AGT

Algebraic & Geometric Topology

Volume 24 (2024)

Representation stability for homotopy automorphisms

ERIK LINDELL

BASHAR SALEH

msp

Algebraic & Geometric Topology **24**:5 (2024) 2673–2705
DOI: 10.2140/agt.2024.24.2673
Published: 19 August 2024

Representation stability for homotopy automorphisms

ERIK LINDELL
BASHAR SALEH

We consider in parallel pointed homotopy automorphisms of iterated wedge sums of finite CW–complexes and boundary-relative homotopy automorphisms of iterated connected sums of manifolds minus a disk. Under certain conditions on the spaces and manifolds, we prove that the rational homotopy groups of these homotopy automorphisms form finitely generated FI–modules, and thus satisfy representation stability for symmetric groups in the sense of Church and Farb. We also calculate explicit bounds on the weights and stability degrees of these FI–modules.

55P62

1. Introduction 2673
2. Representation stability, FI–modules and Lie models of FI–spaces 2676
3. Rational homotopy theory for homotopy automorphisms 2686
4. Homotopy automorphisms of wedge sums 2691
5. Homotopy automorphisms of connected sums 2694
References 2704

1 Introduction

Pointed homotopy automorphisms of iterated wedge sums of spaces and boundary-relative homotopy automorphisms of iterated connected sums of manifolds minus a disk, come with stabilization maps that yield questions of whether the homology groups or the homotopy groups of these homotopy automorphisms stabilize in any sense. Previously Berglund and Madsen [2020] have proven rational homological stability for homotopy automorphisms of iterated connected sums of higher-dimensional tori $S^n \times S^n$ for $n \geq 3$, and these results were later expanded by Grey [2019] and Stoll [2024] for homotopy automorphisms of iterated connected sums of certain manifolds of the form $S^n \times S^m$ for $n, m \geq 3$.

We instead study the rational *homotopy groups* of the homotopy automorphisms in question, which we consider as based spaces with the identity map as the basepoint. These homotopy groups do not stabilize in the traditional sense. Instead, we show that they satisfy a different kind of stability, known as *representation stability*. In the two cases we study here, we consider sequences of rational homotopy groups, which in step n are representations of the symmetric group Σ_n. For such representations there is a consistent way to name the irreducible representations for arbitrary n, and representation stability essentially means that as n tends to infinity, the decomposition into irreducible representations eventually becomes constant.

Representation stability was introduced by Church and Farb [2013] and later further developed by Church, Ellenberg and Farb [Church et al. 2015], who showed that for representations of symmetric groups this notion can be encoded by so-called FI–modules, which are functors from the category of finite sets and injections to the category of vector spaces. The stable range of representation stability corresponds to *stability degree* and *weight* of the corresponding FI–module.

We review FI–modules and representation stability in more detail in Section 2. Our first main result is the following:

Theorem A *Let $(X,*)$ be a pointed simply connected space with the homotopy type of a finite CW–complex and let $X_S := \bigvee^S X$ for any finite set S. For each $k \geq 1$, the functor*

$$S \mapsto \pi_k^{\mathbb{Q}}(\mathrm{aut}_*(X_S))$$

is an FI*–module. If $H_n(X,\mathbb{Q}) = 0$ for $n \geq d$, this* FI*–module is of weight $\leq k+d-1$ and stability degree $\leq k+d$.*

For the analogous theorem for connected sums, we need the notion of a boundary-relative homotopy automorphism of a manifold N (with boundary). A boundary-relative homotopy automorphism of N is a homotopy automorphism of N that preserves the boundary $\partial := \partial N$ pointwise. The boundary-relative homotopy automorphisms of N form a topological monoid, with respect to composition, which we will denote by $\mathrm{aut}_\partial(N)$.

Let $M = M^d$ be a closed oriented d–dimensional manifold. For any finite set S, we let M_S denote the S–fold connected sum of M with itself, with an open d–disk removed: $M_S = \#^S M \setminus \mathring{D}^d$. For $\boldsymbol{n} = \{1,2,\dots,n\}$, we denote $M_{\boldsymbol{n}}$ simply by M_n. A homotopy automorphism of M_n does not extend to a homotopy automorphism of M_{n+1} in any canonical way in general. However, boundary-relative homotopy automorphisms of M_n extend by the identity to a boundary-relative homotopy automorphism of M_{n+1}. In particular, there is a stabilization map

$$s_n \colon \mathrm{aut}_\partial(M_n) \to \mathrm{aut}_\partial(M_{n+1}).$$

By picking some basepoint in the boundary of M_1, there is a deformation retract $M_S \xrightarrow{\simeq} \bigvee^S M_1$ (see eg [Félix et al. 2008, Section 3.1.2]), where the wedge sum is taken along this basepoint. It follows by Theorem A that there is an FI–module given on objects by $S \mapsto \pi_k(\mathrm{aut}_*(M_S)) \cong \pi_k(\mathrm{aut}_*(\bigvee^S M_1))$. For any finite set S we have an obvious inclusion map $\mathrm{aut}_\partial(M_S) \hookrightarrow \mathrm{aut}_*(M_S)$, so we may ask whether we can find an FI–module given by $S \mapsto \pi_k(\mathrm{aut}_\partial(M_S))$ that make these maps into a morphism of FI–modules, ie a natural transformation of functors. We will refer to this as “lifting” the FI–module structure. In our second main theorem, we address this problem:

Theorem B *Let $M = M^d$ be a closed simply connected oriented d–dimensional manifold. With M_S defined as above, we have the following*:

(a) *For each $k \geq 1$, the* FI*–module*

$$S \mapsto \pi_k\Bigl(\mathrm{aut}_*\Bigl(\bigvee^S M_1\Bigr)\Bigr) \cong \pi_k(\mathrm{aut}_*(M_S))$$

lifts to an FI*–module*

$$S \mapsto \pi_k(\mathrm{aut}_\partial(M_S))$$

sending the standard inclusion $\boldsymbol{n} \to \boldsymbol{n+1}$ to the map $\pi_k(\mathrm{aut}_\partial(M_n)) \to \pi_k(\mathrm{aut}_\partial(M_{n+1}))$ induced by the stabilization map s_n.

(b) *The rationalization of this* FI*–module is of weight $\leq k + d - 2$ and stability degree $\leq k + d - 1$.*

Remark 1.1 Theorems A and B are somewhat analogous to those for unordered configuration spaces of manifolds. Rational homological stability for unordered configuration spaces of arbitrary connected manifolds was proven by Church [2012], following integral results for open[1] manifolds by Arnold [1969], McDuff [1975] and Segal [1979]. It was later proven by Kupers and Miller [2018] that the rational *homotopy groups* of unordered configuration spaces on connected, simply connected manifolds of dimension at least 3 satisfy representation stability.

Homotopy automorphisms of iterated wedge sums of *spheres* have been studied by Miller, Patzt and Petersen [Miller et al. 2019]. Using representation stability, they prove that for $d \geq 2$ the sequence $\{B\,\mathrm{aut}(\bigvee_{i=1}^n S^d)\}_{n\geq 1}$ satisfies homological stability with $\mathbb{Z}[\frac{1}{2}]$–coefficients, which proves homological stability with the same coefficients for $\{B\,\mathrm{GL}_n(\mathbb{S})\}_{n\geq 1}$, where $\mathbb{S}$ is the sphere spectrum. These results are neither weaker nor stronger than Theorem A, since on one hand they work with $\mathbb{Z}[\frac{1}{2}]$–coefficients and on the other hand we work with wedge sums of more general CW–complexes than spheres.

For a simply connected d–dimensional manifold M, with boundary $\partial M \cong S^{d-1}$, the rational homotopy theory of $\mathrm{aut}_\partial(M)$ has been thoroughly studied by Berglund and Madsen [2020], whose results we will use.

As a byproduct of the techniques used for proving Theorem B(a) we get the following:

Theorem *Let M be a closed oriented simply connected d–dimensional manifold such that the reduced homology of $M \setminus \mathring{D}^d$ is nontrivial. Given a subspace $A \subseteq \partial M_n$, possibly empty, such that $A \subset M_n$ is a cofibration, then the groups $\pi_0(\mathrm{aut}_A(M_n))$, $\pi_0(\mathrm{Diff}_A(M_n))$ and $\pi_0(\mathrm{Homeo}_A(M_n))$ contain a subgroup isomorphic to Σ_n.*

Structure In Section 2 we review the necessary background on FI–modules. The reader familiar with FI–modules may skip directly to Section 2.8, where we introduce the notion of FI–*Lie models* of pointed FI–spaces, which is of key importance for proving the main theorems. In Section 3 we review rational

[1] Integral homological stability is known not to hold for closed manifolds. A simple counterexample is given already by the 2–sphere S^2, where $H_1(B_n(S^2), \mathbb{Z}) \cong \mathbb{Z}/(2n-2)\mathbb{Z}$; see for example [Birman 1974, Theorem 1.11].

homotopy theory for homotopy automorphisms needed for proving the main theorems. In Section 4 we study homotopy automorphisms of wedge sums and prove Theorem A. In Section 5 we study homotopy automorphisms of connected sums and prove Theorem B.

Acknowledgements The authors would like to thank Dan Petersen and Alexander Berglund for many useful comments and suggestions. We would also like to thank the referees, whose careful reading and useful comments have improved the paper greatly from its first version.

Section 5.1, which treats integral homotopy theory for relative homotopy automorphisms, has developed greatly since the first preprint version of this paper, thanks to many other people. Our decision to consider the integral homotopy groups of the homotopy automorphisms of iterated connected sums is inspired by an answer by Ryan Budney to a question by Saleh at MathOverflow. The method used to prove Theorem B(a) was suggested by Manuel Krannich, who has also provided several other very helpful comments. In addition, he was in the committee for the PhD defense of Lindell, where he, together with Fabian Hebestreit, pointed out several minor errors in the paper and had some very helpful suggestions for improvements.

Saleh was supported by the Knut and Alice Wallenberg Foundation through grant 2019.0521.

2 Representation stability, FI–modules and Lie models of FI–spaces

2.1 Conventions

Throughout the paper, we will use R to denote a commutative ring, which we will assume to be Noetherian for convenience. We will mainly work over the field $\mathbb{Q}$, so unless otherwise specifically stated, all vector spaces are over $\mathbb{Q}$. We will use "dg" to abbreviate the term *differential graded*. FI denotes the category of finite sets with injective maps as morphisms.

If S is a finite set, we will use $|S|$ to denote its cardinality, and we will write $\Sigma(S) := \mathrm{Aut}_{\mathrm{FI}}(S)$ for the symmetric group on S. If $S = \boldsymbol{n} := \{1, 2, \dots, n\}$, we will simply write $\Sigma(S) = \Sigma_n$ for brevity.

Recall that the irreducible $\mathbb{Q}$–representations of Σ_n are indexed by partitions of weight n, ie sequences of nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0 \geq \cdots)$ such that $|\lambda| = \lambda_1 + \lambda_2 + \cdots = n$. We will denote the corresponding $\mathbb{Q}$–representation by V_λ. For any $k \geq n + \lambda_1$, we also define the *padded* partition $\lambda[k] := (k - n \geq \lambda_1 \geq \lambda_2 \geq \cdots)$ and write $V(\lambda)_k := V_{\lambda[k]}$.

2.2 Representation stability

Before we introduce the language of FI–modules, let us recall the original notion of representation stability, which is formulated in terms of consistent sequences of Σ_n–representations.

Definition 2.1 Let R be a commutative ring. A *consistent sequence* of Σ_n–representations over R is a sequence $\{V^n, \phi^n\}$, where V^n is an $R[\Sigma_n]$–module and $\phi^n \colon V^n \to V^{n+1}$ is a Σ_n–equivariant map (where V^{n+1} is considered an $R[\Sigma_n]$–module through the standard inclusion $\Sigma_n \hookrightarrow \Sigma_{n+1}$).

If $R = \mathbb{Q}$, we may define (uniform) *representation stability* for such a sequence as follows:

Definition 2.2 A consistent sequence of rational Σ_n–representations $\{V^n, \phi^n\}$ is said to be uniformly representation stable with stable range $n \geq N$ if, for all $n \geq N$,

(i) the map ϕ^n is injective,

(ii) the image of ϕ^n generates V^{n+1} as a Σ_{n+1}–representation,

(iii) for each partition λ the multiplicity of $V(\lambda)_n$ in V^n is independent of n.

Next, we will introduce FI–modules, and recall how representation stability is encoded in that language.

2.3 FI–modules

We first introduce the notion of an FI–object in an arbitrary category.

Definition 2.3 Let $\mathcal{C}$ be a category. A functor $\mathrm{FI} \to \mathcal{C}$ is called an FI–object in $\mathcal{C}$.

Let us review the kinds of FI–objects that will be of interest to us:

- An FI–object in $(\mathrm{gr})\mathrm{Mod}_R$, the category of ($\mathbb{Z}$–graded) R–modules, is called a (graded) FI–*R–module*. An FI–object in dgMod_R, the category of differential graded R–modules, is called a *dg* FI–R–module. For a dg FI–R–module $\mathcal{V}$, we will write $H_*(\mathcal{V})$ for the composition with the homology functor and refer to it as the homology of $\mathcal{V}$.

- An FI–object in dgLie_R, the category of dg Lie algebras, over R, will be called a dg FI–R–Lie algebra.

- An FI–object in Top_*, the category of pointed topological spaces, will be called a *pointed* FI–*space*. If $\mathcal{P}$ is a property of pointed topological spaces, such as being simply connected, we will say that a pointed FI–space $\mathcal{X}$ has property $\mathcal{P}$ if $\mathcal{X}(S)$ has property $\mathcal{P}$, for every finite set S. If $\mathcal{X}$ is a pointed FI–space with $\pi_1(\mathcal{X}(S))$ being abelian for every finite set S, composing with the (rational) homotopy groups functor π_* (resp. $\pi_*^{\mathbb{Q}}$) naturally gives us a graded FI–$\mathbb{Z}$–module (resp. graded FI–$\mathbb{Q}$–module). We will simply write $\pi_*(\mathcal{X})$ (resp. $\pi_*^{\mathbb{Q}}(\mathcal{X})$) for this composite functor and refer to it as the (rational) homotopy groups of $\mathcal{X}$.

We will generally consider the first two examples for $R = \mathbb{Q}$ and $R = \mathbb{Z}$. If the ring is clear from context, or if the choice of R is not important, we will generally drop it from the notation.

Now let us recall some basics from the theory of FI–modules. Since the category of (graded) R–modules is abelian, the category of (graded) FI–R–modules inherits this structure, which means that there are natural notions of (graded) FI–R–submodules as well as quotients, direct sums and tensor products of (graded) FI–R–modules, all defined pointwise; see [Church et al. 2015, Remark 2.1.2].

Remark 2.4 Any FI–R–module $\mathcal{V}$ gives rise to a consistent sequence $\{V^n := \mathcal{V}(\boldsymbol{n}), \phi^n := \mathcal{V}(\boldsymbol{n} \hookrightarrow \boldsymbol{n+1})\}$ of $R[\Sigma_n]$–modules, where $\boldsymbol{n} \hookrightarrow \boldsymbol{n+1}$ is the standard inclusion.

Remark 2.5 Not every consistent sequence arises from an FI–module; see [loc. cit., Remark 3.3.1].

Sometimes it will be more convenient to work with consistent sequences than with FI–modules. For this purpose, the following lemma is important:

Lemma 2.6 [loc. cit., Remark 3.3.1] *A consistent sequence $\{V^n, \phi^n\}$ is induced by some* FI*–module if and only if every $\sigma \in \Sigma_{n+k}$ with $\sigma|_{\boldsymbol{n}} = \mathrm{id}$ acts trivially on*

$$\mathrm{im}(\phi^{n+k-1} \circ \cdots \circ \phi^n \colon V^n \to V^{n+k}).$$

If two FI*–modules give rise to isomorphic consistent sequences, then the two* FI*–modules are isomorphic.*

The main property of FI–modules that will be of interest to us is *finite generation*, since this is what encodes representation stability:

Definition 2.7 Let $\boldsymbol{n} := \{1, 2, \ldots, n\}$. A (graded) FI–$R$–module $\mathcal{V}$ is said to be *finitely generated* if there exists a finite set $S \subset \bigsqcup_{n\geq 1} \mathcal{V}(\boldsymbol{n})$ such that there is no proper (graded) FI–R–submodule $\mathcal{W}$ of $\mathcal{V}$ such that $S \subset \bigsqcup_{n\geq 1} \mathcal{W}(\boldsymbol{n})$.

Now we can describe how representation stability relates to FI–modules:

Theorem 2.8 [loc. cit., Theorem 1.13] *An* FI–$\mathbb{Q}$*–module $\mathcal{V}$ is finitely generated if and only if the consistent sequence $\{V^n := \mathcal{V}(\boldsymbol{n})\}$ is uniformly representation stable and each V^n is finite-dimensional.*

What makes working with the category of FI–R–modules for any Noetherian ring R particularly useful is that it is *Noetherian*, ie an FI–R–submodule of such a finitely generated FI–R–module is itself finitely generated; see [Church et al. 2015, Theorem 1.3; 2014, Theorem A]. Finite generation is also preserved by tensor products and quotients. This means that to prove that an FI–R–module is finitely generated, it suffices to show that it is a subquotient of a tensor product of some FI–R–modules that are more obviously finitely generated.

Since we want to use rational homotopy theory to prove our results, we need to consider graded FI–modules in our proofs. For this reason we will need the following definition:

Definition 2.9 If $\mathcal{V}$ is a graded FI–R–module and $m \in \mathbb{Z}$, let $\mathcal{V}_m$ be the degree-m part of $\mathcal{V}$, ie the postcomposition with the functor $\mathrm{grVect}_{\mathbb{Q}} \to \mathrm{Vect}_{\mathbb{Q}}$ given by sending a graded vector space to its degree-m part. If $\mathcal{V}_m = 0$ for $m \leq m'$ (resp. $m \geq m'$), we say that $\mathcal{V}$ is concentrated in degrees above (resp. below) m'. Such a graded FI–module is called bounded from below (resp. above).

2.4 Weight and stability degree

For the rest of Section 2 we will assume that $R = \mathbb{Q}$. We have seen how finite generation of FI–$\mathbb{Q}$–modules corresponds to representation stability of the corresponding consistent sequence of rational Σ_n–representations, but in order to make quantitative statements about stability ranges we need to introduce the *weight* and *stability degree* of such FI–modules.

Recall that if V is a Σ_n–representation, $(V)_{\Sigma_n}$ denotes the quotient of *coinvariants* of V. For an FI–module $\mathcal{V}$, this allows us to define a sequence $\{\phi_a(\mathcal{V})^n\}$ of vector spaces and maps between them, for each $a \geq 0$, by $\phi_a(\mathcal{V})^n := (\mathcal{V}(\boldsymbol{a} \sqcup \boldsymbol{n}))_{\Sigma_n}$. Any inclusion $\iota\colon \boldsymbol{n} \hookrightarrow \boldsymbol{n+1}$ gives us an inclusion $\mathrm{id} \sqcup \iota\colon \boldsymbol{a} \sqcup \boldsymbol{n} \hookrightarrow \boldsymbol{a} \sqcup (\boldsymbol{n+1})$, inducing a map $\phi_a(\mathcal{V})^n \to \phi_a(\mathcal{V})^{n+1}$. Since we quotient by Σ_{n+1}, the choice of inclusion ι does not matter.

With this, we can define the *stability degree* of an FI–module:

Definition 2.10 [Church et al. 2015, Definition 3.1.3] The *injectivity degree* inj-deg$(\mathcal{V})$ (resp. *surjectivity degree* surj-deg$(\mathcal{V})$) of an FI–module $\mathcal{V}$ is the smallest $s \geq 0$ such that for all $a \geq 0$, the map $\phi_a(\mathcal{V})^n \to \phi_a(\mathcal{V})^{n+1}$, defined as above, is injective (resp. surjective) for all $n \geq s$ (and if no such s exists we set the degree to ∞). We define the *stability degree* stab-deg(V) of $\mathcal{V}$ to be the maximum of the injectivity and surjectivity degrees.

Definition 2.11 The weight of an FI–module $\mathcal{V}$, which we denote by weight$(\mathcal{V})$, is the maximum weight $|\lambda|$ over all $V(\lambda)_n$ appearing in the Σ_n–representation $\mathcal{V}(\boldsymbol{n})$, if such a maximum exists. If no maximum exists, we set weight$(\mathcal{V}) = \infty$ and if the FI–module is zero we set it to zero.

These definitions are relevant because of their relation to representation stability, which may now be stated as follows:

Proposition 2.12 [loc. cit., Proposition 3.3.3] *Let $\mathcal{V}$ be an* FI*–module. The consistent sequence $\{V^n, \phi^n\}$ determined by $\mathcal{V}$ is uniformly representation stable with stable range $n \geq$* weight$(\mathcal{V})$ + stab-deg$(\mathcal{V})$.

Remark 2.13 This implies that if an FI–module has finite weight and stability degree, it is finitely generated. For this reason we will only be working with weight and stability degree going forward. However, due to the Noetherian property of FI–modules, it is possible to freely take submodules and quotients and preserve finite generation. This is not the case for stability degree, as we will see, making it much easier to prove finite generation than to obtain an explicit bound on stability degree.

Let us recall some useful properties of weight and stability degree. First, the following is immediate from the definitions:

Proposition 2.14 *Let $\mathcal{V}^1$ and $\mathcal{V}^2$ be* FI*–modules. Then* weight$(\mathcal{V}^1 \oplus \mathcal{V}^2) \leq \max($weight$(\mathcal{V}^1)$, weight$(\mathcal{V}^2))$ *and* stab-deg$(\mathcal{V}^1 \oplus \mathcal{V}^2) \leq \max($stab-deg$(\mathcal{V}^1)$, stab-deg$(\mathcal{V}^2))$.

Next, we will recall how weight and stability degree behave under taking tensor products:

Proposition 2.15 *Suppose that $\mathcal{V}^1, \mathcal{V}^2, \dots, \mathcal{V}^k$ are* FI*–modules with stability degrees $\leq r_1, r_2, \dots, r_k$ and weights $\leq s_1, s_2, \dots, s_k$, respectively. Then*

$$\mathrm{weight}(\mathcal{V}^1 \otimes \cdots \otimes \mathcal{V}^k) \leq s_1 + \cdots + s_k$$

and

$$\text{stab-deg}(\mathcal{V}^1 \otimes \cdots \otimes \mathcal{V}^k) \leq \max(r_1 + s_1, \dots, r_k + s_k, s_1 + \cdots + s_k).$$

Proof The first part is [Church et al. 2015, Proposition 3.2.2], while the second part is [Kupers and Miller 2018, Proposition 2.23]. □

We also need to know how stability degree and weight behave when taking submodules and quotients:

Proposition 2.16 *Let $\mathcal{V}$ be an* FI–*module and $\mathcal{W}$ be an* FI–*submodule of $\mathcal{V}$. Then* $\mathrm{weight}(\mathcal{W}) \le \mathrm{weight}(\mathcal{V})$ *and* $\mathrm{weight}(\mathcal{V}/\mathcal{W}) \le \mathrm{weight}(\mathcal{V})$. *If in addition $\mathcal{V}$ is such that $\mathcal{V}(S)$ is finite-dimensional for every finite set S, we have the following*:

(i) $\text{inj-deg}(\mathcal{W}) \le \text{inj-deg}(\mathcal{V})$.

(ii) $\text{surj-deg}(\mathcal{V}/\mathcal{W}) \le \text{surj-deg}\,(V)$.

(iii) *If* $\text{inj-deg}(\mathcal{V}) \le r$ *and* $\text{surj-deg}(\mathcal{W}) \le r$, *then* $\text{inj-deg}(\mathcal{V}/\mathcal{W}) \le r$.

(iv) *If* $\text{surj-deg}(\mathcal{V}) \le r$ *and* $\text{inj-deg}(\mathcal{V}/\mathcal{W}) \le r$, *then* $\text{surj-deg}(\mathcal{W}) \le r$.

Proof The first part follows directly by the definition of weight, and (i) and (ii) are [Church et al. 2015, Lemma 3.1.6].

To prove (iii) and (iv), note that for each $a \ge 0$, ϕ_a defines a functor from the category of FI–modules to the category of sequences of vector spaces and linear maps, and this functor is exact. Thus the following respective propositions from linear algebra suffice to prove (iii) and (iv): if $f: V \to W$ is a linear map of finite-dimensional vector spaces, $V' \subseteq V$ and $W' \subseteq W$ are subspaces, $f': V' \to W'$ is a linear map such that $f'(v) = f(v)$ for all $v \in V'$ and $f/f': V/V' \to W/W'$ is the induced map between the quotients, then

(iii′) if f is injective and f' is surjective, f/f' is injective,

(iv′) if f is surjective and f/f' is injective, f' is surjective.

These are both simple exercises in linear algebra and therefore left to the reader. □

Note however that given *only* the stability degree of an FI–module we can in general not say anything about the stability degree of its FI–submodules or quotients. However, if an FI–module $\mathcal{V}$ is isomorphic to *both* an FI–submodule of an FI–module and a quotient of an FI–module, for both of which we have bounds on the stability degree, we can use the proposition above to determine a bound on $\text{stab-deg}(\mathcal{V})$. This will be the case for an important class of FI–modules that we consider in Section 2.6. Note that in particular, we get the following corollary:

Corollary 2.17 *Suppose $\mathcal{W}$ is an* FI–*module which is a direct summand of another* FI–*module $\mathcal{V}$, ie that there exists a third* FI–*module $\mathcal{U}$ such that $\mathcal{V} \cong \mathcal{W} \oplus \mathcal{U}$. Then* $\text{stab-deg}(\mathcal{W}) \le \text{stab-deg}(\mathcal{V})$.

Finally, we need a way to determine the weight and stability degree in each degree when taking the homology of a differential graded FI–module. We will prove the following more general statement (see [Kupers and Miller 2018, Proposition 2.19]):

Proposition 2.18 *Let $\mathcal{U} \xrightarrow{f} \mathcal{V} \xrightarrow{g} \mathcal{W}$ be a sequence of* FI*–modules and morphisms of* FI*–modules such that $\mathcal{U}(S), \mathcal{W}(S)$ and $\mathcal{V}(S)$ are finite-dimensional for all $S \in \mathrm{FI}$ and $g \circ f = 0$. Then*

$$\mathrm{weight}(\ker(g)/\mathrm{im}(f)) \le \mathrm{weight}(\mathcal{W}),$$

and if all three FI*–modules have stability degree $\le r$, then* $\text{stab-deg}(\ker(g)/\mathrm{im}(f)) \le r$*. In particular, if $\mathcal{V}$ is a dg* FI*–module such that $\mathcal{V}_m(S)$ is finite-dimensional for each m and finite set S then* $\mathrm{weight}(H_m(\mathcal{V})) \le \mathrm{weight}(\mathcal{V}_m)$*, and if* $\text{stab-deg}(\mathcal{V}_i) \le r$ *for $i \in \{m-1, m, m+1\}$, we have* $\text{stab-deg}(H_m(\mathcal{V})) \le r$*.*

Proof The first part follows directly from the first part of Proposition 2.16. We prove the second part by showing that the homology has injectivity and surjectivity degree $\le r$. For injectivity degree, note that $\ker(g)$ has injectivity degree $\le r$ by Proposition 2.16(i), since it is an FI–submodule of $\mathcal{V}$. Furthermore, since the category of FI–modules is abelian, $\mathrm{im}(f) \cong \mathcal{U}/\ker(f)$, which has surjectivity degree $\le r$ by Proposition 2.16(ii). Thus it follows from Proposition 2.16(iii) that $\ker(g)/\mathrm{im}(f)$ has injectivity degree $\le r$.

For surjectivity degree, we argue similarly as follows: The injectivity degree of $\mathrm{im}(g)$ is at most r by Proposition 2.16(i), and since $\mathrm{im}(g) \cong \mathcal{V}/\ker(g)$, we thus get by Proposition 2.16(iv) that $\ker(g)$ has surjectivity degree $\le r$. Thus the quotient $\ker(g)/\mathrm{im}(f)$ does as well, by Proposition 2.16(ii). □

2.5 FI$^{\#}$–modules

Many FI–modules appearing "naturally" actually have additional structure, which may be encoded using the notion of an FI$^{\#}$–module. The category FI$^{\#}$ has the same objects as FI, but the morphisms $S \to T$ are given by a pair of subsets $A \subset S$ and $B \subset T$ and a bijection $A \to B$. We call these *partial injections*. An FI$^{\#}$–object in a category $\mathcal{C}$ is simply a functor $\mathrm{FI}^{\#} \to \mathcal{C}$. Since FI is a subcategory of FI$^{\#}$, any FI$^{\#}$–object has an underlying FI–object, so all the notions defined in the previous sections can be defined for (graded) FI$^{\#}$–modules by simply considering the underlying (graded) FI–module.

We consider FI$^{\#}$–modules because there is a natural way to define *duals* in this category. Note that the category FI$^{\#}$ is naturally isomorphic to its opposite category simply by taking the inverse of the bijection (see the end of [Church et al. 2015, Remark 4.1.3]). This allows us to make the following definition:

Definition 2.19 If $\mathcal{V}\colon \mathrm{FI}^{\#} \to \mathrm{Vect}_{\mathbb{Q}}$, we define the *dual* FI$^{\#}$–module $\mathcal{V}^*$ as the composite functor

$$\mathrm{FI}^{\#} \xrightarrow{\cong} (\mathrm{FI}^{\#})^{\mathrm{op}} \xrightarrow{\mathcal{V}^{\mathrm{op}}} \mathrm{Vect}_{\mathbb{Q}}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}_{\mathbb{Q}}(-,\mathbb{Q})} \mathrm{Vect}_{\mathbb{Q}}.$$

2.6 Schur functors

The graded FI–modules that we will study will be constructed by composing *Schur functors* with simpler graded FI–modules, which is why they are of finite type. In this section we will review what we mean by Schur functors in this context and their properties when composed with graded FI–modules.

If λ is a partition of $k \geq 0$, we define the Schur functor $\mathbb{S}_\lambda\colon \mathrm{grVect}_\mathbb{Q} \to \mathrm{grVect}_\mathbb{Q}$ on objects by

$$V \mapsto S^\lambda \otimes_{\Sigma_k} V^{\otimes k},$$

considering $V^{\otimes k}$ with the standard Σ_k–action and considering S^λ as a graded vector space concentrated in degree 0. Another definition, which gives an isomorphic functor, is that $\mathbb{S}_\lambda(V)$ is given by the composition of the k^{th} tensor power functor with the action of a certain idempotent operator $c_\lambda \in \mathbb{Q}[\Sigma_k]$, known as a *Young symmetrizer*, acting on $V^{\otimes k}$ (see [Fulton and Harris 1991] for a definition). This characterizes $\mathbb{S}_\lambda(V)$ as a subrepresentation of $V^{\otimes k}$.

If W is a finite-dimensional graded Σ_k–representation, we more generally define its associated Schur functor by

$$V \mapsto W \otimes_{\Sigma_k} V^{\otimes k},$$

and denote it by $\mathbb{S}_W$. Note that since W is finite-dimensional, this functor decomposes as a direct sum of Schur functors $\mathbb{S}_\lambda$ (possibly shifted in degree).

Even more generally, given a symmetric sequence $W = (W(1), W(2), \dots)$ of (graded) vector spaces, ie a sequence in which $W(k)$ is a graded Σ_k–representation, we can associate to it the endofunctor $\bigoplus_{k\geq 1} \mathbb{S}_{W(k)} \circ \mathcal{V}$ of $\mathrm{grVect}_\mathbb{Q}$, which we will denote by $\mathbb{S}_W$ and call the Schur functor associated to W.

Schur functors are of interest to us, since they preserve stability degree and weight in the following way:

Proposition 2.20 *Let $W = (W(1), W(2), \dots)$ be a symmetric sequence of graded vector spaces, where each $W(k)$ is finite-dimensional and concentrated in nonnegative degree, and let $\mathcal{V}\colon \mathrm{FI} \to \mathrm{grVect}_\mathbb{Q}$ be a graded* FI*–module such that $\mathcal{V}(S)$ is concentrated in strictly positive degrees for every $S \in \mathrm{FI}$. Suppose that $\mathcal{V}(S)$ is finite-dimensional in each degree and that* $\mathrm{weight}(\mathcal{V}_i) \leq s$ *and* $\text{stab-deg}(\mathcal{V}_i) \leq r$ *for all $i \leq m$. Then* $\mathrm{weight}((\mathbb{S}_W \circ \mathcal{V})_m) \leq ms$ *and* $\text{stab-deg}((\mathbb{S}_W \circ \mathcal{V})_m) \leq \max(r+s, ms)$.

Proof By definition $\mathbb{S}_W \circ \mathcal{V}$ decomposes as the direct sum

$$\bigoplus_{k\geq 1} \mathbb{S}_{W(k)} \circ \mathcal{V},$$

and we may decompose each summand further as

$$\mathbb{S}_{W(k)} \circ \mathcal{V} = \bigoplus_{j\geq 0} \bigoplus_{i\geq 1} W(k)_j \otimes_{\Sigma_k} (\mathcal{V}^{\otimes k})_i.$$

Since $W(k)$ is concentrated in nonnegative degree and $\mathcal{V}$ is concentrated in positive degree, it follows that $\mathbb{S}_{W(k)} \circ \mathcal{V}$ is concentrated in degrees $\geq k$. We thus have

$$(\mathbb{S}_W \circ \mathcal{V})_m = \bigoplus_{k=1}^{m} \bigoplus_{i+j=m} W(k)_j \otimes_{\Sigma_k} (\mathcal{V}^{\otimes k})_i.$$

By Corollary 2.17, it thus suffices to find bounds on the weight and stability degree of $W(k)_j \otimes_{\Sigma_k} (V^{\otimes k})_i$ for all $k \leq m$ and all i and j such that $i + j = m$. By definition, this is a quotient of the FI–module

$W(k)_j \otimes (\mathcal{V}^{\otimes k})_i$. Since $W(k)_j$ is a constant FI–module and $(\mathcal{V}^{\otimes k})_i$ decomposes as a direct sum of summands of the form $\mathcal{V}_{l_1} \otimes \cdots \otimes \mathcal{V}_{l_k}$ such that $l_1 + \cdots + l_k = i$, it follows by Propositions 2.15 and 2.16 that $\text{weight}(W(k)_j \otimes (\mathcal{V}^{\otimes k})_i) \leq is$ and $\text{surj-deg}(W^j \otimes (\mathcal{V}^{\otimes k})_i) \leq \max(r+s, is)$.

By the discussion above, we also have that $W(k)_j \otimes_{\Sigma_k} (\mathcal{V}^{\otimes k})_i$ is isomorphic to a direct sum of graded FI–submodules of $(\mathcal{V}^{\otimes k}[j])_i$ (by decomposing $W(k)_j$ into irreducible Σ_k–representations and applying the corresponding Young symmetrizer for each summand), where $[j]$ denotes a shift of j degrees upwards. Thus we get the same bound on injectivity degree, finishing the proof, since $i \leq m$. □

2.7 Derivation Lie algebras as FI–Lie algebras

Now let us introduce more specific examples of FI–modules that will be of interest to us. Here it will be useful to work with $\mathrm{FI}^\#$–modules. We make the following definition:

Definition 2.21 Let H be a graded vector space. We define a graded $\mathrm{FI}^\#$–module $\mathcal{H}$ by letting $\mathcal{H}(S) := H^{\oplus S}$ for any $S \in \mathrm{FI}$, and for any $A \subset S$, $B \subset T$ and bijection $f\colon A \to B$ we define a linear map $\mathcal{H}(f)\colon \mathcal{H}(S) \to \mathcal{H}(T)$ as the composition

$$\mathcal{H}(S) \twoheadrightarrow \mathcal{H}(A) \to \mathcal{H}(B) \hookrightarrow \mathcal{H}(T),$$

where the first map is the natural projection, the second is the map induced by f and the last is the natural injection.

In the following sections H will be the desuspension of the reduced homology of a simply connected finite CW–complex, so that its homology is finite-dimensional. We then have $\text{weight}(\mathcal{H}) = 1$, since $H^{\oplus S}$ decomposes into a direct sum of trivial and standard representations of $\Sigma(S)$, which correspond to the padded partitions $\lambda[|S|]$ of $\lambda = (1)$ and $\lambda = (0)$, respectively. It is also easily verified that $\text{stab-deg}(\mathcal{H}) = 1$.

Composing with the free graded Lie algebra functor $\mathbb{L}$, we get a new graded $\mathrm{FI}^\#$–module, which we denote by $\mathbb{L}\mathcal{H}$.

Since $\mathcal{H}$ is an $\mathrm{FI}^\#$–module, we may consider its dual $\mathrm{FI}^\#$–module $\mathcal{H}^*$. Let us describe it in some more detail. For a finite set S we simply have $\mathcal{H}^*(S) = \mathcal{H}(S)^* = (H^*)^{\oplus S}$, and if $S \supseteq A \xrightarrow{f} B \subseteq T$ is a partial injection then $\mathcal{H}^*(S \supseteq A \xrightarrow{f} B \subseteq T)$ is the composition

$$\mathcal{H}^*(S) \twoheadrightarrow \mathcal{H}^*(A) \xrightarrow{\circ H(f^{-1})} \mathcal{H}^*(B) \hookrightarrow \mathcal{H}^*(T).$$

Remark 2.22 If we restrict this $\mathrm{FI}^\#$–module to FI and $i\colon S \hookrightarrow T$ is an injection, we can describe the map $\mathcal{H}^*(i)$ as follows: Let $\phi \in \mathcal{H}^*(S)$ and x_α be in the summand of $H^{\oplus T}$ corresponding to $\alpha \in T$. Then

$$(1)\qquad (\mathcal{H}^*(i)(\phi))(x_\alpha) = \begin{cases} 0 & \text{if } \alpha \in T \setminus i(S), \\ (\phi \circ \mathcal{H}(i)^{-1})(x_\alpha) & \text{if } \alpha \in i(S). \end{cases}$$

Just as for $\mathcal{H}$, the following proposition is easily verified:

Proposition 2.23 *If H is a finite-dimensional graded vector space, then the graded $\mathrm{FI}^\#$–module $\mathcal{H}^*$ has weight ≤ 1 and stability degree ≤ 1.*

Next, we will define the graded $\mathrm{FI}^\#$–Lie algebra of derivations on the graded $\mathrm{FI}^\#$–Lie algebra $\mathbb{L}\mathcal{H}$. Recall that if L is a graded Lie algebra, we define a derivation on L as a (graded) linear map $D\colon L \to L$ which satisfies

$$D[x, y] = [Dx, y] + (-1)^{|x||D|}[x, Dy]$$

for all $x, y \in L$. We denote the graded vector space of all derivations by $\mathrm{Der}(L)$.

Definition 2.24 We define the graded FI–module $\mathrm{Der}(\mathbb{L}\mathcal{H})\colon \mathrm{FI} \to \mathrm{grVect}_{\mathbb{Q}}$ by letting $\mathrm{Der}(\mathbb{L}\mathcal{H})(S) = \mathrm{Der}(\mathbb{L}\mathcal{H}(S))$ for $S \in \mathrm{FI}$, and for $i\colon S \hookrightarrow T$ an injection we define $\mathrm{Der}(\mathbb{L}\mathcal{H})(i)$ as follows: Recall that a derivation on $\mathbb{L}\mathcal{H}(T)$ is uniquely determined by its restriction to $\mathcal{H}(T)$. Suppose therefore that $x_\alpha \in \mathcal{H}(T)$ lies in the direct summand of $\mathcal{H}(T)$ corresponding to $\alpha \in T$ and let $D \in \mathrm{Der}(\mathbb{L}(H^{\oplus S}))$. Then $\mathrm{Der}(\mathbb{L}\mathcal{H})(i)D$ is determined by

$$(\mathrm{Der}(\mathbb{L}\mathcal{H})(i)D)(x_\alpha) = \begin{cases} 0 & \text{if } \alpha \in T \setminus i(S), \\ (\mathbb{L}\mathcal{H}(i) \circ D \circ \mathcal{H}(i)^{-1})(x_\alpha) & \text{if } \alpha \in i(S). \end{cases} \tag{2}$$

Remark 2.25 The functor $\mathrm{Der}(\mathbb{L}\mathcal{H})$ may in fact be extended to all of $\mathrm{FI}^\#$ using a similar definition, but since we will not be using this we only consider the simpler functor from $\mathrm{FI}^\#$.

For a graded Lie algebra L, the commutator Lie bracket

$$[D, D'] = D \circ D' - (-1)^{|D||D'|} D' \circ D$$

makes $\mathrm{Der}(L)$ into a graded Lie algebra. A straightforward computation using (2) shows that

$$\mathrm{Der}(\mathbb{L}\mathcal{H})(i)[D, D'] = [\mathrm{Der}(\mathbb{L}\mathcal{H})(i)(D), \mathrm{Der}(\mathbb{L}\mathcal{H})(i)(D')],$$

giving us the following result:

Proposition 2.26 *The functor* $\mathrm{Der}(\mathbb{L}\mathcal{H})\colon \mathrm{FI} \to \mathrm{grVect}_{\mathbb{Q}}$ *of Definition 2.24 factors through the forgetful functor* $\mathrm{grLie}_{\mathbb{Q}} \to \mathrm{grVect}_{\mathbb{Q}}$*, where* $\mathrm{grLie}_{\mathbb{Q}}$ *is the category of graded Lie algebras over* $\mathbb{Q}$*.*

Further, we can determine explicit weights and stability degrees in each degree of this graded FI–module:

Proposition 2.27 *Let H be a finite-dimensional graded vector space concentrated in strictly positive degrees. If the degree of H is bounded strictly below d, for some $d \geq 1$, we have* $\mathrm{weight}(\mathrm{Der}(\mathbb{L}\mathcal{H})_m) \leq m + d$ *and* $\text{stab-deg}(\mathrm{Der}(\mathbb{L}\mathcal{H})_m) \leq m + d$*.*

Proof For every $S \in \mathrm{FI}$, we have an isomorphism of graded vector spaces

$$\Psi_S\colon \mathcal{H}^*(S) \otimes \mathbb{L}\mathcal{H}(S) \xrightarrow{\cong} \mathrm{Der}(\mathbb{L}\mathcal{H})(S),$$

given by sending $\phi \otimes A \in \mathcal{H}^*(S) \otimes \mathbb{L}\mathcal{H}(S)$ to the derivation in $\mathrm{Der}(\mathbb{L}\mathcal{H})(S)$ defined by

$$x \mapsto \phi(x)A$$

on $x \in \mathcal{H}(S)$. We want to prove that this defines a map of graded FI–modules, ie that for every morphism $i\colon S \hookrightarrow T$, the diagram

$$\begin{array}{ccc} \mathcal{H}^*(S) \otimes \mathbb{L}\mathcal{H}(S) & \xrightarrow{\Psi_S} & \mathrm{Der}(\mathbb{L}\mathcal{H})(S) \\ {\scriptstyle \mathcal{H}^*(i)\otimes \mathbb{L}\mathcal{H}(i)}\downarrow & & \downarrow{\scriptstyle \mathrm{Der}(\mathbb{L}\mathcal{H})(i)} \\ \mathcal{H}^*(T) \otimes \mathbb{L}\mathcal{H}(T) & \xrightarrow{\Psi_T} & \mathrm{Der}(\mathbb{L}\mathcal{H})(T) \end{array} \tag{3}$$

is commutative. This can be verified by applying the definitions of Ψ_S and Ψ_T, together with the description of $\mathcal{H}^*(i)$ given by (1) and the description of $\mathrm{Der}(\mathbb{L}\mathcal{H})(i)$ given by (2).

Thus $\mathrm{Der}(\mathbb{L}\mathcal{H}) \cong \mathcal{H}^* \otimes \mathbb{L}\mathcal{H}$, as graded FI–modules. Note that $\mathcal{H}^*$ is concentrated in *negative* degrees, which are bounded from below, by the assumption on H. We thus have

$$(\mathcal{H}^* \otimes \mathbb{L}\mathcal{H})_m = \bigoplus_{i=1}^{d-1} (\mathcal{H}^*)_{-i} \otimes (\mathbb{L}\mathcal{H})_{m+i}.$$

Since both $\mathcal{H}$ and $\mathcal{H}^*$ are of weight and stability degree 1, the same argument as in the proof of Proposition 2.20 shows that $(\mathcal{H}^*)_{-i} \otimes (\mathbb{L}\mathcal{H})_{m+i}$ is simultaneously a quotient of an FI–module of weight and stability degree $\le m+i+1$, so $(\mathcal{H}^* \otimes \mathbb{L}\mathcal{H})_m$ thus has both weight and stability degree $m+d$, due to Propositions 2.16 and 2.17. □

The FI–modules that we consider in Theorem A are the homology groups of graded FI–modules of the type $\mathrm{Der}(\mathbb{L}\mathcal{H})$, with H as above, so it will follow immediately from Proposition 2.18 that we get the claimed bounds on weight and stability degree. In the case of Theorem B, it turns out that we can use Proposition 2.20 more directly, due to results from [Berglund and Madsen 2020].

2.8 FI–Lie models

Now, let us introduce the notion of an FI–*Lie model*, which will be one of our main tools. For the basic theory of Lie models in rational homotopy theory, see for example [Félix et al. 2001].

Definition 2.28 Let $\mathcal{X}$ be a simply connected based FI–space and let $\mathcal{L}$ be a dg FI–Lie algebra. We say that $\mathcal{L}$ is an FI–Lie model for $\mathcal{X}$ if

(i) for every $S \in \mathrm{FI}$, $\mathcal{L}(S)$ is a dg Lie model for the space $\mathcal{X}(S)$, and

(ii) for every morphism $S \hookrightarrow T$ in FI, the dgl map

$$\mathcal{L}(S) \to \mathcal{L}(T)$$

is a model for the map $\mathcal{X}(S) \to \mathcal{X}(T)$.

Remark 2.29 If $\mathcal{L}$ is an FI–Lie model for $\mathcal{X}$, then $H_*(\mathcal{L}) \cong \pi_*^{\mathbb{Q}}(\mathcal{X})$ is an isomorphism of FI–modules.

Remark 2.30 A reader may feel that Definition 2.28 is somewhat unnatural. Indeed, it is not the "philosophically" correct definition of FI–Lie model, seen from a modern homotopy-theoretic perspective. There is an equivalence of ∞–categories

$$(\mathrm{dgLie}_{\mathbb{Q}})_{\geq 1} \cong \mathrm{Top}^{\mathbb{Q}}_{\geq 2},$$

between the ∞–categories of connected dg Lie algebras, localized at the quasi-isomorphisms, and simply connected spaces, localized at the rational homotopy equivalences. The usual definition of dg Lie models in rational homotopy theory is that a connected dg Lie algebra (L, d) is a dg Lie model for a simply connected space X if they are isomorphic under this equivalence. Equivalently, it suffices to require that they are isomorphic under the equivalence between the homotopy categories $h(\mathrm{dgLie}_{\mathbb{Q}})_{\geq 1} \cong h\,\mathrm{Top}^{\mathbb{Q}}_{\geq 2}$. The correct definition of FI–Lie model should therefore be that a dg FI–Lie algebra $\mathcal{L}$ is an FI–Lie model of a simply connected pointed FI–space $\mathcal{X}$ if they are isomorphic under the equivalence of the homotopy categories

$$h\,\mathrm{Fun}(\mathrm{FI}, (\mathrm{dgLie}_{\mathbb{Q}})_{\geq 1}) \cong h\,\mathrm{Fun}(\mathrm{FI}, \mathrm{Top}^{\mathbb{Q}}_{\geq 2}).$$

In contrast, our definition is requiring isomorphism under the equivalence of "ordinary" functor categories

$$\mathrm{Fun}(\mathrm{FI}, h(\mathrm{dgLie}_{\mathbb{Q}})_{\geq 1}) \cong \mathrm{Fun}(\mathrm{FI}, h\,\mathrm{Top}^{\mathbb{Q}}_{\geq 2}).$$

Nevertheless, the naive Definition 2.28 is simpler and sufficient for our purposes here.

3 Rational homotopy theory for homotopy automorphisms

In this section we will review some rational homotopy theory for homotopy automorphisms we will need.

Let X be a simply connected topological space homotopy equivalent to a CW–complex. A homotopy automorphism of X is a self-map $\varphi\colon X \to X$ that is a homotopy equivalence. We denote the topological monoid of unpointed and pointed homotopy automorphisms of X by $\mathrm{aut}(X)$ and $\mathrm{aut}_*(X)$, respectively. Given a subspace $A \subset X$, we denote the topological monoid of A–relative homotopy automorphisms of X, ie the homotopy automorphisms that preserve A pointwise, by $\mathrm{aut}_A(X)$. When A is a point or empty we simply write $\mathrm{aut}_*(X)$ and $\mathrm{aut}(X)$, respectively, and when $X = N$ is a manifold with boundary $A = \partial N$, the monoid of boundary-relative homotopy automorphisms of N is denoted by $\mathrm{aut}_\partial(N)$.

If X is well pointed and $A \subset X$ is a cofibration of cofibrant spaces in the Hurewicz model structure, then all of $\mathrm{aut}(X)$, $\mathrm{aut}_*(X)$ and $\mathrm{aut}_A(X)$ are group-like monoids, and thus equivalent to topological groups. We take the basepoint of a topological monoid G to be the identity element and $\pi_k(G, \mathrm{id})$ is abbreviated by $\pi_k(G)$. We denote the classifying space of G by BG and its universal cover by $\widetilde{BG}$. Moreover, if a topological monoid G is group-like, then G and ΩBG are weakly equivalent as topological monoids. Let $G_\circ \subset G$ denote the connected component of the identity. Then $BG_\circ \simeq \widetilde{BG}$. We observe that

$$\pi_k(G) \otimes \mathbb{Q} \cong \pi_{k+1}(\widetilde{BG}) \otimes \mathbb{Q} \cong \pi_{k+1}(BG_\circ) \otimes \mathbb{Q} \cong H_k(\mathfrak{g}_{BG_\circ})$$

for all $k \geq 1$ and where $\mathfrak{g}_{BG_\circ}$ is any dg Lie algebra model for $BG_\circ$.

The identity component of $\mathrm{aut}_A(X)$ is denoted by $\mathrm{aut}_{A,\circ}(X)$.

Remark 3.1 By [Farjoun 1996], there are functorial and continuous rationalization functors that preserve cofibrations. In particular, given a cofibration $A \subset X$, there is a rationalization functor that induces a group homomorphism $r\colon \pi_0(\mathrm{aut}_A(X)) \to \pi_0(\mathrm{aut}_{A_{\mathbb{Q}}}(X_{\mathbb{Q}}))$.

For $k \geq 1$ we have that

$$\pi_k(\mathrm{aut}_A(X)) \otimes \mathbb{Q} \cong \pi_k(\mathrm{aut}_{A_{\mathbb{Q}}}(X_{\mathbb{Q}})),$$

since $B\,\mathrm{aut}_{A,\circ}(X)_{\mathbb{Q}} \simeq B\,\mathrm{aut}_{A_{\mathbb{Q}},\circ}(X_{\mathbb{Q}})$; see [Berglund and Saleh 2020, Proposition 2.4].

A model for $B\,\mathrm{aut}_{A,\circ}(X)$ is given in terms of dg Lie algebras of derivations.

Definition 3.2 Given a dg Lie algebra L, let $\mathrm{Der}(L)$ denote the dg Lie algebra of derivations of L, where the graded Lie bracket is given by

$$[\theta,\eta] = \theta\circ\eta - (-1)^{|\theta||\eta|}\eta\circ\theta$$

and the differential is given by $\partial = [d_L, -]$ where d_L is the differential of L.

Definition 3.3 Given a chain complex $C = C_*$, the positive truncation of C, denoted by C^+, is given by

$$C_i^+ = \begin{cases} C_i & \text{if } i > 1,\\ \ker(C_1 \xrightarrow{d} C_0) & \text{if } i = 1,\\ 0 & \text{if } i < 1.\end{cases}$$

Definition 3.4 A dg Lie algebra $(\mathbb{L}(V), d)$ is called *quasifree* if its underlying graded Lie algebra structure is a free graded Lie algebra on the graded vector space V.

Definition 3.5 We say that a dg Lie algebra map between two quasifree dg Lie algebras $\phi\colon \mathbb{L}(V) \to \mathbb{L}(U)$ is free if ϕ is injective and $\phi(V) \subseteq U$. In particular U has a decomposition $U \cong V \oplus W$.

Remark 3.6 One can show that the free maps between the quasifree dg Lie algebras are exactly the cofibrant maps between them; see the remark after [Quillen 1969, Proposition 5.5].

Proposition 3.7 (a) *Let X be a simply connected space of the homotopy type of a finite CW–complex with a quasifree dg Lie algebra model $\mathbb{L}_X$. A dg Lie model for $B\,\mathrm{aut}_{*,\circ}(X)$ is given by $\mathrm{Der}^+(\mathbb{L}_X)$.*

(b) *Let $A \subset X$ be a cofibration of simply connected spaces of the homotopy type of finite CW–complexes, and let $\mathbb{L}_A \to \mathbb{L}_X$ be a cofibration (ie a free map) of quasifree dg Lie algebras that models the inclusion $A \subset X$. A dg Lie model for $B\,\mathrm{aut}_{A,\circ}(X)$ is given by the positive truncation of the dg Lie algebra of derivations on $\mathbb{L}_X$ that vanish on $\mathbb{L}_A$, denoted by $\mathrm{Der}^+(\mathbb{L}_X \parallel \mathbb{L}_A)$.*

(c) *The inclusion $\mathrm{Der}^+(\mathbb{L}_X \parallel \mathbb{L}_A) \to \mathrm{Der}^+(\mathbb{L}_X)$ is a model for $B\,\mathrm{aut}_{A,\circ}(X) \to B\,\mathrm{aut}_{*,\circ}(X)$ induced by the inclusion $\mathrm{aut}_{A,\circ}(X) \hookrightarrow \mathrm{aut}_{*,\circ}(X)$.*

Proof For (a), see [Tanré 1983, corollarie VII.4(4)]. For (b), see [Berglund and Saleh 2020, Theorem 1.1]. Statement (c) follows by [loc. cit., Proposition 4.6] and the theory established in [Berglund 2020, Sections 3.4 and 3.5]. □

We recall the notion of geometric realizations of dg Lie algebras. For a detailed account on the subject we refer the reader to [Hinich 1997; Getzler 2009; Berglund 2015; 2020].

Definition 3.8 [Hinich 1997, Definition 2.1.1] Let $\Omega_\bullet = \Omega_\bullet^*$ denote the simplicial commutative dg algebra in which Ω_n^* is the Sullivan–de Rham algebra of polynomial differential forms on the n–simplex. The geometric realization of a positively graded dg Lie algebra L is defined to be the simplicial set $\mathrm{MC}(L \otimes \Omega_\bullet)$ of Maurer–Cartan elements of the simplicial dg Lie algebra $L \otimes \Omega_\bullet$, denoted by $\mathrm{MC}_\bullet(L)$. We recall that the tensor product $L \otimes \Omega$ of a dg Lie algebra L with a commutative dg algebra Ω is again a dg Lie algebra, where $[\ell_1 \otimes c_1, \ell_2 \otimes c_2] = (-1)^{|c_1||\ell_2|}[\ell_1, \ell_2] \otimes c_1 c_2$. A positively graded dg Lie algebra L is a Lie model for a simply connected space X if and only if there exists a zigzag of rational homotopy equivalences between the geometric realization $\mathrm{MC}_\bullet(L)$ and X.

The functor $\mathrm{MC}_\bullet$ takes surjections to Kan fibrations [Getzler 2009, Proposition 4.7] and takes injections to cofibrations (in the classical model structure on simplicial sets). In particular, if $\mathbb{L}_A \to \mathbb{L}_X$ is a free map of dg Lie algebras that models a cofibration $A \subset X$, then the cofibration $\mathrm{MC}_\bullet(\mathbb{L}_A) \hookrightarrow \mathrm{MC}_\bullet(\mathbb{L}_X)$ is a simplicial model for the cofibration $A_\mathbb{Q} \subset X_\mathbb{Q}$. Thus $\mathrm{aut}_{A_\mathbb{Q}}(X_\mathbb{Q})$ and $\mathrm{aut}_{\mathrm{MC}_\bullet(\mathbb{L}_A)}(\mathrm{MC}_\bullet(\mathbb{L}_X))$ are weakly equivalent as topological monoids.

Definition 3.9 The exponential $\exp(\mathfrak{h})$ of a nilpotent Lie algebra $\mathfrak{h}$ concentrated in degree zero is the nilpotent group with underlying set given by $\mathfrak{h}$ and multiplication given by the Baker–Campbell–Hausdorff formula. The exponential of a positively graded dg Lie algebra L, denoted by $\exp_\bullet(L)$, is the simplicial group given by the exponential $\exp(Z_0(L \otimes \Omega_\bullet))$ of the zero cycles in $L \otimes \Omega_\bullet$; see [Berglund 2020].

Proposition 3.10 [loc. cit., Corollary 3.10] *For a positively graded dg Lie algebra L there is an equivalence of topological monoids between $\exp_\bullet(L)$ and the loop space $\Omega\,\mathrm{MC}_\bullet(L)$.*

Definition 3.11 Let $\mathbb{L}(V) \subset \mathbb{L}(V \oplus W)$ be a cofibration of free positively graded dg Lie algebras and let $\mathrm{Der}(\mathbb{L}(V \oplus W) \parallel \mathbb{L}(V))$ denote the dg Lie algebra of derivations on $\mathbb{L}(V \oplus W)$ that vanish on $\mathbb{L}(V)$; the differential is $[d_{\mathbb{L}(V \oplus W)}, -]$. There is a left action of $\exp_\bullet\big(\mathrm{Der}^+(\mathbb{L}(V \oplus W) \parallel \mathbb{L}(V))\big)$ on $\mathrm{MC}_\bullet(\mathbb{L}(V \oplus W))$ given by

$$\Theta.x = \sum_{i \geq 0} \frac{\Theta^i(x)}{i!}. \tag{4}$$

See [Berglund and Saleh 2020, Section 3.2].

Proposition 3.12 [Berglund 2022, Proposition 3.7] *Let $A \subset X$ be a cofibration of simply connected spaces with homotopy types of finite CW–complexes, and let $\iota\colon \mathbb{L}(V) \to \mathbb{L}(V \oplus W)$ be a free map of quasifree dg Lie algebras that models the inclusion $A \subset X$. Then the topological monoid map*

$$F\colon \exp_\bullet\big(\mathrm{Der}^+(\mathbb{L}(V \oplus W) \parallel \mathbb{L}(V))\big) \to \mathrm{aut}_{\mathrm{MC}_\bullet(\mathbb{L}(V)),\circ}\big(\mathrm{MC}_\bullet(\mathbb{L}(V \oplus W))\big) \simeq \mathrm{aut}_{A_\mathbb{Q},\circ}(X_\mathbb{Q}),$$
$$F(\Theta)(x) = \Theta.x,$$

is a weak equivalence.

Proof Note that the action of $\exp_\bullet\bigl(\mathrm{Der}^+(\mathbb{L}(V\oplus W)\,\|\,\mathbb{L}(V))\bigr)$ on $\mathrm{MC}_\bullet(\mathbb{L}(V\oplus W))$ fixes $\mathrm{MC}_\bullet(\mathbb{L}(V))\subset \mathrm{MC}_\bullet(\mathbb{L}(V\oplus W))$ pointwise. In particular, the group action yields a map

$$\exp_\bullet\bigl(\mathrm{Der}^+(\mathbb{L}(V\oplus W)\,\|\,\mathbb{L}(V))\bigr)\to \mathrm{aut}_{\mathrm{MC}_\bullet(\mathbb{L}(V))}\bigl(\mathrm{MC}_\bullet(\mathbb{L}(V\oplus W))\bigr).$$

Moreover, since $\exp_\bullet\bigl(\mathrm{Der}^+(\mathbb{L}(V\oplus W)\,\|\,\mathbb{L}(V))\bigr)$ is connected and F preserves the identity element, we may replace the codomain by

$$\mathrm{aut}_{\mathrm{MC}_\bullet(\mathbb{L}(V)),\circ}\bigl(\mathrm{MC}_\bullet(\mathbb{L}(V\oplus W))\bigr),$$

ie the component of the identity. We proceed by adapting the proof of [loc. cit., Proposition 3.7] to our situation. Given a positively graded dg Lie algebra $\mathfrak{h}$, there is an isomorphism of abelian groups

$$G\colon H_k(\mathfrak{h})\to \pi_k(\exp_\bullet(\mathfrak{h}))$$

where a homology class of a cycle $z\in Z_k(\mathfrak{h})$ is sent to the homotopy class of the k–simplex $z\otimes \nu_k\in Z_0(\mathfrak{h}\otimes\Omega_k^*)$, where ν_k is the class $k!\,dt_1\cdots dt_k$. That G defines an isomorphism is motivated in the proof of [loc. cit., Proposition 3.7].

We have that $\nu_k^2=0$, and consequently

$$F(\theta\otimes\nu_k)=\mathrm{id}+\theta\otimes\nu_k.$$

Let us now analyze $\pi_k\bigl(\mathrm{aut}_{\mathrm{MC}_\bullet(\mathbb{L}(V)),\circ}\bigl(\mathrm{MC}_\bullet(\mathbb{L}(V\oplus W))\bigr)\bigr)$ for $k\ge 1$, as in the proof of [Berglund and Madsen 2020, Theorem 3.6]. In order to simplify notation, $\mathrm{MC}_\bullet(\mathbb{L}(V))$ is denoted by $A_\mathbb{Q}$ and $\mathrm{MC}_\bullet(\mathbb{L}(V\oplus W))$ by $X_\mathbb{Q}$. We have that an element $f\in\pi_k(\mathrm{aut}_{A_\mathbb{Q},\circ}(X_\mathbb{Q}))$ is represented by a map

$$f\colon (S^k\sqcup *)\wedge X_\mathbb{Q}\to X_\mathbb{Q},$$

where $f(*,x)=x$ for every $x\in X_\mathbb{Q}$ and $f(s,a)=a$ for every $a\in A_\mathbb{Q}$ and $s\in S^k$.

A dg Lie algebra model for $(S^k\sqcup *)\wedge X_\mathbb{Q}$ is given by $(\mathbb{L}(U\oplus s^kU),\partial)$ where $U=V\oplus W$ and with a differential determined by the following: Let d be the differential on $\mathbb{L}(U)$. Then $\partial(u)=d(u)$ for every $u\in U$ and $\partial(s^ku)=(-1)^k s^k d(u)$ for every $s^ku\in s^kU$.

Now, $f\colon (S^k\sqcup *)\wedge X_\mathbb{Q}\to X_\mathbb{Q}$ is modeled by some map $\varphi_f\colon \mathbb{L}(U\oplus s^kU)\to\mathbb{L}(U)$ that satisfies $\varphi_f(u)=u$ for every $u\in U$ and $\varphi_f(s^kv)=0$ for every $v\in V\subset U$. Let θ_f be the unique derivation on $\mathbb{L}(U)$ that satisfies $\theta_f(u)=\varphi_f(s^ku)$ for every $u\in U$. Note that θ_f is a cycle and that it vanishes on $\mathbb{L}(V)$, ie $\theta_f\in Z_k\bigl(\mathrm{Der}^+(\mathbb{L}(U)\,\|\,\mathbb{L}(V))\bigr)$. Also note that if $f=\pi_k(F)[\theta\otimes\nu_k]$ then $\theta_f=\theta$. Let $K\colon \pi_k(\mathrm{aut}_{A_\mathbb{Q},\circ}(X_\mathbb{Q}))\to H_k\bigl(\mathrm{Der}^+(\mathbb{L}(U)\,\|\,\mathbb{L}(V))\bigr)$ be given by $K(f)=\theta_f$. It follows from [Berglund and Madsen 2020; Lupton and Smith 2007] that this map is well defined and is an isomorphism.

Set $\mathfrak{h}=\mathrm{Der}^+(\mathbb{L}(U)\,\|\,\mathbb{L}(V))$. The composition

$$H_k(\mathfrak{h})\xrightarrow{G}\pi_k(\exp_\bullet(\mathfrak{h}))\xrightarrow{\pi_k(F)}\pi_k(\mathrm{aut}_{A_\mathbb{Q},\circ}(X_\mathbb{Q}))\xrightarrow{K}H_k(\mathfrak{h})$$

is the identity map, which forces $\pi_k(F)$ to be an isomorphism. This proves that F is a weak equivalence. □

Recall that a topological group G' acts on itself by conjugation $G' \to \mathrm{Aut}(G')$ via $g \mapsto \kappa_g$ where $\kappa_g(h) = ghg^{-1}$. If g and g' belong to the same connected component of G', then κ_g and $\kappa_{g'}$ are homotopic and this induce equal maps on the homotopy groups of G'. This group action restricts to a group action on the identity component $G'_\circ$ of G', which in turn induces an action of G' on $BG'_\circ$. This gives that $\pi_0(G')$ acts on $\pi_*(BG'_\circ)$. Since a group-like monoid G is equivalent to a topological group G', we have that $\pi_0(G)$ acts on $\pi_*(G_\circ)$.

In the rest of this section we discuss the action of $\pi_0(\mathrm{aut}_A(X))$ on $\pi_k(\mathrm{aut}_A(X))$ from a rational homotopy point of view. To do so we recall some theory.

Proposition 3.13 [Espic and Saleh 2020, Theorem 1.3] *Given a map $f\colon \mathbb{L}(V) \to \mathfrak{g}$ of positively graded dg Lie algebras, there exists a minimal relative model $q\colon \mathbb{L}(V \oplus W) \xrightarrow{\sim} \mathfrak{g}$ for f in the following sense:*

(a) *$\mathbb{L}(V)$ is a dg subalgebra of $\mathbb{L}(V \oplus W)$ and $f = q \circ \iota$, where $\iota\colon \mathbb{L}(V) \to \mathbb{L}(V \oplus W)$ is the inclusion.*

(b) *Given a quasi-isomorphism $g\colon \mathbb{L}(V \oplus W) \to \mathbb{L}(V \oplus W)$, where g restricts to an automorphism of $\mathbb{L}(V)$, g is an automorphism.*

Definition 3.14 Let $\iota\colon \mathbb{L}(V) \to \mathbb{L}(V \oplus W)$ be a free map of quasifree dg Lie algebras. We say that an endomorphism $\varphi\colon \mathbb{L}(V \oplus W) \to \mathbb{L}(V \oplus W)$ is ι–relative if $\varphi|_{\mathbb{L}(V)} = \mathrm{id}$, and two ι–relative endomorphisms φ and ψ are ι–equivalent if there exists a homotopy $h\colon \mathbb{L}(V \oplus W) \to \mathbb{L}(V \oplus W) \otimes \Lambda(t, dt)$ from φ to ψ that preserves $\mathbb{L}(V)$ in the sense that $h(v) = v \otimes 1$ for every $v \in \mathbb{L}(V)$; see [Félix et al. 2001, Section 14(a)].

We denote the group of ι–relative automorphisms of $\mathbb{L}(V \oplus W)$ by $\mathrm{Aut}_\iota(\mathbb{L}(V \oplus W))$.

Lemma 3.15 [Espic and Saleh 2020, Corollary 4.6] *Let $\iota\colon \mathbb{L}(V) \to \mathbb{L}(V \oplus W)$ be a minimal relative dg Lie model for a cofibration $A \subset X$ of simply connected spaces. Then there are group isomorphisms*

$$\mathrm{Aut}_\iota(\mathbb{L}(V \oplus W))/\iota\text{–equivalence} \cong \pi_0\big(\mathrm{aut}_{\mathrm{MC}_\bullet(\mathbb{L}(V))}\big(\mathrm{MC}_\bullet(\mathbb{L}(V \oplus W))\big)\big) \cong \pi_0(\mathrm{aut}_{A_\mathbb{Q}}(X_\mathbb{Q})).$$

Remark 3.16 By this lemma, it makes sense to refer to an ι–relative automorphisms of $\mathbb{L}(V \oplus W)$ as an algebraic model for an $A_\mathbb{Q}$–relative homotopy automorphism of $X_\mathbb{Q}$.

Definition 3.17 Consider the group action of $\mathrm{Aut}_\iota(\mathbb{L}(V \oplus W))$ on $\mathrm{Der}(\mathbb{L}(V \oplus W) \,\|\, \mathbb{L}(V))$ given by the following: for $\varphi \in \mathrm{Aut}_\iota(\mathbb{L}(V \oplus W))$ and $\theta \in \mathrm{Der}(\mathbb{L}(V \oplus W) \,\|\, \mathbb{L}(V))$, let

$$\varphi.\theta = \varphi \circ \theta \circ \varphi^{-1}.$$

This induces an action of $\mathrm{Aut}_\iota(\mathbb{L}(V \oplus W))$ on $\exp_\bullet\big(\mathrm{Der}(\mathbb{L}(V \oplus W) \,\|\, \mathbb{L}(V))\big)$.

There is also an action of $\mathrm{Aut}_\iota(\mathbb{L}(V \oplus W))$ on $\mathrm{aut}_{\mathrm{MC}_\bullet(\mathbb{L}(V)),\circ}\big(\mathrm{MC}_\bullet(\mathbb{L}(V \oplus W))\big)$; for $\varphi \in \mathrm{Aut}_\iota(\mathbb{L}(V \oplus W))$ and $f \in \mathrm{aut}_{\mathrm{MC}_\bullet(\mathbb{L}(V)),\circ}\big(\mathrm{MC}_\bullet(\mathbb{L}(V \oplus W))\big)$, let

$$\varphi.f = \mathrm{MC}_\bullet(\varphi) \circ f \circ \mathrm{MC}_\bullet(\varphi^{-1}).$$

Proposition 3.18 *The equivalence*

$$F\colon \exp_\bullet\big(\mathrm{Der}^+(\mathbb{L}(V\oplus W)\,\|\,\mathbb{L}(V))\big)\to \mathrm{aut}_{\mathrm{MC}_\bullet(\mathbb{L}(V)),\circ}\big(\mathrm{MC}_\bullet(\mathbb{L}(V\oplus W))\big)\simeq \mathrm{aut}_{A_\mathbb{Q},\circ}(X_\mathbb{Q})$$

of Proposition 3.12 is $\mathrm{Aut}_\iota(\mathbb{L}(V\oplus W))$*–equivariant with respect to the actions in Definition 3.17.*

Proof This is a straightforward verification left to the reader. □

Corollary 3.19 *Let* $f\in \mathrm{aut}_{A_\mathbb{Q}}(X_Q)$ *and let* $\varphi\in \mathrm{Aut}_\iota(\mathbb{L}(V\oplus W))$ *be a relative model for* f*. The automorphism*

$$\alpha_\varphi\colon \mathrm{Der}(\mathbb{L}(V\oplus W)\,\|\,\mathbb{L}(V))\to \mathrm{Der}(\mathbb{L}(V\oplus W)\,\|\,\mathbb{L}(V)),\quad \alpha_\varphi(\theta)=\varphi\circ\theta\circ\varphi^{-1},$$

is a model for the delooping of the homotopy automorphism

$$\mathrm{Ad}_f\colon \mathrm{aut}_{A_\mathbb{Q}}(X_\mathbb{Q})\to \mathrm{aut}_{A_\mathbb{Q}}(X_\mathbb{Q}),\quad \mathrm{Ad}_f(g)=f\circ g\circ f^{-1},$$

where f^{-1} *is an* $A_\mathbb{Q}$*–relative homotopy inverse to* f*.*

4 Homotopy automorphisms of wedge sums

We fix some notation for this section. Let $(X,*)$ be a fixed simply connected space, homotopy equivalent to a finite CW–complex. For any finite set S, let $X_S:=\bigvee^S X$. For any morphism $S\hookrightarrow T$ in FI, there is an obvious induced basepoint-preserving map $X_S\hookrightarrow X_T$ given by inclusion of wedge summands in the order specified by the injection $S\hookrightarrow T$. Thus the functor $S\mapsto X_S$ is a pointed FI–space, which we will denote by $\mathcal{X}$.

We fix a quasifree dg Lie algebra model $\mathbb{L}(H)=(\mathbb{L}(H),d_{\mathbb{L}(H)})$ for X. A dg Lie model for X_S is given by the S–fold free product of dg Lie algebras

$$\mathbb{L}(H)^{*S}:=\mathbb{L}(H)*\cdots*\mathbb{L}(H)\cong\mathbb{L}(H^{\oplus S}).$$

See [Félix et al. 2001, Section 24(f)]. The association $S\mapsto\mathbb{L}(H^{\oplus S})$ defines a dg FI–Lie algebra $\mathbb{L}\mathcal{H}$. Given a morphism $i\colon S\hookrightarrow T$ in FI, we get an induced inclusion $H^{\oplus S}\hookrightarrow H^{\oplus T}$, which induces an inclusion $\mathbb{L}(H^{\oplus S})\hookrightarrow\mathbb{L}(H^{\oplus T})$ that models the map $\mathcal{X}(i)\colon\mathcal{X}(S)\to\mathcal{X}(T)$; this follows from eg Example 1 in Section 12(c) and Example 2 in Section 24(f) of [loc. cit.]. Thus $S\mapsto\mathbb{L}(H^{\oplus S})$ defines a dg FI–Lie model for the pointed FI–space $S\mapsto\mathcal{X}(S)$.

We proceed and define another pointed FI–space $\mathrm{aut}_*(\mathcal{X})$ (the basepoint is always the identity) as follows: For $S\in\mathrm{FI}$, we let $\mathrm{aut}_*(\mathcal{X})(S):=\mathrm{aut}_*(X_S)$. For $i\colon S\hookrightarrow T$ in FI we get a map $\mathrm{aut}_*(X_S)\hookrightarrow\mathrm{aut}_*(X_T)$, defined, for $x_\alpha\in X_T$ in the wedge summand of X_T corresponding to $\alpha\in T$ and $f\in\mathrm{aut}_*(X_S)$, by

$$(\mathrm{aut}_*(\mathcal{X})(i)f)(x_\alpha)=\begin{cases}x_\alpha & \text{if } \alpha\in T\setminus i(S),\\ (\mathcal{X}(i)\circ f\circ\mathcal{X}(i)^{-1})(x_\alpha) & \text{if } \alpha\in i(S).\end{cases}$$

Note that $\mathrm{aut}_*(\mathcal{X})(i)f$ is in some sense an extension by the identity of f. For instance, if $i_s\colon\boldsymbol{n}\to\boldsymbol{n+1}$ is the standard inclusion, then $\mathrm{aut}_*(\mathcal{X})(i_s)f$ is the homotopy automorphism of $X_{\boldsymbol{n+1}}$ that coincides with f on the first n wedge summands, and is the identity on the last summand.

Restricting to the identity component gives a pointed sub-FI–space $\mathrm{aut}_{*,\circ}(\mathcal{X})$. We are interested in the rational homotopy groups of this FI–space.

Remark 4.1 It is tempting to say that we will construct an FI–Lie model for $\mathrm{aut}_*(\mathcal{X})$. However, this pointed FI–space is generally not simply connected. Instead we take a functorial classifying space construction $B\colon \mathrm{TopMon} \to \mathrm{Top}_*$ from the category of topological monoids to the category of pointed topological spaces and consider the pointed FI–space $B\,\mathrm{aut}_{*,\circ}(\mathcal{X})$, where $\mathrm{aut}_{*,\circ}(X_S)$ is the identity component of $\mathrm{aut}_*(X_S)$ for every $S \in \mathrm{FI}$. For every S we have $B\,\mathrm{aut}_{*,\circ}(X_S) \simeq \widetilde{B\,\mathrm{aut}_*(X_S)}$, and so this is a simply connected pointed FI–space, which enables us to apply our tools from rational homotopy theory. Furthermore, for every $k \geq 1$

$$\pi_k^{\mathbb{Q}}(\mathrm{aut}_*(X_S)) \cong \pi_{k+1}^{\mathbb{Q}}(B\,\mathrm{aut}_{*,\circ}(X_S)),$$

so

$$\pi_k^{\mathbb{Q}}(\mathrm{aut}_*(\mathcal{X})) \cong \pi_{k+1}^{\mathbb{Q}}(B\,\mathrm{aut}_{*,\circ}(\mathcal{X})),$$

as FI–modules.

We have by Proposition 3.7(a) that a model for $B\,\mathrm{aut}_{*,\circ}(X_S)$ is given by $\mathrm{Der}^+(\mathbb{L}(H^{\oplus S}))$, with differential given by $[d_{\mathbb{L}(H^{\oplus S})}, -]$. The inclusion $\mathbb{L}\mathcal{H}(i)\colon \mathbb{L}(H^{\oplus S}) \hookrightarrow \mathbb{L}(H^{\oplus T})$ induces a graded Lie algebra map $\mathrm{Der}^+(\mathbb{L}(H^{\oplus S})) \hookrightarrow \mathrm{Der}^+(\mathbb{L}(H^{\oplus T}))$, as discussed in Proposition 2.26. Moreover, this map commutes with the differential, ie it is a dg Lie algebra map. This, together with Proposition 2.26, yields that we have a dg FI–Lie algebra $(\mathrm{Der}^+(\mathbb{L}\mathcal{H}), [d_{\mathbb{L}\mathcal{H}}, -])$.

We will show that $(\mathrm{Der}^+(\mathbb{L}\mathcal{H}), [d_{\mathbb{L}\mathcal{H}}, -])$ defines an FI–Lie model for the pointed FI–space $B\,\mathrm{aut}_{*,\circ}(\mathcal{X})$.

Proposition 4.2 (a) *Let $i_s\colon \boldsymbol{n} \to \boldsymbol{n+1}$ denote the standard inclusion. Then a dg Lie algebra model for*

$$B\,\mathrm{aut}_{*,\circ}(\mathcal{X})(i_s)\colon B\,\mathrm{aut}_{*,\circ}(X_n) \to B\,\mathrm{aut}_{*,\circ}(X_{n+1})$$

is given by

$$\varphi_n := \mathrm{Der}^+(\mathbb{L}\mathcal{H})(i_s)\colon \mathrm{Der}^+(\mathbb{L}(H^{\oplus n})) \to \mathrm{Der}^+(\mathbb{L}(H^{\oplus n+1})).$$

(b) *The Σ_n–action on $B\,\mathrm{aut}_{*,\circ}(X_n)$ is modeled by the Σ_n–action on $\mathrm{Der}^+(\mathbb{L}(H^{\oplus n}))$.*

Proof (a) To simplify notation, let $\mathbb{L}_k$ denote $\mathbb{L}(H^{\oplus k})$, let $H_l \cong H$ denote the last summand of $H^{\oplus n+1}$ and let $\mathbb{L}_l$ denote $\mathbb{L}(H_l)$. In particular, $\mathbb{L}_{n+1} = \mathbb{L}_n * \mathbb{L}_l$. Let $c_n\colon \mathrm{MC}_\bullet(\mathbb{L}_n) \to \mathrm{MC}_\bullet(\mathbb{L}_{n+1})$ and $c_l\colon \mathrm{MC}_\bullet(\mathbb{L}_l) \to \mathrm{MC}_\bullet(\mathbb{L}_{n+1})$ denote the cofibrations induced by the standard inclusions $\mathbb{L}_n \to \mathbb{L}_n * \mathbb{L}_l$ and $\mathbb{L}_l \to \mathbb{L}_n * \mathbb{L}_l$, respectively.

From Proposition 3.12 we get topological monoid equivalences

$$F_n\colon \exp_\bullet(\mathrm{Der}^+(\mathbb{L}_n)) \to \mathrm{aut}_*(\mathrm{MC}_\bullet(\mathbb{L}_n)).$$

Those maps have adjoints

$$\widetilde{F}_n\colon \exp_\bullet(\mathrm{Der}^+(\mathbb{L}_n)) \times \mathrm{MC}_\bullet(\mathbb{L}_n) \to \mathrm{MC}_\bullet(\mathbb{L}_n).$$

By the explicit formulas for $\{F_n\}$,

$$\widetilde{F}_{n+1} \circ (\exp_\bullet(\varphi_n) \times c_n) = c_n \circ \widetilde{F}_n.$$

In particular, $F_{n+1} \circ \exp_\bullet(\varphi_n)(\Theta)$ is an extension of $F_n(\Theta)$ for $\Theta \in \exp_\bullet(\mathrm{Der}^+(\mathbb{L}_n))$.

We also have that

$$\widetilde{F}_{n+1} \circ (\exp_\bullet(\varphi_n) \times c_l)(g, x) = c_l(x) \quad \text{for all } (g, x) \in \exp_\bullet(\mathrm{Der}^+(\mathbb{L}_n)) \times \mathrm{MC}_\bullet(\mathbb{L}_l).$$

In particular $F_{n+1} \circ \exp_\bullet(\varphi_n)(\Theta)$ restricts to the identity on $\mathrm{MC}_\bullet(\mathbb{L}_l) \subset \mathrm{MC}_\bullet(\mathbb{L}_{n+1})$. That means that $\exp_\bullet(\varphi_n)$ is a simplicial model for $\mathrm{aut}_{*,\circ}(\mathcal{X})(i_s)\colon \mathrm{aut}_{*,\circ}(X_n) \to \mathrm{aut}_{*,\circ}(X_{n+1})$. This gives (a).

(b) This is a direct consequence of Proposition 3.18 and Corollary 3.19. □

Theorem 4.3 *The dg* FI*–Lie algebra* $(\mathrm{Der}^+(\mathbb{L}\mathcal{H}), [d_{\mathbb{L}\mathcal{H}}, -])$ *is an* FI*–Lie model for the pointed* FI*–space* $B\,\mathrm{aut}_{*,\circ}(\mathcal{X})$.

Proof By the second part of Lemma 2.6, an FI–module is completely determined its underlying consistent sequence. By Proposition 4.2, the stabilization maps and the Σ_n–actions defining the consistent sequence for the dg FI–Lie algebra $(\mathrm{Der}^+(\mathbb{L}\mathcal{H}), [d_{\mathbb{L}\mathcal{H}}, -])$ models the stabilization maps and the Σ_n–actions defining the consistent sequence for the FI–space $B\,\mathrm{aut}_{*,\circ}(\mathcal{X})$. From this we conclude that $(\mathrm{Der}^+(\mathbb{L}\mathcal{H}), [d_{\mathbb{L}\mathcal{H}}, -])$ is an FI–Lie model for the pointed FI–space $B\,\mathrm{aut}_{*,\circ}(\mathcal{X})$. □

We now have all the ingredients needed for proving Theorem A.

Theorem A *Let* $(X, *)$ *be a pointed simply connected space with the homotopy type of a finite CW–complex and let* $X_S := \bigvee^S X$ *for any finite set* S*. For each* $k \geq 1$*, the functor*

$$S \mapsto \pi_k^{\mathbb{Q}}(\mathrm{aut}_*(X_S))$$

is an FI*–module. If* $H_n(X, \mathbb{Q}) = 0$ *for* $n \geq d$*, this* FI*–module is of weight* $\leq k + d - 1$ *and stability degree* $\leq k + d$*.*

Proof We will use the established terminology in this section. We have already seen in Theorem 4.3 that $(\mathrm{Der}^+(\mathbb{L}\mathcal{H}), [d_{\mathbb{L}\mathcal{H}}, -])$ is an FI–Lie model for $B\,\mathrm{aut}_{*,\circ}(\mathcal{X})$. Since $H_k(\mathrm{Der}^+(\mathbb{L}\mathcal{H})) \cong \pi_k^{\mathbb{Q}}(\mathrm{aut}_*(\mathcal{X}))$ (see Remark 4.1) it is enough to prove that $H_k(\mathrm{Der}^+(\mathbb{L}\mathcal{H}))$ has the stated bounds on weight and stability degree.

Since $(\mathrm{Der}^+(\mathbb{L}\mathcal{H}), [d_{\mathbb{L}\mathcal{H}}, -])$ defines a dg FI–Lie algebra, it follows that $H_*(\mathrm{Der}^+(\mathbb{L}\mathcal{H}))$ is a graded FI–module. The truncation is defined precisely so that $H_k(\mathrm{Der}^+(\mathbb{L}\mathcal{H})) \cong H_k(\mathrm{Der}(\mathbb{L}\mathcal{H}))$ for all $k \geq 0$. Since $H = s^{-1}\widetilde{H}_*(X)$ and X is assumed to be simply connected, H is finite-dimensional and concentrated in positive degree, and since we have assumed that the homology of X vanishes in degree at least d, H is concentrated in degrees strictly below $d - 1$. The given bounds on stability degree and weight now follow from Propositions 2.27 and 2.18. □

5 Homotopy automorphisms of connected sums

Let M be a closed oriented d–dimensional manifold. For a nonempty finite set S, let $M_S = \left(\#^S M\right) \setminus \mathring{D}^d$ be the space obtained by removing an open d–dimensional disk from the S–fold connected sum of M. If $S = \boldsymbol{n} = \{1, \dots, n\}$ we simply write M_n. We then have a deformation retraction $M_S \xrightarrow{\simeq} \bigvee^S M_1$. Hence there is an FI–module given on objects by $S \mapsto \pi_k(\mathrm{aut}_*(M_S))$, as defined in the previous section.

If we choose a basepoint in the boundary of M_S, there is an inclusion map $\mathrm{aut}_\partial(M_S) \to \mathrm{aut}_*(M_S)$ for every $S \in \mathrm{FI}$. In Section 5.1 we prove that the FI–module $S \mapsto \pi_k(\mathrm{aut}_*(M_S))$ lifts to an FI–module given on objects by $S \mapsto \pi_k(\mathrm{aut}_\partial(M_S))$. In Section 5.2 we prove that $S \mapsto \pi_k^{\mathbb{Q}}(\mathrm{aut}_\partial(M_S))$ is a finitely generated FI–module using certain rational models.

5.1 The integral FI–module structure on the homotopy automorphisms of iterated connected sums

For the purposes of this section, we give an explicit construction of M_n by removing the interiors of n embedded little disks in D^d, which we fix as in Figure 1, left, and gluing n copies of $M \setminus \mathring{D}^d$ along the new boundary components. Note that with this definition, we still have $M_1 = M \setminus \mathring{D}^d$. In Figure 1, right, we see how we can embed M_n into M_{n+1}, and by extending a boundary-relative homotopy automorphism of M_n by the identity thus define a stabilization map

$$s_n : \mathrm{aut}_\partial(M_n) \to \mathrm{aut}_\partial(M_{n+1}).$$

In this section we will define a Σ_n–action on the homotopy groups of $\mathrm{aut}_\partial(M_n)$ and, combining this with the stabilization induced by s_n, we obtain our FI–module structure. Before we do this, we need to introduce some notation:

Definition 5.1 For any pointed space X and any finite set S, let us write $Q_{S,X} : \Sigma(S) \to \pi_0\left(\mathrm{aut}_*\left(\bigvee^S X\right)\right)$ for the group homomorphism given by sending $\sigma \in \Sigma(S)$ to the homotopy class of the automorphism $\mathcal{X}_X(\sigma) : \bigvee^S X \to \bigvee^S X$ described in the beginning of Section 4.

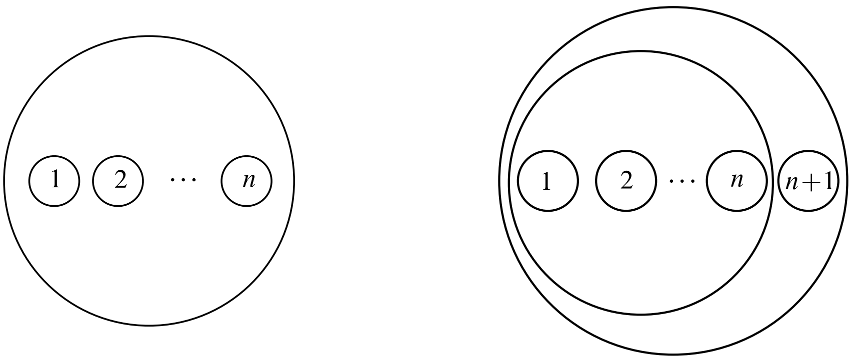

Figure 1: We can define M_n by gluing copies of M_1 into the disks on the left, and we show how to define an embedding $M_n \hookrightarrow M_{n+1}$ on the right.

Remark 5.2 Since $\pi_0\big(\mathrm{aut}_*\big(\bigvee^S X\big)\big)$ acts on $\pi_k\big(\mathrm{aut}_*\big(\bigvee^S X\big)\big)$, we get an induced $\Sigma(S)$–action on $\pi_k\big(\mathrm{aut}_*\big(\bigvee^S X\big)\big)$ by the above. This action coincides with the $\Sigma(S)$–action coming from the FI–module structure discussed in Section 4.

Definition 5.3 The deformation retraction $M_S \to \bigvee^S M_1$ induces an equivalence

$$\mathrm{aut}_*(M_S) \xrightarrow{\sim} \mathrm{aut}_*\Big(\bigvee^S M_1\Big).$$

Composing this map with the inclusion $\mathrm{aut}_\partial(M_S) \hookrightarrow \mathrm{aut}_*(M_S)$ yields a map

$$u\colon \mathrm{aut}_\partial(M_S) \to \mathrm{aut}_*\Big(\bigvee^S M_1\Big)$$

that induces a group homomorphism $\pi_0(u)\colon \pi_0(\mathrm{aut}_\partial(M_S)) \to \pi_0\big(\mathrm{aut}_*\big(\bigvee^S M_1\big)\big)$.

The first thing we will show to construct our FI–module is the following:

Proposition 5.4 *Assuming $d \geq 3$, there is a group homomorphism $\varepsilon_n\colon \Sigma_n \to \pi_0(\mathrm{aut}_\partial(M_n))$ such that Q_{n,M_1} factors as $Q_{n,M_1} = \pi_0(u) \circ \varepsilon_n$.*

Remark 5.5 Since the group $\pi_0(\mathrm{aut}_\partial(M_n))$ acts on the higher homotopy groups $\pi_k(\mathrm{aut}_\partial(M_n))$, this means that there is a Σ_n–action on the higher homotopy groups of $\mathrm{aut}_\partial(M_n)$ which is nontrivial whenever ε_n is nontrivial. This action, together with the stabilization maps, will define our FI–module structure.

We will prove this in a number of steps, so let us first describe the idea: Writing $D := D^d$, we consider the subgroup $G_n \subseteq \mathrm{Diff}_\partial(D)$ consisting of diffeomorphisms which fix the embedded little disks in D from Figure 1, left, up to permutation. There is then a group homomorphism $\pi\colon G_n \to \Sigma_n$, given by sending a diffeomorphism to the permutation it induces on the little disks. We also get a group homomorphism $G_n \to \mathrm{Diff}_\partial(M_n)$, given by constructing M_n as above, and mapping $f \in G_n$ to the boundary-relative diffeomorphism of M_n which is given by f outside the n glued-in copies of M_1, and on $\bigsqcup^n M_1$ is given by $\pi(f)$. We will construct a group homomorphism $\Sigma_n \hookrightarrow \pi_0(G_n)$, which postcomposed with the maps

$$\pi_0(G_n) \to \pi_0(\mathrm{Diff}_\partial(M_n)) \to \pi_0(\mathrm{aut}_\partial(M_n))$$

is the map ε_n described in Proposition 5.4. Let us now give the proof in more detail:

Proof Our choice of embedded disks in D defines an element $e \in \mathrm{Emb}\big(\bigsqcup^n D, D\big)$. Let $\bar{e}$ denote its image in the quotient $\mathrm{Emb}\big(\bigsqcup^n D, D\big)/\Sigma_n$, where we take the quotient of the action permuting the embedded disks. Restricting to the image of e defines a map $\mathrm{Diff}_\partial(D) \to \mathrm{Emb}\big(\bigsqcup^n D, D\big)$, which is a Serre fibration. The quotient map $\mathrm{Emb}\big(\bigsqcup^n D, D\big) \to \mathrm{Emb}\big(\bigsqcup^n D, D\big)/\Sigma_n$ is a covering map, so the composition

$$p\colon \mathrm{Diff}_\partial(D) \to \mathrm{Emb}\Big(\bigsqcup^n D, D\Big)\Big/\Sigma_n \tag{5}$$

is also a Serre fibration. The fiber over $\bar{e}$ consists of the diffeomorphisms which restricted to the image of e are permutations, ie $\operatorname{fib}_p(\bar{e}) = G_n$. We thus get a connecting homomorphism

$$\delta\colon \pi_1(\operatorname{Emb}\Bigl(\bigsqcup^n D, D\Bigr)\Big/\Sigma_n) \to \pi_0(G_n)$$

in the long exact sequence of homotopy groups. We will therefore first show that there is an injective group homomorphism $\Sigma_n \hookrightarrow \pi_1\bigl(\operatorname{Emb}\bigl(\bigsqcup^n D, D\bigr)\bigr)$.

Note that if we let $C_n(\mathring{D})$ denote the ordered configuration space of n points in $\mathring{D}$, there is a map $\hat{\rho}\colon \operatorname{Emb}\bigl(\bigsqcup^n D, D\bigr) \to C_n(\mathring{D})$ given by restricting to the center of each embedded disk. This map also has a section $\hat{s}$, given by sending a configuration to an embedding of n little disks, centered at the respective points and with radii all equal to the minimum distance between the points and between the points and the boundary of D, divided by three. We also get an induced map

$$\rho\colon \operatorname{Emb}\Bigl(\bigsqcup^n D, D\Bigr)\Big/\Sigma_n \to C_n(\mathring{D})/\Sigma_n \tag{6}$$

on orbits, which has a section s defined in the corresponding way. We define $U_n(\mathring{D}) := C_n(\mathring{D})/\Sigma_n$ for brevity. Since we have assumed that $d \geq 3$, we have that $\pi_1(U_n(\mathring{D})) \cong \Sigma_n$ and thus we get a homomorphism $\pi_1(s)\colon \Sigma_n \cong \pi_1(U_n(\mathring{D})) \to \pi_1\bigl(\operatorname{Emb}\bigl(\bigsqcup^n D, D\bigr)/\Sigma_n\bigr)$. Furthermore, note that since s is a section, $\pi_1(\rho) \circ \pi_1(s)$ is the identity on $\pi_1(U_n(\mathring{D}))$ and so $\pi_1(s)$ is injective.

By composing with the connecting homomorphism in the long exact sequence associated to p, we thus get a homomorphism $\Sigma_n \to \pi_0(G_n)$. In order to understand this map better, we describe the connecting homomorphism δ in more detail. If γ is a loop in $\operatorname{Emb}\bigl(\bigsqcup^n D, D\bigr)/\Sigma_n$ based at $\bar{e}$, representing an element of $\pi_1\bigl(\operatorname{Emb}\bigl(\bigsqcup^n D, D\bigr)/\Sigma_n\bigr)$, it lifts to a path $\tilde{\gamma}$ in $\operatorname{Diff}_\partial(D)$ starting at id_D, since p is a Serre fibration. The connecting homomorphism sends the class of γ to the connected component of G_n containing $\tilde{\gamma}(1)$. If we consider the restriction of δ to the image of inclusion $\pi_1(s)$ above, we see that a permutation σ is sent to the isotopy class of some diffeomorphism in G_n which, restricted to the little disks, is precisely σ. If we finally consider the composite map

$$\Sigma_n \to \pi_0(G_n) \to \pi_0(\operatorname{Diff}_\partial(M_n)) \to \pi_0(\operatorname{aut}_\partial(M_n)) \to \pi_0(\operatorname{aut}_*(M_n)) \cong \pi_0\Bigl(\operatorname{aut}_*\Bigl(\bigvee^n M_1\Bigr)\Bigr),$$

it follows by the definition of the map $G_n \to \operatorname{Diff}_\partial(M_n)$ that this takes a permutation to the homotopy class of the homotopy automorphism of $\bigvee^n M_1$ given by permuting the wedge summands in the corresponding way. In other words, the composition is equal to Q_{n,M_1}, so we can simply define ε_n as the composition of the first three maps. □

Remark 5.6 If we assume that M_1 has nontrivial homology, then for any nontrivial permutation σ we have that $\mathcal{X}_{M_1}(\sigma)\colon \bigvee^n M_1 \to \bigvee^n M_1$ is not homotopic to the identity, since it induces a nontrivial permutation of the reduced homology $\widetilde{H}_*\bigl(\bigvee^n M_1\bigr) = \bigoplus_n \widetilde{H}_*(M_1)$, which is different from the identity map whenever $\widetilde{H}_*(M_1)$ is nontrivial. If that is the case, the homomorphism Q_{n,M_1} is injective, so it follows that ε_n is injective as well, and thus both $\pi_0(\mathrm{aut}_\partial(M_n))$ and $\pi_0\big(\mathrm{aut}_*\big(\bigvee^n M_1\big)\big)$ contain a subgroup isomorphic to Σ_n.

Corollary 5.7 *Under the assumptions of Remark 5.6, fix a subspace $A \subseteq \partial M_n$, possibly empty, such that $A \subset M_n$ is a cofibration. Then all of the groups $\pi_0(\mathrm{aut}_A(M_n))$, $\pi_0(\mathrm{Diff}_A(M_n))$ and $\pi_0(\mathrm{Homeo}_A(M_n))$ contain a subgroup isomorphic to Σ_n.*

Proof Suppose that $A \neq \varnothing$ and let us first consider the case of homotopy automorphisms. The map $u\colon \mathrm{aut}_\partial(M_n) \to \mathrm{aut}_*\big(\bigvee^n M_1\big)$ factors as

$$\mathrm{aut}_\partial(M_n) \to \mathrm{aut}_A(M_n) \to \mathrm{aut}_*\left(\bigvee^n M_1\right),$$

proving this case. To get the cases with diffeomorphisms or homeomorphisms, consider the factorization

$$\mathrm{Diff}_\partial(M_n) \to \mathrm{Diff}_A(M_n) \to \mathrm{Homeo}_A(M_n) \to \mathrm{aut}_A(M_n) \to \mathrm{aut}_*\left(\bigvee^n M_1\right).$$

For the case where A is empty, we instead postcompose with the map $\mathrm{aut}_*\big(\bigvee^n M_1\big) \to \mathrm{aut}\big(\bigvee^n M_1\big)$, and the resulting map factors as

$$\mathrm{aut}_\partial(M_n) \to \mathrm{aut}(M_n) \to \mathrm{aut}\left(\bigvee^n M_1\right).$$

The composition of Q_{n,M_1} with the map induced on π_0 by the rightmost map above will still be injective, and from this the case follows. To get the statement for diffeomorphisms and homeomorphisms, we instead use the factorization

$$\mathrm{Diff}_\partial(M_n) \to \mathrm{Diff}(M_n) \to \mathrm{Homeo}(M_n) \to \mathrm{aut}(M_n) \to \mathrm{aut}\left(\bigvee^n M_1\right). \qquad \square$$

Remark 5.8 A referee pointed out that the existence of the homomorphism $\Sigma_n \to \pi_0(\mathrm{aut}_\partial(M_n))$ is likely a consequence of a higher structure. More specifically, it is reasonable to expect that the space $\bigsqcup_{n\geq 1} B\,\mathrm{aut}_\partial(M_n)$ can be endowed with the structure of an E_d–algebra, ie an algebra over the little d–disks operad, in a similar way as, for example, the space $\bigsqcup_{n\geq 1} B\,\mathrm{Diff}_\partial(M_n)$. If this is the case, the E_d–algebra structure maps in particular give us a map

$$E_d(n)/\Sigma_n \to B\,\mathrm{aut}_\partial(M_n),$$

and since $E_d(n)/\Sigma_n \simeq U_n(\mathring{D})$, taking fundamental groups gives us a map $\Sigma_n \to \pi_0(\mathrm{aut}_\partial(M_n))$, which should be precisely ε_n. We expect this to be true, but have elected to use a more hands-on approach, since rigorously constructing the E_d–algebra structure is nontrivial and seems to require using methods from higher homotopy theory that go quite far beyond the scope of this paper. For comparison, what makes this easier in the case of diffeomorphisms is that we have a good model for $B\,\mathrm{Diff}_\partial(M_n)$ as a topological space, in terms of embeddings of M_n into $\mathbb{R}^\infty$ (with certain boundary conditions), modulo the action of $\mathrm{Diff}_\partial(M_n)$. In contrast, it is not clear how to do a similar construction for homotopy automorphisms.

We have now defined the Σ_n–action on the homotopy groups of $\operatorname{aut}_\partial(M_n)$. Next we show that this action is compatible with the stabilization maps s_n.

Proposition 5.9 *There is a commutative diagram*

$$\begin{array}{ccc} \Sigma_n & \longrightarrow & \Sigma_{n+1} \\ {\scriptstyle \varepsilon_n}\downarrow & & \downarrow{\scriptstyle \varepsilon_{n+1}} \\ \pi_0(\operatorname{aut}_\partial(M_n)) & \xrightarrow{\pi_0(s_n)} & \pi_0(\operatorname{aut}_\partial(M_{n+1})) \end{array}$$

where the upper horizontal map is the standard inclusion.

Proof Construct stabilization maps $G_n \to G_{n+1}$, $\operatorname{Emb}\big(\bigsqcup^n D, D\big)/\Sigma_n \to \operatorname{Emb}\big(\bigsqcup^{n+1} D, D\big)/\Sigma_{n+1}$ and $C_n(\mathring{D})/\Sigma_n \to C_{n+1}(\mathring{D})/\Sigma_{n+1}$ in the same way as $s_n : \operatorname{aut}_\partial(M_n) \to \operatorname{aut}_\partial(M_{n+1})$, using Figure 1, right. This gives us a diagram

$$\begin{array}{ccc} \Sigma_n & \longrightarrow & \Sigma_{n+1} \\ \cong\downarrow & & \downarrow\cong \\ \pi_1(U_n(\mathring{D})) & \longrightarrow & \pi_1(U_{n+1}(\mathring{D})) \\ \downarrow & & \downarrow \\ \pi_1\big(\operatorname{Emb}\big(\bigsqcup^n D, D\big)/\Sigma_n\big) & \longrightarrow & \pi_1\big(\operatorname{Emb}\big(\bigsqcup^{n+1} D, D\big)/\Sigma_{n+1}\big) \\ \downarrow & & \downarrow \\ \pi_0(G_n) & \longrightarrow & \pi_0(G_{n+1}) \\ \downarrow & & \downarrow \\ \pi_0(\operatorname{aut}_\partial(M_n)) & \longrightarrow & \pi_0(\operatorname{aut}_\partial(M_{n+1})) \end{array}$$

where the top horizontal arrow is the standard inclusion. The two upper squares, as well as the bottom square, are all commutative by the definition of the stabilization maps. The second square from the bottom can be shown to be commutative simply by once again considering the definition of the connecting homomorphism in detail as above, but we can also reason as follows: Define a map $\operatorname{Diff}_\partial(D) \to \operatorname{Diff}_\partial(D)$ in the same was as we defined the stabilization maps, using Figure 1, right, and extending by the identity (note however that this map is homotopic to the identity), giving us a commutative diagram

$$\begin{array}{ccc} \operatorname{Diff}_\partial(D) & \longrightarrow & \operatorname{Diff}_\partial(D) \\ \downarrow & & \downarrow \\ \operatorname{Emb}\big(\bigsqcup^n D, D\big)/\Sigma_n & \longrightarrow & \operatorname{Emb}\big(\bigsqcup^{n+1} D, D\big)/\Sigma_{n+1} \end{array}$$

which is a map of Serre fibrations. By functoriality, this induces a map between the long exact sequences of homotopy groups, in which the square we consider appears. □

Corollary 5.10 *For $k \geq 1$, the sequence $\{\pi_k(\operatorname{aut}_\partial(M_n)), \pi_k(s_n)\}$ is a consistent sequence of $\mathbb{Z}[\Sigma_n]$–modules.*

Proof Recall that $\pi_0(\mathrm{aut}_\partial(M_n))$ acts on $\pi_k(\mathrm{aut}_\partial(M_n))$ and thus, through the stabilization map

$$\pi_0(s_n)\colon \pi_0(\mathrm{aut}_\partial(M_n)) \to \pi_0(\mathrm{aut}_\partial(M_{n+1})),$$

$\pi_0(\mathrm{aut}_\partial(M_n))$ acts on $\pi_k(\mathrm{aut}_\partial(M_{n+1}))$ as well. By definition of the stabilization map, $\pi_k(s_n)$ is $\pi_0(\mathrm{aut}_\partial(M_n))$–equivariant.

By considering $\pi_k(\mathrm{aut}_\partial(M_n))$ as a $\mathbb{Z}[\Sigma_n]$–module via the homomorphism $\varepsilon_n\colon \Sigma_n \to \pi_0(\mathrm{aut}_\partial(M_n))$, it follows from Proposition 5.9 and the equivariance discussed above that $\{\pi_k(\mathrm{aut}_\partial(M_n)), \pi_k(s_n)\}$ is a consistent sequence of $\mathbb{Z}[\Sigma_n]$–modules. $\square$

Theorem 5.11 *For each $k \geq 1$, the* FI*–module $S \mapsto \pi_k(\mathrm{aut}_*(M_S)) \cong \pi_k\big(\mathrm{aut}_*\big(\bigvee^S M_1\big)\big)$ lifts to an* FI*–module*

$$S \mapsto \pi_k(\mathrm{aut}_\partial(M_S)),$$

where the standard inclusion $\boldsymbol{n} \hookrightarrow \boldsymbol{n+1}$ gives the induced stabilization map

$$\pi_k(s)\colon \pi_k(\mathrm{aut}_\partial(M_n)) \to \pi_k(\mathrm{aut}_\partial(M_{n+1})).$$

Proof We have shown in Corollary 5.10 that the homotopy groups $\{\pi_k(\mathrm{aut}_\partial(M_n))\}_{n\geq 1}$ form a consistent sequence of $\mathbb{Z}[\Sigma_n]$–modules, and from the previous discussion it is clear that the maps $\mathrm{aut}_\partial(M_n) \to \mathrm{aut}_*\big(\bigvee^n M_1\big)$ induce a map of consistent sequences to $\big\{\pi_k\big(\mathrm{aut}_*\big(\bigvee^n M_1\big)\big)\big\}_{n\geq 1}$, which we know comes from an FI–module. Thus, it is sufficient to show that $\{\pi_k(\mathrm{aut}_\partial(M_n))\}_{n\geq 1}$ also comes from an FI–module.

From Lemma 2.6, it suffices to show that if $\sigma \in \Sigma_{n+m}$ is such that $\sigma|_{\boldsymbol{n}} = \mathrm{id}$, it acts trivially on the image of the stabilization map $\pi_k(\mathrm{aut}_\partial(M_n)) \to \pi_k(\mathrm{aut}_\partial(M_{n+m}))$. Embedding M_n in M_{n+m} according to the composition of the embeddings $M_n \hookrightarrow \cdots \hookrightarrow M_{n+m}$ defined by Figure 1, right, we may represent σ by an automorphism $f_\sigma \in \mathrm{aut}_\partial(M_{n+m})$ which is supported completely on $M_m \subset M_{m+n}$ and is thus the identity on $M_n \subset M_{m+n}$. Any homotopy automorphism $g \in \mathrm{im}(s_{n+m-1}\cdots s_n\colon \mathrm{aut}_\partial(M_n) \to \mathrm{aut}_\partial(M_{m+n}))$ is supported on M_n, so $f_\sigma g f_\sigma^{-1} = g$. Hence σ on acts trivially on the image of the stabilization map $\pi_k(\mathrm{aut}_\partial(M_n)) \to \pi_k(\mathrm{aut}_\partial(M_{n+m}))$. $\square$

5.2 Rational representation stability via algebraic models for relative homotopy automorphisms

We will study a certain dg Lie model for $B\,\mathrm{aut}_{\partial,\circ}(M_n)$ constructed in [Berglund and Madsen 2020], and use it to prove that the FI–module $S \mapsto \pi_k^{\mathbb{Q}}(\mathrm{aut}_\partial(M_S)) = \pi_k(\mathrm{aut}_\partial(M_S)) \otimes \mathbb{Q}$ is finitely generated.

We recall that a quasifree dg Lie algebra $(\mathbb{L}(V), d)$ is said to be minimal if $d(V) \subset [\mathbb{L}(V), \mathbb{L}(V)]$. If two minimal dg Lie algebras are quasi-isomorphic then they are isomorphic. Moreover, if $\mathbb{L}(V)$ is a minimal dg Lie algebra model for a nilpotent space X of finite type, then one can show that V is isomorphic to the desuspension of the reduced rational homology of X, which we will denote by $s^{-1}\widetilde{H}_*(X;\mathbb{Q})$.

In this subsection we fix a d–dimensional simply connected oriented closed manifold M, where $M_1 = M \setminus \mathring{D}$ has a nontrivial rational homology. The intersection form on $H_*(M)$ induces a graded symmetric inner product of degree d on the reduced homology $\widetilde{H}_*(M_1)$. This in turn induces a graded antisymmetric inner product of degree $d-2$ on $H = s^{-1}\widetilde{H}^*(M_1)$.

Definition 5.12 Let H be a graded antisymmetric inner product space of degree $d-2$ (eg $s^{-1}\widetilde{H}_*(M_1)$) with a basis $\{\alpha_1, \dots, \alpha_m\}$. The dual basis $\{\alpha_1^\#, \dots, \alpha_m^\#\}$ is characterized by the following property:

$$\langle \alpha_i, \alpha_j^\# \rangle = \delta_{ij}.$$

Let $\omega_H \in \mathbb{L}^2(H)$ be given by

$$\omega_H = \frac{1}{2}\sum_{i=1}^{m}[\alpha_i^\#, \alpha_i].$$

It turns out that ω_H is independent of choice of basis $\{\alpha_1, \dots, \alpha_m\}$; see [Berglund and Madsen 2020] for details.

Remark 5.13 By the same arguments as above, the graded vector space $s^{-1}\widetilde{H}_*(M_n)$ also has a structure of a graded antisymmetric inner product space of degree $d-2$ which coincides with the one given by the direct sum $(s^{-1}\widetilde{H}_*(M_1))^{\oplus n}$.

The next proposition is due to Stasheff [1983, Theorem 2], and is discussed in [Berglund and Madsen 2020, Theorem 3.11].

Proposition 5.14 *Let $M = M^d$ be a closed oriented d–dimensional manifold, let $M_1 = M \setminus \mathring{D}$ and let $H = s^{-1}\widetilde{H}_*(M_1)$. Then the inclusion $S^{d-1} \cong \partial M_1 \hookrightarrow M_1$ is modeled by a dg Lie algebra map*

$$\iota\colon \mathbb{L}(x) \hookrightarrow \mathbb{L}(H), \quad \iota(x) = (-1)^d \omega_H,$$

where $\mathbb{L}(H)$ and $\mathbb{L}(x)$ denote the minimal dgl models for M_1 and S^{d-1}, respectively.

Given a fixed basis $\{\alpha_1, \dots, \alpha_m\}$ for $H = s^{-1}\widetilde{H}_*(M_1)$ we get a basis for $s^{-1}\widetilde{H}_*(M_n) \cong H^{\oplus n}$ which is of the form

$$\{\alpha_i^j \mid 1 \le i \le m, 1 \le j \le n\}.$$

We denote $\omega_{H^{\oplus n}} = \frac{1}{2}\sum_{i,j}[(\alpha_i^j)^\#, \alpha_i^j] \in \mathbb{L}(H^{\oplus n})$ by ω_n. We have that ω_n is invariant under the Σ_n–action on $\mathbb{L}(H^{\oplus n})$ that permutes the summands of $H^{\oplus n}$.

Note that $\iota\colon \mathbb{L}(x) \to \mathbb{L}(H^{\oplus n})$ is not a cofibration. In order to model the inclusion $\partial M_n \subset M_n$ by a cofibration in the model category of dg Lie algebras we need a new model for M_n.

Lemma 5.15 *Let $\mathbb{L}(H^{\oplus n}, x, y)$ be the dg Lie algebra that contains $\mathbb{L}(H^{\oplus n})$ as a dg Lie subalgebra where $|x| = d-2$ and $|y| = d-1$, and where*

$$dx = 0 \quad \text{and} \quad dy = x - (-1)^d \omega_n.$$

Then

$$\hat{\iota}\colon \mathbb{L}(x) \to \mathbb{L}(H^{\oplus n}, x, y), \quad \hat{\iota}(x) = x,$$

is a cofibration that models the inclusion of $\partial M_n \cong S^{d-1}$ into M_n. Moreover this model is a relative minimal model in the sense of [Espic and Saleh 2020].

Proof The dg Lie algebra map $\rho\colon \mathbb{L}(H^{\oplus n}, x, y) \to \mathbb{L}(H^{\oplus n})$ where $\rho|_{H^{\oplus n}} = \mathrm{id}_{H^{\oplus n}}$, $\rho(x) = (-1)^d \omega_n$ and $\rho(y) = 0$ is a quasi-isomorphism. Straightforward computation shows that $\rho \circ \hat{\iota} = \iota$, proving that $\hat{\iota}$ is a model for ι (which is a model for the inclusion of the boundary). Minimality is straightforward verification; see [loc. cit., Section 3]. □

By Proposition 3.7(b), a dg Lie algebra model for $B\,\mathrm{aut}_{\partial,\circ}(M_n)$ is given by $\mathrm{Der}^+(\mathbb{L}(H^{\oplus n}, x, y) \,\|\, \mathbb{L}(x))$. However, we will use another model thanks to the following result:

Proposition 5.16 [Berglund and Madsen 2020, Theorem 3.12] *Let* $\mathrm{Der}(\mathbb{L}(H^{\oplus n}) \| \omega_n)$ *denote the dg Lie algebra of derivations on* $\mathbb{L}(H^{\oplus n})$ *that vanish on* ω_n *and where the differential is given by* $[d_{\mathbb{L}(H^{\oplus n})}, -]$. *Then there is an equivalence of dg Lie algebras*

$$\mathrm{Der}^+(\mathbb{L}(H^{\oplus n}) \,\|\, \omega_n) \to \mathrm{Der}^+(\mathbb{L}(H^{\oplus n}, x, y) \,\|\, \mathbb{L}(x)), \quad \theta \mapsto \hat{\theta},$$

where $\hat{\theta}|_{\mathbb{L}(H^{\oplus n})} = \theta$ *and* $\theta(x) = \theta(y) = 0$.

Remark 5.17 It follows that $\mathrm{Der}^+(\mathbb{L}(H^{\oplus n}) \,\|\, \omega_n)$ is a dg Lie algebra model for $B\,\mathrm{aut}_{\partial,\circ}(M_n)$ and the inclusion $\mathrm{Der}^+(\mathbb{L}(H^{\oplus n}) \| \omega_n) \to \mathrm{Der}^+(\mathbb{L}(H^{\oplus n}))$ is a model for the map $B\,\mathrm{aut}_{\partial,\circ}(M_n) \to B\,\mathrm{aut}_{*,\circ}(M_n)$, induced by the inclusion $\mathrm{aut}_{\partial,\circ}(M_n) \hookrightarrow \mathrm{aut}_{*,\circ}(M_n)$.

Definition 5.18 With the terminology of Section 2, we define a dg FI–Lie algebra $\mathrm{Der}(\mathbb{L}\mathcal{H} \,\|\, \omega_{\mathcal{H}})$ as follows: For $S \in \mathrm{FI}$, we let $\mathrm{Der}(\mathbb{L}\mathcal{H} \,\|\, \omega_{\mathcal{H}})(S) := \mathrm{Der}(\mathbb{L}\mathcal{H}(S) \,\|\, \omega_S)$ be the dg Lie algebra of derivations on $\mathbb{L}\mathcal{H}(S) = \mathbb{L}(H^{\oplus S})$ that vanish on ω_S. For $i : S \hookrightarrow T$ in FI, we get a map

$$\mathrm{Der}(\mathbb{L}\mathcal{H} \,\|\, \omega_{\mathcal{H}})(i) \colon \mathrm{Der}(\mathbb{L}(H^{\oplus S}) \,\|\, \omega_S) \hookrightarrow \mathrm{Der}(\mathbb{L}(H^{\oplus T}) \,\|\, \omega_T),$$

defined as follows: Suppose $x_\alpha \in \mathcal{H}(T)$ lies in the direct summand of $\mathcal{H}(T)$ corresponding to $\alpha \in T$ and let $D \in \mathrm{Der}(\mathbb{L}(H^{\oplus S}) \,\|\, \omega_S)$. Then $\mathrm{Der}(\mathbb{L}\mathcal{H} \,\|\, \omega_{\mathcal{H}})(i)D$ is determined by

$$(\mathrm{Der}(\mathbb{L}\mathcal{H} \,\|\, \omega_{\mathcal{H}})(i)D)(x_\alpha) = \begin{cases} 0 & \text{if } \alpha \in T \setminus i(S), \\ (\mathbb{L}\mathcal{H}(i) \circ D \circ \mathcal{H}(i)^{-1})(x_\alpha) & \text{if } \alpha \in i(S). \end{cases}$$

We conclude from having such a dg FI–Lie algebra the following:

Remark 5.19 The above dg FI–Lie algebra structure induces an FI–module structure on the homology. For $k \geq 1$, we have that $H_k(\mathrm{Der}(\mathbb{L}(H^{\oplus S}) \,\|\, \omega_S)) \cong \pi_k^{\mathbb{Q}}(\mathrm{aut}_\partial(M_S))$, which gives an FI–module structure on $\{\pi_k^{\mathbb{Q}}(\mathrm{aut}_\partial(M_S))\}_{S \in \mathrm{FI}}$. We will show that this FI–module structure coincides with the one obtained by rationalizing the FI–module structure on $\{\pi_k(\mathrm{aut}_\partial(M_S))\}_{S \in \mathrm{FI}}$ defined in Section 5.1.

Proposition 5.20 *A dg Lie algebra model for the stabilization map* $B\,\mathrm{aut}_{\partial,\circ}(M_n) \to B\,\mathrm{aut}_{\partial,\circ}(M_{n+1})$ *is given by*

$$\varphi_n \colon \mathrm{Der}^+(\mathbb{L}(H^{\oplus n}) \,\|\, \omega_n) \to \mathrm{Der}^+(\mathbb{L}(H^{\oplus n+1}) \,\|\, \omega_{n+1}),$$

where $\varphi_n(\theta)$ *coincides with* θ *on the first* n *summands of* $H^{\oplus n+1}$ *and vanishes on the last summand.*

Proof The proof is omitted since it is very similar to the proof of Proposition 4.2. □

Proposition 5.21 *The Σ_n–action on $\pi_*^{\mathbb{Q}}(\mathrm{aut}_\partial(M_n))$ induced by $\varepsilon_n\colon \Sigma_n \to \pi_0(\mathrm{aut}_\partial(M_n))$ is modeled by the Σ_n–action on $H_k(\mathrm{Der}(\mathbb{L}(H^{\oplus n})) \parallel \omega_n)$ induced by the* FI*–module structure from Definition 5.18.*

Proof For every $\sigma \in \Sigma_n$, let $\eta_\sigma \in \mathrm{aut}_\partial(M_n))$ denote a representative for $\varepsilon_n(\sigma) \in \pi_0(\mathrm{aut}_\partial(M_n))$, and define a self-equivalence

$$\mathrm{Ad}_\sigma\colon \mathrm{aut}_\partial(M_n) \to \mathrm{aut}_\partial(M_n), \quad \mathrm{Ad}_\sigma(f) = \eta_\sigma \circ f \circ \eta_{\sigma^{-1}}.$$

This induces a Σ_n–action on $\pi_k(\mathrm{aut}_\partial(M_n))$ given by $\sigma.a = \pi_k(\mathrm{Ad}_\sigma)(a)$ which is precisely the Σ_n–action given by the FI–module structure.

As we saw in Lemma 5.15, $\hat{\iota}\colon \mathbb{L}(x) \to \mathbb{L}(H^{\oplus n}, x, y)$ is a minimal relative model for the inclusion $\partial M_n \hookrightarrow M_n$.

By Lemma 3.15, η_σ is modeled by an $\hat{\iota}$–relative automorphism $\zeta_\sigma \in \mathrm{Aut}_{\hat{\iota}}(\mathbb{L}(H^{\oplus n}, x, y))$, and hence, by Corollary 3.19, the automorphism

$$\alpha_{\zeta_\sigma}\colon \mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y) \parallel \mathbb{L}(x)) \to \mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y) \parallel \mathbb{L}(x)), \quad \alpha_{\zeta_\sigma}(\theta) = \zeta_\sigma \circ \theta \circ \zeta_\sigma^{-1},$$

is a model for the delooping of Ad_σ. In particular, $H_k(\alpha_{\zeta_\sigma})$ is a model for $\pi_k(\mathrm{Ad}_\sigma)$. Moreover, this defines a Σ_n–action on $H_k\big(\mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y) \parallel \mathbb{L}(x))\big)$ given by $\sigma.b = H_k(\alpha_{\zeta_\sigma})(b)$ that models the Σ_n–action on $\pi_k^{\mathbb{Q}}(\mathrm{aut}_\partial(M_n))$ described above.

Since the isomorphism of Lemma 3.15 is not explicit, we do not know what ζ_σ is. However, viewing ζ_σ as a nonrelative automorphism that models pointed homotopy automorphisms, we know that it models the permutation of the summands of $\bigvee_{i=1}^n M_1$ corresponding to $\sigma \in \Sigma_n$. A model for this pointed map is given by $\psi_\sigma\colon \mathbb{L}(H^{\oplus n}, x, y) \to \mathbb{L}(H^{\oplus n}, x, y)$, where $\psi_\sigma(\alpha_i^j) = \alpha_i^{\sigma(j)}$, $\psi_\sigma(x) = x$ and $\psi_\sigma(y) = y$. Since ψ_σ and ζ_σ model the same pointed homotopy class of pointed maps they have to be homotopic as dg Lie algebra maps, and thus α_{ζ_σ} and α_{ψ_σ} induce the same map on the homology of $\mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y))$. In particular, for every cycle $\theta \in Z\big(\mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y))\big)$, the difference $\alpha_{\zeta_\sigma}(\theta) - \alpha_{\psi_\sigma}(\theta)$ is a boundary $\partial\nu$ for some $\nu \in \mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y))$.

Note that ψ_σ is also $\hat{\iota}$–relative, but not necessarily $\hat{\iota}$–equivalent, to ζ_σ. Since ζ_σ and ψ_σ are $\hat{\iota}$–relative, α_{ζ_σ} and α_{ψ_σ} define automorphisms of $\mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y) \parallel \mathbb{L}(x))$. We will show that these automorphisms induce the same map on homology. Given a cycle $\theta \in Z\big(\mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y) \parallel \mathbb{L}(x))\big)$, we have that θ is also a cycle in $\mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y))$, and thus by the above there is some $\nu \in \mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y))$ such that $\alpha_{\zeta_\sigma}(\theta) - \alpha_{\psi_\sigma}(\theta) = \partial\nu$. By this equality $\partial\nu(x) = 0$.

Let $\tilde{\nu} \in \mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y) \parallel \mathbb{L}(x))$ be given by $\tilde{\nu}|_{\mathrm{span}(H^{\oplus n}, y)} = \nu|_{\mathrm{span}(H^{\oplus n}, y)}$ and $\tilde{\nu}(x) = 0$. Now it is straightforward to see that

$$\alpha_{\zeta_\sigma}(\theta) - \alpha_{\psi_\sigma}(\theta) = \partial\nu = \partial\tilde{\nu}.$$

Hence α_{ζ_σ} and α_{ψ_σ} induce the same morphisms on $H_*\big(\mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y)|\mathbb{L}(x))\big)$. From this we conclude that the Σ_n–action on $H_k\big(\mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y)|\mathbb{L}(x))\big)$ given by $\sigma.b = H_k(\alpha_{\psi_\sigma})(b)$ is a model for the Σ_n–action on $\pi_k^{\mathbb{Q}}(\mathrm{aut}_\partial(M_n))$.

Now consider the ω_n–preserving automorphism $\phi_\sigma\colon \mathbb{L}(H^{\oplus n}) \to \mathbb{L}(H^{\oplus n})$ given by $\phi_\sigma = \psi_\sigma|_{\mathbb{L}(H^{\oplus n})}$. This yields an automorphism

$$\alpha_{\phi_\sigma}\colon \mathrm{Der}(\mathbb{L}(H^{\oplus n}) \,\|\, \omega_n) \to \mathrm{Der}(\mathbb{L}(H^{\oplus n}) \,\|\, \omega_n), \quad \alpha_{\phi_\sigma}(\theta) = \phi_\sigma \circ \theta \circ \phi_\sigma^{-1}.$$

The Σ_n–action on $\mathrm{Der}(\mathbb{L}(H^{\oplus n}) \,\|\, \omega_n)$ given by $\sigma.b = \alpha_{\phi_\sigma}(b)$ is the same Σ_n–action coming from the FI–module structure described in Definition 5.18.

The diagram

$$\begin{array}{ccc} \mathrm{Der}^+(\mathbb{L}(H^{\oplus n}) \,\|\, \omega_n) & \xrightarrow{\alpha_{\phi_\sigma}} & \mathrm{Der}^+(\mathbb{L}(H^{\oplus n}) \,\|\, \omega_n) \\ \sim\downarrow & & \downarrow\sim \\ \mathrm{Der}^+(\mathbb{L}(H^{\oplus n}, x, y) \,\|\, \mathbb{L}(x)) & \xrightarrow{\alpha_{\psi_\sigma}} & \mathrm{Der}^+(\mathbb{L}(H^{\oplus n}, x, y) \,\|\, \mathbb{L}(x)) \end{array}$$

where the vertical maps are the quasi-isomorphisms of dg Lie algebras described in Proposition 5.16, is commutative, which gives that the induced Σ_n–action on $H_k(\mathrm{Der}^+(\mathbb{L}(H^{\oplus n}) \,\|\, \omega_n))$ is a model for the Σ_n–action on $H_k\big(\mathrm{Der}(\mathbb{L}(H^{\oplus n}, x, y)|\mathbb{L}(x))\big)$ — which, in turn, is a model for the Σ_n–action on $\pi_k^{\mathbb{Q}}(\mathrm{aut}_\partial(M_n))$. □

We recall that the Lie operad $\mathscr{L}ie$ is a cyclic operad, ie that the Σ_n–action on $\mathscr{L}ie(n)$ extends to a Σ_{n+1}–action. Let $\mathscr{L}ie_c(n+1)$ denote $\mathscr{L}ie(n)$ viewed as a Σ_{n+1}–representation.

Proposition 5.22 [Berglund and Madsen 2020, Proposition 6.6] *There is an isomorphism of* FI*–modules*

$$\mathrm{Der}(\mathbb{L}\mathcal{H} \,\|\, \omega_{\mathcal{H}}) \cong s^{2-d}\mathbb{S}_{\mathscr{L}ie_c}(\mathcal{H}).$$

Proof We will prove that this isomorphism is a special case of the more general isomorphism of Berglund and Madsen, where the authors consider the category of graded antisymmetric inner product spaces of degree $2-d$, with morphisms being linear maps of degree 0 that preserve the inner product. They call this category Sp^{2-d}. An Sp^{2-d}–module is a functor from Sp^{2-d} to the category of graded vector spaces. By [loc. cit., Proposition 6.6], $V \mapsto \mathrm{Der}(\mathbb{L}(V) \,\|\, \omega_V)$ defines an Sp^{2-d}–module that is isomorphic to the Sp^{2-d}–module given by $V \mapsto s^{2-d}\mathbb{S}_{\mathscr{L}ie_c}(V)$.

For any morphism $i\colon S \to T$ in FI, the associated map $\mathcal{H}(i)\colon \mathcal{H}(S) = H^{\oplus S} \to \mathcal{H}(T) = H^{\oplus T}$ is a morphism of Sp^{2-d}–modules. Thus the isomorphism above follows. □

Theorem B *Let $M = M^d$ be a closed simply connected oriented d–dimensional manifold. With M_S defined as above, we have the following*:

(a) *For each $k \geq 1$, the* FI*–module*

$$S \mapsto \pi_k\Big(\mathrm{aut}_*\Big(\bigvee^S M_1\Big)\Big) \cong \pi_k(\mathrm{aut}_*(M_S))$$

lifts to an FI*–module*

$$S \mapsto \pi_k(\mathrm{aut}_\partial(M_S))$$

sending the standard inclusion $\boldsymbol{n} \to \boldsymbol{n+1}$ *to the map* $\pi_k(\mathrm{aut}_\partial(M_n)) \to \pi_k(\mathrm{aut}_\partial(M_{n+1}))$ *induced by the stabilization map* s_n*.*

(b) *The rationalization of this* FI*–module is of weight* $\leq k+d-2$ *and stability degree* $\leq k+d-1$*.*

Proof (a) This is Theorem 5.11.

(b) By the isomorphism in Proposition 5.22,

$$\mathrm{Der}(\mathbb{L}\mathcal{H} \parallel \omega_{\mathcal{H}})_k \cong \mathbb{S}_{\mathscr{L}ie_c}(\mathcal{H})_{k+d-2}.$$

By Proposition 2.20,

$$\mathrm{weight}(\mathrm{Der}(\mathbb{L}\mathcal{H} \parallel \omega_{\mathcal{H}})_k) \leq k+d-2$$

and

$$\text{stab-deg}(\mathrm{Der}(\mathbb{L}\mathcal{H} \parallel \omega_{\mathcal{H}})_k) \leq k+d-2.$$

The weight and the stability degree for the homology follow from Proposition 2.18. □

References

[Arnold 1969] **V I Arnold**, *The cohomology ring of the group of dyed braids*, Mat. Zametki 5 (1969) 227–231 MR Zbl In Russian; translated in Math. Notes 5 (1969) 138–140

[Berglund 2015] **A Berglund**, *Rational homotopy theory of mapping spaces via Lie theory for L_∞–algebras*, Homology Homotopy Appl. 17 (2015) 343–369 MR Zbl

[Berglund 2020] **A Berglund**, *Rational models for automorphisms of fiber bundles*, Doc. Math. 25 (2020) 239–265 MR Zbl

[Berglund 2022] **A Berglund**, *Characteristic classes for families of bundles*, Selecta Math. 28 (2022) art. id. 51 MR Zbl

[Berglund and Madsen 2020] **A Berglund**, **I Madsen**, *Rational homotopy theory of automorphisms of manifolds*, Acta Math. 224 (2020) 67–185 MR Zbl

[Berglund and Saleh 2020] **A Berglund**, **B Saleh**, *A dg Lie model for relative homotopy automorphisms*, Homology Homotopy Appl. 22 (2020) 105–121 MR Zbl

[Birman 1974] **J S Birman**, *Braids, links, and mapping class groups*, Annals of Math. Studies 82, Princeton Univ. Press (1974) MR Zbl

[Church 2012] **T Church**, *Homological stability for configuration spaces of manifolds*, Invent. Math. 188 (2012) 465–504 MR Zbl

[Church and Farb 2013] **T Church**, **B Farb**, *Representation theory and homological stability*, Adv. Math. 245 (2013) 250–314 MR Zbl

[Church et al. 2014] **T Church**, **J S Ellenberg**, **B Farb**, **R Nagpal**, *FI-modules over Noetherian rings*, Geom. Topol. 18 (2014) 2951–2984 MR Zbl

[Church et al. 2015] **T Church**, **J S Ellenberg**, **B Farb**, *FI-modules and stability for representations of symmetric groups*, Duke Math. J. 164 (2015) 1833–1910 MR Zbl

[Espic and Saleh 2020] **H Espic**, **B Saleh**, *On the group of homotopy classes of relative homotopy automorphisms*, preprint (2020) arXiv 2002.12083

[Farjoun 1996] **E D Farjoun**, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Math. 1622, Springer (1996) MR Zbl

[Félix et al. 2001] **Y Félix**, **S Halperin**, **J-C Thomas**, *Rational homotopy theory*, Graduate Texts in Math. 205, Springer (2001) MR Zbl

[Félix et al. 2008] **Y Félix**, **J Oprea**, **D Tanré**, *Algebraic models in geometry*, Oxford Graduate Texts in Math. 17, Oxford Univ. Press (2008) MR Zbl

[Fulton and Harris 1991] **W Fulton**, **J Harris**, *Representation theory*, Graduate Texts in Math. 129, Springer (1991) MR Zbl

[Getzler 2009] **E Getzler**, *Lie theory for nilpotent L_∞–algebras*, Ann. of Math. 170 (2009) 271–301 MR Zbl

[Grey 2019] **M Grey**, *On rational homological stability for block automorphisms of connected sums of products of spheres*, Algebr. Geom. Topol. 19 (2019) 3359–3407 MR Zbl

[Hinich 1997] **V Hinich**, *Descent of Deligne groupoids*, Int. Math. Res. Not. (1997) 223–239 MR Zbl

[Kupers and Miller 2018] **A Kupers**, **J Miller**, *Representation stability for homotopy groups of configuration spaces*, J. Reine Angew. Math. 737 (2018) 217–253 MR Zbl

[Lupton and Smith 2007] **G Lupton**, **S B Smith**, *Rationalized evaluation subgroups of a map, II: Quillen models and adjoint maps*, J. Pure Appl. Algebra 209 (2007) 173–188 MR Zbl

[McDuff 1975] **D McDuff**, *Configuration spaces of positive and negative particles*, Topology 14 (1975) 91–107 MR Zbl

[Miller et al. 2019] **J Miller**, **P Patzt**, **D Petersen**, *Representation stability, secondary stability, and polynomial functors*, preprint (2019) arXiv 1910.05574

[Quillen 1969] **D Quillen**, *Rational homotopy theory*, Ann. of Math. 90 (1969) 205–295 MR Zbl

[Segal 1979] **G Segal**, *The topology of spaces of rational functions*, Acta Math. 143 (1979) 39–72 MR Zbl

[Stasheff 1983] **J Stasheff**, *Rational Poincaré duality spaces*, Illinois J. Math. 27 (1983) 104–109 MR Zbl

[Stoll 2024] **R Stoll**, *The stable cohomology of self-equivalences of connected sums of products of spheres*, Forum Math. Sigma 12 (2024) art. id. e1 MR Zbl

[Tanré 1983] **D Tanré**, *Homotopie rationnelle: modèles de Chen, Quillen, Sullivan*, Lecture Notes in Math. 1025, Springer (1983) MR Zbl

Department of Mathematical Sciences, University of Copenhagen
Copenhagen, Denmark

Department of Mathematics, Stockholm University
Stockholm, Sweden

erikjlindell@gmail.com, basharsaleh1@gmail.com

Received: 7 June 2022 Revised: 17 July 2023

ALGEBRAIC & GEOMETRIC TOPOLOGY
msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US $705/year for the electronic version, and $1040/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
https://msp.org/

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 5 (pages 2389–2970) 2024

Formal contact categories 2389

BENJAMIN COOPER

Comparison of period coordinates and Teichmüller distances 2451

IAN FRANKEL

Topological Hochschild homology of truncated Brown–Peterson spectra, I 2509

GABRIEL ANGELINI-KNOLL, DOMINIC LEON CULVER and EVA HÖNING

Points of quantum SL_n coming from quantum snakes 2537

DANIEL C DOUGLAS

Algebraic generators of the skein algebra of a surface 2571

RAMANUJAN SANTHAROUBANE

Bundle transfer of L–homology orientation classes for singular spaces 2579

MARKUS BANAGL

A reduction of the string bracket to the loop product 2619

KATSUHIKO KURIBAYASHI, TAKAHITO NAITO, SHUN WAKATSUKI and TOSHIHIRO YAMAGUCHI

Asymptotic dimensions of the arc graphs and disk graphs 2655

KOJI FUJIWARA and SAUL SCHLEIMER

Representation stability for homotopy automorphisms 2673

ERIK LINDELL and BASHAR SALEH

The strong Haken theorem via sphere complexes 2707

SEBASTIAN HENSEL and JENNIFER SCHULTENS

What are GT–shadows? 2721

VASILY A DOLGUSHEV, KHANH Q LE and AIDAN A LORENZ

A simple proof of the Crowell–Murasugi theorem 2779

THOMAS KINDRED

The Burau representation and shapes of polyhedra 2787

ETHAN DLUGIE

Turning vector bundles 2807

DIARMUID CROWLEY, CSABA NAGY, BLAKE SIMS and HUIJUN YANG

Rigidification of cubical quasicategories 2851

PIERRE-LOUIS CURIEN, MURIEL LIVERNET and GABRIEL SAADIA

Tautological characteristic classes, I 2889

JAN DYMARA and TADEUSZ JANUSZKIEWICZ

Homotopy types of suspended 4–manifolds 2933

PENGCHENG LI

The braid indices of the reverse parallel links of alternating knots 2957

YUANAN DIAO and HUGH MORTON