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Tautological characteristic classes, I

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We discuss the formalism of tautological characteristic classes of flat bundles. Applied to $\mathrm{PSL}(2, K)$, it yields the Witt class of Nekovář. Applied to $\mathrm{PGL}_+(2n, K)$, the general linear groups with positive determinant over an arbitrary ordered field, it yields (a generalization of) the Euler class.

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Introduction

Take a chain complex C_* and fix the degree k. The identity map $C_k \to C_k$ can be viewed as a cochain of degree k, with values/coefficients in C_k. Usually it is not a cocycle, but we can force it to be, dividing by boundaries, changing coefficients to C_k/B_k. Denote the resulting cocycle by T.

Now suppose that a group G acts on C_*. Clearly T is G–equivariant, in other words it is a cocycle with twisted coefficients. We can force T to be a constant coefficients cocycle simply dividing the coefficients further down to the biggest quotient of the G–module C_k/B_k on which G acts trivially, called coinvariants of G. Denote the resulting image of T by τ.

Besides producing an untwisted cocycle, this construction has an additional crucial advantage: the modules C_k, or even C_k/B_k, are usually very big, while the coinvariants $(C_k/B_k)_G$ are much smaller and sometimes manageable.

It is of interest to go halfway in this procedure: fix a (large) normal subgroup N of G, and take coinvariants $(C_k/B_k)_N$. Then T becomes a (slightly twisted by an action of G/N) cocycle taking values in (sometime still manageable, but bigger) module of N–coinvariants.

One has every reason to expect that this purely algebraic construction has nice functorial properties, and that it carries a significant amount of information about C_* as a G–module. Theorem 1.5 spells out the most natural form of functoriality.

This algebraic construction needs an input. For us such an input comes from a geometry (or, as some will undoubtedly insist, algebra), namely from the complex of geometric configurations. One takes a homogeneous space G/H (for example a projective space over an arbitrary field K) and builds a simplicial complex whose simplices are n–tuples of points "in general position". The notion of general position that we use depends on the situation, and is discussed separately in each case, but the underlying idea is uniform. In all cases the simplicial complexes that we consider have an additional crucial *star property* which is discussed in Section 2. The star property makes the simplicial complex contractible in a strong, geometric sense. The chain complex is just the complex of (alternating or ordered) simplicial chains.

In order to construct characteristic classes of flat G–bundles, we have to address the problem that the G–action on the space of configurations is not free. But this is done in a standard way, by "Borel construction", that we execute on the chain level. We end up with cocycles living in a cochain complex computing group cohomology of G (seen as a discrete group). The star property implies almost immediately that the cohomology class of the cocycle is bounded (see Theorem 3.1). The boundedness of tautological classes is taken with respect to the natural seminorm on the coefficient group. One should keep in mind that this has fairly different overtones from the usual $\mathbb{R}$–coefficients bounded cohomology of Gromov [1982]; compare for example Ghys [1987].

The first interesting case when this construction produces something valuable is that of $\mathrm{PGL}(2, K)$ acting on the projective line. This has been studied by Nekovář, who defined and studied the "Witt class" for $\mathrm{PSL}(2, K)$, with coefficients in the Witt ring of quadratic forms over K. It is an amazing fact that the four-term Witt relation $[a] + [b] = [a + b] + [ab(a + b)]$ is indeed the cocycle relation in the complex of projective point configurations. We review this in detail in Section 7.

The main results of the present paper concern the construction and study of the "Euler class for flat $\mathrm{PGL}(n, K)$–bundles" in the case where K is an arbitrary ordered field and n is even. This class is constructed using the general strategy outlined above. We take the $\mathrm{PGL}(n, K)$–action on the simplicial complex of generic configurations of points in $\mathbb{P}^{n-1}(K)$, the induced action on C_n/B_n, and then we take coinvariants with respect to the group $\mathrm{PGL}_+(n, K)$ of maps with *positive* determinant. (Note that coinvariants with respect to the full projective group are trivial, while coinvariants with respect to $\mathrm{PSL}(n, K)$ are too large for us to handle — for $\mathrm{PGL}_+(n, K)$ we have a nice answer.) The resulting tautological class eu is (an analogue of) the Euler class — for flat $\mathrm{PGL}(n, K)$–bundles. It is twisted by the homomorphism to $\mathbb{Z}/2$ whose kernel consists of maps with positive determinant. The coefficients are $\mathbb{Z}$ for n even and trivial for n odd (see Theorem 8.1).

One can run a parallel construction starting from the $\mathrm{GL}_+(n, K)$–action on the *positive* projective space $\mathbb{P}_+^{n-1}(K)$. The resulting class eu_+ has coefficients in a free abelian group of rank $\lfloor n/2 \rfloor + 1$ (see Theorem 8.1; admittedly, the computation here is somewhat heavy). Consequently, eu_+ can be split into components eu_k that are cohomology classes with $\mathbb{Z}$ coefficients.

We prove several results about the Euler classes eu and eu_+. Theorems 9.1 and 10.1 explain the relation between various components of eu_+. Theorem 11.1 gives a clean formula for the Euler class of a cross product of bundles, while Theorem 12.5 gives a cup product formula for the direct sum. In Section 13 we discuss functoriality. In particular, we relate eu and eu_+ in Theorem 13.1. We also compare the Euler and Witt classes for $\mathrm{PSL}(2, K)$–bundles in Theorem 13.4. Finally, in Theorem 13.6, using the cross-product formula, we show nontriviality of our Euler classes in every even dimension.

Further characteristic classes and more applications are postponed to subsequent papers.

Tautological classes with coefficients in C_k/B_k were defined in a forgotten paper of James Dugundji [1958], where he also proved some results of general nature, like functoriality and universality. The paper was forgotten, probably because the results did not help with actual calculations: modules C_k/B_k are usually very big and unmanageable. We discovered Dugundji's paper when we were already well into our project. Our initial inspiration came from the papers of Nekovář [1990] and Kramer and Tent [2010], where the idea of passing to G–coinvariants is present. With a grain of salt, one may say that the Witt and Maslov classes are constructed in these papers in the tautological way.

Reznikov [1997] noticed that for an ordered field K one has an "Euler class" for $\mathrm{PSL}(2, K)$ with $\mathbb{Z}$ coefficients. In fact, this class is (a multiple of) the image of the Witt class of Nekovář under the signature map from the Witt ring to $\mathbb{Z}$, given by the ordering of K.

The plan of the paper is as follows.

In Part I we discuss the general theory: Section 1 explains definitions and functoriality of tautological classes in a purely algebraic, abstract context; in Section 2 it is shown how actions on simplicial complexes can lead to examples, star-property is recalled, and a method of coefficient calculation for actions on simplicial complexes is described; Section 3 is about (automatic) boundedness of tautological classes; and Section 4 contains a simplicial counterpart of the process of representing classes of flat bundles by pullbacks of invariant forms via sections.

Part II is about $\mathrm{GL}(2)$: in Section 6 we discuss various actions of this group with a view towards investigating the corresponding tautological classes; in Section 7 the Witt group appears as the coefficient group coming from the general formalism applied to the homographic action on the projective line, and the tautological Witt class is defined.

In Part III we define Euler classes for the groups $\mathrm{PGL}(n, K)$ and $\mathrm{PGL}_+(n, K)$, where K is an arbitrary ordered field. In Section 8 actions of these groups on $\mathbb{P}^{n-1}(K)$ and on $\mathbb{P}_+^{n-1}(K)$ are used to define tautological Euler classes eu and eu_+; coefficients are calculated, and eu_+ is decomposed into a direct sum of classes eu_k (with coefficients in $\mathbb{Z}$). In Section 9 we establish a general relation between the classes eu_k, and in Section 10 we express all of them in terms of eu_0 in a weak sense using Smillie's argument. In Sections 11 and 12 we show some multiplicativity properties of eu_0. In Section 13 we further investigate relations between various Euler classes (and the Witt class); we also prove that all these classes are nontrivial (for n even).

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I Generalities

1 Algebraic tautological classes

Chain complexes Let $C_* = (C_n, \partial_n)$ be a chain complex of abelian groups. As usual, we put $Z_n = \ker \partial_n$ (cycles) and $B_n = \operatorname{im} \partial_n$ (boundaries). Let us fix an integer d and consider id_{C_d} as an element of $\mathrm{Hom}(C_d, C_d)$ — the d–cochain group of the complex $\mathrm{Hom}(C_*, C_d)$. This element is usually not a cocycle — yet, if we replace the coefficient group C_d by the quotient C_d/B_d, it becomes one.

Definition 1.1 Let C_* be a chain complex. The tautological cocycle $T^d_{C_*}$ is the d–cycle of the complex $\mathrm{Hom}(C_*, C_d/B_d)$ defined by the quotient map $C_d \to C_d/B_d$. The tautological class $\tau^d_{C_*}$ is the cohomology class of $T^d_{C_*}$ in $H^d(\mathrm{Hom}(C_*, C_d/B_d))$.

The cochain $T^d_{C_*}$ is indeed a cocycle:

$$\delta T^d_{C_*}(c) = T^d_{C_*}(\partial c) = \partial c + B_d = B_d.$$

Notice that $\tau^d_{C_*}$ is functorial, in the following way: Let $f : C_* \to K_*$ be a chain map. Then $f_d : C_d \to K_d$ induces a map $C_d/\ker \partial_d \to K_d/\ker \partial_d$, which in turn induces a map

$$f_* : H^d(\mathrm{Hom}(C_*, C_d/\ker \partial_d)) \to H^d(\mathrm{Hom}(C_*, K_d/\ker \partial_d)).$$

There is also the map $f^* : \mathrm{Hom}(K_*, K_d/\ker \partial_d) \to \mathrm{Hom}(C_*, K_d/\ker \partial_d)$ inducing

$$f^* : H^d(\mathrm{Hom}(K_*, K_d/\ker \partial_d)) \to H^d(\mathrm{Hom}(C_*, K_d/\ker \partial_d)).$$

Clearly, $f^* \tau^d_{K_*} = f_* \tau^d_{C_*}$; indeed, both these classes are represented by the same cocycle

$$C_d \ni c \mapsto f_d(c) + \ker \partial_d \in K_d/\ker \partial_d.$$

***G*–chain complexes** Now suppose that the complex C_* is a G–chain complex, ie it is acted upon by a group G, by chain maps. The group C_d/B_d has the induced G–module structure. The tautological cocycle $T^d_{C_*} : C_d \to C_d/B_d$ is a G–map.

Definition 1.2 Let C_* be a G–chain complex. The tautological class $\tau^d_{C_*,G} \in H^d(\mathrm{Hom}_G(C_*, C_d/B_d))$ (cohomology with twisted coefficients) is the cohomology class of $T^d_{C_*}$.

We have found out that the above class has also been defined and investigated in a forgotten paper of Dugundji [1958].

The G–module C_d/B_d is usually very big. To cut it down in size we will consider its coinvariants group $U_d = (C_d/B_d)_G$ — its largest G–trivial quotient. This group might be either too small to carry information or too big to extract information, yet in some cases it is nontrivial and manageable.

Definition 1.3 Let C_* be a G–chain complex. Let U_d (or $U_d(C_*)$) denote the coinvariants group $(C_d/B_d)_G$. The tautological class $\tau^d_{C_*/G} \in H^d(\mathrm{Hom}_G(C_*, U_d))$ is the cohomology class of $T^d_{C_*/G}$; the cocycle obtained by composing the tautological cocycle $T^d_{C_*}$ with the quotient map $C_d/B_d \to (C_d/B_d)_G$.

Remark (1) The functor of coinvariants is right-exact [Brown 1982, II, Section 2]. Therefore, $U_d = (C_d)_G/(B_d)_G$ (strictly speaking, we divide by the image — not necessarily injective — of $(B_d)_G$ in $(C_d)_G$). Moreover, $\partial\colon C_{d+1} \to C_d$ induces a map $\partial\colon (C_{d+1})_G \to (C_d)_G$, and U_d can also be described as $(C_d)_G/\partial((C_{d+1})_G)$.

(2) If N is a normal subgroup of G then there exists yet another, G/N–twisted tautological class τ in $H^d\big(\mathrm{Hom}_G(C_*, (C_d/B_d)_N)\big)$.

Let us discuss functoriality. Suppose that C_* is a G–complex and that K_* is an H–complex. Assume that $\phi\colon G \to H$ is a homomorphism and that $f: C_* \to K_*$ is a ϕ–equivariant chain map. The group $U_d(K_*)$ acquires a G–module structure via ϕ. We have two maps,

$$H^d\big(\mathrm{Hom}_H(K_*, U_d(K_*))\big) \xrightarrow{f^*} H^d\big(\mathrm{Hom}_G(C_*, U_d(K_*))\big) \xleftarrow{f_*} H^d\big(\mathrm{Hom}_G(C_*, U_d(C_*))\big),$$

the right one induced by the f–induced coefficient map $U_d(C_*) \to U_d(K_*)$. As before, it is straightforward to check that $f^*\tau^d_{K_*/H} = f_*\tau^d_{C_*/G}$ — both of these classes are represented by the cocycle $C_d \ni c \mapsto [f_d(c)] \in U_d(K_*)$.

Acyclic G–chain complexes Let us now assume that C_* is an acyclic G–chain complex. By this we mean that

(1) $C_n = 0$ for $n < 0$;

(2) C_* comes equipped with an augmentation map — a G–homomorphism $\epsilon\colon C_0 \to \mathbb{Z}$, where $\mathbb{Z}$ has the trivial G–module structure;

(3) the augmented complex

$$\cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \xrightarrow{\epsilon} \mathbb{Z} \to 0 \to \cdots$$

is exact.

(In other words: C_* is a resolution of the trivial G–module $\mathbb{Z}$.)

The tautological class $\tau^d_{C_*/G}$ can be used to define a cohomology class of the group G, as follows. Let P_* be a projective resolution of the trivial G–module $\mathbb{Z}$. The cohomology groups $H^*(G, U_d)$ are defined as cohomology groups of the complex $\mathrm{Hom}_G(P_*, U_d)$ [Brown 1982, III, Section 1]. There exists a chain map of resolutions $\psi_{C_*}: P_* \to C_*$ (respecting augmentations, ie extending by identity on $\mathbb{Z}$ to a chain map of the augmented complexes). Moreover, ψ_{C_*} is unique up to chain homotopy [Brown 1982, I, Lemma 7.4].

Definition 1.4 Let C_* be an acyclic G–chain complex, P_* a projective resolution of the trivial G–module $\mathbb{Z}$, and $\psi_{C_*}: P_* \to C_*$ a chain map of resolutions. Let $\psi^*_{C_*}: H^d(\mathrm{Hom}_G(C_*, U_d)) \to H^d(G, U_d)$ be the map on cohomology induced by ψ_{C_*}. We define the tautological class

$$\tau^d_{G,C_*} = \psi^*_{C_*}(\tau^d_{C_*/G}) \in H^d(G, U_d).$$

These classes are functorial just as the previous ones:

Theorem 1.5 *Let C_* be an acyclic G–chain complex, K_* an acyclic H–chain complex, $\phi\colon G \to H$ a group homomorphism, and $f : C_* \to K_*$ a ϕ–equivariant chain map. Consider two maps*

$$H^d(H, U_d(K_*)) \xrightarrow{\phi^*} H^d(G, U_d(K_*)) \xleftarrow{f_*} H^d(G, U_d(C_*));$$

the right one induced by the f–induced coefficient map $U_d(C_*) \to U_d(K_*)$. Then

$$\phi^* \tau^d_{H,K_*} = f_* \tau^d_{G,C_*}.$$

Proof Consider the diagram

$$\begin{array}{ccccc} H^d\big(\mathrm{Hom}_H(K_*, U_d(K_*))\big) & \xrightarrow{f^*} & H^d\big(\mathrm{Hom}_G(C_*, U_d(K_*))\big) & \xleftarrow{f_*} & H^d\big(\mathrm{Hom}_G(C_*, U_d(C_*))\big) \\ \downarrow & & \downarrow & & \downarrow \\ H^d(H, U_d(K_*)) & \xrightarrow{\phi^*} & H^d(G, U_d(K_*)) & \xleftarrow{f_*} & H^d(G, U_d(C_*)) \end{array}$$

There are tautological classes $\tau_{K_*/H} \in H^d\big(\mathrm{Hom}_H(K_*, U_d(K_*))\big)$, defined as the class of the tautological cochain $T_{K_*/G}(k) = [k]$, and a similar $\tau_{C_*/G} \in H^d\big(\mathrm{Hom}_G(C_*, U_d(C_*))\big)$. Their images in the group $H^d\big(\mathrm{Hom}_G(C_*, U_d(K_*))\big)$ coincide, since both are clearly equal to the class of T defined by $T(c) = [f(c)]$. The classes τ_{H,K_*} and τ_{G,C_*} are images of $\tau_{K_*/H}$ and $\tau_{C_*/G}$ (respectively) under the vertical maps. Thus, to prove the theorem, we only need to check that the above diagram is commutative.

Commutativity of the right square: The vertical maps are induced by a (unique up to chain homotopy) G–map of chain complexes $P(G)_* \to C_*$. The horizontal maps are induced by the coefficient map f_*. Since these two maps act on different arguments of the Hom functor, they commute.

Commutativity of the left square: That square is the result of applying a cohomology functor to the diagram

$$\begin{array}{ccc} K_* & \xleftarrow{f_*} & C_* \\ \uparrow & & \uparrow \\ P(H)_* & \xleftarrow{\phi_*} & P(G)_* \end{array}$$

The two compositions to compare are G–maps from $P(G)_*$ to the acyclic chain complex K_* (with the G–structure induced via ϕ). Such a map is unique up to chain-homotopy; hence the compositions are chain-homotopic. After passing to cohomology, they become equal. □

Remark 1.6 The procedure applied in Definition 1.4 to the tautological class works in greater generality, for arbitrary coefficient groups and arbitrary classes. In Theorem 4.4 we will need the following version: Let C_* be an acyclic G–chain complex, A a G–module, $T \in Z^d(\mathrm{Hom}_G(C_*, A))$ an A–valued G–invariant d–cocycle, P_* a projective resolution of the trivial G–module $\mathbb{Z}$, and $\psi_{C_*}\colon P_* \to C_*$ a chain map of resolutions. Let $\psi^*_{C_*}\colon \mathrm{Hom}_G(C_*, A) \to \mathrm{Hom}_G(P_*, A)$ be the cochain map induced by ψ_{C_*}. We define the group cohomology class $\tau \in H^d(G, A)$ associated to T by $\tau := [\psi^*_{C_*}(T)]$.

2 Geometric complexes

Our main source of acyclic G–chain complexes is geometry. Suppose that G acts on an acyclic simplicial complex X by simplicial automorphisms. Then the simplicial chain complex C_*X is an acyclic G–chain complex.

Definition 2.1 Let X be an acyclic simplicial G–complex. The definitions of Section 1 applied to the acyclic simplicial G–chain complex C_*X give rise to

- the coefficient group $U_d = U_d(X) := U_d(C_*X)$;
- the tautological cocycle $T^d_{X/G} := T^d_{C_*X/G}$;
- the tautological class $\tau^d_{X/G} := \tau^d_{C_*X/G}$;
- the tautological group cohomology class $\tau^d_{G,X} := \tau^d_{G,C_*X}$.

In our considerations, the G–complexes X will usually arise as restricted configuration complexes of homogeneous G–spaces. We will typically start from a transitive G–action on a space $\mathbb{P}$. We will use $\mathbb{P}$ as the set of vertices of X, and span simplices of X on tuples of elements of $\mathbb{P}$ satisfying some genericity conditions. (A typical example: $G = \mathrm{SL}(2,K)$, $\mathbb{P} = K^2 \setminus \{0\}$, a tuple of vectors spans a simplex if and only if every two of them are linearly independent.) This scheme applies to many algebraic groups over arbitrary infinite fields.

The acyclicity of these restricted configuration complexes is usually the consequence of the star-property defined below.

Definition 2.2 [Kramer and Tent 2010] A simplicial complex X has the star-property if for any finite subcomplex $Y \subseteq X$ there exists a vertex $v \in X^0 \setminus Y^0$ joinable with every simplex of Y (v is joinable with a k–simplex $\sigma = [y_0, \dots, y_k]$ if $v * \sigma = [v, y_0, \dots, y_k]$ is a $(k+1)$–simplex in X).

Fact 2.3 *If X has the star-property, then it is acyclic.*

Proof Let $z = \sum a_\sigma \sigma$ be a cycle in X. Let Y be the union of all simplices σ that appear in z. Let v be a vertex of X witnessing the star-property for Y. Then $z = \partial\big(\sum a_\sigma v * \sigma\big)$. □

For a complex X with the star-property there is another variant of an acyclic chain complex associated to it; the (nondegenerate) ordered chain complex $C^o_* X$. The group $C^o_k X$ is the free abelian group whose basis is the set of all $(k+1)$–tuples of vertices of X that span k–simplices (in other words, the set of ordered, nondegenerate k–simplices of X). The boundary operator is defined by the usual formula

$$\partial[v_0, \dots, v_k] = \sum_{i=0}^{k} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_k].$$

By the same argument as in Fact 2.3, the complex $C^o_* X$ is acyclic. (Warning: for finite simplicial complexes the nondegenerate ordered chain complex does not calculate homology correctly, eg the complex $C^o_*(\Delta^1)$ is not acyclic.)

If a simplicial complex is acted upon by a group G, one can use the ordered chain complex to define the coefficient group $U^o_d := (C^o_d X/B^o_d X)_G$, the tautological cocycle $T^d_{C^o_* X/G}$ and the tautological class $\tau^d_{C^o_* X/G}$. (If X has the star-property, one can further define the tautological group cohomology class $\tau^d_{G,C^o_* X}$.) There is a natural epimorphic G–chain map $C^o_* X \to C_* X$; it induces an epimorphism $U^o_d \to U_d$. The group U^o_d is usually insignificantly larger than U_d, as we shall see.

The calculations of the groups U_d and U^o_d are often used in this paper; we now explain how they are done. Let $X^{(n)}$ be the set of nondegenerate ordered n–simplices in a simplicial complex X. Let R_n be a set of representatives of orbits of G on $X^{(n)}$. For any $\sigma \in X^{(n)}$ we denote by σ_R the unique element of R_n that is G–equivalent to σ. For chains we put $\left(\sum a_\sigma \sigma\right)_R = \sum a_\sigma \sigma_R$.

Fact 2.4 *Let X be a simplicial G–complex.*

(a) *The group U^o_d is the quotient of the free abelian group with basis R_d by the subgroup spanned by $\{(\partial\rho)_R \mid \rho \in R_{d+1}\}$.*

(b) *The group U_d is the quotient of the free abelian group with basis R_d by the subgroup spanned by $\{(\partial\rho)_R \mid \rho \in R_{d+1}\} \cup \{(t\rho)_R - \operatorname{sgn}(t)\rho \mid \rho \in R_d, t \in S_{d+1}\}$. Moreover, in this description one can change the range of t from the permutation group S_{d+1} to any generating set of this group.*

The proof is based on the formula $U^o_d = (C^o_d X)_G/\partial(C^o_{d+1})_G$ and an analogous formula for U_d. We denote by c_G the image of the chain c in the coinvariants group.

Proof We start with a general remark. Suppose that a group G acts on a set Y. Let $\mathbb{Z}[Y]$ be the free abelian group with basis Y. Then $\mathbb{Z}[Y]$ has a natural G–module structure, and the coinvariants module $\mathbb{Z}[Y]_G$ is the free abelian group with basis Y/G (the orbit space of the G–action on Y). If $R \subseteq Y$ is a set of representatives of G–orbits, then the bijection $R \ni r \mapsto G\cdot r \in Y/G$ induces the natural isomorphism $\mathbb{Z}[R] \to \mathbb{Z}[Y/G] \to \mathbb{Z}[Y]_G$.

Applying this discussion to the G–action on $X^{(n)}$ we see that $(C^o_n X)_G = \mathbb{Z}[X^{(n)}]_G = \mathbb{Z}[X^{(n)}/G] \simeq \mathbb{Z}[R_n]$. This isomorphism $(C^o_n X)_G \to \mathbb{Z}[R_n]$ is clearly given by $c_G \mapsto c_R$. Similarly, $(C^o_{n+1} X)_G$ is isomorphic to $\mathbb{Z}[R_{n+1}]$, which is generated by R_{n+1}. The map $\partial\colon (C^o_{n+1} X)_G \to (C^o_n X)_G$ can be interpreted as the map $\mathbb{Z}[R_{n+1}] \ni c \mapsto (\partial c)_R \in \mathbb{Z}[R_n]$; its image is generated by the images of elements of R_{n+1}, ie by the set $\{(\partial\rho)_R \mid \rho \in R_{n+1}\}$. Part (a) is proved.

For part (b): Let K be the kernel of the epimorphism $C^o_d X \to C_d X$. The group K is generated by $\{t\sigma - (\operatorname{sgn} t)\sigma \mid t \in S_{d+1}, \sigma \in X^{(d)}\}$ (one can change the range of t from S_{d+1} to any generating set of S_{d+1}). Applying the coinvariants functor to the exact sequence $K \to C^o_d X \to C_d X \to 0$ we get the middle row of the following commuting diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & (C^o_{d+1}X)_G & \longrightarrow & (C_{d+1}X)_G & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle\partial} & & \big\downarrow{\scriptstyle\partial} & & \\
K_G & \xrightarrow{\iota_G} & (C^o_dX)_G & \longrightarrow & (C_dX)_G & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \\
 & & U^o_d & \longrightarrow & U_d & & \\
 & & & & \big\downarrow & & \\
 & & & & 0 & &
\end{array}$$

A diagram chase shows that an element of $(C^o_dX)_G$ that maps to 0 in U_d is a sum of images of elements of K_G and $(C^o_{d+1}X)_G$. Consequently,

$$U_d \simeq (C^o_dX)_G/(\partial(C^o_{d+1}X)_G + \iota_G K_G).$$

Therefore, a presentation of U_d can be obtained from the presentation of U^o_d given in (a) by adjoining extra relations generating $\iota_G K_G$. These extra relations are images of generators of K_G under ι_G, ie are of the form $(t\sigma)_G - (\operatorname{sgn} t)\sigma_G$ ($t \in S_{d+1}$, $\sigma \in X^{(d)}$). Under the isomorphism $(C^o_dX)_G \to \mathbb{Z}[R_d]$ this form maps to $(t\sigma)_R - (\operatorname{sgn} t)\sigma_R$. To finish the proof we will check that $(t\sigma)_R = (t\sigma_R)_R$. We have $\sigma_R = g\sigma$ for some $g \in G$. This implies that $t\sigma_R = g(t\sigma)$, and then $(t\sigma_R)_R = (g(t\sigma))_R = (t\sigma)_R$. □

3 Boundedness

A group cohomology class in $H^d(G,\mathbb{R})$ is called bounded if it can be represented by a bounded cocycle $c\colon S_dBG \to \mathbb{R}$ (or, equivalently, a bounded G–invariant $\mathbb{R}$–valued cocycle on S_dEG). Here S_*BG is the singular chain complex of BG; a cocycle c is bounded if there exists $M > 0$ such that for each singular simplex $\sigma\colon \Delta^d \to BG$ we have $|c(\sigma)| \le M$. Instead of $\mathbb{R}$, one can use other groups with seminorm. In particular, if X is a simplicial G–complex, the coefficient group $U = U_d(X)$ carries a natural seminorm, induced by the ℓ^1–norm on C_dX. Explicitly, for $u \in U$ we consider all chains $\sum \alpha_i\sigma_i \in C_dX$ that represent u, and we declare the infimum of $\sum |\alpha_i|$ over all such chains to be $|u|$.

Theorem 3.1 *Suppose that X is an acyclic simplicial G–complex with the star-property. Then the tautological cohomology class $\tau^d_{G,X} \in H^d(G,U)$ is bounded with respect to the seminorm discussed above.*

Proof We will construct a G–chain map $\Psi_*\colon S_*EG \to C_*X$. For each $n \ge 0$ choose a free basis Σ_n of the free G–module S_nEG. We define Ψ_n inductively. For each $\xi_0 \in \Sigma_0$ we choose a vertex $\Psi_0(\xi_0) \in X^{(0)}$; we extend Ψ_0 to S_0EG by G–equivariance and linearity. Once Ψ_{n-1} is defined, we define Ψ_n on Σ_n as follows. For $\xi_n \in \Sigma_n$ we consider $\Psi_{n-1}(\partial\xi_n) = \sum \sigma_i \in C_{n-1}X$. By the star-property, there exists a vertex $v \in X^{(0)}$ joinable to every σ_i; we put $\Psi_n(\xi_n) = \sum v * \sigma_i$, so as to have $\partial\Psi_n(\xi_n) = \Psi_{n-1}(\partial\xi_n)$. Then we extend Ψ_n to S_nEG by G–equivariance and linearity. A straightforward induction shows that for any singular simplex $\xi_n \in S_nEG$ the chain $\Psi_n(\xi_n)$ is a sum of at most $(n+1)!$ simplices.

The class $\tau^d_{G,X}$ is represented by the cocycle $T^d_{X/G} \circ \Psi_d$. The tautological cocycle $T^d_{X/G}$ has norm at most 1 — it maps a simplex to its class in U_d, and that class has norm ≤ 1 by definition of the seminorm. Therefore, for any singular simplex σ_d in EG,

$$|T^d_{X/G}(\Psi_d(\sigma_d))| \le (d+1)!. \qquad \square$$

Remark 3.2 There is a different approach to bounded group cohomology, based on the standard homogeneous resolution of the trivial G–module $\mathbb{Z}$ (see [Brown 1982, I, Section 5]). That approach is equivalent to the one used above, as shown in [Gromov 1982, pages 48–49]; for a more detailed account see [Löh 2010, 2.5.5]. In these references real coefficients are used, but the proof works for coefficients in an arbitrary abelian group with seminorm.

4 Characteristic classes

A cohomology class α of a (discrete) group G can serve as a characteristic class of (flat) G–bundles. Suppose that α is obtained from a G–invariant cocycle on an acyclic G–space X as in Remark 1.6. Then it is possible to describe the characteristic class using the cocycle directly, bypassing α (see Theorem 4.4). This section is organized as follows. We start by recalling the connection between group cohomology and characteristic classes. Next, we describe the classical de Rham version of characteristic classes of flat bundles. Then we discuss auxiliary notions and notation and, finally, we state and prove the main statement, Theorem 4.4. (Recall that we consider G with discrete topology, so that all G–bundles are flat — with locally constant transition functions — and BG is $K(G,1)$.)

Let $\alpha \in H^d(G,A) = H^d(BG,A)$ be a cohomology class of a group G. The space BG is the base of a universal principal G–bundle EG. Every principal G–bundle P over a (paracompact) base space B has a classifying map; a map $f_P \colon B \to BG$ such that $f_P^* EG \simeq P$. The map f_P is unique up to homotopy. Notice that we use $f^*\xi$ to denote the pullback of the bundle ξ via the map f, and we also use $f^*\tau$ and f^*T for the pullback of a cohomology class τ or of a cocycle T. Though occasionally confusing, this dual usage is standard practice in bundle theory.

Definition 4.1 The cohomology class $\alpha(P) := f_P^*(\alpha) \in H^d(B,A)$ is functorial in P, and is called the characteristic class (corresponding to α) of the bundle P.

In this definition the G–module A may have nontrivial G–structure. Then the groups $H^d(G,A)$ and $H^d(B,A)$ are cohomology groups with twisted coefficients, ie with coefficients in a flat G–bundle (local system) with fibre A. For $H^d(G,A) = H^d(BG,A)$ the bundle is $EG \times_G A$; for $H^d(B,A)$ we use $P \times_G A$. We have $P \times_G A = f_P^*(EG \times_G A)$, so the coefficient system used over BG pulls back to the one used over B; therefore we get a map $f_P^* \colon H^d(G,A) \to H^d(B,A)$.

In de Rham theory there is a construction of characteristic classes of flat bundles that does not explicitly refer to BG. In fact, it gives an explicit cocycle representative of the characteristic cohomology class in terms of a section. Suppose that $P \to B$ is a principal flat G–bundle over a manifold B, and that $\omega \in \Omega^d(X)$ is a G–invariant closed form on a contractible G–manifold X. To these data we will associate a class in $H^d_{\mathrm{DR}}(B)$. We start by forming the associated bundle $E = P \times_G X$ with fibre X. Then we choose a section $s\colon B \to E$; it exists and is homotopically unique because X is contractible. Now the idea is that a section s of a flat bundle is an ill-defined — G–ambivalent — map from the base to the fibre. The G–ambivalence is countered by the G–invariance of ω, so the pullback of ω by s is well defined. Let us be more precise. Let $\varphi_U\colon E|_U \to U \times X$ be local trivializations of E. Composing φ_U with $\mathrm{pr}_2\colon U \times X \to X$ we get a map $\psi_U\colon E|_U \to X$. The compositions $\psi_U \circ s|_U\colon U \to X$ are locally defined maps; these maps are not compatible. However, due to the G–invariance of ω, the forms $\omega_U = (\psi_U \circ s|_U)^*\omega \in \Omega^d(U)$ are compatible and define a global closed form in $\Omega^d(B)$. Slightly abusing the notation we denote this form by $s^*\omega$. The cohomology class of $s^*\omega$ in $H^d_{\mathrm{DR}}(B)$ is a characteristic class of the bundle P. An alternative description is to define the global form ω^E on E by gluing the compatible collection of forms $\psi_U^*\omega \in \Omega^d(E|_U)$, and then take $s^*\omega^E$ in the standard sense. (See [Morita 2001, Chapter 2] for more information on these classes.)

Let us pass to the simplicial setting. Let $P \to B$ be a principal G–bundle over a Δ–complex B. (For a basic discussion of Δ–complexes see [Hatcher 2002, Section 2.1].) Let $T \in Z^d(\mathrm{Hom}_G(C_*X, A))$ be an A–valued G–invariant simplicial cocycle on an acyclic simplicial G–complex X, and let $\tau \in H^d(G, A)$ be the associated cohomology class (as in Remark 1.6). The characteristic class of P (corresponding to τ) is the cohomology class $\tau(P) \in H^d(B, A)$ (see Definition 4.1). We will use the strategy explained in the de Rham setting and obtain a cochain on B representing $\tau(P)$ (see Theorem 4.4).

To deal with sections in the simplicial context we introduce a special family of trivializations. Let $P \to B$ be a principal G–bundle over a Δ–complex B. Let X be a simplicial G–complex. Let $E = P \times_G X$ be the associated bundle with fibre X. Consider a simplex $\sigma\colon \Delta \to B$, part of the Δ–complex structure. The bundle σ^*P is a flat principal G–bundle over a simplex; hence it has flat sections. Any such flat section $r\colon \Delta \to \sigma^*P$ induces a trivialization of $\sigma^*E \simeq \sigma^*P \times_G X$ — the map

$$\Delta \times X \ni (p, x) \mapsto [r(p), x] \in \sigma^*P \times_G X$$

is an isomorphism, whose inverse $\varphi_{\sigma,r}$ is a trivialization. We put $\psi_{\sigma,r} = \mathrm{pr}_2 \circ \varphi_{\sigma,r}\colon \sigma^*E \to X$. Notice that all possible flat sections of σ^*P are G–related, and that

$$\psi_{\sigma,rg} = g^{-1}\psi_{\sigma,r}. \tag{4-1}$$

Moreover, if σ_i is a face of σ (say $\sigma_i = \sigma|_{\Delta(i)}$, where $\Delta(i) = [e_0, \dots, \hat{e}_i, \dots, e_n]$), then

$$\psi_{\sigma,r}|_{\sigma_i^*E} = \psi_{\sigma_i, r|_{\Delta(i)}}. \tag{4-2}$$

We will now use the maps $\psi_{\sigma,r}$ to define simplicial sections.

Definition 4.2 Let B be a Δ–complex, X a simplicial G–complex, $P \to B$ a principal G–bundle, and $E = P \times_G X$ the associated bundle over B with fibre X. A section $s\colon B \to E$ is called simplicial if for every simplex $\sigma\colon \Delta \to B$ from the Δ–structure of B, and for any $\psi_{\sigma,r}\colon \sigma^*E \to X$ as described above, the composition $\psi_{\sigma,r} \circ s \circ \sigma \colon \Delta \to X$ is an affine map of Δ onto some simplex of the simplicial structure of X — possibly onto a simplex of dimension smaller than $\dim \Delta$ (the composition $s \circ \sigma$ defines a section of $\sigma^* E$ because, for $p \in \Delta$, we have $(\sigma^* E)_p = E_{\sigma(p)}$).

Remark A simplicial section in uniquely determined by its values at the vertices of the base.

A twisted cochain in $C^d(B, A)$ assigns to a simplex $\sigma \colon \Delta \to B$ a value in $(P \times_G A)_{\sigma(e_0)}$ — the fibre of the coefficient bundle over the initial vertex of σ. This value extends to a (unique) flat section of $\sigma^*(P \times_G A) = \sigma^* P \times_G A$. A flat section of that bundle can be described as $[r, a]$, where r is a section of $\sigma^* P$ and $a \in A$. For each $g \in G$ the pair $[rg, g^{-1}a]$ defines the same section; therefore one can also describe sections as continuous (locally constant) G–maps $\sigma^* P \to A$ — or G–maps from the G–torsor of flat sections of $\sigma^* P$ to A.

Definition 4.3 Let B be a Δ–complex, X a simplicial G–complex, $P \to B$ a principal G–bundle, and $E = P \times_G X$ the associated bundle over B with fibre X, s — a simplicial section of E. Consider a simplex $\sigma \colon \Delta \to B$ from the Δ–structure of B and flat sections r of $\sigma^* P$. Then the expression $T(\psi_{\sigma,r} \circ s \circ \sigma)$ is G–equivariant in r (due to (4-1) and the fact that T is a G–map). The formula

$$s^* T(\sigma) = [r, T(\psi_{\sigma,r} \circ s \circ \sigma)]$$

defines the cochain $s^* T \in C^d(B, A)$ (with twisted coefficients).

(The image of the map $\psi_{\sigma,r} \circ s \circ \sigma$ is a simplex in X, on which we put the orientation corresponding under this map to the standard orientation of the standard simplex; we interpret the argument of T as that oriented simplex. If the image of $\psi_{\sigma,r} \circ s \circ \sigma$ has dimension smaller than d, we interpret the argument of T as the zero chain.)

Remark The fact that $s^* T$ is a cocycle will follow from the proof of the next theorem.

Theorem 4.4 *Let T be an A–valued G–invariant cocycle on an acyclic simplicial G–complex X. Let $\tau \in H^d(G, A)$ be the associated group cohomology class (as in Remark 1.6). Let $P \to B$ be a principal (flat) G–bundle over a Δ–complex B. Let $s \colon B \to P \times_G X$ be a simplicial section. Then the class $\tau(P) \in H^d(B, A)$ — the characteristic class of P corresponding to τ — is represented by the simplicial cocycle $s^* T \in Z^d(B, A)$.*

Proof The total space EG of the universal principal G–bundle $EG \to BG$ is contractible (it is also the universal cover of BG). The G–action on EG is free. Therefore, the singular chain complex $S_* EG$ is a projective (in fact, free) resolution of the trivial G–module $\mathbb{Z}$. Moreover, $(S_* EG)_G \simeq S_* BG$. Let $\Psi = \Psi_{C_* X} \colon S_* EG \to C_* X$ be a resolution map from $S_* EG$ to the simplicial chain complex of X. This map induces the map $\Psi^* \colon \mathrm{Hom}_G(C_* X, A) \to \mathrm{Hom}_G(S_* EG, A)$, and

$$\tau = [\Psi^* T] \in H^d(\mathrm{Hom}_G(S_* EG, A)) = H^d(G, A).$$

Let $f = f_P\colon B \to BG$ the a classifying map of the bundle P, and let $F\colon P \to EG$ be a G–bundle map covering f. Then $\tau(P) = f^*\tau = [f^*\Psi^*T]$. Let us describe the cocycle $f^*\Psi^*T$ explicitly. This cocycle should assign to any simplex $\sigma\colon \Delta \to B$ (from the Δ–structure of B) a value in $(P \times_G A)_{\sigma(e_0)}$; as explained in the paragraph preceding Definition 4.3, the choice of that value is equivalent to the choice of a G–map from the G–set of flat sections r of σ^*P to the G–module A. Suppose that $r\colon \Delta \to \sigma^*P$ is a (flat) section. Then $F \circ r$ is a singular simplex in EG. The map $r \mapsto T(\Psi(F \circ r)) \in A$ is a G–map (since each of F, Ψ and T is a G–map); it defines the value of the cochain $f^*\Psi^*T$ (representing $\tau(P)$) on the simplex σ,

$$f^*\Psi^*T(\sigma) = [r, T(\Psi(F \circ r))]. \tag{4-3}$$

The cochain $f^*\Psi^*T$ depends on several choices:

(1) One can choose the space BG — within the homotopy type.

(2) One can choose $f\colon B \to BG$ — within the homotopy class.

(3) One can choose Ψ — all resolution maps are possible.

Our strategy is to exploit these choices to ensure that $f^*\Psi^*T = s^*T$.

Let $\Sigma(B)$ be the set of all simplices σ forming the Δ–structure of B.

Lemma 4.5 *One can choose the space BG and the map f such that*

(a) *$f^*EG \simeq P$ (ie f is a classifying map of P);*

(b) *all the maps $f \circ \sigma$ for $\sigma \in \Sigma(B)$ are pairwise distinct.*

Proof For dimension d let m_d be the barycentre of the standard simplex Δ^d. For each $\sigma \in \Sigma(B)$ we put $p_\sigma = \sigma(m_{\dim\sigma})$. Then we choose a collection of pairwise different points $(x_\sigma)_{\sigma\in\Sigma(B)}$ in BG. (If BG is too small for that, we change it by wedging it with a contractible space of sufficiently large cardinality.) Finally, we perform a homotopy of f (inductively over skeleta) to ensure $f(p_\sigma) = x_\sigma$. □

Let $f\colon B \to BG$ be a classifying map of P satisfying the conditions of Lemma 4.5. Let $F\colon P \to EG$ be a G–bundle map covering f (the composition of an isomorphism $P \to f^*EG$ with the canonical map $f^*EG \to EG$).

Lemma 4.6 *One can choose the resolution map $\Psi\colon S_*EG \to C_*X$ such that $f^*\Psi^*T = s^*T$.*

Proof Let us discuss how Ψ may be constructed. For each $n \geq 0$ choose a free basis Σ_n of the free G–module S_nEG. Define Ψ_n inductively. The base case is $\Psi_{-1} = \mathrm{id}_{\mathbb{Z}}\colon \mathbb{Z} \to \mathbb{Z}$, with $\mathbb{Z}$ connected to the resolutions by the augmentation maps $\partial\colon S_0EG \to \mathbb{Z}$, $\partial\colon C_0X \to \mathbb{Z}$. Once Ψ_{n-1} is defined, calculate — for every $\sigma \in \Sigma_n$ — the cycle $\Psi_{n-1}(\partial\sigma)$. Since C_*X is acyclic, this cycle is a boundary of some n–chain; pick one such chain and define it to be $\Psi_n(\sigma)$. A crucial remark is that if, for some $\eta \in \Sigma_n$ and some n–simplex ξ in X, we have $\Psi_{n-1}(\partial\eta) = \partial\xi$, then we may put $\Psi_n(\eta) = \xi$. Once Ψ_n is defined on Σ_n, we extend it to S_nEG by G–equivariance and linearity.

For each $(\sigma\colon \Delta \to B) \in \Sigma(B)$ choose a flat section $r(\sigma)\colon \Delta \to \sigma^* P$. Composing this section with the canonical bundle map $\sigma^* P \to P$, and then with $F\colon P \to EG$, we get a singular simplex $F \circ r(\sigma)$ in EG. We denote this simplex by $\widetilde{f \circ \sigma}$ — it is a lift of $f \circ \sigma$. All the lifts $\widetilde{f \circ \sigma}$ are pairwise G–inequivalent, because all $f \circ \sigma$ are pairwise distinct. Therefore we may choose the free bases Σ_n such that they contain all the lifts $\widetilde{f \circ \sigma}$ (for $\sigma \in \Sigma(B)$). We would like to define

$$\Psi(\widetilde{f \circ \sigma}) = \psi_{\sigma, r(\sigma)} \circ s \circ \sigma. \tag{4-4}$$

To be able to do that we need to check that

$$\Psi(\partial(\widetilde{f \circ \sigma})) = \partial(\psi_{\sigma, r(\sigma)} \circ s \circ \sigma). \tag{4-5}$$

Let $\sigma_i = \sigma|_{\Delta(i)}$, where $\Delta(i) = [e_0, \dots, \hat{e}_i, \dots, e_n]$ with $n = \dim \sigma$. (Strictly speaking, we should also use an extra map identifying $\Delta(i)$ with the standard simplex. We will ignore this in order not to overburden the notation.) We have

$$\partial(\widetilde{f \circ \sigma}) = \sum_{i=0}^{n} (-1)^i (\widetilde{f \circ \sigma})|_{\Delta(i)}. \tag{4-6}$$

Observe that $(\widetilde{f \circ \sigma})|_{\Delta(i)}$ is a lift of $f \circ (\sigma|_{\Delta(i)})$; therefore $(\widetilde{f \circ \sigma})|_{\Delta(i)} = (\widetilde{f \circ \sigma_i}) \cdot g(i)$ for some $g(i) \in G$. By induction on the dimension we know that

$$\Psi((\widetilde{f \circ \sigma_i}) \cdot g(i)) = g(i)^{-1} \Psi(\widetilde{f \circ \sigma_i}) = g(i)^{-1} (\psi_{\sigma_i, r(\sigma_i)} \circ s \circ \sigma_i) = \psi_{\sigma_i, r(\sigma_i) g(i)} \circ s \circ \sigma_i,$$

the last equality following from (4-1). We may finally write

$$\Psi(\partial(\widetilde{f \circ \sigma})) = \sum_{i=0}^{n} (-1)^i \psi_{\sigma_i, r(\sigma_i) g(i)} \circ s \circ \sigma_i. \tag{4-7}$$

On the other hand,

$$\partial(\psi_{\sigma, r(\sigma)} \circ s \circ \sigma) = \sum_{i=0}^{n} (-1)^i \psi_{\sigma, r(\sigma)} \circ s \circ \sigma_i. \tag{4-8}$$

Notice that $s \circ \sigma_i$ is a section of $\sigma_i^* E$ (where $E = P \times_G X$); therefore (4-2) applies and yields

$$\psi_{\sigma, r(\sigma)} \circ s \circ \sigma_i = \psi_{\sigma_i, r(\sigma)|_{\Delta(i)}} \circ s \circ \sigma_i. \tag{4-9}$$

The flat sections $r(\sigma)|_{\Delta(i)}$ and $r(\sigma_i)$ of the G–bundle $\sigma_i^* P$ are G–related: $r(\sigma)|_{\Delta(i)} = r(\sigma_i) \cdot g$ for some $g \in G$. Recall that, for any $\eta \in \Sigma(B)$, the singular simplex $\widetilde{f \circ \eta}$ is the composition of $r(\eta)$ with a G–bundle map $\eta^* P \to EG$. It follows that $(\widetilde{f \circ \sigma})|_{\Delta(i)} = (\widetilde{f \circ \sigma_i}) \cdot g$; therefore $g = g(i)$. Consequently,

$$\psi_{\sigma_i, r(\sigma)|_{\Delta(i)}} \circ s \circ \sigma_i = \psi_{\sigma_i, r(\sigma_i) g(i)} \circ s \circ \sigma_i. \tag{4-10}$$

Putting (4-8), (4-9) and (4-10) together we get

$$\partial(\psi_{\sigma, r(\sigma)} \circ s \circ \sigma) = \sum_{i=0}^{n} (-1)^i \psi_{\sigma_i, r(\sigma_i) g(i)} \circ s \circ \sigma_i. \tag{4-11}$$

Comparing this with (4-7) we obtain (4-5). We may therefore define Ψ so that (4-4) holds for all $\sigma \in \Sigma(B)$. Then, for d–dimensional σ we get

$$\begin{aligned} f^*\Psi^*T(\sigma) &= \big[r(\sigma), T\big(\Psi(F \circ r(\sigma))\big)\big] \\ &= [r(\sigma), T(\Psi(\widetilde{f \circ \sigma}))] = [r(\sigma), T(\psi_{\sigma,r(\sigma)} \circ s \circ \sigma)] = s^*T(\sigma). \end{aligned}$$ □

Lemma 4.6 implies $f^*\Psi^*T$ — a cocycle representing $\tau(P)$ — is equal to s^*T; proving Theorem 4.4. □

Remark In our applications of Theorem 4.4 the coefficients will be either untwisted or only mildly twisted (eg a $\mathrm{GL}(2, K)$–module A which is trivial as an $\mathrm{SL}(2, K)$–module).

5 Homological core

Consider an acyclic simplicial G–complex X, the associated coefficient group U_d and the tautological cohomology classes $\tau_{X/G} \in H^d(\mathrm{Hom}_G(C_*X, U_d))$ and $\tau_{G,X} \in H^d(G, U_d)$.

Question Is it possible to represent these tautological classes by cocycles with coefficients in a proper subgroup of U_d?

In general, there is a candidate subgroup. The coefficient group $U_d = (C_dX)_G/\partial(C_{d+1}X)_G$ has a natural homomorphism $\partial\colon U_d \to (C_{d-1}X)_G$ (induced by the usual $\partial\colon C_dX \to C_{d-1}X$).

Definition 5.1 The homological core hU_d of the group U_d is the kernel of the map $\partial\colon U_d \to (C_{d-1}X)_G$.

The following theorem states a weaker property then asked for above, but is quite general.

Theorem 5.2 *Let X be an acyclic simplicial G–complex. Let $\tau_{G,X} \in H^d(G, U_d)$ be the associated tautological class, and let $z \in H_d(BG, \mathbb{Z})$ be a homology class. Then $\langle \tau_{G,X}, z\rangle \in hU_d$.*

Proof The map $\partial\colon U_d \to (C_{d-1}X)_G$ of coefficient groups induces horizontal maps in the commutative diagram

$$\begin{array}{ccc} H^d(\mathrm{Hom}_G(C_*X, U_d)) & \xrightarrow{\partial_*} & H^d\big(\mathrm{Hom}_G(C_*X, (C_{d-1}X)_G)\big) \\ \downarrow & & \downarrow \\ H^d(G, U_d) & \xrightarrow{\partial_*} & H^d(G, (C_{d-1}X)_G) \end{array}$$

The class $\tau_{X/G} \in H^d(\mathrm{Hom}_G(C_*X, U_d))$ is mapped to 0 by ∂_*. Indeed, the class $\tau_{X/G}$ is represented by the tautological cocycle given by $T_{X/G}(\sigma^d) = [\sigma^d]$. Let $t \in \mathrm{Hom}_G(C_{d-1}, (C_{d-1})_G)$ also be tautological: $t(\sigma^{d-1}) = [\sigma^{d-1}]$. Then

$$(\partial_*T_{X/G})(\sigma^d) = \partial[\sigma^d] = [\partial\sigma^d] = t(\partial\sigma^d) = (\delta t)(\sigma^d),$$

that is, $\partial_*T_{X/G} = \delta t$; hence $\partial_*\tau_{X/G} = 0$.

It now follows from the diagram that $\partial_*\tau_{G,X} = 0$ as well. Therefore, $\partial\langle\tau_{G,X}, z\rangle = \langle\partial_*\tau_{G,X}, z\rangle = 0$. □

II GL$(2, K)$

In this part we describe some results of Nekovář [1990] from our point of view. This provides a perfect illustration of the general method.

6 Review of possible actions

The first example where the general approach from Part I gives something interesting is $G = \mathrm{GL}(2, K)$. To proceed we need a simplicial action of G on a complex X with desired properties, one of them being high transitivity. Such X can be constructed by taking as the vertex set a homogeneous space G/S for some S and studying the notion of "generic k–tuple". In $\mathrm{GL}(2, K)$ we have the following interesting subgroups.

(1) If $S = \left(\begin{smallmatrix}1 & * \\ 0 & *\end{smallmatrix}\right)$, then $G/S = K^2 \setminus \{0\}$. A tuple is generic if it consists of pairwise linearly independent vectors. The action of $\mathrm{GL}(2, K)$ is effective and transitive on generic pairs. We will discuss this case later.

(2) If $S = \left(\begin{smallmatrix}* & 0 \\ 0 & *\end{smallmatrix}\right)$, then $G/S = (\mathbb{P}^1(K) \times \mathbb{P}^1(K)) \setminus \Delta$ (where Δ is the diagonal). Generic k–tuples are tuples of pairs (p_i, q_i) with all the points p_i and q_j distinct. Here even the action on pairs is not transitive, because the cross-ratio (p_1, q_1, p_2, q_2) is preserved. The action of $\mathrm{GL}(2, K)$ is not effective; it factors through $\mathrm{PGL}(2, K)$.

(3) If $S = \left(\begin{smallmatrix}* & * \\ 0 & *\end{smallmatrix}\right)$, then $G/S = \mathbb{P}^1(K)$. The action factors through $\mathrm{PGL}(2, K)$. Generic k–tuples are tuples of distinct projective points. The action is triply transitive.

In the first two examples our approach yields very big groups of coefficients. We can compute them (and we do so in the $S = \left(\begin{smallmatrix}1 & * \\ 0 & *\end{smallmatrix}\right)$ case), but we cannot say much about them. In the third case transitivity is higher; hence the coefficient group is smaller and easier to understand. We do the computation in detail (following Nekovář). Actually in this case something interesting happens: while $\mathrm{PGL}(2, K)$ acts transitively on triples, the large normal subgroups $\mathrm{PSL}(2, K)$ acts transitively on pairs, while its orbits on triples are indexed by $\dot{K}/\dot{K}^2$ — the group of square classes. This gives an untwisted cohomology class for $\mathrm{PSL}(2, K)$ and a (slightly) twisted cohomology class for $\mathrm{PGL}(2, K)$.

Now we proceed with the description of the $S = \left(\begin{smallmatrix}1 & * \\ 0 & *\end{smallmatrix}\right)$ case. We do not discuss the $S = \left(\begin{smallmatrix}* & 0 \\ 0 & *\end{smallmatrix}\right)$ case.

GL$(2, K)$–action on nonzero vectors Consider the $\mathrm{GL}(2, K)$–action on $\mathrm{GL}(2, K)/\left(\begin{smallmatrix}1 & * \\ 0 & *\end{smallmatrix}\right) \simeq K^2 \setminus \{0\}$. We declare a tuple of nonzero vectors in K^2 to be generic if its elements are pairwise linearly independent. The complex X has k–simplices spanned on generic $(k+1)$–tuples.

The action of $\mathrm{GL}(2, K)$ is transitive on 1–simplices. However, the action of its large normal subgroup $\mathrm{SL}(2, K)$ does have an invariant: $(v, w) \mapsto \det(v, w) \in \dot{K}$. Thus, we have a potentially nontrivial 1–

dimensional cocycle, with coefficients in a quotient of $\mathbb{Z}[\dot{K}]$. However, any given generic triple can be normalized by an element of $\mathrm{SL}(2,K)$ to

$$\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\a\end{pmatrix},\begin{pmatrix}-x/a\\y\end{pmatrix}\right) \tag{6-1}$$

with nonzero a, x and y; the three determinants are a, y and x. To get the coefficient group we have to divide $\mathbb{Z}[\dot{K}]$ by relations $\langle a\rangle-\langle x\rangle+\langle y\rangle$, for all nonzero a, x and y. The resulting coefficient group is trivial, hence so is the cocycle.

The action of $\mathrm{GL}(2,K)$ on generic triples of vectors has a complete invariant in $\dot{K}\times\dot{K}\times\dot{K}$: the triple of pairwise determinants. The resulting coefficient group is the quotient of $\mathbb{Z}[\dot{K}\times\dot{K}\times\dot{K}]$ by the relation

$$\left(\frac{be-cd}{a},e,c\right)-\left(\frac{be-cd}{a},d,b\right)+(e,d,a)-(c,b,a)=0$$

(all entries assumed nonzero). We skip the details, as they are not dissimilar to ones in the calculation presented later, and the result is not especially meaningful.

$\mathrm{GL}(2,K)$–action on projective line Consider the $\mathrm{GL}(2,K)$–action on $\mathrm{GL}(2,K)/\left(\begin{smallmatrix}*&*\\0&*\end{smallmatrix}\right)\simeq\mathbb{P}^1(K)$. We declare a $(k+1)$–tuple of projective points generic if they are pairwise distinct; we span k–simplices on such tuples. Thus, the complex X is the (infinite) simplex with vertex set $\mathbb{P}^1(K)$.

The $\mathrm{GL}(2,K)$ (in fact, $\mathrm{PGL}(2,K)$) action on X is transitive on 2–simplices. However, the $\mathrm{SL}(2,K)$–action on 2–simplices has an invariant with values in $\dot{K}/\dot{K}^2$ — the set of square classes. Our procedure will produce a cocycle with constant coefficients (in a quotient group of $\mathbb{Z}[\dot{K}/\dot{K}^2]$) for $\mathrm{PSL}(2,K)$, and with twisted coefficients for $\mathrm{PGL}(2,K)$. We discuss this in detail in the next section.

The cross-ration is a complete invariant of (ordered) 3–simplices, ie of 4–tuples of distinct points in $\mathbb{P}^1(K)$, under the action of $\mathrm{PGL}(2,K)$. Thus, our approach yields a 3–cocycle with coefficients in $\mathbb{Z}[\dot{K}\setminus\{1\}]/I$, where I is the subgroup spanned by

$$\left[\frac{\lambda(\mu-1)}{\mu(\lambda-1)}\right]-\left[\frac{\mu-1}{\lambda-1}\right]+\left[\frac{\mu}{\lambda}\right]-[\mu]+[\lambda]$$

with $\lambda,\mu\in\dot{K}\setminus\{1\}$ and $\lambda\neq\mu$. This is related to the dilogarithm function; we do not pursue it further (but see eg [Bergeron et al. 2014]).

7 Action on triples of points in the projective line

In this section we consider the action of $G=\mathrm{PSL}(2,K)$ on the projective line $\mathbb{P}=\mathbb{P}^1(K)$, for an infinite field K. We define X as the (infinite) simplex with vertex set $\mathbb{P}$; in other words, we span simplices of X on tuples of generic (ie pairwise distinct) points in $\mathbb{P}$. The complex X has the star-property and is contractible. To the induced action of G on X the formalism of Part I (see Definition 2.1) associates the coefficient group U_2, the tautological cocycle T, and a tautological cohomology class of G.

Theorem 7.1 *The group U_2 is isomorphic to $W(K)$, the Witt group of the field K.*

Let us briefly recall the definition of $W(K)$ (for details see [Elman et al. 2008, Chapter I]). The isometry classes of nondegenerate symmetric bilinear forms over K form a semiring, with direct sum as addition and tensor product as multiplication. Passing to the Grothendieck group of the additive structure of this semiring we obtain the Witt–Grothendieck ring $\widehat{W}(K)$. The Witt ring $W(K)$ is the quotient of $\widehat{W}(K)$ by the ideal generated by the hyperbolic plane — the form with matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Both rings have explicit presentations in terms of generators and relations; see [Elman et al. 2008, Theorems 4.7 and 4.8].

Proof of Theorem 7.1 We apply Fact 2.4.

Generators We need to find the orbits of the G–action on the set of generic triples of points in $\mathbb{P}$. We denote by $[v]$ the point in $\mathbb{P}$ determined by the vector $v \in K^2$; for $v = \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)$ we shorten $\left[\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\right]$ to $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$.

Lemma 7.2 *Every generic triple $([u],[v],[w])$ of points in $\mathbb{P}$ is G–equivalent to a triple of the form*

$$t_\lambda = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\\lambda\end{bmatrix}\right), \quad \textit{where } \lambda = \det(u,v)\det(v,w)\det(w,u).$$

Triples t_λ and t_μ are equivalent if and only if $\lambda/\mu \in \dot{K}^2$ (the set of squares in $\dot{K}$).

Proof There exists $g \in \mathrm{SL}(2,K)$ such that $gu = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, and $gv = \left(\begin{smallmatrix}0\\\gamma\end{smallmatrix}\right)$ for $\gamma = \det(u,v)$. Then

$$gw = \begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\begin{pmatrix}1\\\beta/\alpha\end{pmatrix}$$

for some $\alpha,\beta \in \dot{K}$, so

$$g\colon ([u],[v],[w]) \mapsto \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\\gamma\end{bmatrix}, \begin{bmatrix}\alpha\\\beta\end{bmatrix}\right) = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\\lambda\end{bmatrix}\right)$$

for $\lambda = \beta/\alpha$. Notice that

(7-1)
$$\begin{aligned}\lambda = \frac{\beta}{\alpha} = \begin{vmatrix}1&0\\0&\gamma\end{vmatrix} \cdot \begin{vmatrix}0&\alpha\\\gamma&\beta\end{vmatrix}^{-1} \cdot \begin{vmatrix}\alpha&1\\\beta&0\end{vmatrix} &= \det(gu,gv)\det(gv,gw)^{-1}\det(gw,gu)\\ &= \det(u,v)\det(v,w)^{-1}\det(w,u).\end{aligned}$$

Notice that the stabilizer of $\left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right)$ in $\mathrm{SL}(2,K)$ consists of diagonal matrices of the form $\left(\begin{smallmatrix}\alpha^{-1}&0\\0&\alpha\end{smallmatrix}\right)$. Such a matrix maps t_λ to

$$\left(\begin{bmatrix}\alpha^{-1}\\0\end{bmatrix}, \begin{bmatrix}0\\\alpha\end{bmatrix}, \begin{bmatrix}\alpha^{-1}\\\alpha\lambda\end{bmatrix}\right) = \left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\\alpha^2\lambda\end{bmatrix}\right) = t_{\alpha^2\lambda}.$$

The last claim of the lemma follows. Finally, the class in $\dot{K}/\dot{K}^2$ of λ given by (7-1) is the same as the class of $\det(u,v)\det(v,w)\det(w,u)$. □

The lemma and Fact 2.4 imply that U_2 is the quotient of $\mathbb{Z}[\dot{K}/\dot{K}^2]$ by two sets of relations (boundary relations and alternation relations). The generator of $\mathbb{Z}[\dot{K}/\dot{K}^2]$ corresponding to t_λ (and the image of this generator in U_2) will be denoted by $[\lambda]$ and called the symbol of the triple.

Alternation relations The transposition of the first two points of a triple maps t_λ to

$$\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\\lambda\end{bmatrix}\right),$$

which can be transformed by $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ to

$$\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\-1\end{bmatrix},\begin{bmatrix}\lambda\\-1\end{bmatrix}\right)=\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\-\lambda^{-1}\end{bmatrix}\right).$$

Since $[-\lambda^{-1}]=[-\lambda]$ in $\dot K/\dot K^2$, the resulting relation can be written as $-[\lambda]=[-\lambda]$. Next, consider the transposition of the last two vectors of a triple; this transposition maps t_λ to

$$\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\\lambda\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right),$$

which can be transformed by $\left(\begin{smallmatrix}1&-\lambda^{-1}\\0&1\end{smallmatrix}\right)$ to

$$\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\\lambda\end{bmatrix},\begin{bmatrix}-\lambda^{-1}\\1\end{bmatrix}\right)=\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\-\lambda\end{bmatrix}\right).$$

Again, we get $-[\lambda]=[-\lambda]$.

Boundary relations A generic quadruple of points in $\mathbb{P}$ can be G–transformed to

$$\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\\lambda\end{bmatrix},\begin{bmatrix}1\\\mu\end{bmatrix}\right),$$

where genericity is equivalent to $\lambda,\mu\in\dot K$, $\lambda\neq\mu$. The boundary of the corresponding 3–simplex is the alternating sum of four triangles — triples obtained from the quadruple by omitting one element. We calculate the symbols of those triples:

Omit $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$: $\left[\begin{vmatrix}0&1\\1&\lambda\end{vmatrix}\cdot\begin{vmatrix}1&1\\\lambda&\mu\end{vmatrix}\cdot\begin{vmatrix}1&0\\\mu&1\end{vmatrix}\right]=[(-1)\cdot(\mu-\lambda)\cdot 1]=[\lambda-\mu].$

Omit $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$: $\left[\begin{vmatrix}1&1\\0&\lambda\end{vmatrix}\cdot\begin{vmatrix}1&1\\\lambda&\mu\end{vmatrix}\cdot\begin{vmatrix}1&1\\\mu&0\end{vmatrix}\right]=[\lambda\cdot(\mu-\lambda)\cdot(-\mu)]=[\lambda\mu(\lambda-\mu)].$

Omit $\left[\begin{smallmatrix}1\\\lambda\end{smallmatrix}\right]$: $[\mu]$.

Omit $\left[\begin{smallmatrix}1\\\mu\end{smallmatrix}\right]$: $[\lambda]$.

The relation is

$$[\lambda-\mu]-[\lambda\mu(\lambda-\mu)]+[\mu]-[\lambda]=0.$$

Putting $a=\lambda-\mu$ and $b=\mu$, we may rewrite this as

$$[a]+[b]=[a+b]+[ab(a+b)].$$

The relation holds for all $a,b\in\dot K$ that satisfy $a+b\neq 0$. This set of relations, plus the alternation relation $[-a]=-[a]$, gives the classical description of the Witt group $W(K)$; see [Elman et al. 2008, Theorem 4.8]. □

Definition 7.3 The tautological second cohomology class of the group $G = \mathrm{PSL}(2, K)$ with coefficients in $U_2 = W(K)$ associated (as in Definition 2.1) to the action of G on X will be called the Witt class and denoted by $w \in H^2(\mathrm{PSL}(2, K), W(K))$.

Remark 7.4 (1) Let T be the tautological ($W(K)$–valued) cocycle associated to the G–action on X. From the proof of Theorem 7.1 it is useful to extract the following explicit formula for the value of T on a 2–simplex in X determined by a triple of pairwise linearly independent vectors $u, v, w \in K^2$:

$$T([u],[v],[w]) = [\det(u,v)\det(v,w)\det(w,u)]. \tag{7-2}$$

(2) One can see from the proof of Theorem 7.1 that the ordered coefficient group U_2^o is isomorphic to the Witt–Grothendieck group $\widehat{W}(K)$ of the field K.

(3) The space $\mathbb{P} = \mathbb{P}^1(K)$ and the complex X are acted upon by the larger group $\mathrm{PGL}(2, K)$. As a result, the Witt class can be interpreted as a twisted cohomology class of $\mathrm{PGL}(2, K)$. The twisting action of $\mathrm{PGL}(2, K)$ on $W(K)$ is easy to see from the formula for λ in Lemma 7.2; the class of $g \in \mathrm{GL}(2, K)$ acts on the symbol $[\lambda]$ mapping it to $[\det(g)\cdot\lambda]$.

(4) For $K = \mathbb{R}$ we have $W(\mathbb{R}) \simeq \mathbb{Z}$. The isomorphism, called the signature map, maps $[\lambda]$ to $+1$ for $\lambda > 0$ and to -1 for $\lambda < 0$. The pullback of the Witt class to $\mathrm{SL}(2,\mathbb{R})$ is a class in $H^2(\mathrm{SL}(2,\mathbb{R}),\mathbb{Z})$; we will relate it to the usual (topological) Euler class (see Theorem 13.4 and Fact 13.5).

(5) For $K = \mathbb{Q}$ the Witt group has a large torsion part which is a direct summand. Computer calculations (using the computer algebra system FriCAS) indicate that the corresponding part of the Witt class is nontrivial.

It is possible to give an explicit formula for a cocycle representing the Witt class. We use the standard homogeneous resolution (see [Brown 1982, II, Section 3]) to describe group cohomology; $W(K)$–valued 2–cocycles are then represented by functions $G \times G \times G \to W(K)$.

Theorem 7.5 *Let us fix an arbitrary nonzero vector $u \in K^2$. The map*

$$G \times G \times G \ni (g_0, g_1, g_2) \mapsto [\det(\tilde g_0 u, \tilde g_1 u)\det(\tilde g_1 u, \tilde g_2 u)\det(\tilde g_2 u, \tilde g_0 u)] \in W(K) \tag{7-3}$$

is a cocycle representing the Witt class $w \in H^2(\mathrm{PSL}(2, K), W(K))$. (By $\tilde g_i$ we denote an arbitrary lift of $g_i \in \mathrm{PSL}(2, K)$ to $\mathrm{SL}(2, K)$. The senseless symbol $[0]$ is interpreted as 0.)

Proof It is straightforward to check that the maps

$$\Psi_n \colon (g_0,\dots,g_n) \mapsto \begin{cases} ([\tilde g_0 u],\dots,[\tilde g_n u]) & \text{if the points } [\tilde g_i u] \text{ are pairwise different,} \\ 0 & \text{otherwise,} \end{cases} \tag{7-4}$$

defines a G–chain map from the homogeneous standard resolution of G to C_*X. (The only subtle case is when $[\tilde g_i u] = [\tilde g_j u]$ for exactly one pair of indices i, j. Then $\Psi_{n-1}\partial(g_0,\dots,g_n)$ has two nonzero summands — however, these summands cancel in the alternating chain complex C_*X.) Composing Ψ_2 with the tautological cocycle T given by (7-2) we obtain the statement of the theorem. □

III Euler class for ordered fields

In this part we define and investigate Euler classes for general linear and projective groups over arbitrary ordered fields.

8 Tautological Euler classes: computation of coefficients

Let K be an ordered field. Let $G = \mathrm{GL}(n, K)$; we will also consider the following closely related groups (where we put $\dot{K} = K \setminus \{0\}$ and $K_+ = \{\lambda \in K \mid \lambda > 0\}$):

$$\begin{aligned} PG &:= \mathrm{PGL}(n, K) = G/\{\lambda I \mid \lambda \in \dot{K}\}, \\ P_+G &:= \mathrm{P_+GL}(n, K) = G/\{\lambda I \mid \lambda \in K_+\}, \\ G_+ &:= \mathrm{GL}_+(n, K) = \{g \in G \mid \det g > 0\}, \\ PG_+ &:= \mathrm{PGL}_+(n, K) = G_+/\{\lambda I \in G_+ \mid \lambda \in K\}, \\ P_+G_+ &:= \mathrm{P_+GL_+}(n, K) = G_+/\{\lambda I \mid \lambda \in K_+\}. \end{aligned}$$

The natural maps between these groups are summarized in the diagram

$$\begin{array}{ccccc} G_+ & \longrightarrow & P_+G_+ & \longrightarrow & PG_+ \\ \downarrow & & \downarrow & & \downarrow \\ G & \longrightarrow & P_+G & \longrightarrow & PG \end{array}$$

The defining action of G on K^n induces actions of P_+G on $\mathbb{P}_+$ and of PG on $\mathbb{P}$; here

$$\mathbb{P}_+ := \mathbb{P}_+^{n-1}(K) = (K^n \setminus \{0\})/K_+, \quad \mathbb{P} := \mathbb{P}^{n-1}(K) = (K^n \setminus \{0\})/\dot{K},$$

where the multiplicative groups K_+ and $\dot{K}$ act on K^n by homotheties.

Next we define simplicial complexes X and X_+ by spanning simplices on generic tuples of points in $\mathbb{P}$ and $\mathbb{P}_+$, respectively. We call a tuple $([v_0], \dots, [v_k])$ generic if every subsequence of $(v_0, \dots, v_k)$ of length $\leq n$ is linearly independent. Ordered fields are infinite, therefore these complexes have the star-property and are contractible. The complex X is acted upon by PG, and X_+ by P_+G. We restrict these actions to PG_+ and to P_+G_+ and we apply the formalism of Part I. We put

$$U := U_n(X), \quad U_+ = U_n(X_+)$$

(see Definitions 2.1 and 1.3), and we define the Euler classes as tautological classes

$$\text{(8-1)} \qquad \mathrm{eu} := \tau^n_{PG_+,X} \in H^n(PG_+, U), \quad \mathrm{eu}_+ := \tau^n_{P_+G_+,X_+} \in H^n(P_+G_+, U_+).$$

Notice that PG_+ is a normal subgroup of PG. Therefore the group U carries the structure of a PG–module, and the class eu can also be considered as a twisted PG–class. (See the second remark after Definition 1.3.) Similarly, the class eu_+ can be regarded as a twisted P_+G–class.

Our first goal is to compute U and U_+, as abelian groups and as PG– and P_+G–modules.

Theorem 8.1 *Let U (resp. U_+) be the coefficient group associated to the action of $\mathrm{PGL}_+(n,K)$ (resp. $\mathrm{P}_+\mathrm{GL}_+(n,K)$) on the complex of generic tuples of points in $\mathbb{P}^{n-1}(K)$ (resp. $\mathbb{P}_+^{n-1}(K)$). Then*

$$U \simeq \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \mathbb{Z} & \text{if } n \text{ is even,} \end{cases} \qquad U_+ \simeq \mathbb{Z}^{\lfloor n/2 \rfloor + 1}.$$

The $\mathrm{PGL}(n,K)$– and $\mathrm{P}_+\mathrm{GL}(n,K)$–structures are given by

$$[g]\cdot u = \begin{cases} u & \text{if } \det g > 0, \\ -u & \text{if } \det g < 0, \end{cases} \qquad (g \in \mathrm{GL}(n,K)).$$

Proof Both calculations are based on Fact 2.4. We denote by $(e_1,\dots,e_n)$ the standard basis of K^n.

Calculation of U Nondegenerate ordered simplices of X correspond to generic tuples of points in $\mathbb{P}$.

Lemma 8.2 *The action of PG_+ on the set of generic $(n+1)$–tuples of points in $\mathbb{P}$ has one orbit for n odd and two orbits for n even.*

Proof Let $p = (p_1,\dots,p_{n+1})$ be a generic $(n+1)$–tuple of points in $\mathbb{P}$. There is an element $g \in G$ (unique up to scaling) that maps p to the standard tuple $e = \bigl([e_1],\dots,[e_n],\bigl[\sum_{i=1}^n e_i\bigr]\bigr)$. If n is odd then $\det(-g) = -\det(g)$; therefore g may be chosen in G_+. It follows that in this case PG_+ acts transitively on the set of generic $(n+1)$–tuples. For n even, all elements g mapping p to e have determinants of the same sign. This sign is a PG_+–invariant of p that we call the sign of p and denote by $\mathrm{sgn}(p)$. Generic tuples of the same sign are PG_+–equivalent: if $\mathrm{sgn}(p) = \mathrm{sgn}(p')$, $gp = e$ and $gp' = e$, then $g^{-1}g'p' = p$ and $\mathrm{sgn}\det(g^{-1}g') = +1$. □

The case of n odd is now straightforward. The image of any n–simplex of X in $(C_nX)_{PG_+}$ is one and the same generator of that cyclic group. The boundary of an $(n+1)$–simplex of X is an alternating sum of an odd number of n–simplices; hence its image in $(C_nX)_{PG_+}$ is again that generator. It follows that $U = 0$ for n odd.

Suppose now that n is even. Lemma 8.2 and Fact 2.4 imply that U is the quotient of the free abelian group with two generators by two sets of relations (boundary relations and alternation relations). The generators correspond to (representatives of) PG_+–orbits on the set of generic $(n+1)$–tuples of points in $\mathbb{P}$; explicitly, we choose

$$\left([e_1],\dots,[e_n],\left[\sum_{i=1}^n e_i\right]\right) \tag{8-2}$$

and denote it by $[+]$ or $[+1]$, and

$$\left([e_1],\dots,[e_{n-1}],[-e_n],\left[\left(\sum_{i=1}^{n-1} e_i\right) - e_n\right]\right) \tag{8-3}$$

and denote it by $[-]$ or $[-1]$. We call $[+]$ and $[-]$ symbols.

The group $\mathbb{Z}^2$ generated by $[+]$ and $[-]$ is isomorphic to $(C_n^o X)_{PG_+}$. The image in this group of an ordered n–simplex of X corresponding to a generic $(n+1)$–tuple $p = (p_1, \dots, p_{n+1})$ of points in $\mathbb{P}$ is $[\mathrm{sgn}(p)]$. In practice, the sign can be calculated as follows: let $p_i = [v_i]$ for $v_i \in K^n$, and let $v_{n+1} = \sum_{i=1}^n \alpha_i v_i$; then

$$\mathrm{sgn}(p) = \mathrm{sgn}(\det(v_1, \dots, v_n) \cdot \alpha_1 \cdots \alpha_n) \tag{8-4}$$

(this is the sign of the determinant of the matrix mapping $(v_1, \dots, v_{n+1})$ to $\bigl(e_1, \dots, e_n, \sum_{i=1}^n e_i\bigr)$; that matrix is the inverse of the product of the matrix with columns $(v_1, \dots, v_n)$ and the diagonal matrix with diagonal entries $(\alpha_1, \dots, \alpha_n)$).

The alternation relations all reduce to $-[+] = [-]$. Indeed, it is straightforward to check that transposing two neighbouring elements in (8-2) or (8-3) changes the sign of the tuple.

We now discuss the boundary relations. We observe that any ordered nondegenerate $(n+1)$–simplex of X — corresponding to a generic $(n+2)$–tuple of points in $\mathbb{P}$ — can be mapped by an element of PG_+ to

$$\Delta = \left([e_1], [e_2], \dots, [e_{n-1}], [se_n], \left[\left(\sum_{i=1}^{n-1} e_i\right) + se_n\right], \left[\left(\sum_{i=1}^{n-1} b_i e_i\right) + sb_n e_n\right]\right).$$

Here $s = \pm 1$ and $b_i \in K$; genericity means that $b_i \neq 0$ and $b_i \neq b_j$. We have

$$\partial \Delta = \sum_{j=1}^{n+2} (-1)^{j-1} [s_j], \tag{8-5}$$

where s_j is the sign of the tuple obtained from Δ by omitting the j^{th} element. We have $s_{n+2} = s$ and $s_{n+1} = s\,\mathrm{sgn}\bigl(\prod b_i\bigr)$. We claim that, for $j < n+1$,

$$s_j = (-1)^j s\,\mathrm{sgn}\left(b_j \prod_{i \neq j} (b_i - b_j)\right). \tag{8-6}$$

Indeed, for $j < n$ we have

$$\mathrm{sgn}\det\left(e_1, \dots, \hat{e}_j, \dots, e_{n-1}, se_n, \sum_{i=1}^{n-1} e_i + se_n\right) = (-1)^{n-j} s = (-1)^j s,$$

$$\sum_{i=1}^{n-1} b_i e_i + sb_n e_n = b_j \left(\sum_{i=1}^{n-1} e_i + se_n\right) + \sum_{i \neq j,n} (b_i - b_j) e_i + (b_n - b_j) se_n,$$

while for $j = n$ we have

$$\mathrm{sgn}\det\left(e_1, \dots, e_{n-1}, \widehat{se}_n, \sum_{i=1}^{n-1} e_i + se_n\right) = s = (-1)^n s,$$

$$\sum_{i=1}^{n-1} b_i e_i + sb_n e_n = b_n \left(\sum_{i=1}^{n-1} e_i + se_n\right) + \sum_{i=1}^{n-1} (b_i - b_n) e_i.$$

Putting (8-5) and (8-6) together, we get

$$\partial\Delta = \sum_{j=1}^{n} (-1)^{j-1}\Big[(-1)^j s \,\mathrm{sgn}\, b_j \prod_{i\neq j} \mathrm{sgn}(b_i - b_j)\Big] + (-1)^n\Big[s\,\mathrm{sgn}\Big(\prod_i b_i\Big)\Big] - [s]. \tag{8-7}$$

We will show that this relation is trivial, ie that all the symbols cancel. We can assume $s=+1$; the case $s=-1$ will automatically follow. Indeed, changing s flips all the symbols: $[+1]\leftrightarrow[-1]$, and trivially transforms a trivial relation to a trivial relation.

Let us artificially put $b_{n+1}=0$; then we may rewrite (8-7) more uniformly as

$$\partial\Delta = \sum_{j=1}^{n+1} (-1)^{j-1}\Big[\mathrm{sgn}\Big(\prod_{i=1}^{j-1}(b_j - b_i)\prod_{i=j+1}^{n+1}(b_i - b_j)\Big)\Big] - [+1]. \tag{8-8}$$

Let $\sigma\in S_{n+1}$ be the permutation ordering the numbers (indices) in the same way that the sequence b does: $\sigma(i)<\sigma(k) \iff b_i<b_k$. We put $\mathrm{inv}(j)=\#\{i \mid (i-j)(\sigma(i)-\sigma(j))<0\}$ (the number of inversions of σ in which j is involved). Then our relation is

$$\sum_{j=1}^{n+1}(-1)^{j-1}[(-1)^{\mathrm{inv}(j)}] - [+1]. \tag{8-9}$$

Lemma 8.3 $$(-1)^{\mathrm{inv}(j)} = (-1)^{\sigma(j)-j}.$$

Proof If exactly k of the indices smaller than j are mapped by σ to indices larger than $\sigma(j)$, then $\sigma(j)-j+k$ of the ones larger than j have to be mapped to values smaller than $\sigma(j)$. Then

$$\mathrm{inv}(j) = k + \sigma(j) - j + k \equiv \sigma(j) - j \pmod 2. \qquad \square$$

Thus, $\mathrm{inv}(j)$ is odd if and only if j and $\sigma(j)$ differ in parity. Since the number of even j's is equal to the number of even $\sigma(j)$'s, this difference in parity appears equally often in each of the two forms: $(j,\sigma(j))=(\mathrm{odd},\mathrm{even})$ and $(j,\sigma(j))=(\mathrm{even},\mathrm{odd})$. In (8-9), pairs of the first kind lead to summands $+[-]$, while pairs of the second kind give $-[-]$. Thus, all the appearances of the symbol $[-]$ cancel. It follows that the sum adds up to $[+]$, which is cancelled by the extra term.

We have shown that the boundary relations are trivial. It follows that U is the quotient of $\mathbb{Z}^2$ by the alternation relations, ie $U\simeq\mathbb{Z}$. The PG–structure description follows from the formula

$$\mathrm{sgn}(gp) = \mathrm{sgn}(\det g)\cdot\mathrm{sgn}(p),$$

valid for $g\in G$ and all generic $(n+1)$–tuples p of points in $\mathbb{P}$.

Calculation of U_+ A generic n–tuple of points in $\mathbb{P}_+$ can be lifted to n linearly independent vectors in K^n. The matrix M with columns given by these vectors is well defined up to multiplication on the right by diagonal matrices with positive entries on the diagonal. The sign of $\det M$ is thus an invariant of the tuple (the *sign of the tuple*); it is also a G_+–invariant of the tuple ($\det gM = \det g\cdot\det M = \det M$

for $g \in G_+$). In fact, this sign is the full G_+–invariant: the tuple of vectors is transformed to the standard basis by M^{-1} if $\det M > 0$, and to the basis $(e_1, \dots, e_{n-1}, -e_n)$ by M^{-1} with negated lowest row if $\det M < 0$. We have shown the following statement.

Lemma 8.4 *The action of G_+ on the set of generic n–tuples of points in $\mathbb{P}_+$ has two orbits, detected by the sign of the tuple.*

We now consider the G_+–action on the set of generic $(n+1)$–tuples. The symbol of such a tuple $(p_1, \dots, p_{n+1})$ is defined to be a sequence of $n+1$ signs, $[s; s_1, \dots, s_n]$. Here s is the sign of $(p_1, \dots, p_n)$. To get the other signs, we first lift each p_i to a vector v_i. Then we express v_{n+1} in terms of the other v_i, $v_{n+1} = \sum_{i=1}^{n} a_n v_n$. Finally, $s_i = \operatorname{sgn} a_i$ (genericity implies $a_i \neq 0$). Clearly, the symbol is G_+–invariant.

Lemma 8.5 *Two generic $(n+1)$–tuples of points in $\mathbb{P}_+$ are G_+–equivalent if and only if they have the same symbol.*

Proof Let the tuples be (p_i) and (q_i), with symbol $[s; s_1, \dots, s_n]$. Then a lift of (p_i) is equivalent to

$$\left(e_1, \dots, e_{n-1}, se_n, \sum_{i=1}^{n} a_i e_i\right)$$

for some $a_i \in \dot{K}$, while a lift of (q_i) is equivalent to

$$\left(e_1, \dots, e_{n-1}, se_n, \sum_{i=1}^{n} b_i e_i\right)$$

for some $b_i \in \dot{K}$; moreover, s_i is the (common) sign of a_i and of b_i (for $i < n$; and s_n is the common sign of sa_n and sb_n). These two representing tuples of vectors are projectively related by the diagonal matrix with positive diagonal entries b_i/a_i. □

The following observation describes the G–action on symbols and allows us to determine the P_+G–structure on U_+.

Fact 8.6 *The symbol of a generic $(n+1)$–tuple is G–equivariant: if $g \in G$ and $\det g < 0$, then the tuples (p_i) and (gp_i) have the opposite leading symbol sign s, and coinciding remaining symbol signs.*

It follows from Lemma 8.5 that the group U_+ is the quotient of the free abelian group spanned by symbols by alternation and boundary relations. We first deal with the alternation relations.

Alternation relations The symbol $[s; s_1, \dots, s_n]$ is represented by the tuple

$$\left(e_1, \dots, se_n, \sum_{i=1}^{n-1} s_i e_i + ss_n e_n\right).$$

Suppose that this tuple is permuted; what happens to the symbol? Since permutation commutes with "linear map applied to each element", we may and will assume $s = +1$ — in our arguments, but not in the final statements. We first treat the case of a permutation σ that fixes the last element. Then the symbol of the permuted tuple is $[\operatorname{sgn}\sigma; (s_{\sigma^{-1}(i)})]$. We get the "usual permutation relation"

$$[s; (s_i)] = \operatorname{sgn}\sigma[s \operatorname{sgn}\sigma; (s_{\sigma^{-1}(i)})].$$

Now let us consider the transposition of k and $n+1$. The new leading sign is

$$\det\left(e_1, \dots, e_{k-1}, \sum_{i=1}^{n} s_i e_i, e_{k+1}, \dots, e_n\right) = s_k.$$

We also have

$$e_k = s_k \sum_{i=1}^{n} s_i e_i + \sum_{i \neq k} (-s_k s_i) e_i,$$

so that the total symbol after transposition is

$$[s_k; -s_k s_1, \dots, -s_k s_{k-1}, s_k, -s_k s_{k+1}, \dots, -s_k s_n].$$

The "transposition relation" is thus

$$[s; (s_i)] = -[s s_k; -s_k s_1, \dots, -s_k s_{k-1}, s_k, -s_k s_{k+1}, \dots, -s_k s_n].$$

In words: If the k^{th} sign is $+$, then we can flip all the other signs (except the leading sign); the resulting symbol will be equal to minus the original. If the k^{th} sign is $-$, we get $[s; (s_i)] = -[-s; (s_i)]$.

There is a difference between the cases $n = 2$ and $n > 2$. In the latter case, for any sequence of n signs there exists a stabilizing transposition; therefore, any sequence of n signs can be ordered (put in the form $++\cdots--$) by an even permutation. Let us begin with the case $n > 2$.

The case $n > 2$ As already mentioned, in this case one can use the usual permutation relation to order the nonleading signs of a symbol without changing the leading sign. To shorten the notation, we will use a^+ for $[+; ++\cdots--]$ (a plus signs after the semicolon), and a^- for $[-; ++\cdots--]$ (a plus signs after the semicolon). For example, when $n = 3$, we put $0^+ = [+; ---]$, $2^+ = [+; ++-]$ and $2^- = [-; ++-]$. The transposition relation (with $s_k = +1$) gives $a^\pm = -(n-a+1)^\pm$ (for $a > 0$). Picking $s_k = -1$ in the transposition relation we get $a^+ = -a^-$ — for $a < n$, but $n^+ = -n^-$ also holds, due to $n^+ = -1^+ = 1^- = -n^-$. To summarize:

Lemma 8.7 *Let $n > 2$. Let $A = \{a^+ \mid 0 \le a \le \lfloor n/2 \rfloor\}$. The quotient of the group $(C_n^o X_+)_{P_+ G_+}$ by the set of alternation relations is the free abelian group with generating set A for n even; for n odd it is the direct sum of the free abelian group generated by A and a $\mathbb{Z}/2$ generated by $\left(\frac{n+1}{2}\right)^+$.*

(The extra $\mathbb{Z}/2$–summand appearing for n odd will eventually get killed by the boundary relations.)

The case $n = 2$ There are eight symbols. The usual permutation relation for $\sigma = (12)$ gives

(8-10) $[+; ++] = -[-; ++]$, $[-; -+] = -[+; +-]$, $[-; +-] = -[+; -+]$, $[+; --] = -[-; --]$.

The transposition relation (for $\sigma=(23)$) is $[s;s_1,s_2]=-[ss_2;-s_2s_1,s_2]$. This gives

(8-11) $-[-;++]=[-;-+],\quad -[+;+-]=[-;+-],\quad -[+;-+]=[+;++],\quad [+;--]=-[-;--].$

We see that all the (signed) symbols appearing in the first three equalities of (8-10) and (8-11) are identified. In particular, $[+,+-]=[+;-+]$, so the $a^{\pm}$ notation still makes sense. Also, the relations $a^++a^-=0$ and $1^++2^+=0$ can be read off from the ones displayed above. Therefore the conclusion of Lemma 8.7 holds for $n=2$.

Boundary relations We will show that they all follow from the alternation relations (with the exception of $\left(\frac{n+1}{2}\right)^+=0$). Let us calculate the boundary of an $(n+1)$–simplex of X_+ represented by a generic $(n+2)$–tuple of vectors. Such a tuple of vectors can be transformed by an element of G_+ to

$$\Delta=\left(e_1,\dots,e_{n-1},se_n,\sum_{i=1}^n s_ie_i,\sum_{i=1}^n s_ib_ie_i\right).$$

The genericity condition (assuming $s=+1$) is that all b_i nonzero and pairwise different. (This will follow from the calculation of $\partial\Delta$.) If $s=-1$, we can transform the tuple by an orientation changing linear map; this will change all leading signs in $\partial\Delta$, and not touch the other signs. Thus, we will assume $s=+1$ — and then double the set of the resulting boundaries by changing the leading signs. If we omit e_j $(i\le n)$ from Δ, then the sign of the determinant of the standardizing matrix is the same as that of

$$\det\left(e_1,\dots,e_{j-1},e_{j+1},\dots,e_n,\sum_{i=1}^n s_ie_i\right)=(-1)^{n-j}s_j.$$

The other signs can be read off from

$$\sum_{i=1}^n s_ib_ie_i=\sum_{i\ne j}(s_ib_i-s_ib_j)e_i+b_j\sum_{i=1}^n s_ie_i.$$

The total symbol (for j omitted) is thus $[(-1)^{n-j}s_j;(s_i\operatorname{sgn}(b_i-b_j))_{i\ne j},\operatorname{sgn}b_j]$.

Omitting the $(n+1)^{\text{st}}$ element gives $[+1;(s_i\operatorname{sgn}b_i)]$.

Omitting the $(n+2)^{\text{nd}}$ element yields $[+1;(s_i)]$. So, finally,

$$\partial\Delta=\sum_{j=1}^n(-1)^{j-1}[(-1)^{n-j}s_j;(s_i\operatorname{sgn}(b_i-b_j))_{i\ne j},\operatorname{sgn}b_j]+(-1)^n[+1;(s_i\operatorname{sgn}b_i)]+(-1)^{n+1}[+1;(s_i)].$$

Let us rewrite this boundary relation. We put (artificially) $b_{n+1}=0$ and $s_{n+1}=-1$. Then we have $\operatorname{sgn}b_j=s_{n+1}\operatorname{sgn}(b_{n+1}-b_j)$, $s_i\operatorname{sgn}b_i=s_i\operatorname{sgn}(b_i-b_{n+1})$ and $(-1)^{n-(n+1)}s_{n+1}=+1$, so

$$\begin{aligned}\partial\Delta&=\sum_{j=1}^{n+1}(-1)^{j-1}[(-1)^{n-j}s_j;(s_i\operatorname{sgn}(b_i-b_j))_{i\ne j}]+(-1)^{n+1}[+1;(s_i)]\\&=\sum_{j=1}^{n+1}(-1)^{n-1}[s_j;(s_i\operatorname{sgn}(b_i-b_j))_{i\ne j}]+(-1)^{n+1}[+1;(s_i)],\end{aligned}$$

where the last equality uses $a^+ = -a^-$. Thus, we need to deduce from our alternation relations (assuming $s_{n+1} = -1$ and $b_{n+1} = 0$) that

$$\sum_{j=1}^{n+1} [s_j; (s_i \operatorname{sgn}(b_i - b_j))_{i \neq j}] + [+1; (s_i)_{i \neq n+1}] = 0.$$

We see that we may change the order of summation to follow the order of the b_i. Indeed, the above summation can be phrased in an index-free way as follows. We have a set of $n+1$ numbers, each with an attached sign (one of these pairs being 0 with $-$). For each element x of the set we form the corresponding symbol k^s, where s is the sign attached to the element x, and k is the number of positive expressions of the form $t(y-x)$, where y runs through our set (and $y \neq x$) and t is the sign attached to y. Finally, there is an extra summand ℓ^+ with ℓ counting all the positive signs.

We may thus renumber the b_i and the s_i (in the same way) so as to have $b_1 > b_2 > \cdots > b_n > b_{n+1}$, with an unknown b equal to zero and the corresponding s equal to -1 and omitted in the extra summand $[+1; (s_i)]$. Let us now consider two consecutive summands (numbered j and $j+1$). They differ at most by the leading sign, s_j versus s_{j+1}, and by the j^{th} nonleading sign, $s_{j+1} \operatorname{sgn}(b_{j+1} - b_j) = -s_{j+1}$ versus $s_j \operatorname{sgn}(b_j - b_{j+1}) = s_j$. Substituting all four possible combinations of (s_j, s_{j+1}) we get:

Claim *Two consecutive summands are of one of the forms*

$$(a^\pm, a^\mp), \quad (a^+, (a+1)^+), \quad (a^-, (a-1)^-).$$

Suppose that k of the s_i are positive. Then the extra summand is k^+, while the first and the last one depend on (s_1, s_{n+1}) and are:

s_1	s_{n+1}	first	last
$+$	$+$	$(n-k+1)^+$	$(k-1)^+$
$+$	$-$	$(n-k+1)^+$	k^-
$-$	$+$	$(n-k)^-$	$(k-1)^+$
$-$	$-$	$(n-k)^-$	k^-

We can append the extra summand k^+ to the sum (while keeping the rule of the claim) and get summation starting from $(n-k+1)^+$ or $(n-k)^-$ and ending at k^+. Then we start cancelling consecutive pairs $(a^\pm, a^\mp)$ (except that we do not cancel the first and the last element) until the sequence becomes monotone (possibly except the first or the last pair). If the final monotone sequence runs from $(n-k+1)^+$ to k^+ then the terms pairwise cancel (first with last, second with last-but-one, etc) if n is even, and $\frac{1}{2}(n+1)$ is left if n is odd. If the sequence starts with $(n-k)^-$, we may put an extra pair $((n-k+1)^+, (n-k+1)^-)$ at the beginning of the sequence, to reduce to the former case — except when $k=0$. If $k=0$, we get a sequence running from n^- to 0^+, ie $(n^-, (n-1)^-, \dots, 1^-, 0^-, 0^+)$. The first n terms cancel in the same manner as before, and $0^- + 0^+ = 0$.

Finally, since the set of permutation relations is invariant under the "exponent sign" flip ($a^\pm \leftrightarrow a^\mp$), the boundary relations obtained from Δ with $s = -1$ are dealt with in the same way.

Fact 8.6 and the relation $a^- = -a^+$ imply the remaining claim (the one describing the P_+G–structure on U_+), completing the proof of Theorem 8.1. □

Remark 8.8 Let T be the tautological (U–valued) n–cocycle associated to the PG_+–action on X, and let T_+ be the tautological (U_+–valued) n–cocycle associated to the P_+G_+–action on X_+. From the proof of Theorem 8.1 it is useful to extract the following explicit description of these cocycles.

(a) Let n be even; then $U \simeq \mathbb{Z}$ is generated by the symbol $[+]$. Suppose that $\sigma = ([v_1], \dots, [v_{n+1}])$ is an n–simplex of X. Then $v_{n+1} = \sum_{i=1}^n \alpha_i v_i$ for some $\alpha_i \in \dot{K}$. We have (see (8-4))

$$T(\sigma) = [\operatorname{sgn}(\det(v_1, \dots, v_n) \cdot \alpha_1 \cdots \alpha_n)].$$

(Recall that $[-] = -[+]$.)

(b) Recall that $U_+ \simeq \mathbb{Z}^{\lfloor n/2 \rfloor + 1}$ with free generating set $A = \{a^+ \mid a = 0, \dots, \lfloor n/2 \rfloor\}$ (see Lemma 8.7). Suppose that $\sigma = ([v_1], \dots, [v_{n+1}])$ is an n–simplex of X_+. Then $v_{n+1} = \sum_{i=1}^n \alpha_i v_i$ for some $\alpha_i \in \dot{K}$. To σ we assign an $(n+1)$–tuple of signs $[s; s_1, \dots, s_n]$, where $s = \operatorname{sgn}\det(v_1, \dots, v_n)$ and $s_i = \operatorname{sgn}\alpha_i$. Next we put $T_+(\sigma) = a^+$ (if $s = +1$ and a of the s_i are $+1$) or $T_+(\sigma) = a^-$ (if $s = -1$ and a of the s_i are $+1$). Finally, we express the symbol in term of the elements of A using the relations $a^+ = -a^-$ and $a^\pm = -(n+1-a)^\pm$ (for $a > 0$).

Definition 8.9 The splitting of $U_+ = \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \mathbb{Z}k^+$ into cyclic summands generated by the elements k^+ ($0 \le k \le \lfloor n/2 \rfloor$) induces the corresponding splittings of cocycles and cohomology classes:

$$T_+ = \bigoplus T_k, \qquad T_k \in Z^n(\operatorname{Hom}_{P_+G_+}(C_*X_+, \mathbb{Z}));$$
$$\mathrm{eu}_+ = \bigoplus \mathrm{eu}_k, \qquad \mathrm{eu}_k \in H^n(P_+G_+, \mathbb{Z});$$
$$\tau_+ = \bigoplus \tau_k, \qquad \tau_k \in H^n(\operatorname{Hom}_{P_+G_+}(C_*X_+, \mathbb{Z})).$$

(In the last formula, τ_+ (τ_k) is the cohomology class of T_+ (T_k).)

Remark 8.10 Suppose that $K < L$ is a field extension, and that on K and on L there are compatible field orders. Then we have the group embedding $\phi\colon P_+G_+(K) \to P_+G_+(L)$, and the natural ϕ–equivariant simplicial complex embedding $X_+(K) \to X_+(L)$ inducing a coefficient group map $f: U_+(K) \to U_+(L)$. But, in our field-independent description of U_+ (see Theorem 8.1 and Remark 8.8) the map f is represented by the identity. Applying Theorem 1.5 to these data we obtain $\phi^*\mathrm{eu}_+(L) = \mathrm{eu}_+(K)$ — the Euler class eu_+ is stable under ordered field restriction. It follows that all eu_k are also stable. Analogous arguments show the same stability statement for eu.

Remark 8.11 It follows from Theorem 3.1 that the classes eu, eu_+ and eu_k are bounded.

9 Relation between the classes eu_k

The classes eu_k defined in Definition 8.9 are related.

Theorem 9.1
$$\sum_{k=0}^{\lfloor n/2\rfloor}(n-2k+1)\mathrm{eu}_k=0.$$

Proof We will see that this relation holds already in $H^n(\mathrm{Hom}_{P_+G_+}(C_*X_+,\mathbb{Z}))$ for the classes τ_k. To prove it, we will find a cochain $c\in C^{n-1}(\mathrm{Hom}_{P_+G_+}(C_*X_+,\mathbb{Z}))$ such that

$$\delta c=\sum_{k=0}^{\lfloor n/2\rfloor}(n-2k+1)T_k \tag{9-1}$$

in $C^n(\mathrm{Hom}_{P_+G_+}(C_*X_+,\mathbb{Z}))$. The boundary map

$$\partial\colon (C_nX_+)_{P_+G_+}\to(C_{n-1}X_+)_{P_+G_+}$$

factors as the composition of the projection $(C_nX_+)_{P_+G_+}\to(C_nX_+/B_nX_+)_{P_+G_+}=U_+$ and a map $\partial'\colon U_+\to(C_{n-1}X_+)_{P_+G_+}$. Each T_k also factors — as the composition of the same projection and the projection T'_k of U_+ on the k^+–summand. Recall that $(C_{n-1}X_+)_{P_+G_+}\cong\mathbb{Z}$ (with generator $[+]$; see Lemma 8.4). We now consider a generator a^+ of U_+ and determine $\partial'(a^+)$. Let

$$v_a=e_1+\cdots+e_a-(e_{a+1}+\cdots+e_n);$$

then $(e_1,\dots,e_n,v_a)$ determines a simplex in X_+ representing a^+. We have

$$\begin{aligned}\partial'(a^+)&=\partial[e_1,\dots,e_n,v_a]\\&=\sum_{j=1}^{n}(-1)^{j+1}[e_1,\dots,\hat e_j,\dots,e_n,v_a]+(-1)^n[e_1,\dots,e_n]\\&=\sum_{j=1}^{n}(-1)^{j+1}(-1)^{n-j}[e_1,\dots,v_a,\dots,e_n]+(-1)^n[+]\\&=\sum_{j=1}^{a}(-1)^{n+1}[+]+\sum_{j=a+1}^{n}(-1)^{n+1}[-]+(-1)^n[+]\\&=(-1)^n((1-a)[+]-(n-a)[-])=(-1)^n(n-2a+1)[+].\end{aligned}$$

Let $c\in C^{n-1}(\mathrm{Hom}_{P_+G_+}(C_*X_+,\mathbb{Z}))=\mathrm{Hom}((C_{n-1}X_+)_{P_+G_+},\mathbb{Z})$ be defined by $c([+])=(-1)^n$. Then $(c\circ\partial')(a^+)=c(\partial'a^+)=(n-2a+1)=\sum_{k=0}^{\lfloor n/2\rfloor}(n-2k+1)T'_k(a^+)$ holds for each a^+. Formula (9-1) follows. □

10 The Smillie argument

The Smillie argument (see [Gromov 1982, Section 1.3]) can be used to show that the classes eu_k are proportional in a weak sense.

Theorem 10.1 *For any $h \in H_n(BP_+G_+, \mathbb{Z})$ (or $h \in H_n(BP_+G_+, \mathbb{Z}/m)$ for m odd) and any $k \le \lfloor n/2 \rfloor$,*

$$\langle \mathrm{eu}_k, h\rangle = (-1)^k \binom{n+1}{k} \langle \mathrm{eu}_0, h\rangle.$$

If n is odd, then $\langle \mathrm{eu}_k, h\rangle = 0$ for all k.

Proof It is well known that there exists a finite simplicial complex Y, a simplicial cycle $Z \in Z_n(Y, \mathbb{Z})$ (or in $Z_n(Y, \mathbb{Z}/m)$), and a map $f : Y \to BG$, such that $f_*[Z] = h$. Let $P = f^*EP_+G_+$ (the pullback of the universal bundle over BP_+G_+). Then

$$\langle \mathrm{eu}_k, h\rangle = \langle \mathrm{eu}_k, f_*[Z]\rangle = \langle f^*\mathrm{eu}_k, [Z]\rangle = \langle \mathrm{eu}_k(P), [Z]\rangle$$

for each k. We will use Theorem 4.4 to compute $\langle \mathrm{eu}_k(P), [Z]\rangle$. Let $E = P \times_{P_+G_+} \mathbb{P}_+$ be the associated bundle.

Pick a generic section $s : Y^{(0)} \to E$. Genericity means that the values of the section at the vertices of any n–simplex of Y, viewed as points in $\mathbb{P}_+$ via a flat trivialization of E over that simplex, form a generic tuple of points. Such a section can be picked vertex-by-vertex. At a vertex y the genericity conditions mean that a certain finite union of proper projective subspaces of E_y is prohibited; since the ordered field K is infinite, that union does not fill out E_y and a generic choice is possible. Any generic section s determines a simplicial section of the associated X_+–bundle over Y, and then Theorem 4.4 may be applied.

For any function $\epsilon : Y^{(0)} \to \{\pm 1\}$ we can form a modified section $\epsilon s : Y^{(0)} \to E$ defined in the obvious way: if $s(v) = (p, [q])$ (for some $q \in K^n \setminus \{0\}$), then $(\epsilon s)(v) = (p, [\epsilon(v)q])$. Every section ϵs is again generic. Theorem 4.4 gives $\mathrm{eu}_+(P) = [(\epsilon s)^* T_+]$, and coefficient splitting allows us to deduce $\mathrm{eu}_k(P) = [(\epsilon s)^* T_k]$; both formulae hold for all functions ϵ. For a given n–simplex σ of X_+ we will average the expression $\langle (\epsilon s)^* T_k, \sigma\rangle$ over all possible functions ϵ.

Let $\sigma = (v_1, \dots, v_n, v_{n+1})$ be one of the n–simplices of Y. Let us choose a flat section r of P over σ, and let $s(v_i) = [r, [q_i]]$, for $q_i \in K^n \setminus \{0\}$. We denote by $s_*\sigma$ the n–simplex of X_+ given by $([q_1], \dots, [q_n], [q_{n+1}])$. This definition depends on the choice of r, but different choices lead to simplices equivalent under the P_+G_+–action. The expression $\langle T_k, s_*\sigma\rangle$ is well defined and equal to $\langle s^*T_k, \sigma\rangle$. Let $\eta = \operatorname{sgn}\det(q_1, \dots, q_n)$, and let $q_{n+1} = \sum_{i \le n} a_i q_i$. Suppose that exactly ℓ of the coefficients a_i are positive — so that the symbol of $s_*\sigma$ is ℓ^η.

For any function ϵ we have $(\epsilon s)_*\sigma = ([\epsilon_1 q_1], \dots, [\epsilon_n q_n], [\epsilon_{n+1} q_{n+1}])$, where $\epsilon_i = \epsilon(v_i)$. We wish to determine all functions ϵ such that $\langle T_k, (\epsilon s)_*\sigma\rangle \ne 0$. This will happen if and only if the decomposition $\epsilon_{n+1} q_{n+1} = \sum_{i \le n} b_i \epsilon_i q_i$ has either k or $n+1-k$ positive coefficients b_i.

Let us first focus on the case of $\epsilon_{n+1} = +1$ and k positive b_i's. We will represent the appropriate functions ϵ in the form $\epsilon'\epsilon''$; the idea is that ϵ' makes all the nonleading signs negative, while ϵ'' changes k of them to $+$. In more detail: $\epsilon'_{n+1} = +1$ and $\epsilon'_i = -\operatorname{sgn} a_i$ for $i \le n$, while ϵ'' is arbitrary with k negative and $n-k$ positive values (plus $\epsilon''_{n+1} = +1$). For such ϵ' and ϵ'' the symbol of $(\epsilon'\epsilon'' s)_*\sigma$ is $k^{\pm}$, where the exponent is $\prod_{i\le n}\epsilon'_i \prod_{i\le n}\epsilon''_i \cdot \operatorname{sgn}\det(q_1,\dots,q_n) = (-1)^{\ell}(-1)^k\eta$. There are $\binom{n}{k}$ appropriate functions ϵ.

For $\epsilon_{n+1} = +1$ and $n+1-k$ positive b_i's we get $\binom{n}{n+1-k} = \binom{n}{k-1}$ possibilities, yielding $(n+1-k)^{\pm}$, with the exponent equal to $(-1)^{\ell}(-1)^{n+1-k}\eta = -(-1)^n(-1)^{\ell}(-1)^k\eta$.

If $\epsilon_{n+1} = -1$ the analysis is analogous. The difference is that ϵ' should now be $\epsilon'_i = \operatorname{sgn} a_i$; therefore, the only change is $(-1)^{n-\ell}$ instead of $(-1)^{\ell}$ in the final exponent sign formulae.

Putting these together, we get (with $N = \#Y^{(0)}$)

$$\begin{aligned}
\Bigl\langle T_k, \sum_{\epsilon}(\epsilon s)_*\sigma\Bigr\rangle &= \Bigl\langle T_k, 2^{N-(n+1)}\Bigl(\binom{n}{k}k^{(-1)^{\ell}(-1)^k\eta} + \binom{n}{k-1}(n+1-k)^{-(-1)^n(-1)^{\ell}(-1)^k\eta} \\
&\qquad + \binom{n}{k}k^{(-1)^{n-\ell}(-1)^k\eta} + \binom{n}{k-1}(n+1-k)^{-(-1)^n(-1)^{n-\ell}(-1)^k\eta}\Bigr)\Bigr\rangle \\
&= 2^{N-(n+1)}\Bigl\langle T_k, \Bigl(\binom{n}{k}((-1)^{\ell}(-1)^k\eta + (-1)^{n-\ell}(-1)^k\eta)k^{+} \\
&\qquad - \binom{n}{k-1}((-1)^n(-1)^{\ell}(-1)^k\eta + (-1)^n(-1)^{n-\ell}(-1)^k\eta)(n+1-k)^{+}\Bigr)\Bigr\rangle \\
&= 2^{N-(n+1)}\Bigl\langle T_k, \Bigl(\binom{n}{k}(1+(-1)^n)(-1)^{\ell}(-1)^k\eta k^{+} \\
&\qquad - \binom{n}{k-1}(1+(-1)^n)(-1)^n(-1)^{\ell}(-1)^k\eta(n+1-k)^{+}\Bigr)\Bigr\rangle.
\end{aligned}$$

For n odd, this is zero. Thus, we assume n even; then the coefficients in the above expression add up to

$$2^{N-n}\Bigl(\binom{n}{k} + \binom{n}{k-1}\Bigr)(-1)^k(-1)^{\ell}\eta = 2^{N-n}\binom{n+1}{k}(-1)^k(-1)^{\ell}\eta.$$

Similarly, if $n+1-\ell$ of the coefficients a_i are positive, we get

$$2^{N-n}\binom{n+1}{k}(-1)^k(-1)^{n+1-\ell}\eta.$$

In both cases, the result can be expressed as

$$2^{N-n}\binom{n+1}{k}(-1)^k\langle(-1)^{\ell}T_{\ell}, s_*\sigma\rangle.$$

Since on any $s_*\sigma$ exactly one of T_{ℓ} is nonzero, we can rewrite this formula as

$$2^{N-n}\binom{n+1}{k}(-1)^k\sum_{\ell}\langle(-1)^{\ell}T_{\ell}, s_*\sigma\rangle,$$

with summation over $\ell \le n/2$. Let us summarize:

$$\left\langle \sum_{\epsilon} (\epsilon s)^* T_k, \sigma \right\rangle = 2^{N-n} \binom{n+1}{k} (-1)^k \sum_{\ell} \langle (-1)^{\ell} s^* T_{\ell}, \sigma \rangle.$$

It follows that

$$\sum_{\epsilon} (\epsilon s)^* T_k = 2^{N-n} \binom{n+1}{k} (-1)^k \sum_{\ell} (-1)^{\ell} s^* T_{\ell}.$$

Now recall that, by Theorem 4.4, each $(\epsilon s)^* T_k$ is a cocycle representing the cohomology class $\mathrm{eu}_k(P)$. Therefore

$$2^N \mathrm{eu}_k(P) = 2^{N-n} \binom{n+1}{k} (-1)^k \sum_{\ell} (-1)^{\ell} \mathrm{eu}_{\ell}(P).$$

Comparing this formula for $k = 0$ and for any other value of k we get the following lemma (which may be regarded as a variant of Theorem 10.1).

Lemma 10.2 *Let P be a flat principal P_+G_+–bundle over a finite simplicial complex Y that has N vertices. Then*

$$2^N \mathrm{eu}_k(P) = 2^N (-1)^k \binom{n+1}{k} \mathrm{eu}_0(P).$$

Evaluating both sides of the formula from the lemma on $[Z]$ concludes the proof of Theorem 10.1. □

Corollary 10.3 *Let P be a $\mathrm{P}_+\mathrm{GL}_+(n, K)$–bundle over an even-dimensional manifold M^n. Then any triangulation of M has at least $2^n |\langle \mathrm{eu}_0(P), [M] \rangle|$ simplices of dimension n.*

Proof Pick a generic section s, over the given triangulation, of the associated bundle with fibre $\mathbb{P}_+$. Then, by Theorem 10.1, $|\langle s^* T_k, [M] \rangle| = |\langle \mathrm{eu}_k(P), [M] \rangle| = \binom{n+1}{k} |\langle \mathrm{eu}_0(P), [M] \rangle|$. Since $|\langle s^* T_k, \sigma \rangle| \le 1$ and the supports of the cocycles $s^* T_k$ are pairwise disjoint, the number of n–simplices of the triangulation is at least

$$\sum_{k=0}^{\lfloor n/2 \rfloor} |\langle s^* T_k, [M] \rangle| = |\langle \mathrm{eu}_0(P), [M] \rangle| \cdot \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{k} = 2^n |\langle \mathrm{eu}_0(P), [M] \rangle|.$$ □

11 Cross product of Euler classes

It will be convenient to put $[n] = \{0, 1, \dots, n\}$. We will use groups $\mathrm{GL}_+(n, K)$ for varying n; therefore we denote U_+ by $U_{n,+}$ in this and the next section.

Theorem 11.1 *Let E and E' be $\mathrm{GL}_+(n, K)$– and $\mathrm{GL}_+(k, K)$–bundles over simplicial complexes X and X' respectively. Let $E \times E'$ be the product bundle over $X \times X'$. For any simplicial cycles $z \in Z_n(X, \mathbb{Z})$ and $z' \in Z_k(X', \mathbb{Z})$,*

$$\langle \mathrm{eu}_0(E), z \rangle \cdot \langle \mathrm{eu}_0(E'), z' \rangle = \langle \mathrm{eu}_0(E \times E'), z \times z' \rangle. \tag{11-1}$$

Proof We first explain the general strategy of the proof. We may and do assume that $X = \operatorname{supp} z$ and $X' = \operatorname{supp} z'$. We triangulate $X \times X'$ subdividing each product of simplices $\sigma \times \sigma'$ in a standard way (to be recalled later). It is also convenient to treat E, E' and $E \times E'$ not as principal bundles, but as vector bundles; eg the fibre E_x will be an n–dimensional vector space over K. We pick generic sections s of E and s' of E', and combine them to a section S of $E \times E'$. To ensure genericity of S we impose stronger than usual, weird-looking genericity conditions on s and on s'. The section s induces a simplicial section s_+ of the associated $X_{n,+}$–bundle, where $X_{n,+}$ is the complex of generic tuples of points in $\mathbb{P}^{n-1}_+(K)$. Then, by Theorem 4.4, we get cocycles $s_+^* T_+$ and $s_+^* T_0$ representing $\mathrm{eu}_+(E)$ and $\mathrm{eu}_0(E)$. For each n–simplex σ in X we have $\langle s_+^* T_+, \sigma\rangle = k^{\pm}$ for some k; if $k = 0$ then $\langle s_+^* T_0, \sigma\rangle = \pm 1$, otherwise $\langle s_+^* T_0, \sigma\rangle = 0$. Similarly, we have cocycles $s_+'^* T_0$ and $S_+^* T_0$ representing $\mathrm{eu}_0(E')$ and $\mathrm{eu}_0(E \times E')$. Suppose that $z = \sum_\sigma n_\sigma \sigma$ and $z = \sum_{\sigma'} n_{\sigma'} \sigma'$; then $z \times z' = \sum_{\sigma,\sigma'} n_\sigma n_{\sigma'} \cdot \sigma \times \sigma'$, where $\sigma \times \sigma'$ denotes the chain representing the standard subdivision of the product of simplices. We have

$$\langle \mathrm{eu}_0(E), z\rangle = \sum_\sigma n_\sigma \langle s_+^* T_0, \sigma\rangle, \quad \langle \mathrm{eu}_0(E'), z'\rangle = \sum_{\sigma'} n_{\sigma'} \langle s_+'^* T_0, \sigma'\rangle,$$

$$\langle \mathrm{eu}_0(E \times E'), z \times z'\rangle = \sum_{\sigma,\sigma'} n_\sigma n_{\sigma'} \langle S_+^* T_0, \sigma \times \sigma'\rangle.$$

Thus, to establish the theorem it is enough to show that

$$\langle s_+^* T_0, \sigma\rangle \cdot \langle s_+'^* T_0, \sigma'\rangle = \langle S_+^* T_0, \sigma \times \sigma'\rangle. \tag{11-2}$$

We do this step-by-step. In Corollary 11.4 we show that if the left-hand side of (11-2) is zero, then so is the right-hand side. In Corollary 11.5 we show that if the left-hand side is nonzero, then in the chain $\sigma \times \sigma'$ there is a unique summand (unique $(n+k)$–simplex) on which $S_+^* T_0$ evaluates to ± 1. Finally, in Lemma 11.6 we check that the sign of that evaluation is consistent with (11-2).

We proceed to the details. First we pick a generic section s of E over $X^{(0)}$. The genericity condition is as follows. For each ℓ–simplex $\sigma = (x_0, \dots, x_\ell)$ of X ($\ell < n$), the vectors $(s(x_0), \dots, s(x_\ell))$ are linearly independent. For each n–simplex $\sigma = (x_0, \dots, x_n)$, if $\sum_{i=0}^n \alpha_i s(x_i) = 0$ is a nontrivial linear relation (projectively unique, because of the previous condition), then $\sum_{i=0}^n \alpha_i \neq 0$. To make sense of these conditions we choose a (flat) trivialization of E over σ.

This kind of section can be chosen vertex-by-vertex. Let $X^{(0)} = (x_1, x_2, \dots, x_N)$. First, we choose any nonzero $s(x_1) \in E_{x_1}$. When $s(x_1), \dots, s(x_{i-1})$ have been chosen, we choose (flat) trivializations over all simplices with vertex x_i. If $\sigma = (x_i, y_1, \dots, y_\ell)$ is an ℓ–simplex of X ($\ell < n$) such that $s(y_1), \dots, s(y_\ell)$ have already been chosen, we use the trivialization of E over σ to transport all $s(y_j)$ to E_{x_i}. There, these vectors span a linear subspace E_i^σ of dimension $\ell < n$. We have to ensure $s(x_i) \notin E_i^\sigma$ in order to fulfill the first genericity condition for σ.

Next, for each simplex $\sigma = (x_i, y_1, \dots, y_n)$ with $s(y_1), \dots, s(y_n)$ already chosen we pick a (flat) trivialization of E over σ and use it to transport the $s(y_j)$ to vectors $s_j^\sigma \in E_{x_i}$. Then we form an affine subspace

$$E_i^\sigma = \{\alpha_1 s_1^\sigma + \dots + \alpha_n s_n^\sigma \mid \alpha_1 + \dots + \alpha_n = 1\}.$$

We have to choose $s(x_i)$ outside of this subspace in order to fulfill the second genericity condition for σ.

A linear space over an ordered (hence infinite) field is not a union of finitely many proper affine subspaces. Therefore, $s(x_i)$ can be suitably chosen. By induction, there exists a generic section s of E over $X^{(0)}$.

With the section s we associate a collection of scalars $\mathscr{A}$. For each n–simplex $\sigma = (x_0, \dots, x_n)$ of X let $\sum_{i=0}^{n} \alpha_i s(x_i) = 0$ be the linear relation with $\sum_{i=0}^{n} \alpha_i = 1$ (in some trivialization of E over σ). For every proper nonempty subset of $[n]$ we sum the corresponding α_i's. The set $\mathscr{A}$ is the collection of all such sums (over all n–simplices).

Now, analogously, we choose a generic section s' of E' over $X'^{(0)}$. It has its own collection of scalars $\mathscr{A}'$. We want $\mathscr{A}$ and $\mathscr{A}'$ to be disjoint. To this end, we perform the above section-choosing procedure for E' with supplementary restrictions. Suppose that we are at step i, choosing $s'(x'_i)$. There is a collection of proper affine subspaces in $E'_{x'_i}$ that we need to avoid; we now describe an additional finite collection, that will enforce our extra "joint genericity" condition. Let $\sigma' = (x'_i, y'_1, \dots, y'_k)$ be a k–simplex of X', such that $s'(y'_j)$ are already chosen, and let $s_j^{\sigma'}$ be $s'(y'_j)$ transported to $E'_{x'_i}$ via a chosen trivialization of E' over σ'. For any generic (in the previous sense) $s'_i = s'(x'_i)$ there is a unique relation $\beta_0 s'_i + \sum_{j=1}^{k} \beta_j s_j^{\sigma'} = 0$ satisfying $\sum_{j=0}^{k} \beta_j = 1$. Pick an $\alpha \in \mathscr{A}$ and a proper nonempty $J \subset [k]$; we want to ensure that $\sum_{j \in J} \beta_j \neq \alpha$. Let us express this as a restriction for the possible position of s'_i. Suppose that $s'_i = \sum_{j=1}^{k} \gamma_j s_j^{\sigma'}$, and that $\sum_{j \in J} \beta_j = \alpha$. Let us express β_j in terms of the γ_j. By the original genericity requirement we know that $\Gamma := -1 + \sum_{j=1}^{k} \gamma_j \neq 0$; therefore

$$-\frac{1}{\Gamma} s'_j + \sum_{i=1}^{k} \frac{\gamma_i}{\Gamma} s_j^{\sigma'} = 0,$$

so $\beta_0 = -1/\Gamma$ and $\beta_j = \gamma_j/\Gamma$. Thus, the condition $\sum_{j \in J} \beta_j = \alpha$ can be rewritten in terms of the γ_j (putting $\gamma_0 = -1$): $\sum_{j \in J} \gamma_j/\Gamma = \alpha$, or $\sum_{j \in J} \gamma_j = \alpha\Gamma$, or finally

$$\sum_{j=0}^{k} (\alpha - \delta_J(j))\gamma_j = 0.$$

Since J is proper and nonempty, regardless of the value of α the set of vectors $\sum_{j=1}^{k} \gamma_j s_j^{\sigma'}$ for γ_j satisfying this condition forms a proper affine subspace of $E'_{x'_i}$. (The two special suspect cases, $J = \{0\}$ with $\alpha = 0$ and $J = \{1, \dots, k\}$ with $\alpha = 1$, are easily seen to be impossible.) Thus, the extra genericity conditions produce a new finite collection of proper affine subspaces to avoid, so that it is possible to fulfill them.

Assume then that we have chosen jointly generic (in the above sense) sections — s of E and s' of E'. We now form a generic section S of $E \times E'$ over $(X \times X')^{(0)}$ by $S(x, x') = (s(x), s'(x'))$. To claim genericity, we need to describe the (standard) triangulation of $X \times X'$. We choose some total orders on $X^{(0)}$ and on $X'^{(0)}$, and order each simplex of $X^{(n)}$ and of $X'^{(k)}$ accordingly. Let $\sigma = (x_0, \dots, x_n) \in X^{(n)}$, and let $\sigma' = (x'_0, \dots, x'_k) \in X'^{(k)}$. Let $x_{(i,j)} = (x_i, x'_j)$. We form the $n \times k$ integer grid — with vertex set $[n] \times [k]$ and edges connecting pairs that differ on exactly one coordinate and exactly by 1. Shortest paths from $(0, 0)$ to (n, k) will be called *admissible*. ("Shortest" is equivalent to "going right or up at each step".) For each admissible path $\gamma\colon [n+k] \to [n]\times[k]$ we span an $(n+k)$–simplex σ_γ in $\sigma\times\sigma'$ on the vertices $(x_{\gamma(j)} \mid j\in[n+k])$. It is well known that the set of all such σ_γ triangulates $\sigma\times\sigma'$ (see [Gelfand and Manin 2003, I.1.5]).

We will call an $(n+1)$–tuple of vectors in an n–dimensional vector space *linearly generic*, if every n of them are linearly independent.

Lemma 11.2 *Vectors $(v_0,\dots,v_n)$ are linearly generic if and only if there is a projectively unique linear relation $\sum_{i=0}^n \alpha_i v_i = 0$, and the coefficients in this relation are all nonzero.*

Proof ($\Leftarrow$) If some n of the v_i's were linearly dependent, a nontrivial linear relation between them could be extended — by adding 0 times the remaining vector — to a nontrivial relation with coefficient 0. This is a contradiction.

($\Rightarrow$) For dimensional reasons, there is a nontrivial linear relation between the v_i's; if some of its coefficients were 0, it would give linear dependence of a proper subset of the v_i's. If the relation was not projectively unique, one could form a linear combination of two nonproportional relations and obtain a nontrivial relation with coefficient 0. □

Observe that for a linearly generic tuple $(v_0,\dots,v_n)$, the class $[(v_0,\dots,v_n)]$ in $U_{n,+}$ is $0^{\pm}$ if and only if all the coefficients in the linear relation $\sum_{i=0}^n \alpha_i v_i = 0$ are of the same sign.

Now we will tackle the question of genericity of the section S (of $E\times E'$ over $(X\times X')^{(0)}$). Let $\sigma=(x_0,\dots,x_n)\in X^{(n)}$ and $\sigma'=(x'_0,\dots,x'_k)\in X'^{(k)}$. Using trivializations of E over σ and of E' over σ', we identify each E_{x_i} with the same vector space V ($\cong K^n$), and each $E'_{x'_j}$ with W ($\cong K^k$). Thus, we put $v_i = s(x_i)\in V$, $w_j = s'(x'_j)\in W$, $V_{(i,j)} = (v_i,w_j) = S(x_{(i,j)})\in V\oplus W$. We would like to show that for each admissible path γ the vectors $(V_{\gamma(j)} \mid j\in[n+k])$ are linearly generic. There are unique scalars α_i and β_j such that

$$\sum_{i=0}^n \alpha_i v_i = 0,\quad \sum_{i=0}^n \alpha_i = 1,\quad \sum_{j=0}^k \beta_j w_j = 0,\quad \sum_{j=0}^k \beta_j = 1.$$

Let $A_u=\sum_{i=0}^u \alpha_i$ and $B_s=\sum_{j=0}^s \beta_j$. For a given path γ, let us arrange these two sequences into one:

$$C_j = \begin{cases} A_i & \text{if } \gamma(j)=(i,*) \text{ and } \gamma(j+1)=(i+1,*),\\ B_i & \text{if } \gamma(j)=(*,i) \text{ and } \gamma(j+1)=(*,i+1).\end{cases}$$

We put $C_{n+k}=A_n=B_k=1$ and $C_{-1}=0$.

Lemma 11.3 *There is a projectively unique linear relation between the vectors $(V_{\gamma(j)} \mid j\in[n+k])$. If we require that the sum of coefficients be 1, this relation is*

$$\sum_{j=0}^{n+k}(C_j - C_{j-1})V_{\gamma(j)} = 0.$$

If all A_u and B_s are distinct, all coefficients of this relation are nonzero.

Proof First, let us prove the formula. Projecting onto the first factor we get

$$\sum_{i=0}^{n}\Big(\sum_{j\in\Gamma_i}(C_j-C_{j-1})\Big)v_i=0, \tag{11-3}$$

where $\Gamma_i=\{j\in[n+k]\mid\gamma(j)=(i,*)\}$ (similarly, we put $\Gamma^i=\{j\in[n+k]\mid\gamma(j)=(*,i)\}$). We have $\Gamma_i=\{u,u+1,\dots,u+\ell\}$ for some integers u, ℓ. Therefore

$$\sum_{j\in\Gamma_i}(C_j-C_{j-1})=C_{u+\ell}-C_{u-1}=A_i-A_{i-1}=\alpha_i.$$

Consequently, (11-3) becomes $\sum_{i=0}^{n}\alpha_i v_i=0$. Similarly, the projection onto W is 0. Thus, the relation stated in the lemma holds.

Suppose now that $\sum_{j=0}^{n+k}\Xi_j V_{\gamma(j)}=0$ is a linear relation with $\sum_{j=0}^{n+k}\Xi_j=1$. Projecting onto the first factor, we get $\sum_{i=0}^{n}\xi_i v_i=0$, where $\xi_i=\sum_{j\in\Gamma_i}\Xi_j$. Since $\sum_{i=0}^{n}\xi_i=\sum_{j=0}^{n+k}\Xi_j=1$ and the vectors v_i are linearly generic, we know that $\xi_i=\alpha_i$. Thus,

$$\sum_{j\in\Gamma_i}\Xi_j=\alpha_i.$$

Similarly,

$$\sum_{j\in\Gamma^i}\Xi_j=\beta_i.$$

Since each j is the largest element of exactly one set Γ_i or Γ^i, these equations recursively and uniquely determine all the Ξ_j.

Consequently, a linear relation between the $V_{\gamma(j)}$ with nonzero sum of coefficients is projectively unique. This implies that there is no nontrivial relation with sum of coefficients 0 — if it existed, it could be added to the one with sum of coefficients 1, contradicting the uniqueness of the latter.

The last claim of the lemma follows directly from the formula. □

Corollary 11.4 *Suppose that the class $[(v_0,\dots,v_n)]$ in $U_{n,+}$, or the class $[(w_0,\dots,w_k)]$ in $U_{k,+}$, is not $0^\pm$. Then, for every admissible path γ, the class $[(V_{\gamma(j)}\mid j\in[n+k])]$ in $U_{n+k,+}$ is not $0^\pm$.*

Proof The assumption can be interpreted as $\alpha_i<0$ for some i, or $\beta_j<0$ for some j. In each case one of the sequences (A_u) or (B_s) is not increasing; therefore, independent of γ, the sequence (C_j) is not increasing. Consequently, the relation between the $V_{\gamma(j)}$ (as in the lemma) cannot have all positive coefficients, while it does have some since their sum is 1. Hence the claim. □

Corollary 11.5 *Suppose that the class $[(v_0,\dots,v_n)]=0^\pm$ in $U_{n,+}$, and the class $[(w_0,\dots,w_k)]=0^\pm$ in $U_{k,+}$. Then there is a unique admissible path γ such that $[(V_{\gamma(j)}\mid j\in[n+k])]=0^\pm$ in $U_{n+k,+}$.*

Proof The sequences (A_u) and (B_s) are increasing. There is a unique γ such that (C_j) is increasing as well — then $[(V_{\gamma(j)}\mid j\in[n+k])]=0^\pm$. For other γ we conclude as in the previous corollary. □

Now we know that (11-2) holds up to sign. To finish the proof it remains to work out the relation between the signs that appear in the exponents in Corollary 11.5, and to check that this relation is consistent with (11-2). The cycle $z \times z'$ contains a triangulated version of the product $\sigma \times \sigma'$, for $\sigma \in \operatorname{supp} z$ and $\sigma' \in \operatorname{supp}(z')$, in the form $\sum_\gamma \operatorname{sgn}(\gamma)\sigma_\gamma$. The summation is over all admissible γ. The sign $\operatorname{sgn}(\gamma)$ equals $(-1)^{A(\gamma)}$, where $A(\gamma)$ is the area of the part of the grid that lies under the image of γ. In particular, if γ goes along the lower edge and the right-hand edge of the grid, the sign is $+1$. If we change γ by moving one $\gamma(j)$ to the opposite vertex of a 1×1 square — and get an admissible γ' — then $\operatorname{sgn}(\gamma') = -\operatorname{sgn}(\gamma)$.

Lemma 11.6 *Suppose that* $[(v_0, \dots, v_n)] = 0^s$ *and* $[(w_0, \dots, w_k)] = 0^{s'}$. *Then*

$$[(V_{\gamma(j)} \mid j \in [n+k])] = \operatorname{sgn}(\gamma) \cdot 0^{ss'}$$

for the γ *from the previous corollary.*

Proof We choose orientations of the bundles E and E'; we get induced orientations of V, W and $V \oplus W$. With respect to some positively oriented bases of V and W we have $\operatorname{sgn}\det(v_0, \dots, v_{n-1}) = s$ and $\operatorname{sgn}\det(w_0, \dots, w_{k-1}) = s'$. We will show that for every admissible γ the sign formula

$$\operatorname{sgn}\det_B(V_{\gamma(j)} \mid j \in [n+k-1]) = \operatorname{sgn}(\gamma)$$

holds, where the determinant is calculated with respect to the basis

$$B = ((v_0, 0), \dots, (v_{n-1}, 0), (0, w_0), \dots, (0, w_{k-1})).$$

(This claim implies the lemma.)

First, for the γ with $A(\gamma) = 0$, the determinant is

$$\begin{vmatrix} 1 & 0 & \cdots & 0 & a_0 & a_0 & \cdots & a_0 \\ 0 & 1 & \cdots & 0 & a_1 & a_1 & \cdots & a_1 \\ & \vdots & & & \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & a_{n-1} & \cdots & a_{n-1} \\ 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ & \vdots & & & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{vmatrix},$$

where all a_i are negative ($v_n = \sum_{i=0}^{n-1} a_i v_i$). To calculate it, we use lower rows to cancel all the a_i except the ones in the $(n+1)^{\text{st}}$ column. Then we use the left columns to cancel all the remaining a_i — this increases the $(n+1, n+1)$–entry. The result is now lower-triangular and positive on the diagonal.

Now let us consider the change of the determinant as $\gamma(j)$ moves across a 1×1 square. This changes one column. That column, and the neighbouring ones, are

$$(\dots, (v_i, w_j), (v_{i+1}, w_j), (v_{i+1}, w_{j+1}), \dots) \leftrightarrow (\dots, (v_i, w_j), (v_i, w_{j+1}), (v_{i+1}, w_{j+1}), \dots).$$

The change, up to sign, can be performed by two column operations:

$$-(v_i, w_{j+1}) = (v_{i+1}, w_j) - (v_i, w_j) - (v_{i+1}, w_{j+1}).$$

Since every admissible γ can be obtained by such operations from the one with $A(\gamma) = 0$, the sign formula holds for all admissible paths. □

This completes the proof of Theorem 11.1. □

12 Cup product of Euler classes

Let E be a (flat) $\mathrm{GL}_+(n, K)$–bundle over a simplicial complex X. We will often trivialize this bundle over simplices of X; to facilitate the use of such trivializations we introduce the following convention. Let $\sigma = (x_0, \dots, x_\ell)$ be a simplex of X. We put $E_\sigma := E_{x_0}$, and we use any (flat) trivialization of E over σ to isomorphically identify all the other E_{x_i} with E_σ. Thus, if $s\colon X^{(0)} \to E$ is a section, we write $s(x_0), \dots, s(x_\ell) \in E_\sigma$.

Definition 12.1 A section $s\colon X^{(0)} \to E$ is called positive, if for every simplex $\sigma = (x_0, \dots, x_\ell)$ of X there is a functional $\phi_\sigma \in E_\sigma^*$ such that $\phi_\sigma(s(x_i)) > 0$ for $i = 0, \dots, \ell$.

If a $\mathrm{GL}_+(n, K)$–bundle E over X admits a generic positive section s, then $\langle \mathrm{eu}_0(E), z\rangle = 0$ for every cycle $z \in Z_n(X, \mathbb{Z})$. Indeed, for every simplex $\sigma \in X^{(n)}$ we have $s_*\sigma \neq 0^\pm$ in $U_{n,+}$, since the values of s at the vertices of σ do not admit a linear relation with all positive coefficients — by positivity of s. It turns out that (over a cycle) every positive section can be perturbed to a generic positive section.

Lemma 12.2 *If a $\mathrm{GL}_+(n, K)$–bundle E over a finite simplicial complex X admits a positive section, then it admits a generic positive section.*

Proof Let s be a positive section, as witnessed by functionals $\phi_\sigma \in E_\sigma^*$ ($\sigma \in X^{(n)}$). We will construct, vertex-by-vertex, a new generic section s', positive with respect to the same collection of functionals. We order the vertices of X, and we start with $s'(x_0) = s(x_0)$. Suppose that $s'(x_\ell)$ have already been chosen for $\ell < i$. Put $V = E_{x_i}$. When choosing $s'(x_i)$ in V, in order to ensure genericity, we need to avoid a finite collection of affine hyperplanes, say defined by equations $(\psi_j(v) = \alpha_j)_{j \in J}$ (where $\psi_j \in V^*$, $\alpha_j \in K$). Also, for each n–simplex σ with vertex x_i, we need to ensure that $\phi_\sigma(s'(x_i)) > 0$ (we identify E_σ with V). Let $w \in V$ be such that $\psi_j(w) \neq 0$ for all $j \in J$; such w exists, since V is not the union of finitely many hyperplanes $(\ker \psi_j)_{j\in J}$. We will find suitable $s'(x_i)$ in the form $v(\alpha) := s(x_i) + \alpha w$, for some scalar α. First, observe that the equation $\psi_j(v(\beta)) = \alpha_j$ has a unique solution $\beta_j = \bigl(\alpha_j - \psi_j(s(x_i))\bigr)/\psi_j(w)$. Let $B := \min\{\beta_j \mid \beta_j > 0\}$. The condition $\phi_\sigma(v(\beta)) > 0$, ie $\phi_\sigma(s(x_i)) + \beta\phi_\sigma(w) > 0$, is equivalent to $\beta > -\phi_\sigma(s(x_i))/\phi_\sigma(w)$ (if $\phi_\sigma(w) > 0$) or to $\beta < -\phi_\sigma(s(x_i))/\phi_\sigma(w)$ (if $\phi_\sigma(w) < 0$). We know that $\beta = 0$ satisfies all these inequalities. Therefore, the scalar $M := \min\{-\phi_\sigma(s(x_i))/\phi_\sigma(w) \mid \phi_\sigma(w) < 0\}$ is positive. We put $\alpha := \frac{1}{4}\min(B, M)$ and $s'(x_i) = v(\alpha)$. □

Corollary 12.3 *Let E and E' be $\mathrm{GL}_+(n,K)$– and $\mathrm{GL}_+(k,K)$–bundles over simplicial complexes X and X', respectively. For any simplicial cycles $z \in Z_{n-\ell}(X,\mathbb{Z})$ and $z' \in Z_{k+\ell}(X',\mathbb{Z})$, where $\ell > 0$,*

$$\langle \mathrm{eu}_0(E \times E'), z \times z' \rangle = 0.$$

Proof We may and do assume that $X = \operatorname{supp} z$ and $X' = \operatorname{supp} z'$. Let s be a generic section of E. For dimensional reasons, the values of s at the vertices of any simplex σ of X are linearly independent; therefore, a functional ϕ_σ can be chosen that evaluates to 1 on each of them. Thus, s is positive. Now define $S\colon (X \times X')^{(0)} \to E \times E'$ by $S(x,x') = (s(x),0)$. Then, for any simplices $\sigma \in X^{(n-\ell)}$ and $\sigma' \in X'^{(k+\ell)}$, and any $(n+k)$–dimensional simplex σ_γ in the standard triangulation of $\sigma \times \sigma'$, we may put $\phi_{\sigma_\gamma} = \phi_\sigma \circ \pi_E$. Then, for every vertex (x,x') of σ_γ we have

$$\phi_{\sigma_\gamma}(S(x,x')) = \phi_\sigma\big(\pi_E(s(x),0)\big) = \phi_\sigma(s(x)) > 0.$$

Therefore, S is a positive section of $E \times E'$ over $\operatorname{supp} z \times z'$. By the lemma above, there exists a generic positive section, and that implies the asserted vanishing. □

Corollary 12.4 *Let E and E' be $\mathrm{GL}_+(n,K)$– and $\mathrm{GL}_+(k,K)$–bundles over simplicial complexes X and X' respectively. For any simplicial cycle $Z \in Z_{n+k}(X \times X',\mathbb{Z})$,*

$$\langle \mathrm{eu}_0(E) \times \mathrm{eu}_0(E'), Z \rangle = \langle \mathrm{eu}_0(E \times E'), Z \rangle.$$

Proof Indeed, by Künneth's formula, an integer multiple of Z is homologous to a combination of cycles of the form $z \times z'$; for the latter, the formula holds either by the previous corollary, or by Theorem 11.1. □

Theorem 12.5 *Let E and E' be $\mathrm{GL}_+(n,K)$– and $\mathrm{GL}_+(k,K)$–bundles over a simplicial complex X. For any simplicial cycle $z \in Z_{n+k}(X,\mathbb{Z})$,*

$$\langle \mathrm{eu}_0(E) \cup \mathrm{eu}_0(E'), z \rangle = \langle \mathrm{eu}_0(E \oplus E'), z \rangle.$$

Proof Let $\Delta\colon X \to X \times X$ be the diagonal map. Then

$$\begin{aligned}\langle \mathrm{eu}_0(E) \cup \mathrm{eu}_0(E'), z \rangle &= \langle \Delta^*(\mathrm{eu}_0(E) \times \mathrm{eu}_0(E')), [z] \rangle \\ &= \langle \mathrm{eu}_0(E) \times \mathrm{eu}_0(E'), \Delta_*[z] \rangle = \langle \mathrm{eu}_0(E \times E'), \Delta_*[z] \rangle \\ &= \langle \Delta^*\mathrm{eu}_0(E \times E'), [z] \rangle = \langle \mathrm{eu}_0(\Delta^*(E \times E')), [z] \rangle \\ &= \langle \mathrm{eu}_0(E \oplus E'), z \rangle.\end{aligned}$$

□

13 Comparison of Euler and Witt classes

We use the functoriality theorem (Theorem 1.5) to compare various tautological classes that we have constructed. We begin with eu and eu_+.

Euler classes We assume n is even. There is a natural map $\mathbb{P}_+ \to \mathbb{P}$; it induces a simplicial (nondegenerate) map $f : X_+ \to X$. The groups P_+G_+ and PG_+ acting on X_+ and X (respectively) are also related by the natural projection homomorphism $\phi\colon P_+G_+ \to PG_+$. The map f is ϕ–equivariant, and induces a coefficient group map $f: U_+ \to U$. Theorem 1.5 applies and gives the diagram

$$H^n(PG_+, U) \xrightarrow{\phi^*} H^n(P_+G_+, U) \xleftarrow{f_*} H^n(P_+G_+, U_+).$$

Recall that $U \simeq \mathbb{Z}$ and $U_+ \simeq \mathbb{Z}^{(n/2)+1}$. The map $f\colon U_+ \to U$ can be described explicitly using Remark 8.8. The generator a^+ of U_+ is represented by the simplex $([e_1], \dots, [e_n], [v_a])$, where

$$v_a = e_1 + \dots + e_a - (e_{a+1} + \dots + e_n).$$

The image of this simplex in X determines in U the symbol $[\operatorname{sgn}(\det(e_1, \dots, e_n) \cdot (-1)^{n-a})] = [(-1)^a]$. Therefore, $f(a^+) = [(-1)^a] = (-1)^a[+]$. It follows that the induced map on cohomology,

$$f_*\colon H^n(P_+G_+, U_+) \to H^n(PG_+, U),$$

maps $\mathrm{eu}_+ = \bigoplus_a \mathrm{eu}_a$ to $\sum_a (-1)^a \mathrm{eu}_a$. Theorem 1.5 implies the following result.

Theorem 13.1 *Let $\phi\colon P_+G_+ \to PG_+$ be the natural projection homomorphism. Then*

$$\phi^*\mathrm{eu} = \sum_{a=0}^{n/2} (-1)^a \mathrm{eu}_a.$$

A (flat) P_+G_+–bundle P over Y determines a PG_+–bundle P' over Y. As is usual in such cases, we put $\mathrm{eu}(P) := \mathrm{eu}(P') \in H^n(Y, \mathbb{Z})$.

Corollary 13.2 *Let P be a (flat) $\mathrm{P}_+\mathrm{GL}_+(n, K)$–bundle over an oriented closed n–manifold M. Then*

$$\langle \mathrm{eu}(P), [M] \rangle = 2^n \langle \mathrm{eu}_0(P), [M] \rangle.$$

Proof Using Theorems 10.1 and 13.1, we calculate

$$\begin{aligned}\langle \mathrm{eu}(P), [M] \rangle &= \left\langle \sum_{k=0}^{n/2} (-1)^k \mathrm{eu}_k(P), [M] \right\rangle \\ &= \sum_{k=0}^{n/2} (-1)^k \langle \mathrm{eu}_k(P), [M] \rangle \\ &= \sum_{k=0}^{n/2} (-1)^k (-1)^k \binom{n+1}{k} \langle \mathrm{eu}_0(P), [M] \rangle = 2^n \langle \mathrm{eu}_0(P), [M] \rangle.\end{aligned}$$ □

Remark 13.3 For $n = 2$, Theorem 9.1 gives $3\mathrm{eu}_0 + \mathrm{eu}_1 = 0$. Theorem 13.1 now implies $\phi^*\mathrm{eu} = 4\mathrm{eu}_0$, ie in this case Corollary 13.2 can be strengthened to equality in $H^2(\mathrm{P}_+\mathrm{GL}_+(2, K), \mathbb{Z})$ — there is no need to evaluate on cycles.

Witt class In Section 7 we discussed the action of $\mathrm{PSL}(2, K)$ on $\mathbb{P}^1$, on the associated complex X, and the resulting Witt class $w \in H^2(\mathrm{PSL}(2, K), W(K))$. In Section 8 we considered the action of $\mathrm{PGL}_+(2, K)$

on the same spaces, and the resulting cohomology class $\mathrm{eu} \in H^2(\mathrm{PGL}_+(2,K),\mathbb{Z})$. Theorem 1.5 may be applied to the identity map $\iota\colon X \to X$ and the injection homomorphism $\phi\colon \mathrm{PSL}(2,K) \to \mathrm{PGL}_+(2,K)$. Before stating the result we compute the coefficient map $\iota\colon W(K) \to \mathbb{Z}$. The symbol $[\lambda]$ is represented by the triple $t_\lambda = \left(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\\lambda\end{smallmatrix}\right]\right)$. To find the symbol of t_λ in $U_2(X,\mathrm{PGL}_+(2,K))$ we write

$$\begin{pmatrix}1\\\lambda\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\end{pmatrix} + \lambda\cdot\begin{pmatrix}0\\1\end{pmatrix}.$$

Then, using Remark 8.8, we get

$$\left[\mathrm{sgn}\left(\begin{vmatrix}1&0\\0&1\end{vmatrix}\cdot 1\cdot\lambda\right)\right] = [\mathrm{sgn}(\lambda)].$$

Therefore, the map $\iota\colon W(K) \to \mathbb{Z}$ is just the signature map σ, given by $\sigma([\lambda]) = \mathrm{sgn}(\lambda)$. The diagram is

$$H^2(\mathrm{PGL}_+(2,K),\mathbb{Z}) \xrightarrow{\phi^*} H^2(\mathrm{PSL}(2,K),\mathbb{Z}) \xleftarrow{\sigma_*} H^2(\mathrm{PSL}(2,K),W(K))$$

and the theorem is as follows.

Theorem 13.4 *Let $\phi\colon \mathrm{PSL}(2,K) \to \mathrm{PGL}_+(2,K)$ be the standard inclusion. Then*

$$\phi^*\mathrm{eu} = \sigma_* w.$$

Furthermore, the pullback of this class to $\mathrm{SL}(2,K)$ is equal to $4\mathrm{eu}_0$.

The last claim of the theorem follows from Remark 13.3.

Nonvanishing Consider a flat vector $\mathrm{SL}(2,\mathbb{R})$–bundle E over a closed oriented surface Σ. The (classical, topological) Euler class $\mathrm{eu}_t(E)$ of E (more precisely, the Euler number $\langle \mathrm{eu}_t(E),[\Sigma]\rangle$) can be computed as the signed number of zeroes of a generic section of E; generic means transversal to the zero section. Consider now a triangulation Y of Σ. Let $s\colon Y^{(0)} \to E$ be a generic section over the set of vertices of Y. Here genericity means that for every 2–simplex σ of Y the values of s at the vertices of σ are pairwise linearly independent (as usual, we compare them using a flat trivialization of E over σ). The section s can be affinely extended to each simplex of Y. Together, these extensions define a generic section of E over Σ in the previous, classical sense. Moreover, the zeroes of this extended section occur exactly in simplices σ on which s^*T_0 (the cocycle representing $\mathrm{eu}_0(E)$; see Remark 8.8 and Definition 8.9) is nonzero, and the sign of the zero in σ is equal to $s^*T_0(\sigma)$. These arguments prove the following statement.

Fact 13.5 *Let E be a flat $\mathrm{SL}(2,\mathbb{R})$–bundle over a closed surface Σ. Then*

$$\langle \mathrm{eu}_0(E),[\Sigma]\rangle = \langle \mathrm{eu}_t(E),[\Sigma]\rangle.$$

We will now prove that all the Euler classes constructed in this paper are nonzero (for n even).

Theorem 13.6 *Let K be an ordered field and let n be even. Then the Euler classes eu, eu_+ and all eu_k are nonzero.*

Proof Recall that an ordered field contains $\mathbb{Q}$ as a subfield, and the order restricted to $\mathbb{Q}$ is standard. Due to field restriction stability of our classes (see Remark 8.10) it is enough to show the theorem for $K = \mathbb{Q}$.

Assume first that $n = 2$. Recall that over a closed oriented surface Σ of genus $g \geq 2$ there are flat vector $\mathrm{SL}(2, \mathbb{R})$–bundles E with nontrivial Euler class eu_t (see [Milnor and Stasheff 1974, Appendix C]). Moreover, Takeuchi proved that $\mathrm{SL}(2, \mathbb{Q})$ can be used as the structure group of such bundles (see [Takeuchi 1971]); let us call such examples (flat $\mathrm{SL}(2, \mathbb{Q})$–bundles with nontrivial eu_t) Takeuchi bundles. Fact 13.5 implies that the Euler class eu_0 is nonzero for Takeuchi bundles. Theorem 10.1 and Corollary 13.2 imply that also eu_1 and eu are nontrivial for them.

For larger even $n = 2k$ we consider the Cartesian product Y of k copies of Σ, and over Y the product bundle $E^{\times k}$ of k–copies of a Takeuchi bundle E. Then Theorem 11.1 shows that

$$\langle \mathrm{eu}_0(E^{\times k}), [Y] \rangle = \langle \mathrm{eu}_0(E), [\Sigma] \rangle^k \neq 0.$$

Again, it follows from Theorem 10.1 and Corollary 13.2 that all eu_k as well as eu are nontrivial on $E^{\times k}$. $\square$

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