AGT

Algebraic & Geometric Topology

Volume 24 (2024)

Asymptotic dimensions of the arc graphs and disk graphs

Koji Fujiwara

Saul Schleimer

msp

Algebraic & Geometric Topology **24**:5 (2024) 2655–2672
DOI: 10.2140/agt.2024.24.2655
Published: 19 August 2024

Asymptotic dimensions of the arc graphs and disk graphs

KOJI FUJIWARA
SAUL SCHLEIMER

We give quadratic upper bounds for the asymptotic dimensions of the arc graphs and disk graphs.

20F65, 20F69, 57K20; 57M60

1 Introduction

The asymptotic dimension, denoted by $\mathsf{dim}_{\mathsf{asym}} X$, of a metric space X was introduced by Gromov [1993, page 29] as a large-scale analogue of the covering dimension. The curve graph, $\mathscr{C}(S)$, for a surface $S = S_{g,b}$ was introduced by Harvey [1981, page 246] as a sort of Bruhat–Tits building for Teichmüller space. It has since been generalised in many ways. Bell and Fujiwara [2008, Corollary 1] first proved that the asymptotic dimension of $\mathscr{C}(S)$ is finite. More recently, Bestvina and Bromberg [2019, Corollary 1.1] proved that $\mathsf{dim}_{\mathsf{asym}} \mathscr{C}(S_g) \leq 4g-4$ (when $g > 1$) and that $\mathsf{dim}_{\mathsf{asym}} \mathscr{C}(S_{g,b}) \leq 4g-3+b = \xi'(S_{g,b})$ when $g > 0$ and $b > 0$ (or $g = 0$ and $b > 2$).

Here we combine the machineries of [Bestvina et al. 2015; Masur and Schleimer 2013] to produce a quasi-isometric embedding of the arc graph $\mathscr{A}(S, \Delta)$ into a finite product of quasitrees of curve complexes. From this we deduce the following:

Corollary 3.11 *Suppose that $S = S_{g,b}$ has nonempty boundary. Suppose that $\Delta \subset \partial S$ is a nonempty union of components. Finally, suppose that $\xi'(S) \geq 1$. Then*

$$\mathsf{dim}_{\mathsf{asym}} \mathscr{A}(S, \Delta) \leq \tfrac{1}{2}(4g+b)(4g+b-3)-2.$$

Sisto (private communication, 2022) suggests that the machineries of [Behrstock et al. 2017, Theorem 5.2; Vokes 2022, Theorem A.2] can be combined to obtain a similar result.

We also obtain the following result for the disk graph $\mathscr{D}(M, S)$ of a compression body:

Corollary 4.18 *Suppose M is a nontrivial spotless compression body with upper boundary $S = S_{g,b}$. Suppose that $\xi'(S) \geq 1$. Then*

$$\mathsf{dim}_{\mathsf{asym}} \mathscr{D}(M, S) \leq \tfrac{1}{2}(4g+b)(4g+b-3)-2.$$

Hamenstädt [2019, Theorem 3.6] has obtained a similar result when M is a handlebody; see Remark 4.20.

Obtaining lower bounds better than linear, if even possible, would require new ideas. Therefore, we end our introduction with the following:

Question 1.1 How tight are the upper bounds of Corollaries 3.11 and 4.18? ◁

2 Background

Suppose that (X, d_X) is a metric space. Suppose that U and V are nonempty subsets of X with bounded diameter. Then we define their distance as

$$d_X(U, V) = \operatorname{diam}_X(U \cup V).$$

This notation is taken from [Masur and Minsky 2000, Formula (2.1), page 916].

We write $p =_{\mathsf{C}} q$ if, for nonnegative numbers p, q and C, we have both $q \le \mathsf{C}p + \mathsf{C}$ and $p \le \mathsf{C}q + \mathsf{C}$. Also, we use the *cut-off* function: $[p]_{\mathsf{C}}$ is equal to p if $p \ge \mathsf{C}$ and is zero otherwise.

We now more-or-less follow the conventions of [Bell and Dranishnikov 2008]. Suppose that X and Y are metric spaces. A relation $f: X \to Y$ is a *coarse map* if there is a constant C such that, for all $x \in X$, the image $f(x)$ is nonempty and has $\operatorname{diam}_Y(f(x)) \le \mathsf{C}$. A coarse map $f: X \to Y$ is a *coarse embedding* if there are functions $F, G: \mathbb{R} \to \mathbb{R}$ such that

- $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} G(t) = \infty$, and
- for all $x, y \in X$, we have

$$F(d_X(x, y)) \le d_Y(f(x), f(y)) \le G(d_X(x, y)).$$

A coarse map $f: X \to Y$ is *coarsely onto* if there is a constant $\mathsf{C} > 0$ such that, for all $y \in Y$, there is a point $x \in X$ with $d_Y(f(x), y) < \mathsf{C}$.

A coarse map $f: X \to Y$ is *coarsely Lipschitz* if there is a constant $\mathsf{C} > 0$ such that, for all $x, y \in X$, we have

$$d_Y(f(x), f(y)) \le \mathsf{C} \cdot d_X(x, y) + \mathsf{C}.$$

That is, we have no lower bound, but we require the upper bound to be affine.

A coarse embedding f is a *quasi-isometric embedding* if the functions F and G are both affine (with positive coefficients). The more usual definition is to require a constant $\mathsf{C} > 0$ such that $d_X(x, y) =_{\mathsf{C}} d_Y(f(x), f(y))$ for all $x, y \in X$. A quasi-isometric embedding f is a *quasi-isometry* if f is also coarsely onto.

2.1 Asymptotic dimension

We now follow [Gromov 1993, Section 1.E]. A metric space X has *asymptotic dimension* $\mathsf{dim}_{\mathsf{asym}}(X)$ at most n if, for every $R > 0$, there is a $\mathsf{D} > 0$ and a cover $\mathcal{U}$ of X such that

- for all $U \in \mathcal{U}$, we have $\operatorname{diam}_X(U) \le \mathsf{D}$, and
- every metric R–ball in X intersects at most $n + 1$ sets in $\mathcal{U}$.

For example, trees have asymptotic dimension at most one. More generally, following [Bell and Dranishnikov 2008, page 1272], a collection of metric spaces $\mathcal{X}$ has asymptotic dimension at most n, *uniformly*, if for every R there is a constant D such that every $X \in \mathcal{X}$ has a cover as above.

If two metric spaces X and Y are quasi-isometric and $\operatorname{dim}_{\mathsf{asym}}(X) \leq n$, then $\operatorname{dim}_{\mathsf{asym}}(Y) \leq n$. In view of this, we say a finitely generated group G has asymptotic dimension at most n if some (and thus all) of its Cayley graphs have asymptotic dimension at most n.

We list two well-known facts about asymptotic dimension.

Fact 2.2 [Bell and Dranishnikov 2008, Theorem 32] *Suppose that U and V are metric spaces. We give $U \times V$ the ℓ^1 product metric. Then*

$$\operatorname{dim}_{\mathsf{asym}} U \times V \leq \operatorname{dim}_{\mathsf{asym}} U + \operatorname{dim}_{\mathsf{asym}} V. \qquad \square$$

Fact 2.3 [Bell and Dranishnikov 2008, Proof of Proposition 22] *If U coarsely embeds into V, then*

$$\operatorname{dim}_{\mathsf{asym}} U \leq \operatorname{dim}_{\mathsf{asym}} V. \qquad \square$$

2.4 Quasitrees of metric graphs

We quickly review the machinery of *quasitrees of metric spaces*, as introduced in [Bestvina et al. 2015, Section 4]. Suppose that $\mathcal{F}$ is a collection of metric graphs. Suppose also that we have, for every pair of distinct graphs $A, B \in \mathcal{F}$, a nonempty subset $\pi_B(A) \subset B$. Also, fix a sufficiently large constant $\mathsf{k} > 0$.

With respect to the data $(\mathcal{F}, \pi, \mathsf{k})$ we require the following axioms:

Axiom 2.5 (bounded projections) *For distinct $A, B \in \mathcal{F}$, we have*

$$\operatorname{diam}_B(\pi_B(A)) \leq \mathsf{k}. \qquad \square$$

For $A, B, C \in \mathcal{F}$, and if $A \neq B$ and $B \neq C$, we adopt the shorthand

$$d_B(A, C) = d_B(\pi_B(A), \pi_B(C)).$$

Axiom 2.6 (Behrstock inequality) *For distinct $A, B, C \in \mathcal{F}$, at most one of the following is greater than k:*

$$d_A(B, C), \quad d_B(A, C), \quad d_C(A, B). \qquad \square$$

Axiom 2.7 (large links) *For distinct $A, C \in \mathcal{F}$, the following set is finite:*

$$\{B \in \mathcal{F} \mid A \neq B,\ B \neq C \text{ and } d_B(A, C) > \mathsf{k}\}. \qquad \square$$

We call these the *BBF axioms*. These are called (P0), (P1) and (P2) in [Bestvina et al. 2015].

Suppose that the data $(\mathcal{F}, \pi, \mathsf{k})$ satisfies the BBF axioms. Then, by [Bestvina et al. 2015, Theorem A], for every sufficiently large $\mathsf{K} \geq \mathsf{k}$ there is a metric graph $\mathcal{C}(\mathcal{F}) = \mathcal{C}_{\mathsf{K}}(\mathcal{F})$, called the *quasitree of graphs*. We denote the metric on $\mathcal{C}(\mathcal{F})$ by $d_{\mathcal{C}}$. We now list several properties of $\mathcal{C}(\mathcal{F})$.

Fact 2.8 [Bestvina et al. 2015, Theorem A and Definition 3.6] *By construction, every $A \in \mathcal{F}$ isometrically embeds into $\mathcal{C}(\mathcal{F})$. The image is totally geodesic. For distinct $A, B \in \mathcal{F}$, their images in $\mathcal{C}(\mathcal{F})$ are disjoint. Finally, these images cover the vertices of $\mathcal{C}(\mathcal{F})$.* □

Thus, we may identify the vertices of A with their images in $\mathcal{C}(\mathcal{F})$. We extend the definition of π_B as follows. If $b \in B$ then we take $\pi_B(b) = b$. If $a \in A$, and A is distinct from B, then we take $\pi_B(a) = \pi_B(A)$. One may think of $\pi_B\colon \mathcal{C}(\mathcal{F}) \to B$ as being a "closest-points projection" map. We adopt the shorthand

$$d_B(a, c) = d_B(\pi_B(a), \pi_B(c)).$$

Fact 2.9 [Bestvina et al. 2015, Definition 4.1] *By construction, there is a constant L (depending only on K) with the following property. Suppose that $A, B \in \mathcal{F}$ are graphs such that the set of Axiom 2.7 is empty. Then $d_{\mathcal{C}}(\pi_A(B), \pi_B(A)) \le \mathsf{L}$.* □

We have the following *distance estimate* for $d_{\mathcal{C}}$:

Theorem 2.10 [Bestvina et al. 2015, Theorem 4.13] *Suppose that $\mathcal{F}$ is a family of metric graphs satisfying the BBF axioms. Suppose that K is sufficiently large. Suppose that $\mathcal{C}_{\mathsf{K}}(\mathcal{F})$ is the quasitree of graphs. Then every K' sufficiently larger than K has the following property: for any $a, c \in \mathcal{C}(\mathcal{F})$, we have*

$$\frac{1}{2}\sum [d_B(a, c)]_{\mathsf{K}'} \le d_{\mathcal{C}(\mathcal{F})}(a, c) \le 6\mathsf{K} + 4\sum [d_B(a, c)]_{\mathsf{K}},$$

where both sums are taken over all $B \in \mathcal{F}$. □

Suppose that G is a group acting on $\mathcal{F}$. We further assume that, for any $g \in G$ and for any $A \in \mathcal{F}$, there is an isometry $g_A\colon A \to g \cdot A$. We suppose that these isometries have the following consistency properties. Suppose that $g, h \in G$ are group elements and $A, B \in \mathcal{F}$ are graphs with $B = g \cdot A$.

- For all $a \in A$, we have $h_B(g_A(a)) = (hg)_A(a)$.
- For any $C \in \mathcal{F}$, we have $g_A(\pi_A(C)) = \pi_B(g \cdot C)$.

From [Bestvina et al. 2015, Section 3.7], we deduce the following: there is an isometric action of G on the quasitree of graphs $\mathcal{C}(\mathcal{F})$ which extends the action of the isometries g_A.

We also have the following control on the asymptotic dimension of $\mathcal{C}(\mathcal{F})$:

Theorem 2.11 [Bestvina et al. 2015, Theorem 4.24] *Suppose that $\mathcal{F}$ is a family of metric graphs satisfying the BBF axioms. Suppose that $\mathcal{F}$ has asymptotic dimension at most D, uniformly. Then $\operatorname{dim}_{\mathsf{asym}}(\mathcal{C}(\mathcal{F})) \le D + 1$.* □

When all of the metric graphs are quasitrees, this can be improved:

Theorem 2.12 [Bestvina et al. 2015, Theorem B(ii)] *Suppose that $\mathcal{F}$ is a family of quasitrees satisfying the BBF axioms. Suppose further that the quasi-isometry constants are uniformly bounded. Then $\operatorname{dim}_{\mathsf{asym}}(\mathcal{C}(\mathcal{F})) \le 1$.* □

The notion of a *quasitree of metric spaces* was introduced and used to prove [Bestvina et al. 2015, Theorem D]: mapping class groups (of connected, compact, oriented surfaces) have finite asymptotic dimension.

2.13 Surfaces, curves and arcs

Let $S = S_{g,b}$ denote the connected, compact, oriented surface of genus g with b boundary components. The *complexity* of S is defined to be $\xi(S) = 3g - 3 + b$. This counts the number of curves in any pants decomposition of S. We will always assume that $\xi(S) \geq 1$. We will also need the *modified complexity* $\xi'(S) = 4g - 3 + b$. If S is closed then we will simply write S_g for $S_{g,0}$.

Suppose that α is an embedded arc or curve in S. We call the embedding *proper* if $\alpha \cap \partial S = \partial\alpha$. A properly embedded arc or curve α in S is *essential* if it does not cut a disk off of S. A properly embedded curve α is *nonperipheral* if it does not cut an annulus off of S.

A *proper isotopy* is an isotopy through proper embeddings. Let $[\alpha]$ denote the proper isotopy class of α. Given α and β, properly embedded arcs or curves, we define their *geometric intersection number*

$$i(\alpha, \beta) = \min\{|\alpha' \cap \beta'| : \alpha' \in [\alpha], \beta' \in [\beta]\}.$$

Note that $i(\alpha, \beta) = 0$ if and only if they have disjoint (proper isotopy) representatives. To lighten the notation, we typically will not distinguish between a curve (or arc) α and its proper isotopy class $[\alpha]$.

A connected, compact subsurface $X \subset S$ is *essential* if every component of ∂X is either a component of ∂S or is essential and nonperipheral in S. If X is essential, we define the *relative boundary* of X to be $\partial_S X = \partial X - \partial S$.

Remark 2.14 If $X \subset Y$ are both essential subsurfaces of S, then $\xi'(X) \leq \xi'(Y)$. Equality holds if and only if X and Y are isotopic. $\lhd$

We say that a properly embedded curve or an arc α *cuts* X if every $\alpha' \in [\alpha]$ intersects X. If α does not cut X, then we say that α *misses* X. Suppose that X and Y are essential, and nonisotopic, subsurfaces of S. We say that X is *nested* in Y if it is (perhaps after an isotopy) contained in Y. We say that X and Y *overlap* if $\partial_S X$ cuts Y and $\partial_S Y$ cuts X.

2.15 Curve and arc graphs

We now define the *curve graph* $\mathscr{C}(S)$. Let $\mathscr{C}^{(0)}(S)$ be the set of proper isotopy classes of essential, nonperipheral curves in S. We have an edge $(\alpha, \beta) \in \mathscr{C}^{(1)}$ exactly when α and β are distinct and $i(\alpha, \beta) = 0$.

We define the *arc graph* $\mathscr{A}(S)$ similarly: $\mathscr{A}^{(0)}(S)$ is the set of proper isotopy classes of essential arcs in S. Again we have an edge $(\alpha, \beta) \in \mathscr{A}^{(1)}$ exactly when α and β are distinct and $i(\alpha, \beta) = 0$. Note that $\mathscr{A}(S)$ is empty when S is closed.

Masur and Schleimer [2013, Definition 7.1] generalise the definition of the arc graph slightly, as follows. Suppose that $\Delta \subset \partial S$ is a nonempty collection of boundary components. We define $\mathscr{A}(S, \Delta)$ to be the subgraph of $\mathscr{A}(S)$ spanned by the arcs having both endpoints in Δ. Note that $\mathscr{A}(S, \partial S) = \mathscr{A}(S)$.

We next define the *arc and curve graph* $\mathscr{AC}(S)$: the zero skeleton is exactly $\mathscr{A}^{(0)}(S) \cup \mathscr{C}^{(0)}(S)$. Edges come from having disjoint representatives, as before. Note that the inclusion of $\mathscr{C}^{(0)}(S)$ into $\mathscr{AC}^{(0)}(S)$ induces a quasi-isometry of graphs.

The definition of the curve complex must be modified when $\xi(S) \le 1$: for $S_{1,1}$ we use $i(\alpha, \beta) = 1$ and for $S_{0,4}$ we use $i(\alpha, \beta) = 2$. For both of these surfaces the graph of curves is a copy of the *Farey graph*. When S is an annulus we define $\mathscr{C}^{(0)}(S)$ to be the set of proper isotopy classes of essential properly embedded arcs, where now isotopies are required to fix boundary points. Two classes span an edge if they have representatives which are disjoint on their interiors.

All of the various curve, arc, and arc and curve graphs are connected when they are nonempty [Masur and Minsky 1999, Lemma 2.1]. We make each of these into a metric graph by decreeing that all edges have length one. It is then a theorem of Masur and Minsky [1999, Theorem 1.1] that, for any surface S with $\xi(S) \ge 1$, the curve complex $\mathscr{C}(S)$ is Gromov hyperbolic. Masur and Schleimer [2013, Theorem 20.3] proved that the same holds for $\mathscr{A}(S, \Delta)$.

2.16 I–bundles

Suppose that F is a connected, compact surface, possibly nonorientable, with nonempty boundary. Let $\rho\colon T \to F$ be an I–bundle. We call F the *base surface* of the bundle. We define $\partial_v T = \rho^{-1}(\partial F)$ to be the *vertical boundary* of T. We define the closure

$$\partial_h T = \overline{\partial T - \partial_v T}$$

to be the *horizontal boundary* of T. Also, we define the curves

$$\partial(\partial_h T) = \partial(\partial_v T)$$

to be the *corners* of T. Finally, there is an involution $\tau\colon \partial_h T \to \partial_h T$ associated to ρ obtained by swapping the ends of interval fibres.

We now define $\rho_F\colon T_F \to F$ to be the *orientation* I–bundle over F. Here the preimage under ρ_F of a simple closed curve α is an annulus or a Möbius band as α is or is not, respectively, orientation-preserving in F. When F is nonorientable, we call T_F *twisted* and so $\partial_h T_F/\tau \cong F$ is nonorientable. If T_F is not twisted then $T_F \cong F \times [-1, 1]$ is a product. In this case, $\tau|_{F\times\{-1\}}$ is a homeomorphism from $F \times \{-1\}$ to $F \times \{1\}$.

2.17 Compression bodies

References on compression bodies, of the type we are interested in here, include [Bonahon 1983, Appendix B; Oertel 2002, Section 1].

Suppose that $S = S_{g,b}$ is a surface. We assume that S is neither a disk nor a sphere. We form $T = S \times I$. We take $\partial^+ T = S \times \{1\}$ and $\partial^- T = S \times \{0\}$ to be the *upper* and *lower* boundaries of T. As before, $\partial_v T = \partial S \times I$ is the *vertical boundary*. We now attach a collection of three-dimensional two- and three-handles to the lower boundary of T to obtain a three-manifold M. We define $\partial^+ M = \partial^+ T$ as well as $\partial_v M = \partial_v T$. Finally, we define

$$\partial^- M = \overline{\partial M - (\partial^+ M \cup \partial_v M)}.$$

Thus, M is a *compression body*. If $\partial^- M$ is not homeomorphic to $\partial^+ M$, then M is *nontrivial*. If $\partial^- M$ has no sphere or disk components, then M is *spotless*. To simplify the notation, we take $S = \partial^+ M$. Note that, if $\partial^- M$ is empty, then M is necessarily a handlebody of positive genus.

We now state the classification of compression bodies.

Theorem 2.18 *Suppose that M and N are compression bodies. Then $(M, \partial_v M)$ is homeomorphic to $(N, \partial_v N)$ if and only if $(\partial^+ M, \partial^- M)$ is homeomorphic to $(\partial^+ N, \partial^- N)$.* □

The proof is similar to that of the classification of surfaces [Farb and Margalit 2012, Theorem 1.1] and of handlebodies [Hempel 1976, Theorem 2.2]. The case of $\partial_v M = \varnothing$ is discussed by Biringer and Vlamis [2017, Corollary 2.3].

2.19 Disk graphs

Suppose that (M, S) is a nontrivial, spotless compression body. Suppose that $(D, \partial D) \subset (M, S)$ is a properly embedded disk. We call D *essential* if ∂D is essential in S. We now define the *disk graph* $\mathcal{D}(M, S)$. The vertices of $\mathcal{D}(M, S)$ are proper isotopy classes of essential disks in (M, S). A pair of distinct vertices D and E give an edge $(D, E) \in \mathcal{D}(M, S)$ if they have disjoint representatives.

2.20 Subsurface projection

We give one of the standard definitions of subsurface projection:

Definition 2.21 [Masur and Schleimer 2013, Definition 4.4] Suppose that X is an essential subsurface, but not a pair of pants, in S. The relation of *subsurface projection*, $\pi_X : \mathcal{AC}(S) \to \mathcal{C}(X)$, is defined as follows. Let $\rho_X : S^X \to S$ be the covering map corresponding to the subgroup $\pi_1(X) < \pi_1(S)$. Note that the Gromov compactification of S^X is homeomorphic to X. This gives an identification of the graphs $\mathcal{C}(S^X)$ and $\mathcal{C}(X)$. For any $\alpha \in \mathcal{AC}(S)$, we define $\alpha^X = \rho_X^{-1}(\alpha)$ to be the full preimage. We define $\alpha|_X$ to be the essential arcs, and essential nonperipheral curves, of α^X. If X is an annulus, then we set $\pi_X(\alpha) = \alpha|_X$. Otherwise, for every $\beta \in \alpha|_X$, we form $N = N(\beta \cup \partial X)$ and we place the essential isotopy classes of $\partial_X N$ into $\pi_X(\alpha)$. ◁

Note that, if α misses X, then $\pi_X(\alpha)$ is empty. Suppose instead that α cuts X. If X is an annulus, then the diameter of $\pi_X(\alpha)$ is at most one. If X is not an annulus, then the diameter of $\pi_X(\alpha)$ is at most two [Masur and Minsky 2000, Lemma 2.3]. If X and Y are essential subsurfaces of S, then we define

$\pi_Y(X) = \pi_Y(\partial_S X)$. If X and Y are disjoint, or if Y is nested in X, then this is empty. We record the following for later use:

Lemma 2.22 *Suppose that X and Y are overlapping essential subsurfaces of S. Then $\pi_Y(X)$ is nonempty and has diameter at most two.* □

We will adopt the following useful shorthand notation. Suppose that α and β are curves or arcs, both cutting an essential subsurface $X \subset S$. Then

$$d_X(\alpha, \beta) = d_{\mathscr{C}(X)}(\pi_X(\alpha), \pi_X(\beta))$$

is the *subsurface projection distance* between α and β in X.

3 Bound for the arc graph

Let $S = S_{g,b}$, where $\xi'(S) > 0$ and $b > 0$. Take $\Delta \subset \partial S$ to be a nonempty union of components. Let $\mathscr{A}(S, \Delta)$ be the graph of essential arcs with endpoints in Δ.

3.1 Witnesses for the arc graph

Definition 3.2 An essential subsurface $X \subset S$ is a *witness* for $\mathscr{A}(S, \Delta)$ if every arc $\alpha \in \mathscr{A}(S, \Delta)$ cuts X. ◁

We repackage a few results [Masur and Schleimer 2013, Lemmas 5.9 and 7.2]:

Lemma 3.3 *Suppose that $X \subset S$ is an essential subsurface, but not an annulus or a pair of pants. The following are equivalent:*

- *X is a witness for $\mathscr{A}(S, \Delta)$.*
- *X contains Δ.*
- *For all arcs $\alpha \in \mathscr{A}(S, \Delta)$, the projection $\pi_X(\alpha)$ is nonempty.*
- *The projection $\pi_X : \mathscr{A}(S, \Delta) \to \mathscr{C}(X)$ is coarsely Lipschitz with a constant of 2.* □

We have a useful corollary:

Corollary 3.4 *Suppose that X and Y are distinct witnesses with $\xi'(X) = \xi'(Y)$. Then X and Y overlap.*

Proof By Lemma 3.3, both X and Y contain Δ; thus, they intersect. Since they have the same modified complexity, by Remark 2.14 they cannot be nested. Thus, they overlap. □

We let $\mathrm{MCG}(S, \Delta)$ be the mapping class group for the pair (S, Δ): the group of mapping classes that preserve Δ setwise. We say that a pair of arcs $\alpha, \beta \in \mathscr{A}(S, \Delta)$ have the same *topological type* (or, more simply, the same *type*) if there is a mapping class $f \in \mathrm{MCG}(S, \Delta)$ such that $f(\alpha) = \beta$.

Lemma 3.5 *The quotient $\mathcal{A}(S,\Delta)/\mathrm{MCG}(S,\Delta)$ has diameter at most two.*

Proof Suppose that $\alpha \in \mathcal{A}(S,\Delta)$ is an arc. We break the proof into cases, depending on the number of components of ∂S and of Δ.

Suppose that Δ has at least two components. Then α is disjoint from some $\gamma \in \mathcal{A}(S,\Delta)$ meeting two components of Δ. Since we made no assumptions on α, we find that, in the quotient, all vertices are at most distance one from $[\gamma]$. Thus, the diameter is at most two.

Suppose that Δ has only one component, but ∂S has at least two. Let δ be some component of $\partial S - \Delta$. Then α is disjoint from some arc $\gamma \in \mathcal{A}(S,\Delta)$ such that γ separates δ from the rest of S. We again obtain a diameter bound of two.

Suppose that ∂S has a single component, which necessarily equals Δ. Then α is disjoint from some arc $\gamma \in \mathcal{A}(S,\Delta)$ such that γ is nonseparating. This gives the diameter bound and finishes the proof. □

3.6 Families of witnesses

Fix a number $c \le \xi'(S)$. The collection

$$\mathcal{F}_c = \{X \subset S \mid X \text{ is a witness for } \mathcal{A}(S,\Delta) \text{ and } \xi'(X) = c\}$$

is called a *family*. We only consider nonempty families.

Suppose that $X, Y, Z \in \mathcal{F}_c$ are witnesses, with Y distinct from both X and Z. Then we define

$$d_Y(X,Z) = d_{\mathcal{C}(Y)}(\pi_Y(X), \pi_Y(Z)).$$

This is well defined by Corollary 3.4. We note that there is an abuse of notation here: the family $\mathcal{F}_c$ consists of witnesses X — that is, surfaces — not metric graphs. However, each witness X gives a metric graph, namely $\mathcal{C}(X)$. We trust this will not cause confusion.

Lemma 3.7 *For every c, the family $\mathcal{F}_c$ satisfies the three BBF axioms given in Section 2.4. Since there are only finitely many of these families, there is a common constant k that works for all of them simultaneously.*

Proof Axiom 2.5 follows from Corollary 3.4 and Lemma 2.22.

Axiom 2.6 follows from Corollary 3.4 and the usual Behrstock inequality [2006, Theorem 4.3]. See [Mangahas 2010, Lemma 2.5] for an elementary proof following ideas of Leininger.

Axiom 2.7 appears as [Masur and Minsky 2000, Lemma 6.2]. See [Bestvina et al. 2015, Lemma 5.3] for a proof giving a concrete bound and avoiding the machinery of hierarchies. □

We now apply the BBF construction, outlined in Section 2.4, to each family $\mathcal{F}_c$. This gives us a quasitree of curve graphs $\mathcal{C}(\mathcal{F}_c)$. We deduce that $\mathcal{C}(\mathcal{F}_c)$ is a hyperbolic metric graph where each of the curve complexes $\mathcal{C}(X)$ for $X \in \mathcal{F}_c$ embeds as a totally geodesic subgraph. From [Bestvina and Bromberg 2019, Corollary 1.1] and Theorem 2.11, we deduce that

$$\mathsf{dim}_{\mathsf{asym}}\, \mathcal{C}(\mathcal{F}_c) \le c + 1.$$

If $c = 1$ then Theorem 2.12 allows us to sharpen the bound:

$$\mathsf{dim}_{\mathsf{asym}}\, \mathcal{C}(\mathcal{F}_1) \le 1.$$

On the other hand, if $c = 4g + b - 3$ then $\mathcal{F}_c = \{S\}$ and we have

$$\mathsf{dim}_{\mathsf{asym}}\, \mathcal{C}(\mathcal{F}_c) \le 4g + b - 3.$$

We now define $\mathcal{P}(S, \Delta)$ to be the product, equipped with the ℓ^1 metric, of the quasitrees of curve graphs $\mathcal{C}(\mathcal{F}_c)$ as c ranges from one to $\xi'(S)$. From the above and from Fact 2.2, we deduce the following:

Corollary 3.8 $$\mathsf{dim}_{\mathsf{asym}}\, \mathcal{P}(S, \Delta) \le \tfrac{1}{2}(4g + b)(4g + b - 3) - 2.$$ □

3.9 Embedding the arc graph

In this section we fix the constants k, K and L. We then state and prove Theorem 3.10.

The constant k is the larger of 13 (as explained in the proof of Lemma 3.13) and the constant given by Lemma 3.7. The constant K is now given by [Bestvina et al. 2015, Theorem A]. Finally, the constant L is provided by Fact 2.9. We take $\mathcal{P}(S, \Delta)$ to be the product of the resulting quasitrees. Here is the statement:

Theorem 3.10 *There is a quasi-isometric embedding ϕ of the arc graph $\mathcal{A}(S, \Delta)$ into the product $\mathcal{P}(S, \Delta)$ of quasitrees of curve graphs. Moreover, ϕ is equivariant with respect to the action of the mapping class group* $\mathrm{MCG}(S, \Delta)$.

From this, and from Fact 2.3, we deduce the following:

Corollary 3.11 *Suppose that $S = S_{g,b}$ has nonempty boundary. Suppose that $\Delta \subset \partial S$ is a nonempty union of components. Finally, suppose that $\xi'(S) \ge 1$. Then*

$$\mathsf{dim}_{\mathsf{asym}}\, \mathcal{A}(S, \Delta) \le \tfrac{1}{2}(4g + b)(4g + b - 3) - 2.$$ □

We now turn to the proof of Theorem 3.10. Fix a modified complexity c.

Definition 3.12 Suppose that $\beta \in \mathcal{A}(S, \Delta)$ is an arc. Suppose that $Y \in \mathcal{F}_c$ is a witness, and β has a representative contained in Y. Then we say that Y *carries* β. ◁

Note that $\pi_Y(\beta) \subset \mathcal{C}(Y)$ is one or two essential, nonperipheral curves in Y. Recall, by Fact 2.8, that $\mathcal{C}(Y)$ embeds into $\mathcal{C}(\mathcal{F}_c)$. We now define a relation $\phi_c \colon \mathcal{A}(S, \Delta) \to \mathcal{C}(\mathcal{F}_c)$ as follows:

$$\phi_c(\alpha) = \{\pi_Y(\beta) \mid d_{\mathcal{A}}(\alpha, \beta) \le 2 \text{ and } Y \in \mathcal{F}_c \text{ carries } \beta\}.$$

Lemma 3.13 *The relation ϕ_c is an equivariant coarse Lipschitz map.*

Proof Equivariance follows from the definition.

The set $\phi_c(\alpha)$ is nonempty by Lemma 3.5. Suppose that Y, Y' and Z lie in $\mathscr{F}_c$. Suppose that β and β' are carried by Y and Y', respectively, and are distance at most two from α. Thus, $d_{\mathscr{A}}(\beta, \beta') \le 4$. We deduce that $d_Z(Y, Y')$ is at most 12.

We now recall our choices (made above) of k, K and L. In particular, we have $\mathsf{k} > 12$. Thus, by Fact 2.9, $\pi_Y(\partial Y')$ and $\pi_{Y'}(\partial Y)$ are distance at most L in $\mathscr{C}(\mathscr{F}_c)$. Thus, $d_{\mathscr{C}}(\beta, \beta') \le \mathsf{L} + 20$, bounding the diameter of $\phi_c(\alpha)$. Thus, ϕ_c is a coarse map.

Furthermore, by [Masur and Minsky 2000, Lemma 2.3], if $d_{\mathscr{A}}(\alpha, \alpha') = 1$ then the distance between $\phi_c(\alpha)$ and $\phi_c(\alpha')$ is also bounded in terms of L. Applying the triangle inequality gives the result. □

Lemma 3.14 *Suppose that $\alpha, \gamma \in \mathscr{A}(S, \Delta)$ are arcs and X is a witness with $\xi'(X) = c$. Then*

$$|d_X(\alpha, \gamma) - d_X(\phi_c(\alpha), \phi_c(\gamma))| \le 12$$

Proof Suppose that $\beta \in \mathscr{A}(S, \Delta)$ has $d_{\mathscr{A}}(\alpha, \beta) \le 2$ and β is carried by some witness Y with $\xi'(Y) = c$. Note that α and $\pi_Y(\beta)$ are distance at most three in $\mathscr{AC}(S)$, the arc and curve complex for S.

Now, if $X = Y$, then

$$d_X(\alpha, \pi_Y(\beta)) = d_X(\alpha, \pi_X(\beta)) = d_X(\alpha, \beta) \le 4$$

by [Masur and Minsky 2000, Lemma 2.3]. If $X \ne Y$, then instead we have

$$d_X(\alpha, \pi_Y(\beta)) = d_{\mathscr{C}(X)}(\pi_X(\alpha), \pi_X(Y)) \le 6.$$

This holds for all β arising in the definition of $\phi_c(\alpha)$. The lemma now follows by applying the triangle inequality twice. □

We now define $\phi\colon \mathscr{A}(S, \Delta) \to \mathscr{P}(S, \Delta)$ by taking

$$\phi(\alpha) = (\phi_c(\alpha))_c.$$

All that remains is to prove that ϕ is a quasi-isometric embedding. Suppose that α and γ are arcs in $\mathscr{A}(S, \Delta)$. We must show that $d_{\mathscr{A}}(\alpha, \gamma)$ and $d_{\mathscr{P}}(\phi(\alpha), \phi(\gamma))$ are coarsely equal.

We first bound $d_{\mathscr{P}}(\phi(\alpha), \phi(\gamma))$ from above. Recall that $\mathscr{P}(S, \Delta)$ is equipped with the ℓ^1 metric and so

$$d_{\mathscr{P}}(\phi(\alpha), \phi(\gamma)) = \sum_c d_{\mathscr{C}(\mathscr{F}_c)}(\phi_c(\alpha), \phi_c(\gamma)).$$

Each of the terms on the right-hand side is bounded in terms of $d_{\mathscr{A}}(\alpha, \gamma)$ by Lemma 3.13 and we are done.

We now bound $d_{\mathscr{A}}(\alpha, \gamma)$ from above. Since $\mathscr{A}(S, \Delta)$ is a *combinatorial complex* in the sense of [Masur and Schleimer 2013, Section 5], we have a corollary of [Masur and Schleimer 2013, Theorems 5.14 and 13.1]:

Theorem 3.15 *Suppose that S and Δ are as above. There is a constant L such that, for any $\mathsf{L}' \geq \mathsf{L}$, there is a constant C with the following property. For any arcs α and γ, we have*

$$d_{\mathscr{A}}(\alpha, \gamma) =_{\mathsf{C}} \sum [d_X(\alpha, \gamma)]_{\mathsf{L}'},$$

where the sum is taken over all witnesses X for $\mathscr{A}(S, \Delta)$. □

Take $\mathsf{K}' > 12$ sufficiently larger than the constants K and L appearing in Theorems 2.10 and 3.15, respectively. Set $\mathsf{L}' = \mathsf{K}' + 12$. Fix a witness X and set $c = \xi'(X)$. If a term $d_X(\alpha, \gamma)$ appears in the upper bound of Theorem 3.15, then, by Lemma 3.14, the term $d_X(\phi_c(\alpha), \phi_c(\gamma))$ appears in the lower bound provided by Theorem 2.10 for the family $\mathscr{F}_c$. Also, $d_X(\alpha, \gamma)$ is at most twice $d_X(\phi_c(\alpha), \phi_c(\gamma))$ (by Lemma 3.14 and because $\mathsf{K}' > 12$). Thus, $d_{\mathscr{A}}(\alpha, \gamma)$ is coarsely bounded above by $d_{\mathscr{P}}(\phi(\alpha), \phi(\gamma))$, as desired. This finishes the proof of Theorem 3.10. □

4 Bound for the disk complex

Suppose that M is a spotless compression body, as defined in Section 2.17. Suppose that $S = S_{g,b} = \partial^+ M$ is the upper boundary. We assume that $\xi'(S) > 0$. Let $\mathscr{D}(M, S)$ be the graph of essential disks with boundary in S.

4.1 Witnesses for the disk complex

Definition 4.2 An essential subsurface $X \subset S$ is a *witness* for $\mathscr{D}(M, S)$ if every disk $D \in \mathscr{D}(M, S)$ cuts X. ◁

Some authors call such an X *disk-busting* [Masur and Schleimer 2013]. We call a witness X *large* if it satisfies

$$\operatorname{diam}_X\big(\pi_X(\mathscr{D}(M, S))\big) > 60.$$

We now record the classification of large witnesses [Masur and Schleimer 2013, Theorems 10.1, 11.10 and 12.1]:

Theorem 4.3 *Suppose that (M, S) is a nontrivial spotless compression body. Suppose that $X \subset S$ is a large witness for $\mathscr{D}(M, S)$. Then we have the following:*

- *X is not an annulus.*
- *If X compresses in M, then there are disks D and E with boundary contained in, and filling, X.*
- *If X is incompressible in M, then there is an orientation I–bundle $\rho_F : T_F \to F$ with $(T_F, \partial_h T_F) \subset (M, S)$ and with X being a component of $\partial_h T_F$. Also, $\partial_v T_F$ is properly embedded in (M, S) and at least one component of $\partial_v T_F$ is isotopic into S. Also, F admits a pseudo-Anosov map.* □

Remark 4.4 Suppose that X is a large incompressible witness, as in the third case. Let F be the base of the associated I–bundle T_F. Let P_F be the collection of annuli, embedded in S, which are isotopic, rel boundary, to components of $\partial_v T_F$. We call these annuli the *paring locus* for T_F. The paring locus P_F is nonempty (Theorem 4.3) and is disk-busting [Masur and Schleimer 2013, Remark 12.17]. We call an essential disk $(D, \partial D) \subset (M, S)$ *vertical* for T_F exactly when $D \cap T_F$ is vertical in T_F. Such disks exist by Theorem 4.3. If D is vertical for T_F, then D meets the paring locus P_F in exactly two essential arcs. Finally, the union $P_F \cup \partial_h T_F$ is a compressible witness for $\mathfrak{D}(M, S)$. ◁

Definition 4.5 If T_F is a product, then $\partial_h T_F = X \sqcup X'$, where $X' \subset S$ is again a large incompressible witness. In this case, we call X and X' *twins*. ◁

Similar to the case of the arc complex (Corollary 3.4), all large witnesses for the disk complex *interfere*, as follows:

Corollary 4.6 *Suppose that X and Y are disjoint large witnesses with $\xi'(X) = \xi'(Y)$. Then either*

(1) *X and Y are twins, or*

(2) *X and Y have twins, X' and Y', respectively, such that X' overlaps with Y and Y' overlaps with X.*

Proof Suppose that X and Y are not twins. Thus, we may apply [Masur and Schleimer 2013, Lemma 12.21] to find that X has a twin X' and X' intersects Y. Since $\xi'(X') = \xi'(Y)$, neither X' nor Y is nested in the other. Thus, X' overlaps with Y. The remainder of the proof is similar. □

As usual, we take $\mathrm{MCG}(M, S)$ to be the mapping class group for the pair (M, S): that is, the group of mapping classes of M that preserve S setwise. We say that a pair of disks $D, E \in \mathfrak{D}(M, S)$ have the same *topological type* (or, more simply, the same *type*) if there is a mapping class $f \in \mathrm{MCG}(M, S)$ such that $f(D) = E$.

Lemma 4.7 *The quotient $\mathfrak{D}(M, S)/\mathrm{MCG}(M, S)$ has diameter at most two.*

Proof There are two cases. Suppose first that $\mathfrak{D}(M, S)$ contains a nonseparating disk, that is, a disk $(D, \partial D) \subset (M, S)$ such that $M - D$ is connected. By Theorem 2.18 (the classification of compression bodies), all nonseparating disks have the same topological type. Suppose that $E \in \mathfrak{D}(M, S)$ is a separating disk. The classification of compression bodies implies that E is disjoint from some nonseparating disk D'. This gives the desired diameter bound.

Suppose instead that all essential disks in (M, S) are separating. Thus, the lower boundary of M is nonempty. Suppose that F is a component of the lower boundary of M. Let D be a separating disk which cuts a copy of $F \times I$ off of M. Now suppose that E is any separating disk. The classification of compression bodies implies that E is disjoint from some disk D' which is a homeomorphic image of D. □

4.8 Families of witnesses

Fix a modified complexity $c \le \xi'(S)$. The collection

$$\mathcal{F}_c = \{X \subset S \mid X \text{ is a large witness for } \mathcal{D}(M,S) \text{ and } \xi'(X) = c\}$$

is called a *complete family*. We now define a *reduced family* as follows. Suppose that X and X' in $\mathcal{F}_c$ are twins. Thus, there is an I–bundle $T = F_T \times I$, where $\partial_h T = X \sqcup X'$. We remove both X and X' from $\mathcal{F}_c$ and replace them by the base surface F_T. We abuse notation and again use $\mathcal{F}_c$ to denote the reduced family. When $A = F_T \in \mathcal{F}_c$ is the base surface associated to T, we abuse notation and define $\partial_S A = \partial_S \partial_h T$.

Lemma 4.9 *Fix (M,S) and c. Suppose that $A, B \in \mathcal{F}_c$ are distinct. Suppose that B is a base surface replacing the twinned surfaces Y and Y'. Then $\partial_S A$ cuts both Y and Y'.*

Proof Let T_B be the product I–bundle associated to B. Let P_B be the paring locus of T_B. Recall that P_B is disk-busting.

Suppose that A is a compressible witness. Thus, P_B cuts A. Suppose that $\partial_S A$ does not cut Y. Thus, Y is (after an isotopy) contained in A. From Remark 2.14, we deduce that Y is isotopic to A. Thus, P_B does not cut A, a contradiction. Thus, $\partial_S A$ cuts Y; a similar argument proves that $\partial_S A$ cuts Y'.

Suppose that A is an incompressible witness, with I–bundle T_A. Let P_A be the paring locus of T_A. There are two subcases as T_A is twisted or a product.

Suppose that T_A is twisted. Thus, $A \cup P_A$ is a compressible witness (Remark 4.4). Again, since P_B is disk-busting, it cuts $A \cup P_A$. If P_B cuts $\partial_S A$, we are done because P_B is parallel into both Y and Y'. If not, then P_B is (after an isotopy) contained in A or contained in P_A. In either case, Y and Y' must cut A. Since $\xi'(Y) = \xi'(Y') = \xi'(A)$, we cannot have Y or Y' contained in A (Remark 2.14). Thus, A overlaps *both* Y and Y', and we are done.

Suppose that T_A is a product. Let X and X' be the twin components of $\partial_h T_A$. Appealing to Corollary 4.6, we may assume that X overlaps with Y. If X overlaps with Y', then we are done. If it does not, then, by Corollary 4.6, we deduce that X' overlaps Y'. Thus, in either case, we are done. □

Definition 4.10 Fix a modified complexity c. Suppose that $A, B \in \mathcal{F}_c$. We now define $\pi_B(A)$. (Note that we are overloading the notation π_B. When the argument is a collection of curves, we use Definition 2.21. When the argument is a witness, we use Definition 4.10.) There are two cases:

1. Suppose that B is not a base surface. Then we define $\pi_B(A) = \pi_B(\partial_S A)$.
2. Suppose that B is a base surface. Suppose that $\rho_B \colon T_B \to B$ is the I–bundle associated to B. Then we isotope $\partial_S A$ to meet $\partial_h T_B$ minimally and we define $\pi_B(A) = \pi_B(\rho_B(T_B \cap \partial_S A))$. ◁

In the above definition, we are considering $\rho_B(T_B \cap \partial_S A)$ as a set of arcs and curves in B. We surger them one at a time to obtain a set of curves in B. Also, in both parts of the definition, Corollary 4.6 implies that $\pi_B(A)$ is nonempty.

Suppose that $A, B, C \in \mathscr{F}_c$. Suppose further that B is distinct from both A and C. Then we define

$$d_B(A, C) = d_{\mathscr{C}(B)}(\pi_B(A), \pi_B(C)).$$

We will now abuse notation: the reduced family $\mathscr{F}_c$ contains surfaces A which index metric spaces, namely the curve graphs $\mathscr{C}(A)$, instead of being metric spaces themselves.

We now verify the three BBF axioms, as stated in Section 2.4. Recall that (M, S) is a spotless compression body, together with its upper boundary. Also, $c \in \mathbb{Z}$ is an integer. Here is the proof of Axiom 2.5:

Lemma 4.11 *There is a constant $k > 0$ such that, for every (M, S), for every c and for every $A, B \in \mathscr{F}_c$, we have that* $\operatorname{diam}_B(\pi_B(A)) \le k$.

Proof Suppose that B is not a base surface. Note that $\partial_S A$ is a disjoint collection of curves. By Corollary 4.6, $\partial_S A$ cuts B. By [Masur and Minsky 2000, Lemma 2.3], the diameter of $\pi_B(A)$ in $\mathscr{C}(B)$ is at most two.

Suppose that B is a base surface. Let D be either a compressing disk for A or a vertical disk for T_A, as provided by Theorem 4.3 or Remark 4.4, respectively. If D is a compressing disk, then ∂D is disjoint from $\partial_S A$. If D is a vertical disk, then ∂D meets $\partial_S A = \partial_S \partial_h T_A$ in exactly four points. We now isotope ∂D to have minimal intersection with $\partial_S B$. Applying [Masur and Schleimer 2013, Lemma 12.20], we deduce that $\pi_B(\rho_B(T_B \cap \partial D))$ has bounded diameter. Thus, by the triangle inequality, $\pi_B(\rho_B(T_B \cap \partial_S A))$ also has bounded diameter, and we are done. □

We adopt the notation $d_Y(A, C) = d_Y(\partial_S A, \partial_S C)$.

Lemma 4.12 *Let k be the constant of Lemma 4.11. Fix (M, S) and c. Suppose that $A, B, C \in \mathscr{F}_c$, where B is a base surface replacing the twinned surfaces Y, Y'. Then*

$$d_Y(A, C) \le d_B(A, C) \le d_Y(A, C) + 2k$$

and the same holds for Y'.

Proof By Lemma 4.9, $d_Y(A, C)$ is defined. The first inequality follows from the definitions. The second inequality follows from two applications of Lemma 4.11 and the triangle inequality. □

Here is the proof of Axiom 2.6:

Lemma 4.13 *Let k be the constant of Lemma 4.11. Fix (M, S) and c. For every $A, B, C \in \mathscr{F}_c$, at most one of the following is greater than $12 + 2k$:*

$$d_A(B, C), \quad d_B(A, C), \quad d_C(A, B).$$

Proof It suffices to assume that $d_B(A, C) > 12 + 2k$ and bound $d_A(B, C)$ from above.

Suppose that neither B nor A is a base surface. Then we may apply the usual Behrstock inequality and deduce that $d_A(B, C) < 10$; see [Mangahas 2010, Lemma 2.5].

Suppose instead B is not a base surface but A is. Let X and X' be the twins over A. By Corollary 4.6, both X and X' overlap B. Applying [Masur and Minsky 2000, Lemma 2.3], $d_B(X,C) > 10+2k$. The usual Behrstock inequality gives $d_X(B,C) < 10$. Applying Lemma 4.12, we deduce that $d_A(B,C) < 10+2k$.

Suppose instead that B is a base surface but A is not. Let Y and Y' be the twins over B. By Lemma 4.12, $d_Y(A,C) \geq 12$ and also $d_{Y'}(A,C) \geq 12$. By Lemma 4.9, both $\partial_S B$ and $\partial_S C$ cut A. Suppose that $\partial_S Y$ cuts A. The usual Behrstock inequality gives $d_A(Y,C) < 10$. We deduce that $d_A(B,C) < 12$, as desired.

Finally, suppose that B and A are base surfaces. Let Y and Y', and X and X', be the twins over B and A, respectively. By Corollary 4.6, we may assume that X and Y overlap. Thus, $d_Y(X,C) > 10$, so $d_X(Y,C) < 10$, and we are done as above. □

Here is the proof of Axiom 2.7:

Lemma 4.14 *Let k be the constant of Lemma 4.11. Fix (M,S) and c. For every $A, C \in \mathscr{F}_c$, the following set is finite:*

$$\{B \in \mathscr{F}_c \mid A \neq B,\ B \neq C \text{ and } d_B(A,C) > 7+2k\}.$$

Proof If A and C are not base surfaces, then this follows from [Bestvina et al. 2015, Lemma 5.3] and Lemma 4.12. If A is a base surface but C is not, then suppose that X and X' are the twins over A. We may repeat the previous argument for the pairs (X,C) as well as (X',C), paying at most an additional two [Masur and Minsky 2000, Lemma 2.3] in each case. When both A and C are base surfaces, there are four such pairs and the cost is at most an additional four in each case. □

Since the axioms hold, as in Section 3.6 we may build the product of quasitrees of spaces $\mathscr{P}(M,S)$ for the disk graph. We obtain the following:

Corollary 4.15 *Suppose that (M,S) is a nontrivial spotless compression body with $S = S_{g,b}$. Suppose that $\xi'(S) \geq 1$. Then*

$$\operatorname{dim}_{\mathsf{asym}} \mathscr{P}(M,S) \leq \tfrac{1}{2}(4g+b)(4g+b-3)-2. \qquad \square$$

4.16 Embedding the disk complex

In this section, we prove the following:

Theorem 4.17 *There is a quasi-isometric embedding ϕ of the disk graph $\mathscr{D}(M,S)$ into the product $\mathscr{P}(M,S)$ of quasitrees of curve graphs. Moreover, ϕ is equivariant with respect to the action of the mapping class group* $\mathrm{MCG}(M,S)$.

We deduce from this, and from Fact 2.3, the following:

Corollary 4.18 *Suppose that (M,S) is a nontrivial spotless compression body with $S = S_{g,b}$. Suppose that $\xi'(S) \geq 1$. Then*

$$\operatorname{dim}_{\mathsf{asym}} \mathscr{D}(M,S) \leq \tfrac{1}{2}(4g+b)(4g+b-3)-2. \qquad \square$$

Remark 4.19 When $g > 1$ and $b = 0$, the upper bound is smaller by one. See [Bestvina and Bromberg 2019, Corollary 1.1]. $\lhd$

Remark 4.20 Hamenstädt [2019, Theorem 3.6] previously showed, in the special case of a handlebody H_g, that $\mathrm{dim}_{\mathsf{asym}}\,\mathscr{D}(H_g, \partial H_g) \le (3g-3)(6g-2)$. Her proof technique is quite different from ours. $\lhd$

The proof of Theorem 4.17 is the same as that of Theorem 3.10, with three changes. First, we replace the diameter bound (Lemma 3.5, for arcs) by Lemma 4.7. Second, we replace the definition of *carries* (Definition 3.12, for arcs) with the following:

Definition 4.21 Suppose that $D \in \mathscr{D}(M, S)$ is a disk and $A \in \mathscr{F}_c$. If $A \subset S$ is compressible (so not a base surface), then A *carries* D exactly when D is a compressing disk for A. If A is a twisted witness, or a base surface, then A *carries* D exactly when D is isotopic to a vertical disk in T_A. $\lhd$

Third and lastly, we replace the distance estimate of Theorem 3.15 with the following:

Theorem 4.22 *Suppose that M and S are as above. There is a constant K such that, for any $\mathsf{K}' \ge \mathsf{K}$, there is a constant C with the following property: for any disks D and E, we have*

$$d_{\mathscr{D}}(D, E) =_{\mathsf{C}} \sum [d_X(D, E)]_{\mathsf{K}'},$$

where the sum is taken over all X in all reduced families for $\mathscr{D}(M, S)$.

Proof The distance estimate [Masur and Schleimer 2013, Theorem 19.9] bounds $d_{\mathscr{D}}(D, E)$ above and below using the sum of projection distances to all witnesses. That is, the sum there is taken over all X in all complete families for $\mathscr{D}(M, S)$. Suppose that B is a base surface for the twins Y and Y'. By [Masur and Schleimer 2013, Theorem 12.20], all three of

$$d_B(D, E), \quad d_Y(D, E) \quad \text{and} \quad d_{Y'}(D, E)$$

are coarsely equal. Here we define $d_B(D, E) = d_B(\pi_B(\partial D), \pi_B(\partial E))$ and $\pi_B(\partial D) = \pi_B(\rho_B(T_B \cap \partial D))$ as in Definition 4.10. This and [Masur and Schleimer 2013, Theorem 19.9] gives the lower bound. The upper bound is proved in the same way, after weakening the constant C by a factor of two. $\square$

Acknowledgement Fujiwara is supported by Grants-in-Aid for Scientific Research from JSPS (20H00114, 15H05739).

References

[Behrstock 2006] **J A Behrstock**, *Asymptotic geometry of the mapping class group and Teichmüller space*, Geom. Topol. 10 (2006) 1523–1578 MR Zbl

[Behrstock et al. 2017] **J Behrstock**, **M F Hagen**, **A Sisto**, *Asymptotic dimension and small-cancellation for hierarchically hyperbolic spaces and groups*, Proc. Lond. Math. Soc. 114 (2017) 890–926 MR Zbl

[Bell and Dranishnikov 2008] **G Bell**, **A Dranishnikov**, *Asymptotic dimension*, Topology Appl. 155 (2008) 1265–1296 MR Zbl

[Bell and Fujiwara 2008] **G C Bell**, **K Fujiwara**, *The asymptotic dimension of a curve graph is finite*, J. Lond. Math. Soc. 77 (2008) 33–50 MR Zbl

[Bestvina and Bromberg 2019] **M Bestvina**, **K Bromberg**, *On the asymptotic dimension of the curve complex*, Geom. Topol. 23 (2019) 2227–2276 MR Zbl

[Bestvina et al. 2015] **M Bestvina**, **K Bromberg**, **K Fujiwara**, *Constructing group actions on quasi-trees and applications to mapping class groups*, Publ. Math. Inst. Hautes Études Sci. 122 (2015) 1–64 MR Zbl

[Biringer and Vlamis 2017] **I Biringer**, **N G Vlamis**, *Automorphisms of the compression body graph*, J. Lond. Math. Soc. 95 (2017) 94–114 MR Zbl

[Bonahon 1983] **F Bonahon**, *Cobordism of automorphisms of surfaces*, Ann. Sci. École Norm. Sup. 16 (1983) 237–270 MR Zbl

[Farb and Margalit 2012] **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton Univ. Press (2012) MR Zbl

[Gromov 1993] **M Gromov**, *Asymptotic invariants of infinite groups*, from "Geometric group theory, II" (G A Niblo, M A Roller, editors), London Math. Soc. Lecture Note Ser. 182, Cambridge Univ. Press (1993) 1–295 MR Zbl

[Hamenstädt 2019] **U Hamenstädt**, *Asymptotic dimension and the disk graph, II*, J. Topol. 12 (2019) 675–684 MR Zbl

[Harvey 1981] **W J Harvey**, *Boundary structure of the modular group*, from "Riemann surfaces and related topics" (I Kra, B Maskit, editors), Ann. of Math. Stud. 97, Princeton Univ. Press (1981) 245–251 MR Zbl

[Hempel 1976] **J Hempel**, *3–Manifolds*, Annals of Math. Studies 86, Princeton Univ. Press (1976) MR Zbl

[Mangahas 2010] **J Mangahas**, *Uniform uniform exponential growth of subgroups of the mapping class group*, Geom. Funct. Anal. 19 (2010) 1468–1480 MR Zbl

[Masur and Minsky 1999] **H A Masur**, **Y N Minsky**, *Geometry of the complex of curves, I: Hyperbolicity*, Invent. Math. 138 (1999) 103–149 MR Zbl

[Masur and Minsky 2000] **H A Masur**, **Y N Minsky**, *Geometry of the complex of curves, II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000) 902–974 MR Zbl

[Masur and Schleimer 2013] **H Masur**, **S Schleimer**, *The geometry of the disk complex*, J. Amer. Math. Soc. 26 (2013) 1–62 MR Zbl

[Oertel 2002] **U Oertel**, *Automorphisms of three-dimensional handlebodies*, Topology 41 (2002) 363–410 MR Zbl

[Vokes 2022] **K M Vokes**, *Hierarchical hyperbolicity of graphs of multicurves*, Algebr. Geom. Topol. 22 (2022) 113–151 MR Zbl

Department of Mathematics, Kyoto University
Kyoto, Japan
Mathematics Institute, University of Warwick
Coventry, United Kingdom

kfujiwara@math.kyoto-u.ac.jp, s.schleimer@warwick.ac.uk

Received: 18 May 2022 Revised: 26 October 2022

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US $705/year for the electronic version, and $1040/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 5 (pages 2389–2970) 2024