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Turning vector bundles

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We define a turning of a rank-$2k$ vector bundle $E \to B$ to be a homotopy of bundle automorphisms ψ_t from $\mathbb{1}_E$, the identity of E, to $-\mathbb{1}_E$, minus the identity, and call a pair (E, ψ_t) a turned bundle. We investigate when vector bundles admit turnings and develop the theory of turnings and their obstructions. In particular, we determine which rank-$2k$ bundles over the $2k$–sphere are turnable.

If a bundle is turnable, then it is orientable. In the other direction, complex bundles are turned bundles and for bundles over finite CW–complexes with rank in the stable range, Bott's proof of his periodicity theorem shows that a turning of E defines a homotopy class of complex structure on E. On the other hand, we give examples of rank-$2k$ bundles over $2k$–dimensional spaces, including the tangent bundles of some $2k$–manifolds, which are turnable but do not admit a complex structure. Hence turned bundles can be viewed as generalisations of complex bundles.

We also generalise the definition of turning to other settings, including other paths of automorphisms, and we relate the generalised turnability of vector bundles to the topology of their gauge groups and the computation of certain Samelson products.

57R22; 55R15, 55R25

1 Introduction

Let $\pi : E \to B$ be a real Euclidean vector bundle over a base space B, which for simplicity we assume is connected. The bundle E has two canonical automorphisms: $\mathbb{1}_E$, the identity, and $-\mathbb{1}_E$, the automorphism which takes a vector to its negative. A *turning* of E is a continuous path ψ_t of bundle automorphisms from $\mathbb{1}_E$ to $-\mathbb{1}_E$: if a turning of E exists, we call E *turnable* and the pair (E, ψ_t) a *turned* vector bundle. The *turning problem* for E is to determine whether E is turnable.

While the turning problem is a natural topological problem amenable to classical methods in algebraic topology, to the best of our knowledge it has not been explicitly discussed in the literature. Our primary interest in turnings stems from the fact that they generalise complex structures. As we explain in Section 5, Bott's original proof of his periodicity theorem shows for bundles over finite CW–complexes that stable turnings are equivalent to stable complex structures. On the other hand, we discovered that there are

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unstable bundles which are turnable but do not admit a complex structure; eg see the discussion following Theorem 1.3. Hence there is a sequence of strict inclusions

$$\{\text{complex bundles}\} \subsetneq \{\text{turned bundles}\} \subsetneq \{\text{stably complex bundles}\}.$$

The turning problem and its generalisations also arise naturally in the study of gauge groups. Turnings are closely related to the group of components and the fundamental group of the gauge group associated to E, and by studying turnings we can gain information about the low-dimensional topology of gauge groups; see eg Theorem 1.9. The generalised turning problem for loops in the structure group is also related to certain Samelson products and our results on turnings lead to new computations of Samelson products, which have implications for the high-dimensional topology of certain gauge groups; see Proposition 3.31.

1.1 Turnability of vector bundles

We begin with some elementary remarks on the turning problem. Given $b \in B$, let $E_b := \pi^{-1}(b)$ be the fibre over b, which is a Euclidean vector space. Since a turning of E restricts to a path from $\mathbb{1}_{E_b}$ to $-\mathbb{1}_{E_b}$, if E is turnable, then the rank of E must be even. Moreover, a turning of E can be used to continuously orient each fibre E_b. Hence if E is turnable, then E is orientable; see the discussion prior to Lemma 2.4. On the other hand, if E admits a complex structure then the map $t \mapsto e^{i\pi t}\mathbb{1}_E$ defines a turning of E and so complex bundles are turned bundles. Since oriented rank-2 bundles are equivalent to complex line bundles, they are turnable and so we assume $k > 1$, unless stated otherwise.

We next discuss the turning problem stably. Suppose that the base space B is (homotopy equivalent to) a finite CW–complex, let $\underline{\mathbb{R}}^j$ denote the trivial rank-j bundle over B and let $E \oplus \underline{\mathbb{R}}^j$ denote the Whitney sum of E and $\underline{\mathbb{R}}^j$. We say that E is *stably turnable* if $E \oplus \underline{\mathbb{R}}^j$ is turnable for some $j \geq 0$ and similarly we say that E admits a stable complex structure if $E \oplus \underline{\mathbb{R}}^j$ admits a complex structure for some $j \geq 0$. Then we have the following result; see Theorem 5.10 for a more refined statement.

Theorem 1.1 *Let $E \to B$ be a vector bundle over a finite-dimensional CW–complex. Then E is stably turnable if and only if E admits a stable complex structure.*

The question of whether E admits a stable complex structure, while in general difficult, can be characterised entirely using the ring structure in real K–theory; see Proposition 5.11. We therefore turn our attention to the turning problem for bundles outside the stable range and in this paper we pay close attention to rank-$2k$ bundles over $2k$–dimensional CW–complexes. Such bundles are just outside the stable range and provide a large class of interesting examples, including the tangent bundles TM of orientable smooth $2k$–manifolds M and rank-$2k$ bundles over S^{2k}. Starting with TS^{2k}, we recall that Kirchoff [12] proved that if TS^{2k} admits a complex structure then TS^{2k+1} is trivial. Shortly afterwards Borel and Serre [3] showed that TS^{2k} admits a complex structure if and only if $k = 0, 1$ or 3 and a little later Bott and Milnor [6] showed that TS^{2k+1} is trivial if and only if $k = 0, 1$ or 3. An important first result on the turning problem is the following strengthening of Kirchoff's theorem; see Theorem 4.3.

Theorem 1.2 *If TS^{2k} is turnable then TS^{2k+1} is trivial.*

Combined with the results of Borel and Serre and Bott and Milnor, Theorem 1.2 shows that TS^{2k} is turnable if and only if it admits a complex structure. However, such a statement does not hold generally, even for rank-$2k$ bundles E over S^{2k}, as Theorem 1.3 below shows.

We next consider the turnability of all oriented rank-$2k$ bundles over S^{2k}. Stable vector bundles over S^{2k} are classified by the real K–theory groups $\widetilde{\mathrm{KO}}(S^{2k})$ which are, respectively, isomorphic to $\mathbb{Z}$, $\mathbb{Z}/2$ and 0 for k respectively even, congruent to 1 mod 4 or congruent to 3 mod 4. Given an oriented rank-$2k$ vector bundle $E \to S^{2k}$, we let $\xi_E \in \widetilde{\mathrm{KO}}(S^{2k})$ denote the reduced real K–theory class defined by E. When $k = 2$, oriented vector bundles over S^4 admit unique homotopy classes of spin structures. By Kervaire [10], the spin characteristic class $p = p_1/2$ defines an isomorphism $p\colon \widetilde{\mathrm{KO}}(S^4) \to H^4(S^4;\mathbb{Z})$ and by Wall [17], $\rho_2(p(\xi_E)) = \rho_2(\mathrm{e}(E))$ for all oriented rank-4 bundles $E \to S^4$, where ρ_d denotes reduction mod d and for any base space B, $\mathrm{e}(E) \in H^{2k}(B;\mathbb{Z})$ denotes the Euler class of an oriented rank-$2k$ bundle $E \to B$. As a corollary of Theorem 4.1 we obtain:

Theorem 1.3 *For $k > 1$, let $E \to S^{2k}$ be an oriented rank-$2k$ bundle. Then E is turnable if and only if one of the following holds*:

(a) $k = 2$ *and* $\rho_4(\mathrm{e}(E) + p(\xi_E)) = 0$ *or* $\rho_4(\mathrm{e}(E) - p(\xi_E)) = 0$.

(b) $k = 3$.

(c) $k > 2$ *is even*, $\rho_4(\mathrm{e}(E)) = 0$ *and* $\rho_2(\xi_E) = 0$.

(d) $k > 3$ *is odd and* $\rho_4(\mathrm{e}(E)) = 0$.

Theorem 1.3 gives many examples of bundles which are turnable but do not admit a complex structure. For example, for $m \geq 2$ let $\tau_m \in \pi_{m-1}(\mathrm{SO}_m)$ denote the homotopy class of the clutching function of TS^m and for $n \in \mathbb{Z}$ let nTS^m denote the bundle corresponding to $n\tau_m \in \pi_{m-1}(\mathrm{SO}_m)$. Then for $m = 4j$, nTS^{4j} is turnable if and only if n is even, whereas by a theorem of Thomas [16, Theorem 1.7], nTS^{4j} admits a complex structure if and only if $n = 0$.

Theorem 1.3 leads to a general result on the turnability of rank-$2k$ bundles over general finite $2k$–dimensional CW–complexes. If J is a complex structure on $E \oplus \underline{\mathbb{R}}^{2j}$ for some $j \geq 0$, denote by $c_k(J) \in H^{2k}(B;\mathbb{Z})$ the k^{th} Chern class of J. We define the subgroup $I^{2k}(B) \subseteq H^{2k}(B;\mathbb{Z}/4)$ by

$$I^{2k}(B) := \begin{cases} ((\times 2) \circ \mathrm{Sq}^2 \circ \rho_2)(H^{2k-2}(B;\mathbb{Z})) & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even,} \end{cases}$$

where Sq^2 is the second Steenrod square and $\times 2$ is the natural map induced by the inclusion of coefficients $\times 2\colon \mathbb{Z}/2 \to \mathbb{Z}/4$.

The following result is a simple consequence of Theorems 6.1 and 6.2.

Theorem 1.4 *Let $E \to B$ be an oriented rank-$2k$ vector bundle over a finite CW–complex of dimension at most $2k$, and if k is even, assume that $H^{2k}(B;\mathbb{Z})$ contains no 2–torsion. Then E is turnable if and only if there is a $j \geq 0$ and a complex structure J on $E \oplus \underline{\mathbb{R}}^{2j}$ such that*

$$[\rho_4(c_k(J))] = \pm[\rho_4(\mathrm{e}(E))] \in H^{2k}(B;\mathbb{Z}/4)/I^{2k}(B).$$

Remark 1.5 When k is even and $H^{2k}(B;\mathbb{Z})$ contains 2–torsion, the condition in Theorem 1.4 remains necessary but is no longer sufficient; see Theorem 6.1 and Example 6.3.

Theorem 1.4 shows that there are many examples of manifolds whose tangent bundles are turnable but do not admit a complex structure. For example, if $M_l = \sharp_l(S^4 \times S^4)$ is the l–fold connected sum of $S^4 \times S^4$ with itself, then for any $j > 0$, the bundle $TM_l \oplus \underline{\mathbb{R}}^{2j}$ admits two homotopy classes of complex structures J, each with $c_4(J) = 0$. But $\mathrm{e}(TM_l) = \pm 2(l+1)$ by the Poincaré–Hopf theorem; see [8, page 113]. It follows that TM_l does not admit a complex structure and that TM_l is turnable if and only if l is odd; eg $T(S^4 \times S^4)$ is turnable but does not admit a complex structure. For a more general statement about when TM is turnable but does not admit a complex structure, see Corollary 6.4.

1.2 The turning obstruction for bundles over suspensions

In order to study the turning problem and obtain most of our results above, we define a complete obstruction to the existence of turnings for bundles over suspensions. For this we need to refine our definition of turning by specifying the homotopy class of the turning in a fibre. If ψ_t is a turning of an oriented rank-$2k$ bundle $E \to B$, then by restricting ψ_t to a fibre E_b we obtain a path of isometries from $\mathbb{1}_{E_b}$ to $-\mathbb{1}_{E_b}$. When E_b is identified with $\mathbb{R}^{2k}$ via an orientation-preserving isomorphism, this path is identified with a path in SO_{2k} from $\mathbb{1}$ to $-\mathbb{1}$, which is well defined up to path homotopy. Given a path γ in SO_{2k} from $\mathbb{1}$ to $-\mathbb{1}$, we shall call a turning a *γ–turning* if its restriction to each fibre is path homotopic to γ. When $k > 1$, as we generally assume, $\pi_1(\mathrm{SO}_{2k}) \cong \mathbb{Z}/2$, so there are precisely two path homotopy classes of possible paths.

Suppose that the base space $B = SX$ is a suspension, ie it is the union of two copies of the cone on X. The restriction of E to each cone admits a γ–turning, which is unique up to homotopy and E admits a γ–turning if and only if the γ–turnings over the cones agree over X up to homotopy. We can then define the *γ–turning obstruction of E* by measuring the difference of the γ–turnings over the cones and there are several equivalent ways to do this, which we present in Section 3.1. Here we discuss what we later call the *adjointed γ–turning obstruction*. Recall that the set of isomorphism classes of oriented rank-$2k$ bundles over SX form a group, which is naturally isomorphic to $[X, \mathrm{SO}_{2k}]$ via the map which sends the isomorphism class of E to the homotopy class of its clutching function $g\colon X \to \mathrm{SO}_{2k}$. We define the adjointed γ–turning obstruction

$$\mathrm{TO}_\gamma\colon [X, \mathrm{SO}_{2k}] \to [SX, \mathrm{SO}_{2k}], \quad [g] \mapsto \big[[x,t] \mapsto g(x)\gamma(t)g(x)^{-1}\big], \tag{1-1}$$

where $[x,t] \in SX$ is the point defined by $(x,t) \in X \times I$. The following result, which follows from Proposition 3.2 and Lemma 3.6, justifies calling TO_γ the γ–turning obstruction.

Proposition 1.6 *Let $E \to SX$ be an oriented rank-$2k$ vector bundle with clutching function $g : X \to \mathrm{SO}_{2k}$. Then E is γ–turnable if and only if $\mathrm{TO}_\gamma([g]) = 0$. Moreover, if X is itself a suspension, then TO_γ is a homomorphism of abelian groups.*

Proposition 1.6 states that the γ–turning obstruction is additive for bundles over double suspensions. This is an essential input to the Theorem 4.1, which largely computes TO_γ for rank-$2k$ bundles over the $2k$–sphere and both homotopy classes of paths γ. Theorem 1.3 above is an immediate corollary of Theorem 4.1.

The final element in the proof of Theorem 4.1 involves generalising the turning problem. The definition of the γ–turning obstruction naturally leads us to consider the turning obstruction for an essential loop $\eta\colon I \to \mathrm{SO}_{2k}$ with $\eta(0) = \eta(1) = \mathbb{1}$. If we replace $\gamma(t)$ by $\eta(t)$ in (1-1) above, we obtain the function

$$\mathrm{TO}_\eta : [X, \mathrm{SO}_{2k}] \to [SX, \mathrm{SO}_{2k}], \quad [g] \mapsto \big[[x,t] \mapsto g(x)\eta(t)g(x)^{-1}\big].$$

If $E \to SX$ is a bundle with clutching function $g\colon X \to \mathrm{SO}_{2k}$, then $\mathrm{TO}_\eta([g])$ is a complete obstruction to finding a loop ψ_t of bundle automorphisms of E based at the identity such that the restriction of ψ_t to a fibre E_b is an essential loop of isometries of E_b. Moreover, if γ is a path in SO_{2k} from $\mathbb{1}$ to $-\mathbb{1}$, then the concatenation of paths $\eta * \gamma$ represents the other path homotopy class of such paths. Hence a bundle $E \to SX$ with clutching function g is turnable if and only if one of $\mathrm{TO}_\gamma([g])$ or $\mathrm{TO}_{\eta*\gamma}([g])$ vanishes. The following result relates TO_η and TO_γ and states that TO_γ is in general 4–torsion; see also Theorem 3.23.

Theorem 1.7 *Let E be an oriented rank-$2k$ vector bundle with clutching function $g : X \to \mathrm{SO}_{2k}$. Then*:

(a) $2\,\mathrm{TO}_\eta([g]) = 0$.

(b) $\mathrm{TO}_{\eta*\gamma}([g]) = \mathrm{TO}_\eta([g]) + \mathrm{TO}_\gamma([g])$.

(c) *If k is even, then* $2\,\mathrm{TO}_\gamma([g]) = 0$.

(d) *If k is odd, then* $2\,\mathrm{TO}_\gamma([g]) = \mathrm{TO}_\eta([g])$ *and* $4\,\mathrm{TO}_\gamma([g]) = 0$.

Remark 1.8 Notwithstanding Theorem 1.7(d), we know of no example of a bundle $E \to SX$ with clutching function g, where $2\,\mathrm{TO}_\gamma([g]) \neq 0$. In particular, by Theorem 4.1, $2\,\mathrm{TO}_\gamma(\tau_{4k+2}) = \mathrm{TO}_\eta([\tau_{4k+2}]) = 0$ for all $k \geq 1$. The proof we give of this result is computational and somewhat surprising to us. It would be interesting to know if there is a space X and a clutching function $g\colon X \to \mathrm{SO}_{2k}$ with $2\,\mathrm{TO}_\gamma([g]) \neq 0$.

1.3 General turnings, the topology of gauge groups and Samelson products

Let $\mathrm{Fr}(E)$ denote the frame bundle of an oriented vector bundle $E \to B$, which is a principal SO_{2k}–bundle over B. The group of automorphisms of E is canonically homeomorphic to the gauge group of $\mathrm{Fr}(E)$, and so the turning problem can be viewed as a problem in the topology of gauge groups: we are asking when a topological feature of the structure group extends to the whole gauge group.

To describe general turning problems, we let G be a path-connected topological group, for example a connected Lie group, and $P \to B$ be a principal G–bundle with gauge group $\mathcal{G}_P$: if $G = \mathrm{SO}_{2k}$ and $P = \mathrm{Fr}(E)$, then we shall write $\mathcal{G}_E$ in place of $\mathcal{G}_{\mathrm{Fr}(E)}$. If $Z(G)$ denotes the centre of G, then multiplication by $z \in Z(G)$ defines an element $z_P \in \mathcal{G}_P$. Given a path $\gamma\colon I \to G$ between elements of $Z(G)$, the *γ–turning problem* for P is to determine whether there is a path ψ_t in $\mathcal{G}_P$ with $\psi(0) = \gamma(0)_P$ and $\psi(1) = \gamma(1)_P$ and whose restriction to a fibre is path-homotopic to γ.

When $B = SX$ is a suspension, then principal G–bundles $P \to SX$ are determined up to isomorphism by their clutching functions $g\colon X \to G$ and the definition and properties of the γ–turning obstruction for vector bundles generalise in the obvious way. The (adjointed) γ–turning obstruction is the map

$$\mathrm{TO}_\gamma\colon [X, G] \to [SX, G], \quad [g] \mapsto \big[[x, t] \mapsto g(x)\gamma(t)g(x)^{-1}\big],$$

and P is γ–turnable if and only if $\mathrm{TO}_\gamma([g]) = 0$; see Remark 3.22. If we allow γ to vary among all paths between central elements of G, the path homotopy classes of the possible paths γ form a groupoid, which is a full subcategory of the fundamental groupoid of G. We call this groupoid the *central groupoid of G* and denote it by $\pi^Z(G)$. If we fix $[g] \in [X, G]$, then we can regard $\mathrm{TO}_\gamma([g])$ as a function of γ. The resulting map

$$\pi^Z(G) \to [SX, G]$$

is a morphism of groupoids (where the group $[SX, G]$ is regarded as a groupoid on one object) and this general point of view allows us to prove Theorem 1.7.

Returning to vector bundles E and the topology of their gauge groups $\mathcal{G}_E$, the η–turning problem has the most direct implications (see Theorem 2.26):

Theorem 1.9 *If $B = SX$ is a suspension and $n_{SX} := |[SX, \mathrm{SO}_{2k}]|$ is finite, then for any rank-$2k$ vector bundle $E \to SX$,*

$$|\pi_0(\mathcal{G}_E)| = \begin{cases} n_{SX} & \text{if } E \text{ is } \eta\text{–turnable}, \\ \frac{1}{2}n_{SX} & \text{if } E \text{ is not } \eta\text{–turnable}. \end{cases}$$

Theorem 1.9 shows that when $[SX, \mathrm{SO}_{2k}]$ is finite, for example when $SX = S^{2k}$, then the η–turnability of a vector bundle E is a homotopy invariant of its gauge group $\mathcal{G}_E$. While the turnability of E is not a priori a homotopy invariant of $\mathcal{G}_E$, recent work of Kishimoto, Membrillo-Solis and Theriault [13] on the homotopy classification of the gauge groups of rank-4 bundles $E \to S^4$, when combined with our results in Theorem 4.1, does show that the turnability of these bundles is a homotopy invariant of their gauge groups; see Proposition 4.8 for a more detailed statement.

We compute TO_η for all rank-$2k$ bundles over S^{2k} in Theorem 4.1. In fact in this case $\mathrm{TO}_\eta([g]) = \langle [g], \eta \rangle$ is the Samelson product of $[g] \in \pi_{2k-1}(\mathrm{SO}_{2k})$ and $\eta \in \pi_1(\mathrm{SO}_{2k})$; see Lemma 3.24. On the other hand, Samelson products are in general delicate to calculate and so the computations of $\mathrm{TO}_\eta([g])$ in Theorem 4.1, which are carried out using the point of view of the turning obstruction, may be of independent interest. For example (see Proposition 3.29), for $\eta_{4j-1}\colon S^{4j} \to S^{4j-1}$ an essential map, we have:

Corollary 1.10 *The Samelson product $\langle \tau_{2k}, \eta \rangle$ is given by $\langle \tau_{4j+2}, \eta \rangle = 0$ and $\langle \tau_{4j}, \eta \rangle = \tau_{4j}\eta_{4j-1} \neq 0$.*

Corollary 1.10 also has implications for the high-dimensional homotopy groups of certain gauge groups; see Proposition 3.31 in Section 3.3.

Organisation The rest of this paper is organised as follows. In Section 2 we set up the necessary preliminaries to discuss the turning problem. We define turnings and γ–turnings, universal bundles which classify turnings and relate the turning problem to the topology of the gauge group. In Section 3 we define the γ–turning obstruction for bundles over suspensions and develop the theory of the γ–turning obstruction, regarded as a map from the central groupoid of a path-connected topological group G. We also show that the η–turning obstruction is given by certain Samelson products. In Section 4, we consider rank-$2k$ vector bundles over the $2k$–sphere and compute their turning obstructions in detail, proving Theorem 1.3. In Section 5, we consider the turning problem for bundles in the stable range and prove Theorem 1.1. Finally, in Section 6 we combine the results of Sections 4 and 5 on rank-$2k$ vector bundles over the $2k$–sphere and stable vector bundles to prove Theorem 1.4 on rank-$2k$ vector bundles over $2k$–dimensional CW–complexes.

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2 Turnings and gauge groups

In this section we set up the necessary definitions and notation for the turning problem and establish some basic results. In Section 2.1 we define turnings and γ–turnings and introduce the terminology to describe the relationship between turnings and orientations of a vector bundle. A more general notion of turning, for principal G–bundles, is defined in Section 2.4. In Section 2.2 we define the associated turning bundle of a vector bundle and construct a universal turned bundle. We also establish some equivalent conditions to turnability in terms of the associated turning bundle and the universal turned bundle. In Sections 2.3 and 2.5 we study the connection between the turnability of a vector bundle and the low-dimensional homotopy groups of its gauge group.

2.1 Turnings

All vector spaces V in this paper are real and Euclidean. The connected component of the group of isometries of V is denoted by $\mathrm{SO}(V)$, $\mathbb{1} \in \mathrm{SO}(V)$ is the identity and $-\mathbb{1}\colon V \to V$ is defined by $-\mathbb{1}(v) = -v$ for all $v \in V$. We use $\mathbb{R}^j$ to denote j–dimensional Euclidean space with its standard metric and as usual we set $\mathrm{SO}_j := \mathrm{SO}(\mathbb{R}^j)$.

All vector bundles $\pi\colon E \to B$ are real and Euclidean and for simplicity we assume that the base space B is connected. We denote the trivial bundle $\mathbb{R}^j \times B \to B$ by $\underline{\mathbb{R}}^j$; the base space will either be specified or clear from the context. We shall use the symbol E to ambiguously denote both the total space of the bundle and the bundle itself. For $b \in B$, $E_b := \pi^{-1}(b)$ is the fibre of E over b, which is a vector space. Let $I := [0,1]$ be the unit interval.

Definition 2.1 Let V be an even-dimensional real vector space, so that $-\mathbb{1} \in \mathrm{SO}(V)$. A *turning* of V is a path $\gamma\colon I \to \mathrm{SO}(V)$ from $\mathbb{1}$ to $-\mathbb{1}$.

In particular, a turning of $\mathbb{R}^{2k}$ is a path in SO_{2k} from $\mathbb{1}$ to $-\mathbb{1}$ and we write $\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ for the mapping space $\mathrm{Map}\big((I,(\{0\},\{1\})),(\mathrm{SO}_{2k},(\{\mathbb{1}\},\{-\mathbb{1}\}))\big)$ consisting of all turnings of $\mathbb{R}^{2k}$, with the compact–open topology. Note that $\Omega\mathrm{SO}_{2k}$, the space of loops based at $\mathbb{1}$, acts freely and transitively on $\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ by pointwise multiplication. Hence choosing $\gamma \in \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ defines a homeomorphism from $\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ to $\Omega\mathrm{SO}_{2k}$ and we will use this homeomorphism to compute the homotopy groups of $\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$.

Definition 2.2 (standard turning of $\mathbb{R}^{2k}$) Let $\mathbb{R}^{2k} = \mathbb{C}^k$ define the standard complex structure on $\mathbb{R}^{2k}$ and let $U_k \subseteq \mathrm{SO}_{2k}$ be the unitary subgroup. The *standard turning* of $\mathbb{R}^{2k}$ is the path

$$\beta\colon I \to \mathrm{SO}_{2k}, \quad t \mapsto e^{i\pi t}\mathbb{1} \in U_k \subseteq \mathrm{SO}_{2k}.$$

If $2k > 2$, then $\pi_0(\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}) \cong \pi_1(\mathrm{SO}_{2k}) \cong \mathbb{Z}/2$, so there are two turnings of $\mathbb{R}^{2k}$ up to homotopy. Indeed, if $\overline{\beta}$ is a representative of the other homotopy class and

$$\eta\colon (I,\{0,1\}) \to (\mathrm{SO}_{2k},\mathbb{1})$$

is a loop representing the generator of $\pi_1(\mathrm{SO}_{2k})$, then $[\overline{\beta}] = [\eta * \beta]$, where $*$ denotes concatenation of paths and $[\gamma]$ denotes the path homotopy class of a path γ. If we pointwise conjugate β with a fixed element of $O_{2k} \setminus \mathrm{SO}_{2k}$, then we obtain a path in $[\overline{\beta}]$; equivalently, an orientation-reversing isomorphism $\mathbb{R}^{2k} \to \mathbb{R}^{2k}$ pulls back β to a turning that is path homotopic to $\overline{\beta}$. Note that the turning defined by the formula $t \mapsto e^{-i\pi t}\mathbb{1}$ is path homotopic to β if k is even and to $\overline{\beta}$ if k is odd.

Let V be a vector space of dimension $2k$ equipped with a turning and an orientation. If $2k > 2$, then we say that the turning and the orientation are compatible if the turning is homotopic to β under an orientation-preserving identification $V \cong \mathbb{R}^{2k}$. If $2k = 2$, then they are compatible if the turning is homotopic to the path $t \mapsto e^{ri\pi t}\mathbb{1}$ for some positive (odd) r under an orientation-preserving identification $V \cong \mathbb{R}^2$. In both cases there is a unique orientation of V which is compatible with a given turning, hence we obtain a well-defined map from the homotopy classes of turnings of V to its orientations. This map is a bijection if $2k > 2$ and surjective if $2k = 2$.

Definition 2.3 (turning, turnable and turned) Let $\pi\colon E \to B$ be a rank-$2k$ vector bundle. A *turning* of E is a path ψ_t in the space of automorphisms of E from $\mathbb{1}_E$ to $-\mathbb{1}_E$. If a turning exists, we say that E is *turnable*, and a *turned* vector bundle is a pair (E,ψ_t), where ψ_t is a turning of E.

Clearly any trivial bundle is turnable and since bundle automorphisms can be pulled back along continuous maps, the pullback of a turnable bundle is turnable. Furthermore, every complex bundle E is turnable via the path $t \mapsto e^{i\pi t}\mathbb{1}_E$.

A turning of a bundle restricts to a turning of each fibre, so by our earlier observations it determines an orientation on each fibre. Therefore we have:

Lemma 2.4 *Every turnable bundle is orientable.* □

If a rank-2 bundle is orientable, then it admits a complex structure, so we have:

Proposition 2.5 *A rank-2 bundle is turnable if and only if it is orientable.* □

From now on we will focus on oriented bundles $\pi : E \to B$ of rank $2k$ and we assume that $2k > 2$ unless otherwise stated. Since we are assuming that B is connected, it follows that an orientable bundle has precisely two possible orientations and we let $\overline{E}$ denote the same bundle with opposite orientation.

Definition 2.6 For a path $\gamma \in \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$, let $\Omega_\gamma \mathrm{SO}_{2k} \subset \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ denote the connected component of γ.

Definition 2.7 (γ–turnable, positive/negative turnable) Let E be a rank-$2k$ bundle and $\gamma \in \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$. We say that E is *γ–turnable* if it has a turning whose restriction to each fibre E_b lies in $\Omega_\gamma \mathrm{SO}_{2k}$ under an orientation-preserving identification $E_b \cong \mathbb{R}^{2k}$.

We also say that E is *positive/negative turnable* if it has a turning which determines the positive/negative orientation on E.

Obviously, if γ is homotopic to γ', then γ–turnability is equivalent to γ'–turnability. By definition positive turnability is equivalent to β–turnability, and negative turnability is equivalent to $\overline{\beta}$–turnability. A bundle is turnable if and only if it is positive turnable or negative turnable. Finally, a bundle E is positive turnable if and only if $\overline{E}$ is negative turnable. For the next definition, recall that E is called *chiral* if E is not isomorphic to $\overline{E}$.

Definition 2.8 (bi-turnable, strongly chiral) If E is both positive and negative turnable, we call it *bi-turnable*. If E is turnable but not bi-turnable, we say that E is *strongly chiral*.

If $E \cong \overline{E}$, then E cannot be strongly chiral. This shows that strong chirality implies chirality.

Definition 2.9 (turning type) The *turning type* of an orientable rank-$2k$ bundle is the property of being either bi-turnable, strongly chiral or not turnable.

2.2 The associated turning bundle

We can also think of a turning of a bundle as a continuous choice of turning in each fibre. To make this precise we define the associated turning bundle below, in analogy with the associated automorphism bundle.

Every oriented rank-$2k$ vector bundle $\pi\colon E \to B$ has an associated principal SO_{2k}–bundle, namely the frame bundle $\mathrm{Fr}(E)$, whose fibres consist of oriented, orthonormal frames of the fibres of E. We will view such frames as linear isomorphisms $\phi_b\colon \mathbb{R}^{2k} \to E_b$. Then SO_{2k} acts on the right on the total space of $\mathrm{Fr}(E)$ via precomposition.

Definition 2.10 (the associated automorphism bundle and the associated turning bundle) For an oriented, rank-$2k$ vector bundle $E \to B$ we define, via the Borel construction,

(a) the associated *automorphism bundle*

$$\mathrm{Aut}(E) := \mathrm{Fr}(E) \times_{\mathrm{SO}_{2k}} \mathrm{SO}_{2k} \to B,$$

where SO_{2k} acts on itself by conjugation, and

(b) the associated *turning bundle*

$$\mathrm{Turn}(E) := \mathrm{Fr}(E) \times_{\mathrm{SO}_{2k}} \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k} \to B,$$

where SO_{2k} acts on $\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ by pointwise conjugation.

Remark 2.11 The fibre of $\mathrm{Aut}(E)$ over $b \in B$ can be identified with $\mathrm{SO}(E_b)$, with the equivalence class $[\phi_b, A] \in \mathrm{Fr}(E) \times_{\mathrm{SO}_{2k}} \mathrm{SO}_{2k}$ corresponding to $\phi_b \circ A \circ \phi_b^{-1}\colon E_b \to E_b$. Similarly, the fibre of $\mathrm{Turn}(E)$ over b consists of the turnings of E_b, with $[\phi_b, \gamma]$ corresponding to the path $t \mapsto \phi_b \circ \gamma(t) \circ \phi_b^{-1}$.

A turning of a bundle E restricts to a turning of each fibre and so determines a section of $\mathrm{Turn}(E)$. In this way we obtain a homeomorphism between the space of turnings of E and the space of sections of $\mathrm{Turn}(E)$. In particular, we have:

Lemma 2.12 *A vector bundle E is turnable if and only if $\mathrm{Turn}(E) \to B$ has a section.* □

Definition 2.13 Fix a model $B\mathrm{SO}_{2k}$ for the classifying space of oriented rank-$2k$ vector bundles and let $V\mathrm{SO}_{2k} \to B\mathrm{SO}_{2k}$ denote the universal rank-$2k$ bundle. We define $BT_{2k} := \mathrm{Turn}(V\mathrm{SO}_{2k})$ to be the total space of the associated turning bundle, and let $\pi_{2k}\colon BT_{2k} \to B\mathrm{SO}_{2k}$ be its projection.

Remark 2.14 The symbol BT_{2k} should be read as a single unit. Defining a topological monoid T_{2k} whose classifying space is the associated turning bundle of the universal bundle $V\mathrm{SO}_{2k} \to B\mathrm{SO}_{2k}$ is an interesting question, but we will not address it in this paper.

Below we explain how BT_{2k} acts as a classifying space for turned vector bundles.

Proposition 2.15 *A rank-$2k$ bundle E over a CW–complex X is turnable if and only if its classifying map $f\colon X \to B\mathrm{SO}_{2k}$ can be lifted over $\pi_{2k}\colon BT_{2k} \to B\mathrm{SO}_{2k}$.*

Proof Since $E \cong f^*(V\mathrm{SO}_{2k})$, we have $\mathrm{Turn}(E) \cong f^*(\mathrm{Turn}(V\mathrm{SO}_{2k}))$; ie there is a pullback diagram

$$\begin{array}{ccc} \mathrm{Turn}(E) & \longrightarrow & BT_{2k} \\ \downarrow & & \downarrow{\scriptstyle \pi_{2k}} \\ X & \xrightarrow{f} & B\mathrm{SO}_{2k} \end{array}$$

It follows from the universal property of pullbacks that f can be lifted to BT_{2k} if and only if $\mathrm{Turn}(E)$ has a section, which is equivalent to the turnability of E. □

We now show how BT_{2k} classifies rank-$2k$ turned vector bundles.

Definition 2.16 Let $VT_{2k} := \pi_{2k}^*(V\mathrm{SO}_{2k}) \to BT_{2k}$.

Note that VT_{2k} has a canonical turning, which we denote by ψ_t^c. Since VT_{2k} is defined as a pullback of $V\mathrm{SO}_{2k}$, its fibre over $x \in BT_{2k}$ can be identified with $(V\mathrm{SO}_{2k})_y$, the fibre of $V\mathrm{SO}_{2k}$ over $y = \pi_{2k}(x)$. By Remark 2.11, x itself can be regarded as a turning of $(V\mathrm{SO}_{2k})_y$ and hence of $(VT_{2k})_x$. That is, each fibre $(VT_{2k})_x$ of VT_{2k} comes equipped with a turning (which varies continuously with x), showing that $\mathrm{Turn}(VT_{2k})$ has a canonical section.

The turned bundle $(VT_{2k} \to BT_{2k}, \psi_t^c)$ is universal in the two senses explained in Theorem 2.18 below.

Definition 2.17 For a space X, let $\mathrm{TB}_{2k}(X)$ be the set of isomorphism classes of rank-$2k$ turned bundles over X: it consists of equivalence classes of turned bundles over X, where two turned bundles are equivalent if there is an isomorphism between them under which their turnings are homotopic.

Theorem 2.18 ((VT_{2k}, ψ_t^c) is a universal rank-$2k$ turned bundle)

(a) *If a rank-$2k$ bundle E over a CW–complex X is equipped with a turning ψ_t, then there is a homotopically unique map $X \to BT_{2k}$ which induces (E, ψ_t) from (VT_{2k}, ψ_t^c).*

(b) *For every CW–complex X there is a bijection* $\mathrm{TB}_{2k}(X) \cong [X, BT_{2k}]$.

Proof (a) A stronger statement holds: the space of pairs $(g, \overline{g})$, where $g\colon X \to BT_{2k}$ is a continuous map and $\overline{g}\colon E \to g^*(VT_{2k})$ is an isomorphism respecting the given turnings, is contractible. To such a pair $(g, \overline{g})$ we assign a pair $(f, \overline{f})$, where $f\colon X \to B\mathrm{SO}_{2k}$ is a continuous map and $\overline{f}\colon E \to f^*(V\mathrm{SO}_{2k})$ is an isomorphism, by letting $f = \pi_{2k} \circ g$ and $\overline{f} = \overline{g}$ (using that $f^*(V\mathrm{SO}_{2k}) = g^*(\pi_{2k}^*(V\mathrm{SO}_{2k})) = g^*(VT_{2k})$). Each pair $(f, \overline{f})$ determines a pullback diagram as in the proof of Proposition 2.15. It follows from the pullback property that f has a unique lift $g\colon X \to BT_{2k}$ corresponding to the given turning of E (section of $\mathrm{Turn}(E)$) and if we regard $\overline{f}$ as an isomorphism $\overline{g}\colon E \to g^*(VT_{2k})$, then this $\overline{g}$ respects the turnings. This shows that the assignment $(g, \overline{g}) \mapsto (f, \overline{f})$ is a homeomorphism. And since $V\mathrm{SO}_{2k}$ is a universal bundle, the space of pairs $(f, \overline{f})$ is contractible.

(b) To a map $g\colon X \to BT_{2k}$ we assign $g^*(VT_{2k})$ with its induced turning. This way we obtain a well-defined map $[X, BT_{2k}] \to \mathrm{TB}_{2k}(X)$, because a homotopy of g induces a bundle over $X \times I$ with a turning and after identifying this bundle with $g^*(VT_{2k}) \times I$ its turning determines a homotopy between the turnings over $X \times \{0\}$ and $X \times \{1\}$. It follows from (a) that this map is surjective.

Suppose that two maps $g_1, g_2\colon X \to BT_{2k}$ determine the same element in $\mathrm{TB}_{2k}(X)$. This means that, after identifying $g_1^*(VT_{2k})$ with $E := g_2^*(VT_{2k})$ via some isomorphism, the induced turnings on E are homotopic, ie there is a homotopy between the corresponding sections of $\mathrm{Turn}(E)$. This homotopy then determines a homotopy (via lifts of $\pi_{2k} \circ g_1\colon X \to B\mathrm{SO}_{2k}$) between g_1 and another lift $g_1'\colon X \to BT_{2k}$ such that under the isomorphism $g_2^*(VT_{2k}) = E \cong g_1^*(VT_{2k}) = (\pi_{2k} \circ g_1)^*(V\mathrm{SO}_{2k}) = (\pi_{2k} \circ g_1')^*(V\mathrm{SO}_{2k}) = (g_1')^*(VT_{2k})$ the same turning is induced on $g_2^*(VT_{2k})$ and $(g_1')^*(VT_{2k})$. By (a), this implies that g_1' is homotopic to g_2. Therefore the map $[X, BT_{2k}] \to \mathrm{TB}_{2k}(X)$ is also injective, hence it is a bijection. □

Remark 2.19 In the constructions of this section, instead of $\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ we could use one of its connected components, $\Omega_\gamma \mathrm{SO}_{2k}$ for a $\gamma \in \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$. Then the turning bundle $\mathrm{Turn}(E)$ would be replaced with its subbundle $\mathrm{Turn}^\gamma(E)$ and BT_{2k} with its connected component $BT_{2k}^\gamma = \mathrm{Turn}^\gamma(V\mathrm{SO}_{2k})$. A bundle E over a CW–complex X is γ–turnable if and only if $\mathrm{Turn}^\gamma(E)$ has a section and if and only if its classifying map $f\colon X \to B\mathrm{SO}_{2k}$ can be lifted to BT_{2k}^γ. Moreover, $VT_{2k}^\gamma := VT_{2k}|_{BT_{2k}^\gamma}$ is universal among bundles equipped with a γ–turning.

2.3 The gauge group

For an oriented rank-$2k$ vector bundle $\pi\colon E \to B$, recall that $\mathrm{Fr}(E)$ denotes the frame bundle of E, which is a principal SO_{2k}–bundle over B. As an elementary exercise in linear algebra shows, the space of automorphisms of a vector bundle $E \to B$ is canonically homeomorphic to the gauge group of $\mathrm{Fr}(E)$, as defined in [9, Chapter 7] and we will use these topological groups interchangeably, denoting them by $\mathcal{G}_E$. In this section we relate the existence of turnings on E to the topology of $\mathcal{G}_E$.

The automorphisms $\mathbb{1}_E$ and $-\mathbb{1}_E$ define elements of $\mathcal{G}_E$. Specifically, since $-\mathbb{1}$ lies in $Z(\mathrm{SO}_{2k})$, the centre of SO_{2k}, we obtain the global map $-\mathbb{1}_E \in \mathcal{G}_E$ given by $p \mapsto p(-\mathbb{1})$. Considering $[\mathbb{1}_E], [-\mathbb{1}_E] \in \pi_0(\mathcal{G}_E)$, we see from Definition 2.3 that E is turnable if and only if $[-\mathbb{1}_E] = [\mathbb{1}_E] \in \pi_0(\mathcal{G}_E)$. Indeed, somewhat more is true, as we now explain.

Fixing a frame $p \in \mathrm{Fr}(E)$ over $b = \pi(p)$ and restricting to the fibre of $\mathrm{Fr}(E) \to B$ over b, we obtain a continuous homomorphism of topological groups $r_p\colon \mathcal{G}_E \to \mathrm{SO}_{2k}$. Replacing SO_{2k} by the mapping cylinder of r_p, we regard r_p as an inclusion and consider the pair $(\mathrm{SO}_{2k}, \mathcal{G}_E)$. A path $\gamma \in \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ defines an element $[\gamma]_{\mathcal{G}} \in \pi_1(\mathrm{SO}_{2k}, \mathcal{G}_E)$, by identifying $\mathbb{1}_E, -\mathbb{1}_E \in \mathcal{G}_E$ with $r_p(\mathbb{1}_E) = \mathbb{1}, r_p(-\mathbb{1}_E) = -\mathbb{1} \in \mathrm{SO}_{2k}$ respectively, and viewing γ as a path in SO_{2k} connecting $\mathbb{1}_E$ and $-\mathbb{1}_E$. Since $r_p\colon \mathcal{G}_E \to \mathrm{SO}_{2k}$ is a homomorphism of topological groups, $\pi_1(\mathrm{SO}_{2k}, \mathcal{G}_E)$ inherits a group structure from the group structures on $\mathcal{G}_E$ and SO_{2k} and we denote the unit by e. Then we have:

Lemma 2.20 *A bundle E is γ–turnable if and only if $[\gamma]_{\mathcal{G}} = e \in \pi_1(\mathrm{SO}_{2k}, \mathcal{G}_E)$.* □

Given the above, it is natural to consider the final segment of the homotopy long exact sequence of the pair $(\mathrm{SO}_{2k}, \mathcal{G}_E)$, which runs as follows:

$$\cdots \to \pi_1(\mathcal{G}_E) \xrightarrow{(r_p)_*} \pi_1(\mathrm{SO}_{2k}) \to \pi_1(\mathrm{SO}_{2k}, \mathcal{G}_E) \to \pi_0(\mathcal{G}_E) \to 0. \tag{2-1}$$

Now $[\bar{\beta}]_{\mathcal{G}} = [\beta]_{\mathcal{G}} + [\eta]$, where $+$ denotes the natural action of $\pi_1(\mathrm{SO}_{2k})$ on $\pi_1(\mathrm{SO}_{2k}, \mathcal{G}_E)$ and $[\eta]$ in $\pi_1(\mathrm{SO}_{2k})$ is the generator. We see that $[\beta]_{\mathcal{G}} = [\bar{\beta}]_{\mathcal{G}}$ if and only if $(r_p)_*\colon \pi_1(\mathcal{G}_E) \to \pi_1(\mathrm{SO}_{2k})$ is onto. For example, in Section 4.2 we shall see that there are rank-4 bundles $E \to S^4$ which are β–turnable but not $\bar{\beta}$–turnable. Applying Lemma 2.20 we see that for these bundles $[\beta]_{\mathcal{G}} \neq [\bar{\beta}]_{\mathcal{G}} \in \pi_1(\mathrm{SO}_{2k}, \mathcal{G}_E)$ and hence the map $(r_p)_*\colon \pi_1(\mathcal{G}_E) \to \pi_1(\mathrm{SO}_{2k})$ is zero. In fact, the homomorphism $\pi_1(\mathcal{G}_E) \to \pi_1(\mathrm{SO}_{2k})$ is closely related to the "turning obstruction for the essential loop in SO_{2k}", and we next discuss turnings in a more general setting.

2.4 The central groupoid and general turnings

In this subsection we generalise the definition of a turning. Let G be a path-connected topological group with centre $Z(G)$. For the computations in this paper the groups SO_{2k}, their double covers Spin_{2k} and their quotients $\mathrm{PSO}_{2k} := \mathrm{SO}_{2k}/\{\pm\mathbb{1}\}$ will be relevant, and we will consider these groups in more detail at the end of this subsection.

Let $\pi\colon P \to B$ be a principal G–bundle over a path-connected space B. The gauge group of P, denoted by $\mathcal{G}_P$, is the group of G–equivariant fibrewise automorphisms of P. Given $z \in Z(G)$, fibrewise multiplication by z defines a central element $z_P \in Z(\mathcal{G}_P)$, where for all $p \in P$,

$$z_P(p) := p \cdot z.$$

We note that if $Z(G)$ is discrete, then the map $Z(G) \to Z(\mathcal{G}_P)$, $z \mapsto z_P$, is an isomorphism. We shall be interested in paths $\gamma\colon I \to G$ which start and end at elements of the centre $Z(G)$, and whether they can be lifted to paths in $\mathcal{G}_P$ which start and end at $\gamma(0)_P$ and $\gamma(1)_P$. Hence we make the following definition:

Definition 2.21 (central groupoid) The *central groupoid* of G is the restriction of the fundamental groupoid of G to paths which start and end in the centre of G. We will use $\pi^Z(G)$ to ambiguously denote the central groupoid of G or the set of its morphisms.

Remark 2.22 Pointwise multiplication gives $\pi^Z(G)$, the set of morphisms of the central groupoid, a group structure, and there is short exact sequence

$$1 \to \pi_1(G, e) \to \pi^Z(G) \to Z(G) \times Z(G) \to 1,$$

where $\pi^Z(G) \to Z(G) \times Z(G)$ is defined by $[\gamma] \mapsto (\gamma(0), \gamma(1))$ and $e \in G$ is the identity. While we do not use this group structure in what follows, it may be helpful for understanding $\pi^Z(G)$; eg it shows that $\pi^Z(\mathrm{SO}_{2k})$ has eight morphisms.

For a point $p \in P$, let $b = \pi(p)$ and $P_b := p \cdot G$ be the fibre of $P \to B$ over b. Define the *restriction map*

$$r_p : \mathcal{G}_P \to G$$

by restricting elements of the gauge group to the fibre over b and using the equation

$$\phi(p) = p \cdot r_p(\phi)$$

for all $\phi \in \mathcal{G}_P$. If we vary $p \in P_b$, then $r_{p\cdot g}(\phi) = g^{-1} r_p(\phi) g$ for all $\phi \in \mathcal{G}_P$ and $g \in G$. Recalling that G and B are path-connected, we see for any path $\phi_t : I \to \mathcal{G}_P$ with $\phi_0 = z_P$ and $\phi_1 = z'_P$, that $[r_p(\phi_t)] \in \pi^Z(G)$ is independent of the choice of p.

Definition 2.23 Let $\phi_t : I \to \mathcal{G}_P$ be a path such that $\phi_0 = z_P$ and $\phi_1 = z'_P$ for some $z, z' \in Z(G)$. We define $r(\phi_t) \in \pi^Z(G)$ to be $[r_p(\phi_t)]$ for any $p \in P$.

Definition 2.24 (γ–turning and γ–turnable) Let $[\gamma] \in \pi^Z(G)$ be represented by a path $\gamma : I \to G$. A *γ–turning* of a principal G–bundle P is a path $\phi_t : I \to \mathcal{G}_P$ with $\phi_0 = \gamma(0)_P, \phi(1) = \gamma(1)_P$ and $r(\phi_t) = [\gamma] \in \pi^Z(G)$. If P admits a γ–turning then P is called *γ–turnable*.

We end this subsection by considering the groups SO_{2k}, Spin_{2k} and PSO_{2k}. When $2k > 2$, we have $\mathrm{PSO}_{2k} = \mathrm{SO}_{2k}/Z(\mathrm{SO}_{2k}) \cong \mathrm{Spin}_{2k}/Z(\mathrm{Spin}_{2k})$ and we list the centres and fundamental groups of these groups in the following tables (where $j \geq 1$), which follow from Lemma 2.25 below:

G	$Z(G)$	$\pi_1(G)$
Spin_{4j}	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$	$\{e\}$
SO_{4j}	$\mathbb{Z}/2$	$\mathbb{Z}/2$
PSO_{4j}	$\{e\}$	$\mathbb{Z}/2 \oplus \mathbb{Z}/2$

G	$Z(G)$	$\pi_1(G)$
Spin_{4j+2}	$\mathbb{Z}/4$	$\{e\}$
SO_{4j+2}	$\mathbb{Z}/2$	$\mathbb{Z}/2$
PSO_{4j+2}	$\{e\}$	$\mathbb{Z}/4$

The next lemma is well known but we include its proof to further illustrate the structure of the central groupoid of SO_{2k}.

Lemma 2.25 *If $k \geq 2$, then $Z(\mathrm{PSO}_{2k}) = \{e\}$, $Z(\mathrm{Spin}_{2k}) \cong \pi_1(\mathrm{PSO}_{2k})$ and*

$$\pi_1(\mathrm{PSO}_{2k}) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } k \text{ is even,} \\ \mathbb{Z}/4 & \text{if } k \text{ is odd.} \end{cases}$$

Proof To see that the centre of PSO_{2k} is trivial, let $x \in \mathrm{SO}_{2k}$ lie in the preimage of $Z(\mathrm{PSO}_{2k})$. Then the commutator $[x, \cdot]$ defines a map $\mathrm{SO}_{2k} \to Z(\mathrm{SO}_{2k})$. Since $Z(\mathrm{SO}_{2k})$ is discrete and $[x, \mathbb{1}] = \mathbb{1}$, this is the constant $\mathbb{1}$ map. Hence $x \in Z(\mathrm{SO}_{2k})$ and thus $Z(\mathrm{PSO}_{2k}) = \{e\}$.

If $q : \mathrm{Spin}_{2k} \to \mathrm{SO}_{2k}$ denotes the nontrivial double covering, we see that $Z(\mathrm{Spin}_{2k}) = q^{-1}(Z(\mathrm{SO}_{2k}))$. Now $Z(\mathrm{SO}_{2k}) = \{\pm\mathbb{1}\}$, the covering $q' : \mathrm{Spin}_{2k} \to \mathrm{PSO}_{2k}$ is the universal covering of PSO_{2k} and it is the composition of q and $\mathrm{SO}_{2k} \to \mathrm{PSO}_{2k}$. It follows that $Z(\mathrm{Spin}_{2k}) = (q')^{-1}([\mathbb{1}]) \cong \pi_1(\mathrm{PSO}_{2k})$.

To compute $\pi_1(\mathrm{PSO}_{2k})$ we first consider the central groupoid $\pi^Z(\mathrm{SO}_{2k})$. It is generated by the morphisms $[\beta], [\bar{\beta}]\colon \mathbb{1} \to -\mathbb{1}$, subject to the relation $([\bar{\beta}]^{-1} \circ [\beta])^2 = \mathrm{Id}_{\mathbb{1}}$; ie $[\bar{\beta}]^{-1} \circ [\beta] = [\eta]$ is the generator of $\pi_1(\mathrm{SO}_{2k}) = \mathbb{Z}/2$. The following diagram shows the named morphisms in $\pi^Z(\mathrm{SO}_{2k})$:

$$-\mathbb{1} \underset{[\bar{\beta}]}{\overset{[\beta]}{\rightleftarrows}} \mathbb{1} \circlearrowright [\eta]$$

The projection $\mathrm{SO}_{2k} \to \mathrm{PSO}_{2k}$ induces a surjective map of groupoids $\pi^Z(\mathrm{SO}_{2k}) \to \pi_1(\mathrm{PSO}_{2k})$, which sends two morphisms $[\gamma], [\gamma'] \in \pi^Z(\mathrm{SO}_{2k})$ into the same element of $\pi_1(\mathrm{PSO}_{2k})$ if and only if $[\gamma'] = [-\gamma]$, where $-\gamma$ denotes the path γ multiplied pointwise by $-\mathbb{1}$. Since

$$[-\beta] = \begin{cases} [\beta]^{-1} & \text{if } k \text{ is even,} \\ [\bar{\beta}]^{-1} & \text{if } k \text{ is odd,} \end{cases}$$

the computation of $\pi_1(\mathrm{PSO}_{2k})$ follows. □

2.5 η–turnings and the path components of $\mathcal{G}_E$

Recall that η denotes the generator of $\pi_1(\mathrm{SO}_{2k})$ and that by definition a rank-$2k$ vector bundle $E \to B$ is η–turnable if and only if the restriction induces a surjective map $\pi_1(\mathcal{G}_E) \to \pi_1(\mathrm{SO}_{2k})$. We return to our discussion of the exact sequence (2-1) from Section 2.3 and first identify it with an isomorphic exact sequence. Assuming that $b \in B$ is nondegenerate, the homomorphism $r_p\colon \mathcal{G}_E \to \mathrm{SO}_{2k}$ is onto and there is a short exact sequence of topological groups

$$\mathcal{G}_{E,0} \to \mathcal{G}_E \xrightarrow{r_p} \mathrm{SO}_{2k}, \tag{2-2}$$

where, by definition, $\mathcal{G}_{E,0} := \mathrm{Ker}(r_p) \subset \mathcal{G}_E$. Regarding (2-2) as a principal $\mathcal{G}_{E,0}$–bundle, it is classified by a map $\mathrm{SO}_{2k} \to B\mathcal{G}_{E,0}$ such that

$$\mathcal{G}_E \xrightarrow{r_p} \mathrm{SO}_{2k} \to B\mathcal{G}_{E,0} \tag{2-3}$$

is a fibration sequence. The homotopy long exact sequence of (2-1) maps isomorphically to the homotopy long exact sequences of (2-3) and (2-2) as follows:

$$\begin{array}{ccccccccc}
\pi_1(\mathcal{G}_E) & \xrightarrow{(r_p)_*} & \pi_1(\mathrm{SO}_{2k}) & \longrightarrow & \pi_1(\mathrm{SO}_{2k}, \mathcal{G}_E) & \longrightarrow & \pi_0(\mathcal{G}_E) & \longrightarrow & 0 \\
\| & & \| & & \downarrow\cong & & \| & & \\
\pi_1(\mathcal{G}_E) & \xrightarrow{(r_p)_*} & \pi_1(\mathrm{SO}_{2k}) & \longrightarrow & \pi_1(B\mathcal{G}_{E,0}) & \longrightarrow & \pi_0(\mathcal{G}_E) & \longrightarrow & 0 \\
\| & & \| & & \downarrow\cong & & \| & & \\
\pi_1(\mathcal{G}_E) & \xrightarrow{(r_p)_*} & \pi_1(\mathrm{SO}_{2k}) & \longrightarrow & \pi_0(\mathcal{G}_{E,0}) & \longrightarrow & \pi_0(\mathcal{G}_E) & \longrightarrow & 0
\end{array} \tag{2-4}$$

Now we fix the base space B and suppose that $B = SX$ is a suspension. For any rank-$2k$ vector bundle $E \to SX$, there is a homotopy equivalence $\mathcal{G}_{E,0} \simeq \mathrm{Map}((SX, *), (\mathrm{SO}_{2k}, \mathbb{1}))$ and in particular the homotopy type of $\mathcal{G}_{E,0}$ does not depend on the vector bundle E. If $|\pi_0(\mathcal{G}_{E,0})| = |[SX, \mathrm{SO}_{2k}]|$ is finite then the exact sequences above in (2-4) show that $|\pi_0(\mathcal{G}_E)|$ depends on the η–turnability of E.

Specifically, we have the following:

Theorem 2.26 *If $B = SX$ is a suspension and $n_{SX} := |[SX, \mathrm{SO}_{2k}]|$ is finite, then for any rank-$2k$ vector bundle $E \to SX$,*

$$|\pi_0(\mathcal{G}_E)| = \begin{cases} n_{SX} & \text{if } E \text{ is } \eta\text{–turnable}, \\ \frac{1}{2} n_{SX} & \text{if } E \text{ is not } \eta\text{–turnable}. \end{cases}$$

In particular, the η–turnability of rank-$2k$ vector bundles over SX with n_{SX} finite is a homotopy invariant of the gauge groups of these bundles. □

Remark 2.27 As one might expect, if a rank-$2k$ vector bundle E admits a spin structure, then the η–turnability of E is equivalent to the γ_{z_0}–turnability of the associated principal Spin_{2k}–bundle for a certain path γ_{z_0} in Spin_{2k}. Let $e \in \mathrm{Spin}_{2k}$ be the identity and define $z_0 \in Z(\mathrm{Spin}_{2k}) \setminus \{e\}$ to be the unique element mapping to $\mathbb{1} \in \mathrm{SO}_{2k}$. Since Spin_{2k} is simply connected, there is a unique path homotopy class of paths from e to z_0, and we denote this path by γ_{z_0}. Given a principal Spin_{2k}–bundle $P \to B$, we can consider the γ_{z_0}–turning problem for P. As Spin_{2k} acts on $\mathbb{R}^{2k}$ via the double covering $\mathrm{Spin}_{2k} \to \mathrm{SO}_{2k}$ and the standard action of SO_{2k}, there is a rank-$2k$ vector bundle $E_P := P \times_{\mathrm{Spin}_{2k}} \mathbb{R}^{2k}$ associated to P and it is not hard to see that the following statements are equivalent:

(1) The principal Spin_{2k}–bundle $P \to B$ is γ_{z_0}–turnable.

(2) The vector bundle E_P associated to P is η–turnable.

(3) The map $(r_p)_* \colon \pi_1(\mathcal{G}_{E_P}) \to \pi_1(\mathrm{SO}_{2k})$ is onto.

3 The turning obstruction

In this section we define the turning obstruction for bundles over suspensions.

First we consider rank-$2k$ oriented vector bundles. Over a suspension SX such a bundle corresponds to an element of $[X, \mathrm{SO}_{2k}]$. Given a path $\gamma \in \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ we first define a map

$$\mathrm{to}_\gamma \colon [X, \mathrm{SO}_{2k}] \to [X, \Omega_\gamma \mathrm{SO}_{2k}]$$

(where $\Omega_\gamma \mathrm{SO}_{2k} \subset \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ denotes the connected component of γ, see Definition 2.6) and prove that it is a complete obstruction to the γ–turnability of a bundle. We also introduce variants

$$\overline{\mathrm{to}}_\gamma \colon [X, \mathrm{SO}_{2k}] \to [X, \Omega_0 \mathrm{SO}_{2k}] \quad \text{and} \quad \mathrm{TO}_\gamma, \overline{\mathrm{TO}}_\gamma \colon [X, \mathrm{SO}_{2k}] \to [SX, \mathrm{SO}_{2k}]$$

and show that they are equivalent to to_γ. Finally we prove that these maps are homomorphisms if X is a suspension.

In Section 3.2 we consider bundles with a path-connected structure group G. Given a path γ between elements of the centre $Z(G)$, we define a generalised turning obstruction map $\mathrm{to}_\gamma \colon [X, G] \to [X, \Omega_\gamma G]$. If we fix an element $[g] \in [X, G]$, then we can regard $\mathrm{to}_\gamma([g])$ as a function of γ and we show that it is compatible with concatenation of paths. We also consider a normalised version of the turning obstruction, $\overline{\mathrm{to}}_\gamma \colon [X, G] \to [X, \Omega_0 G]$. This allows us to compare turning obstructions for different paths and we find

that $\overline{\mathrm{to}}_\gamma = \overline{\mathrm{to}}_{a\gamma}$ for any $a \in Z(G)$ (where $(a\gamma)(t) = a\gamma(t)$). When $Z(G)$ is discrete, we introduce the quotient $PG = G/Z(G)$ and use these observations to show that the turning obstructions are determined by a map $\widehat{\mathrm{to}}_{\,\cdot}(\,\cdot\,)\colon \pi_1(PG)\times[X,G]\to[X,\Omega_0 G]$, which is a homomorphism in the first variable (and also in the second one, if X is a suspension). As an application we prove Theorem 1.7.

3.1 The turning obstruction for vector bundles

Let X be a CW–complex and C_0X and C_1X two copies of the cone on X, so that $SX = C_0X \cup_X C_1X$. By [9, Chapter 8, Theorem 8.2], the set of isomorphism classes of oriented rank-$2k$ bundles over SX is in bijection with $[X, \mathrm{SO}_{2k}]$, the set of homotopy classes of maps from X to SO_{2k}. A bundle E corresponds to its *clutching function* $g\colon X\to \mathrm{SO}_{2k}$ between two local trivialisations $\varphi_i\colon C_iX\times\mathbb{R}^{2k}\to E|_{C_iX}$, defined by $\varphi_0^{-1}|_X\circ\varphi_1|_X(x,v) = (x, g(x)v)$.

Definition 3.1 (the γ–turning obstruction) Let $\gamma\in\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$.

(a) We define the map $\rho_\gamma\colon \mathrm{SO}_{2k}\to\Omega_\gamma\mathrm{SO}_{2k}$ by $\rho_\gamma(A) = (t\mapsto A\gamma(t)A^{-1})$.

(b) For any CW–complex X the *γ–turning obstruction map* is $\mathrm{to}_\gamma := (\rho_\gamma)_*\colon [X,\mathrm{SO}_{2k}]\to[X,\Omega_\gamma\mathrm{SO}_{2k}]$.

Let $0\in[X,\Omega_\gamma\mathrm{SO}_{2k}]$ denote the homotopy class of the constant map. Definition 3.1 is justified by the following:

Proposition 3.2 *Let E be an oriented rank-$2k$ bundle over SX with clutching function $g\colon X\to\mathrm{SO}_{2k}$. Then E is γ–turnable if and only if* $\mathrm{to}_\gamma([g]) = 0\in[X,\Omega_\gamma\mathrm{SO}_{2k}]$.

Proof Recall that γ–turnings of E can be identified with sections of the associated γ–turning bundle $\mathrm{Turn}^\gamma(E) = \mathrm{Fr}(E)\times_{\mathrm{SO}_{2k}}\Omega_\gamma\mathrm{SO}_{2k}$; see Lemma 2.12 and Remark 2.19. The local trivialisations φ_i of E induce local trivialisations $\overline{\varphi}_i\colon C_iX\times\Omega_\gamma\mathrm{SO}_{2k}\to\mathrm{Turn}^\gamma(E)|_{C_iX}$ of $\mathrm{Turn}^\gamma(E)$. By construction, the clutching function $g\colon X\to\mathrm{SO}_{2k}$ is also the clutching function of $\mathrm{Turn}^\gamma(E)$ — recall that SO_{2k} acts on $\Omega_\gamma\mathrm{SO}_{2k}$ by pointwise conjugation.

Since C_iX is contractible and $\Omega_\gamma\mathrm{SO}_{2k}$ is connected, each restriction $\mathrm{Turn}^\gamma(E)|_{C_iX}$ has a unique section $s_i\colon C_iX\to\mathrm{Turn}^\gamma(E)|_{C_iX}$ up to homotopy, given by $s_i(y) = \overline{\varphi}_i(y,\gamma)$. Hence a global section of $\mathrm{Turn}^\gamma(E)$ exists if and only if $s_0|_X$ and $s_1|_X$ are homotopic sections. For $x\in X$ we have

$$s_0(x) = \overline{\varphi}_0(x,\gamma)\quad\text{and}\quad s_1(x) = \overline{\varphi}_1(x,\gamma) = \overline{\varphi}_0\circ\overline{\varphi}_0^{-1}\circ\overline{\varphi}_1(x,\gamma) = \overline{\varphi}_0(x,\gamma^{g(x)}) = \overline{\varphi}_0(x,\rho_\gamma(g(x))),$$

where $\gamma^{g(x)}$ denotes the action of $g(x)\in\mathrm{SO}_{2k}$ on $\gamma\in\Omega_\gamma\mathrm{SO}_{2k}$. These sections are homotopic if and only if $\rho_\gamma\circ g\colon X\to\Omega_\gamma\mathrm{SO}_{2k}$ is homotopic to the constant map with value γ, ie if and only if $\mathrm{to}_\gamma([g]) = 0$. □

The turning obstruction is a map of pointed sets, but $[X,\mathrm{SO}_{2k}]$ is a group and we will show that $[X,\Omega_\gamma\mathrm{SO}_{2k}]$ can also be equipped with a group structure and that to_γ and related maps are often group homomorphisms; see Lemma 3.6.

Let $\Omega_0\mathrm{SO}_{2k} \subset \Omega\mathrm{SO}_{2k}$ denote the component of contractible loops and define the homeomorphism

$$p_\gamma : \Omega_\gamma \mathrm{SO}_{2k} \to \Omega_0 \mathrm{SO}_{2k}, \quad \delta \mapsto (t \mapsto \delta(t)\gamma(t)^{-1}),$$

as well as the commutator map $\overline{\rho}_\gamma := p_\gamma \circ \rho_\gamma$,

$$\overline{\rho}_\gamma : \mathrm{SO}_{2k} \to \Omega_0 \mathrm{SO}_{2k}, \quad A \mapsto (t \mapsto A\gamma(t)A^{-1}\gamma(t)^{-1}).$$

Definition 3.3 (normalised γ–turning obstruction) Let $\gamma \in \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$. For any CW–complex X, the *normalised γ–turning obstruction map* is $\overline{\mathrm{to}}_\gamma := (\overline{\rho}_\gamma)_* : [X, \mathrm{SO}_{2k}] \to [X, \Omega_0\mathrm{SO}_{2k}]$.

Since p_γ is a homeomorphism, the induced map $(p_\gamma)_* : [X, \Omega_\gamma \mathrm{SO}_{2k}] \to [X, \Omega_0\mathrm{SO}_{2k}]$, $[h] \mapsto [p_\gamma \circ h]$, is a bijection which preserves 0, and hence an oriented rank-$2k$ bundle $E \to SX$ with clutching function $g : X \to \mathrm{SO}_{2k}$ is γ–turnable if and only if $\overline{\mathrm{to}}_\gamma([g]) = 0$. For computing to_γ and $\overline{\mathrm{to}}_\gamma$ it is useful to consider their adjointed versions, which we will define below.

Definition 3.4 (forgetful adjoints) Let $h : X \to \Omega_\gamma \mathrm{SO}_{2k}$ and $h' : X \to \Omega_0 \mathrm{SO}_{2k}$ be continuous maps. By taking their adjoints

$$\mathrm{ad}(h) : SX \to \mathrm{SO}_{2k}, \quad [x,t] \mapsto h(x)(t), \qquad \text{and} \qquad \mathrm{ad}(h') : SX \to \mathrm{SO}_{2k}, \quad [x,t] \mapsto h'(x)(t),$$

we define the *forgetful adjoint maps* $\mathrm{ad} : [X, \Omega_\gamma \mathrm{SO}_{2k}] \to [SX, \mathrm{SO}_{2k}]$ and $\mathrm{ad} : [X, \Omega_0 \mathrm{SO}_{2k}] \to [SX, \mathrm{SO}_{2k}]$. (We call these maps "forgetful" because $[\mathrm{ad}(h)]$ and $[\mathrm{ad}(h')]$ are regarded as elements of $[SX, \mathrm{SO}_{2k}]$ rather than of the more restricted sets of homotopy classes on which the inverse adjoint maps $[\mathrm{ad}(h)] \mapsto [h]$ and $[\mathrm{ad}(h')] \mapsto [h']$ are naturally defined.)

If X is connected, then the forgetful adjoint maps are bijections and they preserve 0, the homotopy class of the constant map.

Definition 3.5 (adjointed γ–turning obstructions) Define $\mathrm{TO}_\gamma := \mathrm{ad} \circ \mathrm{to}_\gamma : [X, \mathrm{SO}_{2k}] \to [SX, \mathrm{SO}_{2k}]$ and $\overline{\mathrm{TO}}_\gamma := \mathrm{ad} \circ \overline{\mathrm{to}}_\gamma : [X, \mathrm{SO}_{2k}] \to [SX, \mathrm{SO}_{2k}]$.

Lemma 3.6 (a) *Let X be a CW–complex. Then $\mathrm{TO}_\gamma = \overline{\mathrm{TO}}_\gamma : [X, \mathrm{SO}_{2k}] \to [SX, \mathrm{SO}_{2k}]$.*

(b) *If X is a suspension, then $\mathrm{to}_\gamma, \overline{\mathrm{to}}_\gamma, \mathrm{TO}_\gamma$ and $\overline{\mathrm{TO}}_\gamma$ are each homomorphisms of abelian groups.*

Proof (a) We show that the diagram

$$\begin{array}{ccccc}
 & & [X, \Omega_\gamma \mathrm{SO}_{2k}] & \xrightarrow{\mathrm{ad}} & [SX, \mathrm{SO}_{2k}] \\
 & \overset{\mathrm{to}_\gamma}{\nearrow} & \Big\downarrow{\scriptstyle (p_\gamma)_*} & & \Big\| \\
[X, \mathrm{SO}_{2k}] & & & & \\
 & \underset{\overline{\mathrm{to}}_\gamma}{\searrow} & & & \\
 & & [X, \Omega_0 \mathrm{SO}_{2k}] & \xrightarrow{\mathrm{ad}} & [SX, \mathrm{SO}_{2k}]
\end{array}$$

commutes. The left-hand triangle commutes be definition. For the right-hand square, consider the path of paths $s \mapsto \gamma_s$, where $\gamma_s\colon I \to \mathrm{SO}_{2k}$ is defined by $\gamma_s(t) = \gamma(st)$ for $s, t \in I$. Then the map

$$H\colon SX \times I \to \mathrm{SO}_{2k}, \quad ([x,t],s) \mapsto g(x)\gamma(t)g(x)^{-1}\gamma_s(t)^{-1},$$

is a homotopy from $\mathrm{ad}(\rho_\gamma \circ g)$ to $\mathrm{ad}(\overline{\rho}_\gamma \circ g)$, which proves that the square commutes.

(b) First note that $\Omega_0\mathrm{SO}_{2k}$ is a topological group (via pointwise multiplication of loops), so we can use the homeomorphism p_γ to get a topological group structure on $\Omega_\gamma\mathrm{SO}_{2k}$. Hence for $H = \mathrm{SO}_{2k}$, $\Omega_0\mathrm{SO}_{2k}$ or $\Omega_\gamma\mathrm{SO}_{2k}$ and any space Y the set $[Y, H]$ inherits a group structure from H (and with these group structures $(p_\gamma)_*$ is automatically an isomorphism).

If Y is pointed, then the set $[Y, H]_*$ of homotopy classes of basepoint-preserving maps is also a group and the forgetful map $[Y, H]_* \to [Y, H]$ is an isomorphism. If Y is a suspension, then for any space Z the set $[Y, Z]_*$ has a group structure coming from the co-H-space structure on Y and on the sets $[Y, H]_*$ the two group structures coincide and they are abelian. Since the group structure on $[Y, Z]_*$ is natural in Z, it follows that if X is a suspension, then the maps to_γ and $\overline{\mathrm{to}}_\gamma$ (which are induced by maps of spaces) are homomorphisms (and all groups involved are abelian).

Finally, loop concatenation gives $\Omega_0\mathrm{SO}_{2k}$ an H-space structure and the induced group structure on $[X, \Omega_0\mathrm{SO}_{2k}]$ coincides with the previously defined one. By comparing this with the group structure on $[SX, \mathrm{SO}_{2k}]$ coming from the suspension SX, we obtain that the forgetful adjoint maps and hence TO_γ and $\overline{\mathrm{TO}}_\gamma$, are also homomorphisms. □

Remark 3.7 By Lemma 3.6(b), if $X = SY$ is a suspension, then the set of isomorphism classes of γ–turnable bundles over $SX = S^2Y$ can be identified with a subgroup of $[X, \mathrm{SO}_{2k}]$.

Question 3.8 (a) The sets $[X, \mathrm{SO}_{2k}]$ and $[SX, \mathrm{SO}_{2k}]$ have natural group structures even when X is not a suspension (or co-H-space). Is TO_γ a group homomorphism for an arbitrary X?

(b) Isomorphism classes of rank-$2k$ oriented bundles over a space B are in bijection with $[B, B\mathrm{SO}_{2k}]$, so when B is a suspension, TO_γ can be regarded as a map $[B, B\mathrm{SO}_{2k}] \to [B, \mathrm{SO}_{2k}]$. Is there a similar γ–turning obstruction map for bundles over an arbitrary space B?

We next briefly discuss the behaviour of the turning obstruction under stabilisation: we will return to this topic in greater detail in Section 5. Let $i\colon \mathrm{SO}_{2k} \to \mathrm{SO}_{2k+2}$ denote the standard inclusion and let $S = i_*\colon [X, \mathrm{SO}_{2k}] \to [X, \mathrm{SO}_{2k+2}]$ denote the stabilisation map induced by i. Given a path $\gamma_0 \in \Omega_{\pm\mathbb{1}}\mathrm{SO}_2$, taking the orthogonal sum with γ_0 defines a map $i_{\gamma_0}\colon \Omega_\gamma\mathrm{SO}_{2k} \to \Omega_{\gamma\oplus\gamma_0}\mathrm{SO}_{2k+2}$. It is clear from the definitions that the turning obstructions satisfy $\mathrm{to}_{\gamma\oplus\gamma_0}([i \circ g]) = i_{\gamma_0 *}(\mathrm{to}_\gamma([g]))$ and indeed we have:

Lemma 3.9 *Let X be a CW–complex and $g\colon X \to \mathrm{SO}_{2k}$ a map. Then the adjointed γ–turning obstruction satisfies $\mathrm{TO}_{\gamma\oplus\gamma_0}([i \circ g]) = S(\mathrm{TO}_\gamma([g]))$. In particular, $\mathrm{TO}_\beta([i \circ g]) = \mathrm{TO}_{\overline{\beta}}([i \circ g])$.*

Proof Write $A \oplus B \in \mathrm{SO}_{2k+2}$ for the block sum of matrices $A \in \mathrm{SO}_{2k}$ and $B \in \mathrm{SO}_2$ and consider the path of paths $s \mapsto (\gamma_0)_s$, where $(\gamma_0)_s \colon I \to \mathrm{SO}_2$ is defined by $(\gamma_0)_s(t) = \gamma_0(st)$ for $s, t \in I$. Then

$$H \colon SX \times I \to \mathrm{SO}_{2k+2}, \quad ([x,t],s) \mapsto g(x)\gamma(t)g(x)^{-1} \oplus (\gamma_0)_{(1-s)}(t),$$

is a homotopy from $\mathrm{ad}(\rho_{\gamma\oplus\gamma_0} \circ (i \circ g))$ to $i \circ \mathrm{ad}(\rho_\gamma(g))$, which proves the first statement of the lemma. In particular, $\mathrm{TO}_{\gamma\oplus\gamma_0}([i \circ g])$ is independent of the choice of $\gamma_0 \in \Omega_{\pm\mathbb{1}}\mathrm{SO}_2$. Since for any $\gamma \in \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ the map $\pi_0(\Omega_{\pm\mathbb{1}}\mathrm{SO}_2) \to \pi_0(\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k+2})$ given by $[\gamma_0] \mapsto [\gamma \oplus \gamma_0]$ is surjective, the second statement follows. □

We conclude this subsection with a remark on a related point of view on the turning obstruction.

Remark 3.10 Consider the associated γ–turning bundle of the universal bundle $V\mathrm{SO}_{2k}$ (see Definition 2.13 and Remark 2.19) and the Puppe sequence

$$\cdots \to \Omega B T^\gamma_{2k} \to \Omega B\mathrm{SO}_{2k} \to \Omega_\gamma \mathrm{SO}_{2k} \to B T^\gamma_{2k} \to B\mathrm{SO}_{2k},$$

where Ω denotes the based loops functor. After applying the functor $[X, -]_*$ to this sequence and the adjunction $[X, \Omega Y]_* \cong [\Sigma X, Y]_*$, where ΣX denotes the reduced suspension, we obtain an exact sequence

$$\cdots \to [\Sigma X, BT^\gamma_{2k}]_* \to [\Sigma X, B\mathrm{SO}_{2k}]_* \xrightarrow{\partial} [X, \Omega_\gamma \mathrm{SO}_{2k}]_* \to [X, BT^\gamma_{2k}]_* \to [X, B\mathrm{SO}_{2k}]_*.$$

The arguments in the proof of Proposition 3.2 also show that the γ–turning obstruction can be identified with the boundary map ∂ (note that $[\Sigma X, B\mathrm{SO}_{2k}]_* \cong [X, \mathrm{SO}_{2k}]_* \cong [X, \mathrm{SO}_{2k}]$ and $[X, \Omega_\gamma \mathrm{SO}_{2k}]_* \cong [X, \Omega_\gamma \mathrm{SO}_{2k}]$). So by using the exactness of the sequence we obtain an alternative proof of Proposition 2.15 for bundles over suspensions.

3.2 General turning obstructions

In this subsection we define the turning obstruction for the general turnings of principal bundles, as in Section 2.4. This will help us establish some basic properties of the turning obstruction for vector bundles.

Definition 3.11 Let G be a path-connected topological group, $a, b \in G$ arbitrary elements and $\gamma \colon I \to G$ a path in G. We introduce the following notation:

- ΩG denotes the space of loops in G based at the identity element.
- $\Omega_0 G \subseteq \Omega G$ is the space of nullhomotopic loops, ie the connected component of the constant loop.
- $\Omega_{a,b} G$ is the space of paths in G from a to b.
- $\Omega_\gamma G$ is the space of paths homotopic (rel ∂I) to γ, ie the connected component of γ in $\Omega_{\gamma(0),\gamma(1)} G$.
- $\Omega_Z G = \bigcup_{a,b \in Z(G)} \Omega_{a,b} G$, where $Z(G)$ denotes the centre of G.

Definition 3.12 (parametrised central groupoid of G) Let G be a path-connected topological group. The *parametrised central groupoid of* G has objects the elements of $Z(G)$. The set of all morphisms is $[G, \Omega_Z G]$. The set of morphisms from a to b is $[G, \Omega_{a,b} G]$. Given objects $a, b, c \in Z(G)$ and maps $f_1 : G \to \Omega_{a,b} G$ and $f_2 : G \to \Omega_{b,c} G$, the composition $[f_2] \circ [f_1]$ is represented by $f_1 * f_2 : G \to \Omega_{a,c} G$, defined by $(f_1 * f_2)(x) = f_1(x) * f_2(x)$, where $*$ denotes concatenation of paths.

Let G be a path-connected topological group and recall that for a CW–complex X, isomorphism classes of principal G–bundles over SX with structure group G are in bijection with $[X, G]$.

Definition 3.13 (general γ–turning obstruction) Let $\gamma \in \Omega_Z G$.

(a) We define the map $\rho_\gamma : G \to \Omega_\gamma G$ by $\rho_\gamma(x) = (t \mapsto x\gamma(t)x^{-1})$.

(b) For any CW–complex X the *γ–turning obstruction map* is $\mathrm{to}_\gamma := (\rho_\gamma)_* : [X, G] \to [X, \Omega_\gamma G]$.

Note that the image of ρ_γ is contained in $\Omega_{\gamma(0),\gamma(1)} G$, because $\gamma(0), \gamma(1) \in Z(G)$, and in particular in the component $\Omega_\gamma G$, because G is path-connected and ρ_γ sends the identity element to γ. We let $0 \in [X, \Omega_\gamma G]$ denote the homotopy class of the constant map. The proof of the following proposition is entirely analogous to the proof of Proposition 3.2.

Proposition 3.14 *Suppose that P is a G–bundle over SX with clutching function $g : X \to G$. Then P is γ–turnable if and only if* $\mathrm{to}_\gamma([g]) = 0 \in [X, \Omega_\gamma G]$. □

Definition 3.15 Let $\mathrm{to}_G : \pi^Z(G) \to [G, \Omega_Z G]$ be defined by $\mathrm{to}_G([\gamma]) = [\rho_\gamma]$.

This map is well defined, because a path homotopy between γ and γ' determines a homotopy between the maps ρ_γ and $\rho_{\gamma'}$. Since composition is defined in terms of concatenation both in $\pi^Z(G)$ and $[G, \Omega_Z G]$, it is a map of groupoids (with the identity map on the objects). With this notation,

$$\mathrm{to}_\gamma = \mathrm{to}_G([\gamma])_* : [X, G] \to [X, \Omega_\gamma G] \subseteq [X, \Omega_Z G].$$

Definition 3.16 Let $PG = G/Z(G)$. Let $p_G : \pi^Z(G) \to \pi_1(PG)$ denote the map of groupoids induced by the projection $G \to PG$ (where $\pi_1(PG)$ is regarded as a groupoid on one object).

Every $\gamma \in \Omega_Z G$ determines a homeomorphism $p'_\gamma : \Omega_\gamma G \to \Omega_0 G$ which sends a path δ to the loop $t \mapsto \delta(t)\gamma(t)^{-1}$. If $[\gamma] = [\gamma']$, so that $\Omega_\gamma G = \Omega_{\gamma'} G$, then these homeomorphisms are homotopic; hence they induce a well-defined map $p'_{[\gamma]} : [G, \Omega_\gamma G] \to [G, \Omega_0 G]$. On the other hand, if $[\gamma] \neq [\gamma']$, then $\Omega_\gamma G$ and $\Omega_{\gamma'} G$ (and hence $[G, \Omega_\gamma G]$ and $[G, \Omega_{\gamma'} G]$) are disjoint, so p'_G below is well defined:

Definition 3.17 Let $p'_G : [G, \Omega_Z G] \to [G, \Omega_0 G]$ be the union of the maps $p'_{[\gamma]} : [G, \Omega_\gamma G] \to [G, \Omega_0 G]$.

Since $\Omega_0 G$ is an H-space, $[G, \Omega_0 G]$ is a group, ie a groupoid on one object. For every pair γ, γ' of composable paths in $\Omega_Z G$, the diagram

$$\begin{array}{ccc} \Omega_\gamma G \times \Omega_{\gamma'} G & \xrightarrow{*} & \Omega_{\gamma * \gamma'} G \\ \downarrow{\scriptstyle p'_\gamma \times p'_{\gamma'}} & & \downarrow{\scriptstyle p'_{\gamma*\gamma'}} \\ \Omega_0 G \times \Omega_0 G & \xrightarrow{*} & \Omega_0 G \end{array}$$

commutes. Hence p'_G is a map of groupoids.

Proposition 3.18 *Suppose that G is a path-connected topological group and $Z(G)$ is discrete. Then to_G descends to a homomorphism $\widehat{\mathrm{to}}_G \colon \pi_1(PG) \to [G, \Omega_0 G]$ of abelian groups, ie there is a commutative diagram of groupoids*:

$$\begin{array}{ccc} \pi^Z(G) & \xrightarrow{\mathrm{to}_G} & [G, \Omega_Z G] \\ \downarrow{\scriptstyle p_G} & & \downarrow{\scriptstyle p'_G} \\ \pi_1(PG) & \xrightarrow{\widehat{\mathrm{to}}_G} & [G, \Omega_0 G] \end{array}$$

Proof First we define $\widehat{\mathrm{to}}_G$. Since $Z(G)$ is discrete, the projection $G \to PG$ is a covering, so a loop $\gamma \in \Omega PG$ can be lifted to a path $\widetilde{\gamma} \in \Omega_Z G$ and we define $\widehat{\mathrm{to}}_G([\gamma]) = p'_G \circ \mathrm{to}_G([\widetilde{\gamma}])$. If $\widetilde{\gamma}'$ is another lift of γ, then $\widetilde{\gamma}' = a\widetilde{\gamma}$ for $a = \widetilde{\gamma}'(0)\widetilde{\gamma}(0)^{-1} \in Z(G)$. Then

$$(p'_{\widetilde{\gamma}'} \circ \rho_{\widetilde{\gamma}'}(x))(t) = x\widetilde{\gamma}'(t)x^{-1}\widetilde{\gamma}'(t)^{-1} = xa\widetilde{\gamma}(t)x^{-1}\widetilde{\gamma}(t)^{-1}a^{-1} = x\widetilde{\gamma}(t)x^{-1}\widetilde{\gamma}(t)^{-1} = (p'_{\widetilde{\gamma}} \circ \rho_{\widetilde{\gamma}}(x))(t),$$

and so $p'_G \circ \mathrm{to}_G([\widetilde{\gamma}']) = p'_G \circ \mathrm{to}_G([\widetilde{\gamma}])$. Therefore $\widehat{\mathrm{to}}_G([\gamma])$ does not depend on the choice of the lift $\widetilde{\gamma}$. A homotopy of γ can be lifted to a homotopy of $\widetilde{\gamma}$, so $\widehat{\mathrm{to}}_G([\gamma])$ is also independent of the choice of the representative γ of the homotopy class $[\gamma]$. Therefore $\widehat{\mathrm{to}}_G$ is well defined. By its construction, the diagram commutes. Finally, a lift of the concatenation of two loops in ΩPG is the concatenation of lifts of the loops, so $\widehat{\mathrm{to}}_G$ is a map of groupoids, ie a group homomorphism. □

In light of the above, it is useful define the normalised turning obstruction map in the general setting:

Definition 3.19 Suppose that G is a path-connected topological group and $\gamma \in \Omega_Z G$.

(a) Let $\overline{\rho}_\gamma \colon G \to \Omega_0 G$ be defined by $\overline{\rho}_\gamma(x) = (t \mapsto x\gamma(t)x^{-1}\gamma(t)^{-1})$.

(b) For any CW–complex X, let $\overline{\mathrm{to}}_\gamma = (\overline{\rho}_\gamma)_* \colon [X, G] \to [X, \Omega_0 G]$.

That is, $\overline{\rho}_\gamma = p'_\gamma \circ \rho_\gamma$ and hence $\overline{\mathrm{to}}_\gamma = (p'_\gamma)_* \circ \mathrm{to}_\gamma \colon [X, G] \to [X, \Omega_0 G]$. Since p'_γ is a homeomorphism, $(p'_\gamma)_* \colon [X, \Omega_\gamma G] \to [X, \Omega_0 G]$ is a bijection, so computing $\overline{\mathrm{to}}_\gamma$ is equivalent to computing to_γ. In particular, $\overline{\mathrm{to}}_\gamma$ and to_γ vanish for the same bundles. From our earlier arguments we have:

Proposition 3.20 *Suppose that G is a path-connected topological group and $g \colon X \to G$ is a map.*

(a) *If $\gamma, \gamma' \in \Omega_Z G$ are composable paths, then $\overline{\mathrm{to}}_{\gamma*\gamma'}([g]) = \overline{\mathrm{to}}_\gamma([g]) + \overline{\mathrm{to}}_{\gamma'}([g])$.*

(b) *For any $\gamma \in \Omega_Z G$ and $a \in Z(G)$, we have $\overline{\mathrm{to}}_{a\gamma}([g]) = \overline{\mathrm{to}}_\gamma([g])$.* □

If $Z(G)$ is discrete, then we can use Proposition 3.18 to describe $\overline{\mathrm{to}}_\gamma([g])$, regarded as a two-variable function in γ and $[g]$, in terms of a simpler function. Hence we define

$$\widehat{\mathrm{to}}_{\,\cdot}(\,\cdot\,)\colon \pi_1(PG)\times[X,G]\to[X,\Omega_0 G],\quad \widehat{\mathrm{to}}_{[\gamma]}([g])=\widehat{\mathrm{to}}_G([\gamma])\circ[g].$$

Equivalently, $\widehat{\mathrm{to}}_{[\gamma]}([g])=g^*(\widehat{\mathrm{to}}_G([\gamma]))=\widehat{\mathrm{to}}_G([\gamma])_*([g])$.

Proposition 3.21 *Suppose that G is a path-connected topological group and $Z(G)$ is discrete. For every CW–complex X, the maps $\overline{\mathrm{to}}_{\,\cdot}(\,\cdot\,)$ and $\widehat{\mathrm{to}}_{\,\cdot}(\,\cdot\,)$ satisfy*:

(a) $\overline{\mathrm{to}}_\gamma([g])=\widehat{\mathrm{to}}_{p_G([\gamma])}([g])$ *for every $\gamma\in\Omega_Z G$ and $[g]\in[X,G]$.*

(b) $\widehat{\mathrm{to}}_{\,\cdot}([g])\colon\pi_1(PG)\to[X,\Omega_0 G]$ *is a homomorphism of abelian groups for every $[g]\in[X,G]$.*

(c) *If X is a suspension, then the map $\widehat{\mathrm{to}}_{[\gamma]}\colon[X,G]\to[X,\Omega_0 G]$ is a homomorphism of abelian groups for every $[\gamma]\in\pi_1(PG)$.*

Proof Part (a) follows from the commutativity of the diagram in Proposition 3.18.

Part (b) holds, because $\widehat{\mathrm{to}}_G$ is a homomorphism and the induced map $g^*\colon[G,\Omega_0 G]\to[X,\Omega_0 G]$ is a homomorphism for every $[g]\in[X,G]$, because $\Omega_0 G$ is an H-space.

Part (c) holds, because $\widehat{\mathrm{to}}_{[\gamma]}$ is induced by a map of spaces. □

Remark 3.22 Just as with the turning obstruction for vector bundles, we can take the forgetful adjoints of to_γ and $\overline{\mathrm{to}}_\gamma$ and define

$$\mathrm{TO}_\gamma:=\mathrm{ad}\circ\mathrm{to}_\gamma\colon[X,G]\to[SX,G]\quad\text{and}\quad\overline{\mathrm{TO}}_\gamma:=\mathrm{ad}\circ\overline{\mathrm{to}}_\gamma\colon[X,G]\to[SX,G].$$

We leave the reader to formulate and verify the obvious generalisation of Lemma 3.6. Then by Proposition 3.14, a principal G–bundle $P\to SX$ with clutching function $g\colon X\to G$ is γ–turnable if and only if $\mathrm{TO}_\gamma([g])=0$, equivalently if $\overline{\mathrm{TO}}_\gamma([g])=0$.

We now deduce some consequences of Proposition 3.21. Recall that $[\eta]\in\pi_1(\mathrm{SO}_{2k})$ is the generator.

Theorem 3.23 *Let $\gamma\in\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$, let X be a CW–complex and $E\to SX$ a rank-$2k$ vector bundle with clutching function $g\colon X\to\mathrm{SO}_{2k}$. Then*:

(a) $2\,\overline{\mathrm{to}}_\eta([g])=0$.

(b) $\overline{\mathrm{to}}_{\eta*\gamma}([g])=\overline{\mathrm{to}}_\eta([g])+\overline{\mathrm{to}}_\gamma([g])$.

(c) *If k is even, then $2\,\overline{\mathrm{to}}_\gamma([g])=0$.*

(d) *If k is odd, then $2\,\overline{\mathrm{to}}_\gamma([g])=\overline{\mathrm{to}}_\eta([g])$ and hence $4\,\overline{\mathrm{to}}_\gamma([g])=0$.*

Proof Parts (a) and (b) are direct applications of Proposition 3.20 a).

Parts (c) and (d) follow from Proposition 3.21(a), Proposition 3.21(b) and the fact that $\pi_1(\mathrm{PSO}_{2k})\cong(\mathbb{Z}/2)^2$ when k is even and $\pi_1(\mathrm{PSO}_{2k})\cong\mathbb{Z}/4$ when $k>1$ is odd; see Lemma 2.25. □

3.3 Samelson products and turning obstructions

In this subsection we relate turning obstructions to Samelson products. The Samelson product is a classical operation in algebraic topology [15], and Samelson products can be delicate to compute. First, we show that taking the Samelson product with some loop $[\gamma] \in \pi_1(G)$ coincides with the normalised turning obstruction map $\overline{\mathrm{TO}}_\gamma$ (after suitable identifications); see Lemma 3.24. Second, we show that turning obstructions in G are determined by Samelson products in PG; see Corollary 3.27. As an application we determine some Samelson products based on our calculations of turning obstructions in Section 4.3; see Proposition 3.29. Finally, we show that our results on the η–turning obstruction have consequences for the high-dimensional topology of related gauge groups.

We start by recalling the definition of the Samelson product, in the special case when one of the operands is a loop. Assume that X is connected with $x_0 \in X$ a basepoint and let $g\colon (X, x_0) \to (G, e)$ be a based map. Let $\gamma\colon (S^1, 1) \to (G, e)$ be a map representing $[\gamma] \in \pi_1(G, e)$. Then there is a well-defined map

$$\mathrm{comm}_{g,\gamma}\colon X \wedge S^1 = \Sigma X \to G, \quad [x, t] \mapsto g(x)\gamma(t)g(x)^{-1}\gamma(t)^{-1}.$$

This construction gives rise to the *Samelson product*

$$[X, G]_* \times \pi_1(G, e) \to [\Sigma X, G]_*, \quad ([g], [\gamma]) \mapsto \langle [g], [\gamma] \rangle := [\mathrm{comm}_{g,\gamma}].$$

We can identify the set $[\Sigma X, G]_*$ with $[SX, G]$ via the forgetful map $[\Sigma X, G]_* \to [\Sigma X, G]$ and the map $[\Sigma X, G] \to [SX, G]$ induced by the collapse map $SX \to \Sigma X$ (which are both isomorphisms). The following lemma is a direct consequence of the definitions, where $\overline{\mathrm{TO}}_\gamma$ is defined in Remark 3.22.

Lemma 3.24 *For any $[\gamma] \in \pi_1(G, e)$ and $g\colon (X, x_0) \to (G, e)$, we have*

$$\overline{\mathrm{TO}}_\gamma([g]) = \langle [g], [\gamma] \rangle \in [SX, G]. \qquad \square$$

Lemma 3.24 implies that certain Samelson products can be computed as a special case of turning obstructions. On the other hand, we next show that turning obstructions can be computed from certain Samelson products.

Definition 3.25 Let G be a path-connected topological group. Define the map $\mathrm{sp}_G\colon \pi_1(G) \to [G, \Omega_0 G]$ by $\mathrm{sp}_G([\gamma]) = [x \mapsto (t \mapsto x\gamma(t)x^{-1}\gamma(t)^{-1})]$.

The map sp_G is a homomorphism, because the group structure of $[G, \Omega_0 G]$ can be defined via concatenation in $\Omega_0 G$. This homomorphism encodes the Samelson product (similarly to how to_G encodes the turning obstruction): if $[\gamma] \in \pi_1(G)$ and $g\colon X \to G$, then $\langle [g], [\gamma] \rangle \in [\Sigma X, G]_*$ is the adjoint of $\mathrm{sp}_G([\gamma]) \circ [g] \in [X, \Omega_0 G]_*$.

Let $\pi\colon G \to PG$ denote the projection.

Proposition 3.26 *Suppose that G is a path-connected topological group and $Z(G)$ is discrete. Then there is a commutative diagram of groups*

$$\begin{array}{ccc} \pi_1(PG) & \xrightarrow{\widehat{\mathrm{to}}_G} & [G, \Omega_0 G] \\ {\scriptstyle \mathrm{sp}_{PG}}\downarrow & & \downarrow{\scriptstyle (\Omega_0\pi)_*} \\ [PG, \Omega_0 PG] & \xrightarrow{\pi^*} & [G, \Omega_0 PG] \end{array}$$

where $(\Omega_0\pi)_*$ is an isomorphism.

Proof Let $\gamma \in \Omega PG$. Since $Z(G)$ is discrete, the projection $\pi\colon G \to PG$ is a covering, so γ can be lifted to a path $\widetilde{\gamma} \in \Omega_Z G$. By definition we have $\widehat{\mathrm{to}}_G([\gamma]) = [x \mapsto (t \mapsto x\widetilde{\gamma}(t)x^{-1}\widetilde{\gamma}(t)^{-1})]$. Its image in $[G, \Omega_0 PG]$ is

$$[x \mapsto (t \mapsto \pi(x\widetilde{\gamma}(t)x^{-1}\widetilde{\gamma}(t)^{-1}))] = [x \mapsto (t \mapsto \pi(x)\gamma(t)\pi(x)^{-1}\gamma(t)^{-1})],$$

using that $\pi \circ \widetilde{\gamma} = \gamma$. The image of $\mathrm{sp}_G([\gamma]) = [y \mapsto (t \mapsto y\gamma(t)y^{-1}\gamma(t)^{-1})]$ in $[G, \Omega_0 PG]$ is also $[x \mapsto (t \mapsto \pi(x)\gamma(t)\pi(x)^{-1}\gamma(t)^{-1})]$, therefore the diagram commutes.

Since $\pi\colon G \to PG$ is a covering, every nullhomotopic loop in PG can be lifted to a nullhomotopic loop in G, hence $\Omega_0\pi\colon \Omega_0 G \to \Omega_0 PG$ is a homeomorphism. Moreover, this homeomorphism respects the H-space structures, and therefore $(\Omega_0\pi)_*\colon [G, \Omega_0 G] \to [G, \Omega_0 PG]$ is an isomorphism. □

Recall that by Proposition 3.21, the turning obstruction map $\overline{\mathrm{to}}_\gamma$ can be computed from $\widehat{\mathrm{to}}_G$; namely $\overline{\mathrm{to}}_\gamma([g]) = \widehat{\mathrm{to}}_G(p_G([\gamma])) \circ [g] \in [X, \Omega_0 G]$ for every $\gamma \in \Omega_Z G$ and $g\colon X \to G$. By Proposition 3.26, we have $\widehat{\mathrm{to}}_G(p_G([\gamma])) = [(\Omega_0\pi)^{-1}] \circ \mathrm{sp}_{PG}(p_G([\gamma])) \circ [\pi]$, hence $\overline{\mathrm{to}}_\gamma([g]) = [(\Omega_0\pi)^{-1}] \circ \mathrm{sp}_{PG}(p_G([\gamma])) \circ [\pi \circ g]$. This shows that $\overline{\mathrm{to}}_\gamma$ is determined by $\mathrm{sp}_{PG}(p_G([\gamma]))$; ie turning obstructions in G are determined by Samelson products in PG. We can also express this in terms of the adjointed versions:

Corollary 3.27 *Suppose that G is a path-connected topological group and that $Z(G)$ is discrete. Let $\gamma \in \Omega_Z G$ and $g\colon X \to G$. Then*

$$\overline{\mathrm{TO}}_\gamma([g]) = (\pi_*)^{-1}(\langle [\pi \circ g], p_G([\gamma]) \rangle) \in [SX, G],$$

where $(\pi_*)^{-1}$ is the inverse of the isomorphism $\pi_*\colon [SX, G] \to [SX, PG]$. □

Remark 3.28 If X is simply connected, then $\pi_*\colon [X, G] \to [X, PG]$ is also an isomorphism, which allows us to take the reverse point of view and compute Samelson products in PG from turning obstructions in G: suppose that $\gamma \in \pi_1(PG)$ and $g\colon X \to PG$, then $\langle [g], [\gamma] \rangle = \pi_*(\overline{\mathrm{TO}}_{\widetilde{\gamma}}((\pi_*)^{-1}([g]))) \in [SX, PG]$, where $\widetilde{\gamma} \in \Omega_Z G$ is a lift of γ.

In the next section we will compute various turning obstructions; see Theorem 4.1. By Lemma 3.24 those results give a variety of information about Samelson products $\langle [g], \eta \rangle$ for $\eta \in \pi_1(\mathrm{SO}_{2k})$ the generator, for example we get the following proposition. Recall that $\tau_{2k} \in \pi_{2k-1}(\mathrm{SO}_{2k})$ is the homotopy class of a clutching function of the tangent bundle of S^{2k}, and for $m > 2$ let $\eta_m\colon S^{m+1} \to S^m$ be essential.

Proposition 3.29 *The Samelson product $\langle \tau_{2k}, \eta\rangle \in \pi_{2k}(\mathrm{SO}_{2k})$ is given as follows*:

(a) *If $k = 2j+1$ is odd, $\langle \tau_{4j+2}, \eta\rangle = 0$.*

(b) *If $k = 2j$ is even, $\langle \tau_{4j}, \eta\rangle = \tau_{4j}\eta_{4j-1} \neq 0$.* □

Remark 3.30 Proposition 3.29 can be viewed as an extension of an odd-primary theorem of Hamanaka and Kono [7, Theorem A] to the prime 2.

As another application of Lemma 3.24, we consider the situation where η is not the turning datum in a turning problem, but instead the clutching function of a bundle. Let $E^{2k}_\eta \to S^2$ be a fixed nontrivial oriented rank-$2k$ bundle over S^2. Then $\mathrm{Fr}(E)$ is a nontrivial principal SO_{2k}–bundle over S^2, we write $\mathcal{G}^{2k}_\eta$ for the gauge group of $\mathrm{Fr}(E^{2k}_\eta)$ and consider the fibration sequence (2-2) for $\mathcal{G}^{2k}_\eta$, which we write as $\mathcal{G}^{2k}_{\eta,0} \to \mathcal{G}^{2k}_\eta \to \mathrm{SO}_{2k}$. As discussed in Section 2.5, $\mathcal{G}^{2k}_{\eta,0} \cong \mathrm{Map}((S^2,*),(\mathrm{SO}_{2k},\mathrm{Id}))$. Hence there is a natural isomorphism $\pi_i(\mathcal{G}^{2k}_{\eta,0}) \cong \pi_{i+2}(\mathrm{SO}_{2k})$ and by a theorem of Wockel [19, Theorem 2.3] the boundary map

$$\partial^{2k}_\eta \colon \pi_i(\mathrm{SO}_{2k}) \to \pi_{i-1}(\mathcal{G}^{2k}_{\eta,0}) = \pi_{i+1}(\mathrm{SO}_{2k})$$

in the associated long exact sequence is given by

$$\partial^{2k}_\eta([g]) = -\langle [g], \eta\rangle$$

for all $[g] \in \pi_i(\mathrm{SO}_{2k})$. Combining Lemma 3.24 and Theorem 4.1 therefore gives information about the map ∂^{2k}_η. In particular, for τ_{2k} and η_{4j-1} as in Proposition 3.29 we have:

Proposition 3.31 *The boundary map $\partial^{2k}_\eta \colon \pi_{2k-1}(\mathrm{SO}_{2k}) \to \pi_{2k}(\mathrm{SO}_{2k})$ satisfies*

$$\partial^{4j}_\eta(\tau_{4j}) = \tau_{4j}\eta_{4j-1} \neq 0 \quad \text{and} \quad \partial^{4j+2}_\eta(\tau_{4j+2}) = 0.$$ □

Remark 3.32 If we let E^∞_η denote the stabilisation of the E^{2k}_η, then its frame bundle $\mathrm{Fr}(E^\infty_\eta)$ is a nontrivial principal SO–bundle over S^2 and we let $\mathcal{G}^\infty_\eta$ denote the gauge group of E^∞_η. Since the stable group SO is a homotopy abelian H–space, it follows that there is a weak homotopy equivalence

$$\mathcal{G}^\infty_\eta \simeq \mathrm{Map}(S^2, \mathrm{SO}) \cong \mathrm{Map}_*(S^2, \mathrm{SO}) \times \mathrm{SO}.$$

By comparing the homotopy long exact sequences of the fibrations

$$\mathcal{G}^{4j}_{\eta,0} \to \mathcal{G}^{4j}_\eta \to \mathrm{SO}_{4j} \quad \text{and} \quad \mathcal{G}^\infty_{\eta,0} \to \mathcal{G}^\infty_\eta \to \mathrm{SO},$$

where $\mathcal{G}^\infty_{\eta,0} \subset \mathcal{G}^\infty_\eta$ is the group of gauge transformations which are the identity in the fibre over the basepoint, we see that $\partial^{4j}_\eta \colon \pi_i(\mathrm{SO}_{4j}) \to \pi_{i+1}(\mathrm{SO}_{4j})$ is zero for $i < 4j-2$. When $i = 4j-2$, the domain of ∂^{4j}_η is $\pi_{4j-2}(\mathrm{SO}_{4j}) \cong \pi_{4j-2}(\mathrm{SO}) = 0$, so ∂^{4j}_η vanishes for $i \leq 4j-2$. Hence Proposition 3.31 shows that the first possibly nonzero boundary map in the homotopy long exact sequence of $\mathcal{G}^{4j}_{\eta,0} \to \mathcal{G}^{4j}_\eta \to \mathrm{SO}_{4j}$ is in fact nonzero.

4 Turning rank-$2k$ bundles over the $2k$–sphere

In this section we determine the turning obstructions for oriented rank-$2k$ bundles over the $2k$–sphere for all $k \geq 2$. To state our results, it will be convenient to use the notation $\mathrm{TO}_+ := \mathrm{TO}_\beta$ and $\mathrm{TO}_- := \mathrm{TO}_{\overline{\beta}}$ and when we wish to discuss these obstructions together, we will write $\mathrm{TO}_\pm$. We also define the adjointed η–turning obstruction $\mathrm{TO}_\eta := \mathrm{ad} \circ \mathrm{to}_\eta \colon \pi_{2k-1}(\mathrm{SO}_{2k}) \to \pi_{2k}(\mathrm{SO}_{2k})$. With this notation, the goal of this section is to compute the homomorphisms

$$\mathrm{TO}_\pm \colon \pi_{2k-1}(\mathrm{SO}_{2k}) \to \pi_{2k}(\mathrm{SO}_{2k}) \quad \text{and} \quad \mathrm{TO}_\eta \colon \pi_{2k-1}(\mathrm{SO}_{2k}) \to \pi_{2k}(\mathrm{SO}_{2k}).$$

Thus, if $E \to S^{2k}$ is a rank-$2k$ vector bundle with clutching function $g \colon S^{2k-1} \to \mathrm{SO}_{2k}$, then E is positive turnable if and only if $\mathrm{TO}_+([g]) = 0$, E is negative turnable if and only if $\mathrm{TO}_-([g]) = 0$ and E is η–turnable if and only if $\mathrm{TO}_\eta([g]) = 0$.

In order to state the computations of $\mathrm{TO}_\pm$ and TO_η we record some facts we need about the source and target groups of these homomorphisms, which can be found in [11]. We also introduce notation for generators of these groups. Recall that $\mathrm{e}(\xi) = \mathrm{e}(E_\xi) \in \mathbb{Z}$ is the Euler class and that $\mathrm{e}(\xi)$ is even unless $k = 2, 4$; see [6, Theorem, page 87]. Let $\tau_{2k} \in \pi_{2k-1}(\mathrm{SO}_{2k})$ denote the homotopy class of the clutching function of TS^{2k}. There is an isomorphism

$$\pi_{2k-1}(\mathrm{SO}_{2k}) \cong \mathbb{Z}(\tau_{2k}) \oplus C(\sigma_{2k}),$$

where $C(\sigma_{2k})$ is a cyclic group isomorphic to $\pi_{2k-1}(\mathrm{SO})$ and $S(\sigma_{2k}) \in \pi_{2k-1}(\mathrm{SO})$ is a generator. When $k = 2, 4$, we take $\mathrm{e}(\sigma_{2k}) = 1$, and when $k = 2$, we assume that σ_4 admits a complex structure; see Definition 4.6 and Theorem 4.7. When $k \neq 2, 4$, by Lemma 4.13 below, we assume that $C(\sigma_{2k}) = S(\pi_{2k-1}(\mathrm{SO}_{2k-2}))$; in particular, $\mathrm{e}(\sigma_{2k}) = 0$.

There are isomorphisms

$$\pi_{2k}(\mathrm{SO}_{2k}) \cong \begin{cases} 0 & \text{if } k = 3, \\ \mathbb{Z}/4 & \text{if } k \geq 5 \text{ is odd}, \\ (\mathbb{Z}/2)^2 & \text{if } k \equiv 2 \bmod 4, \\ (\mathbb{Z}/2)^3 & \text{if } k \equiv 0 \bmod 4. \end{cases}$$

When k is odd, we let $\zeta \in \pi_{2k}(\mathrm{SO}_{2k})$ be a generator and note that $\pi_{2k}(\mathrm{SO}) = 0$. When k is even, the stabilisation homomorphism $S \colon \pi_{2k}(\mathrm{SO}_{2k}) \to \pi_{2k}(\mathrm{SO})$ is split onto, where $\pi_{2k}(\mathrm{SO}) = 0$ if $k \equiv 2 \bmod 4$ and $\mathbb{Z}/2$ if $k \equiv 0 \bmod 4$. Moreover, for all $k \neq 3$ there is a short exact sequence

$$0 \to \mathbb{Z}/2(\tau_{2k}\eta_{2k-1}) \to \pi_{2k}(\mathrm{SO}_{2k}) \xrightarrow{\mathrm{ev}_* \oplus S} \pi_{2k}(S^{2k-1}) \oplus \pi_{2k}(\mathrm{SO}) \to 0, \tag{4-1}$$

with $\mathrm{ev} \colon \mathrm{SO}_{2k} \to S^{2k-1}$ given by evaluation at a point in S^{2k-1} and $\eta_{2k-1} \colon S^{2k} \to S^{2k-1}$ essential. The sequence (4-1) is nonsplit when $k \geq 5$ is odd and splits when k is even.

Theorem 4.1 *The turning obstructions* $\mathrm{TO}_\pm \colon \pi_{2k-1}(\mathrm{SO}_{2k}) \to \pi_{2k}(\mathrm{SO}_{2k})$ *satisfy the following*:

(a) *If k is odd, then* $\mathrm{TO}_\pm(\xi) = \mathrm{e}(\xi)\zeta$.

(b) *If $k \equiv 2 \bmod 4$, then* $\mathrm{ev}_*(\mathrm{TO}_\pm(\tau_{2k})) = 1$, $\mathrm{TO}_+(\sigma_{2k}) = 0$ *and* $\mathrm{TO}_-(\sigma_{2k}) = \mathrm{e}(\sigma_{2k})\,\mathrm{TO}_-(\tau_{2k})$.

(c) *If* $k \equiv 0 \bmod 4$, *then* $\mathrm{ev}_*(\mathrm{TO}_\pm(\tau_{2k})) = 1$, $S(\mathrm{TO}_\pm(\tau_{2k})) = 0$ *and* $S(\mathrm{TO}_\pm(\sigma_{2k})) = 1$; *in particular*, $\mathrm{TO}_\pm \otimes \mathrm{Id}_{\mathbb{Z}/2}$ *is injective.*

In particular, if k *is odd, then* $\mathrm{TO}_\eta = \mathrm{TO}_+ - \mathrm{TO}_- = 0$. *If* $k = 2j$ *is even, then* TO_η *satisfies the following*:

(d) $\mathrm{TO}_\eta(\tau_{4j}) = \tau_{4j}\eta_{4j-1} \neq 0$.

(e) *If* $j = 1$ *or* 2, *then* $\mathrm{ev}_*(\mathrm{TO}_\eta(\sigma_{4j})) = 1$ *and* $\mathrm{TO}_\eta \otimes \mathrm{Id}_{\mathbb{Z}/2}$ *is injective.*

(f) *If* $j \geq 3$, *then* $\mathrm{TO}_\eta(\sigma_{4j}) = 0$.

Remark 4.2 Theorem 4.1 shows that unless $k = 2$, for all $[g] \in \pi_{2k-1}(\mathrm{SO}_{2k})$ we have $\mathrm{TO}_+([g]) = 0$ if and only if $\mathrm{TO}_-([g]) = 0$. Hence for $k \neq 2$ rank-$2k$ bundles $E \to S^{2k}$ are either bi-turnable or not turnable and so these bundles are not strongly chiral. On the other hand, when $k = 2$, a bundle $E \to S^4$ is strongly chiral if and only if $\mathrm{e}(E)$ is odd.

The remainder of this section is devoted to the proof of Theorem 4.1. In Section 4.1 we consider the turnability of the tangent bundle of the $2k$–sphere, which is an essential input to the proof. In Section 4.2 we consider the exceptional case of the 4–sphere. In Section 4.3 we consider $\mathrm{TO}_\eta = \mathrm{TO}_+ - \mathrm{TO}_-$. In Section 4.4 we assemble the previous work to prove Theorem 4.1.

4.1 Turning the tangent bundle of the $2k$–sphere

Let TS^n denote the tangent bundle of the n–sphere. We fix the standard orientation on the n–sphere and this orients TS^n. In [12], Kirchoff proved that if TS^{2k} admits a complex structure then TS^{2k+1} is trivial. Later, it was proven in [6] that TS^{2k+1} is trivial if and only if $2k+1 = 1, 3$ or 7. Since elementary calculations show that TS^2 and TS^6 admit complex structures, Kirchoff's theorem implies that TS^{2k} admits a complex structure if and only if TS^{2k+1} is trivial. Here we prove a strengthening of Kirchoff's theorem, which only assumes that TS^{2k} is turnable.

Theorem 4.3 (Kirchoff's theorem for turnings) *If* TS^{2k} *is turnable, then* TS^{2k+1} *is trivial.*

Corollary 4.4 TS^{2k} *is turnable if and only if it admits a complex structure, which is the case if and only if* $2k = 2$ *or* 6. □

Proof of Theorem 4.3 We first recall the following well-known definition of a clutching function c_m for TS^m; see [9, Chapter 8, Corollary 9.9]. Given $x \in S^{m-1}$, write $\mathbb{R}^m = \langle x \rangle \oplus \langle x \rangle^\perp$ as the sum of the line spanned by x and its orthogonal complement and write $v \in \mathbb{R}^m$ as $v = (w, y)$, where $w \in \langle x \rangle$ and $y \in \langle x \rangle^\perp$. Let $c_m \colon S^{m-1} \to O_m$ be the function which assigns to $x \in S^{m-1}$ the reflection to the hyperplane orthogonal to x:

$$c_m(x) \colon \mathbb{R}^m \to \mathbb{R}^m, \quad (w, y) \mapsto (-w, y).$$

Suppose that TS^{2k} is turnable. We will show that the clutching function $c_{2k+1} : S^{2k} \to O_{2k+1}$ is nullhomotopic, proving that TS^{2k+1} is trivial. Using the notation above, we see that

$$TS^{2k} = \{((0, y), x) \mid y \in \langle x \rangle^\perp\} \subset \mathbb{R}^{2k+1} \times S^{2k}.$$

Since TS^{2k} is turnable, there exists a turning α_t on TS^{2k} with $\alpha_0 = \mathbb{1}$ and $\alpha_1 = -\mathbb{1}$. We use α_t to define the homotopy of automorphisms of the trivial bundle given by

$$H\colon (\mathbb{R}^{2k+1} \times S^{2k}) \times I \to \mathbb{R}^{2k+1} \times S^{2k}, \quad (((w, y), x), t) \mapsto ((-w, \alpha_t(y)), x).$$

We see that $H_0 = c_{2k+1}$ and $H_1 = -\mathbb{1}$. Hence H is the required homotopy of clutching functions from c_{2k+1} to a constant map. □

4.2 Rank-4 bundles over the 4–sphere

The set of isomorphism classes of rank-4 bundles over S^4 is in bijection with $\pi_3(\mathrm{SO}_4) \cong \mathbb{Z} \oplus \mathbb{Z}$. We recall the canonical double covering

$$q\colon S^3 \times S^3 \to \mathrm{SO}_4, \quad (x, y) \mapsto (v \mapsto x \cdot v \cdot y),$$

where we regard $x, y \in S^3$ as unit quaternions, $v \in \mathbb{H}$ and $\cdot$ denotes quaternionic multiplication. If we define $g_{(k_1,k_2)}\colon S^3 \to \mathrm{SO}_4$ by $x \mapsto q(x^{k_1}, x^{k_2})$, then the map

$$\mathbb{Z} \oplus \mathbb{Z} \to \pi_3(\mathrm{SO}_4), \quad (k_1, k_2) \mapsto [g_{(k_1,k_2)}],$$

is an isomorphism, which we use as coordinates for $\pi_3(\mathrm{SO}_4)$. By [9, Chapter 8, Proposition 12.10], for example, the map $g_{(1,1)}$ is a clutching function for TS^4 and so $\tau_4 = [g_{(1,1)}]$.

Definition 4.5 For $(k_1, k_2) \in \mathbb{Z}^2$, let $E_{k_1,k_2} \to S^4$ be the oriented rank-4 vector bundle with clutching function $g_{(k_1,k_2)}$; eg $TS^4 \cong E_{1,1}$.

Definition 4.6 We define $\sigma_4 := [g_{(0,1)}]$.

The turning obstructions TO_+, TO_- and TO_η take values in the group $\pi_4(\mathrm{SO}_4)$, and we use the isomorphism $q_*\colon \pi_4(S^3 \times S^3) \to \pi_4(\mathrm{SO}_4)$ to identify $\pi_4(\mathrm{SO}_4) \cong \pi_4(S^3) \oplus \pi_4(S^3) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Theorem 4.7 *The turning obstructions for rank-4 bundles over S^4 are given follows*:

(a) $\mathrm{TO}_+(a, b) = (\rho_2(a), 0)$.

(b) $\mathrm{TO}_-(a, b) = (0, \rho_2(b))$.

(c) $\mathrm{TO}_\eta(a, b) = (\rho_2(a), \rho_2(b))$.

Proof (a) Let $i, j, k \in \mathbb{H}$ be the standard purely imaginary unit quaternions. If we take the standard complex structure on $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ to be given by left multiplication by $i \in \mathbb{H}$, then $g_{(0,1)}(x)$ commutes with i for every $x \in S^3$, and so $g_{(0,1)}(x) \in U_2 \subset \mathrm{SO}_4$ for all $x \in S^3$. Thus $E_{0,1}$ admits a complex structure and so $\mathrm{TO}_+(0, 1) = 0$. By Corollary 4.4, $\mathrm{TO}_\pm(1, 1) \neq 0$, hence $\mathrm{TO}_+(1, 0) \neq 0$.

Consider the map $g_{(1,0)}\colon S^3 \to \mathrm{SO}_4$. In S^3 there is a unique homotopy class $[\gamma]$ of paths from 1 to -1, and we have $(g_{(1,0)})_*([\gamma]) = [\beta]$. Obviously, $(g_{(1,0)})_*([\mathrm{Id}_{S^3}]) = [g_{(1,0)}] = (1, 0) \in \pi_3(\mathrm{SO}_4)$ and $(g_{(1,0)})_*(\pi_4(S^3)) = \mathbb{Z}/2 \oplus 0 \leq \pi_4(\mathrm{SO}_4)$. Since $g_{(1,0)}$ is a continuous group homomorphism, we have $(g_{(1,0)})_*(\mathrm{TO}_\gamma([\mathrm{Id}_{S^3}])) = \mathrm{TO}_\beta([g_{(1,0)}]) = \mathrm{TO}_+(1, 0)$. This implies that $\mathrm{TO}_+(1, 0)$ is a nontrivial element of $\mathbb{Z}/2 \oplus 0$, ie $\mathrm{TO}_+(1, 0) = (1, 0)$. Therefore $\mathrm{TO}_+(a, b) = a\,\mathrm{TO}_+(1, 0) + b\,\mathrm{TO}_+(0, 1) = (\rho_2(a), 0)$.

(b) Since $ji = -k$, right multiplication by i defines a complex structure on $\mathbb{H}$ whose induced orientation is opposite to the standard orientation. Since $g_{(1,0)}(x)$ commutes with right multiplication by i for all $x \in S^3$, we see that $\overline{E}_{1,0}$ admits a complex structure. Hence $\mathrm{TO}_-(1,0) = 0$, and therefore $\mathrm{TO}_-(0,1) \neq 0$.

Let $\overline{S}^3$ denote S^3 with the opposite group structure, then $g_{(0,1)}\colon \overline{S}^3 \to \mathrm{SO}_4$ is a continuous group homomorphism. The unique homotopy class of paths in $\overline{S}^3$ from 1 to -1 maps to $[\overline{\beta}]$ under $g_{(0,1)}$. Therefore, similarly to part (a), we get that $\mathrm{TO}_-(0,1) = (0,1)$.

(c) By Theorem 3.23(b), $\mathrm{TO}_\eta(a,b) = \mathrm{TO}_+(a,b) - \mathrm{TO}_-(a,b)$. □

We now discuss the relationship between the turning type of rank-4 bundles and the homotopy classification of their gauge groups, due to Kishimoto, Membrillo-Solis and Theriault [13]. Following the notation of [13], let $\mathcal{G}_{k_1,k_2}$ denote the gauge group of E_{k_1,k_2}. Let $\{\{a,b\}\}$ denote the multiset consisting of the elements a, b and for integers a and b write (a,b) for their greatest common divisor. Then, for integers r, a and b, write $M^r(a,b)$ for the multiset $\{\{(a,r),(b,r)\}\}$. By [13, Theorem 1.1(b)] if $\mathcal{G}_{k_1,k_2} \simeq \mathcal{G}_{l_1,l_2}$ then $M^4(k_1,k_2) = M^4(l_1,l_2)$.

Recall that the turning type of an orientable bundle is characterised by whether it is either bi-turnable, strongly chiral or not turnable. By Theorem 4.7, E_{k_1,k_1} is bi-turnable if $M^2(k_1,k_2) = \{\{2,2\}\}$, strongly chiral if $M^2(k_1,k_2) = \{\{1,2\}\}$ and not turnable if $M^2(k_1,k_2) = \{\{1,1\}\}$. Hence, combining [13, Theorem 1.1(b)] and Theorem 4.7, we have:

Proposition 4.8 *The turning type of E_{k_1,k_2} is a homotopy invariant of $\mathcal{G}_{k_1,k_2}$.* □

4.3 The turning obstruction TO_η

In this subsection we cover some preliminaries for the computation of $\mathrm{TO}_\eta\colon \pi_{2k-1}(\mathrm{SO}_{2k}) \to \pi_{2k}(\mathrm{SO}_{2k})$.

For any $m \geq 2$, let $V_{m,2}$ be the Stiefel manifold of ordered pairs of orthonormal vectors in $\mathbb{R}^m$. Given $\underline{v} = (v_1, v_2) \in V_{m,2}$ we define $V = \langle v_1, v_2 \rangle$ and write $x \in \mathbb{R}^m$ as $x = (v, w)$ where $v \in V$ and $w \in V^\perp$. The isomorphism $\mathbb{C} \to V$ defined by $1 \mapsto v_1$ and $i \mapsto v_2$ defines a complex structure on V. We define $\gamma_{\underline{v}}$ in $\Omega\mathrm{SO}_m$ by

$$\gamma_{\underline{v}}(t)(v,w) = (e^{2\pi i t}v, w),$$

and we define the map

$$L = L_m\colon V_{m,2} \to \Omega\mathrm{SO}_m, \quad \underline{v} \mapsto \gamma_{\underline{v}}.$$

Next we consider the canonical projection $p\colon \mathrm{SO}_m \to V_{m,2}$ and the composition

$$L \circ p\colon \mathrm{SO}_m \to \Omega\mathrm{SO}_m.$$

It is clear from the definitions that $L \circ p$ is the map ρ_η of Definition 3.13(a), so after the identification $\pi_{m-1}(\Omega\mathrm{SO}_m) = \pi_m(\mathrm{SO}_m)$ we obtain the following:

Lemma 4.9 *For all $[g] \in \pi_{m-1}(\mathrm{SO}_m)$, $(L \circ p)_*([g]) = \mathrm{TO}_\eta([g])$.* □

Remark 4.10 Combining Lemmas 4.9, 3.6 and 3.24, we get

$$(L\circ p)_*([g]) = \mathrm{TO}_\eta([g]) = \overline{\mathrm{TO}}_\eta([g]) = \langle [g], \eta\rangle,$$

and this equation can be generalised to give a method for computing similar Samelson products as follows.

For $2 \le i \le m$, let $V_{m,i}$ denote the Stiefel manifold of mutually orthonormal ordered i–tuples $\underline{v} = (v_1, \dots, v_i)$ of vectors in $\mathbb{R}^m$, set $V = \langle v_1, \dots, v_i\rangle = \mathbb{R}^i$ and write $x \in \mathbb{R}^m$ as $x = (v, w)$, where $v \in V$ and $w \in V^\perp$. Then given any map $\alpha\colon S^{i-1} \to \mathrm{SO}_i$ we define $\alpha_{\underline{v}} \in \Omega^{i-1}\mathrm{SO}_m$, the $(i-1)$–fold based loop space of SO_m, by

$$\alpha_{\underline{v}}(s)(v, w) = (\alpha(s)v, w)$$

for all $s \in S^{i-1}$ and $(v, w) \in \mathbb{R}^m$. Allowing $\underline{v}$ to vary, we obtain the map

$$L(\alpha)\colon V_{m,i} \to \Omega^{i-1}\mathrm{SO}_m, \quad \underline{v} \mapsto \alpha_{\underline{v}},$$

and note that $L\colon V_{m,2} \to \Omega\mathrm{SO}_m$ above is $L(\alpha)$ for the special case of $\alpha\colon S^1 \to \mathrm{SO}_2 = U(1), t \mapsto e^{2\pi i t}$. If $\iota\colon \mathrm{SO}_i \to \mathrm{SO}_m$ denotes the standard inclusion, and $p\colon \mathrm{SO}_m \to V_{m,i}$ the standard projection, then after the identification $\pi_j(\Omega^{i-1}\mathrm{SO}_m) = \pi_{i+j-1}(\mathrm{SO}_m)$, a higher-dimensional version of Lemma 3.6 leads to the equation

$$(L(\alpha)\circ p)_*([g]) = \langle [g], \iota_*([\alpha])\rangle$$

for all $[g] \in \pi_j(\mathrm{SO}_m)$.

Now we consider the case $m = 2k$ and the induced homomorphisms $p_*\colon \pi_{2k-1}(\mathrm{SO}_{2k}) \to \pi_{2k-1}(V_{2k,2})$ and $L_*\colon \pi_{2k-1}(V_{2k,2}) \to \pi_{2k}(\mathrm{SO}_{2k})$. Let $\mathrm{ev}\colon \mathrm{SO}_{2k} \to S^{2k-1}$ be the map defined by evaluation at a point in S^{2k-1}.

Definition 4.11 Define $a_k \in \mathbb{Z}$ by $a_k := 1$ if k is even and $a_k := 2$ if k is odd.

Lemma 4.12 *Let $k \ge 3$. For any isomorphism $\pi_{2k-1}(V_{2k,2}) \to \mathbb{Z}/2 \oplus \mathbb{Z}$ we have*:

(a) $p_*(\tau_{2k}) = (\rho_2(a_k), 2)$.

(b) $L_*(1, 0) = \tau_{2k}\eta_{2k-1}$.

(c) $\mathrm{ev}_*(L_*(0, 1)) = \rho_2(a_k)$.

Proof (a) The sequence $\pi_{2k-1}(S^{2k-2}) \to \pi_{2k-1}(V_{2k,2}) \to \pi_{2k-1}(S^{2k-1})$ is split short exact, with $p_*(\tau_{2k})$ mapping to $2 \in \pi_{2k-1}(S^{2k-1}) = \mathbb{Z}$. It follows that the map $f = p\circ\tau_{2k}\colon S^{2k-1} \to V_{2k,2}$ vanishes on mod 2 cohomology. Also for $x \in H^{2k-2}(V_{2k,2}; \mathbb{Z}/2)$ the generator, $\mathrm{Sq}^2(x) \in H^{2k}(V_{2k,2}; \mathbb{Z}/2) = 0$ and so the functional Steenrod square $\mathrm{Sq}^2_x(g)$ is defined for all maps $g\colon S^{2k-1} \to V_{2k,2}$ which vanish on mod 2 cohomology. Moreover, $g = 2g'$ for some map $g'\colon S^{2k-1} \to V_{2k,2}$ if and only if $\mathrm{Sq}^2_x(g) = 0$.

Now the map $\tau_{2k}\colon S^{2k-1} \to \mathrm{SO}_{2k}$ factors over the double covering $q\colon S^{2k-1} \to \mathbb{R}P^{2k-1}$ and a map $\tau'_{2k}\colon \mathbb{R}P^{2k-1} \to \mathrm{SO}_{2k-1}$. Since q vanishes on mod 2 cohomology and $\mathrm{Sq}^2(t^{2k-2}) = 0$, for $t \in H^1(\mathbb{R}P^{2k-1}; \mathbb{Z}/2)$ a generator, it follows that the functional Steenrod square $\mathrm{Sq}^2_{t^{2k-2}}$ is defined on q.

We consider the composition

$$S^{2k-1} \xrightarrow{q} \mathbb{R}P^{2k-1} \xrightarrow{\tau'_{2k}} \mathrm{SO}_{2k} \xrightarrow{p} V_{2k,2}.$$

A degree argument shows that the map $p \circ \tau'_{2k} \colon \mathbb{R}P^{2k-1} \to V_{2k,2}$ satisfies $(p \circ \tau'_{2k})^*(x) = t^{2k-2}$ and naturality of functional Steenrod squares gives that

$$\mathrm{Sq}^2_x(p \circ \tau_{2k}) = \mathrm{Sq}^2_{t^{2k-2}}(q).$$

But q is the attaching map of the top cell of $\mathbb{R}P^{2k}$ and so $\mathrm{Sq}^2_{t^{2k-2}}(q) = \mathrm{Sq}^2(t^{2k-2}) = \rho_2(a_k)$. Hence we have $\mathrm{Sq}^2_x(p \circ \tau_{2k}) = \rho_2(a_k)$ and so $p_*(\tau_{2k}) = (\rho_2(a_k), 2) \in \pi_{2k-1}(V_{2k,2})$.

(b) Let $\iota_{2k-2} \colon S^{2k-2} \to V_{2k,2}$ be the inclusion of a fibre of the projection $V_{2k,2} \to S^{2k-1}$. Then we have $(1,0) = [\iota_{2k-2} \circ \eta_{2k-2}] \in \pi_{2k-1}(V_{2k,2})$. Hence it suffices to prove that $L_*([\iota_{2k-2}]) = \tau_{2k} \in \pi_{2k-1}(\mathrm{SO}_{2k})$. Now a degree argument shows that $\mathrm{ev}_*(L_*([\iota_{2k-2}])) = 1 + (-1)^{2k-2} = 2 \in \pi_{2k-1}(S^{2k-1}) = \mathbb{Z}$ and we consider the commutative diagram

$$\begin{array}{ccc} V_{2k,2} & \xrightarrow{L_{2k}} & \Omega \mathrm{SO}_{2k} \\ \downarrow & & \downarrow s \\ V_{2k+2,2} & \xrightarrow{L_{2k+2}} & \Omega \mathrm{SO}_{2k+2} \end{array}$$

Since $\pi_{2k-2}(V_{2k+2,2}) = 0$, we see that $S(L_*([\iota_{2k-2}])) = 0$. Hence $L_*([\iota_{2k-2}])$ is the clutching function of a stably trivial bundle with Euler class 2, so $L_*([\iota_{2k-2}]) = \tau_{2k}$, as required.

(c) The standard complex structure on $\mathbb{R}^{2k} = \mathbb{C}^k$ defines a section $s \colon S^{2k-1} \to V_{2k,2}$ of the projection $V_{2k,2} \to S^{2k-1}$ by $s(v) = (v, iv)$. Taking induced maps on π_{2k-1} gives a splitting $\pi_{2k-1}(V_{2k,2}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}$, where $[s] = (0,1)$ and $[\iota_{2k-2} \circ \eta_{2k-2}] = (1,0)$. Since $\mathrm{ev}_*(L_*(1,0)) = 0$, it suffices to prove $\mathrm{ev}_*(L_*([s])) = \rho_2(a_k)$.

It is clear that $L \circ s$ factors as the composition of the Hopf map $H \colon S^{2k-1} \to \mathbb{C}P^{k-1}$ and a map $L' \colon \mathbb{C}P^{k-1} \to \Omega \mathrm{SO}_{2k}$. Another degree argument shows that the adjoint of $\Omega(\mathrm{ev}) \circ L' \colon \mathbb{C}P^{k-1} \to \Omega S^{2k-1}$ has degree one and so the homotopy class of $S^{2k-1} \to \Omega S^{2k-1}$ is determined by the functional Steenrod square $\mathrm{Sq}^2_{z^{k-1}}(H)$, where $z \in H^2(\mathbb{C}P^{k-1}; \mathbb{Z}/2)$ is the generator. Since H is the attaching map of the top cell of $\mathbb{C}P^k$, we have

$$\mathrm{Sq}^2_{z^{k-1}}(H) = \mathrm{Sq}^2(z^{k-1}) = \rho_2(a_k) \in \mathbb{Z}/2 \cong H^{2k}(\mathbb{C}P^k; \mathbb{Z}/2),$$

which completes the proof of part (c). □

Lemma 4.13 *If $k \neq 2, 4$, we may choose $\sigma_{2k} \in S(\pi_{2k-1}(\mathrm{SO}_{2k-2})) \subset \pi_{2k-1}(\mathrm{SO}_{2k})$.*

Proof If $k \neq 2, 4$, then $\sigma_{2k} \in \pi_{2k-1}(\mathrm{SO}_{2k})$ is such that $S(\sigma_{2k})$ generates $\pi_{2k-1}(\mathrm{SO})$ and $\mathrm{e}(\sigma_{2k}) = 0$. The map $\pi_{2k-1}(\mathrm{SO}_{2k-2}) \to \pi_{2k-1}(\mathrm{SO})$ is onto for $k \geq 7$ by [2], which proves the lemma when $k \geq 7$.

For the remaining cases, $k \in \{3,5,6\}$. If $k=3$, then $\pi_{2k}(\mathrm{SO})=0$, and so $\sigma_6=0$. When $k=5,6$ we consider part of the homotopy long exact sequence of the fibration $\mathrm{SO}_{2k-2} \to \mathrm{SO}_{2k-1} \to S^{2k-2}$:

$$\pi_{2k-1}(\mathrm{SO}_{2k-2}) \to \pi_{2k-1}(\mathrm{SO}_{2k-1}) \to \pi_{2k-1}(S^{2k-2}) \to \pi_{2k-2}(\mathrm{SO}_{2k-2}) \to \pi_{2k-2}(\mathrm{SO}_{2k-1}).$$

Applying results of Kervaire [11], we deduce that the boundary map $\pi_{2k-1}(S^{2k-2}) \to \pi_{2k-2}(\mathrm{SO}_{2k-2})$ is injective and so $\pi_{2k-1}(\mathrm{SO}_{2k-2}) \to \pi_{2k-1}(\mathrm{SO}_{2k-1})$ is onto. Since $\mathrm{e}(\sigma_{2k})=0$, $\sigma_{2k} \in S(\pi_{2k-1}(\mathrm{SO}_{2k-1}))$ and so $\sigma_{2k} \in S(\pi_{2k-1}(\mathrm{SO}_{2k-2}))$. □

Lemma 4.14 *If k is odd, then $\mathrm{TO}_\eta([g])=0$ for all $[g] \in \pi_{2k-1}(\mathrm{SO}_{2k})$.*

Proof By Lemma 4.12(a), $p_*(\tau_{2k})=(0,2)$ and by Lemma 4.12(c), $L_*(0,2)=0$. Hence by Lemma 4.9, $\mathrm{TO}_\eta(\tau_{2k})=(L\circ p)_*(\tau_{2k})=0$. By Lemma 4.13, $\sigma_{2k} \in S(\pi_{2k-1}(\mathrm{SO}_{2k-2}))$, ie $\sigma_{2k}=i\circ\sigma'$ for some $\sigma' \in \pi_{2k-1}(\mathrm{SO}_{2k-2})$, with $i\colon \mathrm{SO}_{2k-2} \to \mathrm{SO}_{2k}$ the inclusion. Using the analogue of Lemma 3.9 for closed loops, $\mathrm{TO}_\eta(\sigma_{2k})=\mathrm{TO}_{\mathbb{1}\oplus\eta}(i\circ\sigma')=S(\mathrm{TO}_{\mathbb{1}}(\sigma'))=0$, where $\mathbb{1}$ denotes the constant loop at $\mathbb{1} \in \mathrm{SO}_{2k-2}$. □

4.4 The proof of Theorem 4.1

In this subsection we complete the proof of Theorem 4.1.

Proof of Theorem 4.1 (d) The $j=1$ case follows from Theorem 4.7. If $j \geq 2$, then $p_*(\tau_{4j})=(1,2)$ by Lemma 4.12(a) and since $\pi_{4j}(\mathrm{SO}_{4j})$ is a 2–torsion group, $L_*(1,2)=L_*(1,0)$. By Lemma 4.12(b), $L_*(1,0)=\tau_{4j}\eta_{4j-1}$ and so $\mathrm{TO}_\eta(\tau_{4j})=(L\circ p)_*(\tau_{4j})=\tau_{4j}\eta_{4j-1}$.

(e) If $j=1$, then $\mathrm{TO}_\eta(\sigma_{4j})=(0,1)$ by Theorem 4.7, and $\mathrm{ev}_*(a,b)=a+b$. If $j=2$, then $\mathrm{e}(\sigma_{4j})=1$ and so $p_*(\sigma_{4j})=(\epsilon,1)$ for some $\epsilon \in \mathbb{Z}/2$. Since $\mathrm{ev}_*(L_*(1,0))=0$, Lemma 4.12(c) ensures that $\mathrm{ev}_*(\mathrm{TO}_\eta(\sigma_{4j}))=\mathrm{ev}_*((\epsilon,1))=1$.

(f) If $j \geq 3$, then by Lemma 4.13, $\sigma_{4j} \in S(\pi_{4j-1}(\mathrm{SO}_{4j-2}))$ and so $\mathrm{TO}_\eta(\sigma_{4j})=0$ (as in Lemma 4.14).

(a) If k is odd, the fact that $\mathrm{TO}_+(\xi)=\mathrm{TO}_-(\xi)$ follows from Theorem 3.23(b) and Lemma 4.14. If $k=1,3$, then $\pi_{2k}(\mathrm{SO}_{2k})=0$ and the statement holds trivially. If $k \geq 5$ is odd, then by Corollary 4.4, $\mathrm{TO}_+(\tau_{2k}) \neq 0$. Since k is odd, $2\,\mathrm{TO}_+(\tau_{2k})=\mathrm{TO}_\eta(\tau_{2k})=0$ by Theorem 3.23(d) and Lemma 4.14. Since $\pi_{2k}(\mathrm{SO}_{2k}) \cong \mathbb{Z}/4$ and $\mathrm{e}(\tau_{2k})=2$, the result holds for $\mathbb{Z}(\tau_{2k}) \subseteq \pi_{2k-1}(\mathrm{SO}_{2k})$. If $k \equiv 3 \bmod 4$ then $\pi_{2k-1}(\mathrm{SO}_{2k})=\mathbb{Z}(\tau_{2k})$. If $k \equiv 1 \bmod 4$, then $\pi_{2k-1}(\mathrm{SO}_{2k})=\mathbb{Z}(\tau_{2k})\oplus\mathbb{Z}/2(\sigma_{2k})$, provided $k \geq 5$ as we are assuming. Hence it suffices to show that $\mathrm{TO}_+(\sigma_{2k})=0$. Now $\pi_{2k-1}(U) \to \pi_{2k-1}(\mathrm{SO})$ is onto and so $S(\sigma_{2k})$ is stably complex. By [9, Chapter 20, Corollary 9.8] $S(\sigma_{2k})$ admits a complex structure with $c_k(\sigma_{2k})=(k-1)!$ and since $k \geq 5$, $(k-1)!$ is divisible by 4. Hence $\sigma_{2k}-\frac{1}{2}(k-1)!\tau_{2k}$ admits a complex structure; see Theorem 6.6. Then

$$0=\mathrm{TO}_+\big(\sigma_{2k}-\tfrac{1}{2}(k-1)!\tau_{2k}\big)=\mathrm{TO}_+(\sigma_{2k})-\tfrac{1}{4}(k-1)!(2\,\mathrm{TO}_+(\tau_{2k}))=\mathrm{TO}_+(\sigma_{2k}).$$

(b) The special case $k=2$ is proven in Theorem 4.7. For $k\geq 6$, we first prove that $\mathrm{ev}_*(\mathrm{TO}_\pm(\tau_{2k}))=1$. By Corollary 4.4, $\mathrm{TO}_\pm(\tau_{2k})\neq 0$. By Theorem 3.23(b) and Theorem 4.1(d), we have $\mathrm{TO}_-(\tau_{2k})=\mathrm{TO}_+(\tau_{2k})+\tau_{2k}\eta_{2k-1}$. Since $\mathrm{Ker}(\mathrm{ev}_*)$ is generated by $\tau_{2k}\eta_{2k-1}$, it follows that $\mathrm{ev}_*(\mathrm{TO}_\pm(\tau_{2k}))=1$. As $k\geq 6$, we have $\mathrm{e}(\sigma_{2k})=0$ and we must show that $\mathrm{TO}_\pm(\sigma_{2k})=0$. By Lemma 4.13, $\sigma_{2k}\in S(\pi_{2k-1}(\mathrm{SO}_{2k-2}))$ and so $\mathrm{TO}_\eta(\sigma_{2k})=0$, so it suffices to show that $\mathrm{TO}_+(\sigma_{2k})=0$. The argument is analogous to the case $k\equiv 1 \bmod 4$.

(c) We first prove that $\mathrm{ev}_*(\mathrm{TO}_\pm(\tau_{2k}))=1$. Since τ_{2k} is stably trivial, so is $\mathrm{TO}_\pm(\tau_{2k})$. The proof is now the same as the proof when $k\equiv 2 \bmod 4$. To see that $S(\mathrm{TO}_\pm(\sigma_{2k}))=1$, we note that $S(\sigma_{2k})$ generates $\pi_{2k-1}(\mathrm{SO})\cong\mathbb{Z}$ and that the natural map $\pi_{2k-1}(U)\to\pi_{2k-1}(\mathrm{SO})$ has image the subgroup of index 2 by [4]. Hence σ_{2k} does not admit a stable complex structure and so σ_{2k} is not stably turnable by Theorem 5.10. Now by Lemma 3.9, $S(\mathrm{TO}_\pm(\sigma_{2k}))=\mathrm{TO}_\pm(S(\sigma_{2k}))=1$. □

5 Stable turnings and stable complex structures

In this section we define stable turnings of vector bundles $E\to B$. When B has the homotopy type of a finite CW–complex, we will see that Bott's proof of Bott periodicity shows that the space of stable complex structures on E is weakly homotopy equivalent to the space of stable turnings of E. In particular, in this case E is stably turnable if and only if E admits a stable complex structure.

Recall that $\underline{\mathbb{R}}^j$ denotes the trivial vector bundle over B of rank j.

Definition 5.1 (stably turnable) A rank-n vector bundle $E\to B$ is *stably turnable* if $E\oplus\underline{\mathbb{R}}^j$ is turnable for some $j\geq 0$.

Of course, if E is turnable, then E is stably turnable.

Remark 5.2 In the definition of stably turnable, $n+j$ must be even but n need not be even.

Remark 5.3 It is clear from the definition of turnings that if $E\oplus\underline{\mathbb{R}}^j$ is turnable, then $E\oplus\underline{\mathbb{R}}^{j+2l}$ is turnable for any nonnegative integer l.

For any rank-n vector bundle $E\to B$, recall that $\mathrm{Fr}(E)$, the frame bundle of E, is the principal SO_n–bundle associated to E and for any nonnegative integer j with $n+j$ even,

$$\mathrm{Turn}(E\oplus\underline{\mathbb{R}}^j)=\mathrm{Fr}(E\oplus\underline{\mathbb{R}}^j)\times_{\mathrm{SO}_{n+j}}\Omega_{\pm\mathbb{1}}\mathrm{SO}_{n+j}$$

is the associated turning bundle of $E\oplus\underline{\mathbb{R}}^j$. Now orthogonal sum with the path $\beta\in\Omega_{\pm\mathbb{1}}\mathrm{SO}_2$ defines the injective map $i_\beta\colon\Omega_{\pm\mathbb{1}}\mathrm{SO}_{n+j}\to\Omega_{\pm\mathbb{1}}\mathrm{SO}_{n+j+2}$, which we regard as an inclusion. Thus we regard $\mathrm{Turn}(E\oplus\underline{\mathbb{R}}^j)$ as a subbundle of $\mathrm{Turn}(E\oplus\underline{\mathbb{R}}^{j+2})$ and we set

$$\mathrm{Turn}(E^\infty):=\bigcup_{n+j\ \text{even}}\mathrm{Turn}(E\oplus\underline{\mathbb{R}}^j),$$

which is a fibre bundle over B with fibre $\Omega_{\pm\mathbb{1}}\mathrm{SO}:=\bigcup_{j=1}^\infty\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2j}$.

Lemma 5.4 *Let $E \to B$ be a vector bundle over a space homotopy equivalent to a finite CW–complex. Then E is stably turnable if and only if* $\mathrm{Turn}(E^\infty) \to B$ *admits a section.*

Proof Remark 5.3 and Lemma 2.12 tell us that a vector bundle $E \to B$ being stably turnable implies that $\mathrm{Turn}(E^\infty) \to B$ admits a section. Conversely, noting that the inclusion map $\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2j} \to \Omega_{\pm\mathbb{1}}\mathrm{SO}$ is $(2j-2)$–connected and B is homotopy equivalent to a finite CW–complex, it follows from the obstruction theory (cf [18, Chapter VI, Section 5]) that if $\mathrm{Turn}(E^\infty) \to B$ admits a section, then there must exist a nonnegative integer j such that $\mathrm{Turn}(E \oplus \underline{\mathbb{R}}^j)$ admits a section, which means that $E \to B$ is stably turnable and the proof is complete. □

Given Lemma 5.4, an efficient way to define the notion of a stable turning is via a section of $\mathrm{Turn}(E^\infty)$.

Definition 5.5 (stable turning and the space of stable turnings) A *stable turning* of a vector bundle $E \to B$ is a section of the fibre bundle $\mathrm{Turn}(E^\infty) \to B$. The *space of stable turnings* of E, denoted by $\Gamma(\mathrm{Turn}(E^\infty))$, is the space of sections of $\mathrm{Turn}(E^\infty) \to B$, equipped with the restriction of the compact–open topology.

We next consider *minimal turnings*, which are turnings that restrict to minimal geodesics in each fibre. The manifold SO_{2k} has a canonical Lie invariant metric, which allows us to consider geodesics in SO_{2k}. We write $\Omega^{\min}_{\pm\mathbb{1}}\mathrm{SO}_{2k} \subset \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ for the subspace of paths which are minimal geodesics in SO_{2k} from $\mathbb{1}$ to $-\mathbb{1}$. Since the conjugation action of SO_{2k} on itself is by isometries, it preserves geodesics and so $\Omega^{\min}_{\pm\mathbb{1}}\mathrm{SO}_{2k}$ is an SO_{2k}–subspace of $\Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k}$. For any nonnegative integer j with $n+j$ even, denote by $\mathrm{Turn}^{\min}(E \oplus \underline{\mathbb{R}}^j) \subset \mathrm{Turn}(E \oplus \underline{\mathbb{R}}^j)$ the subbundle of minimal turnings and set

$$\mathrm{Turn}^{\min}(E^\infty) := \bigcup_{n+j \text{ even}} \mathrm{Turn}^{\min}(E \oplus \underline{\mathbb{R}}^j),$$

which is a fibre bundle over B with fibre $\Omega^{\min}_{\pm\mathbb{1}}\mathrm{SO} = \bigcup_{j=1}^{\infty} \Omega^{\min}_{\pm\mathbb{1}}\mathrm{SO}_{2j}$.

Now $\Omega^{\min}_{\pm\mathbb{1}}\mathrm{SO}$ is an SO–subset of $\Omega_{\pm\mathbb{1}}\mathrm{SO}$ and hence $\mathrm{Turn}^{\min}(E^\infty)$ is a subbundle of $\mathrm{Turn}(E^\infty)$. Since the inclusion $\Omega^{\min}_{\pm\mathbb{1}}\mathrm{SO} \to \Omega_{\pm\mathbb{1}}\mathrm{SO}$ is a weak homotopy equivalence [14, Theorem 24.5], part (a) of the next lemma follows immediately. Part (b) follows from Lemma 5.4.

Lemma 5.6 *Let $E \to B$ be a vector bundle over a space homotopy equivalent to a finite CW–complex. Then the following hold*:

(a) *The fibrewise inclusion* $\mathrm{Turn}^{\min}(E^\infty) \to \mathrm{Turn}(E^\infty)$ *is a weak homotopy equivalence.*

(b) *E is stably turnable if and only if* $\mathrm{Turn}^{\min}(E^\infty) \to B$ *admits a section.* □

Now we recall the relationship of complex structures on a bundle and minimal turnings. A complex structure on a rank-$2k$ vector bundle E is an element $J \in \mathcal{G}_E$ such that $J^2 = -\mathbb{1}$. In particular, a complex structure on E endows each fibre of E with the structure of a complex vector space. If E has rank n, then a stable complex structure on E is a complex structure on $E \oplus \underline{\mathbb{R}}^j$ for some $j \geq 0$ with $n+j$ even.

Now let

$$\mathcal{J}_{2k} := \{J \in \mathrm{SO}_{2k} \mid J^2 = -\mathbb{1}\}$$

be the space of (special orthogonal) complex structures on $\mathbb{R}^{2k}$. The space $\mathcal{J}_{2k}$ is an SO_{2k}–space, where SO_{2k} acts on $\mathcal{J}_{2k}$ by conjugation. For any nonnegative integer j with $n+j$ even, define

$$\mathcal{J}(E \oplus \underline{\mathbb{R}}^j) := \mathrm{Fr}(E \oplus \underline{\mathbb{R}}^j) \times_{\mathrm{SO}_{n+j}} \mathcal{J}_{n+j} \subset \mathrm{Aut}(E \oplus \underline{\mathbb{R}}^j)$$

to be the bundle of fibrewise complex structures on $E \oplus \underline{\mathbb{R}}^j$. Regarding $\mathcal{G}_E = \Gamma(\mathrm{Aut}(E))$ we see that a complex structure on E is equivalent to a section of $\mathcal{J}(E) \to B$ and that a stable complex structure on E is equivalent to a section of $\mathcal{J}(E \oplus \underline{\mathbb{R}}^j) \to B$. Letting j tend to infinity, we define

$$\mathcal{J}(E^\infty) := \bigcup_{n+j \text{ even}} \mathcal{J}(E \oplus \underline{\mathbb{R}}^j)$$

to be the bundle of fibrewise stable complex structures on E. The space $\mathcal{J}(E^\infty)$ is the total space of a bundle over B with fibre $\mathcal{J}_\infty := \bigcup_{j=1}^\infty \mathcal{J}_{2j}$, and we have:

Lemma 5.7 *A vector bundle $E \to B$ over a space homotopy equivalent to a finite CW–complex admits a stable complex structure if and only if $\mathcal{J}(E^\infty) \to B$ admits a section.* □

In light of Lemma 5.7, an efficient way to define the notion of a stable complex structure is via a section of $\mathcal{J}(E^\infty)$.

Definition 5.8 (stable complex structure and the space of stable complex structures) A *stable complex structure* on an oriented vector bundle $E \to B$ is a section of the fibre bundle $\mathcal{J}(E^\infty) \to B$. The *space of stable complex structures* of E, denoted by $\Gamma(\mathcal{J}(E^\infty))$, is the space of sections of $\mathcal{J}(E^\infty) \to B$, equipped with the restriction of the compact–open topology.

Now a complex structure on $\mathbb{R}^{2k}$ defines a minimal geodesic in SO_{2k} via complex multiplication with unit complex numbers in the upper half-plane. Explicitly, we define the map

$$\varphi_{2k} : \mathcal{J}_{2k} \to \Omega^{\min}_{\pm\mathbb{1}}\mathrm{SO}_{2k}, \quad J \mapsto (t \mapsto \exp(\pi t J)),$$

where $\exp: T_{\mathbb{1}}\mathrm{SO}_{2k} \to \mathrm{SO}_{2k}$ is the exponential map from the tangent space over the identity. Then φ_{2k} is an SO_{2k}–equivariant homeomorphism; see eg Milnor [14, Lemma 24.1]. It follows for any rank-$2k$ bundle $E \to B$ that φ_{2k} induces a fibrewise homeomorphism $\varphi_{2k}: \mathcal{J}(E) \to \mathrm{Turn}^{\min}(E)$. Stably, we define the map $\varphi_\infty := \lim_{j\to\infty} \varphi_{2j} : \mathcal{J}_\infty \to \Omega^{\min}_{\pm\mathbb{1}}\mathrm{SO}$ and we have:

Lemma 5.9 *Let $E \to B$ be a vector bundle. The map $\varphi_\infty: \mathcal{J}_\infty \to \Omega^{\min}_{\pm\mathbb{1}}\mathrm{SO}$ induces a fibrewise homeomorphism $\varphi_\infty: \mathcal{J}(E^\infty) \to \mathrm{Turn}^{\min}(E^\infty)$.* □

Now the fibre bundle map

$$\mathcal{J}(E^\infty) \to \mathrm{Turn}^{\min}(E^\infty) \to \mathrm{Turn}(E^\infty)$$

induces a map $\Gamma(\mathcal{J}(E^\infty)) \to \Gamma(\mathrm{Turn}(E^\infty))$. By combining Lemmas 5.9, 5.6(a), 5.4 and 5.7 we obtain:

Theorem 5.10 *Let $E \to B$ be a vector bundle over a space homotopy equivalent to a finite CW–complex. The induced map $\Gamma(\mathcal{J}(E^\infty)) \to \Gamma(\mathrm{Turn}(E^\infty))$ from the space of stable complex structures on E to the space of stable turnings on E is a weak homotopy equivalence. Hence E is stably turnable if and only if E admits a stable complex structure.* □

In the remainder of this section we present an alternative proof of the final sentence of Theorem 5.10 using K–theory. Let BU (resp. BO) be the classifying space of the stable unitary group U (resp. stable orthogonal group O). Since O/U is homotopy equivalent to $\Omega^2 BO$ (cf [4]), the canonical fibration

$$O/U \hookrightarrow BU \to BO$$

gives rise to the Bott exact sequence (cf Bott [5, (12.2)] or Atiyah [1, (3.4)])

$$\cdots \to \mathrm{KO}^{-2}(B) \to \widetilde{\mathrm{KU}}(B) \xrightarrow{r} \widetilde{\mathrm{KO}}(B) \xrightarrow{\partial} \widetilde{\mathrm{KO}}^{-1}(B) \to \cdots . \tag{5-1}$$

Here r is the real reduction homomorphism and ∂ is the homomorphism given by $\partial(\xi) = \eta \cdot \xi$, where η is the generator of $\mathrm{KO}^{-1}(\mathrm{pt}) = \mathbb{Z}/2$ and $\cdot$ denotes the product in real K–theory.

Now for a rank-n vector bundle $E \to B$, let $\xi_E \in \widetilde{\mathrm{KO}}(B)$ be the real K–theory class $\xi_E := E \ominus \underline{\mathbb{R}}^n$, which is represented by the virtual bundle obtained as the formal difference of E and the trivial rank-n bundle over B. When B is homotopy equivalent to a finite CW–complex, the bundle E admits a stable complex structure if and only if the real K–theory class ξ_E lies in the image of the real reduction homomorphism r. Hence the next proposition follows from the Bott exact sequence above.

Proposition 5.11 *Let $E \to B$ be a vector bundle over a space homotopy equivalent to a finite CW–complex. Then E admits a stable complex structure if and only if $\eta \cdot \xi_E = 0$.* □

We now relate the boundary map ∂ in (5-1) to turnings of vector bundles. Let $\psi : E \to E$ be an automorphism of a vector bundle $\pi : E \to B$. The *mapping torus* of ψ is the vector bundle $T(\psi) \to B \times S^1$, where

$$T(\psi) := (E \times I)/\simeq$$

with $(v, 0) \simeq (\psi(v), 1)$, and the bundle map is given by $[(v, t)] \mapsto (\pi(v), [t])$. We note that $T(\psi)$ is orientable if and only if ψ is orientation-preserving, in which case $T(\psi)$ inherits an orientation from E. Let $\widehat{\otimes}$ denote the exterior tensor product of vector bundles.

Lemma 5.12 *Let $\underline{\mathbb{R}}$ denote the trivial line bundle over S^1. The bundle $T(\psi)$ is isomorphic to $E \,\widehat{\otimes}\, \underline{\mathbb{R}}$ if and only if ψ is homotopic to the identity.*

Proof The classification of vector bundles [9, Chapter 3, Section 4] shows that a vector bundle isomorphism $E \to E'$ is equivalent to a vector bundle $F \to B \times I$ such that $F|_{B\times\{0\}} = E$ and $F|_{B\times\{1\}} = E'$. Similarly, a vector bundle automorphism $E \to E$ is equivalent to a vector bundle $F \to B \times S^1$ such that $F|_{B\times\{1\}} = E$. In particular, the bundle $E \,\widehat{\otimes}\, \underline{\mathbb{R}}$ corresponds to the identity automorphism $\mathbb{1}_E : E \to E$. It follows that a vector bundle isomorphism $T(\psi) \to E \,\widehat{\otimes}\, \underline{\mathbb{R}}$ is equivalent to a bundle over $B \times S^1 \times I$ and so is equivalent to a path of bundle automorphisms from ψ to $\mathbb{1}_E$. □

An alternative proof of the final sentence of Theorem 5.10 Let $L \to S^1$ be the Möbius bundle, ie the nontrivial rank-1 bundle. Then $\eta \in \mathrm{KO}^{-1}(\mathrm{pt}) = \widetilde{\mathrm{KO}}(S^1)$ is represented by the virtual bundle $L \ominus \underline{\mathbb{R}}$. Let ξ_E be represented by the virtual bundle $E' \ominus \underline{\mathbb{R}}^{2k}$, where $E' \to B$ is a vector bundle of rank $2k$, which is larger than the formal dimension of B. Then $\eta \cdot \xi_E \in \mathrm{KO}^{-1}(B) = \mathrm{KO}(B \wedge S^1)$ is represented by the virtual bundle

$$(E' \mathbin{\widehat{\otimes}} L) \oplus (\underline{\mathbb{R}}^{2k} \mathbin{\widehat{\otimes}} \underline{\mathbb{R}}) \ominus (\underline{\mathbb{R}}^{2k} \mathbin{\widehat{\otimes}} L) \ominus (E' \mathbin{\widehat{\otimes}} \underline{\mathbb{R}})$$

over $B \times S^1$, which is canonically zero over $B \vee S^1$. Since L is the mapping torus of $-\mathbb{1}_{\underline{\mathbb{R}}} \colon \underline{\mathbb{R}} \to \underline{\mathbb{R}}$, it follows that $\underline{\mathbb{R}}^{2k} \mathbin{\widehat{\otimes}} L$ is the mapping torus of $-\mathbb{1}_{\underline{\mathbb{R}}^{2k}} \colon \underline{\mathbb{R}}^{2k} \to \underline{\mathbb{R}}^{2k}$ and so is trivial by Lemma 5.12. Hence $\eta \cdot \xi_E = 0$ if and only if $E' \mathbin{\widehat{\otimes}} L$ is stably isomorphic to $E' \mathbin{\widehat{\otimes}} \underline{\mathbb{R}}$.

Since L is the mapping torus of $-\mathbb{1}_{\underline{\mathbb{R}}} \colon \underline{\mathbb{R}} \to \underline{\mathbb{R}}$, the bundle $E' \mathbin{\widehat{\otimes}} L$ is the mapping tours of $-\mathbb{1}_{E'} \colon E' \to E'$. Now by Lemma 5.12, $E' \mathbin{\widehat{\otimes}} L$ is isomorphic to $E' \mathbin{\widehat{\otimes}} \underline{\mathbb{R}}$ if and only if E' is turnable. Hence $\eta_E \cdot \xi = 0$ if and only if E' is stably turnable and so the final sentence of Theorem 5.10 follows directly from Proposition 5.11. □

6 Turning rank-$2k$ bundles over $2k$–complexes

In Section 5 we saw that a turning of bundle E induces a stable complex structure on E and in Section 4 we computed the turning obstruction for rank-$2k$ bundles over the $2k$–sphere. In this section we combine these results to gain useful information about the turning obstruction for rank-$2k$ bundles over $2k$–dimensional complexes. Throughout this section, B will be a space that is homotopy equivalent to a connected finite CW–complex of dimension $2k$ or less.

Theorem 6.1 below gives a necessary condition for an oriented rank-$2k$ bundle E over B to be positive-turnable. Its statement requires some preliminary definitions. We will say that a complex structure J on $E \oplus \underline{\mathbb{R}}^{2j}$ for some $j \geq 0$ is *compatible with* E if J induces the same orientation on $E \oplus \underline{\mathbb{R}}^{2j}$ as E does. Recall that $\times 2 \colon H^*(B;\mathbb{Z}/2) \to H^*(B;\mathbb{Z}/4)$ is the map induced by the inclusion of coefficients $\times 2 \colon \mathbb{Z}/2 \to \mathbb{Z}/4$ and the subgroup $I^{2k}(B) \subseteq H^{2k}(B;\mathbb{Z}/4)$, which is defined by

$$I^{2k}(B) = \begin{cases} ((\times 2) \circ \mathrm{Sq}^2 \circ \rho_2)(H^{2k-2}(B;\mathbb{Z})) & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Recall also that ρ_4 denotes reduction mod 4.

Theorem 6.1 *Let $E \to B$ be an oriented rank-$2k$ vector bundle. If E is positive-turnable, then for some $j \geq 0$, $E \oplus \underline{\mathbb{R}}^{2j}$ admits a complex structure J such that J is compatible with E and $c_k(J)$ satisfies*

$$[\rho_4(c_k(J))] = [\rho_4(\mathrm{e}(E))] \in H^{2k}(B;\mathbb{Z}/4)/I^{2k}(B).$$

Example 6.3 below shows that there are turnable bundles which do not satisfy the condition of Theorem 6.1. However, this condition is sufficient for the bundle to be turnable if k is odd, and in many cases if k is even.

Theorem 6.2 *Let E be an oriented rank-$2k$ vector bundle over B with either k odd, or k even and $H^{2k}(B;\mathbb{Z})$ 2–torsion free. If E admits a stable complex structure J such that*

$$[\rho_4(c_k(J))] = [\rho_4(\mathrm{e}(E))] \in H^{2k}(B;\mathbb{Z}/4)/I^{2k}(B),$$

then E is positive-turnable.

Example 6.3 Let $M(\mathbb{Z}/2, 4k-1) := S^{4k-1} \cup_2 D^{4k}$ be the mod 2 Moore space with

$$H^{4k}(M(\mathbb{Z}/2, 4k-1);\mathbb{Z}) = \mathbb{Z}/2$$

and let $c\colon M(\mathbb{Z}/2, 4k-1) \to S^{4k}$ be the map collapsing the $(4k-1)$–cell. Since c is the suspension of the map $c'\colon M(\mathbb{Z}/2, 4k-2) \to S^{4k-1}$, which collapses the $(4k-2)$–cell of $M(\mathbb{Z}/2, 4k-2)$, we see that the γ–turning obstruction of $E := c^*(TS^{4k})$ is the pullback $c'^*(\mathrm{TO}_\gamma(\tau_{4k})) \in [M(\mathbb{Z}/2, 4k-2), \mathrm{SO}_{2k}]$. Since $\tau_{4k} \notin 2\pi_{4k-1}(\mathrm{SO}_{4k})$, it follows that $c'^*(\mathrm{TO}_\gamma(\tau_{4k})) \neq 0$. By Proposition 3.2, E is not γ–turnable for any path γ and so E is not turnable. However, E is stably parallelisable and so admits a stable complex structure J with $c_{2k}(J) = 0$. Moreover $\mathrm{e}(E) = 0$, since $\mathrm{e}(TS^{4k}) \in 2H^{4k}(S^{4k};\mathbb{Z})$. Hence E satisfies the condition of Theorem 6.1.

Before proving Theorems 6.1 and 6.2, we give an application of Theorem 6.2 which shows, in particular, that for all $l \geq 1$ there are $8l$–manifolds M whose tangent bundles are turnable but not complex, eg $M = S^4 \times S^4$.

Corollary 6.4 *For $i > 0$, let M be an orientable $4i$–manifold such that the following hold*:

(a) *M is stably parallelisable.*

(b) *$\chi(M) \neq 0$.*

(c) *$\mathrm{KU}(M) \to \mathrm{KO}(M)$ is injective.*

Then TM does not admit a complex structure but TM is turnable if and only if $\chi(M) \equiv 0 \bmod 4$. In particular, for all $m \geq 1$ and $l \geq 0$, the manifolds $M_l := \sharp_l(S^{4m} \times S^{4m})$ are such that TM_l does not admit a complex structure but TM_l is turnable if and only if l is odd.

Proof TM is stably trivial and $\mathrm{KU}(M) \to \mathrm{KO}(M)$ is injective, so E_J, the complex bundle underlying J, is trivial, so $c_{4j}(J) = 0$. On the other hand, $\mathrm{e}(TM) = \chi(M) = 2 + 2l$ by the Poincaré–Hopf theorem [8, page 113]. Hence by Theorem 6.6, TM does not admit a complex structure. However, by Theorems 6.1 and 6.2, TM is turnable if and only if l is odd. □

We now turn to the proofs of Theorems 6.1 and 6.2. Without loss of generality, we may assume that B is a finite CW–complex of dimension at most $2k$. It will be useful to modify a rank-$2k$ bundle $E \to B$ over the $2k$–cells of B and we first describe this process. Let $F \to S^{2k}$ be an oriented rank-$2k$ bundle with clutching function g. Given a $2k$–cell $e^{2k}_\alpha \subset B$ and a $2k$–disc D^{2k}_α embedded in the interior of e^{2k}_α, we define the bundle $E \sharp_\alpha F \to B$ as follows: Write $B = B^\circ \cup_{S^{2k-1}_\alpha} D^{2k}_\alpha$, where $B^\circ := B \setminus \mathrm{Int}(D^{2k}_\alpha)$ and

$$E = E^\circ \cup_\phi (\mathbb{R}^{2k} \times D^{2k}_\alpha),$$

where E° is the restriction of E to B° and $\phi\colon E|_{S^{2k-1}_\alpha} \to \mathbb{R}^{2k} \times S^{2k-1}_\alpha$ is a bundle isomorphism. Define

$$E \sharp_\alpha F := E^\circ \cup_{g\circ\phi} (\mathbb{R}^{2k} \times D^{2k}_\alpha).$$

We let $e^{2k*}_\alpha \in C^{2k}(B)$ be the $2k$–dimensional cellular cochain of B which evaluates to 1 on the $2k$–cell e^{2k}_α, and 0 on every other $2k$–cell. The next result follows from an elementary application of obstruction theory.

Lemma 6.5 *Let E_1 and E_2 be rank-$2k$ oriented bundles over B. Then E_1 is stably equivalent to E_2 if and only if there are $2k$–cells $\alpha_1, \dots, \alpha_n$ of B and integers $j_1, \dots, j_n$ such that*

$$E_2 \cong E_1 \#_{\alpha_1} j_1 TS^{2k} \#_{\alpha_2} \cdots \#_{\alpha_n} j_n TS^{2k}.$$

Moreover, in this case the Euler classes of E_1 and E_2 are represented by cocycles $\mathrm{cc}(\mathrm{e}(E_1))$ and $\mathrm{cc}(\mathrm{e}(E_2))$ such that $\mathrm{cc}(\mathrm{e}(E_2)) = \mathrm{cc}(\mathrm{e}(E_1)) + \sum_{i=1}^n 2j_i e^{2k*}_{\alpha_i}$. □

Lemma 6.5 can be used to prove the following theorem of Thomas.

Theorem 6.6 [16, Theorem 1.7] *Let E be a rank-$2k$ vector bundle over B, where $H^{2k}(B;\mathbb{Z})$ is 2–torsion free. Then E admits a complex structure if and only if E admits a stable complex structure J such that $c_k(J) = \mathrm{e}(E)$.* □

Proof of Theorem 6.1 Recall the universal rank-$2k$ turning bundle

$$BT_{2k} = E\mathrm{SO}_{2k} \times_{\mathrm{SO}_{2k}} \Omega_{\pm\mathbb{1}}\mathrm{SO}_{2k} \to B\mathrm{SO}_{2k}$$

from Definition 2.13 and define $BT := E\mathrm{SO} \times_{\mathrm{SO}} \Omega_{\pm\mathbb{1}}\mathrm{SO} \to B\mathrm{SO}$ to be the universal stable turning bundle. There is a natural map $BT_{2k} \to BT$ and by the results of Section 5, $BT \simeq BU$. Hence we consider the commutative diagram

$$\begin{array}{ccccc}
 & & BU_k & \longrightarrow & BU \\
 & & \downarrow & & \downarrow \simeq \\
 & & BT_{2k} & \xrightarrow{S} & BT \\
 & \overset{\bar f}{\nearrow} & \downarrow & & \downarrow \\
B & \xrightarrow{f} & B\mathrm{SO}_{2k} & \longrightarrow & B\mathrm{SO}
\end{array}$$

where f classifies E and $\bar f$ classifies a positive-turning on E. Since the natural map $BU \to BT$ is a fibre homotopy equivalence over $B\mathrm{SO}$, the turning on E induces a stable complex structure J on E. Consider the lifting problem

$$\begin{array}{ccc}
 & & BU_k \\
 & \overset{S\circ\bar f}{\nearrow} & \downarrow \\
B & \longrightarrow & BT
\end{array}$$

Since the homotopy fibre of $BU_k \to BT$ is $2k$–connected, the map $J := S \circ \bar{f}\colon B \to BT$ has a lift to $J'\colon B \to BU_k$, which is unique up to homotopy. The map J' defines a complex rank-k bundle (E', J') over B, where E' is stably equivalent to E.

By Lemma 6.5, there is a bundle isomorphism

$$\alpha\colon E \to E' \#_{\alpha_1} j_1 TS^{2k} \# \cdots \#_{\alpha_n} j_n TS^{2k}. \tag{6-1}$$

We set $B^\circ := B \setminus \big(\bigcup_{i=1}^n \operatorname{Int}(D^{2k}_{\alpha_i})\big)$, $E^\circ := E|_{B^\circ}$, $E'^\circ := E'|_{B^\circ}$ and $\alpha^\circ := \alpha|_{E^\circ}$. The lift $\bar{f}$ defines a turning ψ_t on E, which restricts to a turning ψ_t° on E°, and the complex structure J' defines a turning ψ'_t on E', which restricts to a turning $\psi_t'^\circ$ on E'° that pulls back along α° to a turning $(\alpha^\circ)^*(\psi_t'^\circ)$ on E°.

If ψ_t° and $(\alpha^\circ)^*(\psi_t'^\circ)$ are equivalent turnings on E°, then the obstruction to extending $(\alpha^\circ)^*(\psi_t'^\circ)$ to E vanishes. On the other hand, the obstruction to extending $\psi_t'^\circ$ to E' vanishes. It follows that the cocycle $\sum_{i=1}^n j_i \operatorname{TO}_+(\tau_{2k}) e^{2k*}_{\alpha_i}$ represents 0 in $H^{2k}(B; \pi_{2k}(\mathrm{SO}_{2k}))$, so the cocycle $\sum_{i=1}^n j_i 2e^{2k*}_{\alpha_i}$ represents 0 in $H^{2k}(B; \mathbb{Z}/4)$ and hence $\rho_4(c_k(J)) = \rho_4(\mathrm{e}(E))$.

If ψ_t° and $(\alpha^\circ)^*(\psi_t'^\circ)$ are not equivalent turnings on E°, then they differ over the $(2k-2)$–cells and $(2k-1)$–cells of B°. If k is even, this variation does not effect the cohomology class represented by $\sum_{i=1}^n j_i \operatorname{ev}_*(\operatorname{TO}_+(\tau_{2k}) e^{2k*}_{\alpha_i})$. If k is odd, then changing the turning can alter the values of the turning obstruction over the cells $e^{2k}_{\alpha_i}$ by any element of $((\times 2) \circ \mathrm{Sq}^2 \circ \rho_2)(H^{2k-2}(B; \mathbb{Z}))$. □

Proof of Theorem 6.2 Suppose that E admits a stable complex structure J. Then, as in the proof of Theorem 6.1, J descends to a complex rank-k bundle (E', J') on B such that E and E' are stably isomorphic. Since (E', J') is a complex bundle, E' is positive-turnable. Consider the bundle isomorphism α from (6-1). Assume first k is even and that $H^{2k}(B; \mathbb{Z})$ contains no 2–torsion. Since J' stabilises to J, $c_k(J') = c_k(J)$ and so $\rho_4(c_k(J')) = \rho_4(c_k(J)) = \rho_4(\mathrm{e}(E))$. It follows that each j_i in (6-1) is even. Now $2\,TS^{2k}$ is positive-turnable by Theorem 4.1. Since E' is positive-turnable, it follows that E is positive-turnable.

When k is odd the argument is similar to the case where k is even, but requires adjustments. We first note that $\rho_2(c_k(J')) = w_{2k}(E) = \rho_2(\mathrm{e}(E))$. Moreover, for k odd, $\operatorname{TO}_+(\tau_{2k}) = \tau_{2k}\eta_{2k-1} \in 2\pi_{2k}(\mathrm{SO}_{2k})$. It follows that if some j_i in (6-1) is odd, then we can modify the turning of E' to ensure that the pullback of its restriction to E'° extends to all of E. □

Proof of Theorem 1.4 and Remark 1.5 These statements follow from Theorems 6.1 and 6.2 and the fact that a turnable bundle over a connected space is either positively turnable or negatively turnable. □

Question 6.7 From the proof of Theorem 6.1 we see that there is a map $BT_{2k} \to BU$, obtained by composing the stabilisation map $BT_{2k} \to BT$ with a homotopy inverse of the natural map $BU \to BT$. This map $BT_{2k} \to BU$ encodes the fact that a turned bundle (E, ψ_t) over a finite CW–complex defines a

stable turning of E and a stable complex structure J on $E \oplus \underline{\mathbb{R}}^j$ for some j. It follows that the universal bundle $VT_{2k} \to BT_{2k}$ has a well-defined homotopy class of stable complex structures and hence there are well-defined Chern classes $c_i(VT_{2k}) \in H^{2i}(BT_{2k}; \mathbb{Z})$.

We pose the following question:

> What special properties, if any, do the Chern classes of VT_{2k} possess?

For example, one would expect that $c_i(VT_{2k}) = 0$ for all $i > k$, and presumably also that there is an equality $[\rho_4(c_k(VT_{2k}))] = [\rho_4(\mathrm{e}(VT_{2k}))] \in H^{2k}(BT_{2k}; \mathbb{Z}/4)/I^{2k}(BT_{2k})$.

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