thus $\bar{\theta}$ is a block-preserving concordance between θ and θ' satisfying

$$\bar{\theta}(X \times I) \cap (\mathbb{R}^s \times B \times 0) = \theta(X) \times 0 \quad \text{and} \quad \bar{\theta}(X \times I) \cap (\mathbb{R}^s \times B \times 1) = \theta'(X) \times 1.$$

There exists a locally finite simplicial complex $\bar{L}$ with subcomplexes $\overline{T}, \bar{L}_\sigma \subset \bar{L}$, for $\sigma \in \overline{K}$, such that

(i) $|\bar{L}| = \mathbb{R}^s \times \overline{B}$,

(ii) the complexes $L \times 0$ and $L' \times 1$ used in constructing ν_θ and $\nu_{\theta'}$ are both subcomplexes of $\bar{L}$ such that

$$|L| = \mathbb{R}^s \times B \times 0 \quad \text{and} \quad |L'| = \mathbb{R}^s \times B \times 1,$$

(iii) $\bar{\theta}(\overline{X}) = |\overline{T}|$,

(iv) for every cell σ in $\overline{K}$,

$$\mathbb{R}^s \times \sigma = |\bar{L}_\sigma|,$$

(v) each simplex of $\bar{L}$ meets $\overline{T}$ in a single face or not at all.

(To achieve the fullness property, (v), it may be necessary to subdivide $L \times 0$ and $L' \times 1$, but we have observed earlier that this does not change the equivalence class of ν_θ or $\nu_{\theta'}$. Thus we may call the subdivisions $L \times 0$ and $L' \times 1$ again.) Let $f\colon L \to [0,1]$ be the unique simplicial map such that $f^{-1}(0) = |T| = \theta(X)$. The disc block bundle ν_θ over T has total space

$$E(\nu_\theta) = N = f^{-1}\big[0, \tfrac{1}{2}\big] \subset \mathbb{R}^s \times B,$$

a regular neighborhood of $\theta(X)$ in $\mathbb{R}^s \times B$. Let $f'\colon L' \to [0,1]$ be the unique simplicial map such that $f'^{-1}(0) = |T'| = \theta'(X)$. The disc block bundle $\nu_{\theta'}$ over T' has total space

$$E(\nu_{\theta'}) = N' = f'^{-1}\big[0, \tfrac{1}{2}\big] \subset \mathbb{R}^s \times B,$$

a regular neighborhood of $\theta'(X)$ in $\mathbb{R}^s \times B$. Let $\bar{f}: \bar{L} \to [0,1]$ be the unique simplicial map such that $\bar{f}^{-1}(0) = |\bar{T}| = \bar{\theta}(X \times I)$. By Proposition 2.1 and its proof, the regular neighborhood

$$\bar{N} := \bar{f}^{-1}\big[0, \tfrac{1}{2}\big] \subset \mathbb{R}^s \times B \times I$$

of $\bar{\theta}(X \times I)$ is the total space $E(\nu_{\bar{\theta}}) = \bar{N}$ of a PL disc block bundle $\nu_{\bar{\theta}}$ over $\bar{T}$ such that

$$\nu_{\bar{\theta}}|_{L \times 0} = \nu_\theta \quad \text{and} \quad \nu_{\bar{\theta}}|_{L' \times 1} = \nu_{\theta'}.$$

Thus, pulling back $\nu_{\bar{\theta}}$ to $X \times I$ along $\bar{\theta}$, we obtain a PL disc block bundle over $X \times I$ whose restriction to $X \times 0$ is ν_θ and whose restriction to $X \times 1$ is $\nu_{\theta'}$. This implies that ν_θ and $\nu_{\theta'}$ are equivalent as disc block bundles. □

The oriented normal block bundle ν_θ provided by Proposition 2.1 is classified by a map

$$X \to \mathrm{B}\widetilde{\mathrm{SPL}}_{s-d}\,.$$

If s is sufficiently large, then by Proposition 2.2 the homotopy class of this map does not depend on the choice of blockwise embedding θ. We denote the resulting disc block bundle equivalence class by ν_ξ and refer to it as the *stable vertical normal block bundle* of ξ. The restriction $s > b + 2d + 1$ ensures that the block bundle ν_ξ is in the stable range, there exists a unique (up to equivalence) oriented PL microbundle μ over X whose underlying block bundle is ν_ξ, and this microbundle is also in the stable range: since $\dim X = d + b < (s - d) - 1$, the natural map

$$[X, \mathrm{BSPL}_{s-d}] \cong [X, \mathrm{B}\widetilde{\mathrm{SPL}}_{s-d}]$$

is a bijection. We will refer to μ as the *stable vertical normal microbundle* of ξ.

Example 2.3 For the trivial F–block bundle ξ with total space $X = F \times B$, we may choose a PL embedding $\theta_F: F \hookrightarrow \mathbb{R}^s$, for s large, and take $\theta: X \hookrightarrow \mathbb{R}^s \times B$ to be $\theta = \theta_F \times \mathrm{id}_B: F \times B \hookrightarrow \mathbb{R}^s \times B$, which is ξ–block preserving. Let ν_F be the (stable) normal disc block bundle of θ_F and μ_F its unique lift to a PL microbundle. Then the stable vertical normal block bundle ν_ξ is represented by $\nu_\theta = \mathrm{pr}_1^* \nu_F$ and the stable vertical normal microbundle is $\mu = \mathrm{pr}_1^* \mu_F$, where $\mathrm{pr}_1: F \times B \to F$ is the factor projection.

Example 2.4 If F is a point, then $X = B$ and we may take $\theta: X = B \hookrightarrow \mathbb{R}^s \times B$ to be $\theta(x) = (0, x)$. The stable vertical normal block bundle ν_ξ and the stable vertical normal microbundle μ are both trivial.

3 The PL Umkehr map

Given an oriented F–block bundle ξ with nonsingular fiber F over a compact polyhedron and a module spectrum E over the Thom spectrum MSPL, we will construct a transfer homomorphism $\xi^!: E_n(B) \to E_{n+d}(X)$. This will be done in Section 5 by composing suspension, the PL Umkehr map $T(\xi)$ and the Thom homomorphism Φ. The Umkehr map will be constructed in the present section, and the Thom homomorphism in the next.

As in Section 2, let F be a closed oriented PL manifold of dimension d and let K be a finite ball complex with associated polyhedron $B = |K|$. Let ξ be an oriented PL F–block bundle over K with total space $X = E(\xi)$. Fix a block-preserving PL embedding $\theta\colon X \hookrightarrow \mathbb{R}^s \times B$ for sufficiently large s, and let us briefly write ν for the vertical normal disc block bundle ν_θ given by Proposition 2.1. As discussed in Section 2, there is a unique PL microbundle μ whose underlying block bundle is ν. The total space $E(\nu) = N$ is a ξ–block-preserving regular neighborhood of $\theta(X)$ in $B \times \mathbb{R}^s$. Let $\dot{\nu}$ denote the sphere block bundle of ν and write ∂N for the total space of $\dot{\nu}$. Let

$$\mathrm{Th}(\nu) := N \cup_{\partial N} \mathrm{cone}(\partial N)$$

be the Thom space of ν. The cone point in $\mathrm{Th}(\nu)$ will be denoted by ∞. Thom spaces of PL microbundles have been constructed by Williamson in [50]. By his construction, we may take $\mathrm{Th}(\mu) = \mathrm{Th}(\nu)$, since the underlying block bundle of μ is ν and the homotopy type of the Thom space depends only on the underlying block bundle (in fact only on the underlying spherical fibration).

We shall construct a PL map

$$T(\xi)\colon S^s B^+ = \mathrm{Th}(\mathbb{R}^s \times B) \to \mathrm{Th}(\nu),$$

called the *Umkehr map*, following the terminology of [8]. Points in $N \subset S^s B^+$ are to be mapped by the identity to points in $N \subset \mathrm{Th}(\nu)$. By Cohen's [17, Theorem 5.3], ∂N is collared in the closure of $(\mathbb{R}^s \times B) - N$. Thus there exists a polyhedral neighborhood V of ∂N in the closure of $(\mathbb{R}^s \times B) - N$ and a PL homeomorphism $h\colon (\partial N) \times [0,1] \cong V$ such that $h(x,0) = x$ for $x \in \partial N$. Now let $T(\xi)$ map those points of V that lie in $h((\partial N) \times \{1\})$ to ∞. Map all points in $S^s B^+ - (N \cup V)$ to ∞. Finally, map the points in V, using the collar coordinate in $[0,1]$, correspondingly along cone lines in $\mathrm{cone}(\partial N) \subset \mathrm{Th}(\nu)$. This concludes the description of the Umkehr map $T(\xi)\colon S^s B^+ \to \mathrm{Th}(\nu)$. Since it sends ∞ to ∞, this is a pointed map.

Example 3.1 We continue Example 2.3 on the trivial F–block bundle ξ. Let $T(F)\colon S^s \to \mathrm{Th}(\nu_F) = \mathrm{Th}(\mu_F)$ be the standard Pontryagin–Thom collapse over a point associated to the embedding $\theta_F : F \hookrightarrow \mathbb{R}^s$. The Umkehr map for ξ is given by

$$T(\xi)\colon S^s \wedge B^+ \xrightarrow{T(F)\wedge \mathrm{id}_{B^+}} \mathrm{Th}(\nu_F) \wedge B^+ = \mathrm{Th}(\nu_\theta).$$

If E is any spectrum, then on reduced E–homology the Umkehr map induces a homomorphism

$$T(\xi)_*\colon \widetilde{E}_{n+s}(S^s B^+) \to \widetilde{E}_{n+s}(\mathrm{Th}(\nu)).$$

The suspension isomorphism provides an identification

$$\sigma\colon E_n(B) = \widetilde{E}_n(B^+) \cong \widetilde{E}_{n+s}(S^s B^+).$$

The composition yields a map

$$T(\xi)_* \circ \sigma\colon E_n(B) \to \widetilde{E}_{n+s}(\mathrm{Th}(\nu)) = \widetilde{E}_{n+s}(\mathrm{Th}(\mu)).$$

Example 3.2 We continue Example 3.1 on the trivial F–block bundle ξ. Let E be a commutative ring spectrum and let $[S^s]_E \in \widetilde{E}_s(S^s)$ denote the image of the unit $1 \in \pi_0(E)$ under $\sigma\colon \widetilde{E}_0(S^0) \xrightarrow{\cong} \widetilde{E}_s(S^s)$. Then the above map $T(\xi)_* \circ \sigma$ has the description

$$T(\xi)_*\sigma(a) = (T(F) \wedge \mathrm{id}_{B+})_*\sigma(a) = (T(F) \wedge \mathrm{id}_{B+})_*([S^s]_E \wedge a) = (T(F)_*[S^s]_E) \wedge a,$$

where $a \in E_n(B)$. Setting $[\mathrm{Th}\,\mu_F]_E = T(F)_*[S^s]_E$, we thus arrive at

$$T(\xi)_*\sigma(a) = [\mathrm{Th}\,\mu_F]_E \wedge a.$$

4 The Thom homomorphism, mock bundles and Witt spaces

We recall the Thom homomorphism Φ associated to an oriented PL microbundle μ. This homomorphism will later be used in the definition of the F–block bundle transfer $\xi^!$ with μ the stable vertical normal PL microbundle of ξ. The Thom map is given by taking the cap product with the Thom class of μ. Therefore, we will recall the homotopy-theoretic description $u_{\mathrm{SPL}}(\mu)$ of this class, as well as its geometric description $u_{\mathrm{BRS}}(\mu_{\mathrm{PLB}})$ in terms of mock bundles, as given by Buoncristiano, Rourke and Sanderson [12], where μ_{PLB} denotes the underlying PL closed disc block bundle of μ. In particular, we take the opportunity to provide a brief review of mock bundle theory. Mock bundles over Witt spaces will play an important role later on. One key fact in the subsequent development is that the total space of a mock bundle over a Witt space is again a Witt space.

Let MSPL be the Thom spectrum associated to PL microbundles (or PL $(\mathbb{R}^m, 0)$–bundles; see Kuiper and Lashof [29]). This is a ring spectrum whose homotopy groups can be identified with the bordism groups of oriented PL manifolds via the Pontryagin–Thom isomorphism. Let γ_m^{SPL} denote the universal oriented rank-m PL bundle over the classifying space BSPL_m. An oriented PL microbundle μ of rank m over a compact polyhedron X is classified by a map $X \to \mathrm{BSPL}_m$, which is covered by a bundle map $\mu \to \gamma_m^{\mathrm{SPL}}$. The induced map on Thom spaces yields a homotopy class

$$u_{\mathrm{SPL}}(\mu) \in [\Sigma^\infty \mathrm{Th}(\mu), \Sigma^m \mathrm{MSPL}] = \widetilde{\mathrm{MSPL}}^m(\mathrm{Th}(\mu)).$$

This class $u_{\mathrm{SPL}}(\mu)$ is the *Thom class* of μ in oriented PL cobordism. It is in fact an MSPL–orientation of μ in Dold's sense. Indeed, every $H\mathbb{Z}$–orientable PL bundle is MSPL–orientable; see Hsiang and Wall [26, Lemma 5, page 357] and Switzer [46, page 308].

Buoncristiano, Rourke and Sanderson give a geometric description of MSPL–cobordism in [12], and use it to obtain in particular a geometric description of the Thom class $u_{\mathrm{SPL}}(\mu)$. The geometric cocycles are given by oriented mock bundles, whose definition we recall here.

Definition 4.1 Let K be a finite ball complex and q an integer (possibly negative). A *q–mock bundle* η^q/K with base K and total space $E(\eta)$ consists of a PL map $p\colon E(\eta) \to |K|$ such that, for each $\sigma \in K$, $p^{-1}(\sigma)$ is a compact PL manifold of dimension $q + \dim(\sigma)$, with boundary $p^{-1}(\partial\sigma)$. The preimage $\eta(\sigma) := p^{-1}(\sigma)$ is called the *block* of η over σ.

The empty set is regarded as a manifold of any dimension; thus $\eta(\sigma)$ may be empty for some cells $\sigma \in K$. Note that if σ^0 is a 0–dimensional cell of K, then $\partial\sigma^0 = \varnothing$ and thus $p^{-1}(\partial\sigma) = \varnothing$. Hence the blocks over 0–dimensional cells are *closed* manifolds. Mock bundles over the same complex are *isomorphic* if there exists a block-preserving PL homeomorphism of total spaces. (The homeomorphism is *not* required to preserve the projections.) For our purposes, we need *oriented* mock bundles, which are defined using incidence numbers of cells and blocks: Suppose that $(M^n, \partial M)$ is an oriented PL manifold and $(N^{n-1}, \partial N)$ is an oriented PL manifold with $N \subset \partial M$. Then an incidence number $\epsilon(N, M) = \pm 1$ is defined by comparing the orientation of N with that induced on N from M; $\epsilon(N, M) = +1$ if these orientations agree and -1 if they disagree. An *oriented cell complex* K is a cell complex in which each cell is oriented. We then have the incidence number $\epsilon(\tau, \sigma)$ defined for codimension-1 faces $\tau < \sigma \in K$.

Definition 4.2 An *oriented mock bundle* is a mock bundle η/K over an oriented (finite) ball complex K in which every block is oriented (ie is an oriented PL manifold) such that for each codimension-1 face τ of $\sigma \in K$, $\epsilon(\eta(\tau), \eta(\sigma)) = \epsilon(\tau, \sigma)$.

Using intersection homology, Witt spaces were introduced by Siegel in [44] as a geometric cycle reservoir representing KO–homology at odd primes. Sources on intersection homology include [21; 22; 27; 20; 3].

Definition 4.3 A *Witt space* is an oriented PL pseudomanifold where the links L^{2k} of odd-codimensional PL intrinsic strata have vanishing middle-perversity degree-k rational intersection homology,

$$\mathrm{IH}_k^{\bar{m}}(L; \mathbb{Q}) = 0.$$

For example, pure-dimensional complex algebraic varieties are Witt spaces, since they are oriented pseudomanifolds and possess a Whitney stratification whose strata all have even codimension. The vanishing condition on the intersection homology of links L^{2k} is equivalent to requiring the canonical morphism from lower middle to upper middle-perversity intersection chain sheaves to be an isomorphism in the derived category of sheaf complexes. Consequently, these middle-perversity intersection chain sheaves are Verdier self-dual, and this induces global Poincaré duality for the middle-perversity intersection homology groups of a Witt space. In particular, Witt spaces X have a well-defined bordism invariant signature and L–classes $L_*(X) \in H_*(X; \mathbb{Q})$ which agree with the Poincaré duals of Hirzebruch's tangential L–classes when X is smooth. The notion of *Witt spaces with boundary* can be introduced as pairs $(X, \partial X)$, where X is a PL space and ∂X a stratum-preservingly collared PL subspace of X such that $X - \partial X$ and ∂X are both compatibly oriented Witt spaces. The following result is [5, Lemma 3.11], which is itself an analog of [12, Lemma 1.2, page 21].

Lemma 4.4 *Let (K, L) be a finite ball complex pair such that the polyhedron $|K|$ is an n–dimensional compact Witt space with (possibly empty) boundary $\partial|K| = |L|$. Orient K in such a way that the sum of oriented n–balls is a cycle rel boundary. (This is possible since $|K|$, being a Witt space, is an oriented pseudomanifold with boundary.) Let η/K be an oriented q–mock bundle over K with projection p. Then the total space $E(\eta)$ is an $(n+q)$–dimensional compact Witt space with boundary $p^{-1}(\partial|K|)$.*

Let (K, L) be any finite ball complex pair. Oriented mock bundles η_0 and η_1 over K, both empty over L, are *cobordant*, if there is an oriented mock bundle η over $K \times I$, empty over $L \times I$, such that $\eta|_{K\times 0} \cong \eta_0$ and $\eta|_{K\times 1} \cong \eta_1$. This is an equivalence relation, and we set

$$\Omega^q_{\mathrm{SPL}}(K, L) := \{[\eta^q/K] : \eta|_L = \varnothing\},$$

where $[\eta^q/K]$ denotes the cobordism class of the oriented q–mock bundle η^q/K over K. Then by the duality theorem [12, Theorem II.3.3] of Buoncristiano, Rourke and Sanderson, Spanier–Whitehead duality, together with the Pontryagin–Thom isomorphism, provides an isomorphism

$$\beta\colon \Omega^{-q}_{\mathrm{SPL}}(K, L) \cong \mathrm{MSPL}^q(|K|, |L|) \tag{5}$$

for compact $|K|$ and $|L|$, which is natural with respect to inclusions $(K', L') \subset (K, L)$; see also [12, Remarks(3), top of page 32]. This is the geometric description of oriented PL cobordism that we use here. The functor $\Omega^*_{\mathrm{SPL}}(-)$ gives rise to a functor on the category of compact polyhedral pairs and homotopy classes of continuous maps, which will be denoted by the same symbol [12, Theorem II.1.1].

Let $\alpha\colon |K| \to \mathrm{B}\widetilde{\mathrm{SPL}}_m$ be an oriented PL closed disc block bundle of rank m over a finite complex K. Let N denote the total space of α and ∂N the total space of the sphere block bundle of α. Then α has a Thom class (see [12, page 26])

$$u_{\mathrm{BRS}}(\alpha) \in \Omega^{-m}_{\mathrm{SPL}}(N, \partial N), \tag{6}$$

which we shall call the BRS–*Thom class* of α, given as follows: Let $i : K \to N$ be the zero section of α. Endow N with the ball complex structure given by taking the blocks $\alpha(\sigma)$ of the bundle α as balls, together with the balls of a suitable ball complex structure on the total space ∂N of the sphere block bundle of α. Then $i : K \to N$ is the projection of an oriented $(-m)$–mock bundle η, and thus determines an element

$$u_{\mathrm{BRS}}(\alpha) = [\eta] \in \Omega^{-m}_{\mathrm{SPL}}(N, \partial N).$$

The block of η over a ball $\alpha(\sigma)$ of N is $\sigma \in K$. The following is [5, Lemma 3.14].

Lemma 4.5 *Let $\alpha\colon |K| \to \mathrm{BSPL}_m$ be an oriented* PL $(\mathbb{R}^m, 0)$*–bundle, with $|K|$ compact. Under the isomorphism β in (5), the* BRS*–Thom class $u_{\mathrm{BRS}}(\alpha_{\mathrm{PLB}})$ of the underlying oriented* PL *block bundle α_{PLB} of α gets mapped to the Thom class $u_{\mathrm{SPL}}(\alpha)$.*

Let E be an MSPL–module spectrum. Then there is a cap product

$$\cap\colon \mathrm{MSPL}^p(X, A) \otimes E_q(X, A) \to E_{q-p}(X).$$

The reduced cobordism group of the Thom space can be expressed as a relative group,

$$\widetilde{\mathrm{MSPL}}^p(\mathrm{Th}(\mu)) \cong \mathrm{MSPL}^p(N, \partial N),$$

where N, as in Section 3, is the total space of the underlying oriented PL closed disc block bundle of μ. Let

$$\rho_*\colon E_*(N) \xrightarrow{\cong} E_*(X)$$

be the inverse of the isomorphism induced on E–homology by the inclusion $X \hookrightarrow N$ of the zero section. Using the cap product

$$\cap\colon \mathrm{MSPL}^m(N,\partial N)\otimes E_q(N,\partial N)\to E_{q-m}(N)\overset{\rho_*}{\cong} E_{q-m}(X),$$

we obtain the *Thom homomorphism*

$$\Phi := \rho_*(u_{\mathrm{SPL}}(\mu)\cap -)\colon \widetilde{E}_q(\mathrm{Th}(\mu))\cong E_q(N,\partial N)\to E_{q-m}(X).$$

Under suitable conditions this map is an isomorphism, for example if X is connected, E is a ring spectrum and $u_{\mathrm{SPL}}(\mu)$ determines an E–orientation of μ. (See [46, page 309, Theorem 14.6]; recall that our X is a finite complex.)

5 Block bundle transfer

Let E be a module spectrum over the Thom spectrum MSPL of oriented PL bundle theory. As in Section 2, F denotes a closed oriented PL manifold of dimension d and K a finite ball complex with associated polyhedron $B = |K|$ of dimension b. Let ξ be an oriented PL F–block bundle over K with total space $X = E(\xi)$. Following Boardman [9] and Becker and Gottlieb [8], we shall construct a *transfer homomorphism*

$$\xi^!\colon E_n(B)\to E_{n+d}(X). \tag{7}$$

Let μ denote the stable oriented vertical normal PL microbundle of ξ whose underlying disc block bundle is ν_θ, the oriented vertical normal disc block bundle of the F–block bundle ξ, associated to a block-preserving embedding θ for ξ. The rank of μ and ν_θ is $m = s-d$ for $d = \dim F$ and s large. The block bundle transfer is defined to be the composition

$$E_n(B)\xrightarrow{T(\xi)_*\circ\sigma}\widetilde{E}_{n+s}(\mathrm{Th}(\mu))\xrightarrow{\Phi}E_{n+d}(X),$$

where σ is the suspension isomorphism, $T(\xi)$ is the Umkehr map of Section 3 and Φ is the Thom homomorphism of μ as described in Section 4.

Remark 5.1 The geometric description of block bundle transfer as provided above is serviceable for the subsequent PL geometric arguments concerning orientation classes of PL pseudomanifolds. We are grateful to a referee for pointing out that the geometric context embeds into a more general homotopy theoretic one as follows: A fibration π with Poincaré complex fiber over a base space B possesses a vertical Spivak fibration ν_π and a canonical map of spectra $\Sigma^\infty B^+\to \mathrm{Th}(\nu_\pi)$, provided one has ∞–categorical functoriality of the Pontryagin–Thom collapse associated to a Poincaré complex; see Carmeli, Cnossen, Ramzi and Yanovski [14] and Klein, Malkiewich and Ramzi [28]. Since the vertical Spivak fibration of an oriented block bundle with manifold blocks has a stable PL bundle reduction (by Hebestreit, Land, Lück and Randal-Williams [25]), it is MSPL–oriented.

We are mainly interested in the case where E is ordinary homology, Ranicki's symmetric $\mathbb{L}^\bullet$–spectrum or Witt bordism. Let us discuss each of these cases in turn.

5.1 Block transfer on ordinary homology

Let $H\mathbb{Z}$ denote the Eilenberg–Mac Lane spectrum of the ring $\mathbb{Z}$. The stable universal Thom class in $H^0(\mathrm{MSPL})$ yields a map $\alpha\colon \mathrm{MSPL} \to H\mathbb{Z}$, and this map is a ring map. Thus α makes the ring spectrum $H\mathbb{Z}$ into an MSPL–module by taking the action map to be

$$\mathrm{MSPL} \wedge H\mathbb{Z} \xrightarrow{\alpha\wedge \mathrm{id}} H\mathbb{Z} \wedge H\mathbb{Z} \xrightarrow{\mu_H} H\mathbb{Z},$$

where μ_H is the product on $H\mathbb{Z}$. The induced map

$$\alpha_*\colon \Omega_n^{\mathrm{SPL}}(Z) \cong \mathrm{MSPL}_n(Z) \to H_n(Z;\mathbb{Z})$$

is the Steenrod–Thom homomorphism sending the bordism class of a singular PL manifold $[f\colon M^n \to Z] \in \Omega_n^{\mathrm{SPL}}(Z)$ to $f_*[M] \in H_n(Z;\mathbb{Z})$. We recall the following standard fact:

Proposition 5.2 (Rudyak [42, Proposition V.1.6]) *Let $\tau\colon D \to E$ be a ring morphism of ring spectra. Let γ be an $(S^n,*)$–fibration equipped with a D–orientation $u_D \in \widetilde{D}^n(\mathrm{Th}\,\gamma)$. Then the image $\tau(u_D) \in \widetilde{E}^n(\mathrm{Th}\,\gamma)$ is an E–orientation of γ.*

We apply this to the ring morphism $\alpha\colon \mathrm{MSPL} \to H\mathbb{Z}$ and to our microbundle μ, which we had already equipped with the MSPL–orientation $u_{\mathrm{SPL}}(\mu)$. By the proposition, the homomorphism

$$\alpha\colon \widetilde{\mathrm{MSPL}}^{s-d}(\mathrm{Th}(\mu)) \to \widetilde{H}^{s-d}(\mathrm{Th}(\mu);\mathbb{Z})$$

sends $u_{\mathrm{SPL}}(\mu)$ to an $H\mathbb{Z}$–orientation

$$u_{\mathbb{Z}}(\mu) := \alpha(u_{\mathrm{SPL}}(\mu)) \in \widetilde{H}^{s-d}(\mathrm{Th}(\mu);\mathbb{Z}). \tag{8}$$

(This is the Thom class of μ in ordinary cohomology.) Another standard fact from stable homotopy theory is:

Lemma 5.3 *Let D and E be ring spectra, and $\tau\colon D \to E$ a ring morphism. We consider E as a D–module via the action map*

$$D \wedge E \xrightarrow{\tau\wedge \mathrm{id}} E \wedge E \xrightarrow{\mu_E} E.$$

This module structure yields a cap product $\cap_{D,E}\colon D^*(X) \otimes E_*(X) \to E_*(X)$. The ring structure on E yields a cap product $\cap_E\colon E^*(X) \otimes E_*(X) \to E_*(X)$. Then the diagram

$$\begin{tikzcd} D^*(X) \otimes E_*(X) \arrow[r, "\cap_{D,E}"] \arrow[d, "\tau\otimes\mathrm{id}"'] & E_*(X) \arrow[d, equal] \\ E^*(X) \otimes E_*(X) \arrow[r, "\cap_E"] & E_*(X) \end{tikzcd}$$

commutes.

By this lemma and (8), the transfer on ordinary homology is given by

$$\begin{aligned} \xi^!(-) &= \rho_*(u_{\mathrm{SPL}}(\mu) \cap_{\mathrm{MSPL},H\mathbb{Z}} T(\xi)_*\sigma(-)) = \rho_*(\alpha(u_{\mathrm{SPL}}(\mu)) \cap_{H\mathbb{Z}} T(\xi)_*\sigma(-)) \\ &= \rho_*(u_{\mathbb{Z}}(\mu) \cap_{H\mathbb{Z}} T(\xi)_*\sigma(-)). \end{aligned}$$

We summarize: the block bundle transfer (7) on ordinary homology $E = H\mathbb{Z}$ is given by

$$\xi^! = \rho_*(u_{\mathbb{Z}}(\mu) \cap T(\xi)_*\sigma(-))\colon H_n(B;\mathbb{Z}) \to H_{n+d}(X;\mathbb{Z}).$$

5.2 Block transfer on Witt bordism

Let $\Omega_*^{\mathrm{Witt}}(-)$ denote Witt bordism theory as defined by Siegel in [44]. Elements of $\Omega_n^{\mathrm{Witt}}(Z)$ are Witt bordism classes of continuous maps $f\colon W^n \to Z$ defined on an n–dimensional closed Witt space W. Let MWITT be the Quinn spectrum associated to the ad-theory of Witt spaces, representing Witt bordism via a natural equivalence

$$\mathrm{MWITT}_*(-) \cong \Omega_*^{\mathrm{Witt}}(-). \tag{9}$$

See Banagl, Laures and McClure [7]. A weakly equivalent spectrum was first considered by Curran [18]. He verified that this spectrum is an MSO–module [18, Theorem 3.6, page 117]. The product of two Witt spaces is again a Witt space. This implies essentially that MWITT is a ring spectrum; for more details see [7]. (There, we focused on the spectrum MIP representing bordism of integral intersection homology Poincaré spaces studied by Goresky and Siegel in [24] and by Pardon in [36], but everything works in an analogous, indeed simpler, manner for $\mathbb{Q}$–Witt spaces.) Every oriented PL manifold is a Witt space. Hence there is a map

$$\phi_W\colon \mathrm{MSPL} \to \mathrm{MWITT},$$

which, using the methods of ad-theories and Quinn spectra employed in [7], can be constructed to be multiplicative. Using this ring map, the spectrum MWITT becomes an MSPL–module with action map

$$\mathrm{MSPL} \wedge \mathrm{MWITT} \to \mathrm{MWITT}$$

given by the composition

$$\mathrm{MSPL} \wedge \mathrm{MWITT} \xrightarrow{\phi_W \wedge \mathrm{id}} \mathrm{MWITT} \wedge \mathrm{MWITT} \to \mathrm{MWITT}.$$

(The product of a Witt space and an oriented PL manifold is again a Witt space.) In particular, there is a cap product

$$\cap\colon \mathrm{MSPL}^j(Z,Y) \otimes \mathrm{MWITT}_n(Z,Y) \to \mathrm{MWITT}_{n-j}(Z) \tag{10}$$

and a transfer

$$\xi^!\colon \mathrm{MWITT}_n(B) \to \mathrm{MWITT}_{n+d}(X),$$

where ξ is our F–block bundle over B and $d = \dim F$.

Let C be any finite ball complex with subcomplex $D \subset C$ and suppose that $Z = |C|$ and $Y = |D|$. By Buoncristiano, Rourke and Sanderson [12], a geometric description of the above cap product (10) is given as follows: One uses the canonical identifications to think of the cap product as a product

$$\cap\colon \Omega_{\mathrm{SPL}}^{-j}(C,D) \otimes \Omega_n^{\mathrm{Witt}}(|C|,|D|) \to \Omega_{n-j}^{\mathrm{Witt}}(|C|).$$

Let us first discuss the absolute case $D = \varnothing$, and then return to the relative one. If C is simplicial, $f: W \to C$ is a simplicial map from an n–dimensional triangulated closed Witt space W to C, and η^q is a q–mock bundle over C (with $q = -j$), then one has (see [12, page 29])

$$[\eta^q/C] \cap [f: W \to |C|] = [g: E(f^*\eta) \to |C|] \in \Omega^{\mathrm{Witt}}_{n-j}(|C|),$$

where g is the diagonal arrow in the cartesian diagram

$$\begin{tikzcd} E(f^*\eta) \arrow[r] \arrow[d] \arrow[dr, "g"] & E(\eta) \arrow[d, "p"] \\ W \arrow[r, "f"] & C \end{tikzcd}$$

Here, one uses the fact (see [12, II.2, page 23]) that mock bundles over simplicial complexes admit pullbacks under simplicial maps. By Lemma 4.4, $E(f^*\eta)$ is a closed Witt space. For the relative case, we observe that if $(W, \partial W)$ is a compact Witt space with boundary, $f: (W, \partial W) \to (|C|, |D|)$ maps the boundary into $|D|$ and $\eta|_D = \varnothing$, then $f^*\eta|_{\partial W} = \varnothing$ and so $\partial E(f^*\eta) = \varnothing$, ie the Witt space $E(f^*\eta)$ is *closed*. Hence it defines an absolute bordism class.

In Section 6, we provide a more direct geometric description of the Witt bordism transfer

$$\xi^!: \mathrm{MWITT}_n(B) \to \mathrm{MWITT}_{n+d}(X)$$

as a pullback transfer $\xi^!_{\mathrm{PB}}: \Omega^{\mathrm{Witt}}_n(B) \to \Omega^{\mathrm{Witt}}_{n+d}(X)$.

5.3 Block transfer on $\mathbb{L}^\bullet$–homology

We write $\mathbb{L}^\bullet = \mathbb{L}^\bullet(\mathbb{Z}) = \mathbb{L}^\bullet\langle 0\rangle(\mathbb{Z})$ for Ranicki's connected symmetric algebraic L–spectrum with homotopy groups $\pi_n(\mathbb{L}^\bullet) = L^n(\mathbb{Z})$, the symmetric L–groups of the ring of integers; see eg [39]. Technically, we shall use the construction of $\mathbb{L}^\bullet$ as the Quinn spectrum of a suitable ad-theory; see Banagl, Laures and McClure [7]. That construction is weakly equivalent to Ranicki's. Localization $\mathbb{Z} \to \mathbb{Q}$ induces a map $\epsilon_{\mathbb{Q}}: \mathbb{L}^\bullet(\mathbb{Z}) \to \mathbb{L}^\bullet(\mathbb{Q})$, and $\pi_n(\mathbb{L}^\bullet(\mathbb{Q})) = L^n(\mathbb{Q})$ with

$$L^n(\mathbb{Q}) \cong \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/_2)^\infty \oplus (\mathbb{Z}/_4)^\infty & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{if } n \not\equiv 0 \pmod 4. \end{cases}$$

The spectra $\mathbb{L}^\bullet(\mathbb{Z})$ and $\mathbb{L}^\bullet(\mathbb{Q})$ are ring spectra. Let MSTOP be the Thom spectrum associated to oriented topological $(\mathbb{R}^n, 0)$–bundles. There is a canonical forget map

$$\phi_F: \mathrm{MSPL} \to \mathrm{MSTOP}.$$

Ranicki [37, page 290] constructed a map

$$\sigma^*: \mathrm{MSTOP} \to \mathbb{L}^\bullet,$$

and in [7], we constructed a map

$$\tau: \mathrm{MWITT} \to \mathbb{L}^\bullet(\mathbb{Q}).$$

(Actually, we even constructed an integral map MIP $\to \mathbb{L}^\bullet$, where MIP represents bordism of integral intersection homology Poincaré spaces, but everything works in the same manner for Witt theory, if one uses the $\mathbb{L}^\bullet$–spectrum with rational coefficients.) This map is multiplicative, ie a ring map, as shown in [7, Section 12], and the diagram

(11)
$$\begin{tikzcd} & \mathrm{MSTOP} \arrow[r, "\sigma^*"] & \mathbb{L}^\bullet(\mathbb{Z}) \arrow[d, "\epsilon_{\mathbb{Q}}"] \\ \mathrm{MSPL} \arrow[ur, "\phi_F"] \arrow[dr, "\phi_W"'] & & \\ & \mathrm{MWITT} \arrow[r, "\tau"] & \mathbb{L}^\bullet(\mathbb{Q}) \end{tikzcd}$$

homotopy commutes, since it comes from a commutative diagram of ad-theories under applying the symmetric spectrum functor $\boldsymbol{M}$ of Laures and McClure [31]. Using the ring map $\tau\phi_W : \mathrm{MSPL} \to \mathbb{L}^\bullet(\mathbb{Q})$, the spectrum $\mathbb{L}^\bullet(\mathbb{Q})$ becomes an MSPL–module with action map

$$\mathrm{MSPL} \wedge \mathbb{L}^\bullet(\mathbb{Q}) \to \mathbb{L}^\bullet(\mathbb{Q})$$

given by the composition

$$\mathrm{MSPL} \wedge \mathbb{L}^\bullet(\mathbb{Q}) \xrightarrow{(\tau\phi_W)\wedge \mathrm{id}} \mathbb{L}^\bullet(\mathbb{Q}) \wedge \mathbb{L}^\bullet(\mathbb{Q}) \to \mathbb{L}^\bullet(\mathbb{Q}).$$

The associated transfer is

$$\xi^! : \mathbb{L}^\bullet(\mathbb{Q})_n(B) \to \mathbb{L}^\bullet(\mathbb{Q})_{n+d}(X),$$

with ξ our F–block bundle over B and $d = \dim F$.

We shall show that the block bundle transfer $\xi^!$ commutes with the passage, under τ_*, from Witt bordism theory to $\mathbb{L}^\bullet(\mathbb{Q})$–homology. The homotopy commutative diagram

$$\begin{tikzcd} \mathrm{MSPL} \wedge \mathrm{MWITT} \arrow[r, "\mathrm{id}\wedge\tau"] \arrow[d, "\phi_W\wedge\mathrm{id}"'] & \mathrm{MSPL} \wedge \mathbb{L}^\bullet(\mathbb{Q}) \arrow[d, "(\tau\phi_W)\wedge\mathrm{id}"] \\ \mathrm{MWITT} \wedge \mathrm{MWITT} \arrow[r, "\tau\wedge\tau"] \arrow[d] & \mathbb{L}^\bullet(\mathbb{Q}) \wedge \mathbb{L}^\bullet(\mathbb{Q}) \arrow[d] \\ \mathrm{MWITT} \arrow[r, "\tau"] & \mathbb{L}^\bullet(\mathbb{Q}) \end{tikzcd}$$

shows that $\tau : \mathrm{MWITT} \to \mathbb{L}^\bullet(\mathbb{Q})$ is an MSPL–module morphism. In the proof of Lemma 5.5, we use the following standard fact:

Lemma 5.4 *If E is a ring spectrum, F and F' are module spectra over E and $\phi\colon F \to F'$ is an E–module morphism, then the diagram*

$$\begin{tikzcd} E^m(X,A) \otimes F_n(X,A) \arrow[r, "\cap"] \arrow[d, "\mathrm{id}\otimes\phi_*"'] & F_{n-m}(X) \arrow[d, "\phi_*"] \\ E^m(X,A) \otimes F'_n(X,A) \arrow[r, "\cap"] & F'_{n-m}(X) \end{tikzcd}$$

commutes: if $u \in E^m(X,A)$ and $a \in F_n(X,A)$, then

$$\phi_*(u \cap a) = u \cap \phi_*(a).$$

Lemma 5.5 *The Thom homomorphisms Φ of an oriented* PL *microbundle μ of rank m over a compact polyhedron X commute with the passage from Witt bordism to $\mathbb{L}^\bullet(\mathbb{Q})$–homology, that is, the diagram*

$$\begin{tikzcd} \widetilde{\mathrm{MWITT}}_n(\mathrm{Th}(\mu)) \arrow[r, "\Phi"] \arrow[d, "\tau_*"'] & \mathrm{MWITT}_{n-m}(X) \arrow[d, "\tau_*"'] \\ \widetilde{\mathbb{L}^\bullet}(\mathbb{Q})_n(\mathrm{Th}(\mu)) \arrow[r, "\Phi"] & \mathbb{L}^\bullet(\mathbb{Q})_{n-m}(X) \end{tikzcd}$$

commutes.

Proof As τ_* is a natural transformation of homology theories, it commutes with the isomorphism ρ_*. Since τ is an MSPL–module morphism, Lemma 5.4 applies to give

$$\tau_*\Phi = \tau_*\rho_*(u \cap -) = \rho_*\tau_*(u \cap -) = \rho_*(u \cap \tau_*(-)) = \Phi\tau_*,$$

where $u = u_{\mathrm{SPL}}(\mu)$. □

Proposition 5.6 *Let F be a closed oriented d–dimensional* PL *manifold and let ξ be an oriented F–block bundle with total space X over the compact polyhedral base B. Then the diagram*

$$\begin{tikzcd} \mathrm{MWITT}_n(B) \arrow[r, "\xi^!"] \arrow[d, "\tau_*"'] & \mathrm{MWITT}_{n+d}(X) \arrow[d, "\tau_*"'] \\ \mathbb{L}^\bullet(\mathbb{Q})_n(B) \arrow[r, "\xi^!"] & \mathbb{L}^\bullet(\mathbb{Q})_{n+d}(X) \end{tikzcd}$$

commutes.

Proof Let μ be the stable vertical normal microbundle of ξ. The right-hand square of the diagram

$$\begin{tikzcd} \mathrm{MWITT}_n(B) \arrow[r, "\cong", "\sigma"'] \arrow[d, "\tau_*"'] & \widetilde{\mathrm{MWITT}}_{n+s}(S^s B^+) \arrow[r, "T(\xi)_*"] \arrow[d, "\tau_*"'] & \widetilde{\mathrm{MWITT}}_{n+s}(\mathrm{Th}(\mu)) \arrow[d, "\tau_*"] \\ \mathbb{L}^\bullet(\mathbb{Q})_n(B) \arrow[r, "\cong", "\sigma"'] & \widetilde{\mathbb{L}^\bullet}(\mathbb{Q})_{n+s}(S^s B^+) \arrow[r, "T(\xi)_*"] & \widetilde{\mathbb{L}^\bullet}(\mathbb{Q})_{n+s}(\mathrm{Th}(\mu)) \end{tikzcd}$$

commutes, as τ_* is a natural transformation of homology theories. The left-hand square, involving the suspension isomorphism σ, commutes for the same reason. The statement now follows from Lemma 5.5. □

An oriented topological $(\mathbb{R}^m, 0)$–bundle α over a CW complex Z, classified by a map $Z \to \mathrm{BSTOP}_m$, possesses a Thom class

$$u_{\mathrm{STOP}}(\alpha) \in \widetilde{\mathrm{MSTOP}}^m(\mathrm{Th}(\alpha))$$

in oriented topological cobordism. The next auxiliary result on compatibility of Thom classes is standard; see eg [5, Lemma 3.7].

Lemma 5.7 *Let α be an oriented* PL $(\mathbb{R}^m, 0)$*–bundle. On cobordism groups, the homomorphism*

$$\phi_F : \widetilde{\mathrm{MSPL}}^m(\mathrm{Th}(\alpha)) \to \widetilde{\mathrm{MSTOP}}^m(\mathrm{Th}(\alpha_{\mathrm{TOP}}))$$

induced by the canonical map $\phi_F : \mathrm{MSPL} \to \mathrm{MSTOP}$ sends the Thom class of α to the Thom class of the underlying oriented topological $(\mathbb{R}^m, 0)$–bundle α_{STOP},

$$\phi_F(u_{\mathrm{SPL}}(\alpha)) = u_{\mathrm{STOP}}(\alpha_{\mathrm{STOP}}).$$

Following [37, pages 290–291], an oriented topological $(\mathbb{R}^m, 0)$–bundle α has a canonical $\mathbb{L}^\bullet$–cohomology orientation

$$u_{\mathbb{L}}(\alpha) \in \widetilde{\mathbb{L}^\bullet}^m(\mathrm{Th}(\alpha)),$$

which we shall also refer to as the $\mathbb{L}^\bullet$–*cohomology Thom class* of α, defined by

$$u_{\mathbb{L}}(\alpha) := \sigma^*(u_{\mathrm{STOP}}(\alpha)). \tag{12}$$

The morphism of spectra $\epsilon_{\mathbb{Q}}\colon \mathbb{L}^\bullet(\mathbb{Z}) \to \mathbb{L}^\bullet(\mathbb{Q})$ coming from localization induces a homomorphism

$$\widetilde{\mathbb{L}^\bullet}^m(\mathrm{Th}(\alpha)) \to \widetilde{\mathbb{L}^\bullet}(\mathbb{Q})^m(\mathrm{Th}(\alpha)).$$

We denote the image of $u_{\mathbb{L}}(\alpha)$ under this map again by $u_{\mathbb{L}}(\alpha) \in \widetilde{\mathbb{L}^\bullet}(\mathbb{Q})^m(\mathrm{Th}(\alpha))$.

Lemma 5.8 *Let α be an oriented* PL $(\mathbb{R}^m, 0)$*–bundle. The homomorphism*

$$\tau\phi_W\colon \widetilde{\mathrm{MSPL}}^m(\mathrm{Th}(\alpha)) \to \mathbb{L}^\bullet(\mathbb{Q})^m(\mathrm{Th}(\alpha))$$

induced by the ring morphism $\tau\phi_W\colon \mathrm{MSPL} \to \mathbb{L}^\bullet(\mathbb{Q})$ sends the MSPL*–cohomology Thom class of α to the $\mathbb{L}^\bullet$–cohomology Thom class of* (*the underlying topological bundle of*) α,

$$\tau\phi_W(u_{\mathrm{SPL}}(\alpha)) = u_{\mathbb{L}}(\alpha).$$

Proof By Lemma 5.7, Ranicki's definition (12) and the homotopy commutativity of (11),

$$\tau\phi_W(u_{\mathrm{SPL}}(\alpha)) = \epsilon_{\mathbb{Q}}\sigma^*\phi_F(u_{\mathrm{SPL}}(\alpha)) = \epsilon_{\mathbb{Q}}\sigma^*(u_{\mathrm{STOP}}(\alpha_{\mathrm{STOP}})) = u_{\mathbb{L}}(\alpha_{\mathrm{STOP}}). \qquad \square$$

Lemma 5.8, together with Lemma 5.3, implies that the F–block bundle transfer on $\mathbb{L}^\bullet(\mathbb{Q})$–homology is given by

$$\xi^! = \rho_*(u_{\mathbb{L}}(\mu) \cap T(\xi)_*\sigma(-))\colon \mathbb{L}^\bullet(\mathbb{Q})_n(B) \to \mathbb{L}^\bullet(\mathbb{Q})_{n+d}(X).$$

Example 5.9 We continue Example 3.2 and compute the transfer for the trivial F–block bundle ξ with total space $X = F \times B$. Let E be a commutative ring spectrum and $\phi\colon \mathrm{MSPL} \to E$ a morphism of ring spectra, equipping E with the structure of an MSPL–module. Recall that we had chosen a PL embedding $\theta_F\colon F \hookrightarrow \mathbb{R}^s$ with s large enough that θ_F has a tubular neighborhood given by a PL microbundle μ_F which represents the stable normal PL microbundle of F. The stable vertical normal bundle of ξ is then given by $\mu = \mathrm{pr}_1^*\mu_F$. Its Thom class $u_{\mathrm{SPL}}(\mu) \in \widetilde{\mathrm{MSPL}}^{s-d}(\mathrm{Th}(\mu)) = \widetilde{\mathrm{MSPL}}^{s-d}(\mathrm{Th}(\mu_F) \wedge B^+)$ is $u_{\mathrm{SPL}}(\mu) = u_{\mathrm{SPL}}(\mu_F) \wedge 1$, since the bundle map $\mu \to \gamma^{\mathrm{SPL}}_{s-d}$ factors as $\mu \to \mu_F \to \gamma^{\mathrm{SPL}}_{s-d}$, where the first map covers the projection $\mathrm{pr}_1\colon F \times B \to F$ and the second map the classifying map for μ_F. The element $\phi(u_{\mathrm{SPL}}(\mu_F))$ is an E–orientation of μ_F [42, Proposition V.1.6] and thus $[F]_E := \rho_{F*}\big(\phi(u_{\mathrm{SPL}}(\mu_F)) \cap [\mathrm{Th}\,\mu_F]_E\big) \in E_d(F)$, with $\rho_{F*}\colon E_d(N_F) \cong E_d(F)$, is an E–homology orientation for the PL manifold F [42, Proposition V.2.8; 46, page 333, Lemma 14.40]. The transfer $\xi^!\colon E_n(B) \to E_{n+d}(F \times B)$ is then given on $a \in E_n(B)$ by

$$\xi^!(a) = [F]_E \times a, \tag{13}$$

as follows from the calculation

$$\begin{aligned}\xi^!(a) &= \Phi T(\xi)_*\sigma(a) = \Phi([\mathrm{Th}\,\mu_F]_E \wedge a) = \rho_*(\phi(u_{\mathrm{SPL}}(\mu)) \cap ([\mathrm{Th}\,\mu_F]_E \wedge a)) \\ &= \rho_*(\phi(u_{\mathrm{SPL}}(\mu_F) \wedge 1) \cap ([\mathrm{Th}\,\mu_F]_E \wedge a)) = \rho_*((\phi(u_{\mathrm{SPL}}(\mu_F)) \wedge 1) \cap ([\mathrm{Th}\,\mu_F]_E \wedge a)) \\ &= \rho_*((\phi(u_{\mathrm{SPL}}(\mu_F)) \cap [\mathrm{Th}\,\mu_F]_E) \times (1 \cap a)) = \rho_{F*}(\phi(u_{\mathrm{SPL}}(\mu_F)) \cap [\mathrm{Th}\,\mu_F]_E) \times a = [F]_E \times a.\end{aligned}$$

6 Geometric pullback transfer on bordism

As in previous sections, F is a closed d–dimensional oriented PL manifold and ξ is an oriented PL F–block bundle with total space X over a finite ball complex K, where $B = |K|$. We shall geometrically construct a pullback transfer

$$\xi^!_{\mathrm{PB}} \colon \Omega^{\mathrm{Witt}}_n(B) \to \Omega^{\mathrm{Witt}}_{n+d}(X)$$

on Witt bordism. Let $f \colon W \to B$ be a continuous map representing an element $[f]$ of $\Omega^{\mathrm{Witt}}_n(B)$. Choose a PL map $g \colon W \to B$ homotopic to f. We follow Casson's method for pulling back F–block bundles [15]. (Note that the pullback of block and mock bundles is not generally defined through cartesian diagrams.) There is a compact polyhedron V and a factorization

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[dr, "g"'] & B \times V \arrow[d, "\mathrm{pr}_1"] \\ & B \end{tikzcd}$$

of g into a PL embedding j followed by a standard projection. Let L be a cell complex with $|L| = V$. The F–block bundle pullback $\mathrm{pr}_1^*\,\xi$ is by definition $\xi \times L$, an F–block bundle over the cell complex $K \times L$ with total space $E(\mathrm{pr}_1^*\,\xi) = X \times V$. Thus the first factor projection $X \times V \to X$ defines a PL map

$$\mathrm{pr}_1 \colon E(\mathrm{pr}_1^*\,\xi) \to X.$$

Let C be the product cell complex $C := K \times L$ and put $\eta := \mathrm{pr}_1^*\,\xi$. Let C' be a subdivision of C such that the subpolyhedron $j(W) \subset V \times B$ is given by $j(W) = |D'|$ for a subcomplex D' of C'. Block bundles can be subdivided, and this does not change the total space [15, page 37]. Let η' over C' be a subdivision of η, $E(\eta') = E(\eta)$. Block bundles can be restricted to subcomplexes. The total space of the restriction is given by the union of the blocks over the cells of the subcomplex. Thus we can restrict η' to the subcomplex D' of C' and obtain an F–block bundle $\eta'|_{D'}$ whose total space is a PL subspace $E(\eta'|_{D'}) \hookrightarrow E(\eta') = E(\eta)$. The composition

$$E(\eta'|_{D'}) \hookrightarrow E(\eta) \to X$$

gives a map

$$\bar{g} \colon E(\eta'|_{D'}) \to X. \tag{14}$$

Let $j^*\eta$ be the F–block bundle over W that corresponds to $\eta'|_{D'}$ under the PL homeomorphism $j \colon W \cong j(W)$. The pullback F–block bundle $g^*\xi$ is then defined to be

$$g^*\xi = j^*\eta = j^*(\mathrm{pr}_1^*\,\xi).$$

Thus (14) is a map
$$\bar{g}\colon E(g^*\xi) \to X.$$
Note that $E(g^*\xi)$ is a compact polyhedron. In the above construction of pullbacks and $\bar{g}$, it was not important that the Witt domain W has empty boundary; everything applies to compact W with boundary as well. Indeed, Casson's pullback applies of course to PL maps with general polyhedral domain. Let ξ and ξ' be F–block bundles over cell complexes K and K' such that $|K| = B = |K'|$. Recall that ξ and ξ' are called *equivalent* if, for some common subdivision K'' of K and K', the subdivision of ξ over K'' is isomorphic to the subdivision of ξ' over K''. (An *isomorphism* of F–block bundles over the same complex is a block-preserving homeomorphism of total spaces.) An equivalence $\phi\colon \xi \cong \xi'$ of F–block bundles over B induces an equivalence
$$g^*\phi\colon g^*\xi \cong g^*\xi'$$
such that
$$\begin{tikzcd} E(g^*\xi) \arrow[r, "\bar{g}"] \arrow[d, "g^*\phi"', "\cong"] & E(\xi) = X \arrow[d, "\cong"', "\phi"] \\ E(g^*\xi') \arrow[r, "\bar{g}'"] & E(\xi') = X' \end{tikzcd}$$
commutes.

Lemma 6.1 *Let $g\colon W \to B$ be a* PL *map defined on a compact Witt space with possibly nonempty boundary ∂W. Then the total space $E(g^*\xi)$ is a closed Witt space with boundary $E((g^*\xi)|_{\partial W})$.*

Proof An F–block bundle is in particular a mock bundle. Thus $g^*\xi$ is a mock bundle over the Witt space W and the result follows from Lemma 4.4. □

By Lemma 6.1, the map $\bar{g}\colon E(g^*\xi) \to X$ represents an element $[\bar{g}] \in \Omega^{\mathrm{Witt}}_{n+d}(X)$.

For future reference and additional clarity in subsequent arguments, let us record explicitly:

Lemma 6.2 *Let W and W' be closed n–dimensional Witt spaces. If $f \simeq f'\colon W \to X$ are homotopic maps, then $[f] = [f'] \in \Omega^{\mathrm{Witt}}_n(X)$. If $\phi\colon W \xrightarrow{\cong} W'$ is a* PL *homeomorphism, and $f\colon W \to X$ and $f'\colon W' \to X$ maps such that $f' \circ \phi = f$, then $[f] = [f'] \in \Omega^{\mathrm{Witt}}_n(X)$.*

Proof The first statement, asserting homotopy invariance, is part of the fact that Witt bordism constitutes a homology theory and is proven by considering a homotopy as a Witt bordism, noting that the cylinder on a closed Witt space is a Witt space with boundary. The bordism required by the second statement is given by taking a cylinder on the domain of the PL homeomorphism and a cylinder on the target of the PL homeomorphism, and then gluing the two cylinders using the homeomorphism. □

Lemma 6.3 *The class*
$$[\bar{g}\colon E(g^*\xi) \to X] \in \Omega^{\mathrm{Witt}}_{n+d}(X)$$
depends only on the Witt class $[g] \in \Omega^{\mathrm{Witt}}_n(B)$.

Proof Let $g_0\colon W_0 \to B$ and $g_1\colon W_1 \to B$ be PL maps such that $[g_0] = [g_1] \in \Omega_n^{\mathrm{Witt}}(B)$. Let $G\colon W \to B$ be a Witt bordism between g_0 and g_1; we may assume G to be PL. Let $i_j\colon W_j \hookrightarrow W$ denote the boundary inclusions for $j = 0, 1$. Since $g_j = G \circ i_j$, there is an equivalence $g_j^*\xi \cong i_j^* G^*\xi$ such that

$$\begin{tikzcd} E(g_j^*\xi) \arrow[d, "\cong"'] \arrow[dr, "\bar{g}_j"] & \\ E(i_j^* G^*\xi) \arrow[r, "\overline{Gi}_j"] & E(\xi) = X \end{tikzcd}$$

commutes. Thus $\bar{g}_j$ and $\overline{Gi}_j$ are Witt bordant, for $j = 0, 1$, by Lemma 6.2. According to Lemma 6.1, $E(G^*\xi)$ is a compact Witt space with boundary $E(i_0^*G^*\xi) \sqcup E(i_1^*G^*\xi)$. The diagram

$$\begin{tikzcd} E(i_j^* G^*\xi) \arrow[d, hook] \arrow[dr, "\overline{Gi}_j"] & \\ E(G^*\xi) \arrow[r, "\overline{G}"] & E(\xi) = X \end{tikzcd}$$

commutes for $j = 0, 1$. Hence, $\overline{G}$ is a Witt bordism between $\overline{Gi}_0$ and $\overline{Gi}_1$. $\square$

We define the *geometric transfer* (or *pullback transfer*)

$$\xi^!_{\mathrm{PB}}\colon \Omega_n^{\mathrm{Witt}}(B) \to \Omega_{n+d}^{\mathrm{Witt}}(X)$$

by

$$\xi^!_{\mathrm{PB}}[g\colon W \to B] = [\bar{g}\colon E(g^*\xi) \to X],$$

where g is a PL representative of the bordism class. Let

$$\xi^!_{\mathrm{BRS}}\colon \Omega_n^{\mathrm{Witt}}(B) \to \Omega_{n+d}^{\mathrm{Witt}}(X)$$

be the map

$$\xi^!_{\mathrm{BRS}}[g] := \rho_*(u_{\mathrm{BRS}}(\nu) \cap T(\xi)_*\sigma[g]),$$

where $\nu = \nu_\xi\colon X \to \mathrm{B}\widetilde{\mathrm{SPL}}_{s-d}$ represents the stable vertical normal PL disc block bundle of ξ. This is a technical intermediary; in terms of their respective definitions, the difference between $\xi^!_{\mathrm{BRS}}$ and $\xi^!$ is that the former uses the Thom class $u_{\mathrm{BRS}}(\nu)$, while the latter uses the Thom class $u_{\mathrm{SPL}}(\mu)$. We will eventually see that $\xi^!_{\mathrm{PB}} = \xi^!_{\mathrm{BRS}} = \xi^!$ on Witt bordism. Towards that goal, let us first investigate the behavior of both the pullback transfer and the BRS–transfer under standard factor projections.

Proposition 6.4 *Let B and D be compact polyhedra. Let $\xi \times D$ denote the F–block bundle over $B \times D$ obtained by pulling back ξ under the projection $\mathrm{pr}_1\colon B \times D \to B$. Then the diagrams*

$$\begin{tikzcd} \Omega_n^{\mathrm{Witt}}(B) \arrow[r, "\xi^!_{\mathrm{PB}}"] & \Omega_{n+d}^{\mathrm{Witt}}(X) \\ \Omega_n^{\mathrm{Witt}}(B \times D) \arrow[u, "\mathrm{pr}_{1*}"] \arrow[r, "(\xi\times D)^!_{\mathrm{PB}}"] & \Omega_{n+d}^{\mathrm{Witt}}(X \times D) \arrow[u, "\mathrm{pr}_{1*}"'] \end{tikzcd} \tag{15}$$

and

(16)
$$\begin{tikzcd} \Omega_n^{\mathrm{Witt}}(B) \arrow[r, "\xi^!_{\mathrm{BRS}}"] & \Omega_{n+d}^{\mathrm{Witt}}(X) \\ \Omega_n^{\mathrm{Witt}}(B \times D) \arrow[u, "\mathrm{pr}_{1*}"] \arrow[r, "(\xi\times D)^!_{\mathrm{BRS}}"] & \Omega_{n+d}^{\mathrm{Witt}}(X \times D) \arrow[u, "\mathrm{pr}_{1*}"'] \end{tikzcd}$$

commute.

Proof We will first establish the commutativity of (15) involving the pullback transfers. Recall that the F–block bundle ξ is given over a cell complex K with $|K| = B$. Let J be a cell complex with polyhedron $|J| = D$. Then $\xi \times D$ is an F–block bundle over the cell complex $K \times J$. Let $[g] \in \Omega_n^{\mathrm{Witt}}(B \times D)$ be any element, represented by a PL map $g\colon W \to B \times D$. Choose a compact polyhedron V and a factorization of g as

(17)
$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[dr, "g"'] & (B \times D) \times V \arrow[d, "\mathrm{pr}_{B\times D}"] \\ & B \times D \end{tikzcd}$$

Let L be a cell complex with $|L| = V$. We will compute $\xi^!_{\mathrm{PB}}\,\mathrm{pr}_{1*}[g]$. The element $\mathrm{pr}_{1*}[g]$ is represented by $\mathrm{pr}_1 \circ g$ with factorization

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[dr, "\mathrm{pr}_1\circ g"'] & B \times D \times V \arrow[d, "\mathrm{pr}_B"] \\ & B \end{tikzcd}$$

The pullback $\mathrm{pr}_B^*\,\xi = \xi \times J \times L$ has total space $E(\mathrm{pr}_B^*\,\xi) = X \times D \times V$ which projects to X via

$$\mathrm{pr}_X\colon E(\mathrm{pr}_B^*\,\xi) = X \times D \times V \to X = E(\xi).$$

Let C be the cell complex $C = K \times J \times L$ and let C' be a subdivision of C such that $j(W)$ is given by $j(W) = |D'|$ for some subcomplex D' of C'. Let $(\mathrm{pr}_B^*\,\xi)'$ be the block bundle over C' obtained by subdivision of $\mathrm{pr}_B^*\,\xi$. Then $(\mathrm{pr}_B^*\,\xi)'$ can be restricted to D', and the total space of this restriction $(\mathrm{pr}_B^*\,\xi)'|_{D'}$ is a subspace of $E((\mathrm{pr}_B^*\,\xi)') = E(\mathrm{pr}_B^*\,\xi)$. The composition of the subspace inclusion with pr_X yields a map

$$\overline{\mathrm{pr}_1 \circ g}\colon E((\mathrm{pr}_B^*\,\xi)'|_{D'}) \subset E(\mathrm{pr}_B^*\,\xi) = X \times D \times V \xrightarrow{\mathrm{pr}_X} X$$

such that

$$\xi^!_{\mathrm{PB}}[\mathrm{pr}_1 \circ g] = [\overline{\mathrm{pr}_1 \circ g}].$$

Let us compute $(\xi \times D)^!_{\mathrm{PB}}[g]$. The relevant factorization is (17); the pullback

$$\mathrm{pr}_{B\times D}^*(\xi \times D) = (\xi \times D) \times L = \xi \times J \times L = \mathrm{pr}_B^*\,\xi$$

has total space $E(\mathrm{pr}_{B\times D}^*(\xi \times D)) = X \times D \times V$ which projects to $X \times D$ via

$$\mathrm{pr}_{X\times D}\colon E(\mathrm{pr}_{B\times D}^*(\xi \times D)) = (X \times D) \times V \to X \times D = E(\xi \times D).$$

Let $(\mathrm{pr}^*_{B\times D}(\xi\times D))'$ be the block bundle over C' obtained by subdivision of $\mathrm{pr}^*_{B\times D}(\xi\times D)$. Then $(\mathrm{pr}^*_{B\times D}(\xi\times D))' = (\mathrm{pr}^*_B\,\xi)'$, and thus

$$(\mathrm{pr}^*_{B\times D}(\xi\times D))'|_{D'} = (\mathrm{pr}^*_B\,\xi)'|_{D'}.$$

Consider the commutative diagram

$$\begin{tikzcd} E((\mathrm{pr}^*_{B\times D}(\xi\times D))'|_{D'}) \arrow[r, hook] \arrow[d, equal] & E(\mathrm{pr}^*_{B\times D}(\xi\times D)) \arrow[r, "\mathrm{pr}_{X\times D}"] \arrow[d, equal] & X\times D \arrow[d, "\mathrm{pr}_1"] \\ E((\mathrm{pr}^*_B\,\xi)'|_{D'}) \arrow[r, hook] & E(\mathrm{pr}^*_B\,\xi) \arrow[r, "\mathrm{pr}_X"] & X \end{tikzcd}$$

The upper horizontal composition is a map $\bar g$ such that

$$(\xi\times D)^!_{\mathrm{PB}}[g] = [\bar g],$$

and the lower horizontal composition is $\overline{\mathrm{pr}_1\circ g}$. Therefore,

$$\mathrm{pr}_{1*}(\xi\times D)^!_{\mathrm{PB}}[g] = [\mathrm{pr}_1\circ\bar g] = [\overline{\mathrm{pr}_1\circ g}] = \xi^!_{\mathrm{PB}}[\mathrm{pr}_1\circ g] = \xi^!_{\mathrm{PB}}\,\mathrm{pr}_{1*}[g].$$

Thus (15) commutes, as claimed.

It remains to establish the commutativity of (16). Let $\nu_\xi = \nu_\theta$ denote the stable vertical normal PL disc block bundle associated to a particular choice of blockwise embedding $\theta\colon X = E(\xi)\hookrightarrow \mathbb{R}^s\times B$ by Proposition 2.1. The PL embedding

$$E(\xi\times D) = X\times D \xhookrightarrow{\theta\times\mathrm{id}_D} (\mathbb{R}^s\times B)\times D = \mathbb{R}^s\times(B\times D)$$

is block preserving over $B\times D$ with respect to the F–blocks $(\xi\times D)(\sigma\times\tau) = \xi(\sigma)\times\tau$ of $\xi\times D$ for $\sigma\in K$ and $\tau\in J$, as

$$(\theta\times\mathrm{id}_D)(\xi(\sigma)\times\tau) = \theta(\xi(\sigma))\times\tau\subset(\mathbb{R}^s\times\sigma)\times\tau = \mathbb{R}^s\times(\sigma\times\tau).$$

Thus the stable vertical normal disc-block bundle of $\xi\times D$ can be computed from the embedding $\theta\times\mathrm{id}_D$, which yields

$$\nu_{\xi\times D} = \nu_{\theta\times\mathrm{id}_D} = \nu_\theta\times D = \nu_\xi\times D,$$

a disc block bundle over $X\times D$. Recall that the Thom space of the block bundle ν_ξ is

$$\mathrm{Th}(\nu_\xi) = N\cup_{\partial N}\mathrm{cone}(\partial N),$$

with $N = E(\nu_\xi)$. Thus, with $N' := E(\nu_{\xi\times D}) = N\times D$, we have

$$\mathrm{Th}(\nu_{\xi\times D}) = N'\cup_{\partial N'}\mathrm{cone}(\partial N').$$

Here $\partial N'$ denotes the total space of the sphere bundle of $\xi\times D$, $\partial N' = (\partial N)\times D$. The projection $\mathrm{pr}_1\colon N' = N\times D\to N$ induces a map

$$\mathrm{Th}(\mathrm{pr}_1)\colon (N\times D)\cup_{(\partial N)\times D}\mathrm{cone}((\partial N)\times D)\to N\cup_{\partial N}\mathrm{cone}(\partial N),$$

ie a map

$$\mathrm{Th}(\mathrm{pr}_1)\colon \mathrm{Th}(\nu_{\xi\times D})\to\mathrm{Th}(\nu_\xi).$$

The suspension of the projection $\mathrm{pr}_1 : B \times D \to B$ is a map $S^s\,\mathrm{pr}_1 : S^s(B \times D)^+ \to S^s B^+$. The F–block bundle $\xi \times D$ has its Umkehr map

$$T(\xi \times D)\colon S^s(B \times D)^+ \to \mathrm{Th}(\nu_{\xi\times D})$$

such that the diagram

$$\begin{tikzcd} S^s B^+ \arrow[r, "T(\xi)"] & \mathrm{Th}(\nu_\xi) \\ S^s(B \times D)^+ \arrow[u, "S^s\,\mathrm{pr}_1"] \arrow[r, "T(\xi\times D)"] & \mathrm{Th}(\nu_{\xi\times D}) \arrow[u, "\mathrm{Th}(\mathrm{pr}_1)"'] \end{tikzcd}$$

commutes up to homotopy. The induced diagram

$$\begin{tikzcd} \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(S^s B^+) \arrow[r, "T(\xi)_*"] & \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(\mathrm{Th}\,\nu_\xi) \\ \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(S^s(B \times D)^+) \arrow[u, "(S^s\,\mathrm{pr}_1)_*"] \arrow[r, "T(\xi\times D)_*"] & \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(\mathrm{Th}\,\nu_{\xi\times D}) \arrow[u, "\mathrm{Th}(\mathrm{pr}_1)_*"'] \end{tikzcd}$$

on reduced Witt bordism commutes. The diagram

$$\begin{tikzcd} \Omega^{\mathrm{Witt}}_n(B) \arrow[r, "\sigma", "\cong"'] & \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(S^s B^+) \\ \Omega^{\mathrm{Witt}}_n(B \times D) \arrow[u, "\mathrm{pr}_{1*}"] \arrow[r, "\sigma", "\cong"'] & \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(S^s(B \times D)^+) \arrow[u, "(S^s\,\mathrm{pr}_1)_*"'] \end{tikzcd}$$

commutes by the naturality of the suspension isomorphism σ.

It remains to show that

$$\begin{tikzcd} \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(\mathrm{Th}\,\nu_\xi) \arrow[r, "u_{\mathrm{BRS}}(\nu_\xi)\cap-"] & \Omega^{\mathrm{Witt}}_{n+d}(E\nu_\xi) \arrow[r, "\rho_*", "\cong"'] & \Omega^{\mathrm{Witt}}_{n+d}(X) \\ \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(\mathrm{Th}\,\nu_{\xi\times D}) \arrow[u, "\mathrm{Th}(\mathrm{pr}_1)_*"] \arrow[r, "u_{\mathrm{BRS}}(\nu_{\xi\times D})\cap-"] & \Omega^{\mathrm{Witt}}_{n+d}(E\nu_{\xi\times D}) \arrow[u, "\mathrm{pr}_{1*}"] \arrow[r, "\rho_*", "\cong"'] & \Omega^{\mathrm{Witt}}_{n+d}(X \times D) \arrow[u, "\mathrm{pr}_{1*}"'] \end{tikzcd} \tag{18}$$

commutes. The right-hand side commutes, since the zero section embedding $X \times D \hookrightarrow E\nu_{\xi\times D} = N \times D$ of $\nu_{\xi\times D}$ is given by $i \times \mathrm{id}_D$, where i is the zero section embedding $i\colon X \hookrightarrow E\nu_\xi = N$ of ν_ξ, so that

$$\begin{tikzcd} N = E\nu_\xi & X \arrow[l, hook', "i"'] \\ N \times D = E\nu_{\xi\times D} \arrow[u, "\mathrm{pr}_1"] & X \times D \arrow[u, "\mathrm{pr}_1"'] \arrow[l, hook', "i\times \mathrm{id}_D"'] \end{tikzcd}$$

commutes. We will prove that the left-hand side commutes as well. The map

$$\mathrm{Th}(\mathrm{pr}_1)\colon \mathrm{Th}(\nu_{\xi\times D}) \to \mathrm{Th}(\nu_\xi)$$

induces a homomorphism

$$\mathrm{Th}(\mathrm{pr}_1)^*\colon \Omega^{d-s}_{\mathrm{SPL}}(E\nu_\xi, \dot{E}\nu_\xi) \to \Omega^{d-s}_{\mathrm{SPL}}(E\nu_{\xi\times D}, \dot{E}\nu_{\xi\times D}),$$

which agrees with the homomorphism

$$\mathrm{pr}_1^*\colon \Omega^{d-s}_{\mathrm{SPL}}(N, \partial N) \to \Omega^{d-s}_{\mathrm{SPL}}((N, \partial N) \times D)$$

induced by the map of pairs $\mathrm{pr}_1 \colon (N, \partial N) \times D \to (N, \partial N)$. By the naturality of the BRS–Thom class [12, top of page 27], this homomorphism maps the BRS–Thom class of ν_ξ to the BRS–Thom class of $\mathrm{pr}_1^* \nu_\xi = \nu_{\xi \times D}$,

$$\mathrm{pr}_1^*(u_{\mathrm{BRS}}(\nu_\xi)) = u_{\mathrm{BRS}}(\nu_{\xi \times D}).$$

Given any element

$$[g] \in \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(\mathrm{Th}\, \nu_{\xi \times D}) = \Omega^{\mathrm{Witt}}_{n+s}((N, \partial N) \times D),$$

the computation

$$u_{\mathrm{BRS}}(\nu_\xi) \cap \mathrm{pr}_{1*}[g] = \mathrm{pr}_{1*}(\mathrm{pr}_1^* u_{\mathrm{BRS}}(\nu_\xi) \cap [g]) = \mathrm{pr}_{1*}(u_{\mathrm{BRS}}(\nu_{\xi \times D}) \cap [g])$$

shows that the left-hand side of (18) commutes. □

The pullback transfer $\xi^!_{\mathrm{PB}}$ on Witt bordism agrees with the transfer $\xi^!_{\mathrm{BRS}}$:

Proposition 6.5 *The diagram*

$$\begin{tikzcd}
\widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(S^s B^+) \arrow[r, "T(\xi)_*"] & \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(\mathrm{Th}(\nu)) \arrow[r, "u_{\mathrm{BRS}}(\nu)\cap-"] & \Omega^{\mathrm{Witt}}_{n+d}(E(\nu)) \arrow[d, "\cong"', "\rho_*"] \\
\Omega^{\mathrm{Witt}}_n(B) \arrow[u, "\sigma", "\cong"'] \arrow[rr, "\xi^!_{\mathrm{PB}}"] & & \Omega^{\mathrm{Witt}}_{n+d}(X)
\end{tikzcd}$$

commutes, that is, $\xi^!_{\mathrm{PB}} = \xi^!_{\mathrm{BRS}}$.

Proof Let $h \colon W^n \to B$ be a continuous map from a closed n–dimensional Witt space W to B, representing an element $[h] \in \Omega^{\mathrm{Witt}}_n(B)$. By simplicial approximation, we may assume that h is PL. We begin by observing that, by Proposition 6.4, it suffices to prove the statement for the case where $h \colon W \to B$ is a PL embedding: Given any PL map $h \colon W \to B$, consider the graph embedding

$$(h, \mathrm{id}_W) \colon W \to B \times W$$

which factors h as

$$\begin{tikzcd}
W \arrow[r, hook, "(h,\mathrm{id}_W)"] \arrow[dr, "h"'] & B \times W \arrow[d, "\mathrm{pr}_1"] \\
& B
\end{tikzcd}$$

Let $\xi \times W$ denote the F–block bundle over $B \times W$ obtained by pulling back ξ under the projection $\mathrm{pr}_1 \colon B \times W \to B$. If the statement is known for embeddings, then

$$(\xi \times W)^!_{\mathrm{PB}}[(h, \mathrm{id}_W)] = (\xi \times W)^!_{\mathrm{BRS}}[(h, \mathrm{id}_W)].$$

Hence by Proposition 6.4 with $D = W$,

$$\begin{aligned}
\xi^!_{\mathrm{PB}}[h] = \xi^!_{\mathrm{PB}}[\mathrm{pr}_1 \circ (h, \mathrm{id}_W)] &= \xi^!_{\mathrm{PB}}\, \mathrm{pr}_{1*}[(h, \mathrm{id}_W)] = \mathrm{pr}_{1*}(\xi \times W)^!_{\mathrm{PB}}[(h, \mathrm{id}_W)] \\
&= \mathrm{pr}_{1*}(\xi \times W)^!_{\mathrm{BRS}}[(h, \mathrm{id}_W)] = \xi^!_{\mathrm{BRS}}\, \mathrm{pr}_{1*}[(h, \mathrm{id}_W)] = \xi^!_{\mathrm{BRS}}[h].
\end{aligned}$$

Consequently, it remains to prove the equality $\xi^!_{\mathrm{PB}} = \xi^!_{\mathrm{BRS}}$ on Witt bordism classes that are represented by PL embeddings.

As in the construction of the Umkehr map $T(\xi)$ in Section 3, let N denote the total space $E(\nu)$ of the stable vertical normal closed disc block bundle $\nu = \nu_\theta$ of ξ associated to a choice of block-preserving embedding $\theta\colon X \hookrightarrow \mathbb{R}^s \times B$, where X is the total space of the given F–block bundle ξ. Thus N is a ξ–block-preserving regular neighborhood of $\theta(X)$ in $\mathbb{R}^s \times B$. Recall that ∂N denotes the total space of the sphere block bundle of ν. Let $D^s \subset \mathbb{R}^s$ be a closed PL ball about the origin which is large enough that $(D^s - \partial D^s) \times B$ contains $N \cup V \subset \mathbb{R}^s \times B$, where V is the outside collar to ∂N used in the construction of the Umkehr map; such a ball exists by compactness of X.

Let $h\colon W \hookrightarrow B$ be a PL embedding of a closed Witt space into B. Recall that K is a cell complex with polyhedron $|K| = B$ and ξ is given over K. By subdivision of K and ξ, we may assume that $h(W) = |K_W|$ for a subcomplex K_W of K. Let L_S be a finite simplicial complex such that

(i) $|L_S| = S^s B^+$,

(ii) there is a subcomplex L of L_S such that $|L| = D^s \times B$,

(iii) for every simplex $\sigma \in K$, there is a subcomplex L_σ of L such that
$$|L_\sigma| = D^s \times \sigma,$$

(iv) there exists a subcomplex L_θ of L such that $|L_\theta| = \theta(X)$,

(v) the stable vertical normal bundle ν is a (disc) block bundle over the complex L_θ such that
$$E(\nu) \cap (D^s \times \sigma) = \bigcup_{\tau \in L_\sigma \cap L_\theta} \nu(\tau),$$
where $\nu(\tau)$ is the disc-block of ν over the simplex τ.

Property (iii) implies that L_τ is a subcomplex of L_σ for every face τ of $\sigma \in K$. Furthermore,
$$D^s \times h(W) = D^s \times |K_W| = \bigcup_{\sigma \in K_W} D^s \times \sigma = \bigcup_{\sigma \in K_W} |L_\sigma| = \Big| \bigcup_{\sigma \in K_W} L_\sigma \Big|,$$
so that
$$L_W := \bigcup_{\sigma \in K_W} L_\sigma$$
is a simplicial subcomplex of L with $D^s \times h(W) = |L_W|$. Since the embedding $\theta\colon X \hookrightarrow \mathbb{R}^s \times B$ is block preserving with respect to the F–blocks of ξ, we have $\theta(\xi(\sigma)) = (D^s \times \sigma) \cap \theta(X)$ for all $\sigma \in K$. So by (iv),
$$\theta(\xi(\sigma)) = |L_\sigma| \cap |L_\theta| = |L_\sigma \cap L_\theta|.$$

Thus the embedded F–blocks $\theta(\xi(\sigma))$ are triangulated by the subcomplex $L_\sigma \cap L_\theta$ of L.

The image $\sigma[h]$ under the suspension isomorphism
$$\sigma\colon \Omega_n^{\mathrm{Witt}}(B) \xrightarrow{\cong} \widetilde{\Omega}_{n+s}^{\mathrm{Witt}}(S^s B^+) = \Omega_{n+s}^{\mathrm{Witt}}((D^s, \partial D^s) \times B)$$
is represented by the closed product PL embedding
$$\mathrm{id} \times h\colon (D^s \times W, \partial(D^s \times W)) \hookrightarrow (D^s \times B, (\partial D^s) \times B).$$

The Umkehr map is a PL map

$$T(\xi)\colon S^s B^+ = \mathrm{Th}(\mathbb{R}^s \times B) = \frac{D^s \times B}{(\partial D^s) \times B} \to \mathrm{Th}(\nu),$$

which is the identity near $\theta(X)$. Composing with it, we obtain a PL map

$$f = T(\xi) \circ (\mathrm{id} \times h)\colon (D^s \times W, \partial(D^s \times W)) \to (\mathrm{Th}(\nu), \infty).$$

Let A be the ball complex with $|A| = N = E(\nu)$ whose balls include the blocks of ν. The rest of the balls come from the sphere block bundle of ν. The BRS–Thom class $u_{\mathrm{BRS}}(\nu) \in \Omega^{-(s-d)}_{\mathrm{SPL}}(N, \partial N)$ is represented by the mock bundle η with projection given by the zero section $i\colon \theta(X) \to A$. Thus the total space of η is $E(\eta) = \theta(X)$. The mock bundle η is an embedded mock bundle in the sense of Buoncristiano, Rourke and Sanderson [12, page 34]: the restriction $i|\colon \eta(\sigma) \to \sigma$ for a ball $\sigma = \nu(\tau) \in A$ is the inclusion $\tau \hookrightarrow \nu(\tau)$, which is locally flat by definition of a disc block bundle. Furthermore, $i|\colon \eta(\sigma) \to \sigma$ is proper, ie $i|^{-1}(\partial\sigma) = \partial\eta(\sigma)$. We wish to compute the cap product

$$u_{\mathrm{BRS}}(\nu) \cap [f\colon (D^s, \partial D^s) \times W \to (\mathrm{Th}(\nu), \infty)] \in \Omega^{\mathrm{Witt}}_{n+d}(E(\nu)).$$

The base complex of η is only known to be a ball complex, not a simplicial complex as required for pulling back a mock bundle via a cartesian square. Thus we need to subdivide simplicially. Let L' be a simplicial subdivision of L and let A' be a simplicial subdivision of A such that A' is a subcomplex of L'. Thus,

$$|A'| = E(\nu) \quad \text{and} \quad |L'| = D^s \times B.$$

The complex L' contains a (simplicial) subcomplex L'_W given by

$$L'_W = \{\tau \in L' : \tau \subset \sigma \text{ for some } \sigma \in L_W\}.$$

This is a subdivision of $L_W \subset L$, and

$$|L'_W| = |L_W| = D^s \times h(W).$$

So the inclusion

$$|L'_W| = D^s \times h(W) \hookrightarrow D^s \times B = |L'|$$

is a *simplicial* map

$$L'_W \hookrightarrow L'.$$

By [12, Theorem 2.1, page 23] (see also [40, subdivision theorem, page 128]), mock bundles can be subdivided: if α is a mock bundle over a ball complex D with total space $E(\alpha)$ and projection $p\colon E(\alpha) \to D$, and D' is a subdivision of D, then there exists a mock bundle α' over D' together with a PL homeomorphism $\phi\colon E(\alpha) \xrightarrow{\cong} E(\alpha')$ which preserves α–blocks over D, and a homotopy

$$F\colon E(\alpha) \times I \to |D| = |D'| \quad \text{with } F_0 = p \text{ and } F_1 = p'\phi,$$

which respects the α–blocks over D. (Here $p'\colon E(\alpha') \to |D'|$ is the projection of α'.) Moreover, if α is an embedded mock bundle, then the subdivision theorem yields again an embedded mock bundle and the homotopy can be taken to be an isotopy which is covered by an ambient isotopy. We apply this to the zero section mock bundle η over A: Since A' is a (simplicial) subdivision of A, there thus exists a correspondingly subdivided mock bundle η' over A'. Since η is an embedded mock bundle $i\colon \theta(X) = E(\eta) \hookrightarrow E(\nu)$, so is η'. Thus the projection map i' of η' may be taken to be a PL embedding $i'\colon E(\eta') \hookrightarrow E(\nu)$. As the zero section i does not touch the sphere bundle of ν (ie η has empty blocks over ∂N), the same is true for the perturbation i'. There exists a PL homeomorphism $\phi\colon \theta(X) = E(\eta) \xrightarrow{\cong} E(\eta')$ which preserves η–blocks over the ball complex A. The maps i and $i'\phi$ are isotopic via an isotopy

$$F\colon \theta(X)\times I \to E(\nu)\times I \quad \text{with } F_0 = i \text{ and } F_1 = i'\phi.$$

This isotopy is covered by an ambient isotopy

$$H\colon E(\nu)\times I \to E(\nu)\times I \quad \text{with } H_0 = \mathrm{id}$$

such that

$$\begin{tikzcd} \theta(X)\times I \arrow[rr, "F_0\times \mathrm{id} = i\times\mathrm{id}"] \arrow[dr, "F"'] & & E(\nu)\times I \arrow[dl, "H"] \\ & E(\nu)\times I & \end{tikzcd}$$

commutes. This implies

$$H_1 \circ i = F_1 = i' \circ \phi. \tag{19}$$

By an induction on the cells $\sigma \in K$, starting with the 0–dimensional cells, F and H can be constructed to preserve blocks over K. More precisely: Let ν_σ denote the restriction of ν to the embedded F–block $\theta(\xi(\sigma)) = \theta(X) \cap (D^s \times \sigma)$. Since ν is a block bundle over the complex L_θ and $\theta(\xi(\sigma))$ is triangulated by $L_\sigma \cap L_\theta$, the total space of ν_σ is given by $E(\nu_\sigma) = \bigcup_\tau \nu(\tau)$, where τ ranges over all simplices of $L_\sigma \cap L_\theta$. Thus by (v) above,

$$E(\nu_\sigma) = E(\nu) \cap (D^s \times \sigma). \tag{20}$$

Then H can be inductively arranged to satisfy

$$H_t(E(\nu_\sigma)) = E(\nu_\sigma) \tag{21}$$

for all $\sigma \in K$ and all $t \in [0,1]$, as follows: Recall that Buoncristiano, Rourke and Sanderson's construction of H in their proof of the mock bundle subdivision theorem proceeds inductively over cells of the base, starting with the 0–cells. In the present context, one organizes their induction as follows: Start with the 0–skeleton A^0 of A. For every 0–cell σ^0 of K, subdivide η over $A^0 \cap D^s \times \sigma^0$ within the manifold $E(\nu) \cap D^s \times \sigma^0$. Extend this subdivision for every 1–cell σ^1 of K to a subdivision over $A^0 \cap D^s \times \sigma^1$ within the manifold $E(\nu) \cap D^s \times \sigma^1$. Continue in this way with 2–cells σ^2, etc, until all cells of K have been used. Then move on to the 1–skeleton A^1 of A. For every 0–cell σ^0 of K, extend the subdivision to a subdivision over $A^1 \cap D^s \times \sigma^0$ within the manifold $E(\nu) \cap D^s \times \sigma^0$. Extend this subdivision for every 1–cell σ^1 of K to a subdivision over $A^1 \cap D^s \times \sigma^1$ within the manifold $E(\nu) \cap D^s \times \sigma^1$, and so on.

The mock bundle η' is defined over the simplicial complex A' with polyhedron $|A'| = E(\nu)$, but using the canonical inclusions $E(\nu) \subset D^s \times B$ and $E(\nu) \subset \mathrm{Th}(\nu)$ we may regard η' as a mock bundle over $D^s \times B$, and as a mock bundle over $\mathrm{Th}(\nu)$. In more detail, the composition

$$E(\eta') \xhookrightarrow{i'} E(\nu) = |A'| \hookrightarrow D^s \times B = |L'|$$

is the projection of a mock bundle over the complex L', whose blocks over simplices in A' are the blocks of η' and blocks over simplices not in A' are taken to be empty. (Here, we are using that η' has empty blocks over the sphere bundle ∂N.) Similarly, after extending the triangulation A' to a triangulation T' of $\mathrm{Th}(\nu)$ by coning off simplices of A' that are in ∂N (and adding the cone point ∞ as a 0–simplex), the composition

$$E(\eta') \xhookrightarrow{i'} E(\nu) = |A'| \hookrightarrow \mathrm{Th}(\nu) = |T'|$$

is the projection of a mock bundle over the complex T', whose blocks over simplices in A' are the blocks of η' and blocks over simplices not in A' are again taken to be empty. In view of the commutative diagram

$$\begin{tikzcd} E(\eta') \arrow[d, hook, "i'"'] & \\ E(\nu) \arrow[d, hook] \arrow[dr, hook] & \\ D^s \times B \arrow[r, "T(\xi)"] & \mathrm{Th}(\nu) \end{tikzcd}$$

the pullback $T(\xi)^*(\eta'/_{T'})$ under the Umkehr map is precisely $\eta'/_{L'}$. Therefore, the mock bundle pullback $f^*(\eta')$ is given by

$$f^*(\eta') = (\mathrm{id} \times h)^* T(\xi)^*(\eta'/_{T'}) = (\mathrm{id} \times h)^*(\eta'/_{L'}).$$

The mock bundle η' (contrary to η, possibly) is defined over a *simplicial* complex L' and, as pointed out above, the inclusion $D^s \times h(W) \hookrightarrow D^s \times B = |L'|$ is a *simplicial* map

$$L'_W \hookrightarrow L'.$$

Therefore, the mock bundle pullback $f^*(\eta') = (\mathrm{id} \times h)^*(\eta')$ is given by the *cartesian* diagram

$$\begin{tikzcd} E((\mathrm{id} \times h)^*\eta') \arrow[r] \arrow[d] \arrow[dr, "g"] & E(\eta') \arrow[d, hook, "i'"] \\ D^s \times h(W) = |L'_W| \arrow[r, hook] & D^s \times B = |L'| \end{tikzcd}$$

It follows that the cap product of the BRS–Thom class with $[f]$ is given by the diagonal arrow

$$u_{\mathrm{BRS}}(\nu) \cap [f] = [g] \in \Omega^{\mathrm{Witt}}_{n+d}(E(\nu)),$$

the total space of the pullback is given by

$$E((\mathrm{id} \times h)^*\eta') = (D^s \times h(W)) \cap i'E(\eta')$$

and g is the subspace inclusion

$$g\colon (D^s \times h(W)) \cap i'E(\eta') \subset i'E(\eta') \subset E(\nu).$$

We show next that the final stage $H_1\colon E(\nu)\to E(\nu)$ of the ambient isotopy H induces a homeomorphism

$$H_1\colon E(\xi|_{K_W}) \xrightarrow{\cong} (D^s\times h(W))\cap i'E(\eta'), \tag{22}$$

where we use θ to identify $X=E(\xi)$ and $\theta(X)$, and to identify $E(\xi|_{K_W})$ and $\theta(X)\cap(D^s\times|K_W|)$. The homeomorphism H_1 restricts to a homeomorphism

$$H_1|\colon \theta(X)\cap(D^s\times|K_W|)\xrightarrow{\cong} H_1(\theta(X)\cap(D^s\times|K_W|)),$$

whose target we shall now compute:

$$\begin{aligned}
H_1(\theta(X)\cap(D^s\times|K_W|)) &= H_1(\theta(X)\cap E(\nu)\cap(D^s\times|K_W|)) = H_1(\theta(X))\cap H_1(E(\nu)\cap D^s\times|K_W|)\\
&= H_1(\theta(X))\cap H_1\Big(E(\nu)\cap\bigcup_{\sigma\in K_W} D^s\times\sigma\Big)\\
&= H_1(\theta(X))\cap H_1\Big(\bigcup_{\sigma\in K_W} E(\nu)\cap(D^s\times\sigma)\Big)\\
&= H_1(\theta(X))\cap \bigcup_{\sigma\in K_W} H_1(E(\nu)\cap(D^s\times\sigma))\\
&= H_1(\theta(X))\cap \bigcup_{\sigma\in K_W} (E(\nu)\cap(D^s\times\sigma)) && \text{(by (20) and (21))}\\
&= H_1(\theta(X))\cap E(\nu)\cap \bigcup_{\sigma\in K_W} (D^s\times\sigma) = H_1(\theta(X))\cap \bigcup_{\sigma\in K_W} (D^s\times\sigma)\\
&= H_1 iE(\eta)\cap \bigcup_{\sigma\in K_W} |L_\sigma|\\
&= i'\phi E(\eta)\cap|L_W| && \text{(by (19))}\\
&= i'E(\eta')\cap(D^s\times h(W)).
\end{aligned}$$

Thus we obtain the homeomorphism (22). In the diagram

$$\begin{tikzcd}
E(\xi|_{K_W}) \arrow[r, hook] \arrow[d, "H_1|"', "\cong"] & E(\xi)=\theta(X) \arrow[r, hook, "i"] & E(\nu) \arrow[d, "\cong"', "H_1"] \\
i'E(\eta')\cap(D^s\times h(W)) \arrow[rr, hook, "g"] & & E(\nu)
\end{tikzcd}$$

all the horizontal arrows are subspace inclusions and thus the diagram commutes. By Lemma 6.2 applied to the PL homeomorphism $H_1|$,

$$[g]=[g\circ H_1|]\in\Omega^{\mathrm{Witt}}_{n+d}(E(\nu)).$$

By commutativity of the diagram,

$$[g\circ H_1|]=[E(\xi|_{K_W})\subset\theta(X)\xrightarrow{H_1 i}E(\nu)]=[E(\xi|_{K_W})\subset\theta(X)\xrightarrow{i'\phi}E(\nu)].$$

By restriction, the isotopy F gives rise to an isotopy

$$\widehat{F}\colon E(\xi|_{K_W})\times I\subset\theta(X)\times I\xrightarrow{F}E(\nu)$$

from
$$\widehat{F}_0 = E(\xi|_{K_W}) \subset \theta(X) \xrightarrow{F_0=i} E(\nu)$$
to
$$\widehat{F}_1 = E(\xi|_{K_W}) \subset \theta(X) \xrightarrow{F_1=i'\phi} E(\nu).$$
By Lemma 6.2,
$$[\widehat{F}_0] = [\widehat{F}_1] \in \Omega^{\mathrm{Witt}}_{n+d}(E(\nu)).$$
Therefore,
$$[g] = [g \circ H_1|] = [\widehat{F}_1] = [\widehat{F}_0] \in \Omega^{\mathrm{Witt}}_{n+d}(E(\nu)).$$
Now the geometric pullback transfer of $[h\colon W \hookrightarrow B]$ is given by
$$\xi^!_{\mathrm{PB}}[h\colon W \hookrightarrow B] = [E(\xi|_{K_W}) \subset E(\xi) = \theta(X)].$$
Hence
$$i_*\xi^!_{\mathrm{PB}}[h\colon W \hookrightarrow B] = [\widehat{F}_0].$$
Finally, since i_* and ρ_* are inverses of each other,
$$\begin{aligned}\xi^!_{\mathrm{PB}}[h\colon W \hookrightarrow B] &= \rho_*[\widehat{F}_0] = \rho_*[g] = \rho_*(u_{\mathrm{BRS}}(\nu) \cap [f]) = \rho_*(u_{\mathrm{BRS}}(\nu) \cap [T(\xi) \circ (\mathrm{id} \times h)])\\ &= \rho_*(u_{\mathrm{BRS}}(\nu) \cap T(\xi)_*\sigma[h]) = \xi^!_{\mathrm{BRS}}[h],\end{aligned}$$
as was to be shown. □

We will refer to the map $\rho_*(u_{\mathrm{BRS}}(\nu) \cap -)$ as the *geometric Thom homomorphism*.

Proposition 6.6 *The homotopy-theoretic Thom homomorphism Φ agrees with the geometric Thom homomorphism, that is, the diagram*

$$\begin{tikzcd} \widetilde{\mathrm{MWITT}}_{n+s}(\mathrm{Th}(\mu)) \arrow[r, "\Phi"] \arrow[d, "\cong"'] & \mathrm{MWITT}_{n+d}(X) \arrow[d, "\cong"] \\ \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(\mathrm{Th}(\nu)) \arrow[r, "\rho_*(u_{\mathrm{BRS}}(\nu)\cap -)"] & \Omega^{\mathrm{Witt}}_{n+d}(X) \end{tikzcd}$$

commutes.

Proof Recall that Φ is given by $\Phi = \rho_*(u_{\mathrm{SPL}}(\mu) \cap -)$. The result follows from Lemma 4.5 applied to μ with underlying oriented block bundle $\mu_{\mathrm{PLB}} = \nu$, together with the geometric description of the cap product given in [12]. □

Proposition 6.7 *Manifold-block bundle transfer on* MWITT*–homology and geometric pullback transfer on Witt bordism agree, that is, the diagram*

$$\begin{tikzcd} \mathrm{MWITT}_n(B) \arrow[r, "\xi^!"] \arrow[d, "\cong"] & \mathrm{MWITT}_{n+d}(X) \arrow[d, "\cong"'] \\ \Omega^{\mathrm{Witt}}_n(B) \arrow[r, "\xi^!_{\mathrm{PB}}"] & \Omega^{\mathrm{Witt}}_{n+d}(X) \end{tikzcd}$$

commutes.

Proof We must show that the outer square of the diagram

$$\begin{tikzcd}
\mathrm{MWITT}_n(B) \arrow[rrrr, "\xi^!"] \arrow[dr, "T(\xi)_*\sigma"] \arrow[ddd, "\cong"] & & & & \mathrm{MWITT}_{n+d}(X) \arrow[ddd, "\cong"] \\
& \widetilde{\mathrm{MWITT}}_{n+s}(\mathrm{Th}(\mu)) \arrow[d, "\cong"] \arrow[urrr, "\Phi"] & & & \\
& \widetilde{\Omega}^{\mathrm{Witt}}_{n+s}(\mathrm{Th}(\nu)) \arrow[drrr, "\rho_*(u_{\mathrm{BRS}}(\nu)\cap -)"] & & & \\
\Omega^{\mathrm{Witt}}_n(B) \arrow[ur, "T(\xi)_*\sigma"] \arrow[rrrr, "\xi^!_{\mathrm{PB}}"] & & & & \Omega^{\mathrm{Witt}}_{n+d}(X)
\end{tikzcd}$$

commutes. The upper part commutes by definition of the F–block bundle transfer $\xi^!$. The left-hand part commutes as the vertical arrows are given by a natural isomorphism of homology theories, while the right-hand part commutes by Proposition 6.6. The lower part of the diagram, involving the pullback transfer $\xi^!_{\mathrm{PB}}$, commutes according to Proposition 6.5. □

A closed n–dimensional Witt space W has a *fundamental class*

$$[W]_{\mathrm{Witt}} \in \Omega^{\mathrm{Witt}}_n(W)$$

in Witt bordism represented by the identity map, $[W]_{\mathrm{Witt}} = [\mathrm{id}\colon W \to W]$. This class corresponds to a unique class $[W]_{\mathrm{Witt}} \in \mathrm{MWITT}_n(W)$ under the natural identification (9).

Proposition 6.8 *Suppose B is a closed Witt space of dimension n. Then the total space X of the oriented F–block bundle ξ over B is a closed Witt space and the geometric pullback transfer*

$$\xi^!_{\mathrm{PB}}\colon \Omega^{\mathrm{Witt}}_n(B) \to \Omega^{\mathrm{Witt}}_{n+d}(X)$$

maps the Witt fundamental class of B to the Witt fundamental class of X,

$$\xi^!_{\mathrm{PB}}[B]_{\mathrm{Witt}} = [X]_{\mathrm{Witt}}.$$

Proof If the base B is Witt, then the total space X is Witt by Lemma 6.1. The Witt fundamental class $[B]_{\mathrm{Witt}}$ is represented by the identity map $g = \mathrm{id}_B\colon B \to B$ (which is PL). Pulling back under this identity map, the map $\bar{g}\colon E(\mathrm{id}^*\xi) \to X$ is the identity $\mathrm{id}\colon E(\mathrm{id}^*\xi) = X \to X$. Therefore,

$$\xi^!_{\mathrm{PB}}[\mathrm{id}\colon B \to B] = [\bar{g}\colon E(\mathrm{id}^*\xi) \to X] = [\mathrm{id}_X] = [X]_{\mathrm{Witt}}.$$ □

Example 6.9 We continue our previous examples on the trivial F–block bundle ξ with total space $X = F \times B$ for B any compact polyhedron. The geometric pullback transfer $\xi^!_{\mathrm{PB}}\colon \Omega^{\mathrm{Witt}}_n(B) \to \Omega^{\mathrm{Witt}}_{n+d}(F \times B)$ is then by construction $\xi^!_{\mathrm{PB}}[g\colon W \to B] = [\mathrm{id}_F \times g\colon F \times W \to F \times B]$. The Witt bordism $\times$–product

$$\times\colon \Omega^{\mathrm{Witt}}_d(F) \times \Omega^{\mathrm{Witt}}_n(B) \to \Omega^{\mathrm{Witt}}_{d+n}(F \times B), \quad [h] \times [g] = [h \times g],$$

can be used to decompose the class $[\mathrm{id}_F \times g]$ as $[F]_{\mathrm{Witt}} \times [g]$. We thus find that

$$\xi^!_{\mathrm{PB}}[g] = [F]_{\mathrm{Witt}} \times [g],$$

which agrees with (13). If $B = W$ is an n–dimensional closed Witt space and g the identity, then $\xi^!_{\mathrm{PB}}[B]_{\mathrm{Witt}} = [F]_{\mathrm{Witt}} \times [B]_{\mathrm{Witt}} = [F \times B]_{\mathrm{Witt}}$, in agreement with Proposition 6.8.

7 Transfer of the $\mathbb{L}^\bullet$–homology fundamental class

In [7], we constructed a canonical $\mathbb{L}^\bullet(\mathbb{Q})$*–homology fundamental class*

$$[X]_{\mathbb{L}} \in \mathbb{L}^\bullet(\mathbb{Q})_n(X)$$

for closed n–dimensional Witt spaces X using the morphism $\tau\colon \mathrm{MWITT} \to \mathbb{L}^\bullet(\mathbb{Q})$ of ring spectra. This class is the image of the Witt theory fundamental class $[X]_{\mathrm{Witt}}$ under the map

$$\tau_*\colon \Omega_n^{\mathrm{Witt}}(X) \cong \mathrm{MWITT}_n(X) \to \mathbb{L}^\bullet(\mathbb{Q})_n(X),$$

ie $[X]_{\mathbb{L}} = \tau_*[X]_{\mathrm{Witt}}$.

Theorem 7.1 *Suppose B is a closed Witt space of dimension n. Then the total space X of the oriented F–block bundle ξ over B is a closed Witt space and the block bundle transfer*

$$\xi^!\colon \mathbb{L}^\bullet(\mathbb{Q})_n(B) \to \mathbb{L}^\bullet(\mathbb{Q})_{n+d}(X)$$

maps the $\mathbb{L}^\bullet(\mathbb{Q})$–homology fundamental class of B to the $\mathbb{L}^\bullet(\mathbb{Q})$–homology fundamental class of X,

$$\xi^![B]_{\mathbb{L}} = [X]_{\mathbb{L}}.$$

Proof By Proposition 6.8, $\xi^!_{\mathrm{PB}}[B]_{\mathrm{Witt}} = [X]_{\mathrm{Witt}}$ for the pullback transfer. Thus, using Proposition 6.7 on the compatibility of block bundle transfer and pullback transfer,

$$\xi^![B]_{\mathrm{Witt}} = \xi^!_{\mathrm{PB}}[B]_{\mathrm{Witt}} = [X]_{\mathrm{Witt}}.$$

Finally, by Proposition 5.6,

$$\xi^![B]_{\mathbb{L}} = \xi^!\tau_*[B]_{\mathrm{Witt}} = \tau_*\xi^![B]_{\mathrm{Witt}} = \tau_*[X]_{\mathrm{Witt}} = [X]_{\mathbb{L}}.$$ □

Example 7.2 We describe the $\mathbb{L}^\bullet(\mathbb{Q})$–homology transfer and illustrate Theorem 7.1 for the trivial F–block bundle ξ with total space $X = F \times B$. We use the notation of the earlier examples on this special case. By Lemma 5.8, $u_{\mathbb{L}}(\mu_F) = \tau\phi_W(u_{\mathrm{SPL}}(\mu_F))$. Hence, using [38, page 552, Proposition 7.1.2],

$$[F]_{\mathbb{L}} = \rho_{F*}\big(\tau\phi_W(u_{\mathrm{SPL}}(\mu_F)) \cap [\mathrm{Th}\,\mu_F]_{\mathbb{L}}\big) \in \mathbb{L}^\bullet(\mathbb{Q})_d(F).$$

See also [39, page 186, Proposition 16.16(c)]. Consequently, (13) applies to yield the description

$$\xi^!(a) = [F]_{\mathbb{L}} \times a$$

for the transfer $\xi^!\colon \mathbb{L}^\bullet(\mathbb{Q})_n(B) \to \mathbb{L}^\bullet(\mathbb{Q})_{n+d}(F \times B)$. When B is a closed n–dimensional Witt space, we obtain

$$\xi^![B]_{\mathbb{L}} = [F]_{\mathbb{L}} \times [B]_{\mathbb{L}} = [F \times B]_{\mathbb{L}}$$

(where the second equality has been established in [7, Theorem 13.1]), in agreement with Theorem 7.1.

8 Behavior of the Cheeger–Goresky–MacPherson L–class under transfer

Rationally, Theorem 7.1 leads to a formula describing the behavior of the Cheeger–Goresky–MacPherson L–class under block bundle transfer.

Theorem 8.1 *Let B be a closed Witt space and let F be a closed oriented* PL *manifold. Let ξ be an oriented* PL *F–block bundle over B with total space X and oriented stable vertical normal* PL *microbundle μ over X. Then the associated block bundle transfer $\xi^!$ sends the Cheeger–Goresky–MacPherson L–class of B to the product*

$$\xi^! L_*(B) = L^*(\mu) \cap L_*(X).$$

Proof By Theorem 7.1, the $\mathbb{L}^\bullet$–homology transfer $\xi^!$ of ξ sends the $\mathbb{L}^\bullet(\mathbb{Q})$–homology fundamental class of B to the $\mathbb{L}^\bullet(\mathbb{Q})$–homology fundamental class of X: $\xi^![B]_{\mathbb{L}} = [X]_{\mathbb{L}}$. It remains to analyze what this equation means after we tensor with $\mathbb{Q}$, ie after we apply the localization morphism

$$\mathbb{L}^\bullet(\mathbb{Q}) \to \mathbb{L}^\bullet(\mathbb{Q})_{(0)} = \bigvee_i S^i H(L^i(\mathbb{Q}) \otimes \mathbb{Q}) = \bigvee_j S^{4j} H\mathbb{Q},$$

which is a ring morphism of ring spectra. By [7, Lemma 11.1],

$$[B]_{\mathbb{L}} \otimes \mathbb{Q} = L_*(B) \quad \text{and} \quad [X]_{\mathbb{L}} \otimes \mathbb{Q} = L_*(X).$$

Let $\Delta_{\mathrm{SPL}} \colon \mathrm{MSPL} \to \mathrm{KO}\big[\tfrac{1}{2}\big]$ be the Sullivan orientation [45]. Using work of Land and Nikolaus [30], we construct in [6, Proposition 2.1] a particular equivalence of $\mathbb{E}_\infty$–ring spectra

$$\kappa \colon \mathrm{KO}\big[\tfrac{1}{2}\big] \xrightarrow{\simeq} \mathbb{L}^\bullet(\mathbb{R})\big[\tfrac{1}{2}\big] = \mathbb{L}^\bullet(\mathbb{Z})\big[\tfrac{1}{2}\big]$$

and show that the composition

$$\mathrm{MSPL} \xrightarrow{\Delta_{\mathrm{SPL}}} \mathrm{KO}\big[\tfrac{1}{2}\big] \xrightarrow{\kappa} \mathbb{L}^\bullet(\mathbb{Z})\big[\tfrac{1}{2}\big]$$

is homotopic to Ranicki's orientation σ^* [6, Proposition 3.3]. Furthermore, using work of Taylor and Williams [47] as well as of Morgan and Sullivan [35], we describe in [6] a particular equivalence

$$\mathbb{L}^\bullet(\mathbb{Z})_{(0)} \xrightarrow{\simeq} \bigoplus_{i \in \mathbb{Z}} H\mathbb{Q}[4i]$$

such that the diagram

$$\begin{tikzcd}
\mathrm{MSPL} \arrow[r, "\Delta_{\mathrm{SPL}}"] \arrow[dr, "\sigma^*"'] & \mathrm{KO}\big[\tfrac{1}{2}\big] \arrow[r, "\mathrm{loc}"] \arrow[d, "\kappa"', "\simeq"] & \mathrm{KO}_{(0)} \arrow[r, "\mathrm{ph}", "\simeq"'] \arrow[d, "\kappa_{(0)}"', "\simeq"] & \bigoplus_{i\in\mathbb{Z}} H\mathbb{Q}[4i] \arrow[d, equal] \\
 & \mathbb{L}^\bullet\big[\tfrac{1}{2}\big] \arrow[r, "\mathrm{loc}"] & \mathbb{L}^\bullet_{(0)} \arrow[r, "\simeq"] & \bigoplus_{i\in\mathbb{Z}} H\mathbb{Q}[4i]
\end{tikzcd}$$

commutes up to homotopy, where ph denotes the Pontryagin character. Now, it is well known that the Pontryagin character of the Sullivan orientation is given by

$$\mathrm{ph}(\mathrm{loc}\,\Delta_{\mathrm{SPL}}) = L^{-1} \cup u \in H^*(\mathrm{MSPL}; \mathbb{Q}),$$

where L is the universal PL L–class $L \in H^*(\mathrm{BSPL}; \mathbb{Q})$ and u the stable Thom class

$$u \in H^0(\mathrm{MSPL}; \mathbb{Z}) = \mathbb{Z}.$$

(See Madsen and Milgram [33, Corollary 5.4, page 102].) Thus by commutativity of the diagram, the rational localization of σ^* is given by $L^{-1} \cup u$. Hence, for our PL microbundle $\mu\colon X \to \mathrm{BSPL}$,

$$u_{\mathbb{L}}(\mu) \otimes \mathbb{Q} = \rho^* L^*(\mu)^{-1} \cup u_{\mathbb{Q}}(\mu),$$

where $u_{\mathbb{Q}}(\mu) \in \widetilde{H}^{s-d}(\mathrm{Th}(\mu);\mathbb{Q})$ is the Thom class of μ in ordinary rational cohomology. (See also Ranicki's [39, Remark 16.2, page 176] for topological block bundles. Note that a PL microbundle has an underlying topological block bundle by composition with $\mathrm{BSPL} \to \mathrm{B}\widetilde{\mathrm{SPL}} \to \mathrm{B}\widetilde{\mathrm{STOP}}$, and that Ranicki omits cupping with $u_{\mathbb{Q}}(\mu)$ in his notation.) Thus

$$\begin{aligned} L_*(X) = [X]_{\mathbb{L}} \otimes \mathbb{Q} &= (\xi^![B]_{\mathbb{L}}) \otimes \mathbb{Q} = \rho_*(u_{\mathbb{L}}(\mu) \cap T(\xi)_*\sigma[B]_{\mathbb{L}}) \otimes \mathbb{Q} \\ &= \rho_*\big(u_{\mathbb{L}}(\mu) \otimes \mathbb{Q} \cap T(\xi)_*\sigma([B]_{\mathbb{L}} \otimes \mathbb{Q})\big) = \rho_*\big((\rho^* L^*(\mu)^{-1} \cup u_{\mathbb{Q}}(\mu)) \cap T(\xi)_*\sigma L_*(B)\big) \\ &= \rho_*\big(\rho^* L^*(\mu)^{-1} \cap (u_{\mathbb{Q}}(\mu) \cap T(\xi)_*\sigma L_*(B))\big) = L^*(\mu)^{-1} \cap \rho_*\big(u_{\mathbb{Q}}(\mu) \cap T(\xi)_*\sigma L_*(B)\big) \\ &= L^*(\mu)^{-1} \cap \xi^! L_*(B). \end{aligned}$$

□

If t is a stable inverse for μ, then t has the interpretation of a stable vertical tangent bundle for ξ, and by Theorem 8.1, the following formula holds:

$$\xi^! L_*(B) = L^*(t)^{-1} \cap L_*(X).$$

Example 8.2 We discuss Theorem 8.1 vis-à-vis (13) in the situation of a trivial F–block bundle ξ over B, using the notation of earlier examples on this case. Let $[F]_{\mathbb{Q}} \in H_d(F;\mathbb{Q})$ denote the fundamental class of the oriented PL manifold F in ordinary rational homology. By (13),

$$\xi^!(a) = [F]_{\mathbb{Q}} \times a$$

for $a \in H_n(B;\mathbb{Q})$. For a closed Witt space B, we obtain in particular

$$\xi^! L_*(B) = [F]_{\mathbb{Q}} \times L_*(B). \tag{23}$$

Let TF denote the tangent PL microbundle of the PL manifold F. Then $\mu_F \oplus TF$ is the trivial microbundle, and hence $L^*(\mu_F)L^*(TF) = L^*(\mu_F \oplus TF) = 1$. Furthermore, the Hirzebruch signature theorem holds for PL manifolds and $L_*(F) = L^*(TF) \cap [F]_{\mathbb{Q}}$; see Madsen and Milgram [33, Chapter 4C] and Thom [48]. According to Theorem 8.1,

$$\begin{aligned} \xi^! L_*(B) = L^*(\mu) \cap L_*(X) &= (L^*(\mu_F) \times 1) \cap (L_*(F) \times L_*(B)) = (L^*(\mu_F) \cap L_*(F)) \times (1 \cap L_*(B)) \\ &= (L^*(\mu_F) \cap L^*(TF) \cap [F]_{\mathbb{Q}}) \times L_*(B) = [F]_{\mathbb{Q}} \times L_*(B), \end{aligned}$$

confirming (23). It is perhaps worthwhile to emphasize that transfer does not in general commute with localization of spectra: if $\xi^!_{\mathbb{Q}}$ denotes the transfer on ordinary rational homology and $\xi^!_{\mathbb{L}}$ the transfer on $\mathbb{L}^\bullet(\mathbb{Q})$–homology, then generally $\xi^!_{\mathbb{Q}}(- \otimes \mathbb{Q}) \neq \xi^!_{\mathbb{L}}(-) \otimes \mathbb{Q}$. For example,

$$\xi^!_{\mathbb{Q}}([B]_{\mathbb{L}} \otimes \mathbb{Q}) = \xi^!_{\mathbb{Q}}(L_*(B)) = [F]_{\mathbb{Q}} \times L_*(B),$$

which contains less information than

$$(\xi^!_{\mathbb{L}}[B]_{\mathbb{L}}) \otimes \mathbb{Q} = [F \times B]_{\mathbb{L}} \otimes \mathbb{Q} = L_*(F \times B) = L_*(F) \times L_*(B).$$

9 Normally nonsingular maps

Let $f: Y \to X$ be a PL map of closed Witt spaces which is the composition

$$\begin{tikzcd} Y \arrow[r, hook, "g"] \arrow[dr, "f"'] & Z \arrow[d, "p"] \\ & X \end{tikzcd}$$

of an oriented normally nonsingular inclusion g with normal bundle ν_g, followed by the projection p of an oriented PL F–fiber bundle ξ with closed PL manifold fiber F and stable vertical normal bundle ν_ξ. Then f is a *normally nonsingular map* in the sense of [22, Definition 5.4.3]. Let c be the codimension of g and d the dimension of F. The bundle transfer $\xi^!$ and the Gysin restriction $g^!$ compose to give a transfer homomorphism

$$H_n(X;\mathbb{Q}) \xrightarrow{\xi^!} H_{n+d}(Z;\mathbb{Q}) \xrightarrow{g^!} H_{n+d-c}(Y;\mathbb{Q}),$$

with $c-d$ the *relative dimension* of f. Combining Theorem 8.1 with [5, Theorem 3.18], we obtain

$$\begin{aligned} g^!\xi^! L_*(X) &= g^!(L^*(\nu_\xi) \cap L_*(Z)) = g^* L^*(\nu_\xi) \cap g^! L_*(Z) = g^* L^*(\nu_\xi) \cap (L^*(\nu_g) \cap L_*(Y)) \\ &= L^*(g^*\nu_\xi \oplus \nu_g) \cap L_*(Y), \end{aligned}$$

at least when Y and Z have even dimensions.

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Received: 8 April 2022 Revised: 9 August 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 5 (pages 2389–2970) 2024