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The braid indices of most links remain unknown as there is no known universal method for determining the braid index of an arbitrary knot. This is also the case for alternating knots. We show that if K is an alternating knot, then the braid index of any reverse parallel link of K can be precisely determined. Specifically, if D is a reduced diagram of K, $v_+(D)$ (resp. $v_-(D)$) is the number of regions in the checkerboard shading of D for which all crossings are positive (resp. negative) and $w(D)$ is the writhe of D, then the braid index of a reverse parallel link of K with framing f, denoted by $\mathbb{K}_f$, is given by the precise formula

$$\boldsymbol{b}(\mathbb{K}_f) = \begin{cases} c(D)+2+a(D)-f & \text{if } f < a(D), \\ c(D)+2 & \text{if } a(D) \le f \le b(D), \\ c(D)+2-b(D)+f & \text{if } f > b(D), \end{cases}$$

where $a(D) = -v_-(D) + w(D)$ and $b(D) = v_+(D) + w(D)$.

57K10, 57K31

1 Introduction

The determination of the braid index of a knot or a link is known to be a challenging problem. To date there is no known method that can be used to determine the precise braid index of an arbitrarily given knot/link. This is also the case when we restrict ourselves to alternating knots and links, although the braid indices of many alternating knots and links can now be determined. For example, all 2–bridge links and all alternating Montesinos links; see Diao, Ernst, Hetyei and Liu [6] and Murasugi [13]. However, we prove a somewhat surprising result: the braid index of any *reverse parallel* link of an alternating knot can be precisely determined. Furthermore, the formula can be derived easily from any reduced diagram of the alternating knot.

Here we study the *reverse parallel* links of alternating knots. A reverse parallel link of a knot consists of the two boundary components of an annulus A embedded in S^3 with the said knot being one of the components and such that the two components are assigned opposite orientations. Let K and K' be the two components of a reverse parallel link induced by an annulus A. Following the convention that has been used in the literature (such as by Nutt [15] and Rudolph [16]), we shall call the linking number f between K and K' when they are assigned parallel orientations the *framing* of K. We note that a reverse

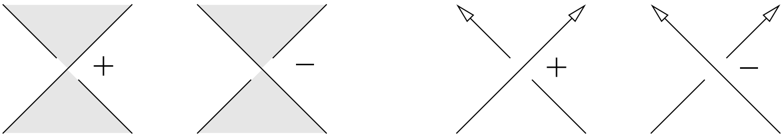


Figure 1: Left: the crossing with respect to a checkerboard shading. Right: the crossing sign with respect to the orientation of the knot.

parallel link of K with framing f is denoted by $K *_f A$ in [15] and by $\mathrm{Bd}_A(K, f)$ in [16]. The framing is independent of the orientation of K, and the ambient isotopy class of A in S^3 depends only on K and the framing. Therefore, the reverse parallel links of K are characterized by the framing f. Since our results (and proofs) only depend on the framing, not the actual annulus A, we shall introduce a new notation $\mathbb{K}_f$ for the reverse parallel link of K with framing f. Keep in mind that the framing f is the linking number of the two components of $\mathbb{K}_f$ with parallel orientations, and hence the linking number of $\mathbb{K}_f$ itself is $-f$.

For a given knot diagram D with a checkerboard shading, a crossing can be assigned a $+$ or a $-$ sign relative to this shading, as shown on the left side of Figure 1. This is not to be confused with the crossing sign with respect to the orientation of the knot which is used in the definition of the writhe of D, as shown on the right side of Figure 1.

Now let K be an alternating knot with a reduced diagram D. It is known that in such a case crossings of D are all positive with respect to one checkerboard shading of D and are all negative with respect to the other checkerboard shading of D. Furthermore, if we let $v_+(D)$ be the number of shaded regions in the shading with respect to which all crossings are positive, and $v_-(D)$ be the number of shaded regions in the complementary shading with respect to which all crossings are negative, then $v_+(D)+v_-(D)-2=c(D)$ where $c(D)$ is the number of crossings in D; see Kauffman [9]. From D we can also obtain its so-called blackboard reverse parallel annulus (resp. framing), which provides a good reference for other choices of annuli (resp. framings) as the other choices come from this one by adding either right-handed or left-handed twists. If the writhe of D is $w(D)$, then the framing of the blackboard reverse parallel is also $w(D)$. If k right-handed (resp. left-handed) twists are added between the two components, then the resulting reverse parallel has framing $w(D)+k$ (resp. $w(D)-k$). See Figure 2 for an illustration.

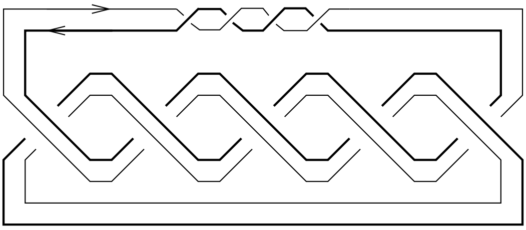


Figure 2: The blackboard reverse parallel of the $(2,5)$ torus knot with two left-handed twists added. The framing of the resulting reverse parallel link (with the added twists) is thus $5+(-2)=3$.

Our main result is the following theorem:

Theorem 1.1 *Let K be an alternating knot and D a reduced diagram of K. Let $c(D)$, $w(D)$, $v_+(D)$ and $v_-(D)$ be as defined above. Then the braid index of $\mathbb{K}_f$, denoted by $\boldsymbol{b}(\mathbb{K}_f)$, is given by the formula*

$$\boldsymbol{b}(\mathbb{K}_f)=\begin{cases} c(D)+2+a(D)-f & \text{if } f<a(D),\\ c(D)+2 & \text{if } a(D)\le f\le b(D),\\ c(D)+2-b(D)+f & \text{if } f>b(D),\end{cases} \tag{1-1}$$

where $a(D)=-v_-(D)+w(D)$ and $b(D)=v_+(D)+w(D)$.

We can summarize Theorem 1.1 pictorially in terms of the blackboard reverse parallel of D:

- The blackboard reverse parallel has braid index $c(D)+2$.
- The braid index remains $c(D)+2$ after adding up to $v_+(D)$ right-handed twists, or up to $v_-(D)$ left-handed twists.
- Each further right or left-handed twist increases the braid index by 1.

So for example, since $v_-(D)=2$ and $v_+(D)=5$ for the $(2,5)$ torus knot, the braid index for the reverse parallel shown in Figure 2 is $c(D)+2=7$. Adding one further left-handed twist would increase the braid index to 8, while we would still have braid index 7 after adding up to 5 right-handed twists to the blackboard parallel.

We shall establish (1-1) by proving that the right side expression is both a lower bound and an upper bound for the $\boldsymbol{b}(\mathbb{K}_f)$. The lower bound is obtained by the Morton–Franks–Williams inequality, while the upper bound is established by direct construction.

2 The lower bound

In this section, we shall prove the following theorem:

Theorem 2.1 *Let $\mathbb{K}_f$ be the reverse parallel link of an alternating knot K with framing f and D a reduced diagram of K. Then*

$$\boldsymbol{b}(\mathbb{K}_f)\ge\begin{cases} c(D)+2+a(D)-f & \text{if } f<a(D),\\ c(D)+2 & \text{if } a(D)\le f\le b(D),\\ c(D)+2-b(D)+f & \text{if } f>b(D),\end{cases} \tag{2-1}$$

where $a(D)=-v_-(D)+w(D)$ and $b(D)=v_+(D)+w(D)$.

2.1 The Homfly and Kauffman polynomials

Before proving this theorem we note some properties of the Homfly and Kauffman polynomials of a link L.

The *Homfly polynomial* $P_L(v,z)\in\mathbb{Z}[v^{\pm1},z^{\pm1}]$ of an oriented link L is determined by the skein relations

$$v^{-1}P_{L^+}-vP_{L^-}=zP_{L^0},$$

where $L^{\pm}$ and L^0 differ only near one crossing as shown below, and takes the value 1 on the unknot:

$L^+ =$ , $L^- =$ , $L^0 =$ .

The *Kauffman polynomial* $F_L(a,z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ for an unoriented link L is defined in [10]. Again it takes the value 1 on the unknot.

When an extra distant unknotted component O is adjoined to the link L to make $L \sqcup O$, each polynomial changes in the following simple way:

$$P_{L \sqcup O}(v,z) = \frac{v^{-1} - v}{z} P_L(v,z), \qquad F_{L \sqcup O}(a,z) = \Big(\frac{a + a^{-1}}{z} - 1\Big) F_L(a,z).$$

Define the *extended Homfly polynomial* EP by

$$\mathrm{EP}_L(v,z) = \frac{v^{-1} - v}{z} P_L(v,z) = P_{L \sqcup O}(v,z) \tag{2-2}$$

and the *extended Kauffman polynomial* EF by

$$\mathrm{EF}_L(a,z) = \Big(\frac{a + a^{-1}}{z} - 1\Big) F_L(a,z) = F_{L \sqcup O}(a,z). \tag{2-3}$$

Remark This extended normalization is often used in the context of quantum invariants, where it allows for more natural specializations of the knot polynomials. It is also more useful in that context to use the Dubrovnik variant of the Kauffman polynomial in place of F.

By plugging in $L = \phi$ on both sides of (2-2) and (2-3), the extended polynomials can be thought of as taking the value 1 on the empty link ϕ.

2.2 Bounds from the Homfly and Kauffman polynomials

The Morton–Franks–Williams inequality [7; 11] gives a lower bound for the braid index $\boldsymbol{b}(L)$ of the link L in terms of the v–spread of the Homfly polynomial $P_L(v,z)$ or its extended version. Explicitly

$$\boldsymbol{b}(L) \geq 1 + \tfrac{1}{2} \operatorname{spr}_v P_L(v,z) = \tfrac{1}{2} \operatorname{spr}_v \mathrm{EP}_L(v,z). \tag{2-4}$$

The a–spread of the Kauffman polynomial is shown by Morton and Beltrami [12] to give a bound for the arc index $\alpha(L)$. Explicitly this is

$$\operatorname{spr}_a F_L(a,z) \leq \alpha(L) - 2.$$

Bae and Park [1] showed that the arc index $\alpha(L)$ is bounded above by $c(L) + 2$, that is, $\alpha(L) \leq c(L) + 2$. Combining these results shows that

$$\operatorname{spr}_a F_L(a,z) \leq c(L). \tag{2-5}$$

2.3 A congruence result

Rudolph [16] relates the Kauffman polynomial of a link L with the Homfly polynomial of the reverse parallels of L.

Notation For Laurent polynomials $A = \sum a_{i,j} v^i z^j$ and $B = \sum b_{i,j} v^i z^j \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ we write $A \cong_{\mathbb{Z}_2} B$ when $a_{i,j} \cong b_{i,j} \bmod 2$ for all i and j.

In the case of a knot K, Rudolph's theorem for the reverse parallel $\mathbb{K}_f$ can then be stated very cleanly in terms of the extended polynomials.

Theorem 2.2 [16, congruence theorem] $\mathrm{EP}_{\mathbb{K}_f}(v,z)-1 \cong_{\mathbb{Z}_2} v^{-2f}\,\mathrm{EF}_K(v^{-2},z^2)$.

2.4 Alternating knots

We can apply these bounds to the case of alternating knots, starting from observations of Cromwell [3] about their Kauffman polynomial.

For any knot K with a diagram D, write the Kauffman polynomial $F_K(a,z)$ of K as

$$F_K(a,z)=a^{-w(D)}\sum_{i,j}a_{i,j}a^iz^j. \tag{2-6}$$

In this form the coefficients $a_{i,j}$ are only nonzero in the range $|i|+j\le c(D)$.

Cromwell extends work of Thistlethwaite [17] to identify two nonzero coefficients $a_{i,j}$ which realize the maximum possible a–spread $c(D)$ for $F_K(a,z)$ in the case of an alternating knot K with reduced diagram D.

Theorem 2.3 [3] *Let K be an alternating knot and D a reduced diagram of K. Then $a_{i,j}=1$ in the two cases $i=1-v_+(D)$, $j=c(D)+i$ and $i=v_-(D)-1$, $j=c(D)-i$.*

Corollary 2.4 *We have $\mathrm{spr}_a\,F_K(a,z)=c(D)$, and $a_{i,j}=0$ in (2-6) unless $1-v_+(D)\le i\le v_-(D)-1$.*

Proof By Theorem 2.3 $\mathrm{spr}_a\,F_K(a,z)\ge v_-(D)-1-(1-v_+(D))=c(D)$, while $\mathrm{spr}_a\,F_K(a,z)\le c(D)$ by (2-5). □

Now set

$$B_D(a,z)=a^{w(D)}\,\mathrm{EF}_K(a,z)=\Big(\frac{a+a^{-1}}{z}-1\Big)\sum_{i,j}a_{i,j}a^iz^j. \tag{2-7}$$

Then $\mathrm{spr}_a\,B_D(a,z)=\mathrm{spr}_a\,F_K(a,z)+2=c(D)+2$. Furthermore, if we write

$$B_D(a,z)=\sum_{i,j}b_{i,j}a^iz^j, \tag{2-8}$$

then $b_{i,j}=0$ unless $-v_+(D)\le i\le v_-(D)$ by Corollary 2.4.

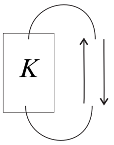
The two critical monomials $a^{-v_+(D)}z^{c(D)-v_+(D)}$ and $a^{v_-(D)}z^{c(D)-v_-(D)}$ in $B_D(a,z)$, which correspond to $i=-v_+(D)$ and $i=v_-(D)$, respectively, both have coefficient $b_{i,j}=1$, by Theorem 2.3. We will use these critical monomials in finding a lower bound for the v–spread of the extended Homfly polynomial of the reverse parallels of D.

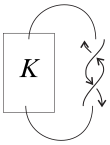
Theorem 2.5 gives a simple formula to calculate the extended Homfly polynomial of $\mathbb{K}_{k+f}$ in terms of the polynomial of $\mathbb{K}_k$.

Theorem 2.5 *For any f and k we have*

$$v^{2f}(\mathrm{EP}_{\mathbb{K}_{k+f}}(v,z)-1)=\mathrm{EP}_{\mathbb{K}_k}(v,z)-1.$$

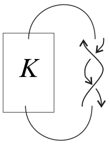
Proof While this is in effect shown by Rudolph [16, Proposition 2(5)] it is easy to give a direct skein theory proof. It is enough to prove it in the case $f = 1$. Now $\mathbb{K}_{k+1}$ is given from $\mathbb{K}_k$ by adding one extra twist in the annulus, as shown:

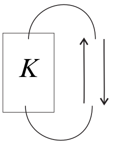
$\mathbb{K}_k =$ 

, $\mathbb{K}_{k+1} =$ 

.

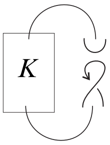
With the reverse parallel orientation on the strings, apply the Homfly skein relation at one of the crossings in the diagram for $\mathbb{K}_{k+1}$. Since this is a negative crossing, $\mathbb{K}_{k+1}$ plays the role of L^-. Switching the crossing gives

$L^+ =$ 

$=$ 

$= \mathbb{K}_k$,

while the smoothed diagram

$L^0 =$ 

is simply an unknotted curve.

The skein relation, in the form

$$\mathrm{EP}_{L^+} = vz\,\mathrm{EP}_{L^0} + v^2\,\mathrm{EP}_{L^-},$$

then gives

$$\mathrm{EP}_{\mathbb{K}_k} = vz\frac{v^{-1}-v}{z} + v^2\,\mathrm{EP}_{\mathbb{K}_{k+1}} = 1 - v^2 + v^2\,\mathrm{EP}_{\mathbb{K}_{k+1}}.$$

Thus

$$v^2(\mathrm{EP}_{\mathbb{K}_{k+1}} - 1) = \mathrm{EP}_{\mathbb{K}_k} - 1. \qquad \square$$

We can now specify a lower bound for the v–spread of the extended Homfly polynomial of the parallels $\mathbb{K}_{w(D)+f}$ as f varies.

Theorem 2.6 *Let K be an alternating knot with reduced diagram D. The framed reverse parallel $\mathbb{K}_{w(D)+f}$ has the following lower bound for the v–spread of its extended Homfly polynomial*:

$$\operatorname{spr}_v \mathrm{EP}_{\mathbb{K}_{w(D)+f}}(v,z) \ge \begin{cases} 2(v_+(D) - f) & \textit{if } f < -v_-(D), \\ 2(v_+(D) + v_-(D)) & \textit{if } -v_-(D) \le f \le v_+(D), \\ 2(f + v_-(D)) & \textit{if } f > v_+(D). \end{cases}$$

Proof Since K is an alternating knot with reduced diagram D, Theorem 2.2 shows that

$$B_D(v^{-2}, z^2) = v^{-2w(D)}\,\mathrm{EF}_K(v^{-2}, z^2) \cong_{\mathbb{Z}_2} \mathrm{EP}_{\mathbb{K}_{w(D)}}(v,z) - 1. \tag{2-9}$$

In $B_D(v^{-2}, z^2) = \sum b_{i,j} v^{-2i} z^{2j}$ there are two critical monomials $v^{-2i} z^{2j}$, one with $i = -v_+(D)$ and $j = c(D) - v_+(D)$, and the other with $i = v_-(D)$ and $j = c(D) - v_-(D)$, where $b_{i,j} = 1$. By (2-9) there are two corresponding critical monomials $v^{-2i} z^{2j}$ in $\mathrm{EP}_{\mathbb{K}_{w(D)}}(v,z) - 1$ whose coefficients are congruent to $b_{i,j}$, and hence are odd. One term has v–degree $-2v_-(D)$ and the other has v–degree $2v_+(D)$.

By Theorem 2.5 we have

$$v^{2f}\,\mathrm{EP}_{\mathbb{K}_{w(D)+f}}(v,z) = (\mathrm{EP}_{\mathbb{K}_{w(D)}}(v,z)-1)+v^{2f}.$$

The v–spread of $\mathrm{EP}_{\mathbb{K}_{w(D)+f}}(v,z)$ is the same as the v–spread of $(\mathrm{EP}_{\mathbb{K}_{w(D)}}(v,z)-1)+v^{2f}$. In this Laurent polynomial consider the appearance of the two critical monomials along with the monomial v^{2f}. Unless one of the two critical monomials $v^{2v_+(D)}z^{2c(D)-2v_+(D)}$ and $v^{-2v_-(D)}z^{2c(D)-2v_-(D)}$ in $B_D(v^{-2},z^2)$ is v^{2f} they will each still have odd coefficients, and the v–spread will be at least $2(v_+(D)+v_-(D))$.

If $f<-v_-(D)$ or $f>v_+(D)$ the monomial v^{2f} has even coefficient in $\mathrm{EP}_{\mathbb{K}_{w(D)}}(v,z)-1$ since it has coefficient 0 in $B_D(v^{-2},z^2)$. In this range of f it then has nonzero coefficient in $(\mathrm{EP}_{\mathbb{K}_{w(D)}}(v,z)-1)+v^{2f}$. This gives the lower bound $2(v_+(D)-f)$ when $f<-v_-(D)$, and $2(v_-(D)+f)$ when $f>v^+(D)$ for $\mathrm{spr}_v\,\mathrm{EP}_{\mathbb{K}_{w(D)+f}}(v,z)$.

To complete the proof of Theorem 2.6 it remains to deal with the cases where v^{2f} is one of the two critical monomials $v^{2v_+(D)}z^{2c(D)-2v_+(D)}$ and $v^{-2v_-(D)}z^{2c(D)-2v_-(D)}$ in $B_D(v^{-2},z^2)$. In the first case this means that $f=v_+(D)$ and $0=c(D)-v_+(D)$. Then $f=c(D)=v_+(D)=n$ and D is the reduced diagram of the $(2,n)$ torus knot. In the other case $-f=c(D)=v_-(D)=n$. Hence D is the reduced diagram of the $(2,-n)$ torus knot.

In the $(2,n)$ case we need to show that the coefficient of v^{2n} in $(\mathrm{EP}_{\mathbb{K}_{w(D)}}(v,z)-1)+v^{2n}$ is nonzero. In Theorem 2.7 we show that this coefficient is 2 by showing that v^{2n} has coefficient 1 in $\mathrm{EP}_{\mathbb{K}_{w(D)}}(v,z)$, where $\mathbb{K}_{w(D)}$ is the blackboard reverse parallel of D.

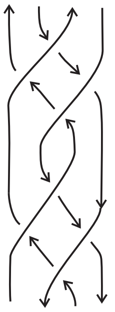
The $(2,-n)$ case follows directly by considering the polynomial of the mirror image. □

The detailed calculation for the special case of the $(2,n)$ torus knot will now be shown.

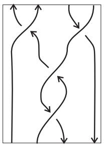
Theorem 2.7 *The blackboard reverse parallel $\mathbb{K}_n$ of the $(2,n)$ torus knot K satisfies*

$$\mathrm{EP}_{\mathbb{K}_n}(v,z) = v^{2n} + \sum_{i<2n,j} a_{i,j}v^iz^j.$$

Proof We can draw a diagram of $\mathbb{K}_n$ as the closure of a 4–strand tangle with two upward and two downward strings, as shown:



It is more convenient to place the upward pair of strings on the left at the top and bottom, and write $\mathbb{K}_n$ as the closure of the tangle T^n, where

$T =$  .

We use the skein relations in the form

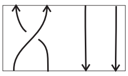
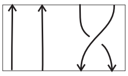
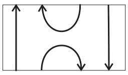
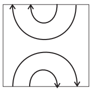
v^{-1}  $- v$  $= z$ 

to write the closure of T^n as a linear combination of the closures of simpler tangles.

Notation The 4–strand tangle U *evaluates to* the extended Homfly polynomial of its closure, which we write as $\mathrm{ev}(U) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$.

Remark Evaluation is linear on tangles and respects the skein relations. It is a sort of trace function in that $\mathrm{ev}(AB) = \mathrm{ev}(BA)$.

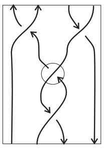
Our first step is to expand T as a combination of the tangles

$\sigma_1 =$  , $\sigma_3 =$  , $h =$  and $H =$  ,

and their products when placed one above the other.

Remark By using the skein relations we are in effect working in a version of the mixed Hecke algebra $H_{2,2}(v, z)$ spanned by tangles with two upward and two downward strings [8].

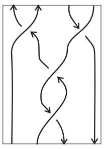
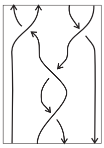
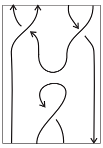
The crossing circled here in

$T =$ 

is a negative crossing, so we can use the skein relation at this crossing in the form

 $= v^{-2}$  $- v^{-1} z$  .

Then

$T =$  $= v^{-2}$  $- v^{-1} z$  $= v^{-2}\sigma_1\sigma_3 - v^{-1} z\sigma_1\sigma_3 h = C + C\tau,$

where for convenience we set $C = c_1 c_3 = (v^{-1}\sigma_1)(v^{-1}\sigma_3)$ and $\tau = (-zv)h$.

Then $T^n = (C + C\tau)^n$. Now C and τ do not commute, so we write

$$T^n = C^n + (C\tau)^n + \sum_{0<k<n} C^{r_1}\tau C^{r_2}\tau \cdots C^{r_k}\tau C^r, \tag{2-10}$$

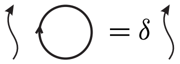
where $r_i \geq 1$, $r \geq 0$ and $r + \sum r_i = n$.

We can estimate the contribution of these terms to the evaluation of T^n.

- The evaluation of C^n only contributes terms up to v–degree 4, by Proposition 2.9.
- The terms in the large sum with weight k in τ evaluate to terms of v–degree at most $2k$. Without changing the evaluation we can assume that $r=0$, since we can cycle C^r from the end to the beginning of the product and amalgamate it with C^{r_1}. The contribution of these terms with $k<n$ to the evaluation of T^n is shown in Proposition 2.10 to have degree no more than $2k$ (and thus at most $2n-2$) in v.
- The most important contribution comes from the evaluation of $(C\tau)^n$, which gives v^{2n}, and no other terms with v–degree $2n$ or larger, as stated in Proposition 2.8.

Before making detailed calculations we note some useful properties, which can be quickly checked diagrammatically:

$$\sigma_1 H = \sigma_3 H, \qquad H\sigma_1 = H\sigma_3, \qquad H = h\sigma_1\sigma_3^{-1}h,$$



$$\text{where } \delta = \frac{v^{-1}-v}{z}, \qquad h^2 = \delta h, \qquad h\sigma_1 h = h.$$

Here are some consequences for our use of $c_1 = v^{-1}\sigma_1$, $c_3 = v^{-1}\sigma_3$, $C = c_1c_3$ and $\tau = (-zv)h$, which follow algebraically:

- $c_1 = c_1^{-1} + zI$ and $c_3 = c_3^{-1} + zI$ (the skein relation, where I stands for the identity tangle).
- $\tau c_1 c_3^{-1}\tau = (-zv)^2 h c_1 c_3^{-1} h = (zv)^2 H$.
- $\tau^2 = (-zv)\delta\tau = (v^2-1)\tau$.
- $\tau c_1 \tau = v^{-1}(-zv)^2 h\sigma_1 h = -z\tau$.
- $\tau C\tau = \tau(c_1c_3^{-1} + zc_1)\tau = (zv)^2 H - z^2\tau$.

Proposition 2.8 *The extended polynomial of the closure of $(C\tau)^n$ is v^{2n} plus lower terms in v for $n>1$, and $1-v^{-2}$ when $n=1$.*

Proof When $n=1$ we have $C\tau = (-zv^{-1})\sigma_1\sigma_3 h$. Now $\sigma_1\sigma_3 h$ closes to a single unknotted curve, so $C\tau$ evaluates to $-zv^{-1}\delta = 1-v^{-2}$.

For $n>1$ write

$$(C\tau)^n = C(\tau C\tau)(C\tau)^{n-2} = (zv)^2 CH(C\tau)^{n-2} - z^2 C\tau(C\tau)^{n-2}.$$

The evaluation of the second term has v–degree at most $2n-2$, by induction on n, so any monomials of larger v–degree must come from the first term.

Now $Hh = \delta H$ and $H\sigma_1 h = H$. We can then write

$$HC\tau = H(c_1c_3^{-1} + zc_1)\tau = H\tau + zHc_1\tau = (-zv)(\delta + zv^{-1})H = (v^2-1-z^2)H.$$

So the first term expands to

$$(zv)^2 CH(C\tau)^{n-2} = (zv)^2(v^2-1-z^2)^{n-2}CH.$$

Now $CH = c_1c_3^{-1}H + zc_1H = H + zv^{-1}\sigma_1 H$. The closure of H is two disjoint unknotted curves, and $\sigma_1 H$ closes to one unknotted curve. These evaluate to δ^2 and δ, respectively. The first term then evaluates to

$$(v^2-1-z^2)^{n-2}(\delta^2(-zv)^2 - z^2(-zv\delta)) = (v^2-1-z^2)^{n-1}(v^2-1).$$

This contributes a single term v^{2n} and no further terms of v–degree larger than $2n-2$. □

The skein relation, in the form $c_1^2 = I + zc_1$, allows us to write c_1^r recursively as a linear combination of c_1 and the identity tangle I,

$$c_1^r = a_r(z)I + b_r(z)c_1,$$

with coefficients which are polynomials in z only. Similarly

$$c_3^r = a_r(z)I + b_r(z)c_3.$$

We can then expand C^r as a linear combination of I, c_1, c_3 and c_1c_3, with coefficients in $\mathbb{Z}[z]$. Explicitly

$$C^r = (a_rI + b_rc_1)(a_rI + b_rc_3).$$

Proposition 2.9 *The term C^n in the expansion of T^n provides terms of degree at most 4 in v, in the evaluation.*

Proof We have

$$C^n = a_n^2 I + a_nb_n(c_1+c_3) + b_n^2c_1c_3 = a_n^2I + a_nb_nv^{-1}(\sigma_1+\sigma_3) + b_n^2v^{-2}\sigma_1\sigma_3.$$

Now I closes to four unknotted curves evaluating to δ^4, σ_1 and σ_3 close to three unknotted curves evaluating to δ^3, and $\sigma_1\sigma_3$ closes to two unknotted curves evaluating to δ^2. The term C^n then contributes $a_n^2\delta^4 + 2a_nb_nv^{-1}\delta^3 + b_n^2v^{-2}\delta^2$ to the evaluation. Since $\delta = (v^{-1}-v)z^{-1}$, and a_n and b_n depend only on z, these terms have v–degree at most 4. □

To complete our proof of Theorem 2.7 we show that the evaluation of the remaining terms in (2-10) has v–degree at most $2n-2$:

Proposition 2.10 *The evaluation of*

$$C^{r_1}\tau\cdots C^{r_i}\tau\cdots C^{r_k}\tau$$

with $r_i \geq 1$ has terms of degree at most $2k$ in v.

Proof We proceed by induction on the number of exponents r_i for which $r_i > 1$.

When $r_i = 1$ for all i this follows from Proposition 2.8.

Otherwise we can cycle the terms in the product without changing its evaluation, and arrange that $r_k = r > 1$. Then

$$\tau C^r\tau = a_r^2\tau^2 + a_rb_r\tau(c_1+c_3)\tau + b_r^2\tau C\tau = a_r^2(v^2-1)\tau - 2za_rb_r\tau + b_r^2\tau C\tau.$$

So

$$C^{r_1}\tau\cdots C^{r_k}\tau = (a_r^2(v^2-1) - 2za_rb_r)C^{r_1}\tau\cdots C^{r_{k-1}}\tau + b_r^2C^{r_1}\tau\cdots C^{r_{k-1}}\tau C\tau.$$

These expressions both have one fewer term C^{r_i} for which $r_i > 1$, so by our induction hypothesis the evaluation of $C^{r_1}\tau\cdots C^{r_{k-1}}\tau C\tau$ has terms of degree at most $2k$ in v while $C^{r_1}\tau\cdots C^{r_{k-1}}\tau$ has terms of degree at most $2k-2$. With the coefficient $a_r^2(v^2-1)-2za_rb_r$ adding 2, in this case all terms in the final evaluation have degree at most $2k$ in v. This establishes the proposition. □

Now all the terms in (2-10) have been dealt with, and Theorem 2.7 for the evaluation of the reverse blackboard parallel of the $(2,n)$ torus knot follows. □

The proof of Theorem 2.6 is then complete. We can now prove Theorem 2.1, which was the goal of this section.

Proof of Theorem 2.1 Using the Morton–Franks–Williams bound (2-4) in Theorem 2.6 immediately gives the lower bound for the braid index of $\mathbb{K}_{w(D)+f}$ as

$$\boldsymbol{b}(\mathbb{K}_{w(D)+f}) \geq \begin{cases} v_+(D)-f & \text{if } f < -v_-(D), \\ v_+(D)+v_-(D) & \text{if } -v_-(D) \leq f \leq v_+(D), \\ f+v_-(D) & \text{if } f > v_+(D). \end{cases}$$

Replacing f by $f-w(D)$ then gives

$$\boldsymbol{b}(\mathbb{K}_f) \geq \begin{cases} v_+(D)-f+w(D) & \text{if } f-w(D) < -v_-(D), \\ v_+(D)+v_-(D) & \text{if } -v_-(D) \leq f-w(D) \leq v_+(D), \\ f-w(D)+v_-(D) & \text{if } f-w(D) > v_+(D). \end{cases}$$

Now $v_+(D)+v_-(D)=c(D)+2$, so after setting $a(D)=w(D)-v_-(D)$ and $b(D)=w(D)+v_+(D)$ this lower bound becomes

$$\boldsymbol{b}(\mathbb{K}_f) \geq \begin{cases} c(D)+2+a(D)-f & \text{if } f < a(D), \\ c(D)+2 & \text{if } a(D) \leq f \leq b(D), \\ c(D)+2-b(D)+f & \text{if } f > b(D), \end{cases}$$

which is the formula (2-1) claimed in Theorem 2.1 □

3 The upper bound

In this section, we shall prove the following theorem, which provides us the desired upper bound for the braid index of $\mathbb{K}_f$.

Theorem 3.1 *If $\mathbb{K}_f$ is a reverse parallel link of an alternating knot K with framing f and D is a reduced diagram of K, then*

$$\boldsymbol{b}(\mathbb{K}_f) \leq \begin{cases} c(D)+2+a(D)-f & \text{if } f < a(D), \\ c(D)+2 & \text{if } a(D) \leq f \leq b(D), \\ c(D)+2-b(D)+f & \text{if } f > b(D), \end{cases} \tag{3-1}$$

where $a(D)=-v_-(D)+w(D)$ and $b(D)=v_+(D)+w(D)$.

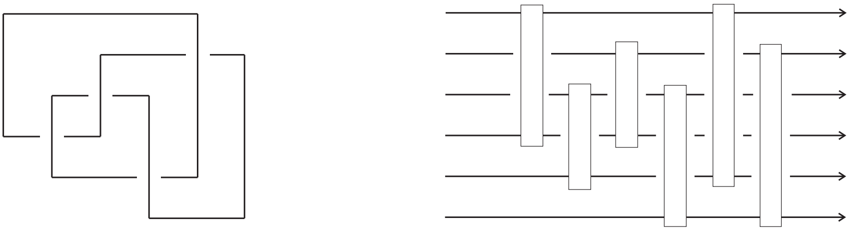


Figure 3: Left: a grid diagram for the figure-eight knot with $\alpha = 6$ arcs. Right: the resulting braid template.

Proof It suffices to show that braid presentations of $\mathbb{K}_f$ can be constructed with the number of strings given in the theorem.

Nutt [15] constructs reverse string satellites of a knot K using an arc presentation for K with α arcs. In its simplest form, Nutt's construction gives an α–string braid presentation of the reverse parallels $\mathbb{K}_f$ for $\alpha + 1$ consecutive values of f.

An arc presentation of K on α arcs provides a grid diagram for K made up of α vertical arcs joined by α horizontal arcs such that the horizontal arcs can only pass under the vertical arcs. Convert this to a braid template by the following procedure. First we extend each horizontal arc (from the points where it is connected to vertical arcs) left and right. If an extended arc runs into a vertical arc, then we make it pass under the vertical arc. Notice that each vertical arc now ends in a $\top$ at its top and in a $\perp$ at its bottom. We then "thicken" each vertical arc slightly to create an empty 2–braid box as shown in Figure 3. In this construction, the horizontal arcs at a $\top$ or $\perp$ of a vertical arc will meet the vertical sides of the corresponding braid box, while any intermediate horizontal arcs pass entirely underneath the braid box.

We can now place a single positive or negative crossing in each braid box, with strings running from left to right, connecting the horizontal arcs that meet the boundary of the box. The other horizontal arcs either do not pass through this crossing, or will pass under it. This gives an α–string closed braid which is a link with two components. Furthermore, one can verify that this closed braid is ambient isotopic to a link within a tubular neighborhood of the grid diagram such that each component runs parallel to K but with opposite orientations. That is, the resulting closed braid presents $\mathbb{K}_f$ for some framing f.

Write $f = a$ for the framing which arises when all the crossings used in the braid boxes are positive. If k of the boxes are filled with negative crossings instead, then we have framing $f = a + k$, so with all crossings negative we have $f = a + \alpha = b$, say.

In the case that $f < a$ or $f > b$, we can present $\mathbb{K}_f$ as a closed braid on $\alpha + \tau$ strings for $f = a - \tau$ or $f = b + \tau$ $(\tau \geq 1)$ by adding τ extra suitably chosen arcs to the grid diagram. This gives us an upper bound for $\boldsymbol{b}(\mathbb{K}_f)$,

$$\boldsymbol{b}(\mathbb{K}_f) \leq \begin{cases} \alpha + a - f & \text{if } f < a, \\ \alpha & \text{if } a \leq f \leq b, \\ \alpha + f - b & \text{if } f > b, \end{cases} \tag{3-2}$$

with $b = a + \alpha$.

An alternating knot K has an arc representation with $c(D)+2$ arcs [1], so we can set $\alpha = c(D)+2$ in (3-2). It remains to show that $a = a(D)$ and $b = b(D)$ in this case.

By our lower bound calculation (2-1), $\mathbb{K}_f$ cannot be presented as a braid on $\alpha = c(D)+2$ strings if $f < a(D)$ or $f > b(D)$. Now $\mathbb{K}_a$ and $\mathbb{K}_b$ are each presented by an α–string braid, so $a \geq a(D)$ and $b \leq b(D)$. Thus $b = c(D)+2+a \geq c(D)+2+a(D) = b(D)$, giving

$$b = b(D), a = a(D)$$

in (3-2). □

We end our paper with the following remarks:

Remark If one desires to use the linking number l of $\mathbb{K}_f$ in the formulation of $\boldsymbol{b}(\mathbb{K}_f)$ instead of the framing f, then the formulation can be easily obtained by substituting f by $-l$ in (1-1). Specifically, (1-1) becomes

$$\boldsymbol{b}(\mathbb{K}_f) = \begin{cases} c(D)+2+a'(D)-l & \text{if } l < a'(D), \\ c(D)+2 & \text{if } a'(D) \leq l \leq b'(D), \\ c(D)+2-b'(D)+l & \text{if } l > b'(D), \end{cases} \tag{3-3}$$

where $a'(D) = -v_+(D) - w(D)$, $b'(D) = v_-(D) - w(D)$ and $l = -f$ is the linking number of $\mathbb{K}_f$. The corresponding formulation of (2-1) matches the one given in [5]. We need to point out that the lower bound formula derived in [5] uses a graph-theoretic approach on the Seifert graphs of D and $\mathbb{K}_f$ constructed from the blackboard reverse parallel of D. However, that approach only works for the special alternating knots, namely those alternating knots which admit a reduced alternating diagram in which the crossings are either all positive or all negative.

Remark The general question of finding the braid index for a satellite of a knot K with some form of reverse string pattern has been considered by Birman and Menasco [2]. Our reverse parallels, along with Whitehead doubles, are the simplest such satellites. Nutt [15] draws on [2] to give lower bounds for the braid index in terms of the arc index of K, as well as the upper bounds which we have used. Coupled with the later work of Bae and Park [1], this would provide our result without the use of Rudolph's congruence.

Some descriptions given by [2] were later found to be incomplete, with Ka Yi Ng [14] providing details of the missing cases. Nutt's lower bound argument needs the analysis in [2] which shows that the arc index of K is a lower bound for the braid index of any reverse string satellite of K. We have not been able to confirm how well the arc index analysis in the original paper extends to Ng's extra cases.

Remark Theorem 1.1 allows us to settle the long-standing conjecture that the ropelength of an alternating knot K is at least proportional to its crossing number. This statement is a consequence of [4, Theorem 3.1] and the fact that the ropelength of K is bounded below by a (fixed) constant multiple of the ropelength of $\mathbb{K}_f$ for some f. The more general case of an alternating link with two or more components remains open.

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