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What are GT-shadows?

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Let B_4 (resp. PB_4) be the braid group (resp. the pure braid group) on 4 strands and $\text{NFI}_{PB_4}(B_4)$ be the poset whose elements are finite-index normal subgroups N of B_4 that are contained in PB_4 . We introduce GT–shadows, which may be thought of as “approximations” to elements of the profinite version \widehat{GT} of the Grothendieck–Teichmüller group (Drinfeld 1991). We prove that GT–shadows form a groupoid whose objects are elements of the underlying set $\text{NFI}_{PB_4}(B_4)$. GT–shadows coming from elements of \widehat{GT} satisfy various additional properties and we investigate these properties. We establish an explicit link between GT–shadows and the group \widehat{GT} . Selected results of computer experiments on GT–shadows are presented. In the appendix we give a complete description of GT–shadows in the abelian setting. We also prove that, in the abelian setting, every GT–shadow comes from an element of \widehat{GT} . Objects very similar to GT–shadows were introduced by D Harbater and L Schneps (1997). A variation of the concept of GT–shadows for the gentle version of \widehat{GT} was studied by P Guillot (2016 and 2018).

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1 Introduction

The absolute Galois group $G_{\mathbb{Q}}$ of the field \mathbb{Q} of rational numbers and the Grothendieck–Teichmüller group \widehat{GT} introduced by V Drinfeld in [7] are among the most mysterious objects in mathematics. A far from complete list of references includes Ellenberg [8], Fresse [9], Harbater and Schneps [14], Ihara [15], Lochak and Schneps [19], Nakamura and Schneps [22], Pop [23] and Schneps [25].

Using the outer action of $G_{\mathbb{Q}}$ on the algebraic fundamental group of $\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$, one can produce a natural group homomorphism

$$(1-1) \quad G_{\mathbb{Q}} \rightarrow \widehat{GT}$$

and, due to Belyi’s theorem [3], this homomorphism is injective. Although both $G_{\mathbb{Q}}$ and \widehat{GT} are uncountable, it is very hard to produce explicit examples of elements in $G_{\mathbb{Q}}$ and in \widehat{GT} . In particular, the famous question on surjectivity of (1-1) posed by Ihara at his ICM address [15] is still open.

The group $G_{\mathbb{Q}}$ can be studied by investigating finite-degree field extensions of \mathbb{Q} . In fact $G_{\mathbb{Q}}$ coincides with the limit of the functor that sends a finite-degree Galois extension K of \mathbb{Q} to the Galois group $\text{Gal}(K/\mathbb{Q})$. The goal of this paper is to propose a loose analog of such a functor for \widehat{GT} .

The most elegant definition of the group \widehat{GT} involves (the profinite completion $\widehat{\text{PaB}}$ of) the operad PaB of parenthesized braids; see Bar-Natan [1], Fresse [9, Chapter 6] and Tamarkin [28]. PaB is an operad in the category of groupoids that is “assembled from” braid groups B_n for all $n \geq 1$. The objects of $\text{PaB}(n)$ are words of the free magma generated by symbols $1, 2, \dots, n$ in which each generator appears exactly once. For example, $\text{PaB}(3)$ has exactly 12 objects: $(12)3, (21)3, (23)1, (32)1, (31)2, (13)2, 1(23), 2(13), 2(31), 3(21), 3(12), 1(32)$. For every $n \geq 2$ and every object τ of PaB , we have

$$\text{Aut}_{\text{PaB}(n)}(\tau) = \text{PB}_n,$$

where PB_n is the pure braid group on n strands.

As an operad in the category of groupoids, PaB is generated by the two morphisms

$$(1-2) \quad \beta := \begin{array}{ccc} & 2 & 1 \\ & \bullet & \bullet \\ & \diagdown & / \\ 1 & & 2 \\ \bullet & & \bullet \end{array} \quad \text{and} \quad \alpha := \begin{array}{ccc} & 1 & (2) & 3 \\ & \bullet & \bullet & \bullet \\ & | & / & | \\ (1) & 2 & & 3 \\ \bullet & \bullet & & \bullet \end{array}.$$

Moreover, any relation on β and α in PaB is a consequence of the pentagon relation and the two hexagon relations; see (A-13), (A-14) and (A-15) in Section A.3. The hexagon relations come from two ways of connecting $(12)3$ to $3(12)$ and two ways of connecting $1(23)$ to $(23)1$ in $\text{PaB}(3)$. Similarly, the pentagon relation comes from two ways of connecting $((12)3)4$ to $1(2(34))$ in $\text{PaB}(4)$. For more details about the operad PaB and its profinite completion $\widehat{\text{PaB}}$, see Appendix A.

By definition, \widehat{GT} is the group $\text{Aut}(\widehat{\text{PaB}})$ of (continuous) automorphisms¹ of the profinite completion $\widehat{\text{PaB}}$ of PaB .

Since the morphisms β and α from (1-2) are topological generators of $\widehat{\text{PaB}}$, every $\widehat{T} \in \widehat{GT}$ is uniquely determined by its values

$$(1-3) \quad \widehat{T}(\beta) \in \text{Hom}_{\widehat{\text{PaB}}}((1, 2), (2, 1)) \quad \text{and} \quad \widehat{T}(\alpha) \in \text{Hom}_{\widehat{\text{PaB}}}((1, 2)3, 1(2, 3)).$$

¹We tacitly assume that our automorphisms act as identity on objects.

Moreover, since $\text{Aut}_{\widehat{\text{PaB}}}((1, 2)3) = \widehat{\text{PB}}_3$, $\text{Aut}_{\widehat{\text{PaB}}}((1, 2)) = \widehat{\text{PB}}_2$ and $\widehat{\text{PB}}_2 \cong \widehat{\mathbb{Z}}$, the underlying set of $\widehat{\text{GT}}$ can be identified with the subset of pairs $(\widehat{m}, \widehat{f}) \in \widehat{\mathbb{Z}} \times \widehat{\text{PB}}_3$ satisfying some relations and technical conditions.

Recall that PB_3 is isomorphic to the direct product $F_2 \times \mathbb{Z}$ of the free group F_2 on two generators and the infinite cyclic group. The F_2 -factor is generated by the two standard generators x_{12} and x_{23} , and the \mathbb{Z} -factor is generated by the element $c := x_{23}x_{12}x_{13}$. In this paper, we tacitly identify F_2 (resp. its profinite completion \widehat{F}_2) with the subgroup $\langle x_{12}, x_{23} \rangle \leq \text{PB}_3$ (resp. the topological closure of $\langle x_{12}, x_{23} \rangle$ in $\widehat{\text{PB}}_3$). Occasionally, we denote the standard generators of F_2 by x and y .

One can show² (see, for example, [Corollary 2.22](#) in [Section 2](#) of this paper) that, for every $\widehat{T} \in \widehat{\text{GT}}$, the corresponding element $\widehat{f} \in \widehat{\text{PB}}_3$ belongs to the topological closure $([\widehat{F}_2, \widehat{F}_2])^{\text{cl}}$ of the commutator subgroup of \widehat{F}_2 .

Remark 1.1 Due to [Proposition 2.19](#), the restriction of every (continuous) automorphism $\widehat{T} \in \text{Aut}(\widehat{\text{PaB}})$ to $\widehat{F}_2 \leq \widehat{\text{PB}}_3 = \text{Aut}_{\widehat{\text{PaB}}}((1, 2)3)$ gives us an automorphism of \widehat{F}_2 . In fact, many authors introduce $\widehat{\text{GT}}$ as the subgroup of (continuous) automorphisms of \widehat{F}_2 of the form

$$x \mapsto x^{\widehat{\lambda}}, \quad y \mapsto \widehat{f}^{-1}y^{\widehat{\lambda}}\widehat{f},$$

where the pair $(\widehat{\lambda}, \widehat{f}) \in \widehat{\mathbb{Z}}^\times \times ([\widehat{F}_2, \widehat{F}_2])^{\text{cl}}$ satisfies certain cocycle relations and the “invertibility condition”. Another equivalent definition of $\widehat{\text{GT}}$ is based on the use of the outer automorphisms of the profinite completions of the pure mapping class groups. For more details about this definition, we refer the reader to [\[14, Main Theorem\]](#).

Remark 1.2 It is known [\[18, Theorem 2\]](#) that, for every $(\widehat{m}, \widehat{f}) \in \widehat{\text{GT}}$, the element \widehat{f} satisfies further, rather subtle, properties. It would be interesting to investigate whether GT-shadows satisfy consequences of these properties.

1.1 The link between $G_{\mathbb{Q}}$ and $\widehat{\text{GT}}$

For completeness, we briefly recall here the link between the absolute Galois group $G_{\mathbb{Q}}$ of rationals and the Grothendieck–Teichmüller group $\widehat{\text{GT}}$.

Applying the basic theory of the algebraic fundamental group (see for instance Grothendieck [\[11\]](#) and Szamuely [\[27, Section 5.6\]](#)) to

$$\mathbb{P}_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\},$$

we get an outer action of the absolute Galois group $G_{\mathbb{Q}}$ on \widehat{F}_2 . Using the fact that this action preserves the inertia subgroups, we can lift this outer action to an honest action of the form

$$(1-4) \quad g(x) = x^{\chi(g)}, \quad g(y) = \widehat{f}_g(x, y)^{-1}y^{\chi(g)}\widehat{f}_g(x, y) \quad \text{for } g \in G_{\mathbb{Q}},$$

²This statement can also be found in many introductory papers on $\widehat{\text{GT}}$.

where $\chi: G_{\mathbb{Q}} \rightarrow \widehat{\mathbb{Z}}^{\times}$ is the cyclotomic character and $\widehat{f}_g(x, y)$ is an element of $([\widehat{F}_2, \widehat{F}_2])^{\text{cl}}$ that depends only on g .

It is known — see Drinfeld [7, Section 4], Ihara [15, Section 3] and Szamuely [27, Theorem 4.7.7 and Fact 4.7.8] — that:

- For all $g \in G_{\mathbb{Q}}$, the pair $(\frac{1}{2}(\chi(g) - 1), \widehat{f}_g(x, y)) \in \widehat{\mathbb{Z}} \times \widehat{F}_2$ defines an element of \widehat{GT} .
- The assignment $g \in G_{\mathbb{Q}} \mapsto (\frac{1}{2}(\chi(g) - 1), \widehat{f}_g(x, y)) \in \widehat{\mathbb{Z}} \times \widehat{F}_2$ defines the group homomorphism (1-1).
- Finally, using Belyi’s theorem [3], one can prove that the homomorphism (1-1) is injective.

For more details, we refer the reader to Ihara [16].

1.2 The groupoid GTSh of GT–shadows and its link to \widehat{GT}

Let us denote by $\text{PaB}^{\leq 4}$ the truncation of the operad PaB up to arity 4, ie

$$\text{PaB}^{\leq 4} := \text{PaB}(1) \sqcup \text{PaB}(2) \sqcup \text{PaB}(3) \sqcup \text{PaB}(4).$$

Moreover, let $\text{NFI}_{\text{PB}_4}(\text{B}_4)$ be the poset of finite-index normal subgroups $N \triangleleft \text{B}_4$ such that $N \leq \text{PB}_4$.

To every $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$, we assign an equivalence relation \sim_N on $\text{PaB}^{\leq 4}$ that is compatible with the structure of the truncated operad and the composition of morphisms. For every $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$, the quotient

$$\text{PaB}^{\leq 4} / \sim_N$$

is a truncated operad in the category of *finite* groupoids.

In this paper, we introduce a groupoid GTSh whose objects are elements of the underlying set $\text{NFI}_{\text{PB}_4}(\text{B}_4)$. Morphisms from \widetilde{N} to N are isomorphisms of truncated operads

$$(1-5) \quad \text{PaB}^{\leq 4} / \sim_{\widetilde{N}} \xrightarrow{\cong} \text{PaB}^{\leq 4} / \sim_N.$$

We call these isomorphisms *GT–shadows*.

Just like PaB , the truncated operad $\text{PaB}^{\leq 4}$ is generated by the braiding $\beta \in \text{PaB}(2)$ and the associator $\alpha \in \text{PaB}(3)$. Hence morphisms of GTSh with the target $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$ are in bijection with pairs

$$(1-6) \quad (m + N_{\text{ord}}\mathbb{Z}, fN_{\text{PB}_3}) \in \mathbb{Z} / N_{\text{ord}}\mathbb{Z} \times \text{PB}_3 / N_{\text{PB}_3}$$

that satisfy appropriate versions of the hexagon relations, the pentagon relation and some technical conditions. Here, the integer N^{ord} and the (finite-index) normal subgroup $N_{\text{PB}_3} \trianglelefteq \text{PB}_3$ are obtained from N via a precise procedure described in Section 2.2.

We denote by $\text{GT}(N)$ the set of such pairs (1-6) and identify them with GT–shadows whose target is N . From now on, we denote by $[(m, f)]$ the GT–shadow represented by a pair $(m, f) \in \mathbb{Z} \times \text{PB}_3$.

A GT-shadow $[(m, f)] \in \text{GT}(\mathbb{N})$ is called *genuine* if there exists an element $\widehat{T} \in \widehat{\text{GT}}$ such that the diagram

$$(1-7) \quad \begin{array}{ccc} \widehat{\text{PaB}}^{\leq 4} & \xrightarrow{\widehat{T}} & \widehat{\text{PaB}}^{\leq 4} \\ \downarrow & & \downarrow \\ \text{PaB}^{\leq 4} / \sim_{\mathbb{N}} & \xrightarrow{\cong} & \text{PaB}^{\leq 4} / \sim_{\mathbb{N}} \end{array}$$

commutes. In (1-7), the lower-horizontal arrow is the isomorphism corresponding to $[(m, f)]$ and the vertical arrows are the canonical projections. If such \widehat{T} does not exist, we say that the GT-shadow $[(m, f)]$ is *fake*.³

In this paper, we show that genuine GT-shadows satisfy additional conditions. For example, every genuine GT-shadow in $\text{GT}(\mathbb{N})$ can be represented by a pair (m, f) with⁴

$$(1-8) \quad f \in [F_2, F_2],$$

where $[F_2, F_2]$ is the commutator subgroup of $F_2 \leq \text{PB}_3$.

A GT-shadow $[(m, f)]$ satisfying all these additional conditions (see Definition 2.20) is called *charming*. In this paper, we show that charming GT-shadows form a subgroupoid of GTSh and we denote this subgroupoid by GTSh^\heartsuit .

The groupoid GTSh^\heartsuit is highly disconnected. However, it is easy to see that, for every $\mathbb{N} \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$, the connected component $\text{GTSh}_{\text{conn}}^\heartsuit(\mathbb{N})$ is a finite groupoid; see Proposition 3.1. In all examples we have considered so far (see Dolgushev [4] and Section 4 of this paper), $\text{GTSh}_{\text{conn}}^\heartsuit(\mathbb{N})$ has at most two objects and, for many of examples of $\mathbb{N} \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$, the groupoid $\text{GTSh}_{\text{conn}}(\mathbb{N})$ has exactly one object, ie $\text{GT}(\mathbb{N})$ is a group. Such elements of $\text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ play a special role and we call them *isolated*. We denote by $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbb{B}_4)$ the subposet of isolated elements of $\text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$.

In this paper, we show that the subposet $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbb{B}_4)$ is cofinal, ie for every $\mathbb{N} \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$, there exists $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbb{B}_4)$ such that $K \leq \mathbb{N}$. We show that the assignment $\mathbb{N} \mapsto \text{GT}(\mathbb{N})$ upgrades to a functor \mathcal{ML} from the poset $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbb{B}_4)$ to the category of finite groups and we prove that the limit of this functor is precisely the Grothendieck–Teichmüller group $\widehat{\text{GT}}$; see Theorem 3.8.

Remark 1.3 Recall from Harbater and Schneps [14] that, omitting the pentagon relation from the definition of $\widehat{\text{GT}}$, we get the gentle version $\widehat{\text{GT}}_0$ of the Grothendieck–Teichmüller group. It is not hard to show that $\widehat{\text{GT}}_0$ is the group of continuous automorphisms of the truncated operad $\widehat{\text{PaB}}^{\leq 3}$ and $\widehat{\text{GT}}$ is a subgroup of $\widehat{\text{GT}}_0$. P Guillot [12; 13] studies a variant of GT-shadows for this gentle version $\widehat{\text{GT}}_0$ of the Grothendieck–Teichmüller group.

³This name was suggested to the authors by David Harbater.

⁴It should be mentioned that, in the computer implementation [4], we only considered GT-shadows of the form $[(m, f)]$ with $f \in F_2 \leq \text{PB}_3$.

1.3 Organization of the paper

In [Section 2](#), we introduce the poset of compatible equivalence relations on the truncated operad $\text{PaB}^{\leq 4}$, and we show that $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ can be identified with the subposet of this poset. We introduce the concept of GT-pair and show that GT-pairs coming from elements of $\widehat{\text{GT}}$ satisfy certain conditions. This consideration motivates the concept of GT-shadow; see [Definition 2.9](#). We prove that GT-shadows form a groupoid GTSh: objects of this groupoid are elements of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ and morphisms are GT-shadows.

In [Section 2](#), we also investigate further conditions on GT-shadows coming from elements of $\widehat{\text{GT}}$, introduce charming GT-shadows and prove that charming GT-shadows form a subgroupoid of GTSh. In this section, we introduce a natural functor $\text{Ch}_{\text{cyclot}}$ from GTSh to the category of finite cyclic groups. We call this functor *the virtual cyclotomic character*.

In [Section 3](#), we introduce an important subposet $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ and construct a functor \mathcal{ML} from $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ to the category of finite groups. In this section, we prove that the limit of the functor \mathcal{ML} is precisely the Grothendieck–Teichmüller group $\widehat{\text{GT}}$.

In [Section 4](#), we present selected results of computer experiments. We outline the basic information about 35 selected elements of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ and the corresponding connected components of the groupoid GTSh. We say a few words about selected remarkable examples. Finally, we discuss two versions of the Furusho property (see [Properties 4.2](#) and [4.3](#)) and list selected open questions.

In [Appendix A](#), we give a brief reminder of (pure) braid groups, the operad PaB and its completion.

In [Appendix B](#), we give a complete description of charming GT-shadows in the abelian setting and we prove that, in this setting, every charming GT-shadow is genuine; see [Theorem B.2](#).

1.4 Notational conventions

For a set X with an equivalence relation and $a \in X$ we will denote by $[a]$ the equivalence class that contains the element a . For a category \mathcal{C} , $\text{Ob}(\mathcal{C})$ denotes the set of objects of \mathcal{C} . Every poset J is naturally a category: $\text{Ob}(J) := J$, for $a, b \in J$, $J(a, b)$ is a singleton if $a \leq b$ and $J(a, b) := \emptyset$ if $a \not\leq b$. For a groupoid \mathcal{G} , the notation $\gamma \in \mathcal{G}$ means that γ is a *morphism* of this groupoid.

Every finite group is tacitly considered with the discrete topology. For a group G , \widehat{G} denotes the profinite completion (see [Ribes and Zalesskii \[24\]](#)) of G . The notation $[G, G]$ is reserved for the commutator subgroup of G . For a normal subgroup $H \trianglelefteq G$ of finite index, we denote by $\text{NFI}_H(G)$ the poset of finite-index normal subgroups N in G such that $N \leq H$. Moreover, $\text{NFI}(G) := \text{NFI}_G(G)$, ie $\text{NFI}(G)$ is the poset of normal finite-index subgroups of a group G .

For a group G and elements $K \leq N$ of the poset $\text{NFI}(G)$, the notation \mathcal{P}_N (resp. $\mathcal{P}_{K,N}$) is reserved for the reduction homomorphism $G \rightarrow G/N$ (resp. $G/K \rightarrow G/N$). The notation $\widehat{\mathcal{P}}_N$ is reserved for the canonical (continuous) homomorphism from \widehat{G} to G/N . Similar notation is used for the canonical functors to finite quotients of a groupoid.

The notation B_n (resp. PB_n) is reserved for the Artin braid group on n strands (resp. the pure braid group on n strands). S_n denotes the symmetric group on n letters. The standard generators of B_n are denoted by $\sigma_1, \dots, \sigma_{n-1}$ and the standard generators of PB_n are denoted by x_{ij} for $1 \leq i < j \leq n$. We will tacitly identify the free group F_2 on two generators with the subgroup $\langle x_{12}, x_{23} \rangle$ of PB_3 .

We will freely use the language of operads as in Dolgushev and Rogers [6, Section 3], Fresse [9, Chapter 1], Loday and Vallette [20], Markl, Shnider and Stasheff [21] and Stasheff [26]. In this paper, we work with operads in the category of sets and in the category of (topological) groupoids. The category of topological groupoids is understood in the “strict sense”. For example, the associativity axioms for the *elementary insertions*⁵ \circ_i (for operads in the category of groupoids) are satisfied “on the nose”.

For an integer $q \geq 1$, a q -truncated operad in the category of groupoids is a collection of groupoids $\{\mathcal{G}(n)\}_{1 \leq n \leq q}$ such that

- for every $1 \leq n \leq q$, the groupoid $\mathcal{G}(n)$ is equipped with an action of S_n ,
- for every triple of integers i, n, m such that $1 \leq i \leq n, n, m, n + m - 1 \leq q$ we have functors

$$(1-9) \quad \circ_i : \mathcal{G}(n) \times \mathcal{G}(m) \rightarrow \mathcal{G}(n + m - 1),$$

- the axioms of the operad for $\{\mathcal{G}(n)\}_{1 \leq n \leq q}$ are satisfied in the cases where all the arities are $\leq q$.

For every operad \mathcal{O} and every integer $q \geq 1$, the disjoint union $\mathcal{O}^{\leq q} := \bigsqcup_{n=0}^q \mathcal{O}(n)$ is clearly a q -truncated operad. In this paper, we only consider 4-truncated operads. So we will simply call them *truncated operads*.

The operad PaB of parenthesized braids, its truncation $\text{PaB}^{\leq 4}$ and its completion $\widehat{\text{PaB}}^{\leq 4}$ play the central role in this paper. See [Appendix A](#) for more details.

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⁵In the literature, elementary insertions are sometimes called *partial compositions*.

2 GT-pairs and GT-shadows

2.1 The poset of compatible equivalence relations on $\text{PaB}^{\leq 4}$

An equivalence relation \sim on the disjoint union of groupoids⁶

$$\text{PaB}^{\leq 4} = \text{PaB}(1) \sqcup \text{PaB}(2) \sqcup \text{PaB}(3) \sqcup \text{PaB}(4)$$

is an equivalence relation on the set of morphisms of $\text{PaB}^{\leq 4}$ such that, if $\gamma \sim \tilde{\gamma}$, then the source (resp. the target) of γ coincides with the source (resp. the target) of $\tilde{\gamma}$. In particular, $\gamma \sim \tilde{\gamma}$ implies that γ and $\tilde{\gamma}$ have the same arity.

Definition 2.1 An equivalence relation \sim on $\text{PaB}^{\leq 4}$ is called *compatible* if

- for every pair of composable morphisms $\gamma, \tilde{\gamma} \in \text{PaB}(n)$, the equivalence class of the composition $\gamma \cdot \tilde{\gamma}$ depends only on the equivalence classes of γ and $\tilde{\gamma}$;
- for every $\gamma, \tilde{\gamma} \in \text{PaB}(n)$ and every $\theta \in S_n$,

$$\gamma \sim \tilde{\gamma} \iff \theta(\gamma) \sim \theta(\tilde{\gamma});$$

- for every tuple of integers $i, n, m, 1 \leq i \leq n, n, m, n + m - 1 \leq 4$, and every $\gamma_1, \tilde{\gamma}_1 \in \text{PaB}(n), \gamma_2, \tilde{\gamma}_2 \in \text{PaB}(m)$, we have

$$\gamma_1 \sim \tilde{\gamma}_1 \implies \gamma_1 \circ_i \gamma_2 \sim \tilde{\gamma}_1 \circ_i \gamma_2 \quad \text{and} \quad \gamma_2 \sim \tilde{\gamma}_2 \implies \gamma_1 \circ_i \gamma_2 \sim \gamma_1 \circ_i \tilde{\gamma}_2.$$

It is clear that, for every compatible equivalence relation \sim on $\text{PaB}^{\leq 4}$, the set

$$(2-1) \quad \text{PaB}^{\leq 4} / \sim$$

of equivalence classes of morphisms in $\text{PaB}^{\leq 4}$ is a truncated operad in the category of groupoids. The set of objects of (2-1) coincides with the set of objects of $\text{PaB}^{\leq 4}$. The action of symmetric groups and the elementary insertions are defined by the formulas

$$\begin{aligned} \theta([\gamma]) &:= [\theta(\gamma)] \quad \text{for } \theta \in S_n \text{ and } \gamma \in \text{PaB}(n), \\ [\gamma_1] \circ_i [\gamma_2] &:= [\gamma_1 \circ_i \gamma_2] \quad \text{for } \gamma_1 \in \text{PaB}(n) \text{ and } \gamma_2 \in \text{PaB}(m). \end{aligned}$$

The conditions of [Definition 2.1](#) guarantee that the composition of morphisms, the action of the symmetric groups on $\text{PaB}(n)/\sim$ and the elementary operadic insertions are well defined. The axioms of the (truncated) operad follow directly from their counterparts for $\text{PaB}^{\leq 4}$.

Compatible equivalence relations on $\text{PaB}^{\leq 4}$ form a poset with the following obvious partial order: we say that $\sim_1 \leq \sim_2$ if \sim_1 is finer than \sim_2 , ie

$$\gamma \sim_1 \tilde{\gamma} \implies \gamma \sim_2 \tilde{\gamma}.$$

⁶Recall that $\text{PaB}(0)$ is the empty groupoid.

It is clear that, for every pair of compatible equivalence relations \sim_1, \sim_2 on $\text{PaB}^{\leq 4}$ such that $\sim_1 \leq \sim_2$, we have a natural onto morphism of truncated operads

$$(2-2) \quad \mathcal{P}_{\sim_1, \sim_2} : \text{PaB}^{\leq 4} / \sim_1 \rightarrow \text{PaB}^{\leq 4} / \sim_2.$$

Moreover, the assignment $\sim \mapsto \text{PaB}^{\leq 4} / \sim$ upgrades to a functor from the poset of compatible equivalence relations to the category of truncated operads.

For every compatible equivalence relation \sim on $\text{PaB}^{\leq 4}$, we denote by \mathcal{P}_{\sim} the natural (onto) morphism of truncated operads:

$$(2-3) \quad \mathcal{P}_{\sim} : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim.$$

2.2 From $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ to the poset of compatible equivalence relations

In this paper, we mostly consider compatible equivalence relations for which the set of morphisms of (2-1) is finite and a large supply of such compatible equivalence relations come from elements of the poset $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$.

For $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, we set

$$(2-4) \quad N_{\text{PB}_3} := \varphi_{123}^{-1}(N) \cap \varphi_{12,3,4}^{-1}(N) \cap \varphi_{1,23,4}^{-1}(N) \cap \varphi_{1,2,34}^{-1}(N) \cap \varphi_{234}^{-1}(N),$$

$$(2-5) \quad N_{\text{PB}_2} := \varphi_{12}^{-1}(N_{\text{PB}_3}) \cap \varphi_{12,3}^{-1}(N_{\text{PB}_3}) \cap \varphi_{1,23}^{-1}(N_{\text{PB}_3}) \cap \varphi_{23}^{-1}(N_{\text{PB}_3}),$$

where $\varphi_{123}, \varphi_{12,3,4}, \varphi_{1,23,4}, \varphi_{1,2,34}$ and φ_{234} are the homomorphisms defined in (A-16), and $\varphi_{12}, \varphi_{12,3}, \varphi_{1,23}$ and φ_{23} are the homomorphisms defined in (A-17); see also the explicit formulas in (A-18) and (A-19).

Proposition 2.2 *For every $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, the subgroup N_{PB_3} (resp. N_{PB_2}) is an element of the poset $\text{NFI}_{\text{PB}_3}(\mathbf{B}_3)$ (resp. the poset $\text{NFI}_{\text{PB}_2}(\mathbf{B}_2)$).*

Proof Since every subgroup of \mathbf{B}_2 is normal and N_{PB_2} has a finite index in PB_2 , the statement about N_{PB_2} is obvious.

It is also easy to see that N_{PB_3} is a subgroup of finite index in PB_3 . So it suffices to prove that

$$gN_{\text{PB}_3}g^{-1} \leq N_{\text{PB}_3} \quad \text{for all } g \in \mathbf{B}_3.$$

Let $h \in N_{\text{PB}_3}$ and $g \in \mathbf{B}_3$. Then

$$(2-6) \quad \varphi_{1,23,4}(g \cdot h \cdot g^{-1}) = \text{ou}(\text{m}(g \cdot h \cdot g^{-1}) \circ_2 \text{id}_{12}),$$

where the map $\text{ou} : \text{PaB}(n) \rightarrow \mathbf{B}_n$ and its right inverse $\text{m} : \mathbf{B}_n \rightarrow \text{PaB}(n)$ are defined in Section A.2.

Using identity (A-11), we get

$$\text{m}(g \cdot h \cdot g^{-1}) = \theta(\text{m}(g)) \cdot \theta(\chi) \cdot \text{m}(g^{-1}),$$

where $\theta = \rho(g)$ and $\chi := \text{m}(h)$.

Therefore

$$(2-7) \quad \mathfrak{m}(g \cdot h \cdot g^{-1}) \circ_2 \text{id}_{12} = (\theta(\mathfrak{m}(g)) \circ_2 \text{id}_{12}) \cdot (\theta(\chi) \circ_2 \text{id}_{12}) \cdot (\mathfrak{m}(g^{-1}) \circ_2 \text{id}_{12}).$$

Combining (2-6) with (2-7), we get

$$(2-8) \quad \varphi_{1,23,4}(g \cdot h \cdot g^{-1}) = \text{ou}(\theta(\mathfrak{m}(g)) \circ_2 \text{id}_{12}) \cdot \text{ou}(\theta(\chi) \circ_2 \text{id}_{12}) \cdot \text{ou}(\mathfrak{m}(g^{-1}) \circ_2 \text{id}_{12}).$$

Since

$$\begin{aligned} \text{ou}(\theta(\mathfrak{m}(g)) \circ_2 \text{id}_{12}) \cdot \text{ou}(\mathfrak{m}(g^{-1}) \circ_2 \text{id}_{12}) &= \text{ou}(\theta(\mathfrak{m}(g)) \circ_2 \text{id}_{12} \cdot \mathfrak{m}(g^{-1}) \circ_2 \text{id}_{12}) \\ &= \text{ou}((\theta(\mathfrak{m}(g)) \cdot \mathfrak{m}(g^{-1})) \circ_2 \text{id}_{12}) \\ &= \text{ou}(\mathfrak{m}(g \cdot g^{-1}) \circ_2 \text{id}_{12}) = \text{ou}(\text{id}_{(1(23))4}) = 1_{B_4}, \end{aligned}$$

the element $\varphi_{1,23,4}(g \cdot h \cdot g^{-1}) \in B_4$ can be rewritten as

$$\varphi_{1,23,4}(g \cdot h \cdot g^{-1}) = \tilde{g} \cdot \text{ou}(\theta(\chi) \circ_2 \text{id}_{12}) \cdot \tilde{g}^{-1}, \quad \text{where } \tilde{g} := \text{ou}(\theta(\mathfrak{m}(g)) \circ_2 \text{id}_{12}).$$

Thus it remains to prove that

$$(2-9) \quad \text{ou}(\theta(\chi) \circ_2 \text{id}_{12}) \in N.$$

For this purpose, we consider the three possible cases: $\theta(1) = 2$, $\theta(2) = 2$ and $\theta(3) = 2$.

- If $\theta(1) = 2$, then $\text{ou}(\theta(\chi) \circ_2 \text{id}_{12}) = \varphi_{12,3,4}(h)$ and (2-9) is a consequence of $h \in \varphi_{12,3,4}^{-1}(N)$.
- If $\theta(2) = 2$, then $\text{ou}(\theta(\chi) \circ_2 \text{id}_{12}) = \varphi_{1,23,4}(h)$ and (2-9) is a consequence of $h \in \varphi_{1,23,4}^{-1}(N)$.
- If $\theta(3) = 2$, then $\text{ou}(\theta(\chi) \circ_2 \text{id}_{12}) = \varphi_{1,2,34}(h)$ and (2-9) is a consequence of $h \in \varphi_{1,2,34}^{-1}(N)$.

We proved that the element ghg^{-1} belongs to $\varphi_{1,23,4}^{-1}(N) \subset \text{PB}_3$. The proofs for the remaining four homomorphisms φ_{123} , $\varphi_{12,3,4}$, $\varphi_{1,2,34}$ and φ_{234} are similar and we omit them. □

It is clear that $N_{\text{PB}_2} = \langle x_{12}^{N_{\text{ord}}} \rangle$, where N_{ord} is the index of N_{PB_2} in PB_2 , ie N_{ord} is the least common multiple of orders of $x_{12}N_{\text{PB}_3}$, $x_{23}N_{\text{PB}_3}$, $x_{12}x_{13}N_{\text{PB}_3}$ and $x_{13}x_{23}N_{\text{PB}_3}$ in $\text{PB}_3/N_{\text{PB}_3}$.

Using the identities $x_{12}x_{13} = x_{23}^{-1}c$ and $x_{13}x_{23} = x_{12}^{-1}c$ involving the generator c (see (A-5)) of the center of PB_3 , it is easy to prove the following statement:

Proposition 2.3 *Let $N_{\text{PB}_2} = \langle x_{12}^{N_{\text{ord}}} \rangle$ be the subgroup of PB_2 defined in (2-5). Then N_{ord} coincides with*

- (1) *the least common multiple of orders of elements $x_{12}N_{\text{PB}_3}$, $x_{23}N_{\text{PB}_3}$ and $x_{12}x_{13}N_{\text{PB}_3}$;*
- (2) *the least common multiple of orders of elements $x_{12}N_{\text{PB}_3}$, $x_{23}N_{\text{PB}_3}$ and $x_{13}x_{23}N_{\text{PB}_3}$; and*
- (3) *the least common multiple of orders of elements $x_{12}N_{\text{PB}_3}$, $x_{23}N_{\text{PB}_3}$ and cN_{PB_3} .* □

Given $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ and the corresponding normal subgroups N_{PB_3} and N_{PB_2} , we will now define an equivalence relation \sim_N on the set of morphisms in $\text{PaB}^{\leq 4}$.

The groupoid $\text{PaB}(1)$ has exactly one object and exactly one (identity) morphism. So $\text{PaB}(1)$ has only one equivalence relation.

Let $n \in \{2, 3, 4\}$ and

$$\gamma, \tilde{\gamma} \in \text{Hom}_{\text{PaB}(n)}(\tau_1, \tau_2).$$

We declare that $\gamma \sim_N \tilde{\gamma}$ if and only if

$$(2-10) \quad \text{ou}(\gamma^{-1} \cdot \tilde{\gamma}) \in N_{\text{PB}_n},$$

where $N_{\text{PB}_4} := N$. In other words, $\gamma \sim_N \tilde{\gamma}$ if and only if

$$\tilde{\gamma} = \gamma \cdot \eta,$$

where $\text{ou}(\eta) \in N_{\text{PB}_n}$ and the source of γ coincides with the target of η .

Proposition 2.4 For every $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, \sim_N is a compatible equivalence relation on $\text{PaB}^{\leq 4}$ in the sense of [Definition 2.1](#). Moreover, the assignment

$$N \mapsto \sim_N$$

upgrades to a functor from the poset $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ to the poset of compatible equivalence relations on $\text{PaB}^{\leq 4}$.

Proof The first property of \sim_N follows from the fact that $N_{\text{PB}_4} := N$ (resp. $N_{\text{PB}_3}, N_{\text{PB}_2}$) is normal in \mathbf{B}_4 (resp. $\mathbf{B}_3, \mathbf{B}_2$).

The second property of \sim_N follows from the obvious identity

$$\text{ou}(\gamma) = \text{ou}(\theta(\gamma)) \quad \text{for all } \gamma \in \text{PaB}(n) \text{ and } \theta \in S_n.$$

The proof of the last property is based on the observation that elementary operadic insertions for PaB can be expressed in terms of the operations $? \circ_i \text{id}_\tau, \text{id}_\tau \circ_i ?$, and the composition of morphisms in the groupoids $\text{PaB}(3)$ and $\text{PaB}(4)$.

Let $n \in \{2, 3\}$ and η be a morphism in $\text{PaB}(n)$ whose target coincides with its source. In particular, $\text{ou}(\eta) \in \text{PB}_n$. Let us prove that, if $\text{ou}(\eta) \in N_{\text{PB}_n}$, then, for every $\tau \in \text{Ob}(\text{PaB}(m))$ with $n + m - 1 \leq 4$, we have

$$(2-11) \quad \text{ou}(\eta \circ_i \text{id}_\tau) \in \text{PB}_{n+m-1} \quad \text{for all } 1 \leq i \leq n,$$

$$(2-12) \quad \text{ou}(\text{id}_\tau \circ_i \eta) \in \text{PB}_{n+m-1} \quad \text{for all } 1 \leq i \leq m.$$

Let $h = \text{ou}(\eta)$. If $m = 2$, then there exists $1 \leq j \leq n$ (resp. $j \in \{1, 2\}$) such that $\text{ou}(m(h) \circ_j \text{id}_{12}) = \text{ou}(\eta \circ_i \text{id}_\tau)$ (resp. $\text{ou}(\text{id}_{12}) \circ_j m(h) = \text{ou}(\text{id}_\tau \circ_i \eta)$).

Thus, if $m = 2$, (2-11) and (2-12) follow directly from the definitions of N_{PB_3} , N_{PB_2} , (2-4), (2-5) and the definitions of the homomorphisms $\varphi_{123}, \dots, \varphi_{12}, \dots$; see (A-16) and (A-17).

If $m = 3$, then there exist $j, k \in \{1, 2\}$ such that

$$\text{ou}(\eta \circ_i \text{id}_\tau) = \text{ou}((\eta \circ_j \text{id}_{12}) \circ_k \text{id}_{12}).$$

For example, if $\eta \in \text{Hom}_{\text{PaB}}((2, 1), (2, 1))$, then $\text{ou}(\eta \circ_2 \text{id}_{2(1,3)}) = \text{ou}((\eta \circ_2 \text{id}_{12}) \circ_3 \text{id}_{12})$.

Thus (2-11) for $m = 3$ follows from (2-11) for $m = 2$. Similarly, (2-12) for $m = 3$ follows from (2-12) for $m = 2$.

We will now use (2-11) and (2-12) to prove the last property of \sim_N .

Consider $\gamma_1, \tilde{\gamma}_1 \in \text{Hom}_{\text{PaB}(n)}(\tau_1, \tau_2)$ and $\gamma_2, \tilde{\gamma}_2 \in \text{Hom}_{\text{PaB}(m)}(\omega_1, \omega_2)$. First suppose $\gamma_1 \sim_N \tilde{\gamma}_1$, so $\tilde{\gamma}_1 = \gamma_1 \cdot \eta$ for some $\eta \in \text{Hom}_{\text{PaB}(n)}(\tau_1, \tau_1)$ such that $\text{ou}(\eta) \in N_{PB_n}$. It follows that

$$\tilde{\gamma}_1 \circ_i \gamma_2 = (\gamma_1 \cdot \eta) \circ_i (\gamma_2 \cdot \text{id}_{\omega_1}) = (\gamma_1 \circ_i \gamma_2) \cdot (\eta \circ_i \text{id}_{\omega_1}).$$

Due to (2-11), $\text{ou}(\eta \circ_i \text{id}_{\omega_1}) \in N_{PB_{n+m-1}}$ and hence $\tilde{\gamma}_1 \circ_i \gamma_2 \sim \gamma_1 \circ_i \gamma_2$.

Now suppose $\gamma_2 \sim_N \tilde{\gamma}_2$, so $\tilde{\gamma}_2 = \gamma_2 \cdot \eta'$ for some $\eta' \in \text{Hom}_{\text{PaB}(m)}(\omega_1, \omega_1)$ such that $\text{ou}(\eta') \in N_{PB_m}$. It follows that

$$\gamma_1 \circ_i \tilde{\gamma}_2 = (\gamma_1 \cdot \text{id}_{\tau_1}) \circ_i (\gamma_2 \cdot \eta') = (\gamma_1 \circ_i \gamma_2) \cdot (\text{id}_{\tau_1} \circ_i \eta').$$

Due to (2-12), $\text{ou}(\text{id}_{\tau_1} \circ_i \eta') \in N_{PB_{n+m-1}}$ and hence $\gamma_1 \circ_i \tilde{\gamma}_2 \sim \gamma_1 \circ_i \gamma_2$.

This completes the proof of the fact that \sim_N is indeed a compatible equivalence relation on $\text{PaB}^{\leq 4}$.

It is clear that, if $\tilde{N}, N \in \text{NFI}_{PB_4}(B_4)$ and $\tilde{N} \leq N$, then $\tilde{N}_{PB_3} \leq N_{PB_3}$ and $\tilde{N}_{PB_2} \leq N_{PB_2}$.

Thus the assignment $N \mapsto \sim_N$ upgrades to a functor from the poset $\text{NFI}_{PB_4}(B_4)$ to the poset of compatible equivalence relations. □

Later, we will need the following technical statement about $\text{NFI}_{PB_4}(B_4)$:

Proposition 2.5 (A) For every $N \in \text{NFI}(PB_3)$, there exists $K \in \text{NFI}_{PB_4}(B_4)$ satisfying the property

$$K_{PB_3} \leq N.$$

(B) For every $N \in \text{NFI}(PB_2)$, there exists $K \in \text{NFI}_{PB_4}(B_4)$ such that $K_{PB_2} \leq N$.

Proof Stronger versions of these statements are proved in Section 3.1; see Proposition 3.9. So we omit the proof of this proposition. □

2.3 The set of GT-pairs $GT_{pr}(N)$

Let $N \in NFI_{PB_4}(B_4)$ and \sim_N be the corresponding compatible equivalence relation on $PaB^{\leq 4}$. Let N_{PB_3} (resp. N_{PB_2}) be the corresponding normal subgroup of PB_3 (resp. PB_2) and N_{ord} be the index of N_{PB_2} in PB_2 .

Since the groupoid $PaB(0)$ is empty, [Theorem A.1](#) implies that the truncated operad $PaB^{\leq 4}$ is generated by morphisms α and β shown in equation (1-2).

Moreover any relation on α and β in $PaB^{\leq 4}$ is a consequence of the pentagon relation

$$(2-13) \quad \begin{array}{ccccc} & & (1(23))4 & \xrightarrow{\alpha \circ_2 id_{12}} & 1((23)4) \\ & id_{12} \circ_1 \alpha \nearrow & & & \searrow id_{12} \circ_2 \alpha \\ ((12)3)4 & & & & & 1(2(34)) \\ & \searrow \alpha \circ_1 id_{12} & & & \nearrow \alpha \circ_3 id_{12} \\ & & (12)(34) & & \end{array}$$

and the hexagon relations

$$(2-14) \quad \begin{array}{ccccc} (12)3 & \xrightarrow{\beta \circ_1 id_{12}} & 3(12) & \xleftarrow{(1,3,2)\alpha} & (31)2 \\ \alpha \downarrow & & & & \uparrow (2,3)(id_{12} \circ_1 \beta) \\ 1(23) & \xrightarrow{id_{12} \circ_2 \beta} & 1(32) & \xrightarrow{(2,3)\alpha^{-1}} & (13)2 \end{array}$$

$$(2-15) \quad \begin{array}{ccccc} 1(23) & \xrightarrow{\beta \circ_2 id_{12}} & (23)1 & \xrightarrow{(1,2,3)\alpha} & 2(31) \\ \alpha^{-1} \downarrow & & & & \uparrow (1,2)(id_{12} \circ_2 \beta) \\ (12)3 & \xrightarrow{id_{12} \circ_1 \beta} & (21)3 & \xrightarrow{(1,2)\alpha} & 2(13) \end{array}$$

Thus morphisms of truncated operads

$$(2-16) \quad T : PaB^{\leq 4} \rightarrow PaB^{\leq 4} / \sim_N$$

are in bijection with pairs

$$(2-17) \quad (m + N_{ord}\mathbb{Z}, fN_{PB_3}) \in \mathbb{Z}/N_{ord}\mathbb{Z} \times PB_3/N_{PB_3}$$

satisfying in B_3/N_{PB_3} the relations

$$(2-18) \quad \sigma_1 x_{12}^m f^{-1} \sigma_2 x_{23}^m f N_{PB_3} = f^{-1} \sigma_1 \sigma_2 (x_{13} x_{23})^m N_{PB_3},$$

$$(2-19) \quad f^{-1} \sigma_2 x_{23}^m f \sigma_1 x_{12}^m N_{PB_3} = \sigma_2 \sigma_1 (x_{12} x_{13})^m f N_{PB_3},$$

and in PB_4/N the relation

$$(2-20) \quad \varphi_{234}(f)\varphi_{1,23,4}(f)\varphi_{123}(f)N = \varphi_{1,2,34}(f)\varphi_{12,3,4}(f)N.$$

More precisely, this bijection sends a pair (2-17) to the morphism $T_{m,f} : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_N$ of truncated operads defined by the formulas

$$T_{m,f}(\alpha) := [\alpha \cdot m(f)] \quad \text{and} \quad T_{m,f}(\beta) := [\beta \cdot m(x_{12}^m)],$$

where m is the map from B_n to $\text{PaB}(n)$ defined in Section A.2.

This observation motivates our definition of a GT-pair:

Definition 2.6 For $N \in \text{NFI}_{\text{PB}_4}(B_4)$, the set $\text{GT}_{\text{pr}}(N)$ consists of pairs

$$(m + N_{\text{ord}}\mathbb{Z}, fN_{\text{PB}_3}) \in \mathbb{Z} / N_{\text{ord}}\mathbb{Z} \times \text{PB}_3 / N_{\text{PB}_3}$$

satisfying (2-18), (2-19) and (2-20). Elements of $\text{GT}_{\text{pr}}(N)$ are called GT-pairs.

We will represent GT-pairs by tuples $(m, f) \in \mathbb{Z} \times \text{PB}_3$. It is straightforward to see that, if relations (2-18), (2-19) and (2-20) are satisfied for a tuple (m, f) , then they are also satisfied for $(m + qN_{\text{ord}}, fh)$, where q is an arbitrary integer and h is an arbitrary element in N_{PB_3} . A GT-pair in $\mathbb{Z} / N_{\text{ord}}\mathbb{Z} \times \text{PB}_3 / N_{\text{PB}_3}$ represented by a tuple $(m, f) \in \mathbb{Z} \times \text{PB}_3$ will be often denoted by

$$[(m, f)].$$

For $N \in \text{NFI}_{\text{PB}_4}(B_4)$ and a tuple (m, f) representing a GT-pair in $\text{GT}_{\text{pr}}(N)$, we denote by $T_{m,f}$ the corresponding morphism of truncated operads,

$$(2-21) \quad T_{m,f} : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_N.$$

It is clear that, for every $n \in \{2, 3, 4\}$, the assignment ou from Section A.2 induces the obvious map

$$(2-22) \quad \text{PaB}(n) / \sim_N \rightarrow B_n / N_{\text{PB}_n}, \quad \text{where } N_{\text{PB}_4} := N.$$

By abuse of notation, we will use the same symbol ou for the map (2-22).

Using (2-22) together with the map $m : B_n \rightarrow \text{PaB}(n)$ from Section A.2 and morphism (2-21), we define group homomorphisms $B_2 \rightarrow B_2 / N_{\text{PB}_2}$, $B_3 \rightarrow B_3 / N_{\text{PB}_3}$ and $B_4 \rightarrow B_4 / N$. Restricting these homomorphisms to PB_2 , PB_3 and PB_4 , we get group homomorphisms $\text{PB}_2 \rightarrow \text{PB}_2 / N_{\text{PB}_2}$, $\text{PB}_3 \rightarrow \text{PB}_3 / N_{\text{PB}_3}$ and $\text{PB}_4 \rightarrow \text{PB}_4 / N$, respectively. More precisely, we have the following statement:

Corollary 2.7 Suppose that $N \in \text{NFI}_{\text{PB}_4}(B_4)$. For every pair $(m + N_{\text{ord}}\mathbb{Z}, fN_{\text{PB}_3}) \in \text{GT}_{\text{pr}}(N)$, and every $n \in \{2, 3, 4\}$, the assignment

$$(2-23) \quad T_{m,f}^{B_n}(g) := \text{ou} \circ T_{m,f} \circ m(g)$$

defines a group homomorphism $B_n \rightarrow B_n / N_{\text{PB}_n}$ (with $N_{\text{PB}_4} := N$). The restriction of $T_{m,f}^{B_n}$ to PB_n gives us a group homomorphism

$$(2-24) \quad T_{m,f}^{\text{PB}_n} : \text{PB}_n \rightarrow \text{PB}_n / N_{\text{PB}_n}.$$

The action of $T_{m,f}^{B_4}$ on the generators of B_4 is given by the formulas

$$(2-25) \quad \begin{aligned} T_{m,f}^{B_4}(\sigma_1) &:= \sigma_1 x_{12}^m N, \\ T_{m,f}^{B_4}(\sigma_2) &:= \varphi_{123}(f)^{-1}(\sigma_2 x_{23}^m) \varphi_{123}(f) N, \\ T_{m,f}^{B_4}(\sigma_3) &:= \varphi_{12,3,4}(f)^{-1}(\sigma_3 x_{34}^m) \varphi_{12,3,4}(f) N. \end{aligned}$$

The action of $T_{m,f}^{B_3}$ on the generators of B_3 are given by the formulas

$$(2-26) \quad T_{m,f}^{B_3}(\sigma_1) := \sigma_1 x_{12}^m N_{PB_3} \quad \text{and} \quad T_{m,f}^{B_3}(\sigma_2) := f^{-1}(\sigma_2 x_{23}^m) f N_{PB_3}.$$

Finally, $T_{m,f}^{B_2}$ sends σ_1 to $\sigma_1 x_{12}^m N_{PB_2}$.

Proof It is clear that, for every two composable morphisms $\gamma_1, \gamma_2 \in \text{PaB}(n)/\sim_N$, we have

$$(2-27) \quad \text{ou}(\gamma_1 \cdot \gamma_2) = \text{ou}(\gamma_1) \cdot \text{ou}(\gamma_2).$$

Then using (A-11) from Section A.2 and the compatibility of $T_{m,f}$ with the structure of the truncated operad we get

$$\begin{aligned} T_{m,f}^{B_n}(g_1 \cdot g_2) &= \text{ou}(T_{m,f}(\mathfrak{m}(g_1 \cdot g_2))) = \text{ou}(T_{m,f}(\rho(g_2)^{-1}(\mathfrak{m}(g_1)) \cdot \mathfrak{m}(g_2))) \\ &= \text{ou}(T_{m,f}(\rho(g_2)^{-1}(\mathfrak{m}(g_1))) \cdot T_{m,f}(\mathfrak{m}(g_2))) = \text{ou}(\rho(g_2)^{-1} T_{m,f}(\mathfrak{m}(g_1)) \cdot T_{m,f}(\mathfrak{m}(g_2))) \\ &= \text{ou}(\rho(g_2)^{-1} T_{m,f}(\mathfrak{m}(g_1))) \cdot \text{ou}(T_{m,f}(\mathfrak{m}(g_2))) = \text{ou}(T_{m,f}(\mathfrak{m}(g_1))) \cdot \text{ou}(T_{m,f}(\mathfrak{m}(g_2))) \\ &= T_{m,f}^{B_n}(g_1) \cdot T_{m,f}^{B_n}(g_2), \end{aligned}$$

whence $T_{m,f}^{B_n}$ is a group homomorphism.

The second statement of the corollary follows immediately from the fact \mathfrak{m} is a right inverse of ou and $T_{m,f}$ acts trivially on objects of PaB .

We will now prove (2-25). The easier cases of $T_{m,f}^{B_3}$ and $T_{m,f}^{B_2}$ are left for the reader.

For the generator σ_1 , we have

$$T_{m,f}^{B_4}(\sigma_1) = \text{ou}(T_{m,f}(\mathfrak{m}(\sigma_1))) = \text{ou}(T_{m,f}(\text{id}_{(12)3} \circ_1 \beta)) = \text{ou}(\text{id}_{(12)3} \circ_1 [\beta \cdot \mathfrak{m}(x_{12}^m)]) = \sigma_1 x_{12}^m N.$$

For the generator σ_2 , we have

$$\begin{aligned} T_{m,f}^{B_4}(\sigma_2) &= \text{ou}(T_{m,f}(\mathfrak{m}(\sigma_2))) = \text{ou}(T_{m,f}((2, 3)(\text{id}_{12} \circ_1 \alpha^{-1}) \cdot (\text{id}_{(12)3} \circ_2 \beta) \cdot (\text{id}_{12} \circ_1 \alpha))) \\ &= \text{ou}((2, 3)(\text{id}_{12} \circ_1 [\mathfrak{m}(f^{-1}) \cdot \alpha^{-1}]) \cdot (\text{id}_{(12)3} \circ_2 [\beta \cdot \mathfrak{m}(x_{12}^m)]) \cdot (\text{id}_{12} \circ_1 [\alpha \cdot \mathfrak{m}(f)])) \\ &= \varphi_{123}(f)^{-1}(\sigma_2 x_{23}^m) \varphi_{123}(f) N. \end{aligned}$$

Finally, for the generator σ_3 , we have

$$\begin{aligned} T_{m,f}^{B_4}(\sigma_3) &= \text{ou}(T_{m,f}(\mathfrak{m}(\sigma_3))) = \text{ou}(T_{m,f}((3, 4)(\alpha^{-1} \circ_1 \text{id}_{12}) \cdot (\text{id}_{(12)3} \circ_3 \beta) \cdot (\alpha \circ_1 \text{id}_{12}))) \\ &= \text{ou}((3, 4)([\mathfrak{m}(f)^{-1} \cdot \alpha^{-1}] \circ_1 \text{id}_{12}) \cdot (\text{id}_{(12)3} \circ_3 [\beta \cdot \mathfrak{m}(x_{12}^m)]) \cdot ([\alpha \cdot \mathfrak{m}(f)] \circ_1 \text{id}_{12})) \\ &= \varphi_{12,3,4}(f)^{-1}(\sigma_3 x_{34}^m) \varphi_{12,3,4}(f) N. \end{aligned}$$

□

Let us now use [Corollary 2.7](#) to prove the following statement.

Corollary 2.8 *Let $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$, $[(m, f)] \in \text{GT}_{\text{pr}}(N)$ and c be the generator of the center of PB_3 ; see [\(A-5\)](#). Then*

$$(2-28) \quad T_{m,f}^{\text{PB}_3}(x_{12}) = x_{12}^{2m+1} N_{\text{PB}_3}, \quad T_{m,f}^{\text{PB}_3}(x_{23}) = f^{-1} x_{23}^{2m+1} f N_{\text{PB}_3},$$

$$(2-29) \quad T_{m,f}^{\text{PB}_3}(x_{13}) = x_{12}^{-m} \sigma_1^{-1} f^{-1} x_{23}^{2m+1} f \sigma_1 x_{12}^m N_{\text{PB}_3}, \quad T_{m,f}^{\text{PB}_3}(c) = c^{2m+1} N_{\text{PB}_3}.$$

Proof The first equation in [\(2-28\)](#) is a simple consequence of the first equation in [\(2-26\)](#).

Using the second equation in [\(2-26\)](#), we get

$$T_{m,f}^{\text{PB}_3}(x_{23}) = T_{m,f}^{\text{PB}_3}(\sigma_2^2) = (f^{-1}(\sigma_2 x_{23}^m) f)^2 N_{\text{PB}_3} = f^{-1}(\sigma_2^2 x_{23}^{2m}) f N_{\text{PB}_3} = f^{-1} x_{23}^{2m+1} f N_{\text{PB}_3}.$$

Thus the second equation in [\(2-28\)](#) is proved.

Using the definition of $x_{13} := \sigma_1^{-1} \sigma_2^2 \sigma_1 = \sigma_1^{-1} x_{23} \sigma_1$, the first equation in [\(2-26\)](#) and the second equation in [\(2-28\)](#), we get

$$T_{m,f}^{\text{PB}_3}(x_{13}) = T_{m,f}^{\text{PB}_3}(\sigma_1^{-1} x_{23} \sigma_1) = x_{12}^{-m} \sigma_1^{-1} f^{-1} x_{23}^{2m+1} f \sigma_1 x_{12}^m N_{\text{PB}_3}.$$

Thus the first equation in [\(2-29\)](#) is also satisfied.

To prove the second equation in [\(2-29\)](#), we use the formulas [\(2-18\)](#), [\(2-19\)](#), [\(2-26\)](#), and the identities $x_{13} x_{23} = x_{12}^{-1} c$, $x_{12} x_{13} = x_{23}^{-1} c$ extensively.

$$\begin{aligned} T_{m,f}^{\text{PB}_3}(c) &= T_{m,f}^{\text{PB}_3}((\sigma_1 \sigma_2)^3) = (T_{m,f}^{\text{B}_3}(\sigma_1 \sigma_2))^3 \\ &= \sigma_1 x_{12}^m f^{-1} \sigma_2 x_{23}^m f \sigma_1 x_{12}^m f^{-1} \sigma_2 x_{23}^m f \sigma_1 x_{12}^m f^{-1} \sigma_2 x_{23}^m f N_{\text{PB}_3} \\ &= f^{-1} \sigma_1 \sigma_2 (x_{13} x_{23})^m \sigma_1 x_{12}^m \sigma_2 \sigma_1 (x_{12} x_{13})^m f f^{-1} \sigma_2 x_{23}^m f N_{\text{PB}_3} \\ &= f^{-1} \sigma_1 \sigma_2 (x_{12}^{-1} c)^m \sigma_1 x_{12}^m \sigma_2 \sigma_1 (x_{23}^{-1} c)^m \sigma_2 x_{23}^m f N_{\text{PB}_3} \\ &= f^{-1} \sigma_1 \sigma_2 x_{12}^{-m} c^m \sigma_1 x_{12}^m \sigma_2 \sigma_1 x_{23}^{-m} c^m \sigma_2 x_{23}^m f N_{\text{PB}_3} \\ &= c^{2m} f^{-1} (\sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2) f N_{\text{PB}_3} = c^{2m+1} N_{\text{PB}_3}. \end{aligned} \quad \square$$

2.4 GT-pairs coming from automorphisms of $\widehat{\text{PaB}}$

Let $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$ and \sim_N be the corresponding compatible equivalence relation. Since the groupoids $\text{PaB}(n)/\sim_N$ for $1 \leq n \leq 4$ are finite, we have a canonical continuous onto morphism of truncated operads

$$(2-30) \quad \widehat{\mathcal{P}}_N : \widehat{\text{PaB}}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_N.$$

Thus, given any continuous automorphism $\widehat{T} : \widehat{\text{PaB}} \rightarrow \widehat{\text{PaB}}$, we can produce the morphism of truncated operads

$$T_N : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_N$$

by setting

$$(2-31) \quad T_N := \widehat{\mathcal{P}}_N \circ \widehat{T} \circ \mathcal{I},$$

where \mathcal{I} is the natural embedding of truncated operads

$$(2-32) \quad \mathcal{I}: \text{PaB}^{\leq 4} \rightarrow \widehat{\text{PaB}}^{\leq 4}.$$

In other words, for every continuous automorphism of $\widehat{\text{PaB}}$ and every $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$, we get a GT-pair $[(m, f)]$ corresponding to T_N . In this situation, we say that the GT-pair $[(m, f)]$ comes from the automorphism \widehat{T} .

GT-pairs coming from automorphisms of $\widehat{\text{PaB}}$ satisfy additional properties. Indeed, since $\mathcal{I}(\text{PaB}^{\leq 4})$ is dense in $\widehat{\text{PaB}}^{\leq 4}$ and the morphism

$$\mathcal{P}_N \circ \widehat{T}: \widehat{\text{PaB}}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_N$$

is continuous and onto, the morphism T_N is also onto.

Thus, if a GT-pair $[(m, f)]$ comes from a (continuous) automorphism of $\widehat{\text{PaB}}$, the group homomorphisms

$$(2-33) \quad T_{m,f}^{\text{PB}_4}: \text{PB}_4 \rightarrow \text{PB}_4 / N,$$

$$(2-34) \quad T_{m,f}^{\text{PB}_3}: \text{PB}_3 \rightarrow \text{PB}_3 / N_{\text{PB}_3},$$

$$(2-35) \quad T_{m,f}^{\text{PB}_2}: \text{PB}_2 \rightarrow \text{PB}_2 / N_{\text{PB}_2},$$

are onto.

GT-pairs satisfying these properties are called GT-shadows. More precisely:

Definition 2.9 Let N be a finite-index normal subgroup of B_4 such that $N \leq \text{PB}_4$. Furthermore, let N_{PB_3} and N_{PB_2} be the corresponding normal subgroups of B_3 and B_2 , respectively, and let N_{ord} be the index of N_{PB_2} in PB_2 . The set $\text{GT}(N)$ consists of GT-pairs $[(m, f)] \in \text{GT}_{\text{pr}}(N)$ for which group homomorphisms (2-33), (2-34) and (2-35) are onto. Elements of $\text{GT}(N)$ are called GT-shadows.

It is easy to see that homomorphism (2-35) is onto if and only if

$$(2-36) \quad (2m + 1) + N_{\text{ord}}\mathbb{Z} \in (\mathbb{Z} / N_{\text{ord}}\mathbb{Z})^\times.$$

We say that a GT-pair $[(m, f)]$ is friendly if m satisfies condition (2-36).

Due to the following proposition, only homomorphisms (2-34) and (2-35) matter.

Proposition 2.10 Let $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$, $[(m, f)] \in \text{GT}_{\text{pr}}(N)$ and $T_{m,f}: \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_N$ be the corresponding map of truncated operads; see (2-21). The following statements are equivalent:

- (1) The pair $[(m, f)]$ is a GT-shadow.
- (2) Group homomorphisms (2-34) and (2-35) are onto.
- (3) The map $T_{m,f}: \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_N$ is onto.

Proof The implication (1) \implies (2) is obvious. It is also clear that if $T_{m,f} : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_{\mathbb{N}}$ is onto, then group homomorphisms (2-33), (2-34) and (2-35) are onto. Thus the implication (3) \implies (1) is also obvious.

It remains to prove the implication (2) \implies (3).

Since group homomorphism (2-35) is onto, there exists $\gamma_2 \in \text{Hom}_{\text{PaB}}(12, 12)$ such that

$$T_{m,f}(\gamma_2) = [\mathfrak{m}(x_{12}^{-m})].$$

Therefore,

$$T_{m,f}(\beta \cdot \gamma_2) = T_{m,f}(\beta) \cdot T_{m,f}(\gamma_2) = [\beta].$$

Since homomorphism (2-35) is onto, there exists $\gamma_3 \in \text{Hom}_{\text{PaB}}((12)3, (12)3)$ such that

$$T_{m,f}(\gamma_3) = [\mathfrak{m}(f^{-1})].$$

Therefore,

$$T_{m,f}(\alpha \cdot \gamma_3) = T_{m,f}(\alpha) \cdot T_{m,f}(\gamma_3) = T_{m,f}(\alpha) \cdot [\mathfrak{m}(f^{-1})] = [\alpha].$$

Since, as a truncated operad in the category of groupoids, $\text{PaB}^{\leq 4}$ is generated by β and α , the truncated operad $\text{PaB}^{\leq 4}/\sim_{\mathbb{N}}$ is generated by the equivalence classes $[\beta] \in \text{PaB}(2)/\sim_{\mathbb{N}}$ and $[\alpha] \in \text{PaB}(3)/\sim_{\mathbb{N}}$.

Using the fact that $[\beta]$ and $[\alpha]$ belong to the image of $T_{m,f}$, we conclude that the morphism of truncated operads $T_{m,f}$ is indeed onto.

Since the implication (2) \implies (3) is established, the proposition is proved. □

2.5 The groupoid GTSh

Let $\mathbb{N} \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ and $[(m, f)] \in \text{GT}(\mathbb{N})$. The morphism of truncated operads

$$T_{m,f} : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_{\mathbb{N}}$$

gives us the obvious compatible equivalence relation \sim_s :

$$(2-37) \quad \gamma_1 \sim_s \gamma_2 \iff T_{m,f}(\gamma_1) = T_{m,f}(\gamma_2).$$

Due to the following proposition we have $\sim_s = \sim_{\mathbb{N}_{m,f}^s}$, where $\mathbb{N}_{m,f}^s := \ker(T_{m,f}^{\text{PB}_4})$ and $\mathbb{N}_{m,f}^s \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$. This will allow us to construct a groupoid GTSh with $\text{Ob}(\text{GTSh}) := \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$. We will see that $\text{GT}(\mathbb{N})$ is the set of morphisms of GTSh with the target \mathbb{N} and, for every morphism $[(m, f)] \in \text{GT}(\mathbb{N})$, its source is $\mathbb{N}_{m,f}^s := \ker(T_{m,f}^{\text{PB}_4})$.

Proposition 2.11 *Let $\mathbb{N} \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$, $[(m, f)] \in \text{GT}(\mathbb{N})$ and*

$$\mathbb{N}^s = \mathbb{N}_{m,f}^s := \ker(T_{m,f}^{\text{PB}_4}) \trianglelefteq \text{PB}_4.$$

Then $\mathbb{N}^s \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ and the compatible equivalence relation \sim_s coincides with $\sim_{\mathbb{N}^s}$.

Proof To prove the first statement, we observe that, since $N \in \text{PB}_4$, the standard homomorphism $\rho: B_4 \rightarrow S_4$ induces a group homomorphism $\tilde{\rho}: B_4/N \rightarrow S_4$. Furthermore, using equations (2-25), it is easy to see that the composition

$$\psi := \tilde{\rho} \circ T_{m,f}^{B_4}: B_4 \rightarrow S_4$$

coincides with ρ . Thus N^5 is the kernel of a group homomorphism $T_{m,f}^{B_4}$ from B_4 to a finite group B_4/N . Hence N^5 is a finite-index normal subgroup of B_4 . Since we also have $N^5 \leq \text{PB}_4$, we conclude that $N^5 \in \text{NFI}_{\text{PB}_4}(B_4)$.

Although the proof of the second statement is rather technical, the main idea is to show that group homomorphisms $T_{m,f}^{\text{PB}_n}$ for $n = 2, 3, 4$ are, in some sense, compatible with the homomorphisms

$$\varphi_{123}, \varphi_{12,3,4}, \varphi_{1,23,4}, \varphi_{1,2,34}, \varphi_{234}, \varphi_{12}, \varphi_{12,3}, \varphi_{1,23}, \varphi_{23}.$$

See equations (A-18) and (A-19). This fact is deduced from the compatibility of $T_{m,f}$ with the structures of truncated operads. Then the desired second statement of Proposition 2.11 is a simple consequence of this compatibility property of homomorphisms $T_{m,f}^{\text{PB}_n}$ for $n = 2, 3, 4$.

Let us consider $h \in \text{PB}_n$ (for $n \in \{2, 3\}$) and denote by \tilde{h} any representative of the coset $T_{m,f}^{\text{PB}_n}(h)$ in $\text{PB}_n/\text{N}_{\text{PB}_n}$. Our first goal is to prove that, for every

$$\varphi \in \begin{cases} \{\varphi_{123}, \varphi_{12,3,4}, \varphi_{1,23,4}, \varphi_{1,2,34}, \varphi_{234}\} & \text{if } n = 3, \\ \{\varphi_{12}, \varphi_{12,3}, \varphi_{1,23}, \varphi_{23}\} & \text{if } n = 2, \end{cases}$$

there exists $g \in \text{PB}_{n+1}/\text{N}_{\text{PB}_{n+1}}$ such that

$$(2-38) \quad g^{-1} T_{m,f}^{\text{PB}_{n+1}}(\varphi(h))g = \varphi(\tilde{h}) \text{N}_{\text{PB}_{n+1}}.$$

Indeed, let $n = 3$ and $\varphi = \varphi_{1,23,4}$. Setting $\eta := \mathfrak{m}(h)$ and using the compatibility of $T_{m,f}$ with operadic insertions and compositions we get

$$(2-39) \quad T_{m,f}(\eta) \circ_2 \text{id}_{12} = T_{m,f}(\eta \circ_2 \text{id}_{12}).$$

Applying ou to the left-hand side of (2-39), we get

$$(2-40) \quad \text{ou}(T_{m,f}(\eta) \circ_2 \text{id}_{12}) = \varphi_{1,23,4}(\tilde{h}) \text{N}_{\text{PB}_4},$$

where \tilde{h} is an element of the coset $T_{m,f}^{\text{PB}_3}(h)$ in $\text{PB}_3/\text{N}_{\text{PB}_3}$.

As for the right-hand side of (2-39), we have

$$\begin{aligned} T_{m,f}(\eta \circ_2 \text{id}_{12}) &= T_{m,f}(\alpha_{((1,2)3)4}^{(1(2,3))4} \cdot \mathfrak{m}(\varphi_{1,23,4}(h)) \cdot \alpha_{(1(2,3))4}^{((1,2)3)4}) \\ &= T_{m,f}(\alpha_{((1,2)3)4}^{(1(2,3))4}) \cdot T_{m,f}(\mathfrak{m}(\varphi_{1,23,4}(h))) \cdot T_{m,f}(\alpha_{(1(2,3))4}^{((1,2)3)4}). \end{aligned}$$

Thus

$$(2-41) \quad \text{ou}(T_{m,f}(\eta \circ_2 \text{id}_{12})) = g^{-1} T_{m,f}^{\text{PB}_4}(\varphi_{1,23,4}(h))g,$$

where $g = \text{ou}(T_{m,f}(\alpha_{(1(2,3))4}^{(1(2,3))4}))$.

Combining (2-40) with (2-41), we conclude that (2-38) holds for $n = 3$ and $\varphi = \varphi_{1,2,3,4}$.

Let us now consider the case when $n = 2$ and $\varphi = \varphi_{12}$.

As above, setting $\eta := m(h)$ and using the compatibility of $T_{m,f}$ with operadic insertions and compositions we get

$$(2-42) \quad \text{id}_{12} \circ_1 T_{m,f}(\eta) = T_{m,f}(\text{id}_{12} \circ_1 \eta).$$

Applying ou to the left-hand side of (2-42), we get

$$(2-43) \quad \text{ou}(\text{id}_{12} \circ_1 T_{m,f}(\eta)) = \varphi_{12}(\tilde{h})N_{\text{PB}_3},$$

where \tilde{h} is an element of the coset $T_{m,f}^{\text{PB}_2}(h)$ in $\text{PB}_2/N_{\text{PB}_2}$.

The right-hand side of (2-42) can be rewritten as

$$T_{m,f}(\text{id}_{12} \circ_1 \eta) = T_{m,f}(m(\varphi_{12}(h))).$$

Hence

$$(2-44) \quad \text{ou}(T_{m,f}(\text{id}_{12} \circ_1 \eta)) = \text{ou}(T_{m,f}(m(\varphi_{12}(h)))) = T_{m,f}^{\text{PB}_3}(\varphi_{12}(h)).$$

Combining (2-43) with (2-44), we conclude that (2-38) holds for $n = 2$ and $\varphi = \varphi_{12}$ with $g = 1_{\text{PB}_3/N_{\text{PB}_3}}$.

The proof of (2-38) for the remaining case proceeds in the similar way.

Let us now prove that, for every $n \in \{2, 3, 4\}$,

$$(2-45) \quad h \in N_{\text{PB}_n}^5 \implies m(h) \sim_5 \text{id}_{((1,2)\dots)},$$

where $((1, 2) \dots)$ denotes 12 (resp. (1, 2)3, ((1, 2)3)4) if $n = 2$ (resp. $n = 3, n = 4$).

For $n = 4$, (2-45) is a straightforward consequence of the definition of N^5 . So let $n = 3$ and \tilde{h} be an element of the coset $T_{m,f}^{\text{PB}_3}(h)$ in $\text{PB}_3/N_{\text{PB}_3}$.

Since $T_{m,f}^{\text{PB}_4}(\varphi(h)) = 1$ in PB_4/N for every $\varphi \in \{\varphi_{123}, \varphi_{12,3,4}, \varphi_{1,2,3,4}, \varphi_{1,2,34}, \varphi_{234}\}$, equation (2-38) implies that

$$\tilde{h} \in \varphi_{123}^{-1}(N) \cap \varphi_{12,3,4}^{-1}(N) \cap \varphi_{1,2,3,4}^{-1}(N) \cap \varphi_{1,2,34}^{-1}(N) \cap \varphi_{234}^{-1}(N).$$

In other words, $\tilde{h} \in N_{\text{PB}_3}$ and hence $T_{m,f}^{\text{PB}_3}(h) = 1$ in $\text{PB}_3/N_{\text{PB}_3}$. Thus (2-45) holds for $n = 3$.

Let us now consider the case $n = 2$ and denote by \tilde{h} an element of the coset $T_{m,f}^{\text{PB}_2}(h)$ in $\text{PB}_2/N_{\text{PB}_2}$.

Since $\varphi(h) \in N_{\text{PB}_3}^5$ for every $\varphi \in \{\varphi_{12}, \varphi_{12,3}, \varphi_{1,2,3}, \varphi_{23}\}$ and implication (2-45) is proved for $n = 3$, we conclude that

$$T_{m,f}^{\text{PB}_3}(\varphi(h)) = 1 \quad \text{for all } \varphi \in \{\varphi_{12}, \varphi_{12,3}, \varphi_{1,2,3}, \varphi_{23}\}.$$

Therefore, equation (2-38) implies that

$$\tilde{h} \in \varphi_{12}^{-1}(N_{\text{PB}_3}) \cap \varphi_{12,3}^{-1}(N_{\text{PB}_3}) \cap \varphi_{1,2,3}^{-1}(N_{\text{PB}_3}) \cap \varphi_{23}^{-1}(N_{\text{PB}_3}).$$

In other words, $\tilde{h} \in N_{\text{PB}_2}$ and hence $T_{m,f}^{\text{PB}_2}(h) = 1$ in $\text{PB}_2/N_{\text{PB}_2}$. Thus implication (2-45) holds for $n = 2$ as well.

Let us now prove that, for every $n \in \{2, 3, 4\}$ and $h \in \text{PB}_n$,

$$(2-46) \quad \mathfrak{m}(h) \sim_s \text{id}_{((1,2)\dots)} \implies h \in \mathbb{N}_{\text{PB}_n}^5.$$

Again, for $n = 4$, (2-46) is a straightforward consequence of the definition of \mathbb{N}^5 . So let $h \in \text{PB}_3$.

Since $\mathfrak{m}(h) \sim_s \text{id}_{(1,2)3}$, $T_{m,f}^{\text{PB}_3}(h)$ is the identity element of $\text{PB}_3/\mathbb{N}_{\text{PB}_3}$. Hence, equation (2-38) implies that $\varphi(h) \in \mathbb{N}^5$ for every $\varphi \in \{\varphi_{123}, \varphi_{12,3,4}, \varphi_{1,23,4}, \varphi_{1,2,34}, \varphi_{234}\}$, or equivalently $h \in \mathbb{N}_{\text{PB}_3}^5$.

Similarly, if $h \in \text{PB}_2$ and $\mathfrak{m}(h) \sim_s \text{id}_{12}$, then $T_{m,f}^{\text{PB}_2}(h)$ is the identity element of $\text{PB}_2/\mathbb{N}_{\text{PB}_2}$. Hence, equation (2-38) implies that $T_{m,f}^{\text{PB}_3}(\varphi(h)) = 1$ in $\text{PB}_3/\mathbb{N}_{\text{PB}_3}$ for every

$$\varphi \in \{\varphi_{12}, \varphi_{12,3}, \varphi_{1,23}, \varphi_{23}\},$$

or, equivalently,

$$\mathfrak{m}(\varphi(h)) \sim_s \text{id}_{((1,2)3)} \quad \text{for all } \varphi \in \{\varphi_{12}, \varphi_{12,3}, \varphi_{1,23}, \varphi_{23}\}.$$

Since implication (2-46) is already proved for $n = 3$, we conclude that

$$\varphi(h) \in \mathbb{N}_{\text{PB}_3}^5 \quad \text{for all } \varphi \in \{\varphi_{12}, \varphi_{12,3}, \varphi_{1,23}, \varphi_{23}\}.$$

Thus $h \in \mathbb{N}_{\text{PB}_2}^5$ and (2-46) is proved for $n = 2$.

Let $n \in \{2, 3, 4\}$, $\tau \in \text{Ob}(\text{PaB}(n))$, $\eta \in \text{Aut}_{\text{PaB}}(\tau)$ and $h := \text{ou}(\eta) \in \text{PB}_n$. Our next goal is to prove that

$$(2-47) \quad h \in \mathbb{N}_{\text{PB}_n}^5 \iff \eta \sim_s \text{id}_\tau.$$

Since $T_{m,f}$ is compatible with the action of the symmetric groups, we may assume, without loss of generality, that the underlying permutation of τ is the identity permutation in S_n .

Therefore

$$(2-48) \quad \eta = \alpha_{((1,2)\dots)}^\tau \mathfrak{m}(h) \alpha_\tau^{((1,2)\dots)}$$

and hence $T_{m,f}(\eta) = \text{id}_\tau$ if and only if $T_{m,f}(\mathfrak{m}(h)) = \text{id}_{((1,2)\dots)}$; the latter is equivalent to $\mathfrak{m}(h) \sim_s \text{id}_{((1,2)\dots)}$.

Thus (2-47) is a consequence of implications (2-45) and (2-46).

Finally, let us use (2-47) to prove the statement of the proposition.

Let $\gamma, \tilde{\gamma} \in \text{PaB}(n)$ (with $n \in \{2, 3, 4\}$) and τ be the source of both morphisms. Clearly, $\gamma \sim_s \tilde{\gamma}$ if and only if $\eta \sim_s \text{id}_\tau$, where $\eta = \gamma^{-1} \cdot \tilde{\gamma}$.

Thus, due to (2-47), $\gamma \sim_s \tilde{\gamma}$ if and only if $\text{ou}(\gamma^{-1} \cdot \tilde{\gamma}) \in \mathbb{N}_{\text{PB}_n}^5$. □

Proposition 2.11 has the following important consequences:

Corollary 2.12 For every GT-shadow $[(m, f)] \in \text{GT}(\mathbb{N})$,

- $|\text{PB}_4 : \mathbb{N}^5| = |\text{PB}_4 : \mathbb{N}|$,
- $|\text{PB}_3 : \mathbb{N}_{\text{PB}_3}^5| = |\text{PB}_3 : \mathbb{N}_{\text{PB}_3}|$, and
- $\mathbb{N}_{\text{ord}}^5 = N_{\text{ord}}$, or equivalently, $\mathbb{N}_{\text{PB}_2}^5 = \mathbb{N}_{\text{PB}_2}$. □

Corollary 2.13 For every GT-shadow $[(m, f)] \in \text{GT}(\mathbb{N})$ the morphism $T_{m,f}: \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_{\mathbb{N}}$ of truncated operads factors as

$$\begin{array}{ccc} \text{PaB}^{\leq 4} & & \\ \mathcal{P}_{\mathbb{N}^5} \downarrow & \searrow T_{m,f} & \\ \text{PaB}^{\leq 4}/\sim_{\mathbb{N}^5} & \xrightarrow{T_{m,f}^{\text{isom}}} & \text{PaB}^{\leq 4}/\sim_{\mathbb{N}} \end{array}$$

where $\mathcal{P}_{\mathbb{N}^5}$ is the canonical projection and $T_{m,f}^{\text{isom}}$ is an isomorphism of truncated operads.

The assignment $[(m, f)] \mapsto T_{m,f}^{\text{isom}}$ gives us a bijection from the set

$$\{[(m, f)] \in \text{GT}(\mathbb{N}) \mid \mathbb{N}^5 = \ker(T_{m,f}^{\text{PB}_4})\}$$

to the set $\text{Isom}(\text{PaB}^{\leq 4}/\sim_{\mathbb{N}^5}, \text{PaB}^{\leq 4}/\sim_{\mathbb{N}})$ of isomorphisms of truncated operads (in the category of groupoids).

Proof Due to Proposition 2.10 and the definition of the equivalence relation \sim_s , we have the commutative diagram of morphisms of truncated operads

$$\begin{array}{ccc} \text{PaB}^{\leq 4} & & \\ \downarrow & \searrow T_{m,f} & \\ \text{PaB}^{\leq 4}/\sim_s & \xrightarrow{T_{m,f}^{\text{isom}}} & \text{PaB}^{\leq 4}/\sim_{\mathbb{N}} \end{array}$$

with $T_{m,f}^{\text{isom}}$ being a bijection⁷ on the level of morphisms.

Thanks to Proposition 2.11, the equivalence relation \sim_s coincides with $\sim_{\mathbb{N}^5}$. Hence $T_{m,f}^{\text{isom}}$ is a morphism of truncated operads

$$(2-49) \quad T_{m,f}^{\text{isom}}: \text{PaB}^{\leq 4}/\sim_{\mathbb{N}^5} \xrightarrow{\cong} \text{PaB}^{\leq 4}/\sim_{\mathbb{N}}.$$

Let us denote by $S_{m,f}^{\text{isom}}: \text{PaB}^{\leq 4}/\sim_{\mathbb{N}} \rightarrow \text{PaB}^{\leq 4}/\sim_{\mathbb{N}^5}$ the inverse of $T_{m,f}^{\text{isom}}$ (viewed as a map of morphisms) and show that $S_{m,f}^{\text{isom}}$ is compatible with the composition of morphisms and with the operadic insertions.

As for the compatibility with operadic insertions, we have

$$\begin{aligned} S_{m,f}^{\text{isom}}([\gamma_1] \circ_i [\gamma_2]) &= S_{m,f}^{\text{isom}}(T_{m,f}^{\text{isom}}([\tilde{\gamma}_1]) \circ_i T_{m,f}^{\text{isom}}([\tilde{\gamma}_2])) = S_{m,f}^{\text{isom}}(T_{m,f}^{\text{isom}}([\tilde{\gamma}_1] \circ_i [\tilde{\gamma}_2])) \\ &= [\tilde{\gamma}_1] \circ_i [\tilde{\gamma}_2] = S_{m,f}^{\text{isom}}([\gamma_1]) \circ_i S_{m,f}^{\text{isom}}([\gamma_2]) \end{aligned}$$

for any $[\gamma_1] \in \text{PaB}(n)/\sim_{\mathbb{N}}$, $[\gamma_2] \in \text{PaB}(k)/\sim_{\mathbb{N}}$ and $k \leq 4$, $1 \leq i \leq n \leq 4$.

The compatibility of $S_{m,f}^{\text{isom}}$ with the composition of morphisms is proved in a similar fashion.

Let us now consider an isomorphism of truncated operads

$$T^{\text{isom}}: \text{PaB}^{\leq 4}/\sim_{\mathbb{N}^5} \xrightarrow{\cong} \text{PaB}^{\leq 4}/\sim_{\mathbb{N}}.$$

⁷We tacitly assume that $T_{m,f}^{\text{isom}}$ acts as the identity on the level of objects.

Pre-composing T^{isom} with the canonical projection $\mathcal{P}_{N^5} : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_{N^5}$ we get an onto morphism $T := T^{\text{isom}} \circ \mathcal{P}_{N^5}$ of truncated operads. Since T is uniquely determined by a GT-shadow $[(m, f)] \in \text{GT}(N)$ such that $\ker(T_{m,f}^{\text{PB}_4}) = N^5$, we conclude that the assignment $[(m, f)] \mapsto T_{m,f}^{\text{isom}}$ is indeed a bijection

$$(2-50) \quad \{[(m, f)] \in \text{GT}(N) \mid N^5 = \ker(T_{m,f}^{\text{PB}_4})\} \xrightarrow{\cong} \text{Isom}(\text{PaB}^{\leq 4} / \sim_{N^5}, \text{PaB}^{\leq 4} / \sim_N).$$

Corollary 2.13 is proved. □

Let us now observe that the assignment

$$(2-51) \quad \text{Hom}(\tilde{N}, N) := \text{Isom}(\text{PaB}^{\leq 4} / \sim_{\tilde{N}}, \text{PaB}^{\leq 4} / \sim_N)$$

upgrades the set $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ to a groupoid. The set of objects of this groupoid is $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ and the set of morphisms from \tilde{N} to N is the set $\text{Isom}(\text{PaB}^{\leq 4} / \sim_{\tilde{N}}, \text{PaB}^{\leq 4} / \sim_N)$ of isomorphisms of truncated operads (in the category of groupoids). Morphisms of this groupoid are composed in the standard way.

The second statement of Corollary 2.13 allows us to tacitly identify (2-51) with the set

$$\{[(m, f)] \in \text{GT}(N) \mid \ker(T_{m,f}^{\text{PB}_4}) = \tilde{N}\}.$$

We will use the identification in the remainder of this paper and we denote by GTSh the resulting *groupoid of GT-shadows*.

The following proposition gives us an explicit formula for the composition of morphisms in GTSh.

Proposition 2.14 *Let $N^{(1)}, N^{(2)}$ and $N^{(3)}$ be elements of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ and*

$$[(m_1, f_1)] \in \text{Hom}_{\text{GTSh}}(N^{(1)}, N^{(2)}), \quad [(m_2, f_2)] \in \text{Hom}_{\text{GTSh}}(N^{(2)}, N^{(3)}).$$

Then their composition $[(m_2, f_2)] \circ [(m_1, f_1)]$ is represented by the pair (m, f) where

$$(2-52) \quad m := 2m_1m_2 + m_1 + m_2 \quad \text{and} \quad fN_{\text{PB}_3}^{(3)} := f_2N_{\text{PB}_3}^{(3)} \cdot T_{m_2, f_2}^{\text{PB}_3}(f_1).$$

Proof Let $[(m_2, f_2)] \in \text{GT}(N^{(3)})$ and $[(m_1, f_1)] \in \text{GT}(N^{(2)})$, where

$$N^{(2)} := \ker(T_{m_2, f_2}^{\text{PB}_4}) \quad \text{and} \quad N^{(1)} := \ker(T_{m_1, f_1}^{\text{PB}_4}).$$

In other words, the GT-shadow $[(m_1, f_1)]$ (resp. $[(m_2, f_2)]$) is a morphism from $N^{(1)}$ to $N^{(2)}$ (resp. a morphism from $N^{(2)}$ to $N^{(3)}$) in GTSh.

By Corollary 2.13, we have the diagram of morphisms of truncated operads

$$(2-53) \quad \begin{array}{ccccc} \text{PaB}^{\leq 4} & & & & \\ \downarrow \mathcal{P}_{N^{(1)}} & \searrow T_{m,f} & & & \\ \text{PaB}^{\leq 4} / \sim_{N^{(1)}} & \xrightarrow{T_{m_1, f_1}^{\text{isom}}} & \text{PaB}^{\leq 4} / \sim_{N^{(2)}} & \xrightarrow{T_{m_2, f_2}^{\text{isom}}} & \text{PaB}^{\leq 4} / \sim_{N^{(3)}} \end{array}$$

where the vertical arrow is the canonical projection.

Formula (2-52) is obtained by looking at the image of the associator $[\alpha] \in \text{PaB}^{\leq 4} / \sim_{N^{(1)}}$ (resp. the braiding $[\beta] \in \text{PaB}^{\leq 4} / \sim_{N^{(1)}}$) under $T_{m_2, f_2}^{\text{isom}} \circ T_{m_1, f_1}^{\text{isom}}$. For $[\alpha]$, we have

$$\begin{aligned} T_{m, f}(\alpha) &= T_{m_2, f_2}^{\text{isom}}(T_{m_1, f_1}^{\text{isom}}[\alpha]) = T_{m_2, f_2}^{\text{isom}}(T_{m_1, f_1}(\alpha)) = T_{m_2, f_2}^{\text{isom}}([\alpha \cdot m(f_1)]) \\ &= T_{m_2, f_2}(\alpha \cdot m(f_1)) = T_{m_2, f_2}(\alpha) \cdot T_{m_2, f_2}(m(f_1)) \\ &= [\alpha \cdot m(f_2)] \cdot m(T_{m_2, f_2}^{\text{PB}_3}(f_1)) = [\alpha \cdot m(f)], \end{aligned}$$

where f is any representative of the coset $f_2 N_{\text{PB}_3}^{(3)} \cdot T_{m_2, f_2}^{\text{PB}_3}(f_1)$ in $\text{PB}_3 / N_{\text{PB}_3}^{(3)}$.

Similarly, computing $T_{m, f}(\beta)$, it is easy to see that $m \equiv 2m_1 m_2 + m_1 + m_2 \pmod{N_{\text{ord}}^{(3)}}$. □

Remark 2.15 Later we will see that it makes sense to focus only on GT–shadows that can be represented by pairs (m, f) with

$$(2-54) \quad f \in F_2 \leq \text{PB}_3.$$

Let us call such GT–shadows *practical*.

Using (2-28) and (2-52), for the composition $[(m, f)] := [(m_2, f_2)] \circ [(m_1, f_1)]$ of practical GT–shadows $[(m_2, f_2)]$ and $[(m_1, f_1)]$ we get the formulas

$$(2-55) \quad \begin{aligned} m &:= 2m_1 m_2 + m_1 + m_2, \\ f(x, y) &:= f_2(x, y) f_1(x^{2m_2+1}, f_2(x, y)^{-1} y^{2m_2+1} f_2(x, y)). \end{aligned}$$

Due to this observation, practical GT–shadows form a subgroupoid of GTSh.

Remark 2.16 The authors *do not know* whether there exists $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$ and an onto morphism of truncated operads $\text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_N$ that *cannot* be represented by a pair $(m, f) \in \mathbb{Z} \times F_2$.

2.5.1 The virtual cyclotomic character Let us observe that to every $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$ we assign the (finite) cyclic group

$$\text{PB}_2 / \langle x_{12}^{N_{\text{ord}}} \rangle \cong \mathbb{Z} / N_{\text{ord}} \mathbb{Z},$$

where N_{ord} is the index of N_{PB_2} in PB_2 . Moreover, if $[(m, f)]$ is a morphism from K to N in the groupoid GTSh, then both N and K give the same quotient $\text{PB}_2 / \langle x_{12}^{N_{\text{ord}}} \rangle$ of PB_2 , ie $K_{\text{ord}} = N_{\text{ord}}$.

Proposition 2.14 implies that the assignment $N \mapsto \mathbb{Z} / N_{\text{ord}} \mathbb{Z}$ upgrades to a functor $\text{Ch}_{\text{cyclot}}$ from GTSh to the category of finite cyclic groups. More precisely:

Corollary 2.17 Let $[(m, f)]$ be a morphism from $N^{(1)}$ to $N^{(2)}$ in the groupoid GTSh. The assignments

$$(2-56) \quad \begin{aligned} N &\mapsto \text{PB}_2 / N_{\text{PB}_2}, \quad [(m, f)] \mapsto \text{Ch}_{\text{cyclot}}(m, f) \in \text{Aut}(\text{PB}_2 / N_{\text{PB}_2}^{(2)}), \\ \text{Ch}_{\text{cyclot}}(m, f)(x_{12} N_{\text{PB}_2}^{(2)}) &:= x_{12}^{2m+1} N_{\text{PB}_2}^{(2)}, \end{aligned}$$

define a functor $\text{Ch}_{\text{cyclot}}$ from the groupoid GTSh to the category of finite cyclic groups.

Proof Since, for every GT shadow $[(m, f)]$, $2m + 1$ represents an invertible element of the ring $\mathbb{Z}/N_{\text{ord}}^{(2)}\mathbb{Z}$, $\text{Ch}_{\text{cyclot}}(m, f)$ is clearly an automorphism of $\text{PB}_2/N_{\text{PB}_2}^{(2)} = \text{PB}_2/N_{\text{PB}_2}^{(1)}$.

Thus it remains to show that $\text{Ch}_{\text{cyclot}}$ is compatible with the composition of GT-shadows.

For this purpose, we consider two composable GT-shadows: $[(m_1, f_1)] \in \text{Hom}_{\text{GTSh}}(N^{(1)}, N^{(2)})$ and $[(m_2, f_2)] \in \text{Hom}_{\text{GTSh}}(N^{(2)}, N^{(3)})$.

Since $N^{(1)}$, $N^{(2)}$ and $N^{(3)}$ belong to the same connected component of GTSh , $N_{\text{PB}_2}^{(1)} = N_{\text{PB}_2}^{(2)} = N_{\text{PB}_2}^{(3)}$ or equivalently, $N_{\text{ord}}^{(1)} = N_{\text{ord}}^{(2)} = N_{\text{ord}}^{(3)}$. So let us set $N_{\text{PB}_2} := N_{\text{PB}_2}^{(1)}$ and $N_{\text{ord}} := N_{\text{ord}}^{(1)}$.

Let $[(m, f)] := [(m_2, f_2)] \circ [(m_1, f_1)]$.

Due to the first equation in (2-52), $m \equiv 2m_1m_2 + m_1 + m_2 \pmod{N_{\text{ord}}}$. Hence

$$\begin{aligned} \text{Ch}_{\text{cyclot}}(m, f)(x_{12}N_{\text{PB}_2}) &= x_{12}^{2(2m_1m_2+m_1+m_2)+1}N_{\text{PB}_2} = x_{12}^{4m_1m_2+2m_1+2m_2+1}N_{\text{PB}_2} \\ &= x_{12}^{(2m_1+1)(2m_2+1)}N_{\text{PB}_2} = (x_{12}^{(2m_1+1)}N_{\text{PB}_2})^{2m_2+1} \\ &= \text{Ch}_{\text{cyclot}}(m_2, f_2) \circ \text{Ch}_{\text{cyclot}}(m_1, f_1)(x_{12}N_{\text{PB}_2}). \end{aligned}$$

Thus $\text{Ch}_{\text{cyclot}}$ is indeed a functor from GTSh to the category of finite cyclic groups. □

We call the functor $\text{Ch}_{\text{cyclot}}$ the *virtual cyclotomic character*. This name is justified by the following remark.

Remark 2.18 Let $N \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$, $g \in G_{\mathbb{Q}}$ and $[(m, f)]$ be the GT-shadow in $\text{GT}(N)$ induced by the element in $\widehat{\text{GT}}$ corresponding to g . Then

$$(2-57) \quad \text{Ch}_{\text{cyclot}}(m, f)(x_{12}N_{\text{PB}_2}) = x_{12}^{\chi(g)N_{\text{ord}}}N_{\text{PB}_2},$$

where $\chi: G_{\mathbb{Q}} \rightarrow \widehat{\mathbb{Z}}^\times \cong \text{Aut}(\widehat{\mathbb{Z}})$ is the cyclotomic character and $\chi(g)N_{\text{ord}}$ represents the image of $\chi(g)$ in $\text{Aut}(\mathbb{Z}/N_{\text{ord}}\mathbb{Z}) \cong (\mathbb{Z}/N_{\text{ord}}\mathbb{Z})^\times$. Equation (2-57) follows from the discussion in [27, Example 4.7.4 and Remark 4.7.5]. See also [16, Proposition 1.6].

2.6 Charming GT-shadows

Recall that PB_3 is isomorphic to $F_2 \times \mathbb{Z}$ where the F_2 -factor is freely generated by x_{12} and x_{23} and the \mathbb{Z} -factor is generated by the central element c given in (A-5). This implies that $\widehat{\text{PB}}_3 \cong \widehat{F}_2 \times \widehat{\mathbb{Z}}$. Due to the following proposition, the action of $\widehat{\text{GT}}$ on $\widehat{\text{PB}}_3$ (viewed as the automorphism group of $(12)3$ in $\widehat{\text{PaB}}$) respects this decomposition:

Proposition 2.19 For every (continuous) automorphism \widehat{T} of $\widehat{\text{PaB}}^{\leq 4}$, its restriction to the subgroup $\widehat{F}_2 \leq \widehat{\text{PB}}_3$ gives us an automorphism⁸ of \widehat{F}_2

$$\widehat{T}|_{\widehat{F}_2}: \widehat{F}_2 \rightarrow \widehat{F}_2$$

defined by the formulas

$$(2-58) \quad \widehat{T}(x) := x^{2\widehat{m}+1} \quad \text{and} \quad \widehat{T}(y) := \widehat{f}^{-1}y^{2\widehat{m}+1}\widehat{f}.$$

⁸In fact, some specialists like to define $\widehat{\text{GT}}$ as a certain subgroup of continuous automorphisms of \widehat{F}_2 .

The restriction of \widehat{T} to the central factor $\widehat{\mathbb{Z}}$ of $\widehat{\text{PB}}_3$ gives us the continuous automorphism of $\widehat{\mathbb{Z}}$ defined by the formula

$$(2-59) \quad \widehat{T}(c) := c^{2\widehat{m}+1}.$$

Proof Due to Proposition 2.5, the action of \widehat{T} on $\widehat{\text{PB}}_3$ is determined by the group homomorphisms

$$T_{m,f}^{\text{PB}_3} : \text{PB}_3 \rightarrow \text{PB}_3/\text{N}_{\text{PB}_3}, \quad \text{where } N \in \text{NFI}_{\text{PB}_4}(\text{B}_4),$$

corresponding to GT–shadows $[(m, f)]$ that come from \widehat{T} .

Combining this observation with equations (2-28) and the second equation in (2-29) and using the fact that c is a central element of PB_3 , we conclude that the restrictions of \widehat{T} to $\widehat{\text{F}}_2$ and to $\widehat{\mathbb{Z}}$ give us group homomorphisms

$$(2-60) \quad \widehat{T}|_{\widehat{\text{F}}_2} : \widehat{\text{F}}_2 \rightarrow \widehat{\text{F}}_2 \quad \text{and} \quad \widehat{T}|_{\widehat{\mathbb{Z}}} : \widehat{\mathbb{Z}} \rightarrow \widehat{\mathbb{Z}},$$

respectively.

Since the restrictions of the inverse of \widehat{T} to $\widehat{\text{F}}_2$ and to $\widehat{\mathbb{Z}}$ give us inverses of the two homomorphisms in (2-60), respectively, the homomorphisms in (2-60) are indeed automorphisms.

Explicit formulas (2-58) and (2-59) are consequences of equations (2-28) and the second equation in (2-29). □

If a GT–shadow $[(m, f)]$ comes from an automorphism of $\widehat{\text{PaB}}$ then it satisfies further conditions. The following definition is motivated by these conditions.

Definition 2.20 Let $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$. A GT–shadow $[(m, f)] \in \text{GT}(N)$ is called *genuine* if it comes from an automorphism of $\widehat{\text{PaB}}$. Otherwise, $[(m, f)]$ is called *fake*. Further, a GT–shadow $[(m, f)] \in \text{GT}(N)$ is called *charming* if

- the coset $f\text{N}_{\text{PB}_3}$ can be represented by $f_1 \in [\text{F}_2, \text{F}_2]$, and
- the group homomorphism

$$(2-61) \quad T_{m,f}^{\text{F}_2} := T_{m,f}^{\text{PB}_3} |_{\text{F}_2} : \text{F}_2 \rightarrow \text{F}_2/(\text{N}_{\text{PB}_3} \cap \text{F}_2)$$

is onto.

Since the intersection $\text{N}_{\text{PB}_3} \cap \text{F}_2$ plays an important role, we will denote it by N_{F_2} ,

$$(2-62) \quad \text{N}_{\text{F}_2} := \text{N}_{\text{PB}_3} \cap \text{F}_2.$$

Clearly, the kernel of the homomorphism $T_{m,f}^{\text{F}_2} : \text{F}_2 \rightarrow \text{F}_2/\text{N}_{\text{F}_2}$ coincides with $\text{N}_{\text{F}_2}^s$ and $|\text{F}_2 : \text{N}_{\text{F}_2}^s| = |\text{F}_2 : \text{N}_{\text{F}_2}|$ for every charming GT–shadow $[(m, f)]$.

Let us prove that:

Proposition 2.21 *Every genuine GT–shadow is charming.*

Proof Let $N \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$ and $[(m, f)] \in \text{GT}(N)$ be a genuine GT-shadow. The element $f \in \text{PB}_3$ can be written uniquely as

$$f = g c^k,$$

where $g \in F_2$, $k \in \mathbb{Z}$ and c is defined in (A-5).

Since N_{PB_3} is a normal subgroup of finite index in PB_3 , the subgroup $N_{F_2} := N_{\text{PB}_3} \cap F_2$ is normal in F_2 and it has a finite index in F_2 . Similarly, the subgroup $N_{\mathbb{Z}} := N_{\text{PB}_3} \cap \mathbb{Z}$ has a finite index in \mathbb{Z} . Therefore, the subgroup $N_{F_2} \times N_{\mathbb{Z}}$ is normal and it has finite index in PB_3 .

Due to Proposition 2.5, there exists $K \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$ such that K_{PB_3} is contained in $N_{F_2} \times N_{\mathbb{Z}}$. Since $[(m, f)]$ is a genuine GT-shadow, there exists $(m_1, f_1) \in \mathbb{Z} \times \text{PB}_3$ such that (m_1, f_1) represents the same GT-shadow $[(m, f)]$ in $\text{GT}(N)$ and $[(m_1, f_1)] \in \text{GT}(K)$.

Thus, without loss of generality, we may assume that $m = m_1$ and $f = f_1$, ie $[(m, f)] \in \text{GT}(K)$.

Using relation (2-18), we have

$$\sigma_1 x_{12}^m f^{-1} \sigma_2 x_{23}^m f K_{\text{PB}_3} = f^{-1} \sigma_1 \sigma_2 (x_{13} x_{23})^m K_{\text{PB}_3}.$$

Next, using (A-5) and the fact that c is a central element of B_3 , we get that

$$x_{12}^m \sigma_2^{-1} \sigma_1^{-1} g(x_{12}, x_{23}) \sigma_1 x_{12}^m g(x_{12}, x_{23})^{-1} \sigma_2 x_{23}^m g(x_{12}, x_{23}) c^{-m+k} \in K_{\text{PB}_3}.$$

Using equations in (A-6) from Section A.1, we deduce that

$$x_{12}^m g(x_{23}^{-1} x_{12}^{-1} c, x_{12}) (x_{23}^{-1} x_{12}^{-1} c)^m g(x_{23}^{-1} x_{12}^{-1} c, x_{23})^{-1} x_{23}^m g(x_{12}, x_{23}) c^{-m+k} \in K_{\text{PB}_3}$$

or

$$x_{12}^m g(x_{23}^{-1} x_{12}^{-1}, x_{12}) (x_{23}^{-1} x_{12}^{-1})^m g(x_{23}^{-1} x_{12}^{-1}, x_{23})^{-1} x_{23}^m g(x_{12}, x_{23}) c^k \in K_{\text{PB}_3}.$$

Since K_{PB_3} is a subgroup of $N_{F_2} \times N_{\mathbb{Z}}$, we have $c^k \in N_{\mathbb{Z}} \subset N_{\text{PB}_3}$. Hence $f c^{-k} N_{\text{PB}_3} = g(x_{12}, x_{23}) N_{\text{PB}_3}$, and so the GT-shadow has a representative of the form (m, f) where $f \in F_2$.

It remains to show that

- $[(m, f)]$ can be represented by a pair (m, f_1) with $f_1 \in [F_2, F_2]$, and
- the homomorphism (2-61) is onto.

Since homomorphism (2-61) does not depend on the choice of the representative of the GT-shadow $[(m, f)]$, we first prove that this homomorphism is indeed onto.

Due to Proposition 2.19, we have the following commutative diagram:

$$\begin{array}{ccc} \widehat{F}_2 & \xrightarrow{\widehat{T}|_{\widehat{F}_2}} & \widehat{F}_2 \\ i \uparrow & & \downarrow \widehat{\Phi}_{N_{F_2}} \\ F_2 & \xrightarrow{T^{m,f}} & F_2/N_{F_2} \end{array}$$

Since F_2 is dense in \widehat{F}_2 , we get that the composition $\widehat{\mathcal{P}}_{N_{F_2}} \circ \widehat{T}|_{\widehat{F}_2} \circ i$ is surjective, whence we conclude $T_{m,f}^{F_2}$ is onto.

Let us now prove that $[(m, f)]$ can be represented by a pair (m, \tilde{f}) with $\tilde{f} \in [F_2, F_2]$.

Let q be the least common multiple of the orders of $x_{12}N_{F_2}$ and $x_{23}N_{F_2}$ in F_2/N_{F_2} and $\psi_x: PB_4 \rightarrow S_q$, $\psi_y: PB_4 \rightarrow S_q$ be the group homomorphisms defined by equations

$$\begin{aligned} \psi_x(x_{12}) &:= (1, 2, \dots, q), & \psi_x(x_{23}) &= \psi_x(x_{13}) = \psi_x(x_{14}) = \psi_x(x_{24}) = \psi_x(x_{34}) := \text{id}_{S_q}, \\ \psi_y(x_{34}) &:= (1, 2, \dots, q), & \psi_y(x_{12}) &= \psi_y(x_{23}) = \psi_y(x_{13}) = \psi_y(x_{14}) = \psi_y(x_{24}) := \text{id}_{S_q}, \end{aligned}$$

respectively.

Let K be an element of $\text{NFI}_{PB_4}(B_4)$ such that

$$(2-63) \quad K \leq N \cap \ker(\psi_x) \cap \ker(\psi_y).$$

Since $[(m, f)]$ is a genuine GT–shadow, there exists a GT–shadow $[(m_1, f_1)] \in \text{GT}(K)$ such that (m_1, f_1) is also a representative of $[(m, f)]$. We can assume, without loss of generality, that $f_1 \in F_2$.

Applying equation (2-20) to f_1 we see that

$$(2-64) \quad f_1^{-1}(x_{13}x_{23}, x_{34})f_1^{-1}(x_{12}, x_{23}x_{24})f_1(x_{23}, x_{34})f_1(x_{12}x_{13}, x_{24}x_{34})f_1(x_{12}, x_{23}) \in K.$$

Inclusions (2-63) and (2-64) imply that

$$\begin{aligned} \psi_x(f_1^{-1}(x_{13}x_{23}, x_{34})f_1^{-1}(x_{12}, x_{23}x_{24})f_1(x_{23}, x_{34})f_1(x_{12}x_{13}, x_{24}x_{34})f_1(x_{12}, x_{23})) &= \text{id}_{S_q}, \\ \psi_y(f_1^{-1}(x_{13}x_{23}, x_{34})f_1^{-1}(x_{12}, x_{23}x_{24})f_1(x_{23}, x_{34})f_1(x_{12}x_{13}, x_{24}x_{34})f_1(x_{12}, x_{23})) &= \text{id}_{S_q}. \end{aligned}$$

Hence the sum s_x of exponents of x_{12} in f_1 and the sum s_y of exponents of x_{23} in f_1 are multiples of q , ie $x_{12}^{-s_x} \in N_{F_2}$ and $x_{23}^{-s_y} \in N_{F_2}$.

Thus $(m, f_1x_{12}^{-s_x}x_{23}^{-s_y})$ is yet another representative of the GT–shadow $[(m, f)]$ in $\text{GT}(N)$ and, by construction, $f_1x_{12}^{-s_x}x_{23}^{-s_y} \in [F_2, F_2]$. □

The following statement can be found in many introductory (and “not so introductory”) papers on the Grothendieck–Teichmüller group $\widehat{\text{GT}}$. Here, we deduce it from Proposition 2.21.

Corollary 2.22 *For every $(\widehat{m}, \widehat{f}) \in \widehat{\text{GT}}$, \widehat{f} belongs to the topological closure of commutator subgroup of \widehat{F}_2 .*

Proof It suffices to show that, for every $N \in \text{NFI}(F_2)$, the element $\widehat{\mathcal{P}}_N(\widehat{f}) \in F_2/N$ can be represented by $f_1 \in [F_2, F_2]$. Let us observe that $N \times \langle c \rangle \in \text{NFI}(PB_3)$.

Due to Proposition 2.5, there exists $K \in \text{NFI}_{PB_4}(B_4)$ such that $K_{PB_3} \leq N \times \langle c \rangle$. Clearly, $K_{F_2} \leq N$.

Since the pair $(\widehat{\mathcal{P}}_{K_{\text{ord}}}(\widehat{m}), \widehat{\mathcal{P}}_{K_{F_2}}(\widehat{f}))$ is a charming GT–shadow in $\text{GT}(K)$, the element $\widehat{\mathcal{P}}_{K_{F_2}}(\widehat{f}) \in F_2/K_{F_2}$ can be represented by $f_1 \in [F_2, F_2]$. Since $K_{F_2} \leq N$, the same element $f_1 \in [F_2, F_2]$ represents the coset $\widehat{\mathcal{P}}_N(\widehat{f}) \in F_2/N$. □

Let us denote by $GT^\heartsuit(N)$ the subset of all charming GT-shadows in $GT(N)$, and prove that $GT(N)$ can be safely replaced by $GT^\heartsuit(N)$ in all the constructions of Section 2.5. More precisely:

Proposition 2.23 *The assignment*

$$(2-65) \quad \text{Hom}_{GTSh^\heartsuit}(\tilde{N}, N) := \{[(m, f)] \in GT^\heartsuit(N) \mid \tilde{N} = \ker(T_{m,f}^{PB_4})\} \quad \text{for } \tilde{N}, N \in \text{NFI}_{PB_4}(B_4)$$

upgrades the set $\text{NFI}_{PB_4}(B_4)$ to a subgroupoid $GTSh^\heartsuit$ of $GTSh$.

Proof Let $[(m_1, f_1)] \in \text{Hom}_{GTSh^\heartsuit}(N^{(1)}, N^{(2)})$ and $[(m_2, f_2)] \in \text{Hom}_{GTSh^\heartsuit}(N^{(2)}, N^{(3)})$. The GT-shadows $[(m_1, f_1)]$ and $[(m_2, f_2)]$ are charming so we may assume, without loss of generality, that $f_1, f_2 \in [F_2, F_2]$. Due to Remark 2.15, the composition $[(m_2, f_2)] \circ [(m_1, f_1)]$ is represented by a pair (m, f) with

$$f = f_2 f_1 (x^{2m_2+1}, f_2^{-1}(x, y) y^{2m_2+1} f_2(x, y)).$$

Since $f_1, f_2 \in [F_2, F_2]$, it is clear that f also belongs to $[F_2, F_2]$.

Since $T_{m,f}^{F_2}: F_2 \rightarrow F_2/N^{(3)}$ is the composition of the onto homomorphism $T_{m_1, f_1}^{F_2}: F_2 \rightarrow F_2/N^{(2)}$ and the isomorphism $T_{m_2, f_2}^{F_2, \text{isom}}: F_2/N^{(2)} \rightarrow F_2/N^{(3)}$, the homomorphism $T_{m,f}^{F_2}$ is also onto.

We proved that the subset of charming GT-shadows is closed under composition.

To prove that the subset of charming GT-shadows is closed under taking inverses, we start with a charming GT-shadow $[(m, f)] \in \text{Hom}_{GTSh^\heartsuit}(N^s, N)$ and assume that $f \in [F_2, F_2]$. Let $[(\tilde{m}, \tilde{f})] \in \text{Hom}_{GTSh}(N, N^s)$ be the inverse of $[(m, f)]$ in $GTSh$. In other words,

$$(2-66) \quad \begin{aligned} 2m\tilde{m} + m + \tilde{m} &\equiv 0 \pmod{N_{\text{ord}}}, \\ f T_{m,f}^{PB_3}(\tilde{f}) &= 1_{PB_3/N_{PB_3}}. \end{aligned}$$

Our goal is to show that the coset $\tilde{f}N_{PB_3}$ can be represented by $g \in [F_2, F_2]$.

Since f^{-1} belongs to $[F_2, F_2]$, we have

$$f^{-1} = [g_{11}, g_{12}][g_{21}, g_{22}] \cdots [g_{r1}, g_{r2}],$$

where each $g_{ij} \in F_2$ and $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$.

Since the homomorphism $T_{m,f}^{F_2}: F_2 \rightarrow F_2/N_{F_2}$ is onto, for every g_{ij} there exists $\tilde{g}_{ij} \in F_2$ such that $T_{m,f}^{F_2}(\tilde{g}_{ij}) = g_{ij}N_{F_2}$. Hence, for

$$g := [\tilde{g}_{11}, \tilde{g}_{12}][\tilde{g}_{21}, \tilde{g}_{22}] \cdots [\tilde{g}_{r1}, \tilde{g}_{r2}] \in [F_2, F_2],$$

we have $T_{m,f}^{PB_3}(g) = f^{-1}N_{PB_3}$, or, equivalently,

$$(2-67) \quad f T_{m,f}^{PB_3}(g) = 1_{PB_3/N_{PB_3}}.$$

Combining (2-66) with (2-67) we conclude that $g^{-1}\tilde{f}$ belongs to the kernel of $T_{m,f}^{PB_3}: PB_3 \rightarrow PB_3/N_{PB_3}$. Thus, due to Proposition 2.11, g also represents the coset $\tilde{f}N_{PB_3}$.

Since, by construction, $g \in [F_2, F_2]$, the desired statement is proved. □

3 The Main Line functor \mathcal{ML} and \widehat{GT}

In this section, we use (charming) GT -shadows to construct a functor \mathcal{ML} from a certain subposet of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ to the category of finite groups. We prove that the limit of the functor \mathcal{ML} is isomorphic to the Grothendieck–Teichmüller group \widehat{GT} .

3.1 Connected components of GTSh^\heartsuit , settled GT -shadows and isolated elements of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$

Since the set $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ is infinite, so is the groupoid GTSh^\heartsuit . Moreover, the groupoid GTSh^\heartsuit is highly disconnected. Indeed, if \tilde{N} and N are connected by a morphism in GTSh^\heartsuit , then they must have the same index in PB_4 .

For $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, we denote by

$$\text{GTSh}_{\text{conn}}^\heartsuit(N)$$

the connected component of N in the groupoid GTSh^\heartsuit . Clearly, an element \tilde{N} of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ is an object of $\text{GTSh}_{\text{conn}}^\heartsuit(N)$ if and only if there exists $[(m, f)] \in \text{GT}^\heartsuit(N)$ such that

$$\tilde{N} = \ker(T_{m,f}^{\text{PB}_4}).$$

We call objects of the groupoid $\text{GTSh}_{\text{conn}}^\heartsuit(N)$ *conjugates* of N .

Since $\text{GT}^\heartsuit(N)$ is a finite set for every $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, it is easy to show that:

Proposition 3.1 *For every $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, the (connected) groupoid $\text{GTSh}_{\text{conn}}^\heartsuit(N)$ is finite.* □

To establish a more precise link between (charming) GT -shadows and the group \widehat{GT} , we will be interested in a certain subposet of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$. Let us start with the following definition:

Definition 3.2 Let $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ and $[(m, f)] \in \text{GT}^\heartsuit(N)$. A charming GT -shadow $[(m, f)]$ is called *settled* if its source N^s coincides with N , ie $\ker(T_{m,f}^{\text{PB}_4}) = N$. An element N of the poset $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ is called *isolated* if every GT -shadow in $\text{GT}^\heartsuit(N)$ is settled.

Clearly, a GT -shadow $[(m, f)] \in \text{GT}^\heartsuit(N)$ is settled if and only if $[(m, f)]$ is an automorphism of the object N in the groupoid GTSh^\heartsuit . Moreover, an element $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ is isolated if and only if the groupoid $\text{GTSh}_{\text{conn}}^\heartsuit(N)$ has exactly one object. In this case, $\text{GT}^\heartsuit(N)$ is the group of automorphisms of the object N in the groupoid GTSh^\heartsuit .

The following proposition gives us a simple way to produce many examples of isolated elements of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$.

Proposition 3.3 For every $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, the normal subgroup

$$(3-1) \quad N^\# := \bigcap_{K \in \text{Ob}(\text{GTSh}_{\text{conn}}^\heartsuit(N))} K$$

is an isolated element of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$.

Proof Let $[(m, f)] \in \text{GT}^\heartsuit(N^\#)$ and $N^{\#,s}$ be the source of the corresponding morphism in GTSh^\heartsuit , ie $N^{\#,s} := \ker(T_{m,f}^{\text{PB}_4})$.

Since $N^\# \leq K$, the same pair $(m, f) \in \mathbb{Z} \times F_2$ represents a GT-shadow in $\text{GT}^\heartsuit(K)$. Moreover, the homomorphism from PB_4 to PB_4/K corresponding to $[(m, f)] \in \text{GT}^\heartsuit(K)$ is the composition $\mathcal{P}_{N^\#,K} \circ T_{m,f}^{\text{PB}_4}$ of $T_{m,f}^{\text{PB}_4}$ with the canonical projection

$$\mathcal{P}_{N^\#,K} : \text{PB}_4/N^\# \rightarrow \text{PB}_4/K.$$

Let $h \in N^\#$, let $\tilde{h} \in \text{PB}_4$ be a representative of $T_{m,f}^{\text{PB}_4}(h)$ and let K^s be the source of the GT-shadow $[(m, f)] \in \text{GT}^\heartsuit(K)$, ie $K^s := \ker(\mathcal{P}_{N^\#,K} \circ T_{m,f}^{\text{PB}_4})$.

Since $N^\# \leq K^s$, we have

$$(3-2) \quad \mathcal{P}_{N^\#,K}(T_{m,f}^{\text{PB}_4}(h)) = 1_{\text{PB}_4/K}.$$

The identity (3-2) implies that $\tilde{h} \in K$ for every $K \in \text{Ob}(\text{GTSh}_{\text{conn}}^\heartsuit(N))$ and hence $\tilde{h} \in N^\#$. Therefore, $T_{m,f}^{\text{PB}_4}(h) = 1_{\text{PB}_4/N^\#}$ or equivalently, $h \in N^{\#,s}$.

We proved that

$$(3-3) \quad N^\# \leq N^{\#,s}.$$

Since these subgroups have the same index in PB_4 , inclusion (3-3) implies that $N^{\#,s} = N^\#$.

Since we started with an arbitrary GT-shadow in $\text{GT}^\heartsuit(N^\#)$, we proved that $N^\#$ is indeed an isolated element of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$. □

Remark 3.4 In all examples we have considered so far (see Section 4 on selected results of computer experiments), $\text{GTSh}_{\text{conn}}^\heartsuit(N)$ has at most two objects. Hence equation (3-1) gives us a practical way to produce examples of isolated elements of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$.

Let us denote by

$$(3-4) \quad \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$$

the subposet of isolated elements of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$.

Since $N^\# \leq N$ for every $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, Proposition 3.3 implies that:

Corollary 3.5 The subposet $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ is **cofinal**. In other words, for every element N of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, there exists $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ such that $K \leq N$. □

Although [Corollary 3.5](#) implies that the poset $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ is directed (it is a cofinal subposet of a directed poset), it is still useful to know that the intersection of two isolated elements of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ is an isolated element of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$:

Proposition 3.6 For every $N^{(1)}, N^{(2)} \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$,

$$N^{(1)} \cap N^{(2)}$$

is also an isolated element of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$.

Proof $K := N^{(1)} \cap N^{(2)}$ is clearly an element of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$. So our goal is to prove that K is isolated.

Let $[(m, f)] \in \text{GT}^\heartsuit(K)$, and let K^\heartsuit be the kernel of the homomorphism $T_{m,f}^{\text{PB}_4}: \text{PB}_4 \rightarrow \text{PB}_4/K$.

Recall that $\mathcal{P}_{K,N^{(1)}}$ (resp. $\mathcal{P}_{K,N^{(2)}}$) is the canonical homomorphism from PB_4/K to $\text{PB}_4/N^{(1)}$ (resp. to $\text{PB}_4/N^{(2)}$). Since $K \leq N^{(1)}$ and $K \leq N^{(2)}$, the pair (m, f) also represents a GT-shadow in $\text{GT}^\heartsuit(N^{(1)})$ and a GT-shadow in $\text{GT}^\heartsuit(N^{(2)})$. Moreover, the compositions $\mathcal{P}_{K,N^{(1)}} \circ T_{m,f}^{\text{PB}_4}$ and $\mathcal{P}_{K,N^{(2)}} \circ T_{m,f}^{\text{PB}_4}$ are the homomorphisms $\text{PB}_4 \rightarrow \text{PB}_4/N^{(1)}$ and $\text{PB}_4 \rightarrow \text{PB}_4/N^{(2)}$ corresponding to these GT-shadows in $\text{GT}^\heartsuit(N^{(1)})$ and $\text{GT}^\heartsuit(N^{(2)})$, respectively.

Let us now consider $h \in K^\heartsuit$. Since $T_{m,f}^{\text{PB}_4}(h) = 1_{\text{PB}_4/K}$, we have

$$(3-5) \quad \mathcal{P}_{K,N^{(1)}} \circ T_{m,f}^{\text{PB}_4}(h) = 1_{\text{PB}_4/N^{(1)}} \quad \text{and} \quad \mathcal{P}_{K,N^{(2)}} \circ T_{m,f}^{\text{PB}_4}(h) = 1_{\text{PB}_4/N^{(2)}}.$$

Since $N^{(1)}, N^{(2)}$ are both isolated, identities (3-5) imply that $h \in N^{(1)}$ and $h \in N^{(2)}$. Hence $h \in K$.

Since we showed that $K^\heartsuit \leq K$ and both subgroups have the same (finite) index in PB_4 , we have the desired equality $K^\heartsuit = K$. □

Recall that, for every isolated element $N \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, the set $\text{GT}^\heartsuit(N)$ is a finite group. More precisely, $\text{GT}^\heartsuit(N)$ is the (finite) group of automorphisms of N in the groupoid GTSh^\heartsuit . Let us denote this finite group by $\mathcal{ML}(N)$, and prove that:

Proposition 3.7 The assignment

$$N \mapsto \mathcal{ML}(N)$$

upgrades to a functor \mathcal{ML} from the poset $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ to the category of finite groups.

Proof Let $K \leq N$ be isolated elements of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$. Our goal is to define a group homomorphism

$$(3-6) \quad \mathcal{ML}_{K,N}: \mathcal{ML}(K) \rightarrow \mathcal{ML}(N)$$

and show that, for every triple of nested elements $N^{(1)} \leq N^{(2)} \leq N^{(3)}$ of $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$,

$$(3-7) \quad \mathcal{ML}_{N^{(2)},N^{(3)}} \circ \mathcal{ML}_{N^{(1)},N^{(2)}} = \mathcal{ML}_{N^{(1)},N^{(3)}}.$$

For this proof, it is convenient to identify GT-shadows $[(m, f)] \in \text{GT}^\heartsuit(\mathbb{K})$ with the corresponding onto morphisms $T_{m,f}: \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_{\mathbb{K}}$ of truncated operads. So let $[(m, f)] \in \text{GT}^\heartsuit(\mathbb{K})$ and $T_{m,f}$ be the corresponding morphism.

Recall that $\mathcal{P}_{\mathbb{K},\mathbb{N}}$ denotes the canonical onto morphism of truncated operads

$$\mathcal{P}_{\mathbb{K},\mathbb{N}}: \text{PaB}^{\leq 4}/\sim_{\mathbb{K}} \rightarrow \text{PaB}^{\leq 4}/\sim_{\mathbb{N}}.$$

Composing $\mathcal{P}_{\mathbb{K},\mathbb{N}}$ with $T_{m,f}$ we get an onto morphism

$$\mathcal{P}_{\mathbb{K},\mathbb{N}} \circ T_{m,f}: \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_{\mathbb{N}},$$

and hence an element of $\text{GT}^\heartsuit(\mathbb{N})$.

We set

$$(3-8) \quad \mathcal{ML}_{\mathbb{K},\mathbb{N}}(T_{m,f}) := \mathcal{P}_{\mathbb{K},\mathbb{N}} \circ T_{m,f}.$$

To prove that $\mathcal{ML}_{\mathbb{K},\mathbb{N}}$ is a group homomorphism from $\mathcal{ML}(\mathbb{K})$ to $\mathcal{ML}(\mathbb{N})$, we recall that, since \mathbb{K} is isolated, every onto morphism of truncated operads $T: \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_{\mathbb{K}}$ factors as

$$\begin{array}{ccc} \text{PaB}^{\leq 4} & & \\ \mathcal{P}_{\mathbb{K}} \downarrow & \searrow T & \\ \text{PaB}^{\leq 4}/\sim_{\mathbb{K}} & \xrightarrow{T^{\text{isom}}} & \text{PaB}^{\leq 4}/\sim_{\mathbb{K}} \end{array}$$

Let us now show that, for every onto morphism of truncated operads $T: \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_{\mathbb{K}}$, the following diagram commutes:

$$(3-9) \quad \begin{array}{ccc} \text{PaB}^{\leq 4} & & \\ \mathcal{P}_{\mathbb{K}} \downarrow & \searrow T & \\ \text{PaB}^{\leq 4}/\sim_{\mathbb{K}} & \xrightarrow{T^{\text{isom}}} & \text{PaB}^{\leq 4}/\sim_{\mathbb{K}} \\ \mathcal{P}_{\mathbb{K},\mathbb{N}} \downarrow & & \mathcal{P}_{\mathbb{K},\mathbb{N}} \downarrow \\ \text{PaB}^{\leq 4}/\sim_{\mathbb{N}} & \xrightarrow{(\mathcal{P}_{\mathbb{K},\mathbb{N}} \circ T)^{\text{isom}}} & \text{PaB}^{\leq 4}/\sim_{\mathbb{N}} \end{array}$$

Since the top triangle of (3-9) commutes by definition of T^{isom} , we only need to prove the commutativity of the square. Let $\gamma \in \text{PaB}^{\leq 4}$ and $[\gamma]_{\mathbb{K}}$ (resp. $[\gamma]_{\mathbb{N}}$) be equivalence classes of γ in $\text{PaB}^{\leq 4}/\sim_{\mathbb{K}}$ (resp. in $\text{PaB}^{\leq 4}/\sim_{\mathbb{N}}$). Since $T^{\text{isom}}([\gamma]_{\mathbb{K}}) = T(\gamma)$, $(\mathcal{P}_{\mathbb{K},\mathbb{N}} \circ T)^{\text{isom}}([\gamma]_{\mathbb{N}}) = \mathcal{P}_{\mathbb{K},\mathbb{N}} \circ T(\gamma)$ and $\mathcal{P}_{\mathbb{K},\mathbb{N}}([\gamma]_{\mathbb{K}}) = [\gamma]_{\mathbb{N}}$, we have

$$\mathcal{P}_{\mathbb{K},\mathbb{N}} \circ T^{\text{isom}}([\gamma]_{\mathbb{K}}) = \mathcal{P}_{\mathbb{K},\mathbb{N}} \circ T(\gamma) = (\mathcal{P}_{\mathbb{K},\mathbb{N}} \circ T)^{\text{isom}}([\gamma]_{\mathbb{N}}) = (\mathcal{P}_{\mathbb{K},\mathbb{N}} \circ T)^{\text{isom}} \circ \mathcal{P}_{\mathbb{K},\mathbb{N}}([\gamma]_{\mathbb{K}}).$$

Thus (3-9) indeed commutes.

Now let T_1 and T_2 be onto morphisms (of truncated operads)

$$T_1, T_2: \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_{\mathbb{K}}.$$

Since

$$T_1^{\text{isom}} \circ T_2 : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_K$$

is the composition of T_1 and T_2 in GTSh^\heartsuit and

$$(\mathcal{P}_{K,N} \circ T_1)^{\text{isom}} \circ (\mathcal{P}_{K,N} \circ T_2) : \text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4} / \sim_N$$

is the composition of $\mathcal{P}_{K,N} \circ T_1$ and $\mathcal{P}_{K,N} \circ T_2$ in GTSh^\heartsuit , our goal is to prove that

$$(3-10) \quad \mathcal{P}_{K,N} \circ (T_1^{\text{isom}} \circ T_2) = (\mathcal{P}_{K,N} \circ T_1)^{\text{isom}} \circ (\mathcal{P}_{K,N} \circ T_2).$$

Due to commutativity of (3-9), for $T = T_1$ we have

$$\mathcal{P}_{K,N} \circ T_1^{\text{isom}} \circ T_2 = (\mathcal{P}_{K,N} \circ T_1)^{\text{isom}} \circ \mathcal{P}_{K,N} \circ T_2.$$

Thus equation (3-10) indeed holds and we proved that $\mathcal{ML}_{K,N}$ is a group homomorphism.

Let us now consider isolated elements $N^{(1)} \leq N^{(2)} \leq N^{(3)}$ of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$. Since

$$\mathcal{P}_{N^{(1)},N^{(3)}} = \mathcal{P}_{N^{(2)},N^{(3)}} \circ \mathcal{P}_{N^{(1)},N^{(2)}},$$

we have

$$\mathcal{ML}_{N^{(2)},N^{(3)}} \circ \mathcal{ML}_{N^{(1)},N^{(2)}}(T_{m,f}) = \mathcal{P}_{N^{(2)},N^{(3)}} \circ \mathcal{P}_{N^{(1)},N^{(2)}} \circ T_{m,f} = \mathcal{P}_{N^{(1)},N^{(3)}} \circ T_{m,f} = \mathcal{ML}_{N^{(1)},N^{(3)}}(T_{m,f})$$

for every $[(m, f)] \in \text{GT}^\heartsuit(N^{(1)})$.

Thus the desired identity (3-7) holds and the proposition is proved. □

We call the functor⁹ \mathcal{ML} the *Main Line functor*.

In the next section, we will prove the following theorem:

Theorem 3.8 *The (profinite version) $\widehat{\text{GT}}$ of the Grothendieck–Teichmüller group is isomorphic to*

$$\text{lim}(\mathcal{ML}).$$

3.2 Proof of Theorem 3.8

We will need the following auxiliary statements:

Proposition 3.9 (A) *For every $N \in \text{NFI}(\text{PB}_3)$, there exists $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ satisfying the property*

$$K_{\text{PB}_3} \leq N.$$

(B) *For every $N \in \text{NFI}(\text{PB}_2)$ there exists $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ such that $K_{\text{PB}_2} \leq N$.*

⁹One of the authors of this paper is trying to live in the sequence of suburbs of Philadelphia called the Main Line. The functor \mathcal{ML} is named after this beautiful sequence of suburbs.

Proof Let $N \in \text{NFI}(\text{PB}_3)$ and ψ be a group homomorphism from PB_3 to S_n such that $\ker(\psi) = N$.

Using relations (A-3) on the generators of PB_4 , it is easy to show that the equations

$$\begin{aligned} \tilde{\psi}(x_{12}) &:= \psi(x_{12}), & \tilde{\psi}(x_{23}) &:= \psi(x_{23}), & \tilde{\psi}(x_{13}) &:= \psi(x_{13}), \\ \tilde{\psi}(x_{14}) &= \tilde{\psi}(x_{24}) = \tilde{\psi}(x_{34}) &:= \text{id}_{S_n}, \end{aligned}$$

define a group homomorphism $\tilde{\psi} : \text{PB}_4 \rightarrow S_n$.

Moreover, the kernel of $\tilde{\psi}$ satisfies the property

$$\varphi_{123}^{-1}(\ker(\tilde{\psi})) = N.$$

Hence

$$(3-11) \quad \varphi_{123}^{-1}(\ker(\tilde{\psi})) \cap \varphi_{12,3,4}^{-1}(\ker(\tilde{\psi})) \cap \varphi_{1,2,3,4}^{-1}(\ker(\tilde{\psi})) \cap \varphi_{1,2,3,4}^{-1}(\ker(\tilde{\psi})) \cap \varphi_{234}^{-1}(\ker(\tilde{\psi})) \leq N.$$

Let \tilde{N} be the normal subgroup of PB_4 obtained by intersecting all normal subgroups of PB_4 of index $|\text{PB}_4 : \ker(\tilde{\psi})|$. Since \tilde{N} is a characteristic subgroup of PB_4 of finite index (in PB_4), we have

$$\tilde{N} \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4).$$

Furthermore, due to Corollary 3.5, there exists an isolated element K of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ satisfying the property $K \leq \tilde{N}$. Combining $K \leq \tilde{N}$ with $\tilde{N} \leq \ker(\tilde{\psi})$ and (3-11), we deduce that

$$K_{\text{PB}_3} \leq N.$$

Thus the desired statement (A) is proved.

Just as for statement (A), we start with a group homomorphism $\kappa : \text{PB}_2 \rightarrow S_n$ whose kernel coincides with N .

It is easy to see that the equations

$$\tilde{\kappa}(x_{12}) := \kappa(x_{12}), \quad \tilde{\kappa}(x_{23}) := \kappa(x_{12})^{-1}, \quad \tilde{\kappa}(x_{13}) := \text{id}_{S_n}, \quad \tilde{\kappa}(x_{14}) = \tilde{\kappa}(x_{24}) = \tilde{\kappa}(x_{34}) := \text{id}_{S_n},$$

define a group homomorphism $\tilde{\kappa} : \text{PB}_4 \rightarrow S_n$.

The kernel of $\tilde{\kappa}$ satisfies the property

$$(3-12) \quad \varphi_{12}^{-1}(\varphi_{123}^{-1}(\ker(\tilde{\kappa}))) = N.$$

Let \tilde{N} be the normal subgroup of PB_4 obtained by intersecting all normal subgroups of PB_4 of index $|\text{PB}_4 : \ker(\tilde{\kappa})|$. Since \tilde{N} is a characteristic subgroup of PB_4 of finite index (in PB_4), we have

$$\tilde{N} \in \text{NFI}_{\text{PB}_4}(\mathbf{B}_4).$$

As above, there exists an isolated element K of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$ satisfying the property $K \leq \tilde{N}$. Combining $K \leq \tilde{N}$ with $\tilde{N} \leq \ker(\tilde{\kappa})$ and (3-12), we deduce that

$$K_{\text{PB}_2} \leq N.$$

Thus statement (B) is also proved. □

Proposition 3.9 allows us to produce a more practical description of $\widehat{\text{PaB}}^{\leq 4}$. To give this description, we note that the assignment $K \mapsto \text{PaB}^{\leq 4}/K$ upgrades to a functor from the poset $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ to the category of truncated operads in finite groupoids. Indeed, for every pair $K_1 \leq K_2$ of elements of $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ we have the obvious morphism of truncated operads

$$\mathcal{P}_{K_1, K_2}: \text{PaB}^{\leq 4}/\sim_{K_1} \rightarrow \text{PaB}^{\leq 4}/\sim_{K_2}.$$

Moreover, for every triple $K_1 \leq K_2 \leq K_3$ of elements of $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$, we have $\mathcal{P}_{K_2, K_3} \circ \mathcal{P}_{K_1, K_2} = \mathcal{P}_{K_1, K_3}$.

Let us denote by

$$(3-13) \quad \widetilde{\text{PaB}}^{\leq 4}$$

the limit of this functor.

More concretely, $\widetilde{\text{PaB}}(n)$ consists of functions

$$\gamma: \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4) \rightarrow \bigsqcup_{K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)} \text{PaB}(n)/\sim_K$$

satisfying these two conditions:

- For every $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$, $\gamma(K) \in \text{PaB}(n)/\sim_K$.
- For every pair $K_1 \leq K_2$ in $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$, $\mathcal{P}_{K_1, K_2}(\gamma(K_1)) = \gamma(K_2)$.

Since for every pair $K_1 \leq K_2$ of elements of $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ we have

$$\mathcal{P}_{K_1, K_2} \circ \widehat{\mathcal{P}}_{K_1} = \widehat{\mathcal{P}}_{K_2},$$

the assignment

$$\Psi(\widehat{\gamma})(K) := \widehat{\mathcal{P}}_K(\widehat{\gamma}) \quad \text{for } \widehat{\gamma} \in \widehat{\text{PaB}}(n)$$

defines a morphism of truncated operads

$$(3-14) \quad \Psi: \widehat{\text{PaB}}^{\leq 4} \rightarrow \widetilde{\text{PaB}}^{\leq 4}.$$

Let us prove:

Corollary 3.10 *The morphism Ψ in (3-14) is an isomorphism of truncated operads in the category of topological groupoids.*

Proof Since the compatibility with the structures of truncated operads and the composition of morphisms is obvious, it suffices to prove that Ψ is a homeomorphism of topological spaces.

Let τ, τ' be objects of $\text{PaB}(n)$ and $\widehat{\gamma}_1, \widehat{\gamma}_2 \in \text{Hom}_{\widehat{\text{PaB}}}(\tau, \tau')$ such that $\Psi(\widehat{\gamma}_1) = \Psi(\widehat{\gamma}_2)$ or equivalently, for every $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$,

$$\Psi(\widehat{\gamma}_2^{-1} \cdot \widehat{\gamma}_1)(K)$$

is the identity automorphism of τ in $\text{PaB}(n)/\sim_K$.

Thus, due to [Proposition 3.9](#), the image of $\widehat{\gamma}_2^{-1} \cdot \widehat{\gamma}_1$ in PB_n/N is the identity element for every $N \in \text{NFI}(\text{PB}_n)$. Therefore $\widehat{\gamma}_2^{-1} \cdot \widehat{\gamma}_1$ is the identity element of $\widehat{\text{PB}}_n$ and hence

$$\widehat{\gamma}_1 = \widehat{\gamma}_2.$$

We have proved that Ψ is one-to-one.

Let $\gamma \in \widehat{\text{PaB}}(n)$, and let τ and τ' be the source and the target of γ , respectively. Let λ be any isomorphism from τ to τ' in $\text{PaB}(n)$. By abuse of notation, we will use symbol λ for its obvious image in $\widehat{\text{PaB}}(n)$ and in $\widehat{\text{PaB}}(n)$.

Due to [Proposition 3.9](#), there exists an element $\widehat{h} \in \widehat{\text{PB}}_n$ such that

$$(3-15) \quad \widehat{\mathcal{P}}_K(\widehat{h}) = (\lambda^{-1} \cdot \gamma)(K) \quad \text{for all } K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4).$$

Equation (3-15) implies that $\Psi(\lambda \cdot \widehat{h}) = \gamma$. Thus we have proved that Ψ is onto.

Since, for every $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$, the composition of Ψ with the canonical projection

$$\widehat{\text{PaB}}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_K$$

coincides with the continuous map

$$\widehat{\mathcal{P}}_K: \widehat{\text{PaB}}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_K,$$

we conclude that Ψ is continuous.

Since Ψ is a continuous bijection from a compact space $\widehat{\text{PaB}}^{\leq 4}$ to a Hausdorff space $\widehat{\text{PaB}}^{\leq 4}$, Ψ is indeed a homeomorphism. □

Due to [Corollary 3.10](#), we can safely replace $\widehat{\text{PaB}}^{\leq 4}$ by $\widehat{\text{PaB}}^{\leq 4}$ in all further considerations. We will also use the same symbol \mathcal{I} (resp. $\widehat{\mathcal{P}}_K$ for $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$) for the canonical embedding $\mathcal{I}: \text{PaB}^{\leq 4} \rightarrow \widehat{\text{PaB}}^{\leq 4}$ and the canonical projection $\widehat{\mathcal{P}}_K: \widehat{\text{PaB}}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_K$.

Recall that, for every $\widehat{T} \in \widehat{\text{GT}}$ and $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$, the formula $T_K := \widehat{\mathcal{P}}_K \circ \widehat{T} \circ \mathcal{I}$ defines an onto morphism of truncated operads $\text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/K$. Since K is an isolated element of $\text{NFI}_{\text{PB}_4}(\mathbf{B}_4)$, [Corollary 2.13](#) implies that the onto morphism T_K factors as

$$(3-16) \quad T_K = T_K^{\text{isom}} \circ \mathcal{P}_K,$$

where T_K^{isom} is an isomorphism of truncated operads

$$T_K^{\text{isom}}: \text{PaB}^{\leq 4}/K \xrightarrow{\cong} \text{PaB}^{\leq 4}/K$$

and \mathcal{P}_K is the canonical projection $\text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/K$.

Proposition 3.11 *For every $\widehat{T} \in \widehat{\text{GT}}$ and for every $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$, the following diagram commutes:*

$$(3-17) \quad \begin{array}{ccc} \widehat{\text{PaB}}^{\leq 4} & \xrightarrow{\widehat{T}} & \widehat{\text{PaB}}^{\leq 4} \\ \widehat{\mathcal{P}}_K \downarrow & & \widehat{\mathcal{P}}_K \downarrow \\ \text{PaB}^{\leq 4}/\sim_K & \xrightarrow{T_K^{\text{isom}}} & \text{PaB}^{\leq 4}/\sim_K \end{array}$$

Proof By definition of T_K^{isom} , $\widehat{\mathcal{P}}_K \circ \widehat{T} \circ \mathcal{I}(\gamma) = T_K^{\text{isom}} \circ \mathcal{P}_K(\gamma)$ for every $\gamma \in \text{PaB}^{\leq 4}$.

Hence

$$(3-18) \quad \widehat{\mathcal{P}}_K \circ \widehat{T}(\mathcal{I}(\gamma)) = T_K^{\text{isom}} \circ \widehat{\mathcal{P}}_K(\mathcal{I}(\gamma)) \quad \text{for all } \gamma \in \text{PaB}^{\leq 4}.$$

Since the image $\mathcal{I}(\text{PaB}^{\leq 4})$ of $\text{PaB}^{\leq 4}$ in $\widetilde{\text{PaB}}^{\leq 4}$ is dense in $\widetilde{\text{PaB}}^{\leq 4}$ and the target $\text{PaB}^{\leq 4}/\sim_K$ of the compositions $\widehat{\mathcal{P}}_K \circ \widehat{T}$ and $T_K^{\text{isom}} \circ \mathcal{P}_K$ is Hausdorff, identity (3-18) implies that diagram (3-17) indeed commutes. □

Proof of Theorem 3.8 Let K and \widetilde{K} be elements of $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$ such that $\widetilde{K} \leq K$ and $\mathcal{P}_{\widetilde{K},K}$ be the canonical projection from $\text{PaB}^{\leq 4}/\sim_{\widetilde{K}}$ to $\text{PaB}^{\leq 4}/\sim_K$. Furthermore, let T_K and $T_{\widetilde{K}}$ be onto morphisms from $\text{PaB}^{\leq 4}$ to $\text{PaB}^{\leq 4}/K$ and $\text{PaB}^{\leq 4}/\widetilde{K}$, respectively, coming from $\widehat{T} \in \widehat{\text{GT}}$.

Since $\widehat{\mathcal{P}}_K = \widehat{\mathcal{P}}_{\widetilde{K},K} \circ \mathcal{P}_{\widetilde{K}}$, the diagram

$$\begin{array}{ccc} \text{PaB}^{\leq 4} & \xrightarrow{T_{\widetilde{K}}} & \text{PaB}^{\leq 4}/\sim_{\widetilde{K}} \\ & \searrow T_K & \downarrow \mathcal{P}_{\widetilde{K},K} \\ & & \text{PaB}^{\leq 4}/\sim_K \end{array}$$

commutes. Hence the assignment $\widehat{T} \mapsto \{T_K\}_{K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)}$ gives us a map

$$(3-19) \quad \widehat{\text{GT}} \rightarrow \lim(\mathcal{ML}).$$

Let us show that the map (3-19) is a group homomorphism.

Indeed, let $\widehat{T}^{(1)}, \widehat{T}^{(2)} \in \widehat{\text{GT}}$, put $\widehat{T} := \widehat{T}^{(1)} \circ \widehat{T}^{(2)}$ and let $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$. Using Proposition 3.11, we get

$$\widehat{\mathcal{P}}_K \circ \widehat{T} = \widehat{\mathcal{P}}_K \circ \widehat{T}^{(1)} \circ \widehat{T}^{(2)} = T_K^{(1), \text{isom}} \circ \widehat{\mathcal{P}}_K \circ \widehat{T}^{(2)} = T_K^{(1), \text{isom}} \circ T_K^{(2), \text{isom}} \circ \widehat{\mathcal{P}}_K.$$

On the other hand, $\widehat{\mathcal{P}}_K \circ \widehat{T} = T_K^{\text{isom}} \circ \widehat{\mathcal{P}}_K$ and hence

$$(3-20) \quad T_K^{\text{isom}} \circ \widehat{\mathcal{P}}_K = T_K^{(1), \text{isom}} \circ T_K^{(2), \text{isom}} \circ \widehat{\mathcal{P}}_K.$$

Since $\widehat{\mathcal{P}}_K: \widetilde{\text{PaB}}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_K$ is onto, identity (3-20) implies that

$$T_K^{\text{isom}} = T_K^{(1), \text{isom}} \circ T_K^{(2), \text{isom}}.$$

Thus the map (3-19) is indeed a group homomorphism.

Our next goal is to show that homomorphism (3-19) is one-to-one and onto.

To prove that (3-19) is one-to-one, we consider $\widehat{T} \in \widehat{\text{GT}}$ such that T_K coincides with the canonical projection

$$\text{PaB}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_K \quad \text{for every } K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4).$$

Hence, for every $\gamma \in \text{PaB}^{\leq 4}$, we have

$$\widehat{\mathcal{P}}_K \circ \widehat{T}(\mathcal{I}(\gamma)) = \widehat{\mathcal{P}}_K \circ \mathcal{I}(\gamma) \quad \text{for all } K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4).$$

This means that the restriction of \hat{T} to the subset $\mathcal{F}(\text{PaB}^{\leq 4}) \subset \widetilde{\text{PaB}}^{\leq 4}$ coincides with the restriction of the identity map $\text{id}: \widetilde{\text{PaB}}^{\leq 4} \rightarrow \widetilde{\text{PaB}}^{\leq 4}$ to the subset $\mathcal{F}(\text{PaB}^{\leq 4})$. Since the subset $\mathcal{F}(\text{PaB}^{\leq 4})$ is dense in $\widetilde{\text{PaB}}^{\leq 4}$ and the space $\widetilde{\text{PaB}}^{\leq 4}$ is Hausdorff, we conclude that \hat{T} is the identity map $\text{id}: \widetilde{\text{PaB}}^{\leq 4} \rightarrow \widetilde{\text{PaB}}^{\leq 4}$. Thus the injectivity of (3-19) is established.

Note that an element of $\text{lim}(\mathcal{ML})$ is a family $\{\mathcal{T}_K^{\text{isom}}\}_{K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)}$ of isomorphisms of truncated operads

$$\mathcal{T}_K^{\text{isom}}: \text{PaB}^{\leq 4}/\sim_K \xrightarrow{\cong} \text{PaB}^{\leq 4}/\sim_K$$

satisfying the following property: for every pair $K \leq \tilde{K}$ in $\text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$, the diagram

$$(3-21) \quad \begin{array}{ccc} \text{PaB}^{\leq 4}/\sim_K & \xrightarrow{\mathcal{T}_K^{\text{isom}}} & \text{PaB}^{\leq 4}/\sim_K \\ \mathcal{P}_{K, \tilde{K}} \downarrow & & \downarrow \mathcal{P}_{K, \tilde{K}} \\ \text{PaB}^{\leq 4}/\sim_{\tilde{K}} & \xrightarrow{\mathcal{T}_{\tilde{K}}^{\text{isom}}} & \text{PaB}^{\leq 4}/\sim_{\tilde{K}} \end{array}$$

commutes.

Due to commutativity of (3-21), the formula

$$(3-22) \quad \hat{T}(\gamma)(K) := \mathcal{T}_K^{\text{isom}}(\gamma(K))$$

defines a morphism of truncated operads in groupoids $\hat{T}: \widetilde{\text{PaB}}^{\leq 4} \rightarrow \widetilde{\text{PaB}}^{\leq 4}$.

To prove that \hat{T} is continuous, we need to show that the composition

$$\hat{\mathcal{P}}_K \circ \hat{T}: \widetilde{\text{PaB}}^{\leq 4} \rightarrow \text{PaB}^{\leq 4}/\sim_K$$

is continuous for every $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$.

By definition of \hat{T} in (3-22),

$$(3-23) \quad \hat{\mathcal{P}}_K \circ \hat{T} = \mathcal{T}_K^{\text{isom}} \circ \hat{\mathcal{P}}_K \quad \text{for every } K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4).$$

Since $\mathcal{T}_K^{\text{isom}}$ is an automorphism of the (finite) groupoid $\text{PaB}^{\leq 4}/\sim_K$ equipped with the discrete topology and $\hat{\mathcal{P}}_K$ is continuous, identity (3-23) implies that the composition $\hat{\mathcal{P}}_K \circ \hat{T}$ is indeed continuous.

Thus equation (3-22) defines a continuous endomorphism of the operad $\widetilde{\text{PaB}}^{\leq 4}$.

To find the inverse of \hat{T} , we denote by $\mathcal{G}_K^{\text{isom}}$ the inverse of $\mathcal{T}_K^{\text{isom}}$ for every $K \in \text{NFI}_{\text{PB}_4}^{\text{isolated}}(\mathbf{B}_4)$. Then it is easy to see that the formula

$$\hat{S}(\gamma)(K) := \mathcal{G}_K^{\text{isom}}(\gamma(K))$$

defines the inverse of \hat{T} .

The proof of surjectivity of (3-19) is complete. □

Let us consider $K, N \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ with $K \leq N$ and a pair $(m, f) \in \mathbb{Z} \times F_2$ that represents a GT-shadow in $\text{GT}^\heartsuit(K)$. Clearly, the same pair (m, f) also represents a GT-shadow in $\text{GT}^\heartsuit(N)$. In other words, if $K \leq N$, then we have a natural map

$$(3-24) \quad \text{GT}^\heartsuit(K) \rightarrow \text{GT}^\heartsuit(N).$$

It makes sense to consider this map even if neither K nor N are isolated.

Definition 3.12 We say that a GT-shadow $[(m, f)] \in \text{GT}^\heartsuit(N)$ *survives into* K if $[(m, f)]$ belongs to the image of the map (3-24). In other words, there exists $(m_1, f_1) \in \mathbb{Z} \times F_2$ such that $[(m_1, f_1)] \in \text{GT}^\heartsuit(K)$, $m_1 \cong m \pmod{N_{\text{ord}}}$ and $f_1 N_{F_2} = f N_{F_2}$.

The following statement is a straightforward consequence of Proposition 3.3 and Theorem 3.8.

Corollary 3.13 Let $N \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ and $[(m, f)] \in \text{GT}^\heartsuit(N)$. The GT-shadow $[(m, f)]$ is genuine if and only if $[(m, f)]$ survives into K for every $K \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ such that $K \leq N$. □

4 Selected results of computer experiments

In the computer implementation [4], an element N of $\text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ is represented by a group homomorphism ψ from PB_4 to a symmetric group such that $N = \ker(\psi)$. Each homomorphism $\psi : \text{PB}_4 \rightarrow S_d$ is, in turn, represented by a tuple of permutations

$$(4-1) \quad (g_{12}, g_{23}, g_{13}, g_{14}, g_{24}, g_{34}) \in (S_d)^6$$

satisfying the relations of PB_4 ; see (A-3).

It should be mentioned that, in [4], we consider only practical GT-shadows; see Remark 2.15. In particular, throughout this section, $\text{GT}(N)$ denotes the set of practical GT-shadows with the target N . Clearly, every charming GT-shadow is practical.

Table 1 presents basic information about 35 selected elements

$$(4-2) \quad N^{(0)}, N^{(1)}, \dots, N^{(34)} \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4).$$

For every $N^{(i)}$ in this list, the quotient $F_2/N_{F_2}^{(i)}$ is nonabelian. Table 1 also shows $N_{\text{ord}}^{(i)} := |\text{PB}_2 : N_{\text{PB}_2}^{(i)}|$, the size of $\text{GT}(N^{(i)})$ (ie the total number of practical GT-shadows with the target $N^{(i)}$) and the size of $\text{GT}^\heartsuit(N^{(i)})$. The last column indicates whether $N^{(i)}$ is isolated or not.

For every nonisolated element N in the list (4-2), the connected component $\text{GTSh}_{\text{conn}}^\heartsuit(N)$ has exactly two objects. More precisely,

- $N^{(4)}$ is a conjugate of $N^{(3)}$ and $N^{(3)} \cap N^{(4)} = N^{(14)}$,
- $N^{(11)}$ is a conjugate of $N^{(10)}$ and $N^{(10)} \cap N^{(11)} = N^{(24)}$,
- $N^{(17)}$ is a conjugate of $N^{(16)}$ and $N^{(16)} \cap N^{(17)} = N^{(30)}$,
- $N^{(27)}$ is a conjugate of $N^{(26)}$ and $N^{(26)} \cap N^{(27)} = N^{(34)}$.

For $N^{(31)}$, $GT(N^{(31)})$ has 588 elements. To find the size of $GT(N^{(31)})$, the computer had to look at $\approx 9 \cdot 10^6$ elements of the group $F_2/N_{F_2}^{(31)}$. For the iMac with the processor 3.4 GHz, Intel Core i5, it took over nine full days to complete this task.

For $N^{(32)}$, $GT(N^{(32)})$ has 800 elements. To find the size of $GT(N^{(32)})$, the computer had to look at over $9 \cdot 10^6$ elements of the group $F_2/N_{F_2}^{(32)}$. For the iMac with the processor 3.4 GHz, Intel Core i5, it took almost 10 full days to complete this task.

Remark 4.1 Recall that the definition of an isolated element of $NFI_{PB_4}(B_4)$ (see Definition 3.2) is based on charming GT-shadows. In principle, it is possible that there exists an isolated element $N \in NFI_{PB_4}(B_4)$ for which $GT(N)$ has a nonsettled element. We *did not* encounter such examples in our experiments.

4.1 Selected remarkable examples

For the 19th example $N^{(19)}$ in Table 1, the quotient $F_2/N_{F_2}^{(19)}$ has order $7776 = 2^5 \cdot 3^5$. Due to the similarity between this order and the historic year 1776, we decided to call the subgroup $N^{(19)}$ the *Philadelphia subgroup* of PB_4 . This subgroup is the kernel of the homomorphism from PB_4 to S_9 that sends the standard generators of PB_4 to the permutations

$$(4-3) \quad \begin{aligned} g_{12} &:= (1, 3, 2)(4, 6, 5), & g_{23} &:= (1, 4, 9)(2, 7, 6), & g_{13} &:= (1, 7, 5)(3, 6, 9), \\ g_{14} &:= (2, 6, 7)(3, 8, 5), & g_{24} &:= (1, 8, 6)(3, 4, 7), & g_{34} &:= (1, 2, 3)(7, 9, 8), \end{aligned}$$

respectively.

Since $N^{(19)}$ is isolated, $GT^\heartsuit(N^{(19)})$ is a group. We showed that $GT^\heartsuit(N^{(19)})$ is isomorphic to the dihedral group $D_6 = \langle r, s \mid r^6, s^2, rsrs \rangle$ of order 12. We also showed that the kernel of the restriction of the virtual cyclotomic character to $GT^\heartsuit(N^{(19)})$ coincides with the cyclic subgroup $\langle r \rangle$ of order 6.

The last element $N^{(34)}$ in (4-2) has the biggest index $762,048 = 2^6 \cdot 3^5 \cdot 7^2$ in PB_4 . This subgroup is the kernel of the homomorphism from PB_4 to S_{18} that sends the standard generators of PB_4 to

$$(4-4) \quad \begin{aligned} g_{12} &:= (1, 3, 5, 7, 9, 2, 4, 6, 8)(10, 12, 14, 16, 18, 11, 13, 15, 17), \\ g_{23} &:= (1, 3, 7, 8, 2, 4, 9, 6, 5)(10, 15, 17, 11, 12, 16, 18, 14, 13), \\ g_{13} &:= (1, 3, 8, 5, 4, 9, 2, 6, 7)(10, 11, 15, 17, 13, 12, 18, 14, 16), \\ g_{14} &:= (1, 3, 7, 8, 2, 4, 9, 6, 5)(10, 15, 17, 11, 12, 16, 18, 14, 13), \\ g_{24} &:= (1, 7, 6, 2, 4, 8, 9, 3, 5)(10, 15, 14, 11, 16, 18, 12, 13, 17), \\ g_{34} &:= (1, 3, 5, 7, 9, 2, 4, 6, 8)(10, 12, 14, 16, 18, 11, 13, 15, 17), \end{aligned}$$

respectively. We call this subgroup the *Mighty Dandy*.

Due to Proposition 3.3, the Mighty Dandy is an isolated element and hence $GT^\heartsuit(N^{(34)})$ is a group. This is what we showed about this group:

- $GT^\heartsuit(N^{(34)})$ has order $486 = 2 \cdot 3^5$.
- The kernel Ker_{34} of the restriction of the virtual cyclotomic character to $GT^\heartsuit(N^{(34)})$ is an abelian subgroup of order $81 = 3^4$; in fact, Ker_{34} is isomorphic to $\mathcal{L}_9 \times \mathcal{L}_9$.
- $GT^\heartsuit(N^{(34)})$ is isomorphic to the semidirect product

$$(\mathcal{L}_2 \times \mathcal{L}_3) \ltimes (\mathcal{L}_9 \times \mathcal{L}_9).$$

- The Sylow 3-subgroup Syl of $GT^\heartsuit(N^{(34)})$ is a nonabelian group of order $3^5 = 243$; Syl is a normal subgroup of $GT^\heartsuit(N^{(34)})$ and it is isomorphic to the semidirect product

$$\mathcal{L}_3 \ltimes (\mathcal{L}_9 \times \mathcal{L}_9).$$

Although every element N in the list (4-2) has the property $|F_2 : N_{F_2}| > |PB_4 : N|$, there are examples $N \in \text{NFI}_{PB_4}(B_4)$ for which $|PB_4 : N|$ is significantly bigger than the index $|F_2 : N_{F_2}|$.

One such example was suggested to us by Leila Schneps. *Leila's subgroup* $N^\mathcal{L}$ of PB_4 is the kernel of a homomorphism from PB_4 to S_{130} and it can be retrieved from one of the storage files in [4]. Here is what we know about $N^\mathcal{L}$:

- The index of $N^\mathcal{L}$ in PB_4 is $2^{29} \cdot 3^{12} = 285\,315\,214\,344\,192$.
- The index of $N^\mathcal{L}_{PB_3}$ in PB_3 is $2^{12} \cdot 3^6 = 2\,985\,984$.
- The index of $N^\mathcal{L}_{F_2}$ in F_2 is $2^{10} \cdot 3^5 = 248\,832$.
- $N^\mathcal{L}_{\text{ord}} = 12$.
- The order of the commutator subgroup of $F_2/N^\mathcal{L}_{F_2}$ is $2^6 \cdot 3^3 = 1728$.
- There are only $48 = 2^4 \cdot 3$ charming GT-shadows for $N^\mathcal{L}$.
- $N^\mathcal{L}$ is an *isolated* element of $\text{NFI}_{PB_4}(B_4)$ and hence $GT^\heartsuit(N^\mathcal{L})$ is a group.

We found that the group $GT^\heartsuit(N^\mathcal{L})$ is isomorphic to the semidirect product

$$(4-5) \quad \mathcal{L}_2 \ltimes (\mathcal{L}_2 \times \mathcal{L}_2 \times \mathcal{L}_2 \times \mathcal{L}_3),$$

where the nontrivial element of \mathcal{L}_2 acts on

$$(4-6) \quad \mathcal{L}_2 \times \mathcal{L}_2 \times \mathcal{L}_2 \times \mathcal{L}_3 = \langle a|a^2 \rangle \times \langle b|b^2 \rangle \times \langle c|c^2 \rangle \times \langle d|d^3 \rangle$$

by the automorphism

$$a \mapsto b, \quad b \mapsto a, \quad c \mapsto c, \quad d \mapsto d^{-1}.$$

The restriction of the virtual cyclotomic character to $GT^\heartsuit(N^\mathcal{L})$ gives us the group homomorphism

$$GT^\heartsuit(N^\mathcal{L}) \rightarrow (\mathbb{Z}/12\mathbb{Z})^\times,$$

and the kernel of this homomorphism is the subgroup of (4-6) generated by ab , c and d .

i	$ \text{PB}_4 : N^{(i)} $	$ F_2 : N_{F_2}^{(i)} $	$ [F_2/N_{F_2}^{(i)}, F_2/N_{F_2}^{(i)}] $	$N_{\text{ord}}^{(i)}$	$ \text{GT}(N^{(i)}) $	$ \text{GT}^\heartsuit(N^{(i)}) $	isolated?
0	8	16	2	4	4	4	True
1	8	16	2	4	8	4	True
2	12	36	4	3	18	6	True
3	21	63	7	3	36	12	False
4	21	63	7	3	36	12	False
5	24	288	8	6	72	12	True
6	24	144	4	6	72	12	True
7	48	144	4	6	72	12	True
8	60	1500	60	5	100	20	True
9	60	900	4	15	360	24	True
10	72	144	18	4	16	8	False
11	72	144	18	4	16	8	False
12	108	972	27	6	72	12	True
13	120	6000	60	10	400	40	True
14	147	441	49	3	216	72	True
15	168	8232	168	7	294	42	True
16	168	1344	168	4	64	32	False
17	168	1344	168	4	64	32	False
18	180	13 500	60	15	600	40	True
19	216	7776	216	6	72	12	True
20	240	6000	60	10	400	40	True
21	324	8748	108	9	486	54	True
22	504	40 824	504	9	486	54	True
23	504	24 696	504	7	294	42	True
24	648	1296	162	4	32	16	True
25	720	54 000	240	15	1800	120	True
26	1512	40 824	504	9	486	54	False
27	1512	40 824	504	9	486	54	False
28	2520	63 000	2520	5	200	40	True
29	2520	45 360	2520	6	144	48	True
30	28 224	225 792	28 224	4	512	256	True
31	181 440	8 890 560	181 440	7	588	84	True
32	181 440	9 072 000	181 440	10	800	160	True
33	181 440	40 824 000	181 440	15	\geq 1800	120	True
34	762 048	20 575 296	254 016	9	\geq 4374	486	True

Table 1: The basic information about selected 35 compatible equivalence relations.

4.2 Is there a charming GT-shadow that is also fake?

Table 1 shows that the set $\text{GT}^\heartsuit(N)$ of charming GT-shadows corresponding to a given $N \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ is typically a *proper subset* of $\text{GT}(N)$. For example, for the Philadelphia subgroup $N^{(19)}$, we have 72 GT-shadows and only 12 of them are charming.

Due to [Proposition 2.21](#), every noncharming GT–shadow is fake. Thus, for a typical N from our list of 35 elements of $\text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$, we have many examples of a fake GT–shadows. For instance, $\text{GT}(N^{(19)})$ contains at least 60 fake GT–shadows.

It is more challenging to find examples of charming GT–shadows that are fake. At the time of writing, we did *not* find a single example of a charming GT–shadow that is also fake.

Here is what we did. In the list [\(4-2\)](#), there are exactly 24 pairs $(N^{(i)}, N^{(j)})$ with $i \neq j$ such that

$$N^{(j)} \leq N^{(i)}.$$

For each such pair, we showed that every GT–shadow in $\text{GT}^\heartsuit(N^{(i)})$ survives into $N^{(j)}$, ie the natural map $\text{GT}^\heartsuit(N^{(j)}) \rightarrow \text{GT}^\heartsuit(N^{(i)})$ is onto. We also looked at other selected examples of elements $K \leq N$ in $\text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ in which N belongs to the list [\(4-2\)](#) and K is obtained by intersecting N with another element of [\(4-2\)](#). In all examples we have considered so far, the natural map $\text{GT}^\heartsuit(K) \rightarrow \text{GT}^\heartsuit(N)$ is onto.

4.3 Versions of the Furusho property and selected open questions

Two versions of the Furusho property are motivated by a remarkable theorem which says roughly that, in the prounipotent setting, the pentagon relation implies the hexagon relations. For a precise statement, we refer the reader to [\[2, Theorem 3.1\]](#) and [\[10, Theorem 1\]](#).

We say that an element $N \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ satisfies *the strong Furusho property* if:

Property 4.2 For every $fN_{F_2} \in F_2/N_{F_2}$ satisfying pentagon relation [\(2-20\)](#) modulo N , there exists $m \in \mathbb{Z}$ such that

- $2m + 1$ represents a unit in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$, and
- the pair (m, f) satisfies hexagon relations [\(2-18\)](#) and [\(2-19\)](#).

Furthermore, we say that an element $N \in \text{NFI}_{\text{PB}_4}(\mathbb{B}_4)$ satisfies *the weak Furusho property* if:

Property 4.3 For every $fN_{F_2} \in [F_2/N_{F_2}, F_2/N_{F_2}]$ satisfying pentagon relation [\(2-20\)](#) modulo N , there exists $m \in \mathbb{Z}$ such that

- $2m + 1$ represents a unit in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$, and
- the pair (m, f) satisfies hexagon relations [\(2-18\)](#) and [\(2-19\)](#).

Using [\[4\]](#), we showed that the following 11 elements of the list [\(4-2\)](#)

$$(4-7) \quad N^{(1)}, \quad N^{(2)}, \quad N^{(3)}, \quad N^{(4)}, \quad N^{(6)}, \quad N^{(7)}, \quad N^{(9)}, \quad N^{(10)}, \quad N^{(11)}, \quad N^{(14)}, \quad N^{(24)}$$

satisfy [Property 4.2](#) and the remaining 24 elements of [\(4-2\)](#) do *not* satisfy [Property 4.2](#).

For instance, for the Philadelphia subgroup $N^{(19)}$, $N_{\text{ord}}^{(19)} = 6$ and there are 216 elements $fN_{F_2}^{(19)}$ in $F_2/N_{F_2}^{(19)}$ that satisfy the pentagon relation modulo $N^{(19)}$. However, for only 36 of these 216 elements, there exists $m \in \{0, 1, \dots, 5\}$ such that $2m + 1$ represents a unit in $\mathbb{Z}/6\mathbb{Z}$ and the pair (m, f) satisfies hexagon relations (2-18) and (2-19) (modulo $N_{\text{PB}_3}^{(19)}$).

Using [4], we also showed that the 13 elements of the list (4-2)

$$(4-8) \quad N^{(0)}, N^{(1)}, N^{(2)}, N^{(3)}, N^{(4)}, N^{(5)}, N^{(6)}, N^{(7)}, N^{(9)}, N^{(10)}, N^{(11)}, N^{(14)}, N^{(24)}$$

satisfy Property 4.3, and the remaining 22 elements of (4-2) do *not* satisfy Property 4.3.

For instance, for the Mighty Dandy $N^{(34)}$, $N_{\text{ord}}^{(34)} = 9$ and there are 4096 elements¹⁰ in $[F_2/N_{F_2}^{(34)}, F_2/N_{F_2}^{(34)}]$ that satisfy the pentagon relation modulo $N^{(34)}$. However, for only 243 of them does there exist some $m \in \{0, 1, \dots, 8\}$ such that $2m + 1$ represents a unit in $\mathbb{Z}/9\mathbb{Z}$ and the pair (m, f) satisfies hexagon relations (2-18) and (2-19) (modulo $N_{\text{PB}_3}^{(34)}$).

We conclude this section with selected open questions. Most of these questions are motivated by our experiments [4].

Question 4.4 Let $N \in \text{NFI}_{\text{PB}_4}(B_4)$ and $(m, f) \in \mathbb{Z} \times F_2$ be a pair satisfying (2-18), (2-19), (2-20) (relative to \sim_N). Recall that, due to Proposition 2.10, if the group homomorphisms $T_{m,f}^{\text{PB}_2}$ and $T_{m,f}^{\text{PB}_3}$ are onto then so is the group homomorphism

$$T_{m,f}^{\text{PB}_4} : \text{PB}_4 \rightarrow \text{PB}_4/N.$$

Using [4], the authors could not find an example of a pair $(m, f) \in \mathbb{Z} \times F_2$ for which $T_{m,f}^{\text{PB}_4}$ is onto but $T_{m,f}^{\text{PB}_2}$ is not onto or $T_{m,f}^{\text{PB}_3}$ is not onto. Can one prove that, if $T_{m,f}^{\text{PB}_4}$ is onto, then so are the group homomorphisms $T_{m,f}^{\text{PB}_2}$ and $T_{m,f}^{\text{PB}_3}$?

Question 4.5 Is it possible to find an example of a nonisolated $N \in \text{NFI}_{\text{PB}_4}(B_4)$ for which the connected component $\text{GTSh}_{\text{conn}}^\heartsuit(N)$ has more than 2 objects? In other words, is it possible to find $N \in \text{NFI}_{\text{PB}_4}(B_4)$ that has > 2 distinct conjugates?

Question 4.6 Is it possible to find $K, N \in \text{NFI}_{\text{PB}_4}(B_4)$ such that $K \leq N$ and the natural map

$$\text{GT}^\heartsuit(K) \rightarrow \text{GT}^\heartsuit(N)$$

is **not** onto? In other words, can one produce an example of a charming GT-shadow that is also fake?

Question 4.7 Is it possible to find $N \in \text{NFI}_{\text{PB}_4}(B_4)$ for which F_2/N_{F_2} is **nonabelian** and we can identify all genuine GT-shadows in the set $\text{GT}^\heartsuit(N)$?

Note that, if F_2/N_{F_2} is abelian, all charming GT-shadows can be described completely and they are *all genuine*. (See Theorem B.2 in Appendix B.)

¹⁰For the iMac with the processor 3.4 GHz, Intel Core i5, it took over 52 hours to find all these elements.

Appendix A The operad PaB and its profinite completion

The operad PaB of parenthesized braids is an operad in the category of groupoids and it was introduced¹¹ by D Tamarkin in [28].

In this appendix, we give a brief reminder of the operad PaB and its profinite completion. For a more detailed exposition, we refer the reader to [9, Chapter 6].

A.1 The groups B_n and PB_n

The Artin braid group B_n on n strands is, by definition, the fundamental group of the orbifold

$$\text{Conf}(n, \mathbb{C})/S_n,$$

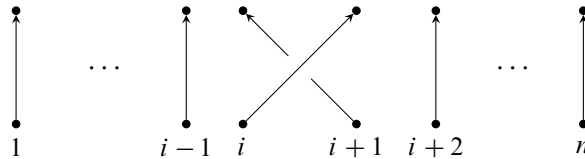
where $\text{Conf}(n, \mathbb{C})$ denotes the configuration space of n (labeled) points on \mathbb{C} , ie

$$\text{Conf}(n, \mathbb{C}) := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}.$$

It is known [17, Chapter 1] that B_n has the presentation

$$(A-1) \quad \langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1} \text{ if } |i - j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1} \text{ for } 1 \leq i \leq n - 2 \rangle,$$

where σ_i is the element depicted below:



Recall that the *pure braid group* PB_n on n strands is the kernel of the standard group homomorphism $\rho: B_n \rightarrow S_n$. This homomorphism sends the generator σ_i to the transposition $(i, i + 1)$.

For $1 \leq i < j \leq n$, we denote by x_{ij} the elements of PB_n given by

$$(A-2) \quad x_{ij} := \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1},$$

and recall [17, Section 1.3] that PB_n has the presentation

$$PB_n \cong \langle \{x_{ij}\}_{1 \leq i < j \leq n} \mid \text{relations} \rangle$$

with the relations

$$(A-3) \quad x_{rs}^{-1} x_{ij} x_{rs} = \begin{cases} x_{ij} & \text{if } s < i \text{ or } i < r < s < j, \\ x_{rj} x_{ij} x_{rj}^{-1} & \text{if } s = i, \\ x_{rj} x_{sj} x_{ij} x_{sj}^{-1} x_{rj}^{-1} & \text{if } r = i < s < j, \\ x_{rj} x_{sj} x_{rj}^{-1} x_{sj}^{-1} x_{ij} x_{sj} x_{rj} x_{sj}^{-1} x_{rj}^{-1} & \text{if } r < i < s < j. \end{cases}$$

¹¹A very similar construction appeared in the beautiful paper [1] by D Bar-Natan.

For example, the standard generators of PB_3 are

$$(A-4) \quad x_{12} := \sigma_1^2, \quad x_{23} := \sigma_2^2, \quad x_{13} := \sigma_2 \sigma_1^2 \sigma_2^{-1}.$$

The element

$$(A-5) \quad c := x_{23} x_{12} x_{13} = x_{12} x_{13} x_{23} = (\sigma_1 \sigma_2)^3 = (\sigma_2 \sigma_1)^3$$

has infinite order; it generates the center of PB_3 and the center of B_3 .

The elements x_{12} and x_{23} generate a free subgroup in PB_3 . Thus PB_3 is isomorphic to $F_2 \times \mathbb{Z}$.

A direction calculation shows that

$$(A-6) \quad \sigma_1^{-1} x_{23} \sigma_1 = x_{13}, \quad \sigma_2^{-1} x_{12} \sigma_2 = x_{23}^{-1} x_{12}^{-1} c, \quad \sigma_2^{-1} x_{13} \sigma_2 = x_{12}.$$

A.2 The groupoid $\text{PaB}(n)$

Objects of $\text{PaB}(n)$ are parenthesizations of sequences $(\tau(1), \tau(2), \dots, \tau(n))$ where τ is a permutation S_n .

For example, $\text{PaB}(2)$ has exactly two objects $(1\ 2)$ and $(2\ 1)$ and $\text{PaB}(3)$ has 12 objects:

$$(12)3, \quad (21)3, \quad (23)1, \quad (32)1, \quad (31)2, \quad (13)2, \quad 1(23), \quad 2(13), \quad 2(31), \quad 3(21), \quad 3(12), \quad 1(32).$$

To define morphisms in $\text{PaB}(n)$, we denote by \mathfrak{p} the obvious projection from the set of objects of $\text{PaB}(n)$ onto S_n . For example,

$$\mathfrak{p}((23)1) := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

For two objects τ_1 and τ_2 of $\text{PaB}(n)$, we set

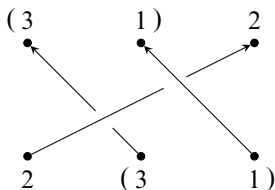
$$(A-7) \quad \text{Hom}_{\text{PaB}}(\tau_1, \tau_2) := \rho^{-1}(\mathfrak{p}(\tau_2)^{-1} \circ \mathfrak{p}(\tau_1)) \subset B_n,$$

where ρ is the standard homomorphism B_n to S_n .

For instance, $\text{Hom}_{\text{PaB}}(2(31), (31)2)$ consist of elements $g \in B_n$ such that

$$\rho(g) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

An example of an isomorphism from $2(31)$ to $(31)2$ in $\text{PaB}(3)$ is shown below:



The composition of morphisms in $\text{PaB}(n)$ comes from the multiplication in B_n . For example, if η is the element of $\text{Hom}_{\text{PaB}}(\tau_1, \tau_2)$ corresponding to $h \in B_n$ and γ is the element of $\text{Hom}_{\text{PaB}}(\tau_2, \tau_3)$ corresponding to $g \in B_n$ then their composition $\gamma \cdot \eta$ is the element of $\text{Hom}_{\text{PaB}}(\tau_1, \tau_3)$ corresponding to $g \cdot h$. We use \cdot for the composition of morphisms in PaB and the multiplication of elements in braid groups.

By definition of morphisms, we have a natural forgetful map

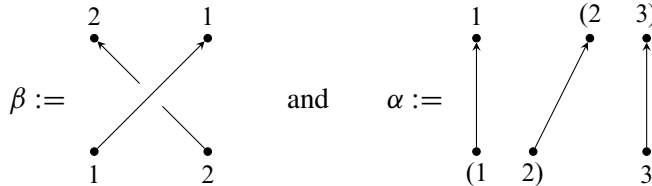
$$(A-8) \quad \text{ou}: \text{PaB}(n) \rightarrow B_n.$$

This map assigns to a morphism $\gamma \in \text{PaB}(n)$ the corresponding element of the braid group B_n . Moreover, since the composition of morphisms in $\text{PaB}(n)$ comes from the multiplication in B_n , we have

$$\text{ou}(\gamma \cdot \eta) = \text{ou}(\gamma) \cdot \text{ou}(\eta)$$

for every pair γ, η of composable morphisms.

The isomorphisms $\alpha \in \text{PaB}(3)$ and $\beta \in \text{PaB}(2)$ depicted as

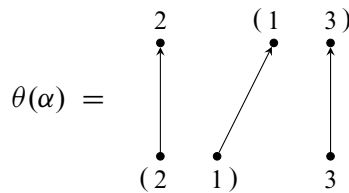


as in (1-2) play a very important role. We call β the *braiding* and α the *associator*. Note that, although α corresponds to the identity element in B_3 , it is not an identity morphism in $\text{PaB}(3)$ because $(12)3 \neq 1(23)$.

The symmetric group S_n acts on $\text{Ob}(\text{PaB}(n))$ in the obvious way. Moreover, for every $\theta \in S_n$ and $\gamma \in \text{Hom}_{\text{PaB}(n)}(\tau_1, \tau_2)$, we denote by $\theta(\gamma)$ the morphism from $\theta(\tau_1)$ to $\theta(\tau_2)$ that corresponds to the same element of the braid group B_n , ie

$$(A-9) \quad \text{ou}(\theta(\gamma)) = \text{ou}(\gamma).$$

For example, if $\theta = (1, 2) \in S_3$ then



For our purposes, it is convenient to assign to every element $g \in B_n$ the corresponding morphism $m(g) \in \text{PaB}(n)$ from $(\dots(1, 2)3) \dots n$ to $(\dots(i_1, i_2)i_3) \dots i_n$, where $i_k := \rho(g)^{-1}(k)$. It is easy to see that the map

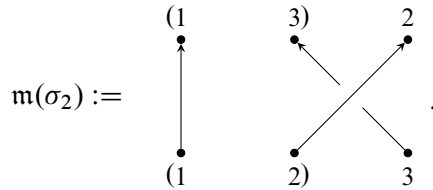
$$(A-10) \quad m: B_n \rightarrow \text{PaB}(n)$$

defined in this way is a right inverse of ou ; see (A-8).

It is also easy to see that, for every pair $g_1, g_2 \in B_n$, we have

$$(A-11) \quad m(g_1 \cdot g_2) = \rho(g_2)^{-1}(m(g_1)) \cdot m(g_2).$$

For example, for $\sigma_1, \sigma_2 \in B_3$, $m(\sigma_1) = \text{id}_{12} \circ_1 \beta$ and



The composition $m(\sigma_2) \cdot m(\sigma_1)$ is not defined because the source of $m(\sigma_2)$ does not coincide with the target of $m(\sigma_1)$. On the other hand, the source of $(1, 2)(m(\sigma_2))$ coincides with the target of $m(\sigma_1)$ and $(1, 2)(m(\sigma_2)) \cdot m(\sigma_1) = m(\sigma_2 \cdot \sigma_1)$.

A.3 The operad structure on PaB

We already explained how the symmetric group S_n acts on the groupoid $\text{PaB}(n)$. Furthermore, it is easy to see that $\{\text{Ob}(\text{PaB}(n))\}_{n \geq 1}$ is the underlying collection of the free operad (in the category of sets) generated by the collection \mathbb{T} with

$$\mathbb{T}(n) := \begin{cases} \{1\,2, 2\,1\} & \text{if } n = 2, \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus the functors

$$(A-12) \quad \circ_i : \text{PaB}(n) \times \text{PaB}(m) \rightarrow \text{PaB}(n + m - 1)$$

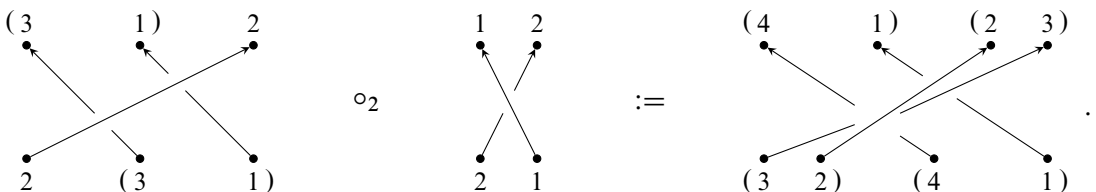
act on the level of objects in the obvious way.

For example,

$$(23)1 \circ_2 12 := ((23)4)1, \quad 21 \circ_1 (23)1 := 4((23)1), \quad 2(3(14)) \circ_3 1(32) := 2((3(54))(16)),$$

where we use the gray color to indicate what happens with the inserted sequence. For instance, in the third example, $1(32) \mapsto (3(54))$.

To define the action of the functor \circ_i on the level of morphisms, we proceed as follows: given $\gamma \in \text{PaB}(n)$, $\tilde{\gamma} \in \text{PaB}(m)$ and $1 \leq i \leq n$, we set $g := \text{ou}(\gamma)$ and $\tilde{g} := \text{ou}(\tilde{\gamma})$; we compute the source and the target of $\gamma \circ_i \tilde{\gamma}$ using the rules of operad $\{\text{Ob}(\text{PaB}(k))\}_{k \geq 1}$. Finally, to get the element of B_{n+m-1} corresponding to $\gamma \circ_i \tilde{\gamma}$, we replace the strand of g that originates at the position labeled by i by a “thin” version of \tilde{g} . For example,



For a more precise definition of operadic multiplications on PaB we refer the reader to [9, Chapter 6].

The (iso)morphisms α and β satisfy the *pentagon relation*

$$(A-13) \quad \begin{array}{ccc} & (1(23))4 & \xrightarrow{\alpha \circ_2 \text{id}_{12}} 1((23)4) \\ \text{id}_{12} \circ_1 \alpha \nearrow & & \searrow \text{id}_{12} \circ_2 \alpha \\ ((12)3)4 & & 1(2(34)) \\ & \searrow \alpha \circ_1 \text{id}_{12} & \nearrow \alpha \circ_3 \text{id}_{12} \\ & (12)(34) & \end{array}$$

and the two *hexagon relations*

$$(A-14) \quad \begin{array}{ccccc} (12)3 & \xrightarrow{\beta \circ_1 \text{id}_{12}} & 3(12) & \xleftarrow{(1,3,2)\alpha} & (31)2 \\ \alpha \downarrow & & & & \uparrow (2,3)(\text{id}_{12} \circ_1 \beta) \\ 1(23) & \xrightarrow{\text{id}_{12} \circ_2 \beta} & 1(32) & \xrightarrow{(2,3)\alpha^{-1}} & (13)2 \end{array}$$

$$(A-15) \quad \begin{array}{ccccc} 1(23) & \xrightarrow{\beta \circ_2 \text{id}_{12}} & (23)1 & \xrightarrow{(1,2,3)\alpha} & 2(31) \\ \alpha^{-1} \downarrow & & & & \uparrow (1,2)(\text{id}_{12} \circ_2 \beta) \\ (12)3 & \xrightarrow{\text{id}_{12} \circ_1 \beta} & (21)3 & \xrightarrow{(1,2)\alpha} & 2(13) \end{array}$$

It is known [9, Theorem 6.2.4] that:¹²

Theorem A.1 *As the operad in the category of groupoids, PaB is generated by morphisms α and β of (1-2). Moreover, any relation on α and β in PaB is a consequence of (A-13), (A-14) and (A-15).*

A.4 The cosimplicial homomorphisms for pure braid groups in arities 2, 3, 4

The collection $\{\text{PB}_n\}_{n \geq 1}$ of pure braid groups can be equipped with the structure of a cosimplicial group. For our purposes we will need the cofaces of this cosimplicial structure only in arities 2, 3 and 4.

Let τ_1 and τ_2 be objects of $\text{PaB}(n)$ which differ only by parenthesizations, ie $p(\tau_1) = p(\tau_2)$. For such objects, we denote by $\alpha_{\tau_1}^{\tau_2}$ the isomorphism from τ_1 to τ_2 given by the identity element of B_n . For example, the associator α is precisely $\alpha_{(12)3}^{1(23)}$ and α^{-1} is precisely $\alpha_{1(23)}^{(12)3}$.

Using the identity morphism $\text{id}_{12} \in \text{PaB}(2)$, the maps ou , m (see (A-8) and (A-10)) and the operadic insertions, we define the following maps from PB_3 to PB_4 and the maps from PB_2 to PB_3 :

$$(A-16) \quad \begin{aligned} \varphi_{123}(h) &:= \text{ou}(\text{id}_{12} \circ_1 \text{m}(h)), & \varphi_{12,3,4}(h) &:= \text{ou}(\text{m}(h) \circ_1 \text{id}_{12}), \\ \varphi_{1,2,34}(h) &:= \text{ou}(\text{m}(h) \circ_3 \text{id}_{12}), & \varphi_{234}(h) &:= \text{ou}(\text{id}_{12} \circ_2 \text{m}(h)), \\ \varphi_{1,23,4}(h) &:= \text{ou}(\text{m}(h) \circ_2 \text{id}_{12}), \end{aligned}$$

¹²A very similar statement is proved in [1]. See Claim 2.6 in loc. cit. It goes without saying that Theorem A.1 can be thought of as a version of Mac Lane’s coherence theorem for braided monoidal categories.

$$(A-17) \quad \begin{aligned} \varphi_{12}(h) &:= \text{ou}(\text{id}_{12} \circ_1 \mathfrak{m}(h)), & \varphi_{23}(h) &:= \text{ou}(\text{id}_{12} \circ_2 \mathfrak{m}(h)), \\ \varphi_{12,3}(h) &:= \text{ou}(\mathfrak{m}(h) \circ_1 \text{id}_{12}), & \varphi_{1,23}(h) &:= \text{ou}(\mathfrak{m}(h) \circ_2 \text{id}_{12}). \end{aligned}$$

Proposition A.2 *The equations in (A-16) (resp. in (A-17)) define group homomorphisms from PB_3 (resp. PB_2) to PB_4 (resp. PB_3).*

Proof Let us consider the map $\varphi_{1,23,4}: \text{PB}_3 \rightarrow \text{PB}_4$. For elements $h, \tilde{h} \in \text{PB}_3$, we set

$$\gamma := \mathfrak{m}(h) \quad \text{and} \quad \tilde{\gamma} := \mathfrak{m}(\tilde{h}).$$

Since PaB is an operad in the category of groupoids, we have

$$(\gamma \cdot \tilde{\gamma}) \circ_2 \text{id}_{12} = (\gamma \circ_2 \text{id}_{12}) \cdot (\tilde{\gamma} \circ_2 \text{id}_{12}).$$

Hence

$$\begin{aligned} \varphi_{1,23,4}(h) \cdot \varphi_{1,23,4}(\tilde{h}) &= \text{ou}(\gamma \circ_2 \text{id}_{12}) \cdot \text{ou}(\tilde{\gamma} \circ_2 \text{id}_{12}) = \text{ou}((\gamma \circ_2 \text{id}_{12}) \cdot (\tilde{\gamma} \circ_2 \text{id}_{12})) \\ &= \text{ou}((\gamma \cdot \tilde{\gamma}) \circ_2 \text{id}_{12}) = \varphi_{1,23,4}(h \cdot \tilde{h}), \end{aligned}$$

where the last identity is a consequence of $\gamma \cdot \tilde{\gamma} = \mathfrak{m}(h \cdot \tilde{h})$.

The proofs for the remaining eight maps are very similar and we leave it to the reader. □

Since all nine maps in (A-16) and (A-17) are group homomorphisms, they are uniquely determined by their values on generators of PB_3 and PB_4 , respectively. It is easy to see that

$$(A-18) \quad \begin{aligned} \varphi_{123}(x_{12}) &= x_{12}, & \varphi_{123}(x_{23}) &= x_{23}, & \varphi_{123}(x_{13}) &= x_{13}, \\ \varphi_{234}(x_{12}) &= x_{23}, & \varphi_{234}(x_{23}) &= x_{34}, & \varphi_{234}(x_{13}) &= x_{24}, \\ \varphi_{12,3,4}(x_{12}) &= x_{13}x_{23}, & \varphi_{12,3,4}(x_{23}) &= x_{34}, & \varphi_{12,3,4}(x_{13}) &= x_{14}x_{24}, \end{aligned}$$

$$(A-19) \quad \begin{aligned} \varphi_{1,23,4}(x_{12}) &= x_{12}x_{13}, & \varphi_{1,23,4}(x_{23}) &= x_{24}x_{34}, & \varphi_{1,23,4}(x_{13}) &= x_{14}, \\ \varphi_{1,2,34}(x_{12}) &= x_{12}, & \varphi_{1,2,34}(x_{23}) &= x_{23}x_{24}, & \varphi_{1,2,34}(x_{13}) &= x_{13}x_{14}, \\ \varphi_{12}(x_{12}) &= x_{12}, & \varphi_{23}(x_{12}) &= x_{23}, & \varphi_{12,3}(x_{12}) &= x_{13}x_{23}, & \varphi_{1,23}(x_{12}) &= x_{12}x_{13}. \end{aligned}$$

A.5 The profinite completion $\widehat{\text{PaB}}$ of PaB

Let \mathcal{G} be a connected groupoid with finitely many objects and G be the group that represents the isomorphism class of $\text{Aut}(a)$ for some object a of \mathcal{G} . We tacitly assume that the group G is residually finite. Following [5], an equivalence relation \sim on \mathcal{G} is called *compatible* if:

- (1) $\gamma_1 \sim \gamma_2$ implies the source (resp. the target) of γ_1 coincides with the source (resp. the target) of γ_2 .
- (2) $\gamma_1 \sim \gamma_2$ implies $\gamma_1 \cdot \gamma \sim \gamma_2 \cdot \gamma$ and $\tau \cdot \gamma_1 \sim \tau \cdot \gamma_2$ (if the compositions are defined).
- (3) The set \mathcal{G}/\sim of equivalence classes is finite.

It is clear that, for every compatible equivalence relation \sim on \mathcal{G} , the quotient \mathcal{G}/\sim is naturally a finite groupoid (with the same set of objects).

Compatible equivalence relations on \mathcal{G} form a directed poset and the assignment $\sim \mapsto \mathcal{G}/\sim$ gives us a functor from this poset to the category of finite groupoids. In [5], the profinite completion $\widehat{\mathcal{G}}$ of the groupoid \mathcal{G} is defined as the limit of this functor.

In [5], it was also shown that compatible equivalence relations on \mathcal{G} are in bijection with finite-index normal subgroups N of G . This gives us the following “pedestrian” way of thinking about morphisms in $\widehat{\mathcal{G}}(a, b)$: choose¹³ $\lambda \in \mathcal{G}(a, b)$, then every morphism in $\gamma \in \widehat{\mathcal{G}}(a, b)$ can be uniquely written as

$$\gamma = \lambda \cdot h,$$

where $h \in \widehat{G}$.

In [5], we also proved that the assignment $\mathcal{G} \mapsto \widehat{\mathcal{G}}$ upgrades to a functor from the category of groupoids to the category of topological groupoids. Moreover, this is a symmetric monoidal functor.

Thus, “putting hats” over $\text{PaB}(n)$ for every $n \geq 0$ gives us an operad $\widehat{\text{PaB}}$ in the category of topological groupoids.

Appendix B Charming GT–shadows in the abelian setting: examples of genuine GT–shadows

Let us prove the following statement:

Proposition B.1 *For $N \in \text{NF}_{\text{PB}_4}(\mathbb{B}_4)$, the following conditions are equivalent:*

- (a) *The quotient group PB_4/N is abelian.*
- (b) *The quotient group $\text{PB}_3/N_{\text{PB}_3}$ is abelian.*
- (c) *The quotient group F_2/N_{F_2} is abelian.*

Proof Implications (a) \implies (b) and (b) \implies (c) are straightforward, so we leave them to the reader.

Let us assume that the quotient group F_2/N_{F_2} is abelian. Then the images of x_{12} and x_{23} in $\text{PB}_3/N_{\text{PB}_3}$ commute. Furthermore, since the image of c in $\text{PB}_3/N_{\text{PB}_3}$ is obviously in the center of $\text{PB}_3/N_{\text{PB}_3}$ and $\text{PB}_3 = \langle x_{12}, x_{23}, c \rangle$, we conclude that the quotient group $\text{PB}_3/N_{\text{PB}_3}$ is also abelian.

To show that the generators $\bar{x}_{ij} := x_{ij}N$ for $1 \leq i < j \leq 4$ of PB_4/N commute with each other, we consider the group homomorphisms from PB_3 to PB_4 given by formulas (A-18).

Note that, for every homomorphism $\varphi: \text{PB}_3 \rightarrow \text{PB}_4$ in the set

$$(B-1) \quad \{\varphi_{234}, \varphi_{12,3,4}, \varphi_{1,23,4}, \varphi_{1,2,34}, \varphi_{234}\},$$

we have $N_{\text{PB}_3} \leq \varphi^{-1}(N) \leq \text{PB}_3$. Therefore, since the quotient $\text{PB}_3/N_{\text{PB}_3}$ is abelian, the quotient $\text{PB}_3/\varphi^{-1}(N)$ is also abelian.

¹³ $\mathcal{G}(a, b)$ is nonempty because \mathcal{G} is connected.

Applying these observations to every φ in (B-1), we deduce that:

- The elements $\bar{x}_{12}, \bar{x}_{23}, \bar{x}_{13}$ commute with each other.
- The elements $\bar{x}_{23}, \bar{x}_{34}, \bar{x}_{24}$ commute with each other.
- The elements $\bar{x}_{13}\bar{x}_{23}, \bar{x}_{34}$ and $\bar{x}_{14}\bar{x}_{24}$ commute with each other.
- The elements $\bar{x}_{12}, \bar{x}_{23}\bar{x}_{24}$ and $\bar{x}_{13}\bar{x}_{14}$ commute with each other.
- The elements $\bar{x}_{14}, \bar{x}_{12}\bar{x}_{13}$ and $\bar{x}_{24}\bar{x}_{34}$ commute with each other.

Using these observations one can show that $[\bar{x}_{ij}, \bar{x}_{kl}] = 1_{\text{PB}_4/\text{N}}$ for every pair in the set

$$\{(i, j), (k, l) \mid 1 \leq i < j \leq 4, 1 \leq k < l \leq 4\} - \{(1, 2), (3, 4)\}, \{(1, 3), (2, 4)\}, \{(2, 3), (1, 4)\}.$$

Luckily, due to (A-3), we have

$$x_{12}x_{34} = x_{34}x_{12}, \quad x_{23}x_{14} = x_{14}x_{23}, \quad x_{13}^{-1}x_{24}x_{13} = [x_{14}, x_{34}]x_{24}[x_{14}, x_{34}]^{-1}.$$

Thus all generators \bar{x}_{ij} of PB_4/N commute with each other. □

If one of the three equivalent conditions of Proposition B.1 is satisfied then we say that we are in the *abelian setting*.

We can now prove the following analog of the Kronecker–Weber theorem:

Theorem B.2 *Let $\text{N} \in \text{NFI}_{\text{PB}_4}(\text{B}_4)$. If the quotient group PB_4/N is abelian, then*

$$(B-2) \quad \text{GT}^\heartsuit(\text{N}) = \{(m + N_{\text{ord}}\mathbb{Z}, \bar{1}) \mid 0 \leq m \leq N_{\text{ord}} - 1, \text{gcd}(2m + 1, N_{\text{ord}}) = 1\},$$

where $\bar{1}$ is the identity element of $\text{F}_2/\text{N}_{\text{F}_2}$. Furthermore, every GT-shadow in (B-2) is genuine.

Proof Since $\bar{1}$ can be represented by the identity element of F_2 , every element of the set

$$(B-3) \quad X_{\text{N}} := \{(m + N_{\text{ord}}\mathbb{Z}, \bar{1}) \mid 0 \leq m \leq N_{\text{ord}} - 1, \text{gcd}(2m + 1, N_{\text{ord}}) = 1\}$$

satisfies the pentagon relation (2-20).

For every element of X_{N} , the hexagon relations (2-18) and (2-19) boil down to

$$(B-4) \quad \sigma_1 x_{12}^m \sigma_2 x_{23}^m \text{N}_{\text{PB}_3} = \sigma_1 \sigma_2 (x_{13} x_{23})^m \text{N}_{\text{PB}_3},$$

$$(B-5) \quad \sigma_2 x_{23}^m \sigma_1 x_{12}^m \text{N}_{\text{PB}_3} = \sigma_2 \sigma_1 (x_{12} x_{13})^m \text{N}_{\text{PB}_3}.$$

Equation (B-4) follows easily from the identity

$$\sigma_2^{-1} x_{12} \sigma_2 = x_{23}^{-1} x_{13} x_{23}$$

and the fact that the quotient $\text{PB}_3/\text{N}_{\text{PB}_3}$ is abelian.

Similarly, equation (B-5) follows easily from the identity

$$\sigma_1^{-1}x_{23}\sigma_1 = x_{13}$$

and the fact that the quotient $\text{PB}_3/\text{N}_{\text{PB}_3}$ is abelian.

We proved that every element of X_N is a GT-pair for N . Moreover, since $2m + 1$ represents a unit in the ring $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$, every GT-pair in X_N is friendly, ie the group homomorphism $T_{m,1}^{\text{PB}_2} : \text{PB}_2 \rightarrow \text{PB}_2/\text{N}_{\text{PB}_2}$ is onto.

Due to (2-28) and the second identity in (2-29), for every $m \in \mathbb{Z}$ we have

$$T_{m,1}^{\text{PB}_3}(x_{12}) = x_{12}^{2m+1}\text{N}_{\text{PB}_3}, \quad T_{m,1}^{\text{PB}_3}(x_{23}) = x_{23}^{2m+1}\text{N}_{\text{PB}_3}, \quad T_{m,1}^{\text{PB}_3}(c) = c^{2m+1}\text{N}_{\text{PB}_3}.$$

Since the orders of the elements $x_{12}\text{N}_{\text{PB}_3}$, $x_{23}\text{N}_{\text{PB}_3}$ and $c\text{N}_{\text{PB}_3}$ divide N_{ord} and $2m + 1$ represents a unit in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$, all three cosets $x_{12}\text{N}_{\text{PB}_3}$, $x_{23}\text{N}_{\text{PB}_3}$ and $c\text{N}_{\text{PB}_3}$ belong to the image of $T_{m,1}^{\text{PB}_3}$. Thus, due to Proposition 2.10, every element of X_N is a GT-shadow.

Furthermore, every GT-shadow in X_N is charming. The first condition of Definition 2.20 is clearly satisfied and the second one follows from the fact that $2m + 1$ represents a unit in $\mathbb{Z}/N_{\text{ord}}\mathbb{Z}$ and the orders of the elements $x_{12}\text{N}_{\text{F}_2}$, $x_{23}\text{N}_{\text{F}_2}$ divide N_{ord} .

Since the inclusion $\text{GT}^\heartsuit(N) \subset X_N$ is obvious, the first statement of Theorem B.2 is proved.

Let us now show that every GT-shadow in $\text{GT}^\heartsuit(N)$ is genuine.

By Remark 2.18 and the surjectivity of the cyclotomic character, we know that, for every $\bar{\lambda} \in (\mathbb{Z}/N_{\text{ord}}\mathbb{Z})^\times$ there should exist at least one genuine GT-shadow $[(m, f)] \in \text{GT}^\heartsuit(N)$ such that

$$(B-6) \quad 2\bar{m} + \bar{1} = \bar{\lambda}.$$

Let us assume that N_{ord} is odd. In this case $\bar{2} \in (\mathbb{Z}/N_{\text{ord}}\mathbb{Z})^\times$ and hence, for every fixed $\bar{\lambda} \in (\mathbb{Z}/N_{\text{ord}}\mathbb{Z})^\times$, equation (B-6) has exactly one solution $\bar{m} \in \mathbb{Z}/N_{\text{ord}}\mathbb{Z}$.

Since, for every $\bar{\lambda} \in (\mathbb{Z}/N_{\text{ord}}\mathbb{Z})^\times$, we have exactly one GT-shadow $(\bar{m}, \bar{1})$ in $\text{GT}^\heartsuit(N)$ such that $2\bar{m} + \bar{1} = \bar{\lambda}$, the surjectivity of the cyclotomic character implies that every GT-shadow in $\text{GT}^\heartsuit(N)$ is genuine.

The case when $N_{\text{ord}} = 2k$ (for $k \in \mathbb{Z}_{\geq 1}$) requires more work. In this case, equation (B-6) has exactly two solutions for every $\bar{\lambda} \in (\mathbb{Z}/2k\mathbb{Z})^\times$. More precisely, if $2\bar{m} + \bar{1} = \bar{\lambda}$ then the solution set for (B-6) is $\{\bar{m}, \bar{m} + \bar{k}\}$.

The proof of the desired statement about $\text{GT}^\heartsuit(N)$ is based on the fact that the integers $2m + 1$ and $2m + 2k + 1$ represent two distinct units in the ring $\mathbb{Z}/4k\mathbb{Z}$.

Let K be an element of $\text{NFI}_{\text{PB}_4}(\text{B}_4)$ such that

- $K \leq N$,
- PB_4/K is abelian, and
- $4k$ divides $K_0 := |\text{PB}_2 : K_{\text{PB}_2}|$.

One possible way to construct such K is to define a group homomorphism $\psi : \text{PB}_4 \rightarrow S_{4k}$ by the formulas

$$(B-7) \quad \psi(x_{ij}) := (1, 2, \dots, 4k) \quad \text{for all } 1 \leq i < j \leq 4,$$

and set $K := N \cap \ker(\psi)$.

Since the natural group homomorphism

$$(\mathbb{Z}/K_0\mathbb{Z})^\times \rightarrow (\mathbb{Z}/4k\mathbb{Z})^\times$$

is onto, there exist $\bar{\lambda}_1 \neq \bar{\lambda}_2$ in $(\mathbb{Z}/K_0\mathbb{Z})^\times$ whose images in $(\mathbb{Z}/4k\mathbb{Z})^\times$ are the two distinct units represented by $2m+1$ and $2m+2k+1$, respectively.

Therefore there exist genuine GT-shadows $[(m_1, 1)]$ and $[(m_2, 1)]$ in $\text{GT}^\heartsuit(K)$ such that

$$2m_1 + 1 \equiv \lambda_1 \pmod{K_0} \quad \text{and} \quad 2m_2 + 1 \equiv \lambda_2 \pmod{K_0}.$$

Consequently, m_1 and m_2 satisfy these congruences mod $4k$:

$$2m_1 + 1 \equiv 2m + 1 \pmod{4k} \quad \text{and} \quad 2m_2 + 1 \equiv 2m + 2k + 1 \pmod{4k}.$$

Thus the images of the genuine GT-shadows $[(m_1, 1)]$ and $[(m_2, 1)]$ in $\text{GT}(N)$ are $[(m, 1)]$ and $[(m+k, 1)]$. \square

Remark B.3 In the abelian setting, every charming GT-shadow comes from an element of $G_{\mathbb{Q}}$. The authors *do not know* whether there is a genuine GT-shadow (in the nonabelian setting) that does not come from an element of $G_{\mathbb{Q}}$. Of course, if such a GT-shadow exists then the homomorphism (1-1) is not onto. (Some mathematicians believe that, in modern mathematics, there are no tools for tackling this question.)

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