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Homotopy types of suspended 4–manifolds

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Given a closed, smooth, connected, orientable 4–manifold M whose integral homology groups can have 2–torsion, we determine the homotopy decomposition of the double suspension $\Sigma^2 M$ as wedge sums of some elementary $\boldsymbol{A}_3^3$–complexes which are 2–connected finite complexes of dimension at most 6. Furthermore, we utilize the Postnikov square (or equivalently Pontryagin square) to find sufficient conditions for the homotopy decompositions of $\Sigma^2 M$ to desuspend to that of ΣM.

55P15, 55P40, 57N65

1 Introduction

Recently, research on the homotopy properties of manifolds has emerged in two directions. The first direction is the loop homotopy of manifolds, which can be traced back to Beben and Wu's work [6] in 2011. After them, many people made efforts to promote the development of this project, such as Beben, Theriault and Huang [4; 5; 15]. On the other hand, as exhibited by So and Theriault [19], the suspension homotopy of manifolds has rich applications in some important objects of geometry and physics, such as gauge groups and current groups. Hereafter, this research direction has been widely studied, for instance in Huang [11; 12; 13], Cutler and So [8] and Huang and Li [14].

This paper contributes to further research on the suspension homotopy of manifolds. In the above related literature, due to some intractable obstructions, the authors usually avoid handling 2–torsions of the integral homology groups of the manifolds. For example, So and Theriault [19] required the 4–manifolds are 2–torsion-free in integral homology, Huang [13] restricts to 6–manifolds with integral homology groups containing no 2– or 3–torsions, while Cutler and So [8] and Huang and Li [14] respectively studied the suspension homotopy of simply connected 6–manifolds and 7–manifolds after localization away from 2.

In this paper we developed new techniques and tools in homotopy theory to obtain *complete* classification of the homotopy types of suspended 4–manifolds which can have 2–torsion in homology. For instance, we successfully apply certain homotopy properties of some $\boldsymbol{A}_n^3$*–complexes* (defined below) to obtain the homotopy decompositions of $\Sigma^2 M$. Moreover, the Postnikov squaring operation (1-1) and the Pontryagin squaring operation (1-2) appear to be powerful in the characterizations of the homotopy type of ΣM; see Section 5.

To make sense of the introduction, we need the following notions and notation. Let G be an abelian group and let n be a positive integer. Denote by $H_n(X;G)$ (resp. $H^n(X;G)$) the n^{th} (singular) homology (resp. cohomology) group of X with coefficients in G, and denote by $P^n(G)$ the n–dimensional Peterson space (see Neisendorfer [18]) which admits a unique nontrivial reduced integral cohomology group G in dimension n. In particular, for integers $n,k\geq 2$, we denote by $\mathbb{Z}/k=\mathbb{Z}/k\mathbb{Z}$ the group of integers modulo k. Recall the Peterson spaces have the cell structure

$$P^n(k)=P^n(\mathbb{Z}/k)=S^{n-1}\cup_k e^n,$$

which admits the obvious inclusion i_{n-1} of the bottom sphere S^{n-1} into $P^n(k)$ and the pinch map q_n onto S^n. For each $n\geq 3$, there is a generator $\widetilde{\eta}_r\in\pi_{n+1}(P^n(2^r))$ satisfying the formula

$$q_n\widetilde{\eta}_r\simeq\eta_n;$$

see Lemma 2.1, where $\eta_n\colon S^{n+1}\to S^n$ is the iterated suspensions of the Hopf map $\eta\colon S^3\to S^2$. For a homomorphism $\phi\colon G\to G'$ of groups, $\ker(\phi)$ and $\mathrm{im}(\phi)$ denote the kernel and the image subgroups of ϕ, respectively.

A finite CW–complex X is called an $\boldsymbol{A}_n^k$*–complex* if X is $(n-1)$–connected and has dimension at most $n+k$. It is well known that elementary (or called indecomposable) $\boldsymbol{A}_n^1$–complexes consist of spheres S^n, S^{n+1} and the Moore spaces $P^{n+1}(p^r)$ with p odd primes and $r\geq 1$. One may consult Zhu, Li and Pan [24; 16; 23; 25] and Baues and Hennes [3] for more homotopy theory of such complexes. We need the following elementary $\boldsymbol{A}_n^3$–complexes with $n\geq 3$ and $r,s\geq 1$:

$$\begin{aligned}
C_\eta^{n+2}&=S^n\cup_\eta \boldsymbol{C}S^{n+1}=\Sigma^{n-2}\mathbb{C}P^2, & C_r^{n+2}&=P^{n+1}(2^r)\cup_{i_n\eta}\boldsymbol{C}S^{n+1},\\
C^{n+2,s}&=S^n\cup_{\eta q_{n+1}}\boldsymbol{C}P^{n+1}(2^s), & C_r^{n+2,s}&=P^{n+1}(2^r)\cup_{i_n\eta q_{n+1}}\boldsymbol{C}P^{n+1}(2^s),\\
A^{n+3}(\eta^2)&=S^n\cup_{\eta^2}\boldsymbol{C}S^{n+2}, & A^{n+3}(\widetilde{\eta}_r)&=P^{n+1}(2^r)\cup_{\widetilde{\eta}_r}\boldsymbol{C}S^{n+2},\\
A^{n+3}(2^r\eta^2)&=P^{n+1}(2^r)\cup_{i_n\eta^2}\boldsymbol{C}S^{n+2}.
\end{aligned}$$

Here the first four $\boldsymbol{A}_n^2$–complexes are the *elementary Chang complexes* (due to Chang [7]), and the last two spaces are the only two $\boldsymbol{A}_n^3$–complexes with the homology groups

$$H_n\cong\mathbb{Z}/2^r,\qquad H_{n+3}=H_0\cong\mathbb{Z},\qquad H_i=0\quad\text{for } i\neq 0,n,n+3.$$

Compare [2, Theorem 10.3.1]. Note that all of the above $\boldsymbol{A}_n^3$–complexes desuspend: they can be defined for $n\geq 2$.

To deal with 2–torsions in $H_*(M;\mathbb{Z})$, we shall employ the following cohomology operations. Let X be a connected CW–complex. For each $r\geq 1$, there are unstable cohomology operations: the *Postnikov square*

$$\mathfrak{P}_0\colon H^1(X;\mathbb{Z}/2^r)\to H^3(X;\mathbb{Z}/2^{r+1}) \tag{1-1}$$

and the *Pontryagin square*

$$\mathfrak{P}_1\colon H^2(X;\mathbb{Z}/2^r)\to H^4(X;\mathbb{Z}/2^{r+1}). \tag{1-2}$$

These two operations were carefully studied by Whitehead [21; 22]. Given a cohomology operation C which maps $H^r(X,A;G_1)\to H^s(X,A;G_2)$ for each pair (X,A), the *suspension operation* $S(C)$ is the composition (see [20, Section 3])

$$H^{r-1}(Y;G_1)\xrightarrow{\sigma} H^r(\Sigma Y;G_1)\xrightarrow{C} H^s(\Sigma Y;G_2)\xrightarrow{\sigma^{-1}} H^{s-1}(Y;G_2),$$

where σ is the suspension isomorphism. Note that $\mathfrak{P}_0$ is the suspension operation of $\mathfrak{P}_1$:

$$\sigma\mathfrak{P}_0=\mathfrak{P}_1\sigma; \tag{1-3}$$

see [20, Theorem I(i)]. The Adem relations

$$\mathrm{Sq}^3=\mathrm{Sq}^1\,\mathrm{Sq}^2 \quad \text{and} \quad \mathrm{Sq}^3\,\mathrm{Sq}^1+\mathrm{Sq}^2\,\mathrm{Sq}^2=0$$

yield the secondary operation Θ_n based on the relation $\varphi_n\theta_n=0$ with

$$\begin{aligned}\theta_n&=\begin{pmatrix}\mathrm{Sq}^2\,\mathrm{Sq}^1\\ \mathrm{Sq}^2\end{pmatrix}\colon K_n\to K_{n+3}\times K_{n+2},\\ \varphi_n&=(\mathrm{Sq}^1,\mathrm{Sq}^2)\colon K_{n+3}\times K_{n+2}\to K_{n+4},\end{aligned} \tag{1-4}$$

where $n\ge 1$, $K_m=K_m(\mathbb{Z}/2)$ denotes the Eilenberg–Mac Lane space of type $(\mathbb{Z}/2,m)$. For a space X, the secondary operation $\Theta_n\colon S_n(X)\to T_n(X)$ is the induced homomorphism with

$$\begin{aligned}S_n(X)&=\ker(\theta_n)_\sharp=\ker(\mathrm{Sq}^2)\cap\ker(\mathrm{Sq}^2\,\mathrm{Sq}^1),\\ T_n(X)&=\mathrm{coker}(\Omega\varphi_n)_\sharp=H^{n+3}(X;\mathbb{Z}/2)/\mathrm{im}(\mathrm{Sq}^1+\mathrm{Sq}^2).\end{aligned}$$

The secondary operation Θ_n detects the map $\eta^2=\eta_n\eta^{n+1}\colon S^{n+2}\to S^n$; see [9, page 96] or Lemma 2.7. For each $r\ge 1$, the higher-order Bockstein operations

$$\beta_r\colon H^*(X;\mathbb{Z}/2)\dashrightarrow H^{*+1}(X;\mathbb{Z}/2) \tag{1-5}$$

are inductively defined by setting β_1 as the usual Bockstein homomorphism associated to the short exact sequence

$$0\to\mathbb{Z}/2\to\mathbb{Z}/4\to\mathbb{Z}/2\to 0;$$

for $r\ge 2$, β_r is defined on the intersection of $\ker(\beta_i)$, $i<r$, and takes values in the quotient by the $\mathrm{im}(\beta_i)$, $i<r$. This is also indicated by the dashed arrow in (1-5). See [9, Section 5.2] for more details. Note that the higher Bocksteins β_r and the sequence $\Theta=\{\Theta_n\}_{n\ge 1}$ are both *stable* (cf [9, 4.2.2]):

$$\Omega\beta_r=\beta_r,\quad \Omega\Theta_{n+1}=\Theta_n.$$

Let M be a closed, smooth, connected, orientable 4–manifold. By Poincaré duality and the universal coefficient theorem for cohomology, the homology groups $H_*(M;\mathbb{Z})$ are given by Table 1, where m,d are nonnegative integers, and T is a finitely generated torsion abelian group. Denote the 2–primary component of T by

$$T_2=\bigoplus_{j=1}^{n}\mathbb{Z}/2^{r_j}.$$

Now we are prepared to state our first main theorem.

i	0, 4	1	2	3	≥ 5
$H_i(M;\mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}^m \oplus T$	$\mathbb{Z}^d \oplus T$	$\mathbb{Z}^m$	0

Table 1: $H_*(M;\mathbb{Z})$.

Theorem 1.1 *Let M be a closed, smooth, connected, orientable 4–manifold with integral homology $H_*(M;\mathbb{Z})$ given by Table 1.*

(1) *Suppose that M is spin, then $\Sigma^2 M$ has two possible homotopy types:*

(a) *If $\Theta(H^1(M;\mathbb{Z}/2)) = 0$, then there is a homotopy equivalence*

$$\Sigma^2 M \simeq \Big(\bigvee_{i=1}^{m}(S^3 \vee S^5)\Big) \vee \Big(\bigvee_{i=1}^{d} S^4\Big) \vee P^4(T) \vee P^5(T) \vee S^6.$$

(b) *If $\Theta(H^1(M;\mathbb{Z}/2)) \neq 0$, then*

$$\Sigma^2 M \simeq \Big(\bigvee_{i=1}^{m}(S^3 \vee S^5)\Big) \vee \Big(\bigvee_{i=1}^{d} S^4\Big) \vee P^4\Big(\frac{T}{\mathbb{Z}/2^{r_{j_0}}}\Big) \vee P^5(T) \vee A^6(2^{r_{j_0}}\eta^2),$$

where j_0 is the maximum of the indices $j \leq n$ such that

$$\Theta(x) \neq 0 \quad \text{and} \quad \beta_{r_j}(x) \neq 0 \quad \text{for } x \in H^1(M;\mathbb{Z}/2).$$

(2) *Suppose that M is nonspin. Then the suspension $\Sigma^i M$ has the following possible homotopy types:*

(a) *If for any $u \in H^4(\Sigma^2 M;\mathbb{Z}/2)$ with $\mathrm{Sq}^2(u) \neq 0$ and any $v \in \ker(\mathrm{Sq}^2)$ it holds that*

$$\beta_r(u+v) = 0 \quad \text{and} \quad u+v \notin \mathrm{im}(\beta_s) \quad \text{for all } r,s \geq 1,$$

then there is a homotopy equivalence

$$\Sigma^2 M \simeq \Big(\bigvee_{i=1}^{m}(S^3 \vee S^5)\Big) \vee \Big(\bigvee_{i=1}^{d-1} S^4\Big) \vee P^4(T) \vee P^5(T) \vee C^6_\eta.$$

(b) *Suppose that for any $u \in H^2(M;\mathbb{Z}/2)$ with $\mathrm{Sq}^2(u) \neq 0$ and any $v \in \ker(\mathrm{Sq}^2)$, it holds that*

$$u+v \notin \mathrm{im}(\beta_s) \quad \text{for all } s \geq 1,$$

while there exist $u' \in H^2(M;\mathbb{Z}/2)$ with $\mathrm{Sq}^2(u') \neq 0$ and $v' \in \ker(\mathrm{Sq}^2)$ such that

$$\beta_r(u'+v') \neq 0 \quad \text{for some } r \geq 1.$$

Then there is a homotopy equivalence

$$\Sigma^2 M \simeq \bigvee_{i=1}^{m}(S^3 \vee S^5) \vee \bigvee_{i=1}^{d} S^4 \vee P^4(T) \vee P^5\Big(\frac{T}{\mathbb{Z}/2^{r_{j_1}}}\Big) \vee C^6_{r_{j_1}},$$

where j_1 is the maximum of the indices $j \leq n$ such that

$$\mathrm{Sq}^2(u') \neq 0 \quad \text{and} \quad \beta_r(u'+v') \neq 0.$$

(c) *Suppose that there exist* $u \in H^2(M;\mathbb{Z}/2)$ *with* $\mathrm{Sq}^2(u) \neq 0$ *and* $v \in \ker(\mathrm{Sq}^2)$ *such that* $u+v \in \mathrm{im}(\beta_r)$ *for some* r.

(i) *If* $\Theta(H^1(M;\mathbb{Z}/2)) = 0$, *then there is a homotopy equivalence*

$$\Sigma^2 M \simeq \Big(\bigvee_{i=1}^{m}(S^3 \vee S^5)\Big) \vee \Big(\bigvee_{i=1}^{d} S^4\Big) \vee P^5(T) \vee P^4\Big(\frac{T}{\mathbb{Z}/2^{r_{j_2}}}\Big) \vee A^6(\tilde{\eta}_{r_{j_2}}),$$

where j_2 *is the minimum of the indices* $j \leq n$ *such that* $u+v \in \mathrm{im}(\beta_{r_j})$.

(ii) *If* $\Theta(H^1(M;\mathbb{Z}/2)) \neq 0$ *and* $T_2 \cong \mathbb{Z}/2^{r_{j_2}}$, *then there is a homotopy equivalence*

$$\Sigma^2 M \simeq \Big(\bigvee_{i=1}^{m}(S^3 \vee S^5)\Big) \vee \Big(\bigvee_{i=1}^{d} S^4\Big) \vee P^5(T) \vee P^4\Big(\frac{T}{\mathbb{Z}/2^{r_{j_2}}}\Big) \vee A^6_{\varepsilon}(\tilde{\eta}_{r_{j_2}}),$$

where $A^6_{\varepsilon}(\tilde{\eta}_{r_{j_2}})$ *is the homotopy cofiber of* $\tilde{\eta}_{r_{j_2}} + \varepsilon \cdot i_3\eta^2$ *with* $\varepsilon \in \{0,1\}$.

(iii) *If* $\Theta(H^1(M;\mathbb{Z}/2)) \neq 0$ *and* $n \geq 2$ (ie T_2 has at least 2 direct summands), *then there is a homotopy equivalence*

$$\Sigma^3 M \simeq$$
$$\Big(\bigvee_{i=1}^{m}(S^4 \vee S^6)\Big) \vee \Big(\bigvee_{i=1}^{d} S^5\Big) \vee P^6(T) \vee A^7_{\varepsilon}(\tilde{\eta}_{r_{j_2}}) \vee P^5\Big(\frac{T}{\mathbb{Z}/2^{r_{j_2}} \oplus \mathbb{Z}/2^{r_{j_0'}}}\Big) \vee A^7(2^{r_{j_0'}}\eta^2),$$

where $A^7_{\varepsilon}(\tilde{\eta}_{r_{j_2}}) = \Sigma A^6_{\varepsilon}(\tilde{\eta}_{r_{j_2}})$, *the index* j_2 *the minimum of the indices* $j \leq n$ *such that* $u+v \in \mathrm{im}(\beta_{r_j})$, *and* j_0' *is the maximum of the indices* $j \leq n$ *with* $j \neq j_2$ *such that*

$$\Theta(x) \neq 0 \quad \text{and} \quad \beta_{r_j}(x) \neq 0 \quad \text{for all } x \in H^3(C_\varphi;\mathbb{Z}/2).$$

From the above complete discussion we see that when M is nonspin, the nontriviality of the secondary operation Θ on $H^1(M;\mathbb{Z}/2)$ only affects case when $u+v \in \mathrm{im}(\beta_r)$ for some r. In the last case (iii) we made one more suspension to cancel the possible nontrivial Whitehead products in k'–invariant of the homology decomposition of the $\Sigma^2 M$.

We also study the homotopy type of the suspension ΣM in terms of the Postnikov square $\mathfrak{P}_0$ (or equivalently the Pontryagin square $\mathfrak{P}_1$).

Theorem 1.2 *Let* M *be a closed, smooth, connected, orientable* 4*–manifold with* $H_*(M;\mathbb{Z})$ *given by Table 1. If the Postnikov square*

$$\mathfrak{P}_0 \colon H^1(M;\mathbb{Z}/2^{r_j}) \to H^3(M;\mathbb{Z}/2^{r_j+1})$$

is trivial for each $j = 1, 2, \dots, n$, *then the desuspensions of the homotopy decompositions of* $\Sigma^2 M$ *in Theorem 1.1 yield the homotopy decompositions of* ΣM.

If $H_*(M;\mathbb{Z})$ contains no 2–torsion (ie $T_2 = 0$), then the homotopy decomposition $\Sigma M \simeq \bigvee_{i=1}^{m} S^2 \vee \Sigma W$ (4-1) implies that the Pontryagin square

$$\mathfrak{P}_1 \colon H^1(\Sigma M;\mathbb{Z}/2^{r_j}) \to H^3(\Sigma M;\mathbb{Z}/2^{r_j+1})$$

is trivial, hence so is $\mathfrak{P}_0$ by (1-3). Hence Theorem 1.2 extends So and Theriault's results [19, Theorem 1.1].

However, the author didn't find any other 4–manifolds M satisfying conditions in Theorem 1.2. This is also why we arrange the above theorem after Theorem 1.1.

The paper is organized as follows. In Section 2 we review some homotopy theory of partial elementary $\boldsymbol{A}_n^3$–complexes and list some technical lemmas about the Pontryagin or Steenrod square operations. Section 3 introduces the main analysis methods adopted in this paper, including a useful criterion to determine the homotopy type of suspensions and the matrix method to determine the homotopy type of homotopy cofibers of certain maps. Section 4 simply analyses the homology decomposition of the suspension ΣM. In Section 5 we utilize the methods developed in Section 3 to give a detailed discussion on the homotopy decompositions of our suspended four-manifolds. At the end, we prove Theorems 1.1 and 1.2, respectively.

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2 Some technical lemmas

In this section we recall some homotopy groups of mod 2^r Moore spaces and prove some lemmas about the Pontryagin or Steenrod square operations.

Throughout, all spaces $X, Y, \dots$ are based connected CW–complexes, and $[X, Y]$ is the set of based homotopy classes of based maps from X to Y. We identify a map f with its homotopy class in notation. For composable maps g and f, denote by gf or $g \circ f$ the composition of g with f. Unless otherwise specified, $\boldsymbol{C}X$ denotes the reduced mapping cone of a space X, and C_f denotes the homotopy cofiber of a given map $f\colon X \to Y$. For a cyclic group G, we write $G\langle x\rangle$ to mean x is a generator of G.

2.1 Some homotopy theory of mod 2^r Moore spaces

Let $n, k \geq 2$. There is a homotopy cofibration for the mod k Moore space $P^n(k)$:

$$S^{n-1} \xrightarrow{k} S^{n-1} \xrightarrow{i_{n-1}} P^n(k) \xrightarrow{q_n} S^n,$$

where i_{n-1} and q_n are the canonical inclusion and projection, respectively. Recall that if 2 doesn't divide k, then

$$\pi_n(P^n(k)) = \pi_{n+1}(P^n(k)) = 0 \quad \text{for all } n \geq 3.$$

For each $r, s \geq 1$, let $\rho_r\colon \mathbb{Z} \to \mathbb{Z}/2^r$ be the reduction mod 2^r with $1_r = \rho_r(1)$, let $\chi^r_s\colon \mathbb{Z}/2^r \to \mathbb{Z}/2^s$ be the homomorphism given by

$$\chi^r_s(1_r) = \begin{cases} 1_s & \text{if } r \geq s, \\ 2^{s-r}1_s & \text{if } r < s. \end{cases} \tag{2-1}$$

For each $n \geq 3$, there exists a map (with n omitted in notation)

$$B(\chi_s^r)\colon P^{n+1}(2^r) \to P^{n+1}(2^s)$$

such that

$$H_n(B(\chi_s^r)) = \chi_s^r \quad \text{and} \quad \Sigma B(\chi_s^r) = B(\chi_s^r).$$

Moreover, $B(\chi_s^r)$ satisfies the relation formulas (cf [3])

$$B(\chi_s^r) i_n = \begin{cases} i_n & \text{if } r \geq s, \\ 2^{s-r} i_n & \text{if } r \leq s, \end{cases} \quad \text{and} \quad q_{n+1} B(\chi_s^r) = \begin{cases} 2^{r-s} q_{n+1} & \text{if } r \geq s, \\ q_{n+1} & \text{if } r \leq s. \end{cases} \tag{2-2}$$

Note that a multiple $t\alpha$ (written also as $t \cdot \alpha$) of an element $\alpha \in \pi_k(X)$ coincides with the composite $\alpha \circ t$.

Lemma 2.1 *Let $r \geq 1$ and $n \geq 3$ be integers.*

(1) $\pi_{n-1}(P^n(2^r)) \cong \mathbb{Z}/2^r \langle i_{n-1} \rangle$.

(2) $\pi_3(P^3(2^r)) \cong \mathbb{Z}/2^{r+1}\langle i_2\eta\rangle$, $\pi_{n+1}(P^{n+1}(2^r)) \cong \mathbb{Z}/2\langle i_n\eta\rangle$.

(3) *There are isomorphisms*

$$\pi_{n+1}(P^n(2^r)) \cong \begin{cases} \mathbb{Z}/4\langle \tilde{\eta}_1 \rangle & \text{if } r = 1, \\ \mathbb{Z}/2\langle \tilde{\eta}_r\rangle \oplus \mathbb{Z}/2\langle i_{n-1}\eta^2\rangle & \text{if } r \geq 2, \end{cases}$$

where $\tilde{\eta}_r$ satisfies the formulas

$$\tilde{\eta}_r = B(\chi_r^1)\tilde{\eta}_1, \quad q_n\tilde{\eta}_r = \eta, \quad 2\tilde{\eta}_1 = i_{n-1}\eta^2. \tag{2-3}$$

(4) *Dually, there are isomorphisms*

$$\pi^n(P^{n+2}(2^r)) \cong \begin{cases} \mathbb{Z}/4\langle \bar{\eta}_1 \rangle & \text{if } r = 1, \\ \mathbb{Z}/2\langle \bar{\eta}_r\rangle \oplus \mathbb{Z}/2\langle \eta^2 q_{n+2}\rangle & \text{if } r \geq 2, \end{cases}$$

where $\bar{\eta}_r$ satisfies the formula

$$\bar{\eta}_r i_{n+1} = \eta_n, \quad 2\bar{\eta}_1 = \eta^2 q_{n+2}.$$

Proof (1) The isomorphism holds by the Hurewicz theorem.

(2) By [2, bottom of page 19, top of page 20], it holds that

$$\pi_n(P^n(2^r)) \cong \begin{cases} \Gamma(\mathbb{Z}/2^r) \cong \mathbb{Z}/2^{r+1} & \text{if } n = 3, \\ \mathbb{Z}/2^r \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2 & \text{if } n \geq 4. \end{cases}$$

Here $\Gamma(\mathbb{Z}/2^r)$ is the Whitehead quadratic group; see [2] or [21]. The composite $i_{n-1}\eta$ is clearly a generator of $\pi_n(P^n(2^r))$.

(3) By [2, Proposition 11.1.12], $\pi_4(P^3(2^r))$ is isomorphic to the stable homotopy group $\pi_4^s(P^3(2^r))$, whose generators and the relations (2-3) refer to [3].

(4) The isomorphisms and the relation formulas follow by (3) under the Spanier–Whitehead duality:

$$\pi^n(P^{n+2}(2^r)) \cong \pi_{n+2}(P^{n+1}(2^r)). \qquad \square$$

For simplicity we still denote $\widetilde{\eta}_r\colon S^{n+1}\to P^n(2^r)$ the iterated suspensions of the generator $\widetilde{\eta}_r$ of $\pi_4(P^3(2^r))$. Combining (2-2) and (2-3), we have:

Corollary 2.2 *Let $r,s\geq 1$. There hold relations*

$$B(\chi^r_s)\widetilde{\eta}_r=\begin{cases}\widetilde{\eta}_s & \text{if } s\geq r,\\ 2^{r-s}\widetilde{\eta}_s & \text{if } s\leq r.\end{cases}$$

2.2 Whitehead's quadratic functor

Recall the *Whitehead quadratic functor*

$$\Gamma\colon \mathbf{Ab}\to\mathbf{Ab}$$

on the category **Ab** of abelian groups [21; 1]. The functor Γ is characterized by the following property: a function $\varphi\colon G\to G'$ between abelian groups is called *quadratic* if $\varphi(x)=\varphi(-x)$ and the function $G\times G\to G'$ with $(x,y)\mapsto\varphi(x+y)-\varphi(x)-\varphi(y)$ is bilinear. For each abelian group G, there is a *universal quadratic function*

$$\gamma=\gamma_G\colon G\to\Gamma(G)$$

such that for any quadratic function $\varphi\colon G\to G'$, there is a unique homomorphism $\varphi^{\square}\colon\Gamma(G)\to G'$ such that $\varphi=\varphi^{\square}\circ\gamma$. It follows that for a homomorphism $\phi\colon G\to G'$, there is a unique induced homomorphism $\Gamma(\phi)\colon\Gamma(G)\to\Gamma(G')$ such that $\Gamma(\phi)\circ\gamma_G=\gamma_{G'}\circ\phi$. The universal quadratic function $\gamma=\gamma_G$ induces the bilinear pairing

$$[1,1]\colon G\otimes G\to\Gamma(G),\quad [1,1](x,y)=\gamma(x+y)-\gamma(x)-\gamma(y). \tag{2-4}$$

Lemma 2.3 (cf [2]) *Let G be an abelian group and let $n\geq 0$.*

(1) *For the cyclic group $G=\mathbb{Z}/n$ we have*

$$\Gamma(\mathbb{Z}/n)\cong\mathbb{Z}/(n^2,2n),$$

where $\mathbb{Z}/0=\mathbb{Z}$ and $(n^2,2n)$ is the greatest common divisor. The group is generated by $\gamma(1_n)$ with $1_n=1+n\mathbb{Z}$.

(2) *For any $x\in G$, there holds $\gamma(nx)=n^2\gamma(x)$.*

2.3 Squaring operations

For an abelian group G, the Pontryagin square

$$\mathfrak{P}_1\colon H^2(X;G)\to H^4(X;\Gamma(G))$$

is a *quadratic* function with respect to the cup product $\smile$:

$$\mathfrak{P}_1(-x)=\mathfrak{P}_1(x),\quad \mathfrak{P}_1(nx)=n^2\mathfrak{P}_1(x),\quad \mathfrak{P}_1(x+y)=\mathfrak{P}_1(x)+\mathfrak{P}_1(y)+[1,1]_*(x\smile y), \tag{2-5}$$

where $[1,1]_*$ is induced by the coefficient homomorphism (2-4). The Pontryagin square is natural with respect to maps $X\to Y$ between spaces and with respect to homomorphisms $G\to G'$ between groups.

Let X be an A_2^2–complex and let

$$C_4(X) \xrightarrow{d} C_3(X) \xrightarrow{d} C_2(X)$$

be its cellular chain complex. Represent a cohomology class $x \in H^2(X;G)$ by a cocycle $\hat{x}\colon C_2(X) \to G$, which induces a unique homomorphism

$$\tilde{x}\colon H_2(X) = C_2(X)/dC_3(X) \to G,$$

and therefore a unique homomorphism

$$\Gamma(\tilde{x})\colon \Gamma(H_2(X)) \to \Gamma(G).$$

By the universal coefficient theorem, there is an isomorphism

$$\mu\colon H^2(X;H_2(X)) \xrightarrow{\cong} \mathrm{Hom}(H_2(X), H_2(X)).$$

Let $\iota_2 \in H^2(X;H_2(X))$ be given such that $\mu(\iota_2)$ is the identity on $H_2(X)$. By [1, Chapter I] we know that the Pontryagin square

$$\mathfrak{P}_1\colon H^2(X;G) \to H^4(X;\Gamma(G))$$

is completely determined by the *Pontryagin element*

$$\mathfrak{P}_1(\iota_2) \in H^4(X;\Gamma(H_2(X)))$$

in the sense that there holds an formula

$$\mathfrak{P}_1(x) = \Gamma(\tilde{x})_*(\mathfrak{P}_1(\iota_2)),$$

where $\Gamma(\tilde{x})_*$ is induced by the coefficient homomorphism.

Let $C_r(t\eta)$ be the homotopy cofiber of $t \cdot i_2\eta\colon S^3 \to P^3(2^r)$, where $r \geq 1$ and $t \in \mathbb{Z}/2^{r+1}$. Note that $C_r(t\eta)$ is an A_2^2–polyhedron and has the *A_2^2–form*

$$f = (t\eta, 2^r)\colon S^3 \vee S^2 \to S^2, \tag{2-6}$$

ie $C_r(t\eta)$ is the homotopy cofiber of the attaching map f between spheres.

Lemma 2.4 *Let $t \in \mathbb{Z}/2^{r+1}$ and $r \geq 1$. The Pontryagin square*

$$\mathfrak{P}_1\colon H^2(C_r(t\eta);\mathbb{Z}/2^r) \to H^4(C_r(t\eta);\mathbb{Z}/2^{r+1})$$

is trivial if and only if $t = 0$.

Proof Let $\iota_2 \in H^2(C_r(t\eta);\mathbb{Z}/2^r)$ be the generator which corresponds to the identity on $H_2(C_r(t\eta))$. By [1, Chapter I, Proposition 7.6] and the A_2^2–form (2-6), the Pontryagin element $\mathfrak{P}_1(\iota_2)$ is represented by the cocycle

$$t \cdot \Gamma(\rho_r)\gamma = \Gamma(\rho_r)(t\gamma)\colon \mathbb{Z} \xrightarrow{t\gamma} \Gamma(\mathbb{Z}) \xrightarrow{\Gamma(\rho_r)} \Gamma(\mathbb{Z}/2^r).$$

Note that $\Gamma(\rho_r)\gamma = \gamma\rho_r$ represents a generator of $H^4(C_r(t\eta);\Gamma(\mathbb{Z}/2^r))$ by the universal coefficient theorem. Then it follows by Lemma 2.3 that $\mathfrak{P}_1 = 0$ if and only if $t = 0$. □

Recall that the Steenrod square

$$\mathrm{Sq}^2\colon H^n(-;\mathbb{Z}/2)\to H^{n+2}(-;\mathbb{Z}/2)$$

is a stable cohomology operation such that $\mathrm{Sq}^2(x)=x^2$ for any cohomology class x of dimension 2; see [10, Section 4.L].

Lemma 2.5 (cf [24]) *For any $n\geq 3$, the Steenrod square*

$$\mathrm{Sq}^2\colon H^n(C;\mathbb{Z}/2)\to H^{n+2}(C;\mathbb{Z}/2)$$

is an isomorphism for each (elementary) Chang complex C.

Lemma 2.6 *For each $n\geq 2$ and $r\geq 1$, the Steenrod square*

$$\mathrm{Sq}^2\colon H^{n+1}(A^{n+3}(\tilde{\eta}_r);\mathbb{Z}/2)\to H^{n+3}(A^{n+3}(\tilde{\eta}_r);\mathbb{Z}/2)$$

is an isomorphism.

Proof By (2-3) there is a homotopy commutative diagram of homotopy cofibrations (in which rows and columns are all homotopy cofibrations):

$$\begin{array}{ccccc}
* & \longrightarrow & S^n & = & S^n \\
\downarrow & & \downarrow{\scriptstyle i_n} & & \downarrow \\
S^{n+2} & \xrightarrow{\tilde{\eta}_r} & P^{n+1}(2^r) & \longrightarrow & A^{n+3}(\tilde{\eta}_r) \\
\| & & \downarrow{\scriptstyle q_{n+1}} & & \downarrow{\scriptstyle d} \\
S^{n+2} & \xrightarrow{\eta} & S^{n+1} & \longrightarrow & C_\eta^{n+3}
\end{array}$$

It follows that $d^*\colon H^k(C_\eta^{n+3};\mathbb{Z}/2)\to H^k(A^{n+3}(\tilde{\eta}_r);\mathbb{Z}/2)$ is an isomorphism for $k=n+1,n+3$. The isomorphism in the lemma then follows by Lemma 2.5 and the commutative square

$$\begin{array}{ccc}
H^{n+1}(C_\eta^{n+3};\mathbb{Z}/2) & \xrightarrow[\cong]{\mathrm{Sq}^2} & H^{n+3}(C_\eta^{n+3};\mathbb{Z}/2) \\
\cong\downarrow{\scriptstyle d^*} & & \cong\downarrow{\scriptstyle d^*} \\
H^{n+1}(A^{n+3}(\tilde{\eta}_r);\mathbb{Z}/2) & \xrightarrow{\mathrm{Sq}^2} & H^{n+3}(A^{n+3}(\tilde{\eta}_r);\mathbb{Z}/2)
\end{array}$$

□

2.4 Higher-order cohomology operations

Recall the secondary cohomology operations

$$\Theta_n\colon S_n(X)\to T_n(X) \tag{2-7}$$

based on the relation $\varphi_n\theta_n=0$ of (1-4), where

$$S_n(X)=\ker(\theta_n)_\sharp=\ker(\mathrm{Sq}^2)\cap\ker(\mathrm{Sq}^2\,\mathrm{Sq}^1),$$
$$T_n(X)=\mathrm{coker}(\Omega\varphi_n)_\sharp=H^{n+3}(X;\mathbb{Z}/2)/\mathrm{im}(\mathrm{Sq}^1+\mathrm{Sq}^2).$$

Lemma 2.7 *Let $n \geq 2$ and $r \geq 1$. For $X = A^{n+3}(\eta^2)$ or $A^{n+3}(2^r\eta^2)$, the secondary operation Θ_n acts nontrivially on $H^n(X;\mathbb{Z}/2)$; that is,*

$$0 \neq \Theta_n\colon H^n(X;\mathbb{Z}/2) \to H^{n+3}(X;\mathbb{Z}/2).$$

Proof For $X = A^{n+3}(\eta^2)$ or $A^{n+3}(2^r\eta)$, we compute that

$$S_n(X) = H^n(X;\mathbb{Z}/2) \cong \mathbb{Z}/2 \quad \text{and} \quad T_n(X) = H^{n+3}(X;\mathbb{Z}/2) \cong \mathbb{Z}/2.$$

The proof of $\Theta_n \neq 0$ for $X = A^{n+3}(\eta^2)$ refers to [9, page 96]. There is a homotopy cofibration

$$S^n \xrightarrow{i_n \circ 2^r} A^{n+3}(\eta^2) \xrightarrow{j} A^{n+3}(2^r\eta),$$

which induces the commutative square

$$\begin{array}{ccc} H^n(A^{n+3}(2^r\eta);\mathbb{Z}/2) & \xrightarrow{\Theta_n} & H^{n+3}(A^{n+3}(2^r\eta);\mathbb{Z}/2) \\ \cong\big\downarrow j^* & & \cong\big\downarrow j^* \\ H^n(A^{n+3}(\eta^2);\mathbb{Z}/2) & \xrightarrow{\Theta_n \neq 0} & H^{n+3}(A^{n+3}(\eta^2);\mathbb{Z}/2) \end{array}$$

Thus $\Theta \neq 0$ for $X = A^{n+3}(2^r\eta)$. □

The higher-order Bocksteins (1-5)

$$\beta_r\colon H^n(X;\mathbb{Z}/2) \dashrightarrow H^{n+1}(X;\mathbb{Z}/2)$$

are helpful to detect torsion elements of $H_*(X;\mathbb{Z})$ or $H^*(X;\mathbb{Z})$.

Lemma 2.8 (cf [17, pages 173 and 61]) *The following statements hold:*

(1) *The higher Bockstein β_r detects the degree 2^r map on S^n; in other words, for each $r \geq 1$, there is exactly one nontrivial higher Bockstein*

$$\beta_r\colon H^{n-1}(P^n(2^r);\mathbb{Z}/2) \to H^n(P^n(2^r);\mathbb{Z}/2).$$

(2) *For each $r \geq 1$, elements of $H^*(X;\mathbb{Z}/2)$ coming from free integral homology class lie in* $\ker(\beta_r)$ *and not in* $\mathrm{im}(\beta_r)$.

(3) *If $z \in H^{n+1}(X;\mathbb{Z})$ generates a direct summand $\mathbb{Z}/2^r$ for some r, then there exist generators $z' \in H^n(X;\mathbb{Z}/2)$ and $z'' \in H^{n+1}(X;\mathbb{Z}/2)$ such that*

$$\beta_r(z') = z'' \qquad \text{and} \qquad \beta_i(z') = \beta_i(z'') = 0 \quad \text{for } i < r.$$

3 Analysis methods

In this section we list some auxiliary lemmas that simplify the proof arguments in the next section. These lemmas appear to be applicable to other similar problems as well, so we leave them in a separate section.

We say that a map $f : X \to Y$ is *homologically trivial* if the induced homomorphism $f_* \colon H_i(X) \to H_i(Y)$ is trivial for each i.

Lemma 3.1 [10, Theorem 4H.3] *Suppose that X is a simply connected space of dimension N. Write $H_i = H_i(X)$. Then there is a sequence $X_2 \subseteq X_3 \subseteq \cdots \subseteq X_m$ of subcomplexes X_j of X such that*

1. $i_* \colon H_j(X_n) \cong H_j(X)$ *for* $j \le n$ *and* $H_j(X_n) = 0$ *for* $j > n$,
2. $X_2 = M_2(H_2)$ *and* $X_N = X$,
3. *there is a principal homotopy cofibration*

$$M_n(H_{n+1}) \xrightarrow{k_n} X_n \xrightarrow{i_n} X_{n+1} \to M_{n+1}(H_{n+1})$$

with k_n homologically trivial.

Note that we have the canonical inclusions $X^n \subseteq X_n \subseteq X^{n+1}$, where X^k denotes the k–skeleton of X. The map k_n above is called the n^{th} *k'–invariant*, and plays a key role in the homology decomposition of X. For instance, k_n is null-homotopic if and only if $X_n \simeq X_{n-1} \vee M_n(H_n X)$.

Lemma 3.2 *Let $f : \bigvee_{i=1}^m A_i \to \bigvee_{j=1}^n B_j$ be a map which induces trivial homomorphism in cohomology groups with coefficients in abelian groups G and G'. Let*

$$f_\jmath = p_\jmath \circ f \quad \text{and} \quad f_{\imath,\jmath} = f_\jmath \circ i_\imath = p_\jmath \circ f \circ i_\imath,$$

where $i_\imath \colon A_\imath \to \bigvee_{i=1}^m A_i$ and $p_\jmath \colon \bigvee_{j=1}^n B_j \to B_\jmath$ are respectively the canonical inclusion and projection, with $1 \le \imath \le m$ and $1 \le \jmath \le n$.

1. *If $H^*(C_f; G)$ contains no nontrivial cup products, then so do $H^*(C_{f_\jmath}; G)$ and $H^*(C_{f_{\imath,\jmath}}; G)$ for all $\imath$ and $\jmath$.*
2. *If the cohomology operation $\mathbb{O} \colon H^k(C_f; G) \to H^l(C_f; G')$ is trivial, then so are the operations*

$$\mathbb{O}_\jmath \colon H^k(C_{f_\jmath}; G) \to H^l(C_{f_\jmath}; G') \quad \text{and} \quad \mathbb{O}_{\imath\jmath} \colon H^k(C_{f_{\imath\jmath}}; G) \to H^l(C_{f_{\imath\jmath}}; G').$$

where $\mathbb{O}_\jmath$ and $\mathbb{O}_{\imath\jmath}$ are the cohomology operation of the same type as $\mathbb{O}$.

Proof (1) The statement (1) is due to [19, Lemma 4.2].

(2) By the proof of [19, Lemma 4.2], for any integer $k \ge 0$ and any coefficient group G there are monomorphisms

$$d_\jmath^* \colon H^k(C_{f_\jmath}; G) \to H^k(C_f; G)$$

and epimorphisms

$$d_{\imath\jmath}^* \colon H^k(C_{f_\jmath}; G) \to H^k(C_{f_{\imath\jmath}^*}; G).$$

Consider the commutative diagrams

$$\begin{array}{ccccc} H^k(C_f;G) & \xleftarrow{d_\jmath^*} & H^k(C_{f_\jmath};G) & \xrightarrow{d_{\imath,\jmath}^*} & H^k(C_{f_{\imath,\jmath}};G) \\ \downarrow{\scriptstyle \mathbb{O}} & & \downarrow{\scriptstyle \mathbb{O}_\jmath} & & \downarrow{\scriptstyle \mathbb{O}_{\imath,\jmath}} \\ H^l(C_f;G') & \xleftarrow{d_\jmath^*} & H^l(C_{f_\jmath};G') & \xrightarrow{d_{\imath,\jmath}^*} & H^k(C_{f_{\imath,\jmath}};G') \end{array}$$

It follows that $\mathbb{O}_\jmath$ is the restriction of $\mathbb{O}$, and $\mathbb{O}_{\imath\jmath}$ is induced by $\mathbb{O}_\jmath$. Thus if $\mathbb{O}$ is trivial, so are $\mathbb{O}_\jmath$ and $\mathbb{O}_{\imath,\jmath}$. □

The following lemma is useful to determine the homotopy type of a suspension; see [14, Lemma 6.4] or [19, Lemma 5.6].

Lemma 3.3 *Let $S \xrightarrow{f} \left(\bigvee_{i=1}^n A_i\right) \vee B \xrightarrow{g} \Sigma C$ be a homotopy cofibration of simply connected CW–complexes. For $j = 1, \dots, n$, let $p_j : \bigvee_i A_i \to A_j$ be the canonical projection onto the wedge summand A_j. Suppose that each composition*

$$f_j : S \xrightarrow{f} \bigvee_i A_i \xrightarrow{p_j} A_j$$

is null-homotopic. Then there is a homotopy equivalence

$$\Sigma C \simeq \bigvee_{i=1}^n A_i \vee D,$$

where D is the homotopy cofiber of the composition $S \xrightarrow{f} \left(\bigvee_i A_i\right) \vee B \xrightarrow{q_B} B$, with q_B the obvious projection.

Let $X = \Sigma X'$ and $Y_i = \Sigma Y_i'$ be suspensions, for $i = 1, 2, \dots, n$. Let

$$i_l : Y_l \to \bigvee_{j=i}^n Y_i \quad \text{and} \quad p_k : \bigvee_{i=1}^n Y_i \to Y_k$$

be, respectively, the canonical inclusions and projections, for $1 \le k, l \le n$. By the Hilton–Milnor theorem, we may write a map $f : X \to \bigvee_{i=1}^n Y_i$ as

$$f = \sum_{k=1}^n i_k \circ f_k + \theta,$$

where $f_k = p_k \circ f : X \to Y_k$ and θ satisfies $\Sigma\theta = 0$. The first part $\sum_{k=1}^n i_k \circ f_k$ is usually represented by a vector

$$u_f = (f_1, f_2, \dots, f_n)^t.$$

We say that f is completely determined by its components f_k if $\theta = 0$; in this case, write $f = u_f$. Let $h = \sum_{k,l} i_l h_{lk} p_k$ be a self-map of $\bigvee_{i=1}^n Y_i$ which is completely determined by its components $h_{kl} = p_k \circ h \circ i_l : Y_l \to Y_k$. Write

$$M_h := (h_{kl})_{n\times n} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n1} & \cdots & h_{nn} \end{pmatrix}.$$

Then the composition law $h(f+g)\simeq hf+hg$ implies that the product

$$M_h(f_1,f_2,\dots,f_n)^t$$

given by the matrix multiplication represents the composite $h\circ f$. Two maps $f=u_f$ and $g=u_g$ are called *equivalent*, and we write

$$(f_1,f_2,\dots,f_n)^t\sim(g_1,g_2,\dots,g_n)^t,$$

if there is a self-homotopy equivalence h of $\bigvee_{i=1}^n Y_i$, which can be represented by the matrix M_h, such that

$$M_h(f_1,f_2,\dots,f_n)^t\simeq(g_1,g_2,\dots,g_n)^t.$$

Recall that the above matrix multiplication refers to elementary row operations in matrix theory; and note that the homotopy cofibers of the maps $f=u_f$ and $g=u_g$ are homotopy equivalent if f and g are equivalent.

The following lemma serves as an example of the above matrix method.

Lemma 3.4 *Define X by the homotopy cofibration*

$$S^4\xrightarrow{(f_1,f_2,\dots,f_n)^t}\bigvee_{j=1}^n V_j\longrightarrow X,$$

where $f_j\colon S^4\to V_j$ for $j=1,\dots,n$.

(1) *If $V_j=S^3$ for $j=1,2,\dots,n$ and $f_{j_0}=\eta$ for some j_0, then there is a homotopy equivalence*

$$X\simeq C_\eta^5\vee\bigvee_{j\neq j_0}S^3.$$

(2) *If $V_j=P^4(2^{r_j})$ for $j=1,2,\dots,n$, and $f_j=i_3\eta$ for some j, then there is a homotopy equivalence*

$$X\simeq C_{r_{j_1}}^5\vee\bigvee_{j\neq j_1}P^4(2^{r_j}),$$

where $j_1=\max\{1\le j\le n\mid f_j=i_3\eta\}$.

Proof (1) If there is a unique $f_{j_0}=\eta$, the statement clearly holds. We may assume that $f_1=\eta$ and $f_i=\varepsilon_i\cdot\eta$, with $\varepsilon_i\in\{0,1\}$. Then

$$\begin{pmatrix}1&0&\cdots&0\\-\varepsilon_2&1&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\-\varepsilon_n&0&\cdots&1\end{pmatrix}\begin{pmatrix}\eta\\\varepsilon_2\cdot\eta\\\vdots\\\varepsilon_n\cdot\eta\end{pmatrix}\simeq\begin{pmatrix}\eta\\0\\\vdots\\0\end{pmatrix}.$$

It follows that there exists a self-homotopy equivalence e_S of $\bigvee_{j=1}^m S^3$ such that

$$e_Sf\sim(\eta,0,\dots,0)^t,$$

and hence there is a homotopy equivalence

$$X = C_f \simeq C_{e_S f} \simeq C_\eta^5 \vee \bigvee_{j=2}^{m} S^3.$$

(2) The statement clearly holds if there is a unique j such that $f_j = i_3\eta$. Let j_1 be defined in the lemma. If there is an index j_2 such that

$$f_{j_2} = i_3\eta \in \pi_4(P^4(2^{r_{j_2}})),$$

then the matrix multiplication

$$\begin{pmatrix} 1_P & 0 \\ -B(\chi_s^r) & 1_P \end{pmatrix} \begin{pmatrix} i_3\eta \\ i_3\eta \end{pmatrix} \simeq \begin{pmatrix} i_3\eta \\ 0 \end{pmatrix}$$

implies $(f_{j_1}, f_{j_2})^t \sim (f_{j_1}, 0)^t$. By induction, it follows that there exists a self-homotopy equivalence e_P of $\bigvee_{j=1}^m P^4(2^{r_j})$ such that

$$e_P \circ (f_1, f_2, \dots, f_n)^t \simeq (0, \dots, 0, i_3\eta, 0, \dots, 0)^t,$$

where $i_3\eta$ in the latter vector lies in the j_1^{th} position. Thus we have a homotopy equivalence

$$X = C_f \simeq C_{r_{j_1}}^5 \vee \bigvee_{j \neq j_1} P^4(2^{r_j}). \qquad \square$$

4 Homology decomposition of ΣM

By Table 1 and [19, Lemma 5.1], there is a homotopy equivalence

$$\Sigma M \simeq \Big(\bigvee_{i=1}^{m} S^2\Big) \vee \Sigma W, \tag{4-1}$$

where W is a CW–complex with integral homology given by Table 2. By Lemma 3.1 and Table 2, there are homotopy cofibrations

$$\bigvee_{i=1}^{d} S^2 \vee P^3(T) \xrightarrow{k_3} P^3(T) \to W_3, \quad \bigvee_{i=1}^{m} S^3 \xrightarrow{k_4} W_3 \to W_4, \quad S^4 \xrightarrow{k_5} W_4 \to \Sigma W, \tag{4-2}$$

where k_3, k_4, k_5 are homologically trivial maps. Let $T_2 = \bigoplus_{j=1}^n \mathbb{Z}/2^{r_j}$ be the 2–primary component of T and write $T = T_2 \oplus T_{\neq 2}$. For each $k \geq 3$, there are homotopy equivalences (cf [18])

$$P^k(T) \simeq P^k(T_2) \oplus P^k(T_{\neq 2}) \simeq \Big(\bigvee_{j=1}^{n} P^k(2^{r_j})\Big) \vee P^k(T_{\neq 2}).$$

i	$0, 4$	1	2	3	≥ 5
$H_i(W)$	$\mathbb{Z}$	T	$\mathbb{Z}^d \oplus T$	$\mathbb{Z}^m$	0

Table 2: $H_*(W;\mathbb{Z})$.

Lemma 4.1 *There is a homotopy equivalence*

$$W_3 \simeq \Big(\bigvee_{i=1}^{d} S^3\Big) \vee P^3(T) \vee P^4(T).$$

Proof By (4-2), there is a homotopy cofibration

$$\Big(\bigvee_{i=1}^{d} S^2\Big) \vee P^3(T) \xrightarrow{f} P^3(T) \to W_3,$$

where f is a homologically trivial map with its two components of the types

$$f_1^S : \Big(\bigvee_{i=1}^{d} S^2\Big) \hookrightarrow \Big(\bigvee_{i=1}^{d} S^2\Big) \vee P^3(T) \xrightarrow{f} P^3(T) \quad \text{and} \quad f_2^T : P^3(T) \hookrightarrow \Big(\bigvee_{i=1}^{d} S^2\Big) \vee P^3(T) \xrightarrow{f} P^3(T).$$

Here the arrows $\hookrightarrow$ denote the obvious inclusions. Clearly f_1^S and f_2^T are both homologically trivial. Set $T = \bigoplus_{k=1}^{l} p_k^{r_k}$ with p_k primes. Then the Hurewicz isomorphism $\pi_2(P^3(T)) \cong H_2(P^3(T))$ implies that both f_1^S and the composite

$$S_T = \bigvee_{k=1}^{l} S^2 \xrightarrow{j} P^3(T) \xrightarrow{f_2^T} P^3(T)$$

are null-homotopic, where j is the canonical inclusion. Let

$$m_T = \bigvee_{k} p_k^{r_k} : S_T \to S_T$$

be the attaching map of $P^3(T)$. There is a homotopy commutative diagram of homotopy cofibrations

$$\begin{array}{ccccc}
S_T & \longrightarrow & * & \longrightarrow & \Sigma S_T \\
\downarrow{\scriptstyle m_T} & & \downarrow & & \downarrow{\scriptstyle i_2\circ(\Sigma m_T)} \\
S_T & \xrightarrow{0} & P^3(T) & \longrightarrow & P^3(T) \vee \Sigma S_T \\
\downarrow{\scriptstyle i} & & \| & & \downarrow \\
P^3(T) & \xrightarrow{f_2^T} & P^3(T) & \longrightarrow & C_{f_2^T}
\end{array}$$

in which rows are columns are homotopy cofibrations. It follows that

$$C_{f_2^T} \simeq P^3(T) \vee P^4(T),$$

and hence there is a homotopy equivalence

$$W_3 \simeq \bigvee_{i=1}^{d} S^3 \vee C_{f_2^T} \simeq \Big(\bigvee_{i=1}^{d} S^3\Big) \vee P^3(T) \vee P^4(T).$$

□

Lemma 4.2 *There is a homotopy equivalence*

$$W_4 \simeq \Big(\bigvee_{i=1}^{d} S^3\Big) \vee P^4(T) \vee C_{g_2}$$

for some homologically trivial map $g_2 : \bigvee_{i=1}^{m} S^3 \to P^3(T)$*. Moreover, there is a homotopy equivalence*

$$\Sigma W_4 \simeq \Big(\bigvee_{i=1}^{d} S^4\Big) \vee P^4(T) \vee P^5(T) \vee \bigvee_{i=1}^{m} S^5.$$

Proof By (4-2) and Lemma 4.1, there is a homotopy cofibration

$$\bigvee_{i=1}^{m} S^3 \xrightarrow{g} \Big(\bigvee_{i=1}^{d} S^3\Big) \vee P^3(T) \vee P^4(T) \to W_4,$$

with g a homologically trivial map. The map g is determined by the following components:

$$g_1\colon S^3 \to \bigvee_{i=1}^{m} S^3 \xrightarrow{g} \Big(\bigvee_{i=1}^{d} S^3\Big) \vee P^3(T) \vee P^4(T) \to \bigvee_{i=1}^{d} S^3 \to S^3,$$

$$g_2\colon S^3 \to \bigvee_{i=1}^{m} S^3 \xrightarrow{g} \Big(\bigvee_{i=1}^{d} S^3\Big) \vee P^3(T) \vee P^4(T) \to P^3(T),$$

$$g_3\colon S^3 \to \bigvee_{i=1}^{m} S^3 \xrightarrow{g} \Big(\bigvee_{i=1}^{d} S^3\Big) \vee P^3(T) \vee P^4(T) \to P^4(T).$$

Here the unlabeled maps are the obvious inclusions and projections. The maps g_1, g_2, g_3 are all homologically trivial. The Hurewicz theorem implies that both g_1 and g_3 are null-homotopic. Then by Lemma 3.3 we get the first statement.

To prove the second homotopy equivalence, it suffices to show that if $f : S^4 \to P^4(T)$ is homologically trivial, then f is null-homotopic. Consider the following homologically trivial components of f:

$$f_1\colon S^4 \xrightarrow{f} P^4(T) \to P^4(T_{\neq 2}),$$

$$f_2^j : S^4 \xrightarrow{f} P^4(T) \to P^4(T_2) \to P^4(2^{r_j}) \quad \text{for } j = 1, 2, \dots, n.$$

The map f_1 is clearly null-homotopic, because $\pi_4(P^4(p^r)) = 0$ for odd primes p. Observe that $W_4 = \Sigma W^4$ is a suspension, the Steenrod square Sq^2 acts trivially on $H^2(W_4; \mathbb{Z}/2)$. By Lemma 3.2(2), Sq^2 acts trivially on $H^3(C_{f_2^j}; \mathbb{Z}/2)$. Since $\pi_4(P^4(2^{r_j})) \cong \mathbb{Z}/2\langle i_3\eta\rangle$ (Lemma 2.1), we may set

$$f_2^j = \varepsilon_j \cdot i_3\eta, \quad \text{where } \varepsilon_j \in \mathbb{Z}/2.$$

Note that the homotopy cofiber of $i_3\eta \in \pi_4(P^4(2^{r_j}))$ is the Chang complex $C^5_{r_j}$, by Lemma 2.5 we then get that $\varepsilon_j = 0$, or equivalently f_2^j is null-homotopic for each $j = 1, 2, \dots, n$. Thus f is null-homotopic, by Lemma 3.3. □

5 Proofs of Theorems 1.1 and 1.2

By (4-2) and Lemma 4.2, there is a homotopy cofibration

$$S^5 \xrightarrow{h} \Big(\bigvee_{i=1}^{d} S^4\Big) \vee P^4(T) \vee P^5(T) \vee \bigvee_{i=1}^{m} S^5 \to \Sigma^2 W,$$

where h a homologically trivial map, $T \cong T_2 \oplus T_{\neq 2}$ with $T_2 \cong \bigoplus_{j=1}^{n} \mathbb{Z}/2^{r_j}$. Since $\pi^5(P^4(p^r)) = \pi_5(P^5(p^s)) = 0$ for any odd primes p, Lemma 3.3 indicates that there is a homotopy equivalence

$$\Sigma^2 W \simeq P^4(T_{\neq 2}) \vee P^5(T_{\neq 2}) \vee \Big(\bigvee_{i=1}^{m} S^5\Big) \vee C_\varphi, \tag{5-1}$$

where $\varphi\colon S^5 \to \big(\bigvee_{i=1}^d S^4\big) \vee P^4(T_2) \vee P^5(T_2)$ is a homologically trivial map. The map φ has the following three types of components:

$$\varphi_1\colon S^5 \xrightarrow{\varphi} \Big(\bigvee_{i=1}^d S^4\Big) \vee P^4(T_2) \vee P^5(T_2) \to S^4,$$

$$\varphi_2^j\colon S^5 \xrightarrow{\varphi} \Big(\bigvee_{i=1}^d S^4\Big) \vee P^4(T_2) \vee P^5(T_2) \to P^4(T_2) \to P^4(2^{r_j}),$$

$$\varphi_3^j\colon S^5 \xrightarrow{\varphi} \Big(\bigvee_{i=1}^d S^4\Big) \vee P^4(T_2) \vee P^5(T_2) \to P^5(T_2) \to P^5(2^{r_j}),$$

where $j = 1, 2, \dots, n$ and the unlabeled maps are the obvious projections.

Proposition 5.1 *If* $\mathrm{Sq}^2(H^4(\Sigma^2 W;\mathbb{Z}/2)) = 0$, *then the homotopy type of* $\Sigma^2 W$ *is determined by the secondary operation* Θ *of equation* (2-7) *and the higher Bockstein* β_r. *Explicitly, if* $\Theta(H^3(C_\varphi;\mathbb{Z}/2)) = 0$, *then there is a homotopy equivalence*

$$C_\varphi \simeq \Big(\bigvee_{i=1}^d S^4\Big) \vee P^4(T_2) \vee P^5(T_2) \vee S^6.$$

Otherwise we have

$$C_\varphi \simeq \Big(\bigvee_{i=1}^d S^4\Big) \vee P^4\Big(\frac{T_2}{\mathbb{Z}/2^{r_{j_0}}}\Big) \vee P^5(T_2) \vee A^6(2^{r_{j_0}}\eta^2),$$

where j_0 *is the maximum of the indices* j *satisfying*

$$\Theta(x) \neq 0 \quad \text{and} \quad \beta_{r_j}(x) \neq 0 \quad \text{for all } x \in H^3(C_\varphi;\mathbb{Z}/2).$$

Proof By assumption and (5-1), Sq^2 acts trivially on $H^4(C_\varphi;\mathbb{Z}/2)$, and hence so does Sq^2 on $H^4(C_{\varphi_1};\mathbb{Z}/2)$ and $H^4(C_{\varphi_k^j};\mathbb{Z}/2)$ for each $k = 2, 3$ and $j = 1, 2, \dots, n$, by Lemma 3.2(2). It follows by Lemmas 2.5 and 2.6 that φ_1 and φ_3^i are null-homotopic, and

$$\varphi_2^j = y_j \cdot i_3\eta^2 \quad \text{for all } y_j \in \mathbb{Z}/2 \text{ and } j = 1, 2, \dots, n.$$

By Lemma 2.7, the coefficients y_j can be detected by the secondary operation Θ. There are possibly many such indices j; however, similar arguments to those in the proof of Lemma 3.4 show that there exists a homotopy equivalence e of $P^4(T_2)$ such that

$$e(\varphi_2^1, \varphi_2^2, \dots, \varphi_2^n)^t \simeq (0, \dots, 0, \varphi_2^{j_0} = i_3\eta^2, 0, \dots, 0)^t,$$

with j_0 described in the proposition. The proof is then completed by applying Lemma 3.3. □

Proposition 5.2 *Suppose that* $\mathrm{Sq}^2(H^4(\Sigma^2 W;\mathbb{Z}/2)) \neq 0$. *Then the homotopy types of* C_φ *or* ΣC_φ *can be characterized as follows*:

(1) *Suppose that for any $u \in H^4(\Sigma^2 M;\mathbb{Z}/2)$ with $\mathrm{Sq}^2(u)\neq 0$ and any $v\in\ker(\mathrm{Sq}^2)$, it holds that*

$$\beta_r(u+v)=0 \quad \text{and} \quad u+v\notin \mathrm{im}(\beta_s) \quad \text{for all } r,s\geq 1.$$

Then there is a homotopy equivalence

$$C_\varphi \simeq \Big(\bigvee_{i=1}^{d-1} S^4\Big)\vee P^5(T_2)\vee P^4(T_2)\vee C^6_\eta.$$

(2) *Suppose that for any $u \in H^4(\Sigma^2 M;\mathbb{Z}/2)$ with $\mathrm{Sq}^2(u)\neq 0$ and any $v\in\ker(\mathrm{Sq}^2)$, it holds that*

$$u+v\notin \mathrm{im}(\beta_s) \quad \text{for all } s\geq 1,$$

while there exist $u' \in H^4(\Sigma^2 M;\mathbb{Z}/2)$ with $\mathrm{Sq}^2(u')\neq 0$ and $v'\in\ker(\mathrm{Sq}^2)$ such that

$$\beta_r(u'+v')\neq 0 \quad \text{for some } r\geq 1.$$

Then there is a homotopy equivalence

$$C_\varphi \simeq \Big(\bigvee_{i=1}^{d} S^4\Big)\vee P^5\Big(\frac{T_2}{\mathbb{Z}/2^{r_{j_1}}}\Big)\vee P^4(T_2)\vee C^6_{r_{j_1}},$$

with j_1 the maximum of indices j such that

$$\mathrm{Sq}^2(u')\neq 0 \quad \text{and} \quad \beta_{r_{j_1}}(u'+v')\neq 0.$$

(3) *Suppose that there exist $u \in H^4(\Sigma^2 M;\mathbb{Z}/2)$ with $\mathrm{Sq}^2(u)\neq 0$ and $v\in\ker(\mathrm{Sq}^2)$ such that*

$$u+v\in \mathrm{im}(\beta_r) \quad \text{for some } r\geq 1.$$

(a) *If $\Theta(H^3(C_\varphi;\mathbb{Z}/2))=0$, then there is a homotopy equivalence*

$$C_\varphi \simeq \Big(\bigvee_{i=1}^{d} S^4\Big)\vee P^5(T_2)\vee P^4\Big(\frac{T_2}{\mathbb{Z}/2^{r_{j_2}}}\Big)\vee A^6(\tilde{\eta}_{r_{j_2}}),$$

with j_2 the minimum of the indices j such that $u+v\in\mathrm{im}(\beta_{r_j})$.

(b) *If $\Theta(H^3(C_\varphi;\mathbb{Z}/2))\neq 0$ and $T_2\cong \mathbb{Z}/2^{r_{j_2}}$, then*

$$C_\varphi \simeq \Big(\bigvee_{i=1}^{d} S^4\Big)\vee P^5(T_2)\vee P^4\Big(\frac{T_2}{\mathbb{Z}/2^{r_{j_2}}}\Big)\vee A^6_\varepsilon(\tilde{\eta}_{r_{j_2}}),$$

where $A^6_\varepsilon(\tilde{\eta}_{r_{j_2}})$ is the homotopy cofiber of $\tilde{\eta}_{r_{j_2}}+\varepsilon\cdot i_3\eta^2$ with $\varepsilon\in\{0,1\}$.

(c) *If $\Theta(H^3(C_\varphi;\mathbb{Z}/2))\neq 0$ and T_2 has at least two direct summands, then*

$$\Sigma C_\varphi \simeq \Big(\bigvee_{i=1}^{d} S^5\Big)\vee P^6(T_2)\vee P^5\Big(\frac{T_2}{\mathbb{Z}/2^{r_{j_2}}\oplus \mathbb{Z}/2^{r_{j'_0}}}\Big)\vee A^7_\varepsilon(\tilde{\eta}_{r_{j_2}})\vee A^7(2^{r_{j'_0}}\eta^2),$$

where $A^7_\varepsilon(\tilde{\eta}_{r_{j_2}})=\Sigma A^6_\varepsilon(\tilde{\eta}_{r_{j_2}})$, the index j_2 the minimum of the indices $j\leq n$ such that $u+v\in\mathrm{im}(\beta_{r_j})$, and j'_0 is the maximum of the indices $j\leq n$, $j\neq j_2$ such that

$$\Theta(x)\neq 0 \quad \text{and} \quad \beta_{r_j}(x)\neq 0 \quad \text{for all } x\in H^3(C_\varphi;\mathbb{Z}/2).$$

Proof By the Hilton–Milnor theorem and Lemmas 2.1 and 2.7, we can put

$$\varphi = \sum_{i=1}^{d} x_i \cdot \eta + \sum_{j=1}^{n} y_j \cdot i_4\eta + \sum_{k=1}^{n} z_k \cdot \tilde{\eta}_{r_k} + \sum_{l=1}^{n} w_l \cdot i_3\eta^2 + \theta, \tag{5-2}$$

where θ is a linear combination of Whitehead products in $\pi_5(P^4(T_2))$.

By Lemmas 2.5 and 2.6, the condition $\mathrm{Sq}^2(H^4(\Sigma^2 W;\mathbb{Z}/2)) \neq 0$ enforces that at least one of these coefficients x_i, y_j, z_k is nonzero.

(1) Under the conditions in (1), we deduce from Lemma 2.8 that u comes from a free integral homology class. It follows that

$$y_j = z_k = 0 \qquad \text{and} \qquad x_i = 1 \quad \text{for some } i$$

in the expression (5-2). By Lemma 3.4(1), we may assume that there is exactly one index i such that $x_i = 1$. Consider the homotopy type of the map

$$\begin{pmatrix} \eta \\ i_3\eta^2 \end{pmatrix} : S^5 \to S^4 \vee P^4(2^{r_{j_0}}).$$

Clearly we have an equivalence

$$\begin{pmatrix} \eta \\ i_3\eta^2 \end{pmatrix} \sim \begin{pmatrix} \eta \\ 0 \end{pmatrix}.$$

After composing with self-homotopy equivalences of $\left(\bigvee_{i=1}^{d} S^4\right) \vee P^4(T_2) \vee P^5(T_2)$, we may assume that the above x_i is the unique nonzero coefficient, which completes the proof of the homotopy equivalence in (1) by Lemma 3.3.

(2) The arguments are similar to (1). The conditions (2) imply that

$$x_i = z_k = 0 \qquad \text{and} \qquad y_j = 1 \quad \text{for some } j,$$

while Lemma 3.4(2) guarantees that we may assume that there is exactly one such j, which is equal to j_1, as described in the proposition. By Lemma 2.1(4), we have

$$(i_3\bar{\eta}_{r_{j_1}})(i_4\eta) = i_3(\bar{\eta}_{r_{j_1}} i_4)\eta = i_3\eta^2,$$

hence

$$\begin{pmatrix} i_4\eta \\ i_3\eta^2 \end{pmatrix} \sim \begin{pmatrix} i_4\eta \\ 0 \end{pmatrix}.$$

Thus we may assume that all coefficients $w_l = 0$ and the homotopy equivalence in (2) then follows.

(3) The conditions (3) imply that

$$z_k \equiv 1 \pmod 2 \quad \text{for some } k.$$

By Lemma 3.4, we may firstly assume that

$$x_1 = \epsilon \in \mathbb{Z}/2, \qquad x_i = 0 \quad \text{for } i > 1,$$
$$y_{j_0} = \varepsilon \in \mathbb{Z}/2, \qquad y_j = 0 \quad \text{for } j \neq j_0.$$

Note that $(\epsilon,\varepsilon)\neq(1,1)$, because

$$\begin{pmatrix}\eta\\ i_4\eta\end{pmatrix}\sim\begin{pmatrix}\eta\\ 0\end{pmatrix}: S^5\to S^4\vee P^5(2^r).$$

By the relation $q_4\tilde{\eta}_{r_k}=\eta$ in (2-3), we have

$$\begin{pmatrix}\eta\\ \tilde{\eta}_{r_k}\end{pmatrix}\sim\begin{pmatrix}0\\ \tilde{\eta}_{r_k}\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}i_4\eta\\ \tilde{\eta}_{r_k}\end{pmatrix}\sim\begin{pmatrix}0\\ \tilde{\eta}_{r_k}\end{pmatrix}.$$

It follows that $z_k\equiv 1 \pmod 2$ implies that $\epsilon=\varepsilon=0$. By Corollary 2.2 we have

$$\begin{pmatrix}1 & 0\\ -B(\chi^r_s) & 1\end{pmatrix}\begin{pmatrix}\tilde{\eta}_r\\ \tilde{\eta}_s\end{pmatrix}\simeq\begin{pmatrix}\tilde{\eta}_r\\ 0\end{pmatrix}\quad\text{for } r\leq s.$$

Thus up to homotopy we may assume that $x_i=y_j=0$ and there exists exactly one $z_{k_0}\equiv 1 \pmod 2$ with $k_0=j_2$ as described in the proposition.

If $\Theta(H^3(C_\varphi;\mathbb{Z}/2))=0$, then Lemma 2.7 implies that $w_l=0$ for all l. The first homotopy equivalence in (3) then follows by Lemma 3.3.

If $\Theta(H^3(C_\varphi;\mathbb{Z}/2))\neq 0$, by Lemma 2.7 we have $w_l\neq 0$ for at least one $l\leq n$. It reduces to considering the homotopy type of the homotopy cofiber of the component

$$\varphi_2\colon S^5\to P^4(T_2).$$

Note that when composing a self-homotopy equivalence of $P^4(T_2)$ to get z_{j_2}, it happens that $w_{j_2}=0$ or $w_{j_2}=1$.

If $n=1$ and $T_2\cong\mathbb{Z}/2^{r_{j_2}}$, the second homotopy equivalence in (3) then follows by Lemma 3.3. If $n\geq 2$, there are indices $l\neq j_2$, then similar arguments to that in the proof of Proposition 5.1 show that there is an equivalence

$$(i_3\eta^2,\dots,i_3\eta^2,\dots)^t\sim(0,\dots,0,i_3\eta^2,0,\dots,0)^t,$$

where the unique $i_3\eta^2$ appears at the maximal j_0' among indices $l\leq n$, with $l\neq j_2$, such that

$$\Theta(x)\neq 0\quad\text{and}\quad\beta_{r_{j_0'}}(x)\neq 0\quad\text{for all } x\in H^3(C_\varphi;\mathbb{Z}/2).$$

Thus we get a homotopy equivalence $C_{\varphi_2}\simeq C_{\varphi_2'+\theta}$, where

$$\varphi_2'=(0,\dots,0,\tilde{\eta}_{r_{j_2}}+\varepsilon\cdot i_3\eta^2,0,\dots,0,i_3\eta^2,0,\dots,0)^t.$$

After one suspension the possible Whitehead product θ becomes trivial. Thus we get

$$\Sigma C_{\varphi_2'}\simeq P^5\left(\frac{T_2}{\mathbb{Z}/2^{r_{j_2}}\oplus\mathbb{Z}/2^{j_0'}}\right)\vee A^7_\varepsilon(\tilde{\eta}_{r_{j_2}})\vee A^7(2^{r_{j_0'}}\eta^2),$$

and therefore $\Sigma C_\varphi\simeq\left(\bigvee_{i=1}^d S^5\right)\vee P^6(T_2)\vee\Sigma C_{\varphi_2'}$. □

Proof of Theorem 1.1 It is well known that a closed, smooth, connected, orientable 4–manifold M is spin if and only if the Steenrod square Sq^2 acts trivially on $H^2(M;\mathbb{Z}/2)$. The homotopy types of $\Sigma^2 M$ in Theorem 1.1 then are obtained by (4-1), (5-1) and Propositions 5.1 and 5.2. □

Next, we give a proof of Theorem 1.2. By (1-3), there hold equivalence relations

$$\mathfrak{P}_0(H^1(M;\mathbb{Z}/2^r)) = 0 \iff \mathfrak{P}_1(H^2(\Sigma M;\mathbb{Z}/2^r)) = 0.$$

Lemma 5.3 *If the Pontryagin square $\mathfrak{P}_1$ acts trivially on $H^2(\Sigma M;\mathbb{Z}/2^r)$, then so does $\mathfrak{P}_1$ on $H^2(W_4;\mathbb{Z}/2^r)$.*

Proof By Lemma 3.1 and the universal coefficient theorem for cohomology, the canonical inclusion $i\colon W_4 \to \Sigma W$ induces isomorphisms

$$i^*\colon H^2(\Sigma W;\mathbb{Z}/2^r) \xrightarrow{\cong} H^2(W_4;\mathbb{Z}/2^r) \quad \text{and} \quad i^*\colon H^4(\Sigma W;\mathbb{Z}/2^{r+1}) \xrightarrow{\cong} H^2(W_4;\mathbb{Z}/2^{r+1}).$$

If $\mathfrak{P}_1$ acts trivially on $H^2(\Sigma M;\mathbb{Z}/2^r)$, then so does $\mathfrak{P}_1$ on $H^2(\Sigma W;\mathbb{Z}/2^r)$, by (4-1). The commutative diagram

$$\begin{array}{ccc} H^2(\Sigma W;\mathbb{Z}/2^r) & \xrightarrow{\mathfrak{P}_1=0} & H^4(\Sigma W;\mathbb{Z}/2^{r+1}) \\ \cong\downarrow i^* & & \cong\downarrow i^* \\ H^2(W_4;\mathbb{Z}/2^r) & \xrightarrow{\mathfrak{P}_1} & H^4(W_4;\mathbb{Z}/2^{r+1}) \end{array}$$

then implies $\mathfrak{P}_1 = 0$ on the second row. □

Lemma 5.4 *If the Pontryagin square $\mathfrak{P}_1$ acts trivially on $H^2(\Sigma M;\mathbb{Z}/2^{r_j})$ for each $j = 1, 2, \dots, n$, then there is a homotopy equivalence*

$$W_4 \simeq \Big(\bigvee_{i=1}^{d} S^3\Big) \vee \Big(\bigvee_{i=1}^{m} S^4\Big) \vee P^3(T) \vee P^4(T).$$

Proof By Lemma 4.2 there is a homotopy equivalence

$$W_4 \simeq \Big(\bigvee_{i=1}^{d} S^3\Big) \vee P^4(T) \vee C_{g_2}$$

for some homologically trivial map $g_2\colon \bigvee_{i=1}^m S^3 \to P^3(T)$. It suffices to show the homologically trivial component

$$g_2\colon S^3 \to P^3(T)$$

is null-homotopic. By Lemma 3.3, it suffices to show that the components

$$g_2^j\colon S^3 \xrightarrow{g_2} P^3(T_2) \xrightarrow{p_j} P^3(2^{r_j})$$

are null-homotopic for each $j = 1, 2, \dots, n$.

Since $\pi_3(P^3(2^r)) \cong \mathbb{Z}/2^{r+1}$, for all $j = 1, 2, \dots, n$ we may set

$$g_2^j = t_j \cdot i_2\eta$$

for some $t_j \in \mathbb{Z}/2^{r_j+1}$. The assumption and Lemma 5.3 imply that the Pontryagin square

$$\mathfrak{P}_1 \colon H^2(W_4; \mathbb{Z}/2^{r_j}) \to H^4(W_4; \mathbb{Z}/2^{r_j+1})$$

is trivial. By the universal coefficient theorem for cohomology, g_2^j induces trivial homomorphism in mod 2^{r_j} or mod 2^{r_j+1} cohomology, and hence by Lemma 3.2(2), the Pontryagin square

$$\mathfrak{P}_1 \colon H^2(C_{g_2^j}; \mathbb{Z}/2^{r_j}) \to H^4(C_{g_2^j}; \mathbb{Z}/2^{r_j+1})$$

is trivial for each j. Then it follows by Lemma 2.4 that $t_j = 0$, or equivalently g_2^j is null-homotopic for each $j = 1, 2, \dots, n$. □

Proof of Theorem 1.2 By Lemma 5.4 and (4-2), there is a homotopy cofibration

$$S^4 \xrightarrow{k_5} W_4 \simeq \Big(\bigvee_{i=1}^{d} S^3\Big) \vee \Big(\bigvee_{i=1}^{m} S^4\Big) \vee P^3(T) \vee P^4(T) \to \Sigma W,$$

with k_5 homologically trivial. Since $\pi_4(P^3(p^r)) = \pi_4(P^4(p^r)) = 0$, Lemma 3.3 implies that there is a homotopy equivalence

$$\Sigma W \simeq \Big(\bigvee_{i=1}^{m} S^4\Big) \vee P^3(T_{\neq 2}) \vee P^4(T_{\neq 2}) \vee C_\phi,$$

where $\phi \colon S^4 \to \big(\bigvee_{i=1}^{d} S^3\big) \vee P^3(T_2) \vee P^4(T_2)$ is a homologically trivial map. Compare (5-1). The discussion on the homotopy type of ΣW is totally parallel to that of $\Sigma^2 W$ in the proofs of Propositions 5.1 and 5.2. The proof is then completed by (4-1). □

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