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*Algebraic & Geometric
Topology*

Volume 24 (2024)

Rigidification of cubical quasicategories

PIERRE-LOUIS CURIEN

MURIEL LIVERNET

GABRIEL SAADIA



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We construct a cubical analogue of the rigidification functor from quasicategories to simplicial categories present in the work of Joyal and Lurie. We define a functor \mathcal{C}^\square from the category $cSet$ of cubical sets of Doherty, Kapulkin, Lindsey, and Sattler to the category $sCat$ of (small) simplicial categories. We show that this rigidification functor establishes a Quillen equivalence between the Joyal model structure on $cSet$ (as it is called by the four authors) and Bergner’s model structure on $sCat$. We follow the approach to rigidification of Dugger and Spivak, adapting their framework of necklaces to the cubical setting.

18N60; 06A07, 20B30

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Introduction

The last decades have seen an explosion of the use of ∞ -categories in various fields such as algebraic topology, algebraic geometry and homotopy type theory. In the early 2000s, various definitions of ∞ -categories have emerged, starting from the notion of quasicategories developed by Joyal [2008] and Lurie [2009] based on the definition of Boardman and Vogt [1973]. Other definitions have been explored such as enriched categories in spaces (or Kan complexes), or complete Segal spaces to name a few. By model of ∞ -categories we mean a category with a Quillen model category structure whose fibrant-cofibrant objects are the ∞ -categories in consideration and whose notion of weak equivalence corresponds to a good notion of equivalence of ∞ -categories. Bergner’s book [2018] clearly explains these different models and the Quillen equivalences relating them.

For instance, the model for quasicategories is the Joyal model category structure on the category $sSet$ of simplicial sets, while the model for categories enriched in Kan complexes is the Bergner model structure on the category $sCat$ of (small) simplicial categories. There exists a so-called rigidification functor \mathfrak{C}^Δ from $sSet$ to the category $sCat$ which is a Quillen equivalence between these two models. This functor is called rigidification because simplicial categories have a strict composition of 1–morphisms, as opposed to quasicategories where only weak compositions exist. The construction of the rigidification as well as the proof that it yields a Quillen equivalence have been achieved first in an unpublished manuscript of Joyal [2007], then by Lurie [2009], and then by Dugger and Spivak [2011a]. Dugger and Spivak build their rigidification functor using a technical tool, necklaces, and prove that there is a zigzag of weak equivalences of simplicial categories between their construction and Lurie’s one. The key idea of this construction is the following: given an ordered simplicial set X , the simplicial set $\mathfrak{C}^\Delta(X)(a, b)$ is the nerve of a poset whose objects are directed paths and relations are generated by 2–simplices in X . If $X = \Delta^n$, one obtains the subset lattice of an ordered set.

Cubical sets have been often considered as an alternative to simplicial sets in combinatorial topology, including in the early work of Kan and Serre (see eg [Serre 1951]). It has been also developed in computer science, in particular in concurrency theory (see eg [Fajstrup et al. 2016; Gaucher 2008; Pratt 1991]) and in homotopy type theory (see eg [Cohen et al. 2018]). Based on the work of Kapulkin, Lindsey and Wong [Kapulkin et al. 2019], Doherty, Kapulkin, Lindsey, and Sattler [Doherty et al. 2024] defined a notion of cubical quasicategory, and have constructed a model category structure on cubical sets, analogous to the Joyal structure on simplicial sets, whose fibrant-cofibrant objects are cubical quasicategories. They also show that the categories $cSet$ of cubical sets and $sSet$ are related by two adjunctions. The first one is $T \dashv U$, where $T: cSet \rightarrow sSet$ is a triangulation functor, and the second one is $Q \dashv \int$, where $Q: sSet \rightarrow cSet$ is a “cubification” functor implementing simplices as cubes with some degenerate faces. Both give rise to Quillen equivalences between these model category structures, so cubical quasicategories provide another definition for the notion of ∞ –category. By plugging together the Quillen equivalences $T: cSet \rightarrow sSet$ and $\mathfrak{C}^\Delta: sSet \rightarrow sCat$ of *triangulation* and rigidification, we get a Quillen equivalence between Joyal model structure of $cSet$ and Bergner’s model structure on $sCat$.

The goal of this paper is to build a different, direct Quillen equivalence $\mathfrak{C}^\square: cSet \rightarrow sCat$ using directed paths in the spirit of Dugger and Spivak. Note that the same notion of directed path is used in directed homotopy theory with applications to computer science. We refer the interested reader to the papers by Ziemiański [2017; 2020]. In particular, for a representable cubical set \square^n and for two vertices a and b , the simplicial set $\mathfrak{C}^\square(\square^n)(a, b)$ is the nerve of a poset whose objects are directed paths from a to b in the n –cube, and relations are generated by 2–cubes in \square^n . We prove a result of independent interest, namely that this poset is isomorphic to the weak Bruhat order on a symmetric group. Following closely the techniques developed by Dugger and Spivak, we prove that the functor \mathfrak{C}^\square is the left adjoint of a Quillen equivalence between two models of ∞ –categories, making use this time of the *cubification* equivalence of [Doherty et al. 2024], by showing that \mathfrak{C}^Δ factorises through cubification via

our rigidification, up to natural homotopy, ie $\mathcal{C}^\square \circ Q \xrightarrow{\simeq} \mathcal{C}^\Delta$, and then concluding by the two-out-of-three property.

Plan of the paper

In Section 1, we recall Bergner’s model structure and the material from [Doherty et al. 2024] needed for our purposes. Section 2 is devoted to the study of paths and necklaces (adapted from [Dugger and Spivak 2011a; 2011b]). We define our rigidification functor and study its properties in Section 3. The Quillen equivalence is established in Section 4. Appendices A and B deal with relevant categorical and combinatorial matters, respectively.

1 Recollection and notation

1.1 Simplicial rigidification

We recall the *Bergner model structure* of $s\mathcal{C}at$, where $s\mathcal{C}at$ denotes the category of simplicial categories, that is, categories enriched in simplicial sets. We recall that given a simplicial category \mathcal{C} , the category $\pi_0(\mathcal{C})$ has as objects those of \mathcal{C} and that (for all a, b) $\pi_0(\mathcal{C})(a, b)$ is the set of connected components of the simplicial set $\mathcal{C}(a, b)$.

The Bergner model structure of $s\mathcal{C}at$ is the enriched model structure coming from the usual Kan–Quillen model structure on $sSet$.

- Weak equivalences are *Dwyer–Kan equivalences*, that is, functors $F : \mathcal{C} \rightarrow \mathcal{D}$ such that
 - $\pi_0(F) : \pi_0(\mathcal{C}) \rightarrow \pi_0(\mathcal{D})$ is essentially surjective, and
 - for all $x, y \in \text{Ob}(\mathcal{C})$, the map $F_{x,y} : \mathcal{C}(x, y) \rightarrow \mathcal{D}(Fx, Fy)$ is a Kan–Quillen equivalence.
- Fibrations are *Dwyer–Kan fibrations*, that is, functors $F : \mathcal{C} \rightarrow \mathcal{D}$ such that
 - $\pi_0(F) : \pi_0(\mathcal{C}) \rightarrow \pi_0(\mathcal{D})$ is an isofibration between categories, and
 - for all $x, y \in \text{Ob}(\mathcal{C})$, the map $F_{x,y} : \mathcal{C}(x, y) \rightarrow \mathcal{D}(Fx, Fy)$ is a Kan fibration.

See [Bergner 2018] for more details.

Note that if F happens to be a bijection on objects, then $\pi_0(F)$ is a fortiori essentially surjective. This will be the case in our main result, so we will have to focus only on the second condition for DK-equivalences.

For the next proposition we use notation of Section A.1.

Proposition 1.1.1 *The following two functors form a Quillen adjunction*

$$sSet \begin{array}{c} \xrightarrow{\Sigma} \\ \perp \\ \xleftarrow{\text{Hom}} \end{array} s\mathcal{C}at_{*,*}$$

where

- $s\mathcal{C}at_{*,*}$ stands for the category of bipoined (small) simplicial categories, with the model structure induced by the Bergner structure on $s\mathcal{C}at$,

- the model structure on $sSet$ is the Joyal structure,
- $\Sigma(S)$ is the simplicial category with two objects, α and ω , and with only one nontrivial mapping space $\text{Hom}(\alpha, \omega) = S$, and
- $\text{Hom}(\mathcal{C}_{x,y}) = \mathcal{C}(x, y)$.

Proof The functor Hom is a right adjoint and sends fibrations and acyclic fibrations to fibrations and acyclic fibrations for the Kan–Quillen model category structure on $sSet$. Since the Kan–Quillen model category structure is a left Bousfield localisation of the Joyal model category structure, we get the result. \square

The simplicial rigidification functor $\mathfrak{C}^\Delta : sSet \rightarrow sCat$ is obtained as a left Kan extension along the Yoneda functor. On the representables, \mathfrak{C}^Δ is defined as follows:

- $\text{Ob}(\mathfrak{C}^\Delta(\Delta^n)) = \{0, \dots, n\}$.
- For $i \leq j$, $\mathfrak{C}^\Delta(\Delta^n)(i, j)$ is the nerve of the poset $\mathcal{P}(]i, j[)$, where $]i, j[$ is the set

$$\{i + 1, i + 2, \dots, j - 1\}.$$

The poset structure is given by subset inclusion. Note that this is the one-point simplicial set if $j = i$ or $j = i + 1$. For $i > j$, $\mathfrak{C}^\Delta(\Delta^n)(i, j) = \emptyset$.

- Composition $N(\mathcal{P}(]j, k[)) \times N(\mathcal{P}(]i, j[)) \rightarrow N(\mathcal{P}(]i, k[))$ is induced by the function mapping Y, X to $(X \cup \{j\} \cup Y) \setminus \{i, k\}$.

The nerve functor $N : Cat \rightarrow sSet$ being monoidal, it induces a functor from categories enriched in categories to categories enriched in simplicial sets (see [Riehl 2014, Chapter 3] for basics on enriched category theory). We also call this functor the nerve functor and denote it by N . In particular, the simplicial category $\mathfrak{C}^\Delta(\Delta^n)$ is obtained as the nerve of a poset-enriched category.

Remark 1.1.2 The simplicial rigidification functor is built by left Kan extension and so is cocontinuous, which implies in particular that the set of objects of $\mathfrak{C}^\Delta(X)$ is in bijection with X_0 . This is a general fact. Indeed, for a functor $F : I \rightarrow sCat$, the set of objects of the simplicial category $\text{colim } F$ is in bijection with the colimit (in Set) of the object functor $\text{Ob} \circ F : I \rightarrow Set$, since the object functor is cocontinuous: it is left adjoint to the codiscrete functor $Set \xrightarrow{\text{codisc}} sCat$ sending a set X to the simplicial category whose set of objects is X and whose simplicial set of morphisms between any two objects is Δ^0 .

Theorem 1.1.3 [Bergner 2018, Corollary 7.8.17] *The functor $\mathfrak{C}^\Delta : sSet \rightarrow sCat$ is the left adjoint of a Quillen equivalence between the Joyal model structure on $sSet$ and the Bergner model structure on $sCat$.*

1.2 Cubical quasicategories

We next present the material of [Doherty et al. 2024] needed for our purposes. There are different notions of cubical sets depending on whether one considers all or part of the negative and positive connections, the diagonals, and the symmetries. In this paper we consider the category of cubical sets with negative connections only, and shall denote it simply \square . Note that as in [Doherty et al. 2024], our results certainly

hold if we consider the category of cubical sets with positive connections instead, or both connections. We refer the reader interested by the reason why at least one connection is needed for the constructions of the Quillen functors to the introductions of [Doherty et al. 2024; Maltiniotis 2009], as well as Liang Ze Wong’s slides [2020].

The category \square is the subcategory of the category of posets whose objects are $[1]^n, n \geq 0$, and whose morphisms are generated by

- the faces $\partial_{i,\epsilon}^n : [1]^{n-1} \rightarrow [1]^n$ ($1 \leq i \leq n$ and $\epsilon \in \{0, 1\}$), consisting in inserting ϵ at the i -coordinate,
- the degeneracies $\sigma_i^n : [1]^n \rightarrow [1]^{n-1}$ ($1 \leq i \leq n$), consisting in forgetting the i -coordinate, and
- the negative connections $\gamma_{i,0}^n : [1]^n \rightarrow [1]^{n-1}$ ($1 \leq i \leq n - 1$), mapping (x_1, \dots, x_n) to

$$(x_1, \dots, x_{i-1}, \max(x_i, x_{i+1}), x_{i+2}, \dots, x_n).$$

Adapting [Grandis and Mauri 2003, Theorem 5.1] (see also [Maltiniotis 2009]) to our case, we have that every map in the category \square can be factored uniquely as a composite

$$(\partial_{c_1,\epsilon_1} \cdots \partial_{c_r,\epsilon_r})(\gamma_{b_1,0} \cdots \gamma_{b_q,0})(\sigma_{a_1} \cdots \sigma_{a_p})$$

with $1 \leq a_1 < \cdots < a_p, 1 \leq b_1 < \cdots < b_q$ and $c_1 > \cdots > c_r \geq 1$.

In particular, it factors uniquely as an epimorphism followed by a monomorphism. Relying on this factorisation, one can give an alternative presentation of \square by generators (as above) and relations given by cubical identities, as listed in [Doherty et al. 2024] just before Proposition 1.16.

The category of presheaves on \square is called the category of cubical sets and denoted by $cSet$. The representable presheaves are denoted by \square^n , and are called the n -cubes.

In addition, the factorisation of Grandis and Mauri in \square induces the existence of the standard form of an n -cube x in a cubical set S . We recollect here [Doherty et al. 2024, Proposition 1.18 and Corollaries 1.19 and 1.20], where, as usual, “nondegenerate” stands for “not in the image of a degeneracy or a connection”.

Proposition 1.2.1 *Let S and T be two cubical sets.*

- (1) *For any n -cube $x : \square^n \rightarrow S$, there exists a unique decomposition $x = y \circ \varphi$, where $\varphi : \square^n \rightarrow \square^m$ is an epimorphism and $y : \square^m \rightarrow S$ is a nondegenerate m -cube.*
- (2) *Any map $\varphi : S \rightarrow T$ in $cSet$ is determined by its action on nondegenerate cubes.*
- (3) *A map $\varphi : S \rightarrow T$ is a monomorphism if and only if it maps nondegenerate cubes of S to nondegenerate cubes of T and does so injectively.*

A *vertex* of a cubical set S is an element of S_0 (where $S_0 = S([1]^0)$). The vertices of \square^n are in one-to-one correspondence with the n -tuples (a_1, \dots, a_n) of $[1]^n$, or equivalently with the subsets of $\{1, \dots, n\}$. We will use either point of view, depending on the context.

Notation 1.2.2 The order of $[1]^n$ induces an order \preceq on the vertices of \square^n :

$$(a_1, \dots, a_n) \preceq (b_1, \dots, b_n) \iff a_i \leq b_i \text{ for all } 1 \leq i \leq n.$$

It is isomorphic to the subset lattice of $\{1, \dots, n\}$ via $(a_1, \dots, a_n) \mapsto \{i \mid a_i = 1\}$. Hence it has a least element $\alpha = \emptyset$ and a greatest element $\omega = \{1, \dots, n\}$ (or $\alpha = (0, \dots, 0)$ and $\omega = (1, \dots, 1)$).

For $a \preceq b$, let $d(a, b)$ be the cardinality of $b \setminus a$ and let $\iota_{a,b}^n$ be the face map $\square^{d(a,b)} \hookrightarrow \square^n$ satisfying $\iota_{a,b}^n(\alpha) = a$ and $\iota_{a,b}^n(\omega) = b$ (see Lemma 1.2.4).

Example 1.2.3 If $n = 5$, $a = (1, 0, 0, 0, 0)$ and $b = (1, 0, 1, 0, 1)$, then $b \setminus a = \{3, 5\}$, and $\iota_{a,b}^5: \square^2 \rightarrow \square^5$ is given by $\iota_{a,b}^5(x, y) = (1, 0, x, 0, y)$.

Lemma 1.2.4 A map $\varphi: \square^n \rightarrow \square^m$ satisfies $d(\varphi(\alpha), \varphi(\omega)) \leq n$. The map φ is a monomorphism if and only if $d(\varphi(\alpha), \varphi(\omega)) = n$, and in this case φ is determined by $\varphi(\alpha)$ and $\varphi(\omega)$. In particular, if $n = 1$, then $\varphi(\alpha) = \varphi(\omega)$ or $\varphi(\omega) \setminus \varphi(\alpha) = \{i\}$ for some i .

Proof We decompose $\varphi = u \circ v$ with $v: \square^n \rightarrow \square^p$ a composition of degeneracies and connections and $u: \square^p \rightarrow \square^m$ a composition of faces. We have $p \leq n$, and $v(\alpha) = \alpha$ and $v(\omega) = \omega$. A composition of faces inserts some 0 and 1 at some places and thus leaves the distance between two vertices invariant. In particular $d(u(\alpha), u(\omega)) = p \leq n$. In addition, since degeneracies and connections always decrease $d(\alpha, \omega)$ strictly, we get that φ is a monomorphism if and only if φ is a composition of faces, if and only if $d(\varphi(\alpha), \varphi(\omega)) = n$. A face is uniquely determined by its value on α and ω , so is a composition of faces. The second part of the statement is immediate. □

We next recall two model category structures on cubical sets. The first one, the Grothendieck model structure, models homotopy types and is described by Cisinski [2014], and the second one models $(\infty, 1)$ -categories and is described in [Doherty et al. 2024].

Definition 1.2.5 We recall here some useful definitions of [Doherty et al. 2024, Section 4].

- The boundary of \square^n , that is, the union of all the faces of \square^n , is denoted by $\partial \square^n$ and the canonical inclusion by $\partial^n: \partial \square^n \rightarrow \square^n$.
- The union of all the faces except $\partial_{i,\epsilon}$ is denoted by $\square_{i,\epsilon}^n$, and the inclusion $\square_{i,\epsilon}^n \rightarrow \square^n$ is called an *open box inclusion*.
- Given a face $\partial_{i,\epsilon}$ of \square^n , its *critical edge* $e_{i,\epsilon}$ is the unique edge of \square^n that is adjacent to $\partial_{i,\epsilon}$ and contains the vertex α or ω which is not in $\partial_{i,\epsilon}$. Namely, this is the edge between the vertices $(1 - \epsilon, \dots, 1 - \epsilon)$ and $(1 - \epsilon, \dots, 1 - \epsilon, \epsilon, 1 - \epsilon, \dots, 1 - \epsilon)$, where ϵ is placed at the i -coordinate. Equivalently, for $\epsilon = 1$, this is the edge from α to $\{i\}$ and if $\epsilon = 0$ this is the edge from $\{1, \dots, n\} \setminus \{i\}$ to ω .
- For $n \geq 2$, quotienting by the critical edge results in the (i, ϵ) -inner cube $\widehat{\square}_{i,\epsilon}^n$, the (i, ϵ) -inner open box $\widehat{\square}_{i,\epsilon}^n$, and the (i, ϵ) -inner open box inclusion $h_{i,\epsilon}^n: \widehat{\square}_{i,\epsilon}^n \rightarrow \widehat{\square}_{i,\epsilon}^n$.

- A (cubical) *Kan fibration* is a map having the right lifting property with respect to all open box inclusions.
- A (cubical) *inner fibration* is a map having the right lifting property with respect to all inner open box inclusions.
- A *cubical quasicategory* is a cubical set X such that $X \rightarrow *$ is an inner fibration.

Theorem 1.2.6 (Cisinski [Doherty et al. 2024, Theorem 1.34]) *The category $cSet$ carries a cofibrantly generated model structure, referred to as the Grothendieck model structure, in which*

- *cofibrations are the monomorphisms, and*
- *fibrations are Kan fibrations.*

We next sum up [Doherty et al. 2024, Theorems 4.2 and 4.16] for the Joyal model structure.

Theorem 1.2.7 *The category $cSet$ carries a cofibrantly generated model structure, referred to as the Joyal model structure, in which*

- *cofibrations are the monomorphisms, and*
- *fibrant objects are cubical quasicategories.*

Moreover, fibrations between fibrant objects are inner fibrations having the right lifting property with respect to the two endpoint inclusions $j_0: \{0\} \rightarrow K$ and $j_1: \{1\} \rightarrow K$, where K is the cubical set

$$\begin{array}{ccccc}
 1 & \longrightarrow & 0 & \xlongequal{\quad} & 0 \\
 \parallel & & \downarrow & & \parallel \\
 1 & \xlongequal{\quad} & 1 & \longrightarrow & 0
 \end{array}$$

We next recall the notion of equivalence and of special open box; see [Doherty et al. 2024, Section 4].

Definition 1.2.8 Let X be a cubical set.

- An edge $f: \square^1 \rightarrow X$ is an *equivalence* if it factors through the inclusion of the middle edge $\square^1 \rightarrow K$.
- For $n \geq 2$, $1 \leq i \leq n$ and $\epsilon \in \{0, 1\}$, a *special open box* in X is a map $\square_{i,\epsilon}^n \rightarrow X$ which sends the critical edge $e_{i,\epsilon}$ to an equivalence.

Intuitively, in reference to the above drawing of K , the definition of equivalence says that f has a left and a right inverse (the images of the nondegenerate horizontal edges), witnessed as such by the images of the two 2-cubes.

Finally, we collect results of [Doherty et al. 2024, Sections 5 and 6] on the comparison between cubical and simplicial models.

Definition 1.2.9 We can construct a monoidal product $\otimes: cSet \times cSet \rightarrow cSet$ by taking the left Kan extension of the monoidal product on \square given by $[1]^n \times [1]^m \mapsto [1]^{n+m}$ postcomposed with the Yoneda morphism. In particular $\square^n \otimes \square^m \cong \square^{n+m}$.

Note that this monoidal product is not symmetric.

In [Doherty et al. 2024] the authors provide four different but analogous functors $sSet \rightarrow cSet$, each of them labelled by a face of the 2-cube. We choose the one labelled by the face $\partial_{2,1}$ (corresponding to $Q_{R,1}$ in [Doherty et al. 2024]) and denote it by Q throughout the paper. Note that one of these constructions appeared first in [Kapulkin et al. 2019]. It is obtained as a left Kan extension of a functor $\Delta \rightarrow cSet$, which we describe in the following definition.

Definition 1.2.10 Let $n \geq 0$. The cubical set Q^n is the quotient of the n -cube \square^n obtained as the pushout

$$\begin{array}{ccc} (\square^0 \otimes \square^{n-1}) \sqcup (\square^1 \otimes \square^{n-2}) \sqcup \dots \sqcup (\square^{n-1} \otimes \square^0) & \xrightarrow{(\partial_{1,1}, \partial_{2,1}, \dots, \partial_{n,1})} & \square^n \\ \downarrow & & \downarrow \pi_n \\ \square^{n-1} \sqcup \square^{n-2} \sqcup \dots \sqcup \square^0 & \xrightarrow{\quad \quad \quad \Gamma \quad \quad \quad} & Q^n \end{array}$$

The family $(Q^n)_{n \geq 0}$ assembles to a functor $Q: \Delta \rightarrow cSet$, where faces and degeneracies are induced by the generating maps of \square as follows:

- the i^{th} face $Q(d_i): Q^{n-1} \rightarrow Q^n$ is the map induced by $\square^{n-1} \xrightarrow{\partial_{i,0}} \square^n$ if $i > 0$ and by $\partial_{1,1}$ if $i = 0$, and
- the i^{th} degeneracy $Q(s_i): Q^{n+1} \rightarrow Q^n$ is the map induced by $\square^{n+1} \xrightarrow{\gamma_{i,0}} \square^n$ if $i > 0$ and by σ_1 if $i = 0$.

Lemma 1.2.11 *The set of vertices of Q^n is in bijection with the set $\{0, \dots, n\}$ and the map π_n sends $a \subseteq \{1, \dots, n\}$ to $\sup a$, setting $\sup \emptyset = 0$. Furthermore, the action of the faces and degeneracies on the vertices of Q^n coincides with the action on the vertices of the simplicial set Δ^n .*

Proof Since colimits in $cSet$ are computed dimensionwise, the set $(Q^n)_0$ of vertices is obtained as the pushout of the diagram above evaluated at $[1]^0$. We claim that the set $\{0, \dots, n\}$, together with the map

$$\pi_n: \mathcal{P}(\{1, \dots, n\}) \rightarrow \{0, \dots, n\}, \quad a \mapsto \sup a,$$

satisfies the universal property of the pushout. Consider $I_1 \subseteq \{1, \dots, i-1\}$ and $I_2 \subseteq \{i+1, \dots, n\}$. Then (I_1, I_2) is mapped horizontally to $I_1 \cup \{i\} \cup I_2$ and vertically to I_2 ; hence $I_1 \cup \{i\} \cup I_2$ is identified to $I'_1 \cup \{i\} \cup I_2$ for any other $I'_1 \subseteq \{1, \dots, i-1\}$. The claim follows easily from this observation. The rest of the statement is also checked easily. \square

The left Kan extension of $Q: \Delta \rightarrow cSet$ along the Yoneda morphism is also denoted by $Q: sSet \rightarrow cSet$ and admits a right adjoint \int defined as $(\int S)_n := \text{Hom}_{cSet}(Q^n, S)$. We have the following Quillen equivalences [Doherty et al. 2024, Corollary 6.24 and Proposition 6.25].

Theorem 1.2.12 *The adjunction $Q : sSet \rightleftarrows cSet : f$ is both a Quillen equivalence*

- *between the Joyal model structure on $sSet$ and the Joyal model structure on $cSet$, and*
- *between the Kan–Quillen model structure on $sSet$ and the Grothendieck model structure on $cSet$.*

2 Necklaces and paths

In this section and the following one, we follow closely the steps taken by Dugger and Spivak [2011a] in order to understand more concretely the simplicial rigidification functor. We adapt their approach to define the simplicial rigidification of cubical sets.

2.1 Necklaces

Let $cSet_{*,*} = \partial \square^1 \downarrow cSet$ be the category of double pointed cubical sets. Given a cubical set S and two vertices $a, b \in S_0$, the notation $S_{a,b}$ stands for the double pointed cubical set corresponding to the morphism $(\partial \square^1 \rightarrow S) \in cSet_{*,*}$ mapping 0 to a and 1 to b . We refer to Section A.1 for general constructions. When there is no ambiguity on the double pointing, we omit the indices and write $S \in cSet_{*,*}$. For example, the cube \square^n is naturally double pointed by α and ω (see Notation 1.2.2), and if not specified otherwise we will consider this double pointing.

Definition 2.1.1 • A (cubical) necklace is an object T of $cSet_{*,*}$ of the form $\square^{n_1} \vee \dots \vee \square^{n_k}$, for some sequence (n_1, \dots, n_k) of positive integers. The double pointing is induced by $\alpha \in \square^{n_1}$ and $\omega \in \square^{n_k}$. The empty sequence corresponds to the necklace $T = \square^0$ and it is the unique one satisfying $\alpha = \omega$.

- For $k \geq 1$, the canonical morphism $B_i : \square^{n_i} \rightarrow T$ in $cSet$ is called the i^{th} bead of T , so that $\text{id}_T = B_1 * \dots * B_k$ (see Definition A.1.2 for the notation).
- We denote by Nec the full subcategory of $cSet_{*,*}$, whose objects are cubical necklaces. Objects will be identified with sequences (n_1, \dots, n_k) of positive integers. Note that if S is a necklace and $(a, b) \neq (\alpha, \omega)$, then an object of the slice category $Nec \downarrow S_{a,b}$, ie a morphism $T \rightarrow S_{a,b}$ with T a necklace, is not a morphism in Nec since the double pointing in Nec is given by (α, ω) .
- Given two sequences (n_1, \dots, n_k) and (m_1, \dots, m_l) , their concatenation is the sequence

$$(n_1, \dots, n_k, m_1, \dots, m_l).$$

- A decomposition of a nonempty sequence (n_1, \dots, n_k) in l blocks is a collection (A_1, \dots, A_l) of nonempty sequences such that their concatenation is (n_1, \dots, n_k) .

The following proposition describes the morphisms in Nec .

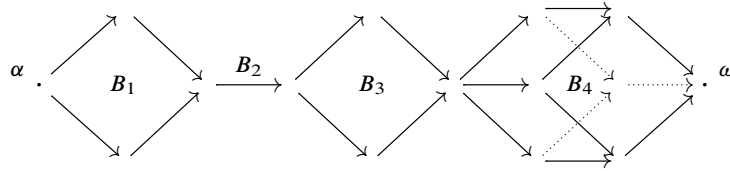


Figure 1: The necklace associated to the sequence (2, 1, 2, 3).

Proposition 2.1.2 (1) *In the category Nec , a morphism φ from (n_1, \dots, n_k) to (m) decomposes uniquely as $\varphi = \varphi_1 * \dots * \varphi_k$, where all $\varphi_i : \square^{n_i} \rightarrow \square^m$ are morphisms in \square , and satisfy*

$$\begin{aligned} \varphi_1(\alpha) &= \alpha, \\ \varphi_i(\omega) &= \varphi_{i+1}(\alpha) \quad \text{for all } 1 \leq i \leq k - 1, \\ \varphi_k(\omega) &= \omega. \end{aligned}$$

In particular, $m \leq n_1 + \dots + n_k$.

(2) *Given a morphism $f : (n_1, \dots, n_k) \rightarrow (m_1, \dots, m_l)$ in Nec , there is a decomposition (A_1, \dots, A_l) of the sequence (n_1, \dots, n_k) into l parts and morphisms $f_j : A_j \rightarrow (m_j)$ in Nec such that*

$$f = f_1 \vee \dots \vee f_l.$$

This decomposition is unique if, for any $1 \leq i \leq k$, the restriction of f to the bead (n_i) is not constant.

Proof The first part of the proposition is a direct consequence of the definition of the concatenation. Note that a map from (n_1, \dots, n_k) to (m) yields a chain $\alpha = a_0 \leq a_1 \leq \dots \leq a_{k-1} \leq a_k = \omega$ in $(\square^m)_0$ with $d(a_{i-1}, a_i) \leq n_i$, by Lemma 1.2.4. Hence $m = d(\alpha, \omega) \leq n_1 + \dots + n_k$.

Let us prove the second part. For the sake of clarity, we denote by (α_i, ω_i) the initial and terminal vertices of \square^{m_i} . For any cubical set S and any $n \geq 0$, we denote by $\sigma : S_0 \rightarrow S_n$ the map induced by the unique map $\square^n \rightarrow \square^0$ in \square . Let $T = \square^{m_1} \vee \dots \vee \square^{m_l}$. By definition of concatenation, the set of n -cubes in the cubical set T is the quotient of the disjoint union of the n -cubes of \square^{m_i} by the relation $\sigma(\omega_i) = \sigma(\alpha_{i+1})$ for $1 \leq i \leq l - 1$. Let $\varphi : (n_1, \dots, n_k) \rightarrow T$ be a morphism in Nec and let $\varphi_i : (n_i) \rightarrow (m_1, \dots, m_l)$ be its components, that is, φ_i is an n_i -cube of T , and $\varphi = \varphi_1 * \dots * \varphi_k$. Since $\varphi_1(\alpha) = \alpha_1$, necessarily φ_1 is an n_1 -cube of \square^{m_1} . Since $\varphi_1(\omega) = \varphi_2(\alpha)$, there are two possibilities: either $\varphi_1(\omega) \neq \omega_1$ and then φ_2 is an n_2 -cube of \square^{m_1} , or $\varphi_1(\omega) = \omega_1$ and φ_2 is an n_2 -cube of \square^{m_2} . Inductively, we get a decomposition (A_1, \dots, A_l) , where A_i is a sequence of consecutive n_j such that φ_j is an n_j -cube of \square^{m_i} . The decomposition is not unique in general. For example, if above we had $\varphi_2 = \sigma(\omega_1)$, then we could have chosen to keep n_2 in A_1 . But under the assumption that each φ_i is not constant, we do have uniqueness, since we can identify unambiguously in which component of T it lies. □

Example 2.1.3 (i) There is no morphism from $(2, 1, 3)$ to $(3, 2, 1)$: there is a unique decomposition of $(2, 1, 3)$ into three parts and, by Lemma 1.2.4, there is no morphism in Nec from \square^2 to \square^3 .

- (ii) A morphism $f : (2, 1, 3) \rightarrow (2, 1)$ in Nec is either of the form $g \vee h$ with $g : (2, 1) \rightarrow (2)$ and $h : (3) \rightarrow (1)$ in Nec (first type), or is $\text{id}_{(2)} \vee (\text{id}_{(1)} * c_\omega)$, where $c_\omega : (3) \rightarrow (1)$ is the constant map with value ω . Indeed, there are two decompositions of $(2, 1, 3)$ into two blocks. The decomposition $((2, 1), (3))$ gives the first decomposition. The decomposition $((2), (1, 3))$ gives $f = k \vee (l_1 * l_2)$, where $k : (2) \rightarrow (2) \in Nec$ is necessarily the identity and $l_1 * l_2 : (1, 3) \rightarrow (1)$. The only maps $l_1 : (1) \rightarrow (1)$ such that $l_1(\alpha) = \alpha$ are c_α and the identity. If $l_1 = c_\alpha$, then $(k * c_\omega) \vee l_2 : (2, 1) \vee (3) \rightarrow (2) \vee (1)$ is also a decomposition of f of the first type. If $l_1 = \text{id}_{(1)}$, then $l_2 = c_\omega$.

Notation 2.1.4 If $T = (n_1, \dots, n_k)$ is a cubical necklace, then $T_0 = (\square^{n_1})_0 \vee \dots \vee (\square^{n_k})_0$ is a bounded poset (see Definition A.2.1). We will also denote the order in T_0 by \preceq . Moreover, setting $n = n_1 + \dots + n_k$, any monomorphism in Nec from T to \square^n is a morphism of posets on vertices, which justifies the notation \preceq .

The following lemma is an easy consequence of Lemma 1.2.4.

Lemma 2.1.5 Let $T = (n_1, \dots, n_k)$ be a cubical necklace.

- Any monomorphism $\varphi : T \hookrightarrow \square^n$ in Nec is uniquely determined by a sequence $a_0 \prec a_1 \prec \dots \prec a_k$ of vertices in \square^n satisfying $a_0 = \alpha, a_k = \omega$ and $d(a_{i-1}, a_i) = n_i$. The sequence

$$\emptyset \prec \{1, \dots, n_1\} \prec \dots \prec \{1, \dots, n_1 + \dots + n_{k-1}\} \prec \{1, \dots, n\}$$

corresponds to an embedding $T \hookrightarrow \square^n$ that we will call the **standard embedding**.

- If $n_1 = \dots = n_k = 1$, then any morphism $\varphi : T \rightarrow \square^n$ in Nec is uniquely determined by a sequence $a_0 \preceq a_1 \preceq \dots \preceq a_k$ of vertices in \square^n satisfying $a_0 = \alpha, a_k = \omega$ and $d(a_{i-1}, a_i) \leq 1$.
- If $n_1 = \dots = n_k = 1$, then any monomorphism $\varphi : T \hookrightarrow \square^n$ in Nec is uniquely determined by a sequence $a_0 \prec a_1 \prec \dots \prec a_k$ of vertices in \square^n satisfying $a_0 = \alpha, a_k = \omega$ and $d(a_{i-1}, a_i) = 1$.

Definition 2.1.6 Let $T = (n_1, \dots, n_k)$ be a necklace and $a \preceq b$ be vertices of T . We define the cubical set $T_{[a,b]}$ and the morphism $\iota_{a,b}^T : T_{[a,b]} \hookrightarrow T$ in $cSet$ as follows:

- If $a, b \in \square^{n_i}$ then $T_{[a,b]} := \square^{d(a,b)}$ and $\iota_{a,b}^T := B_i \circ \iota_{a,b}^{n_i}$ (cf Notation 1.2.2).
- If $a \in \square^{n_i}$ and $b \in \square^{n_j}$ with $i < j$, then $T_{[a,b]} := \square^{d(a,\omega)} \vee \square^{n_{i+1}} \vee \dots \vee \square^{n_{j-1}} \vee \square^{d(\alpha,b)}$ and $\iota_{a,b}^T := (B_i \circ \iota_{a,\omega}^{n_i}) * B_{i+1} * \dots * B_{j-1} * (B_j \circ \iota_{\alpha,b}^{n_j})$.

Hence $T_{[a,b]}$ is a necklace and $\iota_{a,b}^T : T_{[a,b]} \hookrightarrow T$ is a monomorphism in $cSet$. We call $T_{[a,b]}$ the **subnecklace** of T between a and b .

Remark 2.1.7 This is well defined as \square^0 is the unit of the monoidal product \vee , so the construction of $T_{[a,b]}$ does not depend on the bead chosen for containing a or b . Recall that $T_{a,b}$ denotes a double pointed version of T while $T_{[a,b]}$ is a necklace whose underlying cubical set is different from T . The next proposition makes the link between these two cubical sets.

Proposition 2.1.8 Let T be a necklace and $a \preceq b \in T_0$. The object $\iota_{a,b}^T: T_{[a,b]} \hookrightarrow T$ is terminal in $\text{Nec} \downarrow T_{a,b}$.

Proof Let $f: X_{\alpha,\omega} \rightarrow T_{a,b}$ be a map in $c\text{Set}_{*,*}$ with X a necklace. Proceeding like in the proof of Proposition 2.1.2, we get that f factors uniquely through $T_{[a,b]}$ as $f = \iota_{a,b}^T \circ \hat{f}$ with $\hat{f}: X_{\alpha,\omega} \rightarrow T_{[a,b]}$ a morphism in Nec . \square

The next lemma states properties that will be needed in the proof of Proposition 3.3.2.

Lemma 2.1.9 Let S be a cubical subset of \square^n . Let $a \preceq b$ be two vertices of S .

- (1) An m -cube $x: \square^m \rightarrow S$ is nondegenerate if and only if x is a monomorphism.
- (2) A map $f: T \rightarrow S$ with T a necklace is a monomorphism if and only if it is a monomorphism on every bead.
- (3) Any object $f: T \rightarrow S_{a,b}$ of $\text{Nec} \downarrow S_{a,b}$ factors uniquely as $f = \iota(f)\pi(f)$, where $\pi(f): T \rightarrow T^f$ is an epimorphism in Nec and $\iota(f): T^f \rightarrow S_{a,b}$ is a monomorphism in $c\text{Set}$.

Proof We make use of Proposition 1.2.1.

(1) If x is nondegenerate in S , then x is nondegenerate in \square^n , hence a monomorphism, and so is x . Conversely, if x is a monomorphism, we can factor uniquely $x = ip$ with p an epimorphism and i a nondegenerate map. In particular p is a monomorphism, thus an isomorphism of cubes, that is, it is the identity.

(2) Let $B_i: \square^{n_i} \rightarrow T = (n_1, \dots, n_k)$ be the inclusion of the i^{th} bead of T . If f is a monomorphism, then fB_i is. For the converse, we use Proposition 1.2.1(3). Let $x: \square^m \rightarrow T$ be a nondegenerate m -cube of T . There exists a bead B_i such that x factors through it; thus $f(x) = (fB_i)(x)$ is nondegenerate. Let x and y be two nondegenerate cubes in T such that $f(x) = f(y)$. Assume x factors through B_i , and y through B_j , with $i < j$. Then $fB_i(\alpha_i) < fB_i(\omega_i) \preceq fB_j(\alpha_j)$. The inequalities hold because S is a cubical subset of \square^n and the left one is strict because fB_i is a monomorphism. Hence $f(x) = f(y)$ is not possible. Therefore, x and y factor through the same bead, and hence $x = y$ by our assumption.

(3) For every bead B_i of T , fB_i factors uniquely as $\iota_i(f) \circ \pi_i(f)$ with $\pi_i(f)$ an epimorphism and $\iota_i(f)$ a monomorphism (by (1)). Setting $\pi(f) = \pi_1(f) \vee \dots \vee \pi_k(f)$ and $\iota(f) = \iota_1(f) * \dots * \iota_k(f)$, we get the desired factorisation (by (2)). It is unique since f writes uniquely as $f_1 * \dots * f_k$ and each f_i factors uniquely. \square

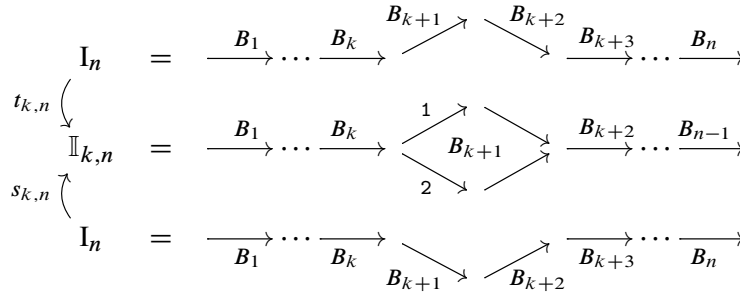
2.2 The path category of a cubical set

In this section we associate to a cubical set S a category enriched in prossets (ie preordered sets) $\mathfrak{C}_{\text{path}}^{\square}(S)$ in Proset-Cat . The idea is that $\mathfrak{C}_{\text{path}}^{\square}(S)(a, b)$ has for objects concatenations of nondegenerate 1-cubes joining a to b and that the preorder is induced by the 2-cubes of S .

Notation 2.2.1 For $n \geq 0$, let I_n be the necklace $(\square^1)^{\vee n}$. For $n \geq 2$ and $0 \leq k \leq n - 2$, let $\mathbb{I}_{k,n}$ be the concatenation $I_k \vee \square^2 \vee I_{n-2-k}$. The source and target maps $s_{k,n}, t_{k,n} : I_n \rightarrow \mathbb{I}_{k,n}$ are the morphisms in Nec defined by

$$s_{k,n} = \text{id}^{\vee k} \vee (\partial_{1,0} * \partial_{2,1}) \vee \text{id}^{\vee n-2-k} \quad \text{and} \quad t_{k,n} = \text{id}^{\vee k} \vee (\partial_{2,0} * \partial_{1,1}) \vee \text{id}^{\vee n-2-k}$$

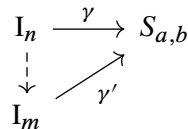
as presented in the following diagram:



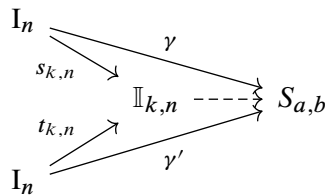
Definition 2.2.2 Let $S_{a,b} \in cSet_{*,*}$. The set $\mathcal{C}_{\text{path}}^{\square}(S)(a, b)$ of paths joining a to b is defined as

$$\mathcal{C}_{\text{path}}^{\square}(S)(a, b) = \bigcup_n \text{Hom}(I_n, S_{a,b}) / \sim,$$

where \sim is the equivalence relation generated by $\gamma \sim \gamma'$ if there exists a factorisation in $cSet_{*,*}$



For $\gamma : I_n \rightarrow S_{a,b}$ and $\gamma' : I_m \rightarrow S_{a,b}$, we write $[\gamma] \rightsquigarrow [\gamma']$ if there exists $0 \leq k \leq n - 2$ and a factorisation in $cSet_{*,*}$



We then define the preorder structure $\mathcal{C}_{\text{path}}^{\square}(S)(a, b)$ as the reflexive transitive closure of \rightsquigarrow , which we also denote by \rightsquigarrow .

The next proposition lifts the definition at the level of categories enriched in preordered sets, that is, Proset-categories.

Proposition 2.2.3 Any cubical set S gives rise to a Proset-category $\mathcal{C}_{\text{path}}^{\square}(S)$ whose objects are the vertices of S and whose homsets are given by Definition 2.2.2, with composition given by concatenation of paths. In addition, the assignment $S \mapsto \mathcal{C}_{\text{path}}^{\square}(S)$ upgrades to a functor $\mathcal{C}_{\text{path}}^{\square} : cSet \rightarrow \text{Proset-Cat}$.

Proof For $\gamma: I_n \rightarrow S_{a,b}$ and $\beta: I_m \rightarrow S_{b,c}$, the class of the concatenation $\gamma * \beta: I_{n+m} = I_n \vee I_m \rightarrow S_{a,c}$ does not depend on the choice of the representatives γ and β . This defines a composition

$$\mathfrak{C}_{\text{path}}^{\square}(S)(b, c) \times \mathfrak{C}_{\text{path}}^{\square}(S)(a, b) \rightarrow \mathfrak{C}_{\text{path}}^{\square}(S)(a, c)$$

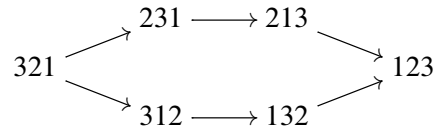
by $([\beta], [\gamma]) \mapsto [\gamma * \beta]$. Similarly if $[\gamma] \rightsquigarrow [\gamma']$ and $[\beta] \rightsquigarrow [\beta']$ then $[\gamma * \beta] \rightsquigarrow [\gamma' * \beta']$, and everything is functorial in S . □

2.3 The path category of a necklace

Let A be a totally ordered set with k elements. An element in the set of bijections Σ_A of A is represented by a sequence (a_1, \dots, a_k) such that $\{a_1, \dots, a_k\} = A$. We consider the (reverse right) weak Bruhat order on Σ_A , that is, the order generated by

$$(a_1, \dots, a_i, a_{i+1}, \dots, a_k) \rightsquigarrow_B (a_1, \dots, a_{i+1}, a_i, \dots, a_k) \quad \text{if } 1 \leq i < k \text{ and } a_i > a_{i+1}.$$

For example, for $A = \{1, 2, 3\}$ the Hasse diagram of \rightsquigarrow_B is given by



Note that, given two disjoint subsets A and B of $\{1, \dots, n\}$, the concatenation $*$: $\Sigma_A \times \Sigma_B \rightarrow \Sigma_{A \sqcup B}$ of sequences is a map of posets (for the order \rightsquigarrow_B), where the total orders on A , B and $A \sqcup B$ are induced by that of $\{1, \dots, n\}$. We refer to the book by Björner [1984] for more on orders on Coxeter groups.

Lemma 2.3.1 For all n , a and b , each element in $\mathfrak{C}_{\text{path}}^{\square}(\square^n)(a, b)$ has a unique representative γ , which is a monomorphism, corresponding to a sequence $a_0 = a < a_1 < \dots < a_l = b$, with $d(a_i, a_{i+1}) = 1$. The same holds replacing \square^n with any cubical subset S of \square^n .

Proof Let $[\gamma']$ be an element of $\mathfrak{C}_{\text{path}}^{\square}(\square^n)(a, b)$. By Lemma 2.1.5, γ' corresponds to some sequence $s' = (a'_0 = a \leq x_1 \leq \dots \leq a'_k = b)$ such that $d(a'_i, a'_{i+1}) \leq 1$. We claim that the desired γ is the monomorphism corresponding to the sequence $\kappa(s') = (a_0 = a < a_1 < \dots < a_l = b)$ obtained by eliminating the repetitions in the sequence s' . This is a consequence of the following two easy facts:

- (i) for γ as just defined, $[\gamma] \sim [\gamma']$, and
- (ii) if $[\gamma'_1] \sim [\gamma'_2]$, with corresponding sequences s'_1 and s'_2 , then $\kappa(s'_1) = \kappa(s'_2)$.

The last part of the statement follows from the observation that if s' above lies in S , then so does $\kappa(s')$. □

Proposition 2.3.2 For every pair of vertices $a \leq b$ in \square^n , there is an isomorphism of preordered sets

$$\mathfrak{C}_{\text{path}}^{\square}(\square^n)(a, b) \rightarrow \Sigma_{b \setminus a},$$

compatible with concatenation. As a consequence, the preorder \rightsquigarrow on paths of a cube is a partial order, isomorphic to the weak Bruhat order \rightsquigarrow_B on the symmetric group.

Proof Assume $a \leq b$. With the notation of Lemma 2.3.1, we can associate with each $[\gamma] \in \mathfrak{C}_{\text{path}}^{\square}(\square^n)(a, b)$ a sequence $a_0 = a < a_1 < \dots < a_l = b$, with $d(a_i, a_{i+1}) = 1$. We denote by x_i the unique element in $a_{i+1} \setminus a_i = \{x_i\}$, so that $\{x_1, \dots, x_l\} = b \setminus a$. Then the map $\Psi: \mathfrak{C}_{\text{path}}^{\square}(\square^n)(a, b) \rightarrow \Sigma_{b \setminus a}$ sending $[\gamma]$ to the sequence (x_1, \dots, x_l) in $\Sigma_{b \setminus a}$ is well defined and bijective.

Let $f: \mathbb{I}_{k,m} \rightarrow \square_{a,b}^n$ in $c\mathcal{S}et_{*,*}$, witnessing $[f \circ s_{k,m}] \rightsquigarrow [f \circ t_{k,m}]$. Let

$$\begin{aligned} a_0 &\leq \dots \leq a_k \leq a_{k+1} \leq a_{k+2} \leq \dots \leq a_m, \\ a_0 &\leq \dots \leq a_k \leq a'_{k+1} \leq a_{k+2} \leq \dots \leq a_m \end{aligned}$$

be the sequences corresponding to $f \circ s_{k,m}$ and $f \circ t_{k,m}$, respectively. If $d(a_k, a_{k+2}) = 2$, then there exists $u < v$ such that $a_{k+2} \setminus a_k = \{u, v\}$, $a_{k+1} \setminus a_k = \{v\}$ and $a'_{k+1} \setminus a_k = \{u\}$, hence $\Psi(f \circ s_{k,m}) \rightsquigarrow_B \Psi(f \circ t_{k,m})$. If $d(a_k, a_{k+2}) < 2$, then

$$[f \circ s_{k,m}] = [\gamma] = [f \circ t_{k,m}],$$

where γ corresponds to $a_0 \leq \dots \leq a_k < a_{k+2} \leq \dots \leq a_m$. Hence Ψ is a morphism of preordered sets. Similarly, and even more straightforwardly, we see that Ψ^{-1} is also a morphism of posets, which in particular implies that $\mathfrak{C}_{\text{path}}^{\square}(\square^n)(a, b)$ is a poset. \square

We also observe that if $a \not\leq b$, there is no morphism $\mathbb{I}_m \rightarrow (\square^n)_{a,b}$, and that the constant path is the unique path in $\mathfrak{C}_{\text{path}}^{\square}(\square^n)(a, a)$. Hence $\mathfrak{C}_{\text{path}}^{\square}(\square^n)$ is a P -shaped poset-category, with P the subset lattice of $\{1, \dots, n\}$. We refer to Section A.2 for this notion, and for the description of the concatenation product \vee on such categories used in the following proposition.

Corollary 2.3.3 *Let $T = \square^{n_1} \vee \dots \vee \square^{n_k}$ be a necklace. Then*

- (1) *for $a \leq b \in T_0$, the inclusion $T_{[a,b]} \subseteq T_{a,b}$ induces an isomorphism of posets*

$$\mathfrak{C}_{\text{path}}^{\square}(T)(a, b) \cong \mathfrak{C}_{\text{path}}^{\square}(T_{[a,b]})(\alpha, \omega),$$

- (2) *if $T = U \vee V$, the composition in the poset category $\mathfrak{C}_{\text{path}}^{\square}(T)$ provides a morphism*

$$\mathfrak{C}_{\text{path}}^{\square}(V)(\alpha_V, \omega_V) \times \mathfrak{C}_{\text{path}}^{\square}(U)(\alpha_U, \omega_U) \rightarrow \mathfrak{C}_{\text{path}}^{\square}(T)(\alpha_T, \omega_T),$$

which is an isomorphism of posets, and

- (3) *$\mathfrak{C}_{\text{path}}^{\square}(T)$ is a poset-category and we have an isomorphism of poset-categories*

$$\mathfrak{C}_{\text{path}}^{\square}(T) \cong \mathfrak{C}_{\text{path}}^{\square}(\square^{n_1}) \vee \dots \vee \mathfrak{C}_{\text{path}}^{\square}(\square^{n_k}).$$

Proof (1) By Definition 2.1.6 and Proposition 2.1.8, a path γ joining a to b in T is equivalent to a morphism $\mathbb{I}_n \rightarrow T_{[a,b]}$, where $T_{[a,b]}$ is a necklace. By Proposition 2.1.8, any map $\mathbb{I}_{k,n} \rightarrow T_{a,b}$ factorises through $T_{[a,b]}$, hence the result.

- (2) Let us prove that the morphism of posets induced by composition/concatenation

$$\mathfrak{C}_{\text{path}}^{\square}(\square^{n_2})(\alpha_2, \omega_2) \times \mathfrak{C}_{\text{path}}^{\square}(\square^{n_1})(\alpha_1, \omega_1) \rightarrow \mathfrak{C}_{\text{path}}^{\square}(\square^{n_1} \vee \square^{n_2})(\alpha_1, \omega_2)$$

is an isomorphism of prosets. By Lemma 2.3.1, and viewing $\square^{n_1} \vee \square^{n_2}$ as a cubical subset of $\square^{n_1+n_2}$ via the standard embedding, any element in the right-hand side admits a unique representative

$$\gamma : I_l \rightarrow \square^{n_1} \vee \square^{n_2},$$

which is a monomorphism.

Since γ preserves α and ω , we have $l = n_1 + n_2$ and thus $\gamma = \gamma_1 \vee \gamma_2$ is the unique decomposition provided by Proposition 2.1.2. Hence the morphism is a bijection. We have to prove that $[\gamma] \rightsquigarrow [\gamma']$ implies $[\gamma_1] \rightsquigarrow [\gamma'_1]$ and $[\gamma_2] \rightsquigarrow [\gamma'_2]$. It is enough to prove it for the “elementary moves” that generate the relation \rightsquigarrow by reflexive and transitive closure. Any $f : \mathbb{I}_{k,m} \rightarrow \square^{n_1} \vee \square^{n_2}$ factors as $f = f_1 \vee f_2$, where either f_1 or f_2 is a path. It implies that if $[\gamma] \rightsquigarrow [\gamma']$ then either $[\gamma_1] \rightsquigarrow [\gamma'_1]$ and $[\gamma_2] = [\gamma'_2]$, or $[\gamma_2] \rightsquigarrow [\gamma'_2]$ and $[\gamma_1] = [\gamma'_1]$. In conclusion, (2) holds, since the left-hand side of the morphism is a poset.

(3) It is clear that this generalises to any finite wedge product of cubes. In particular, if

$$T_{[a,b]} := \square^{d(a,\omega_i)} \vee \square^{n_{i+1}} \vee \dots \vee \square^{n_{j-1}} \vee \square^{d(\alpha_j,b)},$$

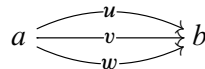
then

$$\begin{aligned} \mathfrak{C}_{\text{path}}^\square(T)(a,b) &= \mathfrak{C}_{\text{path}}^\square(T_{[a,b]})(\alpha, \omega) && \text{(by (1))} \\ &\cong \mathfrak{C}_{\text{path}}^\square(\square^{d(a,\omega_i)})(\alpha_i, \omega_i) \times \dots \times \mathfrak{C}_{\text{path}}^\square(\square^{d(\alpha_j,b)})(\alpha_j, \omega_j) && \text{(by (2))} \\ &= \mathfrak{C}_{\text{path}}^\square(\square^{n_i})(a, \omega_i) \times \dots \times \mathfrak{C}_{\text{path}}^\square(\square^{n_j})(\alpha_j, b) && \text{(by (1))} \\ &\cong (\mathfrak{C}_{\text{path}}^\square(\square^{n_i}) \vee \dots \vee \mathfrak{C}_{\text{path}}^\square(\square^{n_j}))(a, b) && \text{(by Proposition A.2.5).} \end{aligned}$$

Finally, we note that having established this isomorphism a fortiori implies that $\mathfrak{C}_{\text{path}}^\square(T)$ is poset-enriched, since all $\mathfrak{C}_{\text{path}}^\square(\square^{n_i})$ are. □

Applying the nerve functor from Proset-categories to simplicial categories, we get the functor $N \circ \mathfrak{C}_{\text{path}}^\square : cSet \rightarrow sCat$. Unfortunately it is not cocontinuous, as we show in Example 2.3.4, so that it cannot serve as a left functor in a Quillen equivalence. The next section is devoted to build such a functor and to study its properties.

Example 2.3.4 In this example, all simplicial categories involved have only one (possibly) nontrivial mapping space, and hence reduce to simplicial sets. Consider the quotient $\tilde{\square}^2$ of \square^2 obtained by collapsing the edges between $(0, 0)$ and $(0, 1)$, and between $(1, 0)$ and $(1, 1)$, and consider the pushout X of the two horizontal inclusions $\partial_{2,0}, \partial_{2,1} : \square^1 \rightarrow \tilde{\square}^2$. The cubical set X can be represented as



with two nondegenerate 2–cubes inducing $u \rightsquigarrow v \rightsquigarrow w$ in $\mathfrak{C}_{\text{path}}^\square(X)$. It follows that $N(\mathfrak{C}_{\text{path}}^\square(X))$ is not 1–skeletal. However $N(\mathfrak{C}_{\text{path}}^\square(\square^1))$ and $N(\mathfrak{C}_{\text{path}}^\square(\tilde{\square}^2))$ are 1–skeletal. But a pushout of 1–skeletal simplicial sets is 1–skeletal, so the pushout of $N(\mathfrak{C}_{\text{path}}^\square(\partial_{2,0}))$ and $N(\mathfrak{C}_{\text{path}}^\square(\partial_{2,1}))$ cannot be $N(\mathfrak{C}_{\text{path}}^\square(X))$. The point of this counterexample is that taking a colimit in $cSet$ may result in merging of orders: X sees $u \rightsquigarrow v \rightsquigarrow w$, while each copy of $\tilde{\square}^2$ sees only $u \rightsquigarrow v$ or $v \rightsquigarrow w$. By applying the functor $N \circ \mathfrak{C}_{\text{path}}^\square$ before the colimit functor, we lose this piece of magic!

3 The rigidification functor \mathfrak{C}^\square

In this section, we define rigidification as a left Kan extension of the restriction of $N \circ \mathfrak{C}_{\text{path}}^\square$ to the cubes \square^n , and provide concrete descriptions of its simplicial homsets, making an essential use of necklaces (see Remark 3.2.5).

3.1 Definition of the rigidification

The rigidification functor is defined as the left Kan extension along the Yoneda functor $Y : \square \rightarrow cSet$ of the composition

$$\square \xrightarrow{Y} cSet \xrightarrow{\mathfrak{C}_{\text{path}}^\square} \text{Proset-Cat} \xrightarrow{N} sCat.$$

By usual means, we obtain an adjunction $\mathfrak{C}^\square : cSet \rightleftarrows sCat : N^\square$. The simplicial category $\mathfrak{C}^\square(S)$ is obtained as the colimit over the category of elements of S of some $\mathfrak{C}^\square(\square^n)$.

Lemma 3.1.1 *For every cubical set S , the set of objects of the simplicial category $\mathfrak{C}^\square(S)$ is in bijection with S_0 , and thus will be identified with it.*

Proof By Remark 1.1.2, the functor $sCat \xrightarrow{\text{Ob}} Set$ is cocontinuous. We conclude since the statement holds on cubes by definition (cf Proposition 2.2.3). \square

Notation 3.1.2 The previous lemma implies that the rigidification functor lifts to a functor

$$\mathfrak{C}^\square : cSet_{*,*} \rightarrow sCat_{*,*}.$$

We denote by \mathfrak{C}_t^\square the functor from $cSet_{*,*}$ to $sSet$ defined on objects by

$$\mathfrak{C}_t^\square(S_{a,b}) = \mathfrak{C}^\square(S)(a, b) \quad \text{for all } a, b \in S_0.$$

Lemma 3.1.3 *The space $\mathfrak{C}_t^\square(\square^n)$ is contractible.*

Proof We have $\mathfrak{C}_t^\square(\square^n) = \mathfrak{C}^\square(\square^n)(\alpha, \omega) \cong N(\Sigma_{\{1, \dots, n\}})$ by Proposition 2.3.2 with the weak Bruhat order on $\Sigma_{\{1, \dots, n\}}$. The latter is a bounded poset (see Definition A.2.1); hence its nerve is contractible. \square

A direct application of Corollaries 2.3.3 and A.2.6 is the following theorem.

Theorem 3.1.4 *For all necklaces T , there is an isomorphism of simplicial categories*

$$\mathfrak{C}^\square(T) \cong N(\mathfrak{C}_{\text{path}}^\square(T)).$$

In addition, if $T = U \vee V$, the composition in the simplicial category $\mathfrak{C}^\square(T)$ provides a morphism $\mathfrak{C}_t^\square(V) \times \mathfrak{C}_t^\square(U) \rightarrow \mathfrak{C}_t^\square(T)$, which is an isomorphism of simplicial sets.

Corollary 3.1.5 *Let T be a necklace. For every $a \preceq b \in T_0$, the simplicial set $\mathfrak{C}^\square(T)(a, b)$ is contractible. In particular, the simplicial set $\mathfrak{C}_t^\square(T)$ is contractible.*

Proof By Corollary 2.3.3, $\mathfrak{C}_{\text{path}}^\square(T)(a, b)$ is a product of bounded posets; hence its nerve is contractible. We conclude using Theorem 3.1.4. \square

3.2 Computing the rigidification functor

The construction by left Kan extension gives us a way to express $\mathfrak{C}^\square(S)$ as a colimit in $s\mathcal{C}at$, which is difficult to compute. In this section, we use necklaces as “paths of higher dimension” to obtain a handy way to compute $\mathfrak{C}^\square(S)$. Indeed, we follow step by step the techniques developed by Dugger and Spivak [2011a, Proposition 4.3].

Each object $f : T \rightarrow S_{a,b}$ in $Nec \downarrow S_{a,b}$ induces a morphism $\mathfrak{C}_t^\square(T) \rightarrow \mathfrak{C}_t^\square(S_{a,b})$ in $sSet$ and this construction gives a morphism $\text{colim}_{Nec \downarrow S_{a,b}} \mathfrak{C}_t^\square(T) \rightarrow \mathfrak{C}_t^\square(S_{a,b})$ in $sSet$. We prove that it is an isomorphism, so that $\mathfrak{C}^\square(S)(a, b) = \mathfrak{C}_t(S_{a,b})$ is computed as a colimit in $sSet$.

Notation 3.2.1 Let $S_{a,b} \in cSet_{*,*}$.

- We set $E_t(S_{a,b}) := \text{colim}_{Nec \downarrow S_{a,b}} \mathfrak{C}_t^\square(T)$.
- Let $(\alpha_t)_{S_{a,b}} : E_t(S_{a,b}) \rightarrow \mathfrak{C}_t^\square(S_{a,b})$ be the structure map from the colimit to the cocone $\mathfrak{C}_t^\square(S_{a,b})$.

The following facts are left to the reader.

- The definition of E_t is functorial, hence defines a functor $E_t : cSet_{*,*} \rightarrow sSet$.
- The morphisms $(\alpha_t)_{S_{a,b}}$ in $sSet$ form a natural transformation $\alpha_t : E_t \Rightarrow \mathfrak{C}_t^\square$.

Our goal is to prove that α_t is a natural isomorphism. In fact, we will prove that E_t can be upgraded to a functor $E : cSet \rightarrow s\mathcal{C}at$ that is naturally isomorphic to \mathfrak{C}^\square .

Proposition 3.2.2 *There exists a functor $E : cSet \rightarrow s\mathcal{C}at$ and a natural transformation $\alpha : E \Rightarrow \mathfrak{C}^\square$ such that*

- $\text{Ob}(E(S)) = S_0$,
- $E(S)(a, b) = E_t(S_{a,b})$, and
- α_S is the identity on objects and $\alpha_S(a, b) = (\alpha_t)_{S_{a,b}} : E(S)(a, b) \rightarrow \mathfrak{C}^\square(S)(a, b)$.

Proof We take $* \rightarrow S_{a,a} \in Nec \downarrow S_{a,a}$ as identity morphism $\text{id}_a \in E(S)(a, a)$. The composition in the simplicial category $E(S)$ is defined to be the composite featured as the left arrow in the diagram

$$\begin{array}{ccc}
 E_t(S_{b,c}) \times E_t(S_{a,b}) = \text{colim}_{V \rightarrow S_{b,c}} \mathfrak{C}_t^\square(V) \times \text{colim}_{U \rightarrow S_{a,b}} \mathfrak{C}_t^\square(U) & \xleftarrow{\cong} & \text{colim}_{\substack{V \rightarrow S_{b,c} \\ U \rightarrow S_{a,b}}} (\mathfrak{C}_t^\square(V) \times \mathfrak{C}_t^\square(U)) \\
 & \downarrow \circ E & \cong \downarrow \text{Theorem 3.1.4} \\
 E_t(S_{a,c}) = \text{colim}_{T \rightarrow S_{a,c}} \mathfrak{C}_t^\square(T) & \xleftarrow{\quad \quad \quad} & \text{colim}_{\substack{V \rightarrow S_{b,c} \\ U \rightarrow S_{a,b}}} \mathfrak{C}_t^\square(U \vee V)
 \end{array}$$

where the top arrow is invertible, as \times is cocontinuous in $sSet$. Then the monoidal structure of $(cSet_{*,*}, \vee, *)$ ensures that $E(S)$ with identities and composition as above is a simplicial category. The functoriality of E comes from that of E_t .

Let us check that the $(\alpha_t)_{S_{a,b}}$ induce an enriched functor $\alpha_S: E(S) \rightarrow \mathfrak{C}^\square(S)$, which is the identity on objects. We have to prove that the following diagram commutes:

$$\begin{array}{ccc}
 \text{colim}_{V \rightarrow S_{b,c}} \mathfrak{C}_t^\square(V) \times \text{colim}_{U \rightarrow S_{a,b}} \mathfrak{C}_t^\square(U) & \xleftarrow{\cong} & \text{colim}_{\substack{V \rightarrow S_{b,c} \\ U \rightarrow S_{a,b}}} (\mathfrak{C}_t^\square(V) \times \mathfrak{C}_t^\square(U)) & \xrightarrow{\cong} & \text{colim}_{\substack{V \rightarrow S_{b,c} \\ U \rightarrow S_{a,b}}} \mathfrak{C}_t^\square(U \vee V) & \rightarrow & \text{colim}_{T \rightarrow S_{a,c}} \mathfrak{C}_t^\square(T) \\
 \downarrow (\alpha_t)_{S_{b,c}} \times (\alpha_t)_{S_{a,b}} & \swarrow & & \searrow & & \downarrow (\alpha_t)_{S_{a,c}} \\
 \mathfrak{C}_t^\square(S_{b,c}) \times \mathfrak{C}_t^\square(S_{a,b}) & \xrightarrow{\hspace{10em}} & & & & \mathfrak{C}_t^\square(S_{a,c})
 \end{array}$$

It suffices to notice that for all $V \rightarrow S_{b,c}$ and $U \rightarrow S_{a,b}$, $\mathfrak{C}^\square(U \vee V) \rightarrow \mathfrak{C}^\square(S)$ is a simplicially enriched functor and so the following square commutes:

$$\begin{array}{ccc}
 \mathfrak{C}^\square(U \vee V)(\alpha_V, \omega_V) \times \mathfrak{C}^\square(U \vee V)(\alpha_U, \omega_U) & \longrightarrow & \mathfrak{C}^\square(U \vee V)(\alpha_U, \omega_V) \\
 \downarrow & & \downarrow \\
 \mathfrak{C}^\square(S)(b, c) \times \mathfrak{C}^\square(S)(a, b) & \longrightarrow & \mathfrak{C}^\square(S)(a, c)
 \end{array}$$

Theorem 3.1.4 applied to our case gives

$$\begin{aligned}
 \mathfrak{C}^\square(U \vee V)(\alpha_V, \omega_V) &= \mathfrak{C}^\square(V)(\alpha_V, \omega_V), \\
 \mathfrak{C}^\square(U \vee V)(\alpha_U, \omega_U) &= \mathfrak{C}^\square(U)(\alpha_U, \omega_U),
 \end{aligned}$$

so the diagram becomes

$$\begin{array}{ccc}
 \mathfrak{C}_t^\square(V) \times \mathfrak{C}_t^\square(U) & \longrightarrow & \mathfrak{C}_t^\square(U \vee V) \\
 \downarrow & & \downarrow \\
 \mathfrak{C}_t^\square(S_{b,c}) \times \mathfrak{C}_t^\square(S_{a,b}) & \longrightarrow & \mathfrak{C}_t^\square(S_{a,c})
 \end{array}$$

and we conclude by universality of colimits. The naturality of $\alpha_S: E(S) \rightarrow \mathfrak{C}^\square(S)$ comes from the naturality of α_t . □

Proposition 3.2.3 *The natural transformation $\alpha: E \Rightarrow \mathfrak{C}^\square$ is a natural isomorphism.*

Proof Let $S \in cSet$. We know that α_S is the identity on objects. Assume first that $S = T$ is a necklace. Let $a, b \in T_0$. If $a \preceq b$, by Proposition 2.1.8, $\iota_{a,b}^T: T_{[a,b]} \hookrightarrow T$ is terminal in $Nec \downarrow T_{a,b}$; hence $E(T)(a, b) \cong \mathfrak{C}_t^\square(T_{[a,b]}) = \mathfrak{C}^\square(T)(a, b)$, and the isomorphism is precisely induced by α_T . If $a \not\preceq b$, then both categories are empty; hence the result holds for necklaces.

We prove that for all vertices a and b of S , the morphism $\alpha_S(a, b): E(S)(a, b) \rightarrow \mathfrak{C}^\square(S)(a, b)$ is an isomorphism of simplicial sets, by providing an inverse $\beta_S(a, b)$. Recall that

$$\alpha_S(a, b) = \alpha_t(S_{a,b}): E_t(S_{a,b}) \rightarrow \mathfrak{C}_t^\square(S_{a,b})$$

is the (unique) map from the colimit to the cocone $\mathfrak{C}_t^\square(S_{a,b})$. Define $(\beta_t)_{S_{a,b}} : \mathfrak{C}_t^\square(S_{a,b}) \rightarrow E_t(S_{a,b})$ as the composite

$$\begin{array}{ccc} E_t(S_{a,b}) & \longleftarrow & (\operatorname{colim}_{\square^k \rightarrow S} E(\square^k))(a, b) \\ \uparrow (\beta_t)_{S_{a,b}} \text{ (dashed)} & & \cong \downarrow \text{ (}\square^k \text{ is a necklace)} \\ \mathfrak{C}_t^\square(S_{a,b}) & \xleftarrow{\cong} & (\operatorname{colim}_{\square^k \rightarrow S} \mathfrak{C}_t^\square(\square^k))(a, b) \end{array}$$

By naturality of all the maps involved in the diagram, the family $(\beta_t)_{S_{a,b}}$ assembles to a natural transformation $\beta_t : \mathfrak{C}_t^\square \Rightarrow E_t$, giving rise to $\beta : \mathfrak{C}^\square \Rightarrow E$.

We show, in this order, that β is a right inverse, and a left inverse of α . To show the former, it is enough to show $\alpha_S \circ \beta_S \circ j_f = j_f$, for all $f : \square^k \rightarrow S_{a,b}$, where j_f is the characteristic map $E(\square^k) \rightarrow \operatorname{colim}_{\square^k \rightarrow S} E(\square^k)$, and where we identify $\mathfrak{C}^\square(S)$ with $\operatorname{colim}_{\square^k \rightarrow S} E(\square^k)$. Indeed, we have

$$\begin{aligned} \alpha_S \circ \beta_S \circ j_f &= \alpha_S \circ E(f) && \text{(by definition of } \beta) \\ &= \mathfrak{C}^\square(f) \circ \alpha_{\square^k} && \text{(by naturality of } \alpha) \\ &= j_f && \text{(by the identification above).} \end{aligned}$$

We prove now that $(\beta_t)_{S_{a,b}} \circ (\alpha_t)_{S_{a,b}}$ is the identity in a similar way. By the universal property of the colimit, it is enough to prove $(\beta_t)_{S_{a,b}} \circ (\alpha_t)_{S_{a,b}} \circ i_f = i_f$ for all $T \xrightarrow{f} S_{a,b}$, where $i_f : \mathfrak{C}_t^\square(T) \rightarrow E_t(S_{a,b})$ is the characteristic morphism. This comes from the commutative diagram

$$\begin{array}{ccc} & & E_t(S_{a,b}) \\ & \nearrow i_f \text{ (dashed)} & \downarrow (\alpha_t)_{S_{a,b}} \\ \mathfrak{C}_t^\square(T) & \xrightarrow{\mathfrak{C}_t^\square(f)} & \mathfrak{C}_t^\square(S_{a,b}) \\ \downarrow (\beta_t)_T \cong & & \downarrow (\beta_t)_{S_{a,b}} \\ E_t(T) & \xrightarrow{E_t(f)} & E_t(S_{a,b}) \\ \downarrow (\alpha_t)_T \cong & \nearrow i_f \text{ (dashed)} & \\ \mathfrak{C}_t^\square(T) & & \end{array}$$

id \curvearrowright

(the above triangle commutes by definition of α_t , the middle square commutes by naturality of β_t , and the bottom triangle commutes by definition of $E_t(f)$). □

As a direct corollary, we get the main theorem of the section.

Theorem 3.2.4 *Let S be a cubical set and $a, b \in S_0$. We have the following isomorphism of simplicial sets:*

$$\mathfrak{C}_t^\square(S_{a,b}) = \mathfrak{C}_t^\square(S)(a, b) \cong \operatorname{colim}_{(T \rightarrow S_{a,b}) \in \operatorname{Nec} \downarrow S_{a,b}} \mathfrak{C}_t^\square(T).$$

In the following remark, we point out that necklaces were instrumental in getting the characterisation given in the previous theorem.

Remark 3.2.5 We do not have an isomorphism between $\mathfrak{C}^\square(S)(a, b)$ and the colimit above restricted to cubes. This already fails when S is a necklace. Consider $S = (1, 1)$ and $(a, b) = (\alpha, \omega)$. There is no morphism from a cube to $S_{\alpha, \omega}$ in $cSet_{*,*}$, whereas there is one from the necklace S to itself (the identity morphism). Hence the restricted colimit is empty, while $\mathfrak{C}^\square(S)(a, b)$ is not.

Remark 3.2.6 It can be shown, using the result above, that $\pi_0(\mathfrak{C}^\square(S)) \cong \pi_0(\mathfrak{C}_{\text{path}}^\square(S))$ for any cubical set S .

3.3 Case of cubical subsets of a cube

Definition 3.3.1 Let $S_{a,b}$ in $cSet_{*,*}$, with S a cubical subset of an n -cube. The subcategory of $Nec \downarrow S_{a,b}$ whose objects are monomorphisms $T \rightarrow S_{a,b}$ and arrows are monomorphisms between necklaces is denoted $\text{SubNeck}(S_{a,b})$. The category $\text{SubNeck}(S_{a,b})$ is actually a poset, as shown in Proposition B.1.1.

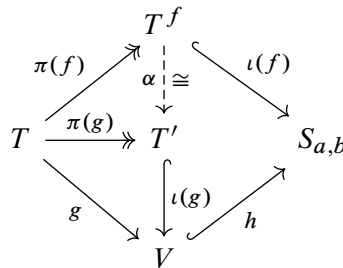
Proposition 3.3.2 Let $S_{a,b} \in cSet_{*,*}$, with S a cubical subset of an n -cube. The rigidification functor has the expression

$$\mathfrak{C}^\square(S)(a, b) = \mathfrak{C}_t^\square(S_{a,b}) \cong \text{colim}_{\text{SubNeck}(S_{a,b})} \mathfrak{C}_t^\square(T).$$

Proof We use Lemma 2.1.9 and its notation. Recall from Theorem 3.2.4 that

$$\mathfrak{C}_t^\square(S_{a,b}) \cong \text{colim}_{Nec \downarrow S_{a,b}} \mathfrak{C}_t^\square(T).$$

Consider the inclusion functor $U : \text{SubNeck}(S_{a,b}) \hookrightarrow Nec \downarrow S_{a,b}$ and fix $f \in Nec \downarrow S_{a,b}$. The category $f \downarrow U$ has $\pi(f) : f \rightarrow \iota(f)$ as object, and hence is not empty. Let $g : f \rightarrow h$ be an object in $f \downarrow U$, so that $hg = f = \iota(f)\pi(f)$. The morphism g in Nec admits the factorisation $g = \iota(g)\pi(g)$. By the unique decomposition of f as an epimorphism followed by a monomorphism, there exists an isomorphism $\alpha : T^f \rightarrow T'$, as illustrated by the diagram



In conclusion, the morphism $\iota(g)\alpha$ is a monomorphism, hence a morphism in $f \downarrow U$ from $\pi(f)$ to g . Thus the category $f \downarrow U$ is connected. We have proved that U is a final functor, from which the statement follows (cf [Mac Lane 1998, Section IX.3]). □

4 Quillen equivalence

In this section, we prove that the adjunction $\mathfrak{C}^\square \dashv N^\square$ is a Quillen equivalence between the Joyal model structure on $cSet$ and the Bergner model structure on $sCat$.

4.1 Properties of the functor \mathfrak{C}^\square

Proposition 4.1.1 *The functor \mathfrak{C}^\square preserves cofibrations.*

Proof Since cofibrations of $cSet$ are generated by $\partial^n : \partial \square^n \rightarrow \square^n$, it is enough to prove that

$$\mathfrak{C}^\square(\partial^n) : \mathfrak{C}^\square(\partial \square^n) \rightarrow \mathfrak{C}^\square(\square^n)$$

is a cofibration. We claim that the diagram

$$\begin{array}{ccc} \Sigma(\mathfrak{C}_t^\square(\partial \square^n)) & \longrightarrow & \mathfrak{C}^\square(\partial \square^n) \\ \Sigma(\mathfrak{C}_t^\square(\partial^n)) \downarrow & & \downarrow \mathfrak{C}^\square(\partial^n) \\ \Sigma(\mathfrak{C}_t^\square(\square^n)) & \longrightarrow & \mathfrak{C}^\square(\square^n) \end{array}$$

where the horizontal morphisms are given by the counit of the adjunction $\Sigma \dashv \text{Hom}$ of Proposition 1.1.1, is a pushout diagram. The set of objects of the simplicial categories on the right-hand side of the diagram is in bijection with $(\square^n)_0 = (\partial \square^n)_0$, which is a bounded poset. Let a and b be two such objects. If $(a, b) \neq (\alpha, \omega)$, then $\square^{d(a,b)} \subseteq \partial \square^n$, so that the functor $Nec \downarrow (\partial \square^n)_{a,b} \rightarrow Nec \downarrow (\square^n)_{a,b}$ is an isomorphism of categories. Theorem 3.2.4 implies then that the map $\mathfrak{C}^\square(\partial^n)(a, b) : \mathfrak{C}^\square(\partial \square^n)(a, b) \rightarrow \mathfrak{C}^\square(\square^n)(a, b)$ is an isomorphism. We conclude by Proposition A.2.7. We show next that $\mathfrak{C}_t^\square(\partial \square^n) \rightarrow \mathfrak{C}_t^\square(\square^n)$ is a cofibration. Indeed $\partial \square^n$ is a cubical subset of \square^n ; hence $\mathfrak{C}_t^\square(\partial \square^n) \cong \text{colim}_{T \in \text{SubNeck}(\partial \square^n)} \mathfrak{C}_t^\square(T)$ by Proposition 3.3.2, so $\mathfrak{C}_t^\square(\partial \square^n) \rightarrow \mathfrak{C}_t^\square(\square^n)$ is a cofibration by Lemma B.1.5. The functor Σ preserves cofibrations by Proposition 1.1.1, and so do pushout diagrams. □

We refer to Definition 1.2.5 for the notation in the next lemma.

Lemma 4.1.2 *In the diagram*

$$\begin{array}{ccc} \mathfrak{C}^\square(\Gamma_{i,\epsilon}^n)(a, b) & \xrightarrow[\cong]{\mathfrak{C}^\square(p)} & \mathfrak{C}^\square(\widehat{\Gamma}_{i,\epsilon}^n)(pa, pb) \\ \downarrow & & \downarrow \\ \mathfrak{C}^\square(\square^n)(a, b) & \xrightarrow[\cong]{\mathfrak{C}^\square(p)} & \mathfrak{C}^\square(\widehat{\square}_{i,\epsilon}^n)(pa, pb) \end{array}$$

the horizontal arrows (where p is the quotient map) are isomorphisms of simplicial sets when $a \neq \{i\}$ if $\epsilon = 1$, and $b \neq \{1, \dots, n\} \setminus \{i\}$ if $\epsilon = 0$.

Proof We prove the isomorphism $\mathfrak{C}^\square(p): \mathfrak{C}^\square(\square^n) \rightarrow \mathfrak{C}^\square(\widehat{\square}_{i,\epsilon}^n)$ with $\epsilon = 1$ and $i = 1$, so that the critical edge $e_{1,1}: \square^1 \rightarrow \square^n$ corresponds to the edge from $\alpha = \emptyset$ to $\{1\}$. The other cases are similar. Since \mathfrak{C}^\square preserves colimits, we have the pushout diagram

$$\begin{array}{ccc} \mathfrak{C}^\square(\square^1) & \xrightarrow{\mathfrak{C}^\square(e_{1,1})} & \mathfrak{C}^\square(\square^n) \\ \downarrow & & \downarrow \\ \mathfrak{C}^\square(\square^0) & \longrightarrow & \mathfrak{C}^\square(\widehat{\square}_{1,1}^n) \end{array}$$

Let us define the following simplicial category S . The set of objects of S is identified with that of $\widehat{\square}_{1,1}^n$. We denote by $\bar{\alpha} = p(\alpha) = p(\{1\})$. Define $S(\bar{\alpha}, \bar{\alpha}) = *$, $S(\bar{\alpha}, b) = \mathfrak{C}^\square(\square^n)(\alpha, b)$ for $b \neq \bar{\alpha}$ and $S(a, b) = \mathfrak{C}^\square(\square^n)(a, b)$ for $a \neq \bar{\alpha}$. The composition is induced by that of $\mathfrak{C}^\square(\square^n)$. Let $\pi: \mathfrak{C}^\square(\square^n) \rightarrow S$ be the map which coincides with p on objects, and is the identity on morphisms except for the case $\pi: \mathfrak{C}^\square(\square^n)(\{1\}, b) \rightarrow S(\bar{\alpha}, b)$, for which we use the composite

$$\mathfrak{C}^\square(\square^n)(\{1\}, b) \rightarrow \mathfrak{C}^\square(\square^n)(\{1\}, b) \times \mathfrak{C}^\square(\square^n)(\alpha, \{1\}) \xrightarrow{\circ} \mathfrak{C}^\square(\square^n)(\alpha, b) \cong S(\bar{\alpha}, b),$$

which is well defined since $\mathfrak{C}^\square(\square^n)(\alpha, \{1\}) = *$. One checks easily that S satisfies the universal pushout property; hence $\mathfrak{C}^\square(\widehat{\square}_{1,1}^n) \cong S$. We conclude, since by definition of S ,

$$\mathfrak{C}^\square(\widehat{\square}_{1,1}^n)(pa, pb) \cong S(pa, pb) = \mathfrak{C}^\square(\square^n)(a, b) \quad \text{if } a \neq \{1\}.$$

The side condition is needed since, for $a = \{1\}$ and $b > 1$,

$$S(p(\{1\}), b) = S(\bar{\alpha}, b) = \mathfrak{C}^\square(\square^n)(\alpha, b) \not\cong \mathfrak{C}^\square(\square^n)(\{1\}, b).$$

The proof for the inner open box is exactly the same since $\mathfrak{C}^\square(\square^n)(\alpha, \{1\}) = *$. □

Proposition 4.1.3 $\mathfrak{C}^\square(h_{i,\epsilon}^n): \mathfrak{C}^\square(\widehat{\square}_{i,\epsilon}^n) \rightarrow \mathfrak{C}^\square(\widehat{\square}_{i,\epsilon}^n)$ is an acyclic cofibration.

Proof The proof is analogous to the proof of Proposition 4.1.1. Note that the set of vertices of $\widehat{\square}_{i,\epsilon}^n$ coincides with that of $\widehat{\square}_{i,\epsilon}^n$ and is a bounded poset. By Proposition A.2.7, in order to establish that the diagram

$$\begin{array}{ccc} \Sigma(\mathfrak{C}_t^\square(\widehat{\square}_{i,\epsilon}^n)) & \longrightarrow & \mathfrak{C}^\square(\widehat{\square}_{i,\epsilon}^n) \\ \Sigma(\mathfrak{C}_t^\square(h_{i,\epsilon}^n)) \downarrow & & \downarrow \mathfrak{C}^\square(h_{i,\epsilon}^n) \\ \Sigma(\mathfrak{C}_t^\square(\widehat{\square}_{i,\epsilon}^n)) & \longrightarrow & \mathfrak{C}^\square(\widehat{\square}_{i,\epsilon}^n) \end{array}$$

is a pushout diagram, we have to show that the maps

$$\mathfrak{C}^\square(h_{i,\epsilon}^n)(pa, pb): \mathfrak{C}^\square(\widehat{\square}_{i,\epsilon}^n)(pa, pb) \rightarrow \mathfrak{C}^\square(\widehat{\square}_{i,\epsilon}^n)(pa, pb)$$

are isomorphisms, for every (a, b) such that $(pa, pb) \neq (p\alpha, p\omega)$, or equivalently using Lemma 4.1.2, that $\mathfrak{C}^\square(h_{i,\epsilon}^n)(a, b): \mathfrak{C}^\square(\square^n)(a, b) \rightarrow \mathfrak{C}^\square(\square^n)(a, b)$ is an isomorphism. The latter is established exactly

as in the proof of Proposition 4.1.1, noticing that $\square^{d(a,b)} \subseteq \Pi_{i,\epsilon}^n$ for such (a, b) . What remains to prove is that $\mathfrak{C}_t^\square(\widehat{\Pi}_{i,\epsilon}^n) \rightarrow \mathfrak{C}_t^\square(\widehat{\square}_{i,\epsilon}^n)$ is an acyclic cofibration, or equivalently, by Lemma 4.1.2 again, that $\mathfrak{C}_t^\square(\Pi_{i,\epsilon}^n) \rightarrow \mathfrak{C}_t^\square(\square_{i,\epsilon}^n)$ is an acyclic cofibration for the Kan–Quillen structure on simplicial sets. We already know that it is a cofibration, since \mathfrak{C}^\square preserves cofibrations by Proposition 4.1.1 and so does \mathfrak{C}_t^\square . Since $\mathfrak{C}_t^\square(\square^n)$ is contractible (Lemma 3.1.3), showing that $\mathfrak{C}_t^\square(\Pi_{i,\epsilon}^n) \rightarrow \mathfrak{C}_t^\square(\square^n)$ is acyclic amounts to proving that $\mathfrak{C}_t^\square(\Pi_{i,\epsilon}^n)$ is contractible. From Proposition 3.3.2, we have

$$\mathfrak{C}_t^\square(\Pi_{i,\epsilon}^n) = \operatorname{colim}_{T \in \operatorname{SubNeck}(\Pi_{i,\epsilon}^n)} \mathfrak{C}_t^\square(T).$$

Furthermore, we show in Section B.1 that the category $\operatorname{SubNeck}(\Pi_{i,\epsilon}^n)$ is direct (and hence is Reedy). The diagram $(T \rightarrow \Pi_{i,\epsilon}^n) \mapsto \mathfrak{C}_t^\square(T)$ is Reedy cofibrant: for every $T \in \operatorname{SubNeck}(\Pi_{i,\epsilon}^n)$, the latching morphism

$$\operatorname{colim}_{\substack{U \in \operatorname{SubNeck}(T) \\ U \neq T}} \mathfrak{C}_t^\square(U) \rightarrow \mathfrak{C}_t^\square(T)$$

is a monomorphism by Lemma B.1.5. It follows from [Hirschhorn 2003, Theorem 19.9.1], and from the fact that any direct category has fibrant constants, that the natural map

$$\operatorname{hocolim}_{T \in \operatorname{SubNeck}(\Pi_{i,\epsilon}^n)} \mathfrak{C}_t^\square(T) \rightarrow \operatorname{colim}_{T \in \operatorname{SubNeck}(\Pi_{i,\epsilon}^n)} \mathfrak{C}_t^\square(T)$$

is a weak equivalence of simplicial sets. In conclusion,

$$\mathfrak{C}_t^\square(\Pi_{i,\epsilon}^n) \sim \operatorname{hocolim}_{T \in \operatorname{SubNeck}(\Pi_{i,\epsilon}^n)} \mathfrak{C}_t^\square(T) \sim \operatorname{hocolim}_{T \in \operatorname{SubNeck}(\Pi_{i,\epsilon}^n)} * \sim N(\operatorname{SubNeck}(\Pi_{i,\epsilon}^n)) \sim *$$

since $\mathfrak{C}_t^\square(T)$ and $N(\operatorname{SubNeck}(\Pi_{i,\epsilon}^n))$ are contractible, by Corollary 3.1.5 and Proposition B.2.1. \square

4.2 Quillen adjunction

We shall use the following result of Joyal [2008, E.2.14].

Proposition 4.2.1 *An adjunction $L \dashv R$ between two model categories is Quillen if and only if L preserves cofibrations and R preserves fibrations between fibrant objects.*

In view of this proposition, what remains to prove is that N^\square sends fibrations between fibrant categories to fibrations between cubical quasicategories. We shall use two lemmas, which we now present. We refer to Definition 1.2.8 for the definition of equivalence and that of special open box. The first lemma is tautological.

Lemma 4.2.2 *Let \mathcal{A} be a fibrant simplicial category, $v: \mathfrak{C}^\square(\square^1) \rightarrow \mathcal{A}$ and $\tilde{v}: \square^1 \rightarrow N^\square(\mathcal{A})$ its transpose. Since $\mathfrak{C}^\square(\square^1)$ is the simplicial category with only one nontrivial arrow, we see v as an arrow in \mathcal{A} . Then \tilde{v} is an equivalence in the cubical quasicategory $N^\square(\mathcal{A})$ if and only if $\pi_0(v)$ is an isomorphism in $\pi_0(\mathcal{A})$.*

Lemma 4.2.3 *Let $p: X \rightarrow Y$ be an inner fibration between cubical quasicategories and $\square_{i,\epsilon}^2 \rightarrow X$ a special open box. Any commutative square of the following form has a lift:*

$$\begin{array}{ccc} \square_{i,\epsilon}^2 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \square^2 & \longrightarrow & Y \end{array}$$

Proof It is a special case of [Doherty et al. 2024, Lemma 4.14]. We give a proof for the case $(i, \epsilon) = (1, 1)$, the other cases being similar. We represent the map $\square_{1,1}^2 \rightarrow X$ on the left below, with f an equivalence. The 1-cube f being an equivalence, the middle diagram below exists. Gluing the two diagrams, we get the partially filled 3-cube in X (on the right)

$$\begin{array}{ccc} \xrightarrow{f} & \xrightarrow{\quad} & \xrightarrow{\quad} \\ \downarrow u & \downarrow f & \downarrow u \\ \xrightarrow{v} & \xrightarrow{g} & \xrightarrow{v} \end{array}$$

where our conventions for the coordinates in dimensions 2 and 3 are

$$\begin{array}{ccc} & \longrightarrow & 1 \\ & \downarrow & \\ 2 & & \end{array} \qquad \begin{array}{ccc} & & 1 \\ & \nearrow & \\ & \longrightarrow & 2 \\ & \downarrow & \\ & & 3 \end{array}$$

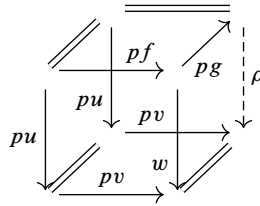
The map $B: \square^2 \rightarrow Y$ is represented by the following 2-cube in Y :

$$\begin{array}{ccc} \xrightarrow{pf} & & \\ \downarrow pu & & \downarrow w \\ \xrightarrow{pv} & & \end{array}$$

The proof goes in three steps. For the first step, we assume $Y = *$. We complete the above 3-cube progressively, as follows:

The top face is full by hypothesis, the left and the bottom faces are given by degeneracies. Because X is a cubical quasicategory, the back face (picture in the middle), then the right face (picture on the right), and finally the whole cube, and hence a fortiori the front face, can be filled. In consequence any special open box in a cubical quasicategory X can be filled by a 2-cube in X .

The second step consists in filling the following 3–cube in Y , where the front, top, left and bottom faces are already filled:



By [Doherty et al. 2024, Lemma 2.6], g is an equivalence because f is and so is pg . Hence the right face is a special open box in the cubical quasicategory Y and thus can be filled by step 1. Then the whole cube is filled because the critical edge associated to the back face is the identity.

For the last step, we resume the filling of the 3–cube of the first step (with the same pictures as above) in X , but now in the general case. The aim is to fill in the front face of the 3–cube in X by a 2–cube A satisfying $pA = B$. Because p is an inner fibration, its back face can be filled by a 2–cube such that the dashed arrow in the picture in the middle is sent to ρ by p . Then the same is true for its right face, so the dashed arrow in the picture on the right is sent to w by p . Finally, the whole cube is filled and sent by p to the 3–cube in Y , and a fortiori its front face A satisfies $pA = B$. \square

Proposition 4.2.4 *The functor N^\square sends fibrations between fibrant simplicial categories to fibrations between cubical quasicategories.*

Proof Using Proposition 4.1.3, we conclude by adjunction that $N^\square(\mathcal{C})$ is a cubical quasicategory if \mathcal{C} is a fibrant simplicial category, and that if $f : \mathcal{C} \rightarrow \mathcal{D}$ is a DK-fibration between fibrant simplicial categories, then $N^\square(f)$ is an inner fibration between cubical quasicategories. By Theorem 1.2.7, we are left to show that $N^\square(f)$ has the right lifting property with respect to the endpoint inclusions $j_0 : \{0\} \rightarrow K$ and $j_1 : \{1\} \rightarrow K$. These cases being similar, we only treat the first one. Consider a commutative square

$$\begin{array}{ccc} \{0\} & \xrightarrow{\bar{a}} & N^\square \mathcal{C} \\ \downarrow j_0 & & \downarrow N^\square f \\ K & \longrightarrow & N^\square \mathcal{D} \end{array}$$

We shall first lift the middle vertical edge of K . By Lemma 4.2.2, its image in $N^\square(\mathcal{D})$ corresponds to some arrow $v \in \mathcal{D}_0(a, b)$, which is an isomorphism in $\pi_0(\mathcal{D})$. The same will have to be true for its image in $N^\square(\mathcal{C})$ through the lifting. The object \bar{a} of \mathcal{C} satisfies $f(\bar{a}) = a$. We proceed as follows.

- Since $\pi_0(f)$ is an isofibration of categories, there exists some $\bar{b} \in \text{Ob}(\mathcal{C})$ and some $v' \in \mathcal{C}_0(\bar{a}, \bar{b})$ such that $f(\bar{b}) = b$, $\pi_0(f(v')) = \pi_0(v)$, and $\pi_0(v')$ is an isomorphism in $\pi_0(\mathcal{C})$.
- Since $\mathcal{D}(a, b)$ is a Kan complex, we can find a 1–simplex $\delta \in \mathcal{D}_1(a, b)$ such that $\partial_1 \delta = v$ and $\partial_0 \delta = f(v')$.
- Since $f_{\bar{a}, \bar{b}} : \mathcal{C}(\bar{a}, \bar{b}) \rightarrow \mathcal{D}(a, b)$ is a Kan fibration, we can lift $\delta \in \mathcal{D}_1(a, b)$ to some $\bar{\delta} \in \mathcal{C}_1(\bar{a}, \bar{b})$ satisfying $\partial_0 \bar{\delta} = v'$.

Then $\bar{v} = \partial_1 \bar{\delta} \in \mathcal{C}_0(\bar{a}, \bar{b})$ meets our goal, ie satisfies $f(\bar{v}) = v$, and $\pi_0(\bar{v}) = \pi_0(v')$ is an isomorphism in $\pi_0(\mathcal{C})$ such that \bar{v} , seen as an edge in $N^\square(\mathcal{A})$, is an equivalence, by Lemma 4.2.2. Therefore, the two open boxes in

$$\begin{array}{ccc} 1 & & 0 \equiv 0 \\ \parallel & \searrow \bar{v} & \parallel \\ 1 \equiv 1 & & 0 \end{array}$$

are special. Calling this diagram \bar{v} , our lifting problem reduces now to

$$\begin{array}{ccc} K' & \xrightarrow{\bar{v}} & N^\square \mathcal{C} \\ \downarrow & & \downarrow N^\square f \\ K & \longrightarrow & N^\square \mathcal{D} \end{array}$$

where K' is K without its 2-cubes and without its horizontal nondegenerate 1-cubes. This is performed by applying Lemma 4.2.3 to each of the two special boxes above. □

Proposition 4.2.5 *The adjunction $\mathcal{C}^\square \dashv N^\square$ is Quillen.*

Proof This follows from Propositions 4.1.1 and 4.2.4, thanks to Joyal’s characterisation recalled in Proposition 4.2.1. □

4.3 Quillen equivalence

In order to prove that the Quillen adjunction $\mathcal{C}^\square \dashv N^\square$ is a Quillen equivalence, we first compare it with the simplicial rigidification \mathcal{C}^Δ using the functor Q of Section 1.2 and then use the Quillen equivalences induced by Q and \mathcal{C}^Δ .

Lemma 4.3.1 *There exists a morphism $\phi_n: \mathcal{C}^\square(Q^n) \rightarrow \mathcal{C}^\Delta(\Delta^n)$ in $sCat$, which is a bijection on objects and is natural in $[n] \in \Delta$, ie a natural transformation $\phi: \mathcal{C}^\square \circ Q \Rightarrow \mathcal{C}^\Delta \circ Y$.*

Proof We start by defining a family of morphisms $\psi_n: \mathcal{C}^\square(\square^n) \rightarrow \mathcal{C}^\Delta(\Delta^n)$ in $sCat$. On objects, we set $\psi_n(a) = \sup a$ (cf Lemma 1.2.11). If $a \preceq b$ ($a \subseteq b$), then $\sup a \leq \sup b$, and we define a map $\psi_n(a, b): \mathcal{C}^\square(\square^n)(a, b) \rightarrow \mathcal{C}^\Delta(\Delta^n)(\sup a, \sup b)$ in $sSet$, as follows. Since $\mathcal{C}^\square(\square^n)(a, b)$ is the nerve of the poset $\Sigma_{b \setminus a}$ with the weak order \sim_B (see Section 2.3), and $\mathcal{C}^\Delta(\Delta^n)(\sup a, \sup b)$ is the nerve of the poset $\mathcal{P}(\lceil \sup a, \sup b \rceil)$ with the inclusion order, we define this map at the level of the underlying posets. Let $k = d(a, b)$ and $(x_1, \dots, x_k) \in \Sigma_{b \setminus a}$. We set

$$\tilde{\psi}_n(a, b)(x_1, \dots, x_k) = \{x_l \mid x_p < x_l \text{ for all } p < l\} \cap \lceil \sup a, \sup b \rceil.$$

The map $\tilde{\psi}_n(a, b)$ is a morphism of posets. Assume $x_r > x_{r+1}$ for some r . Then

$$x := (x_1, \dots, x_k) \sim_B (x_1, \dots, x_{r+1}, x_r, \dots, x_k) =: y.$$

We write $A(x) = \{x_l \mid x_p < x_l \text{ for all } p < l\}$. Then we observe that $A(x) \setminus \{x_r, x_{r+1}\} = A(y) \setminus \{x_r, x_{r+1}\}$, that $x_{r+1} \notin A(x)$, and that $(x_r \in A(x)) \Rightarrow (x_r \in A(y))$. It follows that $A(x) \subseteq A(y)$, and hence $\tilde{\psi}_n(a, b)(x) \subseteq \tilde{\psi}_n(a, b)(y)$.

We next show that $\tilde{\psi}_n$ preserves the concatenation product. Assume $a \preceq b \preceq c$. Let

$$x := (x_1, \dots, x_k) \in \Sigma_{b \setminus a}, \quad y := (y_1, \dots, y_{k'}) \in \Sigma_{c \setminus b}.$$

We set $z := (x_1, \dots, x_k, y_1, \dots, y_{k'})$. We have to prove that if $\sup b \notin \{\sup a, \sup c\}$ (equivalently $\sup a < \sup b < \sup c$), then

$$\tilde{\psi}_n(a, c)(z) = \tilde{\psi}_n(a, b)(x) \cup \{\sup b\} \cup \tilde{\psi}_n(b, c)(y).$$

We observe that $A(z)$ splits as $A(x) \cup B$, where $B \subseteq A(y)$ and $A(y) \cap]\sup b, \sup c[\subseteq B$. This settles the left-to-right inclusion, as well as the inclusions $\tilde{\psi}_n(a, b)(x) \subseteq \tilde{\psi}_n(a, c)(z)$ and $\tilde{\psi}_n(b, c)(y) \subseteq \tilde{\psi}_n(a, c)(z)$. Since $\sup b > \sup a$, we have $\sup(b \setminus a) = \sup b$. Thus there exists l such that $x_l = \sup b$ and $x_p \leq \sup b$ for all $p \in \{1, \dots, k\}$, and a fortiori $\{\sup b\} \subseteq \tilde{\psi}_n(a, c)(z)$ holds.

In conclusion, setting $\psi_n = N(\tilde{\psi}_n)$, we have shown that $\psi_n : \mathfrak{C}^\square(\square^n) \rightarrow \mathfrak{C}^\Delta(\Delta^n)$ is an enriched functor of simplicial categories.

Let us prove that ψ_n factors through the map $\mathfrak{C}^\square(\pi_n) : \mathfrak{C}^\square(\square^n) \rightarrow \mathfrak{C}^\square(Q^n)$, where π_n is the quotient map of Definition 1.2.10. As \mathfrak{C}^\square is cocontinuous, from Definition 1.2.10, the following diagram is a pushout:

$$\begin{array}{ccc} \mathfrak{C}^\square(\square^0 \otimes \square^{n-1}) \sqcup \mathfrak{C}^\square(\square^1 \otimes \square^{n-2}) \sqcup \dots \sqcup \mathfrak{C}^\square(\square^{n-1} \otimes \square^0) & \xrightarrow{(\mathfrak{C}^\square(\partial_{1,1}), \mathfrak{C}^\square(\partial_{2,1}), \dots, \mathfrak{C}^\square(\partial_{n,1}))} & \mathfrak{C}^\square(\square^n) \\ \downarrow & & \downarrow \mathfrak{C}^\square(\pi_n) \\ \mathfrak{C}^\square(\square^{n-1}) \sqcup \mathfrak{C}^\square(\square^{n-2}) \sqcup \dots \sqcup \mathfrak{C}^\square(\square^0) & \longrightarrow & \mathfrak{C}^\square(Q^n) \end{array}$$

So, by universality, all we need to get our factorisation is a commutative square

$$\begin{array}{ccc} \mathfrak{C}^\square(\square^0 \otimes \square^{n-1}) \sqcup \mathfrak{C}^\square(\square^1 \otimes \square^{n-2}) \sqcup \dots \sqcup \mathfrak{C}^\square(\square^{n-1} \otimes \square^0) & \xrightarrow{(\mathfrak{C}^\square(\partial_{1,1}), \mathfrak{C}^\square(\partial_{2,1}), \dots, \mathfrak{C}^\square(\partial_{n,1}))} & \mathfrak{C}^\square(\square^n) \\ \downarrow & & \downarrow \psi_n \\ \mathfrak{C}^\square(\square^{n-1}) \sqcup \mathfrak{C}^\square(\square^{n-2}) \sqcup \dots \sqcup \mathfrak{C}^\square(\square^0) & \dashrightarrow & \mathfrak{C}^\Delta(\Delta^n) \end{array}$$

ie for all $1 \leq i \leq n$, we want a lift in the diagram

$$\begin{array}{ccc} \mathfrak{C}^\square(\square^{i-1} \otimes \square^{n-i}) & \xrightarrow{\mathfrak{C}^\square(\partial_{i,1})} & \mathfrak{C}^\square(\square^n) \xrightarrow{\psi_n} \mathfrak{C}^\Delta(\Delta^n) \\ \downarrow & \nearrow \gamma_{i,n} & \\ \mathfrak{C}^\square(\square^{n-i}) & & \end{array}$$

We first define a map $\tilde{\gamma}_{i,n}$ between the underlying poset-enriched categories, and then we will set $\gamma_{i,n} = N(\tilde{\gamma}_{i,n})$. Let $a \otimes a'$ be a vertex of $\square^{i-1} \otimes \square^{n-i}$. We note that $(\psi_n \circ \partial_{i,1})(a \otimes a') = \sup(a \cup \{i\} \cup a')$ is

independent of a and we set $\tilde{\gamma}_{i,n}(a') = \sup(\{i\} \cup a')$. Let $a \otimes a'$ and $b \otimes b'$ be two objects of $\square^{i-1} \otimes \square^{n-i}$, with $a \preceq b$ and $a' \preceq b'$, and $(x_1, \dots, x_k; y_1, \dots, y_{k'})$ be an element of $\Sigma_{b \setminus a} \times \Sigma_{b' \setminus a'}$. We have

$$\begin{aligned}
 (\psi_n \circ \partial_{i,1})(a \otimes a'; b \otimes b')(x_1, \dots, x_k; y_1, \dots, y_{k'}) \\
 = \{y_l \mid y_p < y_l \text{ for all } p < l\} \cap]\sup(\{i\} \cup a'), \sup(\{i\} \cup b')[,
 \end{aligned}$$

so $\tilde{\gamma}_{i,n}$ is a well-defined morphism of posets, and hence $\gamma_{i,n} = N(\tilde{\gamma}_{i,n})$ provides the required lifting.

As a consequence, there is a well-defined morphism $\phi_n: \mathfrak{C}^\square(Q^n) \rightarrow \mathfrak{C}^\Delta(\Delta^n)$ in $s\mathcal{C}at$ for each $[n]$, which is a bijection on objects. It remains to show that it yields a natural transformation $\phi: \mathfrak{C}^\square \circ Q \Rightarrow \mathfrak{C}^\Delta \circ Y$. Namely, given $u: [n] \rightarrow [m]$ in Δ and denoting the induced map by $u_*: Q^n \rightarrow Q^m$, we have to prove the commutativity of the diagram

$$\begin{array}{ccc}
 \mathfrak{C}^\square(Q^n) & \xrightarrow{\mathfrak{C}^\square(u_*)} & \mathfrak{C}^\square(Q^m) \\
 \phi_n \downarrow & & \downarrow \phi_m \\
 \mathfrak{C}^\Delta(\Delta^n) & \xrightarrow{\mathfrak{C}^\Delta(u)} & \mathfrak{C}^\Delta(\Delta^m)
 \end{array}$$

Lemma 1.2.11 implies that it is commutative at the level of objects. It is enough to check it for u a face d_j or a degeneracy s_j (left to the reader). □

Remark 4.3.2 Note that $\mathfrak{C}^\Delta \circ Y \not\cong \mathfrak{C}^\square \circ Q$.

Proposition 4.3.3 *The natural transformation of Lemma 4.3.1 induces a natural DK-equivalence*

$$\phi: \mathfrak{C}^\square \circ Q \Rightarrow \mathfrak{C}^\Delta.$$

Proof We follow closely the proof of [Doherty et al. 2024, Proposition 6.21]. Every simplicial set S is a colimit of representables, and the functors $\mathfrak{C}^\square \circ Q$ and \mathfrak{C}^Δ are left adjoint, hence preserve colimits. It follows that we can upgrade the natural transformation of Lemma 4.3.1 as $\phi: \mathfrak{C}^\square \circ Q \Rightarrow \mathfrak{C}^\Delta$, which is componentwise a bijection on objects by Lemma 3.1.1. We prove first that if S is k -skeletal for some k , then ϕ_S is a DK-equivalence. If $k = 0$ or $k = 1$, this is an isomorphism of simplicial categories. Assume it is true for every $k < n$. The functors $\mathfrak{C}^\square \circ Q$ and \mathfrak{C}^Δ preserve cofibrations, as well as Joyal weak equivalences (since every object of $sSet$ is cofibrant). Given any $1 \leq i \leq n - 1$, the cofibration $\Lambda_i^n \rightarrow \Delta^n$ is a Joyal weak equivalence of simplicial sets and Λ_i^n is $(n-1)$ -skeletal, so that ϕ_{Δ^n} is a DK-equivalence in $s\mathcal{C}at$, by the two-out-of-three property. Given an $(n-1)$ -skeletal simplicial set X , let us consider a pushout diagram of the form \mathcal{X} :

$$\begin{array}{ccc}
 \partial\Delta^n & \longrightarrow & X \\
 \downarrow & & \downarrow \\
 \Delta^n & \longrightarrow & X'
 \end{array}$$

The left vertical arrow is a cofibration. Applying our functors, we get a cube diagram, where the front face is $(\mathfrak{C}^\square \circ Q)(\mathcal{X})$, and the back face is $\mathfrak{C}^\Delta(\mathcal{X})$. Both are pushout diagrams, and their left vertical arrow is a cofibration. By the induction hypothesis and by the proof above, the morphisms $\phi_{\partial\Delta^n}$, ϕ_{Δ^n} and ϕ_X

are DK-equivalences. We can thus apply [Hirschhorn 2003, Proposition 15.10.10] and conclude that $\phi_{X'}$ is a DK-equivalence. Since an n -skeletal simplicial set is obtained by transfinite composition of pushouts from its $(n-1)$ -skeleton, we obtain that, for any n -skeletal simplicial set X , ϕ_X is a Dwyer–Kan equivalence. Finally, we observe that any simplicial set S is a sequential colimit of cofibrations (the family of inclusions of the n -skeleton into the $(n+1)$ -skeleton), preserved by the two functors and thus entailing that ϕ_S is a DK-equivalence, by [Hirschhorn 2003, Proposition 15.10.12]. \square

Corollary 4.3.4 *The adjunction $\mathcal{C}^\square \dashv N^\square$ is a Quillen equivalence.*

Proof The proposition above implies that the total left derived functor $\mathbb{L}(\mathcal{C}^\square \circ Q)$ is isomorphic to $\mathbb{L}(\mathcal{C}^\Delta)$. But \mathcal{C}^Δ is a Quillen equivalence (Theorem 1.1.3); hence $\mathcal{C}^\square \circ Q$ is also a Quillen equivalence. We conclude by Theorem 1.2.12 and the two-out-of-three property for Quillen equivalences. \square

Appendix A Tools in category theory

In this section, we collect some categorical and enriched categorical tools that are needed in the paper.

A.1 Wedge sum and concatenation

Let \mathcal{C} be a category with a distinguished object $*$. Let X be an object of \mathcal{C} . A point x in X is a map $x: * \rightarrow X$. We say also that X is pointed by x . If x is a point in X and y is a point in Y , we define the wedge sum $X \vee Y$ as the pushout (if it exists) of the diagram

$$X \xleftarrow{x} * \xrightarrow{y} Y.$$

Example A.1.1 Taking the category of posets as \mathcal{C} , the terminal singleton poset as distinguished object, P a poset with a maximal element ω , Q a poset with a minimal element α , and pointing P and Q by ω and α respectively, the wedge sum $P \vee Q$ is the poset obtained by “placing Q to right of P ”: $P \vee Q$ as a set is the pushout in the category of sets, and every element of P is less than every element of Q . Note that $P \rightarrow P \vee Q$ is an embedding of posets.

We will consider a similar construction in $\mathcal{C}_{*,*} = * \sqcup * \downarrow \mathcal{C}$, the category of double pointed objects in \mathcal{C} . We will denote an object $(a, b): * \sqcup * \rightarrow X$ in this category by $X_{a,b}$.

Definition A.1.2 Let $X_{a,b}$ and $Y_{u,v}$ be two objects of $\mathcal{C}_{*,*}$.

- The wedge sum $X \vee Y$ of the pointed sets $b: * \rightarrow X$ and $u: * \rightarrow Y$ is naturally double pointed by $(a, v): * \sqcup * \rightarrow X \vee Y$.
- For $f: X_{a,b} \rightarrow X'_{a',b'}$ and $g: Y_{u,v} \rightarrow Y'_{u',v'}$, we denote by $f \vee g: (X \vee Y)_{a,v} \rightarrow (X' \vee Y')_{a',v'}$ the double pointed map induced by the universal property of the pushout and the natural maps $X' \rightarrow X' \vee Y'$ and $Y' \rightarrow X' \vee Y'$. It endows $\mathcal{C}_{*,*}$ with a monoidal structure, with unit $*$ (doubled pointed by itself). We call this product the *concatenation product*.

Let X be an object in \mathcal{C} and let u, v and w be points in X . For any maps $f: S_{a,b} \rightarrow X_{u,v}$ and $g: T_{a',b'} \rightarrow X_{v,w}$ in $\mathcal{C}_{*,*}$, we write $f * g: (S \vee T)_{a,b'} \rightarrow X_{u,w}$ for the corresponding structure map out of the pushout.

A.2 P -shaped categories

We introduce the notion of P -shaped category, for P a poset.

Definition A.2.1 A *bounded poset* is a poset P having a least and greatest element denoted respectively by α_P and ω_P .

Remark A.2.2 Given two bounded posets P and Q , the poset $P \vee Q$ (see Example A.1.1), is also bounded, by α_P and ω_Q .

In this section we fix $(\mathcal{V}, \otimes, I)$ a symmetric monoidal category, with initial object denoted by \emptyset . We assume that $\emptyset \otimes X \cong \emptyset$ for all objects X in \mathcal{V} .

Definition A.2.3 Let P be a poset. A \mathcal{V} -enriched category \mathcal{C} is *P -shaped* if

- the set of objects of \mathcal{C} is in bijection with P ,
- $\mathcal{C}(p, p) = I$ for all $p \in P$, and
- $\mathcal{C}(p, q) \neq \emptyset$ implies $p \leq q$ for all $p, q \in P$.

It is double pointed by $\alpha_P, \omega_P: * \rightarrow \mathcal{C}$, where $*$ denotes the \mathcal{V} -enriched category with one object and homset I .

Note that the condition imposed above is implicitly used in this definition. Suppose that $p < q \leq r$. The \mathcal{V} -enriched structure implies that there is a composition morphism $\mathcal{C}(r, p) \otimes \mathcal{C}(q, r) \rightarrow \mathcal{C}(q, p)$, which by the P -shape axioms is a morphism $\emptyset \otimes \mathcal{C}(q, r) \rightarrow \emptyset$, which we take to be the identity up to the identification $\emptyset \otimes \mathcal{C}(q, r) \cong \emptyset$.

Example A.2.4 Any poset P gives rise to a \mathcal{V} -enriched category \hat{P} : the objects are the elements of P , and for every p and q in P we have $\hat{P}(p, q) = I$ if $p \leq q$, and $\hat{P}(p, q) = \emptyset$ otherwise. Hence \hat{P} is P -shaped.

The next proposition computes the wedge sum $\mathcal{C} \vee \mathcal{D}$ of a P -shaped \mathcal{V} -category \mathcal{C} and a Q -shaped \mathcal{V} -category \mathcal{D} along ω_P and α_Q .

Proposition A.2.5 Let P and Q be two bounded posets. Let \mathcal{C} be a P -shaped \mathcal{V} -category and \mathcal{D} be a Q -shaped \mathcal{V} -category. The wedge sum $\mathcal{E} = \mathcal{C} \vee \mathcal{D}$ along ω_P and α_Q , in the category of \mathcal{V} -categories, exists and is $(P \vee Q)$ -shaped. It is described as follows:

- the set of objects of \mathcal{E} is in bijection with $P \vee Q$, so we can identify objects of \mathcal{E} with elements of $P \vee Q$,

- $$\mathcal{E}(x, y) = \begin{cases} \mathcal{C}(x, y) & \text{if } x, y \in P, \\ \mathcal{D}(x, y) & \text{if } x, y \in Q, \\ \mathcal{D}(\alpha_Q, y) \otimes \mathcal{C}(x, \omega_P) & \text{if } x \in P \text{ and } y \in Q, \\ \emptyset & \text{otherwise,} \end{cases}$$

and the composition is the obvious one.

Proof Note that $\mathcal{C}(\omega_P, \omega_P) = I = \mathcal{D}(\alpha_Q, \alpha_Q)$ implies that the definition above is consistent. Note also that \mathcal{E} is $(P \vee Q)$ -shaped. We prove that \mathcal{E} satisfies the required universal property in the category of \mathcal{V} -categories, taking $*$ to be the \mathcal{V} -category with one object, with homset I . Denote by $\iota_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{E}$ the natural morphism and similarly for $\iota_{\mathcal{D}}$. Given a \mathcal{V} -category \mathcal{F} and two \mathcal{V} -functors $F: \mathcal{C} \rightarrow \mathcal{F}$ and $G: \mathcal{D} \rightarrow \mathcal{F}$ satisfying $F(\omega_P) = G(\alpha_Q)$, we prove that there exists a unique \mathcal{V} -functor $H: \mathcal{E} \rightarrow \mathcal{F}$ such that $H\iota_{\mathcal{C}} = F$ and $H\iota_{\mathcal{D}} = G$. The functor H is clearly uniquely defined on objects, and on most of the morphisms. For $x \in P \setminus \{\omega_P\}$ and $y \in Q \setminus \{\alpha_Q\}$, we (have to) define H as the composite

$$\mathcal{D}(\alpha_Q, y) \otimes \mathcal{C}(x, \omega_P) \xrightarrow{G \otimes F} F(G(\alpha_Q), G(y)) \otimes F(F(x), F(\omega_P)) \xrightarrow{\circ} \mathcal{F}(F(x), G(y)),$$

and we check easily that this defines an enriched functor. □

Corollary A.2.6 *Let P and Q be two bounded posets. If \mathcal{C} is a P -shaped poset-category and \mathcal{D} is a Q -shaped poset-category, then $N(\mathcal{C}) \vee N(\mathcal{D})$ is isomorphic to $N(\mathcal{C} \vee \mathcal{D})$ as simplicial categories.*

Proof It follows directly from the explicit description of $\mathcal{C} \vee \mathcal{D}$ in Proposition A.2.5 and from the fact that $N(A \times B) \cong N(A) \times N(B)$ for any posets A and B . □

Proposition A.2.7 *Let \mathcal{C} be a P -shaped simplicial category, with P a bounded poset. Let*

$$\varphi: \mathcal{C}(\alpha_P, \omega_P) \rightarrow Y$$

be a morphism of simplicial sets. Denote by \mathcal{D} the colimit of the pushout diagram

$$\Sigma Y \xleftarrow{\Sigma \varphi} \Sigma \mathcal{C}(\alpha_P, \omega_P) \rightarrow \mathcal{C},$$

where the right arrow is the counit of the adjunction of Proposition 1.1.1. The simplicial category \mathcal{D} is P -shaped and has simplicial sets of morphisms $\mathcal{D}(a, b)$ if $(a, b) \neq (\alpha_P, \omega_P)$, and Y if $(a, b) = (\alpha_P, \omega_P)$.

For $a \leq b \leq c$, the composition $\mathcal{D}(b, c) \otimes \mathcal{D}(a, b) \rightarrow \mathcal{D}(a, c)$ is that of \mathcal{C} if $(a, c) \neq (\alpha_P, \omega_P)$ and is the composition in \mathcal{C} followed by φ if $(a, c) = (\alpha_P, \omega_P)$.

Proof Note that since the functor $\text{Ob}: s\text{Cat} \rightarrow \text{Set}$ is cocontinuous, the set of objects of the colimit of the diagram is in bijection with P . Moreover the description of the hom sets of \mathcal{D} shows that the category \mathcal{D} is P -shaped. We check that the proposed simplicial category \mathcal{D} satisfies the universal property

of pushouts. Let \mathcal{E} be a simplicial category with $F: \mathcal{C} \rightarrow \mathcal{E}$ and $g: Y \rightarrow \mathcal{E}(F(\alpha_P), F(\omega_P))$ such that $g \circ \varphi = F_{\alpha_P, \omega_P}$. There is a unique way to build $H: \mathcal{D} \rightarrow \mathcal{E}$ making the diagrams commute. It coincides with F on objects as well as on morphisms $\mathcal{D}(a, b) \rightarrow \mathcal{E}(F(a), F(b))$ for $(a, b) \neq (\alpha_P, \omega_P)$. In addition, we have $H_{\alpha_P, \omega_P} = g$. It is easy to check that H is a morphism of simplicial categories, using the equation relating g and F . \square

Remark A.2.8 Proposition A.2.7 holds more widely for all P -shaped \mathcal{V} -categories. All it uses is the adjunction $\Sigma \vdash \text{Hom}$ of Proposition 1.1.1. But the latter (as a mere adjunction) holds in the general setting of bipointed P -shaped \mathcal{V} -categories. Moreover, when restricted to $[1]$ -shaped \mathcal{V} -categories, the adjunction is an equivalence.

Appendix B Combinatorics

B.1 The category $\text{SubNeck}(T)$, with T a necklace

Proposition B.1.1 *The category $\text{SubNeck}(\square^n)$ is a poset. It is in bijection with the poset of ordered partitions of $\{1, \dots, n\}$, where the order $A \leq_r B$ is the refinement inverse order, defined as the reflexive and transitive closure of*

$$(A_1; \dots; A_k) \leq_r (A_1; \dots; A_i \cup A_{i+1}; \dots; A_k).$$

In particular, it has a greatest element, the partition with 1 block $(\{1, \dots, n\})$, and the set of minimal elements is naturally in bijection with the symmetric group Σ_n . This poset admits all least upper bounds.

Proof (See also [Ziemiański 2017, Section 7; 2020, Section 7], where elements of $\text{SubNeck}(\square^n)$ are identified with cube chains.) As seen in Lemma 2.1.5, an object $\varphi: \square^{n_1} \vee \dots \vee \square^{n_k} \hookrightarrow \square^n$ is determined

- by a sequence $s_\varphi = (\alpha = a_0 < a_1 < \dots < a_k = \omega)$, with $a_i \in (\square^n)_0$, $d(a_{i-1}, a_i) = n_i$ and $n_1 + \dots + n_k = n$, or, equivalently,
- by an ordered partition of $\{1, \dots, n\}$: setting $A_i = a_i \setminus a_{i-1}$, we get $A_\varphi := (A_1; \dots; A_k)$.

We denote the set $\{a_0, \dots, a_k\}$ arising from a sequence s as above by \bar{s} . The following easy verifications are left to the reader:

- Given φ and ψ in $\text{SubNeck}(\square^n)$, there is a morphism from φ to ψ if and only if $\bar{s}_\psi \subseteq \bar{s}_\varphi$, and this morphism is unique.
- We have $\bar{s}_\psi \subseteq \bar{s}_\varphi$ if and only if $A_\varphi \leq_r A_\psi$.

In particular, if A is a set of sequences $\{s_1, \dots, s_l\}$ then its least upper bound is the sequence s associated to $\bigcap_{i=1}^l \bar{s}_i$. \square

The following corollary is a direct consequence of the previous proposition and of Proposition 2.1.2.

Corollary B.1.2 Let $T = \square^{n_1} \vee \dots \vee \square^{n_k}$ be a necklace. The category $\text{SubNeck}(T)$ is a poset, whose poset relation is denoted by \leq_r . It is the product of the categories $\text{SubNeck}(\square^{n_i})$. In particular, it has a greatest element and admits all least upper bounds.

Definition B.1.3 Let (P, \leq) be a poset. A subset A of P is *downward closed* if for all $y \leq x$ in P , $x \in A$ implies $y \in A$. It is *upward closed* if it is downward closed in the opposite poset of P .

Lemma B.1.4 Let (P, \leq) be a poset and A and B upward closed subsets of P . Then

$$N(A \cup B) \cong \text{colim}(N(A) \leftrightarrow N(A \cap B) \hookrightarrow N(B)).$$

Proof We claim that

$$\begin{array}{ccc} N(A \cap B) & \longrightarrow & N(A) \\ \downarrow & & \downarrow \\ N(B) & \longrightarrow & N(A \cup B) \end{array}$$

is a pushout diagram in the category $sSet$. Since colimits in $sSet$ are computed dimensionwise, it is enough to prove this claim for n -simplices. An element in $N_n(A \cup B)$ is a sequence $x = (x_0 \leq \dots \leq x_n)$ with $x_i \in A \cup B$ for all i . Since A and B are upward closed subsets of P , if $x_0 \in A$, then $x_i \in A$ for all i , and similarly if $x_0 \in B$. Let X be a simplicial set and $f_A: N(A) \rightarrow X$ and $f_B: N(B) \rightarrow X$ be two morphisms of simplicial sets that coincide on $N(A \cap B)$. We define $g: N_n(A \cup B) \rightarrow X_n$ as $g(x) = f_A(x)$ if $x_0 \in A$ and $g(x) = f_B(x)$ if $x_0 \in B$. It yields a well-defined map of simplicial sets, since if $x_0 \in A \cap B$ then $x_i \in A \cap B$ for all i , and $f_A = f_B$ on $N(A \cap B)$. \square

Lemma B.1.5 Let T be a necklace. If $\mathcal{A} \subseteq \text{SubNeck}(T)$ is downward closed (for the order \leq_r), then the canonical morphism $\text{colim}_{U \in \mathcal{A}} \mathfrak{C}_t^\square(U) \rightarrow \mathfrak{C}_t^\square(T)$ is a monomorphism of simplicial sets.

Proof We only need to examine the situation of two n -simplices u and v coming from different subnecklaces U and V in \mathcal{A} and whose images in $\mathfrak{C}_t^\square(T)$ are identified:

$$\begin{array}{ccc} \Delta^n & \xrightarrow{u} & \mathfrak{C}_t^\square(U) \\ \downarrow v & & \downarrow \\ \mathfrak{C}_t^\square(V) & \longrightarrow & \mathfrak{C}_t^\square(T) \end{array}$$

The composite map $\varphi: \Delta^n \rightarrow \mathfrak{C}_t^\square(T)$ gives a set A_φ of $n + 1$ paths of T (by Theorem 3.1.4) with values both in U and V . Let W be the upper bound of A_φ in $\text{SubNeck}(T)$, provided by Corollary B.1.2. Since U and V are upper bounds of A_φ , we have $W \leq_r U$ and $W \leq_r V$ in $\text{SubNeck}(T)$. Hence there is a factorisation in the diagram

$$\begin{array}{ccc} \Delta^n & \xrightarrow{u} & \mathfrak{C}_t^\square(U) \\ \downarrow v & \dashrightarrow w & \downarrow \\ \mathfrak{C}_t^\square(V) & \xrightarrow{\quad} & \mathfrak{C}_t^\square(T) \end{array}$$

$\mathfrak{C}_t^\square(W)$

Moreover, $W \in \mathcal{A}$ since \mathcal{A} is downward closed. Thus the diagram says that u, v and w are identified in the colimit, which completes the proof. \square

B.2 On the homotopy type of $\text{SubNeck}(\Pi_{i,\epsilon}^n)$

To simplify the notation in this section, for $n \geq 1$, we will denote by P_n the poset of ordered partitions of $\{1, \dots, n + 1\}$, that is, $P_n = \text{SubNeck}(\square^{n+1})$. Similarly, we set $\partial P_n = \text{SubNeck}(\partial \square^{n+1})$ and $\sqcap P_n = \text{SubNeck}(\Pi_{n+1,0}^{n+1})$ (see Definition 3.3.1). Note that

$$\partial P_n = P_n \setminus (\{1, \dots, n + 1\}) \quad \text{and} \quad \sqcap P_n = \partial P_n \setminus (\{1, \dots, n\}; \{n + 1\}).$$

The nerve of P_n is contractible, since P_n has a greatest element.

The next proposition is what we need for Proposition 4.1.3.

Proposition B.2.1 *For every $n \geq 2$, the nerve of $\text{SubNeck}(\Pi_{i,\epsilon}^n)$ is contractible.*

Proof We first prove that for $n \geq 1$, the nerve of $\sqcap P_n$ is contractible. For $n = 1$, the poset $\sqcap P_1$ is a singleton, namely the ordered partition $(\{2\}; \{1\})$, hence is contractible. Assume $n \geq 2$. Let us fix some notation:

- For an ordered partition $x = (A_1; \dots; A_k) \in P_n$, and $0 \leq l \leq k$, we set

$$m_l(x) := \#(A_1 \cup \dots \cup A_l),$$

with the convention that $m_0(x) = 0$. Note that $m_k(x) = n + 1$.

- For an ordered partition $x = (A_1; \dots; A_k) \in P_n$ with $n + 1 \in A_l$, we set

$$\alpha(x) := m_{l-1}(x) \quad \text{and} \quad \beta(x) := m_l(x).$$

- We have $\partial P_n = T_0 \cup T_1 \cup \dots \cup T_n$ with

$$T_i := \{x \in \partial P_n \mid \alpha(x) \leq i < \beta(x)\}.$$

- For every $0 \leq i \leq n - 1$ we set

$$T_{i,i+1} := T_i \cap T_{i+1} = (T_0 \cup \dots \cup T_i) \cap T_{i+1}.$$

We note that $\sqcap P_n = T_0 \cup T_1 \cup \dots \cup T_{n-1} \cup T'_n$ with $T'_n = T_n \cap (\sqcap P_n)$. We also note (by a simple case analysis) that T_0, \dots, T_{n-1} and T'_n are upward closed, allowing us to use Lemma B.1.4 repeatedly and get

$$N(\sqcap P_n) \cong \text{colim} \left(\begin{array}{ccccccc} N(T_0) & & N(T_1) & \cdots & \cdots & \cdots & N(T_{n-1}) & & N(T'_n) \\ & \swarrow & \nearrow & \swarrow & & \searrow & \nearrow & \swarrow & \nearrow \\ & & N(T_{0,1}) & & \cdots & & N(T_{n-1,n}) & & \end{array} \right).$$

We claim that the colimit of this diagram is Kan–Quillen equivalent to its homotopy colimit. Indeed, let D be the underlying category of the diagram, which has objects x_i for $0 \leq i \leq n$ and $y_{i,i+1}$ for $0 \leq i < n$

and morphisms from $y_{i,i+1}$ to x_i and x_{i+1} . We endow D with the following Reedy structure: x_i has degree $2i$ and $y_{i,i+1}$ has degree $2i + 1$. Then D has fibrant constants. This follows from [Hirschhorn 2003, Proposition 15.10.2], noting that, for every object α of D , the matching category at D is either empty or the one point category. Moreover, the diagram above is Reedy cofibrant; hence the claim above follows from [Hirschhorn 2003, Proposition 19.9.1]. In consequence, we focus our attention on the homotopy type of $N(T_i)$ for $0 \leq i < n$, $N(T'_n)$ and $N(T_{i,i+1})$ for $0 \leq i < n$.

Let $\pi: P_n \rightarrow P_{n-1}$ be the poset morphism removing $(n + 1)$ from the ordered partition. Note that $\pi(T_i) \subset \partial P_n$ for every $1 \leq i \leq n - 1$ since

$$\pi^{-1}(\{1, \dots, n\}) = \{(\{1, \dots, n + 1\}), (\{n + 1\}; \{1, \dots, n\}), (\{1, \dots, n\}; \{n + 1\})\},$$

and since none of the elements in this set lie in T_i for $1 \leq i \leq n - 1$. Note that, given a partition $x = (A_1; \dots; A_k)$ of P_{n-1} and $0 \leq i \leq n - 1$, there exists a unique $l \in \{0, \dots, k - 1\}$ such that $m_l(x) \leq i < m_{l+1}(x)$. We leave it to the reader to check the following facts:

- For $0 \leq i < n$, the induced map $T_{i,i+1} \xrightarrow{\pi} \partial P_{n-1}$ is an isomorphism of posets with inverse

$$x = (A_1; \dots; A_k) \in \partial P_{n-1} \mapsto (A_1; \dots; A_{l+1} \cup \{n + 1\}; \dots; A_k),$$

where $m_l(x) \leq i < m_{l+1}(x)$.

- For $0 < i < n$, the map $T_i \xrightarrow{\pi} \partial P_{n-1}$ is an adjunction of posets with a section σ given for $x = (A_1; \dots; A_k) \in \partial P_{n-1}$ by

$$\sigma(x) = \begin{cases} (A_1; \dots, A_l; \{n + 1\}; A_{l+1}; \dots; A_k) & \text{if } m_l(x) = i, \\ (A_1; \dots; A_{l+1} \cup \{n + 1\}; \dots; A_k) & \text{if } m_l(x) < i < m_{l+1}(x). \end{cases}$$

(Indeed, $\pi \circ \sigma = \text{id}$ and $\sigma \circ \pi \leq \text{id}$.)

- Similarly, the maps $T_0 \xrightarrow{\pi} P_{n-1}$ and $T'_n \xrightarrow{\pi} \partial P_{n-1}$ are adjunctions of posets with the respective sections

$$(A_1; \dots; A_k) \in P_{n-1} \mapsto (\{n + 1\}; A_1; \dots; A_k),$$

$$(A_1; \dots; A_k) \in \partial P_{n-1} \mapsto (A_1; \dots; A_k; \{n + 1\}).$$

Putting everything together, and using the fact that an adjunction of posets gives rise to a homotopy equivalence, and hence to a Kan–Quillen equivalence between their nerves, we have

$$\begin{aligned} N(\square P_n) &\sim \text{hocolim} \left(\begin{array}{ccccccc} N(P_{n-1}) & & N(\partial P_{n-1}) & \cdots & N(\partial P_{n-1}) & & N(\partial P_{n-1}) \\ & \swarrow & \parallel & & \parallel & \swarrow & \parallel \\ & & N(\partial P_{n-1}) & \cdots & & N(\partial P_{n-1}) & \end{array} \right) \\ &\sim \text{hocolim} \left(\begin{array}{c} N(P_{n-1}) \\ \uparrow \\ N(\partial P_{n-1}) \end{array} \right) \sim \text{colim} \left(\begin{array}{c} N(P_{n-1}) \\ \uparrow \\ N(\partial P_{n-1}) \end{array} \right) \sim N(P_{n-1}). \end{aligned}$$

Hence the nerve of $\square P_n$ is contractible.

Finally, we have

$$\text{SubNeck}(\sqcap_{i,0}^{n+1}) = \partial P_n \setminus (\{1, \dots, n+1\} \setminus \{i\}; \{i\}),$$

$$\text{SubNeck}(\sqcap_{i,1}^{n+1}) = \partial P_n \setminus (\{i\}; \{1, \dots, n+1\} \setminus \{i\}),$$

and the proof of the contractibility of these posets is similar to that of $\sqcap P_n$. \square

Remark B.2.2 One can find in [Baues 1980] a geometric interpretation linking cellular strings on the cubes and the permutohedra, the original idea being attributed to Milgram. Related results relative to P_n and ∂P_n can be found in, say, [Ziemiański 2017] or in [Ziegler 1995] (where they are put to use to establish connections between cubical sets and higher dimensional automata, through directed path spaces): the geometric realisations of the ordered partition posets are the permutohedra, which are homeomorphic to a ball. Moreover, the pair $(|N(P_n)|, |N(\partial P_n)|)$ is homeomorphic to the pair (D^n, S^{n-1}) .

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CNRS, IRIF, Université Paris Cité
Paris, France

Université Paris Cité, Institut de Mathématiques de Jussieu–Paris Rive Gauche, CNRS, Sorbonne Université, DMA,
ENS-PSL
Paris, France

Department of Mathematics, Stockholm University
Stockholm, Sweden

curien@irif.fr, muriel.livernet@imj-prg.fr, gabriel.saadia@math.su.se

Received: 26 January 2023 Revised: 12 June 2023

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Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr		


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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<https://msp.org/>

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ALGEBRAIC & GEOMETRIC TOPOLOGY

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