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Definition of the cord algebra of knots using Morse thoery

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The cord algebra of a knot K is isomorphic to the string homology and the Legendrian contact homology of K. The proof of the isomorphism of string homology and cord algebra uses a retraction of broken strings (which are consecutive paths beginning and ending on the knot) on words in linear cords. This suggests a reformulation of the cord algebra using linear cords, which we present. We will define a Morse function such that the binormal linear cords of K are the critical points of degree 0, 1 and 2 of this function. These critical points give rise to a chain complex of K. Then the cord algebra of K is the degree zero homology of K.

57M25, 57M27; 57R17

1 Introduction

The cord algebra is a knot invariant first introduced in 2005 by Lenhard Ng in [11, Definition 1.2]. The topological definition of the cord algebra given there is inspired by the Legendrian contact homology (see Ng [10, Definition 4.1]). Ng extended the definition in 2008 in [12, Definition 2.4] and in 2014 in [13, Definition 4.4]. In 2013, Tobias Ekholm, John B Etnyre, Ng and Michael G Sullivan showed that the cord algebra is isomorphic to the Legendrian contact homology in degree zero [5, Corollary 1.2]. In 2017, Kai Cieliebak, Ekholm, Janko Latschev and Ng developed in [2, Section 2.2] a noncommutative refinement of the original definition, including a basepoint of the knot and a framing. This results in four additional generators in the cord algebra. In [2, Theorem 1.2 and Proposition 1.3], another proof of the isomorphism of the cord algebra to the Legendrian contact homology (in degree zero) is given. This is done with the help of string homology by first showing the isomorphism of string homology and the cord algebra and then the isomorphism of Legendrian contact homology and string homology. To define the string homology (in degree zero), a chain complex is defined which contains broken strings or 1–parameter families of broken strings as generators in degree 0 or 1, respectively. A broken string consists of pieces which are paths beginning and ending on the knot, and whose codomain alternates between the ambient space of the knot and its conormal bundle (there is also a relation of tangent vectors at the switching points). In the proof of the isomorphism of string homology and cord algebra, a retraction of the broken strings on words in (linear) cords is used, where (linear) cords are linear curves in space with start and end points on the knot. Therefore, it makes sense to describe the cord algebra with the help of a suitable complex of (linear) cords. We will do this in this paper. For this purpose we will define a suitable chain complex which contains the critical points of index 0 and 1 of a Morse function in degree 0 and 1, respectively, and the four additional generators mentioned above in degree 0. Further degrees of the chain complex are not needed. These critical points correspond to binormal (linear) cords on the knot. In order to construct a suitable differential, we let the binormal (linear) cords of index 1 flow along their unstable manifolds until they, taking into account four relations, reach (linear) cords of index 0. These four relations are used in [2, Definition 2.6] to define the cord algebra, where two of them were already defined by Ng in [11, after Definition 1.1] and a third was introduced in [12, after Definition 2.3]. We will adapt them here only slightly to the changed concept.

We will need several results from differential topology and Morse theory. The statements and proofs can be found in the literature referred to.

Important statements from differential topology which we will use are the transversality theorem and the jet transversality theorem as well as the relative versions of these two theorems.

To define the cord algebra, we will consider a gradient vector field. This vector field has to satisfy the Smale condition. For this it may be necessary to change the gradient to a pseudogradient. From Morse theory we also need the statement that in a generic 1–parameter family of vector fields without nonconstant periodic orbits only birth-death type degeneracies occur.

At the beginning of Section 2 we will first explain the noncommutative definition of the cord algebra from [2]. A cord is a continuous path in space that has a start and end point on the knot K and does not intersect the knot in any other point. We also need a basepoint on K and a framing which is a slightly shifted copy of K. Four relations are defined which specify relations between different cords, partly taking into account the basepoint and the framing. These relations generate an ideal $\mathscr{I}$ in a noncommutative unital ring $\mathscr{A}$, where $\mathscr{A}$ is generated by homotopy classes of cords and four other generators $\lambda^{\pm 1}, \mu^{\pm 1}$ modulo the relations $\lambda \cdot \lambda^{-1} = \lambda^{-1} \cdot \lambda = \mu \cdot \mu^{-1} = \mu^{-1} \cdot \mu = 1$ and $\lambda \cdot \mu = \mu \cdot \lambda$. The cord algebra of K, $\mathrm{Cord}(K)$, is then defined as the quotient ring $\mathscr{A}/\mathscr{I}$.

In Section 2.1 we will use chords and a (slightly different definition of) framing to be able to define the cord algebra using Morse theory. A chord of K is a straight line in space with start and end points on the knot, ie a special kind of cord. A framing is now a smooth map that assigns a unit normal vector to each point of the knot. The four relations from the original definition of the cord algebra are adapted accordingly. We'll look at the energy function $E\colon K \times K \to \mathbb{R}$, $(x, y) \mapsto \frac{1}{2}|x-y|^2$, and change it to a Morse function. The critical points of Morse index 1 of this Morse function are binormal chords whose unstable manifolds are one-dimensional. These chords are moved along their unstable manifolds and, taking into account the relations, generate an ideal I in an R–algebra C_0 which is generated by critical points of index 0. Here $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ is the commutative ring over $\mathbb{Z}$ which is generated by $\lambda, \lambda^{-1}, \mu$ and μ^{-1} modulo the relations $\lambda \cdot \lambda^{-1} = \mu \cdot \mu^{-1} = 1$. We then define in Section 2.3 the cord algebra as $\mathrm{Cord}(K) := C_0/I$.

In Section 2.1, we discuss, for a generic representative of the knot, various properties of the subsets of chords (viewed as subsets of $K \times K$ via the end point map) where one of three special things happen:

- one of the end points coincides with the basepoint,
- the chord intersects the knot in its interior,
- the chord intersects the framing.

In the definition of the cord algebra we use the Seifert framing as a canonical framing. To determine the cord algebra of a knot, however, it is easier to use the blackboard framing. In order to obtain the cord algebra with respect to the Seifert framing, certain transformations must be applied. We will discuss these transformations in Section 2.4.

In Section 2.5 we will determine the cord algebras for the unknot and the right-handed trefoil knot.

The main result of this paper is:

Theorem 1.1 *The cord algebra defined in terms of Morse theory* (*see Definition 2.30*) *is a knot invariant.*

The proof of Theorem 1.1 will be given in Section 3. We consider two knots K_0 and K_1 which are connected by a smooth isotopy of knots. So it must be shown that the cord algebras of the two knots are the same. First of all, it is necessary to note that in the course of the isotopy there are only a finite number of knots that are not generic. Therefore, we can assume that only one nongeneric knot occurs during the isotopy and show that the cord algebra does not change in the course of the isotopy. Since there are several cases of degeneracies, we have to go through all these cases. Furthermore, we will also show that the cord algebra does not change in the course of the isotopy if no nongeneric knot occurs.

In Section 4 we will prove that the definition of the cord algebra using Morse theory is isomorphic to the topological definition given in [2, Definition 2.6].

Acknowledgements

A thousand thanks to Kai Cieliebak for many inspiring conversations.

2 Definition of the cord algebra

The cord algebra is a knot invariant developed only a few years ago [11; 12; 13]. In [2, Definition 2.6] a noncommutative refinement of the original definition is presented. This refined version is briefly explained in the following. For this we need the terms framing and cord. However, in Section 2.1 we will use chords (instead of cords) and a slightly different definition of framing to define the cord algebra using Morse theory.

Remark 2.1 For the purpose of this overview we use the following interpretation of framing:

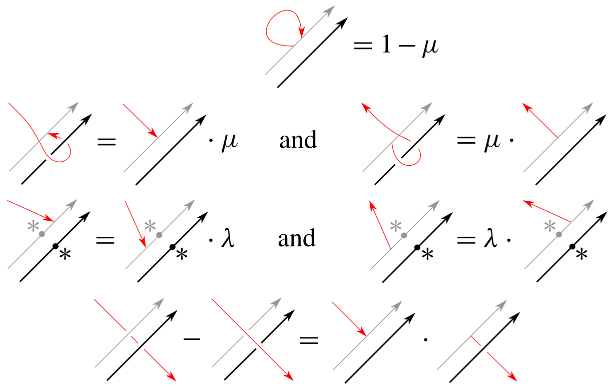

Figure 1: Relations for cords. Here K is depicted in black, K' in gray, and cords are drawn in red.

Let $K \subset \mathbb{R}^3$ be a knot of length L and $\gamma\colon [0, L] \to \mathbb{R}^3$ be an arclength parametrization of K. Let $\nu\colon [0, L] \to S^2$ be a smooth map, where $\nu(t)$ is a unit normal vector to K at the point $\gamma(t)$ for all $t \in [0, L]$. Let $\varepsilon > 0$ be small enough such that the strip $\{\gamma(t) + \alpha\nu(t) : t \in [0, L], \alpha \in [0, \varepsilon]\}$ has no self-intersections. A *framing* of K is the set $K' := \{\gamma(t) + \varepsilon\nu(t) : t \in [0, L]\}$.

Let $K \subset \mathbb{R}^3$ be an oriented knot equipped with a framing K'. Choose a basepoint $*$ on K and a corresponding basepoint $*$ on K' (in fact only the basepoint on K' will be needed).

Definition 2.2 A *cord* of K is a continuous map $\alpha\colon [0, 1] \to \mathbb{R}^3$ such that $\alpha([0, 1]) \cap K = \varnothing$ and $\alpha(0), \alpha(1) \in K' \setminus \{*\}$. Two cords are *homotopic* if they are homotopic through cords.

We now construct a noncommutative unital ring $\mathscr{A}$ as follows: as a ring, $\mathscr{A}$ is freely generated by homotopy classes of cords and four extra generators $\lambda^{\pm 1}, \mu^{\pm 1}$, modulo the relations

$$\lambda \cdot \lambda^{-1} = \lambda^{-1} \cdot \lambda = \mu \cdot \mu^{-1} = \mu^{-1} \cdot \mu = 1, \quad \lambda \cdot \mu = \mu \cdot \lambda.$$

Thus, $\mathscr{A}$ is generated as a $\mathbb{Z}$–module by (noncommutative) words in homotopy classes of cords and powers of λ and μ (and the powers of λ and μ commute with each other, but not with any cords).

Definition 2.3 The *cord algebra* of K is the quotient ring

$$\mathrm{Cord}(K) = \mathscr{A}/\mathscr{I},$$

where $\mathscr{I}$ is the two-sided ideal of $\mathscr{A}$ generated by the relations shown in Figure 1.

Remark 2.4 [2, Remark 2.7] The relations in Figure 1 depict cords in space that agree outside of the drawn region (except in (iv), where either of the two cords on the left-hand side of the equation splits into the two on the right). Thus, the second relation states that appending a meridian to the beginning or end of a cord multiplies that cord by μ on the left or right; μ is also called *meridian*. The third relation imposes that crossing the basepoint multiplies the cord by λ from the left or right; λ is also called *longitude*. The

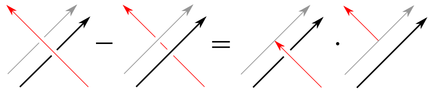

Figure 2: Equivalent representation of the fourth relation.

fourth relation is equivalent to the relation shown in Figure 2. Applying the second and fourth relation in a suitable way to a contractible cord c, we get the equation

$$(c-(1-\mu))c=0.$$

The first relation means that the expression in the brackets already vanishes.

2.1 The energy function for generic knots

In this paper we want to define the cord algebra using Morse theory. For this we need a framing as above. In contrast to the original definition, we will use chords instead of cords.

Definition 2.5 Let $K\subset\mathbb{R}^3$ be a knot of length L and $\gamma\colon[0,L]\to\mathbb{R}^3$ be an arclength parametrization of K.

(i) A *framing* of K is a smooth map $\nu\colon[0,L]\to S^2$, where $\nu(t)$ is a unit normal vector to K at the point $\gamma(t)$ for all $t\in[0,L]$.

(ii) Given a Seifert surface for a knot K, a *Seifert framing* is a framing such that the normal vector $\nu(t)$ is tangent to the Seifert surface and pointing inwards for all $t\in[0,L]$.

Remark 2.6 Assume that $\ddot\gamma$ vanishes nowhere. A framing can also be understood as a map $\nu\colon S^1\to S^1$ by representing $\nu(t)$ for each $t\in S^1$ in local coordinates of the normal plane $N(t)=\operatorname{span}(\ddot\gamma(t),\dot\gamma(t)\times\ddot\gamma(t))$ at the point $\gamma(t)$.

Given a knot K, we take a look at the set of framings of K modulo homotopy:

$$\{\text{framings of } K\}/{\sim}=C(S^1,S^1)/{\sim}=\pi_1(S^1)\cong\mathbb{Z}.$$

As a consequence, the Seifert framing is unique up to homotopy, since the linking number of K and a copy of K which is shifted slightly in the direction of the Seifert framing is always zero.

Let $K\subset\mathbb{R}^3$ be a generic oriented knot of length L and $\gamma\colon[0,L]\to K$ be an arclength parametrization of K. Also, let K be equipped with a framing. For this we use the Seifert framing (obtained by the Seifert algorithm) as a canonical framing. To facilitate the determination of the cord algebra of a knot, however, we will use a different framing. This and the necessary transformations to obtain the cord algebra with respect to the Seifert framing will be discussed in Section 2.4. Furthermore, we choose a basepoint $*$ on K.

Definition 2.7 Let $K\subset\mathbb{R}^3$ be a knot (or a link). A *chord* of K is a curve $\alpha\in C^2([0,1],\mathbb{R}^3)$ such that $\alpha(0),\alpha(1)\in K$ and $\ddot\alpha\equiv0$, ie α is a straight line in $\mathbb{R}^3$ starting and ending on the knot.

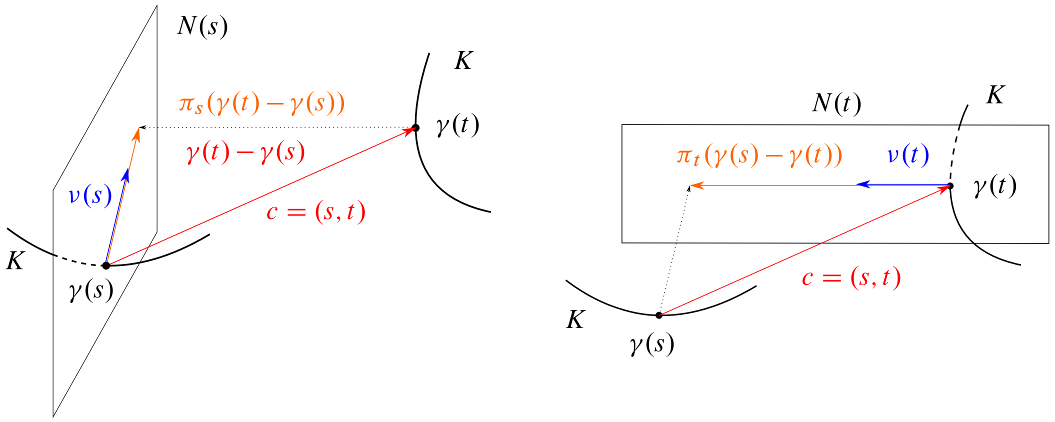

Figure 3: A chord c intersects the framing. Left: c intersects the framing at its starting point. Right: c intersects the framing at its endpoint.

The space of these chords can be canonically identified with $K \times K$ by associating to each chord its endpoints on K: for $s, t \in [0, L]$,

$$c = (\gamma(s), \gamma(t)) \in K \times K$$

is a chord with starting point $\gamma(s)$ and endpoint $\gamma(t)$. The space $K \times K$ can be canonically identified with the torus T^2 by using the identification $S^1 \cong \mathbb{R}/L\mathbb{Z}$. The start and endpoint of a chord $c = (\gamma(s), \gamma(t))$ can be identified with $s \in S^1$ and $t \in S^1$, respectively, and thus we can simply write

$$c = (s, t).$$

We also assign an orientation to each chord: a chord $c = (\gamma(s), \gamma(t))$ is oriented from its start $\gamma(s)$ to its endpoint $\gamma(t)$.

Definition 2.8 Let $K \subset \mathbb{R}^3$ be an oriented knot of length L equipped with a framing ν. Let $\gamma\colon [0, L] \to \mathbb{R}^3$ be an arclength parametrization of K. Let $N(\tau) \subset \mathbb{R}^3$ be the normal plane to K at the point $\gamma(\tau)$ and $\pi_\tau\colon \mathbb{R}^3 \to N(\tau)$ be the orthogonal projection onto $N(\tau)$. Let $c = (s, t) \in K \times K$ be a chord of K.

We say c *intersects the framing* if one of the following conditions is satisfied (see Figure 3):

- $\pi_s(\gamma(t) - \gamma(s)) = \alpha\nu(s)$ for an $\alpha > 0$ (c *intersects the framing at its starting point*), or
- $\pi_t(\gamma(s) - \gamma(t)) = \alpha\nu(t)$ for an $\alpha > 0$ (c *intersects the framing at its endpoint*).

Remark 2.9 From this definition it follows that if a chord $(s, t) \in K \times K$ intersects the framing, so does the reverse oriented chord (t, s). Thus, the set of chords intersecting the framing is symmetric with respect to the involution on $K \times K$ that interchanges the factors.

Remark 2.10 Let $\varepsilon > 0$ be such that the strip $\{\gamma(t) + \alpha\nu(t) : t \in [0, L], \alpha \in [0, \varepsilon]\}$ has no self-intersections, ie the map

$$[0, L] \times [0, \varepsilon] \to \mathbb{R}^3, \quad (t, \alpha) \mapsto \gamma(t) + \alpha\nu(t),$$

is injective. In diagrams we will draw the set $K' := \{\gamma(t) + \varepsilon\nu(t) : t \in [0, L]\}$ to visualize the framing; K' is also called *framing*. An intersection of a chord with the framing according to Definition 2.8 corresponds approximately to an intersection of the chord with the set K' if the curvature of K' is not too strong in a neighborhood of this intersection. Also, there can be very short chords that do not intersect K', but intersect the framing according to Definition 2.8. Therefore, K' is to be understood only as visualization of the framing. In order to determine intersections of chords with the framing, it may be necessary to use the above definition.

Let

$$E : K \times K \to \mathbb{R}, \quad (x, y) \mapsto \tfrac{1}{2}|x - y|^2,$$

be the *energy function* on the space of chords, where x and y, as described above, are the start and endpoint, respectively, of a chord (cf [2, Section 7.4]). Using the parametrization γ we can write

$$E(s, t) = \tfrac{1}{2}|\gamma(s) - \gamma(t)|^2.$$

Furthermore, we will need the following subsets of $K \times K$:

- Let $S \subset K \times K$ be the set of chords that intersect the knot K in their interior.
- Let $F \subset K \times K$ be the set of chords that intersect the framing. Since F is symmetric with respect to the diagonal in $K \times K$ according to Remark 2.9, F can be split into

  $$F = F^s \cup F^e$$

  where F^s and F^e are the sets of chords that intersect the framing at their start and endpoint, respectively. Then

  $$F^e = \{(s, t) \in K \times K : (t, s) \in F^s\}.$$

- Let $B \subset K \times K$ be the set of chords that begin or end at the basepoint $*$. If we choose a parametrization with $\gamma(0) = *$, then $B = (K \times \{0\}) \cup (\{0\} \times K)$.

Lemma 2.11 [2, Lemma 7.10] *For a generic knot $K \subset \mathbb{R}^3$ the following holds for the space $K \times K$ of chords* (*see* Figure 4):

(i) *E attains its minimum* 0 *along the diagonal*, *which is a Bott nondegenerate critical manifold*; *the other critical points are nondegenerate binormal chords of index* 0, 1 *and* 2.

(ii) *The subset $S \subset K \times K$ of chords meeting K in their interior is an immersed curve with boundary consisting of finitely many chords tangent to K at one endpoint*, *and with finitely many transverse self-intersections consisting of chords meeting K twice in their interior.*

(iii) *The negative gradient $-\nabla E$ is not pointing into S at the boundary points.*

The proof of this lemma is given in [2].

From now on, we assume that the representative of the knot has been chosen to have the properties listed in Lemma 2.11.

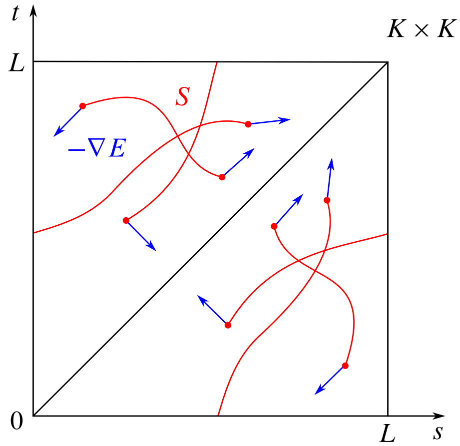

Figure 4: The space $K \times K$ of chords.

Remark 2.12 The Morse Bott function E can be converted into a Morse function by adding a smooth function $f\colon K \times K \to \mathbb{R}$ to E that satisfies the following properties:

(i) f has exactly two critical points along the diagonal of $K \times K$, a minimum m and a maximum M.

(ii) Outside the diagonal, the function f is smoothly extended in such a way that it vanishes outside a small neighborhood of the diagonal, especially at all other critical points.

In the following, we assume that E is a Morse function since this can be achieved by adding an arbitrarily small perturbation f.

Remark 2.13 The set S is obviously symmetric with respect to the diagonal in $K \times K$: if the chord (s,t) intersects the knot in its interior, so does the reverse oriented chord (t,s); the same also holds for the boundary of S. So ∂S can be split into

$$\partial S = \partial^s S \cup \partial^e S$$

where $\partial^s S$ and $\partial^e S$ are the sets of chords that are tangent to K at their start and endpoint, respectively.

Lemma 2.14 *For a generic knot K and a generic framing $\nu\colon S^1 \to S^2$ the following holds for the space $K \times K$ of chords:*

The set $F^s \subset K \times K$ of chords that intersect the framing at their starting point is a one-dimensional submanifold with boundary and $\partial F^s = \partial^s S$.

The proof of this lemma is given in the appendix.

Remark 2.15 Lemma 2.14 implies that F^s has no self-intersections. Since F is symmetric, an analogous statement holds for F^e. So the self-intersections of F are the intersections of F^s and F^e. These are either located on the diagonal of $K \times K$ or occur in pairs symmetrically to the diagonal and contain the chords that intersect the framing at their start and endpoint.

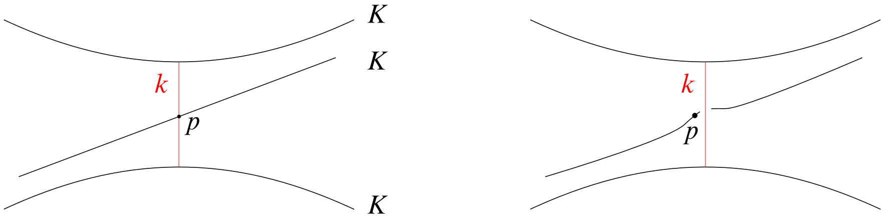

Figure 5: Perturbation of the knot in a neighborhood of p such that the chord k does not intersect the knot in its interior anymore.

Definition 2.16 Let K be a knot and $E\colon K \times K \to \mathbb{R}$ be the energy function that has been perturbed to a Morse function as described above. Denote by $\operatorname{Crit}_k(E)$ the set of critical points of Morse index k of the function E.

Lemma 2.17 *For a generic knot K, a generic basepoint and a generic framing, the following holds for the space $K \times K$ of chords:*

$$\operatorname{Crit}_{0,1} \cap (B \cup F \cup S) = \varnothing$$

where $\operatorname{Crit}_{0,1} := \operatorname{Crit}_0 \cup \operatorname{Crit}_1$.

Proof According to Lemma 2.11(i), $\operatorname{Crit}_{0,1}$ is a finite set. The sets B, F and S can be considered separately from each other:

If $\operatorname{Crit}_{0,1} \cap B \neq \varnothing$, an arbitrarily small shift of the basepoint is sufficient to achieve $\operatorname{Crit}_{0,1} \cap B = \varnothing$.

If $\operatorname{Crit}_{0,1} \cap F^s \neq \varnothing$, let $k \in \operatorname{Crit}_{0,1} \cap F^s$. If the framing is changed in an arbitrarily small neighborhood of the starting point of the chord k by an arbitrarily small perturbation, k no longer intersects the framing at its starting point. Thus, also the reverse oriented chord, which was an element of F^e, does not intersect the framing at its endpoint anymore. This procedure is necessary only finitely many times because $\operatorname{Crit}_{0,1} \cap F$ is finite.

If $\operatorname{Crit}_{0,1} \cap S \neq \varnothing$, let $k \in \operatorname{Crit}_{0,1} \cap S$ and p be the intersection point of k with the knot in the interior of k. If the knot is changed in an arbitrarily small neighborhood of p by an arbitrarily small perturbation in a suitable direction (see Figure 5), k no longer intersects the knot in its interior. Since $\operatorname{Crit}_{0,1} \cap S$ is finite, only finitely many such perturbations have to be made. □

From now on, we assume that the representative of the knot has been chosen such that it satisfies the statement of Lemma 2.17, the energy function E is a Morse function according to Remark 2.12 and the framing satisfies Lemma 2.14.

We will show more generic properties of the function E. For this it may be necessary to perturb the function E with the help of the (jet) transversality theorem, ie to vary E in the function space $C^n(T^2, \mathbb{R})$ for an $n > 0$. Such a variation can be considered as the addition of another smooth function $g\colon K \times K \to \mathbb{R}$ to E. Denote by

$$E_g := E + g$$

the perturbed function. The function g can always be chosen so that it is arbitrarily small (in the C^n sense) (according to the (jet) transversality theorem) and vanishes in a small neighborhood of the diagonal in $K \times K$ and small neighborhoods of all critical points of E (since E does not have to be perturbed in these neighborhoods, because the properties shown in Lemmas 2.20 and 2.23 for generic knots are already satisfied on the diagonal and because, according to Lemma 2.17, $\mathrm{Crit}_{0.1} \cap (B \cup F \cup S) = \varnothing$). Thus, all critical points remain unchanged, ie the following holds for $k = 0, 1, 2$,

$$\mathrm{Crit}_k(E_g) = \mathrm{Crit}_k(E).$$

Likewise, the sets S, F and B do not change, since the knot itself is not changed.

We will consider the flow φ^s of the negative gradient $-\nabla E$, in the perturbed case the flow φ_g^s of $-\nabla E_g$. The stable and unstable manifolds of a critical point k of E are

$$W^s(k) = \{x \in T^2 : \lim_{s \to +\infty} \varphi^s(x) = k\}, \quad W^u(k) = \{x \in T^2 : \lim_{s \to -\infty} \varphi^s(x) = k\},$$

respectively. Similarly, the stable and unstable manifolds of critical points k of E_g are denoted by $W_g^s(k)$ and $W_g^u(k)$, respectively.

To be able to show some generic properties of the function E_g, we need the following statement: if we perturb the unstable manifold of a critical point of index 1 a little bit, we can find a function $\tilde{g}$ such that the perturbed unstable manifold is realized by the flow of the negative gradient of the function $E_{\tilde{g}}$. First, we show the following lemma.

Lemma 2.18 (Cieliebak and Volkov [4, Lemma 3.7]) *For all $\delta, \varepsilon > 0$ there exists a smooth function $\chi : \mathbb{R} \to [0, 1]$ such that*

(i) *χ is nondecreasing on $\mathbb{R}_-$ and nonincreasing on $\mathbb{R}_+$,*

(ii) *χ is constant 1 in a neighborhood of 0,*

(iii) *χ is constant 0 on $\mathbb{R} \setminus (-\delta, \delta)$,*

(iv) *for all $x \in \mathbb{R}$, we have $|x\chi'(x)| < \varepsilon$.*

Proof We consider the function $f : [0, \delta] \to \mathbb{R}$ defined by $x \mapsto \varepsilon \log(\delta/x)$. Notice that f satisfies $xf'(x) = -\varepsilon$, $f(\delta) = 0$ and $f(\delta e^{-1/\varepsilon}) = 1$. Now shift the function $\max(f, 1)$ slightly to the left, extend by 0 to the right, smoothen it and mirror it on the y–axis. This gives us the function χ we are looking for. □

Lemma 2.19 *Let $a, b, c, d \in \mathbb{R}$ with $a < 0 < b$ and $c < d$. Let $f \in C^\infty([c, d], (a, b))$ be a function such that*

(i) *$f(c) = f(d) = 0$,*

(ii) *for $1 \le k \le \infty$, we have $f^{(k)}(c) = f^{(k)}(d) = 0$.*

Let $\tilde{f} : [c, d] \to (a, b) \times [c, d]$, $y \mapsto (f(y), y)$, be the graph of f over the y–axis, and let $g : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto y$, be the projection to the second coordinate.

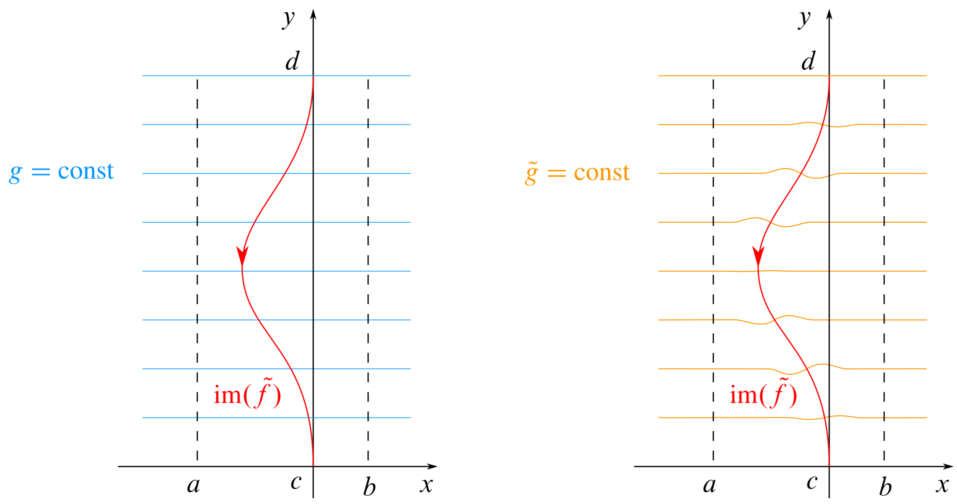

Figure 6: Adjustment of the level sets. Left: $\text{im}(\tilde f)$ as a graph over the y–axis and the level sets of g. Right: changed level sets.

Then there exists a function $\tilde g\colon \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$\tilde g = g \quad \text{on } \mathbb{R}^2 \setminus ((a,b)\times(c,d)),$$
$$\text{im}(\tilde f) = \{\varphi^s_{\tilde g}((0,d)) : 0 \le s \le \tilde t\},$$

where $\varphi^s_{\tilde g}$ *is the flow along the vector field* $-\nabla \tilde g$ *and* $\varphi^{\tilde t}_{\tilde g}((0,d)) = (0,c)$.

Proof Figure 6, left, shows $\text{im}(\tilde f)$ and the level sets of g in $R := [a,b]\times[c,d]$. We now want to change g on R to $\tilde g$ so that the level sets of $\tilde g$ are perpendicular to $\text{im}(\tilde f)$ (see Figure 6, right), but outside of R coincide with the level sets of g. To do this we use a diffeomorphism

$$\Phi\colon R \to R, \quad (x,y) \mapsto (x, y + \Psi(x,y)),$$

where $\Psi \in C^\infty(R,\mathbb{R})$ satisfies

(i) $\Psi(x,y) = 0$ for all $(x,y) \in \partial R$,

(ii) $\Psi(f(y),y) = 0$,

(iii) $\frac{\partial \Psi}{\partial x}(f(y),y) = -f'(y)$.

With this function Φ we set

$$\tilde g := g \circ \Phi^{-1}.$$

Thus, the level sets of $\tilde g$ are as required. The flow of the negative gradient runs perpendicular to the level sets and is unique. So we have

$$\text{im}(\tilde f) = \{\varphi^s_{\tilde g}((0,d)) : 0 \le s \le \tilde t\}$$

with $\tilde t > 0$ where $\varphi^{\tilde t}_{\tilde g}((0,d)) = (0,c)$.

It remains to show that such a function Ψ exists so that Φ is a diffeomorphism. We choose

$$\Psi\colon R \to \mathbb{R}, \quad (x,y) \mapsto -f'(y)(x-f(y))\chi(x-f(y)),$$

where $\chi\colon \mathbb{R} \to \mathbb{R}$ is a smooth cutoff function with the properties

(i) $\chi(0) = 1$,

(ii) $\operatorname{supp}\chi \subset [-\delta, \delta]$, where $\delta > 0$ must be chosen so that $a+\delta < f(y) < b-\delta$ for all $y \in [c,d]$,

(iii) $|z\chi'(z)| < \varepsilon$ for all $z \in \mathbb{R}$ and an $\varepsilon > 0$, which will be determined more precisely later.

According to Lemma 2.18, such a function χ exists for any $\delta, \varepsilon > 0$. All properties required for Ψ are satisfied. Moreover,

$$\begin{aligned}\frac{\partial\Psi}{\partial y}(x,y) &= -f''(y)(x-f(y))\chi(x-f(y)) + \underbrace{(f'(y))^2\chi(x-f(y))}_{\geq 0} + (f'(y))^2(x-f(y))\chi'(x-f(y))\\ &\geq -|f''(y)(x-f(y))\chi(x-f(y))| - |(f'(y))^2(x-f(y))\chi'(x-f(y))|.\end{aligned}$$

The first term in the last line does not vanish only if $|x-f(y)| < \delta$. After possibly decreasing δ, $|f''(y)(x-f(y))\chi(x-f(y))| < \frac{1}{2}$ can be achieved. Additionally, we can get $|(x-f(y))\chi'(x-f(y))| < \varepsilon$. So by choosing ε small enough, we can achieve $|(f'(y))^2(x-f(y))\chi'(x-f(y))| < \frac{1}{2}$. Altogether, $\frac{\partial\Psi}{\partial y}(x,y) > -1$ can be guaranteed for all $(x,y) \in R$.

Now we can show that the above defined function Φ is a diffeomorphism. First, we consider

$$D_{(x,y)}\Phi = \begin{pmatrix} 1 & 0 \\ \frac{\partial\Psi}{\partial x}(x,y) & 1 + \frac{\partial\Psi}{\partial y}(x,y) \end{pmatrix}.$$

Since $1 + \frac{\partial\Psi}{\partial y}(x,y) > 0$ for all $(x,y) \in R$, $D_{(x,y)}\Phi$ is an isomorphism for all $(x,y) \in R$.

It also follows from $1 + \frac{\partial\Psi}{\partial y}(x,y) > 0$ that the map $y \mapsto y + \Psi(x,y)$ is strictly increasing. Thus, Φ is injective.

Since Φ is continuous and $\Phi|_{\partial R} = \mathrm{id}$, the surjectivity of Φ follows.

Thus, Φ is a diffeomorphism and this proves the lemma. □

Now the following generic properties of the function E_g can be shown:

Lemma 2.20 *For a generic function g the following holds for the space $K \times K$ of chords:*

(i) *For all critical points k of the function E_g of index 1, $W_g^u(k) \pitchfork B$, $W_g^u(k) \pitchfork S$, and $W_g^u(k) \pitchfork F^s$ as well as $W_g^u(k) \pitchfork F^e$. (Note that there is no condition on avoiding the intersections of F^s and F^e.)*

(ii) *For all critical points k of the function E_g of index 1, $W_g^u(k) \cap \partial S = \varnothing$, $W_g^u(k) \cap \partial F = \varnothing$, and $W_g^u(k) \cap S_2 = \varnothing$ where $S_2 \subset S$ is the set of self-intersections of S.*

(iii) *For all critical points k of the function E_g of index 1, $W_g^u(k) \cap B \cap S = \varnothing$ and $W_g^u(k) \cap F \cap S = \varnothing$.*

(iv) *There is no trajectory along the vector field $-\nabla E_g$ between two critical points of index 1.*

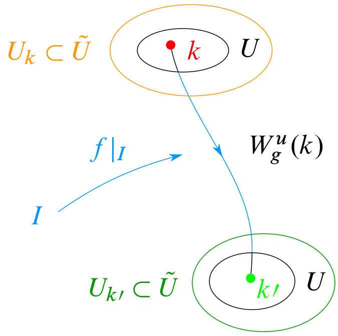

Figure 7: Neighborhoods of critical points k and k' of index 1 and 0, respectively, and the unstable manifold of k.

Remark 2.21 The set of self-intersections of F does not have to be considered, because at these finitely many points a chord is multiplied by μ or μ^{-1} from the left and from the right. This is not a problem. The set S_2 must be avoided because at these points a chord would be divided into three parts. However, this situation is not covered by any relation.

Proof According to Lemma 2.17, we have $\mathrm{Crit}_{0,1} \cap (B \cup S \cup F) = \varnothing$. Thus, there exist open neighborhoods U_k of all $k \in \mathrm{Crit}_{0,1}$ such that $U_k \cap (B \cup S \cup F) = \varnothing$. Consequently, for all $k \in \mathrm{Crit}_1$,

$$(W_g^u(k) \cap \widetilde{U}) \pitchfork (B \cup S \cup F),$$

where $\widetilde{U} := \bigcup_{k' \in \mathrm{Crit}_{0,1}} U_{k'}$. We shrink $\widetilde{U}$ to U such that $\overline{U} \subset \widetilde{U}$ holds for the closure $\overline{U}$. Then in the following it is sufficient to perturb $W_g^u(k)$ to $W_{\tilde{g}}^u(k)$ for all $k \in \mathrm{Crit}_1$ on the subset $W_g^u(k) \setminus U$ with the help of the relative version of the jet transversality theorem (see Hirsch [7, Theorem 2.8] and Golubitsky and Guillemin [6, Corollary 4.12(b)]), so that

$$W_{\tilde{g}}^u(k) \cap (\widetilde{U} \setminus U) = W_g^u(k) \cap (\widetilde{U} \setminus U)$$

and

$$(W_{\tilde{g}}^u(k) \setminus U) \pitchfork (B \cup S \cup F).$$

To prove (i) and (ii), let k be a critical point of index 1 and $f \in C^\infty(\mathbb{R}, T^2)$ such that f is a parametrization of one of the two flow lines which make up $W_g^u(k)$. Let $I \subset \mathbb{R}$ be the compact interval for which

$$\mathrm{im}(f|_I) = W_g^u(k) \setminus U$$

(see Figure 7). Now suppose $W_g^u(k) \cap S \neq \varnothing$ and $p \in W_g^u(k) \cap S$.

Case 1 If $p \in \partial S$, then we construct a map $\tilde{f} : \mathbb{R} \to T^2$ as follows:

- $\tilde{f}|_{\mathbb{R}\setminus I} = f|_{\mathbb{R}\setminus I}$.
- ∂S is a submanifold of T^2. Using the relative version of the transversality theorem (Golubitsky and Guillemin [6, Corollary 4.12(b)]) we can perturb $f|_I$ in the space $C^n(I, T^2)$, for n big enough, to a map $\tilde{f}|_I$ such that $\tilde{f}|_I \pitchfork \partial S$ and $\mathrm{im}(\tilde{f}|_I) \cap (\widetilde{U} \setminus U) = \mathrm{im}(f|_I) \cap (\widetilde{U} \setminus U)$.

Thus, we get $\tilde{f} \pitchfork \partial S$. It follows that $\tilde{f}^{-1}(\partial S) = \varnothing$ since $\operatorname{codim}(\partial S \subset T^2) = 2$. In suitable local coordinates $\operatorname{im}(f)$ can be represented as a subset of the y–axis in $\mathbb{R}^2$. Since $\tilde{f}$ can be chosen arbitrarily close to f, we can guarantee that in these coordinates $\tilde{f}$ is a graph over the y–axis. According to Lemma 2.19, there exists a function $\tilde{g}$ for which $W^u_{\tilde{g}}(k) \cap U = W^u_g(k) \cap U$ and $W^u_{\tilde{g}}(k) \setminus U = \operatorname{im}(\tilde{f}|_I)$. Thus, we get $W^u_{\tilde{g}}(k) \cap \partial S = \varnothing$.

Case 2 Since the set $S_2 \subset S$ of self-intersections of S is finite according to Lemma 2.11(ii), it follows analogously to case 1 that $W^u_{\tilde{g}}(k) \cap S_2 = \varnothing$.

Case 3 If $p \in \widehat{S} := S \setminus (\partial S \cup S_2)$, then the set of all 1–jets from $\mathbb{R}$ to T^2 is

$$J^1(\mathbb{R}, T^2) = \mathbb{R} \times T^2 \times \mathbb{R}^2.$$

Consider the map

$$h \colon \mathbb{R} \to J^1(\mathbb{R}, T^2), \quad t \mapsto (t, f(t), \dot{f}(t)).$$

We construct a map $\tilde{f} \colon \mathbb{R} \to T^2$ as follows:

- $\tilde{f}|_{\mathbb{R}\setminus I} = f|_{\mathbb{R}\setminus I}$.
- The tangent bundle $T\widehat{S}$ is a submanifold of $T(T^2) = T^2 \times \mathbb{R}^2$. So $\mathbb{R} \times T\widehat{S}$ is a submanifold of $\mathbb{R} \times T(T^2) = \mathbb{R} \times T^2 \times \mathbb{R}^2$. Using the relative version of the jet transversality theorem (see [6, Corollary 4.12(b); 7, Theorem 2.8]) we can perturb $f|_I$ in the space $C^n(I, T^2)$, for n big enough, to a map $\tilde{f}|_I$ such that $\operatorname{im}(\tilde{f}|_I) \cap (\widetilde{U} \setminus U) = \operatorname{im}(f|_I) \cap (\widetilde{U} \setminus U)$ and $\tilde{h}|_I \pitchfork T\widehat{S}$ with the map $\tilde{h}|_I$, $t \mapsto (t, \tilde{f}(t), \dot{\tilde{f}}(t))$, which is also perturbed.

Thus, we get $\tilde{h} \pitchfork \mathbb{R} \times T\widehat{S}$. From $\operatorname{codim}(\mathbb{R} \times T\widehat{S} \subset \mathbb{R} \times T(T^2)) = 2$ it follows that $\tilde{h}^{-1}(\mathbb{R} \times T\widehat{S}) = \varnothing$. So $\operatorname{im}(\tilde{f}) \pitchfork \widehat{S}$ since for all $t \in \mathbb{R}$ with $\tilde{f}(t) \in \widehat{S}$ we have $\dot{\tilde{f}}(t) \notin T_{\tilde{f}(t)}\widehat{S}$. In suitable local coordinates $\operatorname{im}(f)$ can be represented as a subset of the y–axis in $\mathbb{R}^2$. Since $\tilde{f}$ can be chosen arbitrarily close to f, we can guarantee that in these coordinates $\tilde{f}$ is a graph over the y–axis. According to Lemma 2.19, there exists a function $\tilde{g}$ for which $W^u_{\tilde{g}}(k) \setminus U = \operatorname{im}(\tilde{f}|_I)$ and $W^u_{\tilde{g}}(k) \cap U = W^u_g(k) \cap U$.

Altogether, we get $W^u_{\tilde{g}}(k) \pitchfork S$.

Since Crit_1 is a finite set, finitely many (arbitrarily small) perturbations suffice to guarantee transversality to S for all unstable manifolds of critical points of index 1.

The proof of the statements $W^u_g(k) \pitchfork B$, $W^u_g(k) \pitchfork F^s$, $W^u_g(k) \pitchfork F^e$ and $W^u_g(k) \cap \partial F = \varnothing$ is analogous to that of the statements $W^u_g(k) \pitchfork S$ and $W^u_g(k) \cap \partial S = \varnothing$.

The two statements of (iii) can be shown by similar transversality arguments as before.

For (iv), let k be a critical point of index 1 and $f \in C^\infty(\mathbb{R}, T^2)$ with $\operatorname{im}(f) = W^u_g(k)$. According to Lemma 2.11(i), Crit_1 is a submanifold of T^2 with $\operatorname{codim}(\mathrm{Crit}_1 \subset T^2) = 2$. Therefore, the statement follows analogously to (ii). □

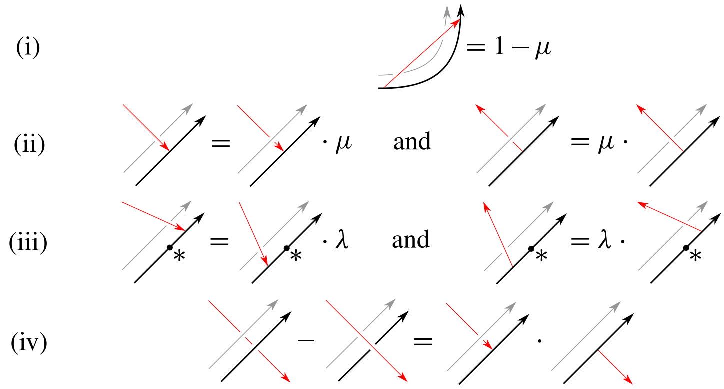

Figure 8: Relations for chords moving along the vector field $-\nabla E$.

Let $R := \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ be the commutative ring over $\mathbb{Z}$, generated by $\lambda, \lambda^{-1}, \mu$ and μ^{-1} modulo the relations $\lambda \cdot \lambda^{-1} = \mu \cdot \mu^{-1} = 1$. Let C_1 be the $\mathbb{Z}$–vector space generated by $\mathrm{Crit}_1(E)$ and C_0 the noncommutative R–algebra generated by $\mathrm{Crit}_0(E)$, ie λ and μ commute with each other, but not with any elements of $\mathrm{Crit}_0(E)$. These in turn do not commute at all. In the following we will let flow chords along the negative gradient $-\nabla E$. It can happen that a chord converges to a point, intersects the basepoint or the framing or intersects the knot in its interior. For these cases we define the relations in Figure 8. There the knot K is drawn in black, the framing K' in gray and the chords in red. Analogous to Remark 2.4, only the relevant parts of the knot are drawn in the pictures. The diagrams each show a small region of the knot. The first line refers to any contractible chord. In the second, third and fourth relation, the diagrams agree outside the drawn region. In the fourth line, the first two diagrams show a chord that runs once over and once under a strand of the knot. At the transition from the first to the second diagram, or vice versa, with the help of a homotopy, the chord intersects the knot. At this point, the chord is split into two parts, which is shown in the other two diagrams. All diagrams are to be understood as objects in three-dimensional space. Therefore, relation (iv) is for example equivalent to the relation shown in Figure 9.

Assume a chord $c \in W_g := \bigcup_{k \in \mathrm{Crit}_1} W_g^u(k) \subset K \times K$. If c is split into two parts according to relation (iv) during the movement along the negative gradient $-\nabla E_g$, both parts and the original chord flow further along $-\nabla E_g$. Each of these chords may be split again according to relation (iv), and so on.

Lemma 2.22 *For a generic function g the following holds for the perturbed function E_g: the recursive splittings as described above can happen only finitely many times, ie relation* (iv) *is applied only finitely many times.*

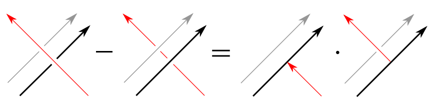

Figure 9: Equivalent representation of relation (iv).

Proof (1) Lemma 2.11(iii) implies that there exists a neighborhood $U \subset K \times K$ of the finitely many chords ∂S that are tangent to K at one endpoint and an $\varepsilon > 0$ with the property that each chord in $U \cap S$ decreases in length by more than ε under the flow of $-\nabla E_g$ before it meets S again, and the same holds for the longer chord resulting from the splitting according to relation (iv). On the other hand, if a string $s \in S \setminus U$ is split at its intersection with the knot, both pieces are shorter than s by at least a fixed length $\delta > 0$ since s can be neither tangent nor "almost" tangent to K. In total, each piece will be at least $\min(\varepsilon, \delta)$ shorter than the original chord. Since its length is finite, however, only finitely many intersections can happen (see [2, proof of Proposition 7.14]).

(2) According to Lemma 2.20(i), we have $W_g \pitchfork S$ for a generic function g. It follows that there exist only finitely many intersections of W_g with S since W_g is compact. The chords arising from splitting at these intersections according to relation (iv) are, according to (1), at least $\min(\varepsilon, \delta)$ shorter than the original chords. Let $k \in \mathrm{Crit}_1$. At the first intersection of $W_g^u(k)$ with S, starting from k in one of the two possible directions, the chord is split into two chords k_1 and k_2. If k_1 lies now on $W_g^u(k)$, nothing more has to be done, because $W_g^u(k)$ is already transverse to S. Otherwise the trajectory of k_1 along the negative gradient $-\nabla E_g$ can be perturbed outside of $W_g^u(k)$ according to Lemma 2.19 such that this trajectory becomes transverse to S. This is possible because $W_g^u(k)$ and the trajectory of k_1 are disjoint outside the critical points.

The same procedure is used for k_2. If k_2 is neither on $W_g^u(k)$ nor on the trajectory starting at k_1 that may have been perturbed, we perturb the trajectory starting at k_2 with the help of Lemma 2.19 outside of $W_g^u(k)$ and the trajectory starting at k_1 so that it becomes transverse to S.

All other intersections of $W_g^u(k)$ with S are handled in the same way; likewise the unstable manifolds of all other chords from Crit_1.

Since W_g has only finitely many intersections with S, the use of relation (iv) also results in a finite number of chords. The ε from (1) may change due to the necessary perturbations of g. But since the perturbed function is arbitrarily close to g, the new ε is arbitrarily close to the original one. Therefore, it can be achieved that all resulting chords are still at least $\min(\varepsilon, \delta)$ shorter than the original chords.

The newly created chords are now handled in the same way as the unstable manifolds of the critical points, and the function g may be perturbed accordingly. Since the trajectories of these chords are transverse to S, they intersect S only finitely many times. This process is continued as long as further intersections of a trajectory with S occur and relation (iv) is applied. Since each chord has finite length and becomes shorter at a splitting by at least $\min(\varepsilon, \delta)$, this process ends after finitely many steps and the function g has to be adapted only to finitely many trajectories, which are pairwise disjoint outside the critical points.

All in all: relation (iv) is used only finitely many times. □

Now we can show further properties of a generic function E_g:

Let $k \in \mathrm{Crit}_1$. Denote by $\widehat{W}_g^u(k)$ the "extended" unstable manifold of k, ie all flow lines of chords arising from the iterative splitting according to relation (iv).

Lemma 2.23 *For a generic function g the following holds for the function E_g, the space $K \times K$ of chords and all chords $c \in \widehat{W}_g^u(k)$ for all $k \in \mathrm{Crit}_1$:*

- (i) $\varphi_g^s(c) \pitchfork B$, $\varphi_g^s(c) \pitchfork S$, *and* $\varphi_g^s(c) \pitchfork F$ *for* $s \geq 0$.
- (ii) *For all $s \geq 0$, we have $\varphi_g^s(c) \cap \partial S = \varnothing$, $\varphi_g^s(c) \cap \partial F = \varnothing$, and $\varphi_g^s(c) \cap S_2 = \varnothing$ where S_2 is the set of self-intersections of S.*
- (iii) *For all $s \geq 0$, we have $\varphi_g^s(c) \cap B \cap S = \varnothing$ and $\varphi_g^s(c) \cap F \cap S = \varnothing$.*
- (iv) $c \notin (B \cup F \cup S)$.
- (v) $\lim_{s\to\infty} \varphi_g^s(c) \notin \mathrm{Crit}_1$, *ie no such chord runs along $-\nabla E_g$ to a critical point of index* 1.
- (vi) $c \notin \mathrm{Crit}_k$ *for* $k = 0, 1, 2$.

Proof The statements (i)–(v) follow analogously to the proofs of Lemmas 2.17 and 2.20, since Crit_1 is a finite set and relation (iv) is applied only finitely many times according to Lemma 2.22.

For (vi), according to Lemma 2.22, the set of all such chords c is a finite set; likewise Crit_k is finite for $k = 0, 1, 2$. Therefore, the statement can be reached by a small perturbation g, analogous to the proof of Lemma 2.20(ii). □

2.2 Representation of knots as closed braids

Every knot can be represented as a closed braid (Livingston [8, Section 7.3]). To define the chord algebra of a knot, we draw the knot as a closed braid along an ellipse in the following way:

The braid should be positioned in such a way that the binormal chords do not intersect the knot in their interior. This can be achieved by arranging the strands of the braid one above the other, ie from the drawing plane, with each strand being placed above a slightly larger ellipse than the one below. In addition, the strands should have a very small distance from each other. This ensures that no unwanted binormal chords are created that run from the area of the crossings of the knot to "opposite" strands. In order to be able to distinguish the strands in the diagram well, however, they are drawn with larger distance. They are also numbered from the outside to the inside.

By an arbitrarily small perturbation it can be achieved that the torsion vanishes only at finitely many points.

All crossings of the knot are drawn in a quarter of the ellipse so that they lie between two endpoints of the main axes. In the remaining three quarters the strands run parallel to each other. In Figure 10 this is shown by the example of the right-handed trefoil knot, equipped with the Seifert framing. The knot is drawn in black and the framing in gray.

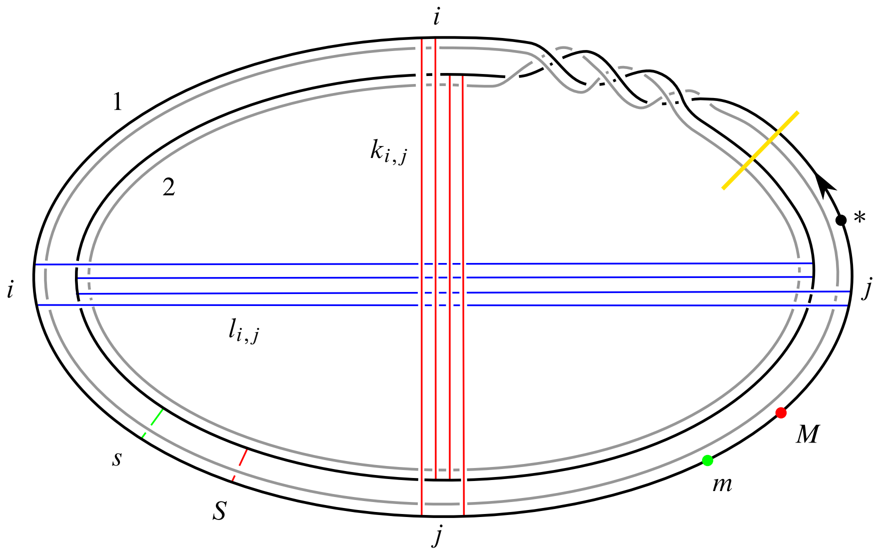

Figure 10: Critical points of the function E on the right-handed trefoil.

The nontrivial binormal chords are, as is easy to see, the chords that run parallel to the main axes of the ellipse, as well as the very short chords that connect the strands together. However, the way of drawing the knot produces a one-dimensional critical submanifold for the latter. By a small perturbation of the knot this can be cleared up in such a way that exactly two critical points arise, one of the index 0, marked with s, and one of the index 1, marked with S. In addition, if there are more than two strands, there may be other short binormal chords between the strands in the area of the crossings. In the case of more than two strands, they are placed in such a way that the arrangement as shown in Figure 11 is achieved. This guarantees that the very short chords connecting the strands do not intersect the knot in their interior. To ensure that no binormal chord intersects the framing, a small perturbation of the framing may be necessary. Another small perturbation of the framing may be necessary to satisfy the conditions listed in Remark A.1 (these conditions are necessary for the proof of Lemma 2.14).

The trivial binormal chords are all points on the diagonal of $K \times K$, ie chords which start and end at the same point on K. As mentioned in Remark 2.12(i), the energy function is perturbed such that we get exactly two critical points along the diagonal: one minimum, denoted by m (which has Morse index 0), and one maximum, denoted by M (which has Morse index 1).

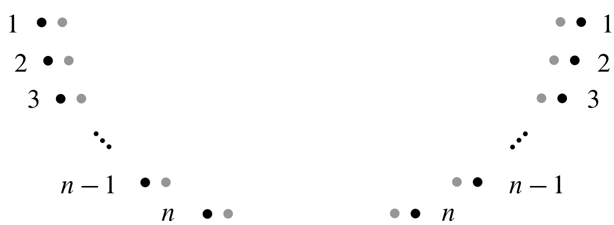

Figure 11: Arrangement of n strands in the cross section.

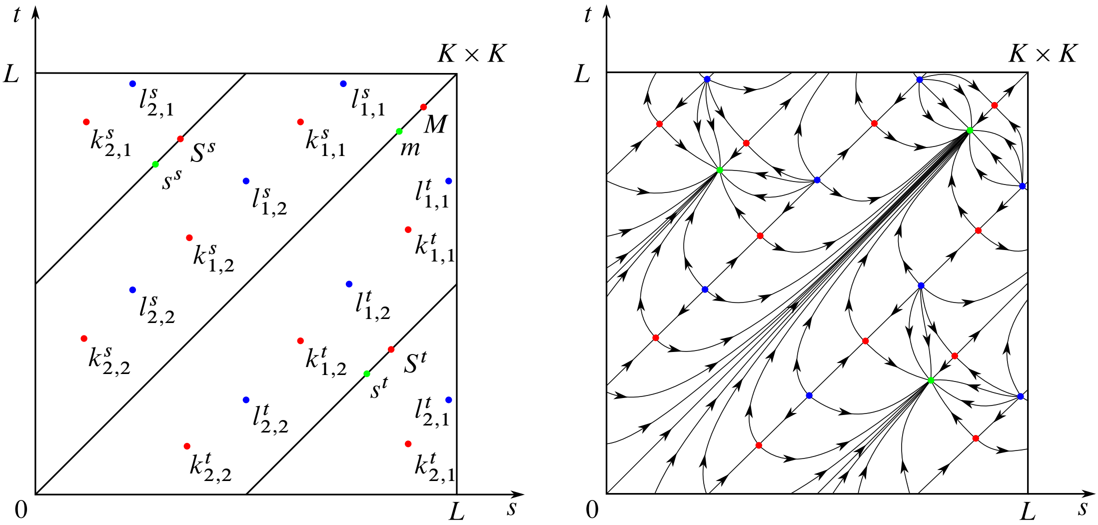

Figure 12: Critical points in $K \times K$ for the knot K in Figure 10. Left: labeling of the critical points. Right: gradient flow $-\nabla E$.

In Figure 10 the critical points, ie the binormal chords, are distinguished by color:

- Critical points of index 0 are marked green, in this example m and s.
- Critical points of index 1 are marked red. Here, M, S and the chords labeled with $k_{i,j}$. The indexing means that the chord runs from strand i to strand j. In order to determine on which strand an endpoint of the chord lies, one moves, starting from this endpoint, along the knot until one reaches the numbering of the strands. The yellow mark must not be exceeded. The numbering is therefore unique.
- Critical points of index 2 are marked blue, here $l_{i,j}$. These are only mentioned for the sake of completeness and will not be required further.

Since the knot K is parametrized by γ, we get a canonical orientation of the knot, which is marked by an arrow. The position of the basepoint is chosen as shown in Figure 10. This choice ensures that no binormal chord has the basepoint as its start or endpoint.

Each chord $c = (s, t) \in K \times K$ can also be assigned an orientation as described above: c is oriented from s to t. If we consider the orientation, all chords that are not on the diagonal of $K \times K$ occur in pairs: once with orientation in one direction and once with reverse orientation. In order to indicate the orientation of a chord in its labeling, another index is introduced, which is noted in the upper right of the name. Denote by $c^s = (s', t)$ the chord c for which $s' < t$ holds, and by $c^t = (s, t')$ the chord c with $t' < s$. For example, in Figure 10 $k^s_{1,1}$ is the longest chord of index 1 with orientation from top to bottom. This indexing is not necessary for points on the diagonal of $K \times K$, because then we have $s = t$, the corresponding chords have vanishing length and occur only once.

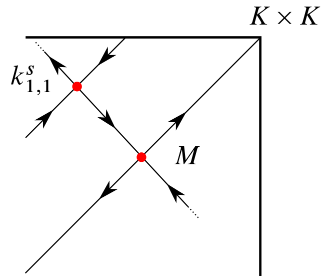

Figure 13: Wrong position of M.

Figure 12 shows the same critical points as Figure 10, but here their positions are shown on $K \times K \cong T^2$. On the left side the critical points are labeled with their names. The arrows on the right indicate the direction of the gradient flow of the function E.

The figure shows that the position of M on the diagonal can be chosen almost arbitrarily: M must not be chosen in such a way that the gradient flow of a critical point of index 1 runs in the direction of M, as shown in Figure 13. If M is placed as shown in Figure 13, the Smale condition is not satisfied (see Audin and Damian [1, Section 2.2.b]).

S is positioned analogously.

For all other critical points the Smale condition is satisfied because the unstable manifolds of points of index 2 and the stable manifolds of points of index 0 are already two-dimensional. So the tangent spaces at these manifolds are two-dimensional and the transversality is ensured.

Thus, $-\nabla E$ represents a gradient field that satisfies the Smale condition.

2.3 Definition of the cord algebra

Before defining the cord algebra of K, we will first define a map $\hat{D}\colon (K \times K) \setminus A \to C_0$, where the exceptional set $A \subset K \times K$ contains all points $c \in K \times K$ that satisfy at least one of the following properties:

(1) $c \in \mathrm{Crit}_2(E) \cup \big(\bigcup_{k \in \mathrm{Crit}_1(E)} W^s(k)\big) \cup \mathrm{Crit}_0(E)$,

(2) $c \in (S \cup F \cup B)$,

(3) $\varphi^s(c) \not\pitchfork B$, $\varphi^s(c) \not\pitchfork F$, or $\varphi^s(c) \not\pitchfork S$ for $s \geq 0$,

(4) $\varphi^s(c) \cap \partial S \neq \varnothing$, $\varphi^s(c) \cap \partial F \neq \varnothing$, or $\varphi^s(c) \cap S_2 \neq \varnothing$ for $s \geq 0$,

(5) $\varphi^s(c) \in S$ for an $s > 0$ and one of the properties (1) to (4) is true for one of the chords resulting from (possibly multiple) splitting according to relation (iv).

So now we can define the map $\hat{D}$ as

$$\hat{D}\colon (K \times K) \setminus A \to C_0, \quad c \mapsto \partial(c) + \delta(c).$$

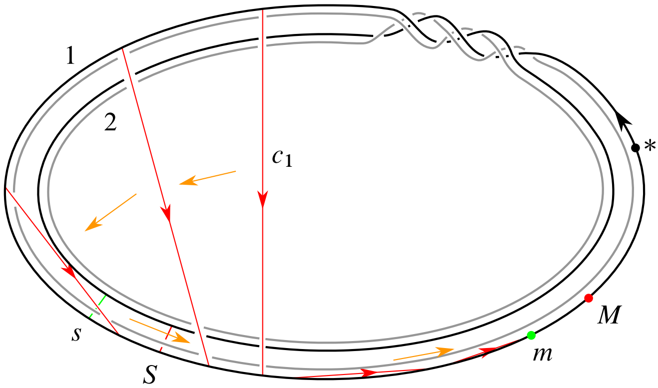

Figure 14: Movement of c_1 along the gradient flow.

The two maps ∂ and δ are described below:

First, let ∂ be the map

$$\partial\colon (K \times K) \setminus A \to C_0, \quad c \mapsto \sum_{d \in \mathrm{Crit}_0(E)} n(c,d)\lambda^{\alpha_1}\mu^{\beta_1} d \lambda^{\alpha_2}\mu^{\beta_2},$$

where

- $n(c,d) \in \{0,1\}$ is the number of trajectories along the vector field $-\nabla E$ from c to d;
- $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}$ are such that the intersections with the framing or the basepoint occurring during the movement of the chord c along $-\nabla E$ are taken into account according to relations (ii) or (iii), respectively.

The map ∂ is well defined since each point $c \in (K \times K) \setminus A$ lies on exactly one trajectory along the vector field $-\nabla E$. This trajectory ends at a critical point of index 0, so there is exactly one $d \in \mathrm{Crit}_0(E)$ with $n(c,d) = 1$, and we have $n(c,d) = 0$ for all other $d \in \mathrm{Crit}_0(E)$. The exponents α_1, α_2, β_1 and β_2 are uniquely determined by the relations (ii) and (iii).

To illustrate the procedure for determining $\partial(c)$ for a chord $c \in (K \times K) \setminus A$, we will consider two examples. The movement of the chord along the gradient flow is indicated by the orange arrows in the figure.

Example 2.24 We want to determine $\partial(c_1)$ with c_1 as in Figure 14. During its movement, the chord neither intersects the basepoint nor the framing and ends at m, so it is contractible. According to relation (i), we get

$$\partial(c_1) = 1 - \mu.$$

Example 2.25 In the second example we want to determine $\partial(c_2)$ with c_2 as in Figure 15. The movement of the chord is again indicated by orange arrows. As can be seen from the figure, the chord first intersects the framing at its endpoint, which results in a multiplication by μ from the right according to relation (ii), and then at its starting point (results in multiplication by μ^{-1} from the left). Then the endpoint of the

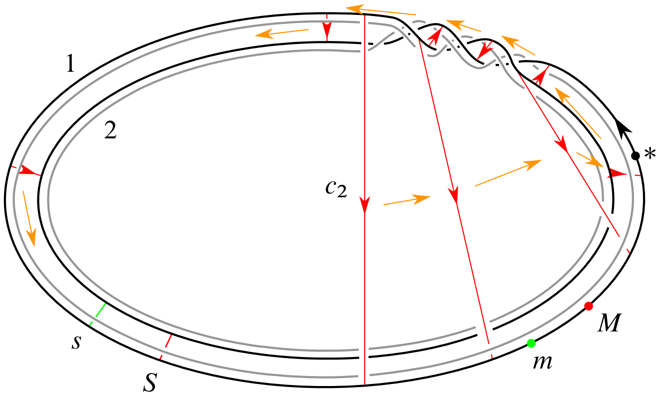

Figure 15: Movement of c_2 along the gradient flow.

chord crosses the basepoint in the direction of the orientation of the knot, which, according to relation (iii), yields a contribution of $\cdot\lambda^{-1}$. At the crossings of the knot, the chord first intersects the framing twice at its endpoint (yields $\cdot\mu^{-2}$), then twice at its starting point (yields $\mu^2\cdot$), and finally twice at its endpoint (yields $\cdot\mu^{-2}$). Then the chord runs without further intersections to the chord s^t of index 0. So, in total,

$$\partial(c_2) = \mu^2\mu^{-1}s^t\mu\lambda^{-1}\mu^{-2}\mu^{-2} = \mu s^t\lambda^{-1}\mu^{-3}$$

because λ and μ commute with each other, but not with s^t.

We now discuss the map δ. When determining $\partial(c)$ for a chord $c \in (K \times K) \setminus A$ it may happen that the chord intersects the knot K in its interior during the movement along the gradient flow. Let P be the set of all these intersections. According to relation (iv), at each $p \in P$ the chord is split into two chords, $c_{p,1}$ and $c_{p,2}$, which are multiplied by each other. Since the algebra C_0 is not commutative, the order of the factors must be chosen according to the orientation of c. Let $c_{p,1}$ be the first and $c_{p,2}$ be the second part in the direction of the orientation of c. Then we determine $\hat{D}(c_{p,1})$ and $\hat{D}(c_{p,2})$. Now we can define the map δ recursively:

$$\delta\colon (K \times K) \setminus A \to C_0, \quad c \mapsto \sum_{p \in P} \operatorname{sign}(c, p)\lambda^{\alpha_1(p)}\mu^{\beta_1(p)}\hat{D}(c_{p,1})\hat{D}(c_{p,2})\lambda^{\alpha_2(p)}\mu^{\beta_2(p)},$$

where

- $\alpha_1(p), \alpha_2(p), \beta_1(p), \beta_2(p) \in \mathbb{Z}$ are analogous to the above definition such that the intersections with the framing or the basepoint occurring during the movement of the chord c along $-\nabla E$ up to the point p are taken into account according to the relations (ii) and (iii), respectively, and
- $\operatorname{sign}(c, p) \in \{-1, 1\}$ is the sign in front of the product of the resulting chords obtained by applying relation (iv) to the chord c at the intersection point p.

Remark 2.26 When splitting c into the chords $c_{p,1}$ and $c_{p,2}$ the exact position of the framing relative to the two chords, as shown in relation (iv), must be taken into account. For this it may be necessary to

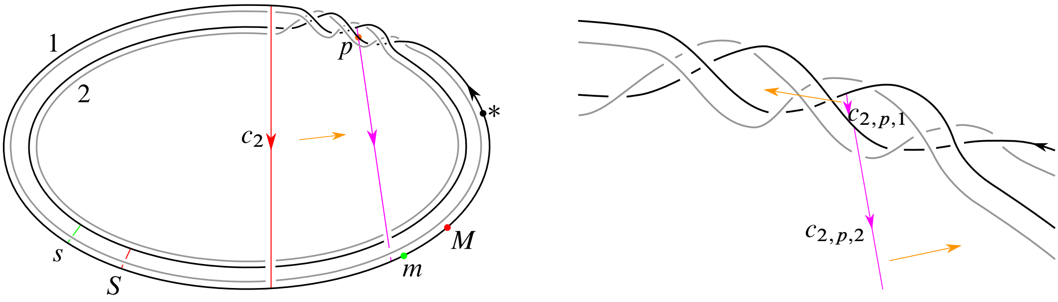

Figure 16: Determination of $\delta(c_2)$.

bend the framing a little bit in one direction. Then the framing is brought back to its original position, and one of the parts $c_{p,1}$ or $c_{p,2}$ is intersected and multiplied by μ or μ^{-1} from left or right according to relation (ii).

The map δ is well defined:

- When determining $\widehat{D}(c)$ for a chord $c \in (K \times K) \setminus A$ relation (iv) is applied only finitely many times according to Lemma 2.22. Thus, both the sum and the recursion depth are finite.
- The way in which the exceptional set A is constructed prevents the resulting chords from being in A when a chord is split according to relation (iv).

Example 2.27 We look again at c_2 from Figure 15 to determine $\delta(c_2)$. Together with the result from Example 2.25 we get $\widehat{D}(c_2)$. If c_2 is moved along the negative gradient, there is a single intersection with the knot in the interior of the chord at point p. There relation (iv) is applied and the chord is split into $c_{2,p,1}$ and $c_{2,p,2}$, as shown in Figure 16. The position of the framing relative to the chord $c_{2,p,2}$ results from the representation of relation (iv) in Figure 9. This figure also shows $\operatorname{sign}(c_2, p) = -1$. According to Example 2.25, we get $\alpha_1(p) = \alpha_2(p) = 0$ because the chord c_2 does not cross the basepoint before reaching the point p, and with relation (ii) we get $\beta_1(p) = -1$, $\beta_2(p) = 1$ because the chord intersects the framing once at its starting point and once at its endpoint. Thus, we get

$$\delta(c_2) = -\mu^{-1}\widehat{D}(c_{2,p,1})\widehat{D}(c_{2,p,2})\mu$$

and we have to determine $\widehat{D}(c_{2,p,1})$ and $\widehat{D}(c_{2,p,2})$:

The chord $c_{2,p,1}$ intersects the framing once at its starting point and twice at its endpoint, but not the basepoint or the knot in its interior, and then runs to the chord s^t. Thus, the application of relation (ii) results in

$$\widehat{D}(c_{2,p,1}) = \mu s^t \mu^{-2}.$$

For the chord $c_{2,p,2}$ we get, according to the relations (i), (ii) and (iii),

$$\partial(c_{2,p,2}) = \mu^{-1}(1-\mu)\lambda^{-1}\mu^{-1} = \lambda^{-1}\mu^{-2} - \lambda^{-1}\mu^{-1}$$

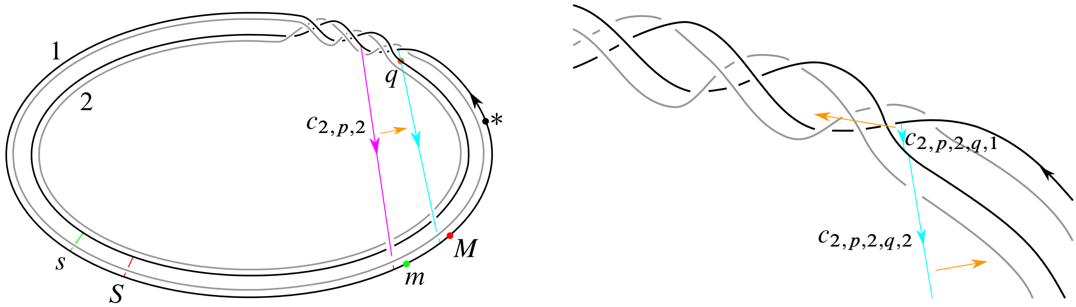

Figure 17: Determination of $\delta(c_{2,p,2})$.

and we get an intersection of the chord with the knot in its interior at the point q. There the chord is split into the parts $c_{2,p,2,q,1}$ and $c_{2,p,2,q,2}$ as shown in Figure 17. Since these two chords have no intersections with the knot in their interior during their movement along the gradient flow, we get

$$\begin{aligned}\delta(c_{2,p,2}) &= -\mu^{-1}\widehat{D}(c_{2,p,2,q,1})\widehat{D}(c_{2,p,2,q,2})\\ &= -\mu^{-1}(\mu\mu^2 s^s\mu^{-2})(\mu^2 s^t\mu^{-2}\mu^{-2}\lambda^{-1}) = -\mu^2 s^s s^t\lambda^{-1}\mu^{-4}.\end{aligned}$$

Now we can determine $\delta(c_2)$:

$$\begin{aligned}\delta(c_2) &= -\mu^{-1}\widehat{D}(c_{2,p,1})\widehat{D}(c_{2,p,2})\mu\\ &= -\mu^{-1}(\mu s^t\mu^{-2})(\lambda^{-1}\mu^{-2} - \lambda^{-1}\mu^{-1} - \mu^2 s^s s^t\lambda^{-1}\mu^{-4})\mu\\ &= -s^t\lambda^{-1}\mu^{-3} + s^t\lambda^{-1}\mu^{-2} + s^t s^s s^t\lambda^{-1}\mu^{-3}.\end{aligned}$$

Together with the result from Example 2.25, we get

$$\widehat{D}(c_2) = \partial(c_2) + \delta(c_2) = \mu s^t\lambda^{-1}\mu^{-3} - s^t\lambda^{-1}\mu^{-3} + s^t\lambda^{-1}\mu^{-2} + s^t s^s s^t\lambda^{-1}\mu^{-3}.$$

This example clearly shows that even for simple knots K determining $\widehat{D}(c)$ for a chord $c \in (K\times K)\setminus A$ can be laborious and has to be done very carefully.

To define the cord algebra of a knot K, we first choose two points $k_+, k_- \in W^u(k)$ for each critical point $k \in \mathrm{Crit}_1(E)$ as follows:

We move k a little bit in the direction of its unstable manifold. Choose a point k_+ near k such that none of the sets S, F, and B intersect the unstable manifold of k between k and k_+. Analogously choose k_- on the other side of k. Examples to illustrate the choice of k_+ and k_- are shown in Figure 18, on the left-hand side as chords on the knot and on the right-hand side as points in $K\times K$.

Now we define the linear map D on generators:

$$D\colon C_1 \to C_0, \quad k \mapsto \widehat{D}(k_+) - \widehat{D}(k_-).$$

Note that D is well defined up to a sign, since the roles of k_+ and k_- can be interchanged. But this is not a problem for the definition of the cord algebra as we will see in Remark 2.31.

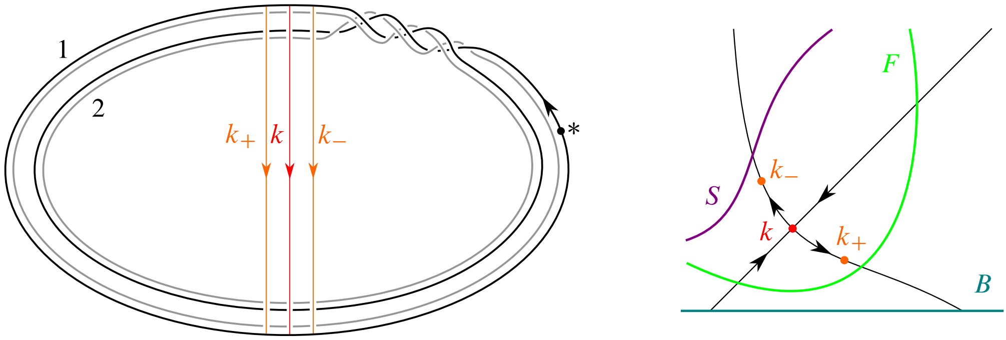

Figure 18: Choice of k_+ and k_-.

Example 2.28 We want to determine $D(k^s_{1,1})$ for the chord $k^s_{1,1}$ of the right-handed trefoil knot as in Figure 10. As can easily be seen, $(k^s_{1,1})_+$ can be chosen as the chord c_1 in Figure 14 and $(k^s_{1,1})_-$ can be chosen as the chord c_2 in Figure 15. With the above results we get

$$D(k^s_{1,1}) = \widehat{D}(c_1) - \widehat{D}(c_2) = 1 - \mu - \mu s^t \lambda^{-1} \mu^{-3} + s^t \lambda^{-1} \mu^{-3} - s^t \lambda^{-1} \mu^{-2} - s^t s^s s^t \lambda^{-1} \mu^{-3}$$

because the chord c_1 does not intersect the knot in its interior during its movement along the negative gradient.

Remark 2.29 On the diagonal of $K \times K$ there are exactly two critical points, m of index 0 and M of index 1 (see Remark 2.12). Both sides of the unstable manifold of M, which corresponds to the diagonal Δ in $K \times K$, end at m. According to relation (i), we have $m = 1 - \mu$. To determine $D(M)$, we select M_+ and M_- as shown in Figure 19. Since the set S does not intersect the diagonal, we get

$$\delta(M_+) = \delta(M_-) = 0.$$

But the diagonal intersects B and possibly F. When determining $\partial(M_\pm)$ the relations (ii) and (iii) must therefore be taken into account. Since the starting point and the endpoint of the chord M coincide, both intersect the basepoint and the framing. In the example of Figure 19 the following therefore holds:

$$\partial(M_+) = \lambda(1-\mu)\lambda^{-1} = 1 - \mu$$

and, depending on the exact course of the framing (ie depending on how relation (ii) is applied),

$$\partial(M_-) = \mu(1-\mu)\mu^{-1} = 1 - \mu \quad \text{or} \quad \partial(M_-) = \mu^{-1}(1-\mu)\mu = 1 - \mu.$$

The same result is obtained if the diagonal intersects the set F several times.

These considerations are independent of the concrete knot. Therefore, we get that in total the following holds for all knots:

$$D(M) = \widehat{D}(M_+) - \widehat{D}(M_-) = (1-\mu) - (1-\mu) = 0.$$

By an analogous consideration we get for a chord c, whose trajectory runs close enough to the diagonal and ends at m (see Figure 19); also here both start and endpoint of the chord intersect the basepoint or the framing if at all such an intersection takes place. Therefore, $\partial(c) = 1 - \mu$ follows here as well.

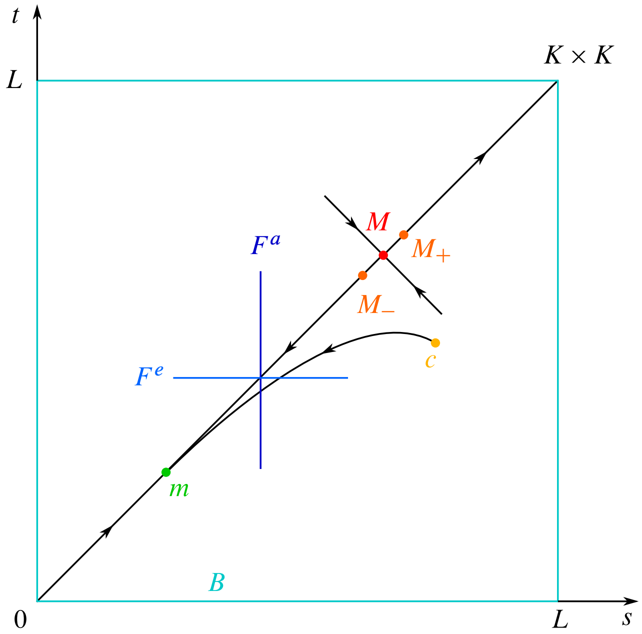

Figure 19: Determination of $D(M)$.

If the trajectory of a chord runs close enough to the diagonal in $K \times K$, the further intersections of this trajectory with B and F can be ignored.

Now we can define the cord algebra of a knot:

Definition 2.30 Let $K \subset \mathbb{R}^3$ be a generic oriented knot, equipped with a Seifert framing and a basepoint. The *cord algebra* of K is

$$\mathrm{Cord}(K) := C_0(K)/I_K,$$

where $I_K = \langle D(C_1(K))\rangle \subset C_0(K)$ is the two-sided ideal generated by the image of $C_1(K)$ under the map D.

Remark 2.31 As mentioned before, D is well defined up to a sign. But as can be seen in Definition 2.30, we get the same result in the quotient, regardless of which chord we designate with k_+ and k_-. Therefore, the cord algebra of a knot is well defined.

2.4 Change of framing

In the definition of the cord algebra the knot K is equipped with the Seifert framing. However, the determination of $D(C_1(K))$ is more convenient if a framing ν is used instead of the Seifert framing such that the associated set K' is a vertically shifted copy of K. For this purpose the strands of the knot are arranged one above the other and each strand is drawn over a slightly larger ellipse than the one below as described above. Then K' as a copy of K can be moved vertically upwards by an $\varepsilon > 0$ which is small enough such that the strip spanned by K and K' has no self-intersections. For each $s \in S^1$ then $\nu(s)$ is the

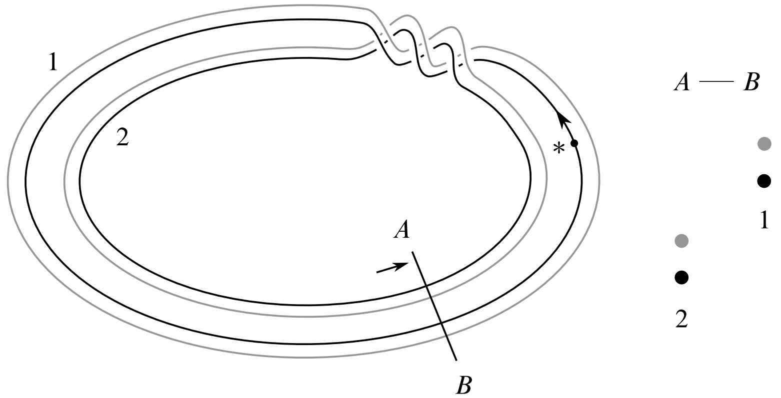

Figure 20: Blackboard framing.

unit normal vector pointing from the origin of the normal plane $N(s)$ in the direction of the intersection of $N(s)$ with K'. In the diagram, however, the framing is drawn slightly outside the corresponding strand in order to clearly distinguish it from the strand. Such a framing is called a *blackboard framing*. In Figure 20 the right-handed trefoil knot with the blackboard framing is shown on the left side; on the right side a section through the knot from A to B with view direction in the direction of the arrow. This kind of representation ensures that the chords corresponding to critical points in $K \times K$ do not intersect the framing.

Using this framing we can now determine $D(C_1(K))$ as described above. However, the following adjustments will have to be made in order to get the cord algebra with respect to the Seifert framing:

To change the framing, n additional windings of the framing around the knot K are added. If we look in the direction of the orientation of the knot, these additional windings run in a mathematically positive or negative direction (ie counterclockwise or clockwise, respectively) around the knot and are taken into account with a sign $+1$ or -1, respectively. So we have $n \in \mathbb{Z}$. If we have determined $D(C_1(K))$ for a framing K'_1, we get $D(C_1(K))$ for a framing K'_2 by applying various transformations. The first transformation is $\lambda \mapsto \lambda\mu^n$, where n is the number of additional windings, with sign, of K'_2 compared to K'_1 (see [2, after Remark 2.3]). This results from the following consideration: the additional windings will first be added near the basepoint; thus, when crossing the basepoint each chord gets the factor $\mu^{\pm n}$ according to relation (ii) in addition to the factor $\lambda^{\pm 1}$ according to relation (iii).

These additional windings change the linking number of the knot and its framing and we get

$$\mathrm{lk}(K, K'_2) = \mathrm{lk}(K, K'_1) - n$$

since the addition of a winding in the positive direction around the knot creates two additional left-handed crossings, and thus decreases the linking number by 1. Let $K'_1 = K'_b$ be the blackboard framing of the knot K and $K'_2 = K'_S$ be the Seifert framing. Then we have $n = \mathrm{lk}(K, K'_b)$ since $\mathrm{lk}(K, K'_S) = 0$. Thus, the first transformation is given by

$$\lambda \mapsto \lambda\mu^{\mathrm{lk}(K, K'_b)}.$$

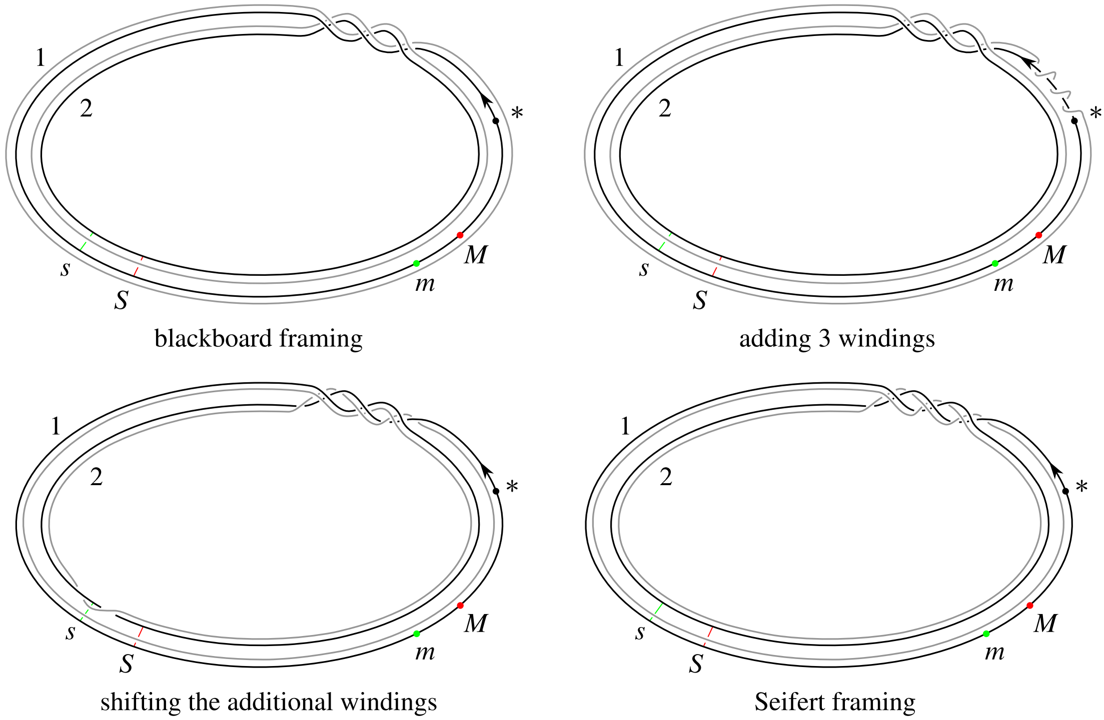

Figure 21: Changing from the blackboard framing to the Seifert framing in the example of the right-handed trefoil knot.

We want to achieve a Seifert framing which is obtained by the Seifert algorithm (for example see Figure 21, bottom right). In the final step of the Seifert algorithm, the Seifert circles are connected by twisted bands. So we shift the added windings along the knot such that these windings match the twisted bands. It can happen that one of these windings intersects a generator of $C_0(K)$ at its start or endpoint. These intersections must now be taken into account in further transformations according to relation (ii), as explained in the following example.

Example 2.32 For the right-handed trefoil knot K we have $\operatorname{lk}(K, K'_b) = 3$. Thus, the first transformation is

$$\lambda \mapsto \lambda\mu^3.$$

If the three windings of the framing added in the neighborhood of the basepoint are moved to the correct places, ie to the crossings of the knot, one of these windings intersects the chord s during the movement, ie the chords s^s and s^t if the orientation is taken into account; see Figure 21. According to relation (ii), the further transformations are

$$s^s \mapsto \mu^{-1}s^s, \quad s^t \mapsto s^t\mu.$$

Remark 2.33 The blackboard framing may have to be changed by a small perturbation in such a way that the assumptions of Remark A.1 for a generic framing are satisfied.

2.5 Examples: unknot and right-handed trefoil

Example 2.34 First, we want to determine the cord algebra of the unknot U. We draw the unknot with the blackboard framing and choose a basepoint as shown in Figure 22. Let $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ be the ring as described above. The critical points of index 0 and 1, and thus C_0 and C_1, can be easily determined from the figure:

$$C_0 = \langle m \rangle_R \overset{\text{rel. (i)}}{=} \langle 1 - \mu \rangle_R = R, \quad C_1 = \langle M, k^s, k^t \rangle_{\mathbb{Z}}.$$

Since the two nontrivial chords of index 1, k^s and k^t, when moving along the gradient flow, intersect neither the framing nor the knot in their interior, relations (ii) and (iv) need not be considered. The image of C_1 under D is therefore very easy to determine:

$$\begin{aligned} D(M) &= 0 \quad \text{(see Remark 2.29)},\\ D(k^s) &= 1 - \mu - (1-\mu)\lambda^{-1},\\ D(k^t) &= -(1-\mu) + \lambda(1-\mu), \end{aligned}$$

No transformation is necessary at the transition to the Seifert framing. Thus, in the quotient C_0/I, where $I = \langle D(C_1) \rangle$ is the two-sided ideal generated by C_1 under the map D, we get the only relation

$$(\lambda - 1)(\mu - 1) = 0$$

and hence

$$\operatorname{Cord}(U) = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]/((\lambda - 1)(\mu - 1)).$$

Example 2.35 To determine the cord algebra of the right-handed trefoil (also called Torus$(3,2)$), the knot is equipped with the blackboard framing and a basepoint is chosen; see Figure 23. The critical points of index 0 and 1 can be determined from the figure. As before, $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$. We get

$$C_0 = \langle m, s^s, s^t \rangle_R, \quad C_1 = \langle M, S^s, S^t, k^s_{1,1}, k^t_{1,1}, k^s_{1,2}, k^t_{1,2}, k^s_{2,1}, k^t_{2,1}, k^s_{2,2}, k^t_{2,2} \rangle_{\mathbb{Z}}.$$

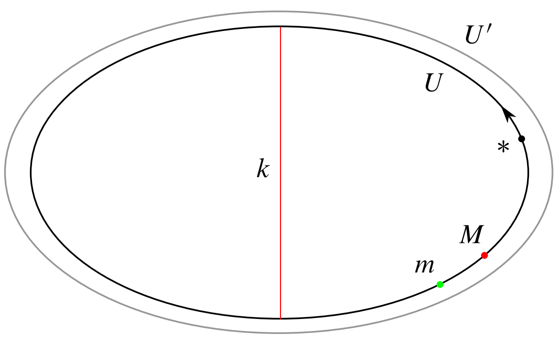

Figure 22: The unknot equipped with the blackboard framing.

Now we can get the image of C_1 under the map D by applying D to the generators of C_1:

$$\begin{aligned}
D(M) &= 0 \quad \text{(see Remark 2.29)},\\
D(S^s) &= -s^s + \mu s^t \mu^{-1}\mu^{-1}\lambda^{-1} = -s^s + \mu s^t \lambda^{-1}\mu^{-2},\\
D(S^t) &= -s^t + \lambda\mu\mu s^s \mu^{-1} = -s^t + \lambda\mu^2 s^s \mu^{-1},\\
D(k^s_{1,1}) &= (1-\mu) - \big(\mu\mu s^t \mu^{-1}\mu^{-1}\lambda^{-1} - \mu s^t\mu^{-1}\mu^{-1}((1-\mu)\lambda^{-1} - \mu\mu s^s\mu^{-1}\mu^{-1}\mu\mu s^t\mu^{-1}\mu^{-1}\lambda^{-1})\big)\\
&= 1-\mu-\mu^2 s^t\lambda^{-1}\mu^{-2} + \mu s^t\lambda^{-1}\mu^{-2} - \mu s^t\lambda^{-1}\mu^{-1} - \mu s^t s^s s^t\lambda^{-1}\mu^{-2},\\
D(k^t_{1,1}) &= -(1-\mu) + \lambda\mu\mu s^s\mu^{-1}\mu^{-1} + (\lambda(1-\mu) + \lambda\mu\mu s^s\mu^{-1}\mu^{-1}\mu s^t\mu^{-1}\mu^{-1})\mu s^s\mu^{-1}\\
&= -1+\mu+\lambda\mu^2 s^s\mu^{-2} + \lambda\mu s^s\mu^{-1} - \lambda\mu^2 s^s\mu^{-1} + \lambda\mu^2 s^s\mu^{-1}s^t\mu^{-1}s^s\mu^{-1},\\
D(k^s_{1,2}) &= -\mu s^s + (1-\mu) + (\mu\mu s^t\mu^{-1}\mu^{-1} + (1-\mu)\mu s^t\mu^{-1}\mu^{-1})\mu s^s\mu^{-1} = -\mu s^s + 1 - \mu + \mu s^t\mu^{-1}s^s\mu^{-1},\\
D(k^t_{1,2}) &= s^t\mu^{-1} - \big((1-\mu) - \mu s^t\mu^{-1}\mu^{-1}(\mu\mu s^s\mu^{-1}\mu^{-1} - \mu\mu s^s\mu^{-1}\mu^{-1}(1-\mu))\big) = s^t\mu^{-1} - 1 + \mu + \mu s^t s^s\mu^{-1},\\
D(k^s_{2,1}) &= \mu s^s - \big((1-\mu)\lambda^{-1} - \mu\mu s^s\mu^{-1}\mu^{-1}\mu\mu s^t\mu^{-1}\mu^{-1}\lambda^{-1}\\
&\qquad - \mu s^s\mu^{-1}\big(\mu\mu s^t\mu^{-1}\mu^{-1}\lambda^{-1} - \mu s^t\mu^{-1}\mu^{-1}((1-\mu)\lambda^{-1} - \mu\mu s^s\mu^{-1}\mu^{-1}\mu\mu s^t\mu^{-1}\mu^{-1}\lambda^{-1})\big)\big)\\
&= \mu s^s - \lambda^{-1} + \lambda^{-1}\mu + \mu^2 s^s s^t\lambda^{-1}\mu^{-2} + \mu s^s\mu s^t\lambda^{-1}\mu^{-2}\\
&\qquad - \mu s^s s^t\lambda^{-1}\mu^{-2} + \mu s^s s^t\lambda^{-1}\mu^{-1} + \mu s^s s^t s^s s^t\lambda^{-1}\mu^{-2},\\
D(k^t_{2,1}) &= -s^t\mu^{-1} + \lambda(1-\mu) + \lambda\mu\mu s^s\mu^{-1}\mu^{-1}\mu s^t\mu^{-1}\mu^{-1}\\
&\qquad + \big(\lambda\mu\mu s^s\mu^{-1}\mu^{-1} + (\lambda(1-\mu) + \lambda\mu\mu s^s\mu^{-1}\mu^{-1}\mu s^t\mu^{-1}\mu^{-1})\mu s^s\mu^{-1}\big)s^t\mu^{-1}\\
&= -s^t\mu^{-1} + \lambda - \lambda\mu + \lambda\mu^2 s^s\mu^{-1}s^t\mu^{-2} + \lambda\mu^2 s^s\mu^{-2}s^t\mu^{-1} + \lambda\mu s^s\mu^{-1}s^t\mu^{-1}\\
&\qquad - \lambda\mu^2 s^s\mu^{-1}s^t\mu^{-1} + \lambda\mu^2 s^s\mu^{-1}s^t\mu^{-1}s^s\mu^{-1}s^t\mu^{-1},\\
D(k^s_{2,2}) &= (1-\mu) - \big(\mu\mu s^s\mu^{-1}\mu^{-1} - \mu\mu s^s\mu^{-1}\mu^{-1}(1-\mu)\\
&\qquad - \mu s^s\mu^{-1}\big((1-\mu) - \mu s^t\mu^{-1}\mu^{-1}(\mu\mu s^s\mu^{-1}\mu^{-1} - \mu\mu s^s\mu^{-1}\mu^{-1}(1-\mu))\big)\big)\\
&= 1-\mu-\mu^2 s^s\mu^{-1} + \mu s^s\mu^{-1} - \mu s^s - \mu s^s s^t s^s\mu^{-1},\\
D(k^t_{2,2}) &= -(1-\mu) + \mu\mu s^t\mu^{-1}\mu^{-1} + (1-\mu)\mu s^t\mu^{-1}\mu^{-1}\\
&\qquad + \big((1-\mu) + (\mu\mu s^t\mu^{-1}\mu^{-1} + (1-\mu)\mu s^t\mu^{-1}\mu^{-1})\mu s^s\mu^{-1}\big)s^t\mu^{-1}\\
&= -1+\mu+\mu s^t\mu^{-2} + s^t\mu^{-1} - \mu s^t\mu^{-1} + \mu s^t\mu^{-1}s^s\mu^{-1}s^t\mu^{-1}.
\end{aligned}$$

According to Example 2.32, the transformations to change from the blackboard framing to the Seifert framing are

$$\lambda \mapsto \lambda\mu^3, \quad s^s \mapsto \mu^{-1}s^s, \quad s^t \mapsto s^t\mu.$$

If these transformations are applied to the above results, the formulas for $D(S^s)$, $D(S^t)$, $D(k^s_{1,2})$ and $D(k^t_{1,2})$ in the quotient $C_0/\langle D(C_1)\rangle$ result in the equations

$$-\mu^{-1}s^s + \mu s^t\lambda^{-1}\mu^{-4} = 0, \tag{2-1}$$

$$-s^t\mu + \lambda\mu^4 s^s\mu^{-1} = 0, \tag{2-2}$$

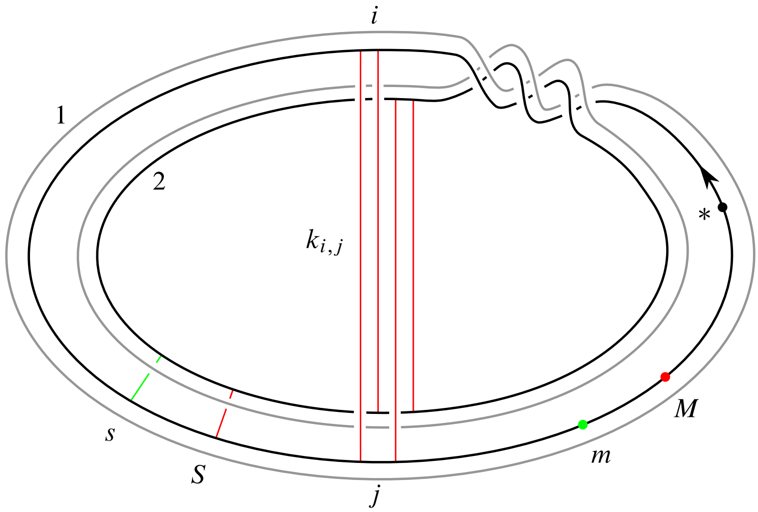

Figure 23: The right-handed trefoil equipped with the blackboard framing.

$$-s^s + 1 - \mu + \mu s^t \mu^{-1} s^s \mu^{-1} = 0, \tag{2-3}$$

$$s^t - 1 + \mu + \mu s^t s^s \mu^{-1} = 0. \tag{2-4}$$

Therefore,

$$(2\text{-}1) \iff s^t = \mu^{-2} s^s \lambda \mu^4 \tag{2-5}$$

$$(2\text{-}2) \iff s^t = \lambda \mu^4 s^s \mu^{-2} \tag{2-6}$$

In the quotient only one generator exists because s^t can be expressed by a term with s^s. For simplicity, in the following we denote s^s by s. If we now eliminate s^t from (2-5) and (2-6), we get

$$s\lambda\mu^6 = \lambda\mu^6 s.$$

We put (2-6) into (2-3) and (2-4) and get the relations

$$1 - \mu - s + \lambda\mu^5 s\mu^{-3} s\mu^{-1} = 0, \quad -1 + \mu + \lambda\mu^4 s\mu^{-2} + \lambda\mu^5 s\mu^{-2} s\mu^{-1} = 0.$$

If the transformations are applied to the remaining results from $D(C_1)$, the resulting equations do not yield new relations. All in all,

$$\mathrm{Cord}(\mathrm{Torus}(3,2)) = \langle s\rangle_R / (s\lambda\mu^6 - \lambda\mu^6 s, 1 - \mu - s + \lambda\mu^5 s\mu^{-3} s\mu^{-1}, -1 + \mu + \lambda\mu^4 s\mu^{-2} + \lambda\mu^5 s\mu^{-2} s\mu^{-1}).$$

3 The cord algebra as a knot invariant

The cord algebra in its original definition, as explained at the beginning of Section 2, is a knot invariant [2, Remark 2.22]. Now we will prove that the cord algebra in our definition using Morse theory is also a knot invariant. So we have to show that for generic knots K_0 and K_1 for which there exists a smooth isotopy $(K_r)_{r\in[0,1]}$ of knots,

$$\mathrm{Cord}(K_0)\cong\mathrm{Cord}(K_1).$$

So let K_0 and K_1 be two knots that are connected by a smooth isotopy $(K_r)_{r\in[0,1]}$ of knots. For all $r\in[0,1]$ let $\gamma_r\colon[0,L_r]\to\mathbb{R}^3$ be an arclength parametrization of K_r such that the map $r\mapsto\gamma_r(0)$ is continuous. Let $\gamma_r(0)$ be the basepoint of K_r. Thus, the set $B=(S^1\times\{0\})\cup(\{0\}\times S^1)\subset T^2\cong K_r\times K_r$ is the same for all K_r. Denote by $S_r\subset K_r\times K_r$ and $F_r\subset K_r\times K_r$ the sets S and F, respectively, with respect to the knot K_r. Likewise, let $E_r\colon K_r\times K_r\to\mathbb{R}$, $(x,y)\mapsto\frac{1}{2}|x-y|^2$, be the energy function with respect to the knot K_r, let $X_r\colon K_r\times K_r\to\mathbb{R}^2$, $(s,t)\mapsto-\nabla E_r(s,t)$, be the associated gradient vector field on $K_r\times K_r$, let φ_r^s be the flow of X_r, and let $W_r^s(k_r)$ and $W_r^u(k_r)$ be the stable and unstable manifold, respectively, of a critical point k_r of the function E_r. Furthermore, let $\widehat{D}_r\colon(K_r\times K_r)\setminus A_r\to C_0(K_r)$ and $D_r\colon C_1(K_r)\to C_0(K_r)$ be the maps belonging to K_r, where $A_r\subset K_r\times K_r$ is the exceptional set belonging to K_r.

3.1 Preparations for the proof of Theorem 1.1

First, we need that only finitely many knots in this isotopy are nongeneric. These finitely many knots each violate exactly one of the properties shown in Lemmas 2.17, 2.20 and 2.23 for generic knots or at these knots pairs of critical points appear or disappear. This statement is formulated in the following lemma. This lemma can be proven by extending the arguments used in the lemmata mentioned above to 1–parameter families of knots and energy functions.

Lemma 3.1 *For a generic smooth isotopy $(K_r)_{r\in[0,1]}$ of knots, where K_0 and K_1 are generic, the following holds:*

(i) *K_r is generic for all $r\in[0,1]\setminus\{r_1,\dots,r_n\}$ for an $n\in\mathbb{N}$ and for $\{r_1,\dots,r_n\}\subset[0,1]$.*

(ii) *For $i\in\{1,\dots,n\}$ the knot K_{r_i} is generic except for the occurrence of exactly one of the following phenomena:*

(1) *$W_{r_i}^u(k)\not\pitchfork B$ for a critical point k of index 1, and $W_{r_i}^u(k)$ is tangent to B at most of order 1.*

(2) *$W_{r_i}^u(k)\not\pitchfork S_{r_i}$ for a critical point k of index 1, and $W_{r_i}^u(k)$ is tangent to S_{r_i} at most of order 1.*

(3) *$W_{r_i}^u(k)\not\pitchfork F_{r_i}$ for a critical point k of index 1, and $W_{r_i}^u(k)$ is tangent to F_{r_i} at most of order 1.*

(4) *$W_{r_i}^u(k)\cap\partial S_{r_i}\neq\varnothing$, and thus $W_{r_i}^u(k)\cap\partial F_{r_i}\neq\varnothing$ according to Lemma 2.14, for a critical point k of index 1.*

(5) *$W_{r_i}^u(k)\cap S_{2,r_i}\neq\varnothing$ for a critical point k of index 1, where $S_{2,r_i}\subset S_{r_i}$ is the set of chords that intersect the knot K_{r_i} twice in their interior.*

(6) *$W_{r_i}^u(k)\cap B\cap S_{r_i}\neq\varnothing$ for a critical point k of index 1.*

(7) *$W_{r_i}^u(k)\cap F_{r_i}\cap S_{r_i}\neq\varnothing$ for a critical point k of index 1.*

(8) *$k\in B$ for a critical point k of index 0 or 1.*

(9) *$k\in S_{r_i}$ for a critical point k of index 0 or 1.*

(10) *$k\in F_{r_i}$ for a critical point k of index 0 or 1.*

(11) *There exists a trajectory between two critical points of index* 1 *along the vector field* $-\nabla E_{r_i}$.

(12) *One of the cases* (1) *to* (10) *holds for a chord which is generated by the application of relation* (iv) (*where in the statements* (1) *to* (7) *the unstable manifolds are to be replaced by trajectories along the vector field* $-\nabla E_{r_i}$, *starting at this chord*).

(13) *A chord which is generated by the application of relation* (iv) *runs along the vector field* $-\nabla E_{r_i}$ *to a critical point of index* 1.

(14) *Creation/cancellation of a pair of critical points of index* 0 *and* 1 *or of index* 1 *and* 2.

Remark 3.2 In this 1–parameter family of gradient vector fields only degenerations of birth-death type occur (see Cieliebak and Eliashberg [3, Lemma 9.7]). Therefore, case (ii)(14) is the only one dealing with degeneracies of the family of vector fields.

To show that the cord algebra is a knot invariant, we first need the following lemma:

Lemma 3.3 *For a generic knot* K *the following holds*:

Let $q \in (K \times K) \setminus A$, *where* A *is the exceptional set defined at the beginning of Section 2.3. Then there exists an open neighborhood* $V \subset K \times K$ *of* q *with* $\hat{D}(x) = \hat{D}(q)$ *for all* $x \in V$.

Proof Since K is generic, all properties from Lemmas 2.17, 2.20 and 2.23 are satisfied.

There exist only finitely many critical points of index 0. Denote these by $g_1, \dots, g_n$ for an $n \in \mathbb{N}$.

Then there exist $\varepsilon_1, \dots, \varepsilon_n > 0$ such that for all $i = 1, \dots, n$,

$$\lim_{s \to \infty} \varphi^s(x) = g_i \quad \text{for all } x \in \mathrm{B}_{2\varepsilon_i}(g_i), \tag{3-1}$$

$$\mathrm{B}_{2\varepsilon_i}(g_i) \cap (S \cup B \cup F) = \varnothing. \tag{3-2}$$

We can achieve (3-1) since $\dim W^s(g_i) = 2$, and (3-2) can be achieved since $g_i \notin (S \cup B \cup F)$ (according to Lemma 2.17) and this is an open condition.

Since $\hat{D}(q) \in C_0(K)$ according to the assumption, we have $q \in W^s(g_{i_q})$ for an $i_q \in \{1, \dots, n\}$. Therefore, a $T_q \geq 0$ exists with

$$\varphi^{T_q}(q) \in \mathrm{B}_{\varepsilon_{i_q}}(g_{i_q}).$$

The solution of a differential equation depends continuously on the initial conditions. So,

$$\forall T < \infty \ \forall \varepsilon > 0 \ \exists \delta(T, \varepsilon) > 0 \ \forall x \in K \times K \ \forall s \in [0, T] \quad |x - q| < \delta \implies |\varphi^s(x) - \varphi^s(q)| < \varepsilon. \tag{3-3}$$

Choose $T = T_q$ and $\varepsilon = \varepsilon_{i_q}$. Then, according to the statement (3-3), there exists a $\delta_q > 0$ such that, for all $x \in \mathrm{B}_{\delta_q}(q)$,

$$\varphi^{T_q}(x) \in \mathrm{B}_{2\varepsilon_{i_q}}(g_{i_q}). \tag{3-4}$$

So far, we've shown that each $x \in \mathrm{B}_{\delta_q}(q)$ flows along the vector field $X = -\nabla E$ to the same critical point of index 0 as q, namely to g_{i_q}.

To prove the lemma, we have to consider the four relations that we need to determine $D(q)$:

Relation (i): Due to the properties (3-1) and (3-4) relation (i) does not lead to different results when determining $\widehat{D}(x)$ for all $x \in \mathrm{B}_{\delta_q}(q)$.

Relation (ii): Let $F_q := \{x \in F : \exists T > 0 : \varphi^T(q) = x\}$. Since K is generic, we have $\varphi^s(q) \pitchfork F$ and $|F_q| < \infty$. So let $F_q = \{f_1, \dots, f_{m_F}\}$ for an $m_F \in \mathbb{N}_0$. For $j = 1, \dots, m_F$ let T_{f_j} be the time for which $\varphi^{T_{f_j}}(q) = f_j$. Because of $\varphi^s(q) \pitchfork F$, there exists an $\varepsilon_F > 0$ with the following properties:

(i) For all $j = 1, \dots, m_F$ and all $x \in \mathrm{B}_{\delta_q}(q)$ there exists at most one $T_{j,x} > 0$ such that

$$\varphi^{T_{j,x}}(x) \in \mathrm{B}_{\varepsilon_F}(f_j) \cap F, \tag{3-5}$$

ie the trajectory starting at x intersects F in $\mathrm{B}_{\varepsilon_F}(f_j)$ at most once.

(ii) For all $x \notin \left(\bigcup_{j=1}^{m_F} \mathrm{B}_{\varepsilon_F}(f_j)\right) \cap F$,

$$\min_{s \in \mathbb{R}} |\varphi^s(q) - x| < \varepsilon_F \implies x \notin F, \tag{3-6}$$

ie outside of $\mathrm{B}_{\varepsilon_F}(f_j)$, F is no closer than ε_F to the trajectory starting from q.

For all $j = 1, \dots, m_F$ there exists, according to statement (3-3) with $T = T_{f_j}$ and $\varepsilon = \varepsilon_F$, a $\delta_{f_j} > 0$ such that for all $x \in \mathrm{B}_{\delta_{f_j}}(q)$,

$$\varphi^{T_{f_j}}(x) \in \mathrm{B}_{\varepsilon_F}(f_j).$$

So after reducing δ_{f_j} if necessary, for all $x \in \mathrm{B}_{\delta_{f_j}}(q)$,

$$\varphi^s(x) \in \mathrm{B}_{\varepsilon_F}(f_j) \cap F \quad \text{for an } s > 0.$$

Let $\delta_F := \min_j \delta_{f_j}$.

Relation (iii) can be treated analogously to relation (ii). With a corresponding notation (B instead of F and b instead of f) we get $\delta_B := \min_j \delta_{b_j}$.

Relation (iv): Let $S_q := \{x \in S : \varphi^s(q) = x \text{ for an } s > 0\} \subset K \times K$. According to Lemma 2.22, there exist only finitely many intersections of $\varphi^s(q)$ and S, ie $|S_q| < \infty$. So let $S_q = \{q_1, \dots, q_{m_S}\}$ for an $m_S \in \mathbb{N}_0$. For all $q_j \in S_q$ let $T_{q_j} > 0$ be the time for which $\varphi^{T_{q_j}}(q) = q_j$. Let $S_2 \subset S$ be the (according to Lemma 2.11(ii) finite) set of self-intersections of S, ie the set of chords that intersect K twice in their interior.

Then there exists an $\varepsilon_S > 0$ such that for all $j = 1, \dots, m_S$,

$$\mathrm{B}_{\varepsilon_S}(q_j) \cap \partial S = \varnothing, \tag{3-7}$$

$$\mathrm{B}_{\varepsilon_S}(q_j) \cap S_2 = \varnothing, \tag{3-8}$$

$$\text{for all } x \in \mathrm{B}_{\delta_q}(q), \text{ there exists at most one } T_{j,x} > 0 \text{ such that } \varphi^{T_{j,x}}(x) \in \mathrm{B}_{\varepsilon_S}(q_j) \cap S, \tag{3-9}$$

$$\text{for all } x \notin \left(\bigcup_{j=1}^{m_S} \mathrm{B}_{\varepsilon_S}(q_j)\right) \cap S, \text{ we have } \min_{s \in \mathbb{R}} |\varphi^s(q) - x| < \varepsilon_S \implies x \notin S. \tag{3-10}$$

Such an ε_S exists since

(i) $S_q \cap \partial S = \varnothing$,

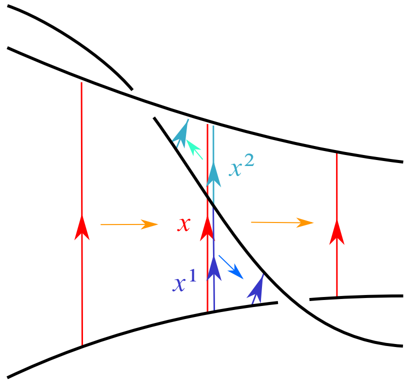

Figure 24: Splitting of a chord according to relation (iv).

(ii) $S_q \cap S_2 = \varnothing$,

(iii) $\varphi^s(q) \pitchfork S$,

(iv) $\varphi^s(q) \cap S = S_q$ and $\varphi^s(q) \pitchfork S$.

and (i)–(iii) are open conditions.

Therefore, the following holds for $j = 1, \dots, m_S$: according to statement (3-3) with $T = T_{q_j}$ and $\varepsilon = \varepsilon_S$, there exists a $\delta_{q_j} > 0$ such that, for all $x \in \mathrm{B}_{\delta_{q_j}}(q)$,

$$\varphi^{T_{q_j}}(x) \in \mathrm{B}_{\varepsilon_S}(q_j).$$

Let $\delta_S := \min_j \delta_{q_j}$.

Every $x \in S \setminus S_2$ splits according to relation (iv) in two chords x^1 and x^2 — see Figure 24 — and we can see that

- x^1 has the same starting point as x,
- x^2 has the same endpoint as x,
- the endpoint of x^1 is the starting point of x^2.

Now consider the maps

$$\Psi_1 \colon S \setminus S_2 \to K \times K, \quad x \mapsto x^1,$$
$$\Psi_2 \colon S \setminus S_2 \to K \times K, \quad x \mapsto x^2.$$

For $j = 1, \dots, m_S$, let

$$\Psi_{q_j^1} := \Psi_1(\mathrm{B}_{\varepsilon_S}(q_j) \cap S), \quad \Psi_{q_j^2} := \Psi_2(\mathrm{B}_{\varepsilon_S}(q_j) \cap S),$$

which are well defined since $\mathrm{B}_{\varepsilon_S}(q_j) \cap S_2 = \varnothing$. In addition, they are connected sets, since the sets $\mathrm{B}_{\varepsilon_S}(q_j) \cap S$ are connected, the above observation concerning the start and endpoints of x^1 and x^2 holds, and K is smooth. See also Figure 25.

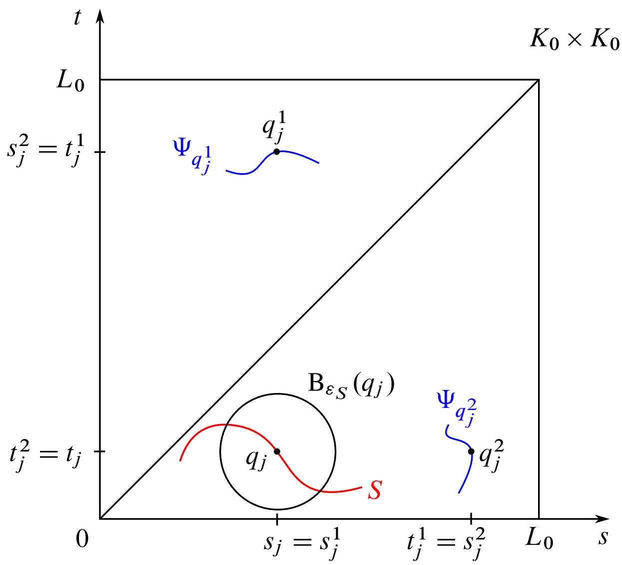

Figure 25: Splitting of chords according to relation (iv).

Now consider the sets $\Psi_{q_j^1}$ and $\Psi_{q_j^2}$ under the gradient flow analogous to the consideration of $B_{\delta_q}(q)$ under the gradient flow where $q_j^1 = \Psi_1(q_j)$ and $q_j^2 = \Psi_2(q_j)$, respectively, take the role of q. Again, the relations (i) to (iv) must be taken into account. Thus, for $i = 1, 2$ and $j = 1, \dots, m_S$ we get, analogous to the above considerations,

- $\delta_{q_j^i,F}$ such that for all $x \in B_{\delta_{q_j^i,F}}(q_j^i) \cap \Psi_{q_j^i}$, we have that $\varphi^s(x)$ and $\varphi^s(q_j^i)$ intersect the set F for $s \geq 0$ so that relation (ii) produces the same result with respect to $\mu^{\pm 1}$;
- $\delta_{q_j^i,B}$ such that for all $x \in B_{\delta_{q_j^i,B}}(q_j^i) \cap \Psi_{q_j^i}$, we have that $\varphi^s(x)$ and $\varphi^s(q_j^i)$ intersect the set B for $s \geq 0$ so that relation (iii) produces the same result with respect to $\lambda^{\pm 1}$;
- $\delta_{q^i_{j,k^i}}$ for $k^i = 1, \dots, m_S^i$, with $m_S^i \in \mathbb{N}$, where q^i_{j,k^i} denote the intersections of $\varphi^s(q_j^i)$, for $s \geq 0$, with S, such that for all $x \in B_{\delta_{q^i_{j,k^i}}}(q_j^i) \cap \Psi_{q_j^i}$, $\varphi^s(x)$ intersects S in an ε_S^i–neighborhood of q^i_{j,k^i} exactly once, but not within an ε_S^i–tube around $\varphi^s(q_j^i)$ outside these neighborhoods.

According to Lemma 2.22, this process ends after finitely many steps.

For $i = 1, 2$ let $\hat{\delta}_{q_j^i} := \min\{\delta_{q_j^i,F}, \delta_{q_j^i,B}, \delta_{q_j^i,1}, \dots, \delta_{q_j^i,m_S^i}\}$. Let $\varepsilon_{q_j^i} > 0$ be such that

$$B_{\varepsilon_{q_j^i}}(q_j) \cap S \subset \Psi_i^{-1}\big(B_{\hat{\delta}_{q_j^i}}(\Psi_i(q_j)) \cap \Psi_{q_j^i}\big).$$

We next prove that Ψ_1 and Ψ_2 are injective. Suppose Ψ_1 is not injective. Then there exist $x, y \in S \setminus S_2$ with $x \neq y$ and $\Psi_1(x) = \Psi_1(y)$, ie $x^1 = y^1$. Thus, x and y have the same starting point and a common intersection point with the knot in their interior. But since $x \neq y$, there are only the two possible positions shown in Figure 26. It follows that $x \in S_2$ or $y \in S_2$, but this contradicts the definition of Ψ_1. Similarly, the injectivity of Ψ_2 can be shown.

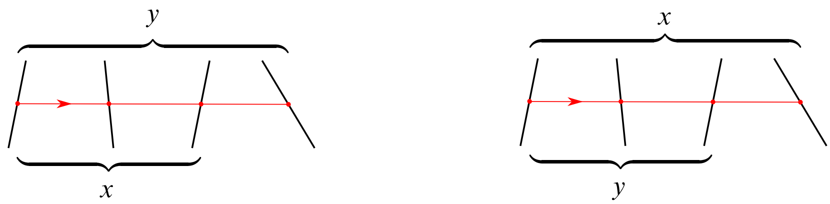

Figure 26: Possible positions of the chords x and y.

We get $\Psi_i^{-1}\big(\mathrm{B}_{\hat{\delta}_{q_j^i}}(\Psi_i(q_j)) \cap \Psi_{q_j^i}\big) \subset \mathrm{B}_{\varepsilon_S}(q_j)$ by the definition of $\Psi_{q_j^i}$ and the injectivity of Ψ_1 and Ψ_2, and thus $\mathrm{B}_{\varepsilon_{q_j^i}}(q_j) \cap S \subset \mathrm{B}_{\varepsilon_S}(q_j)$.

According to statement (3-3) with $T = T_{q_j}$ and $\varepsilon = \varepsilon_{q_j^i}$, there exists a $\delta_{q_j^i} > 0$ such that for all $x \in \mathrm{B}_{\delta_{q_j^i}}(q)$,

$$\varphi^{T_{q_j}}(x) \in \mathrm{B}_{\varepsilon_{q_j^i}}(q_j).$$

The same procedure is used for all intersections q^i_{j,k^i}, for $i = 1, 2,\ j = 1, \dots, m_S$ and $k^i = 1, \dots, m^i_S$; the δ's obtained thereby having to be pulled back by multiple applications of Ψ_i^{-1} and statement (3-3) into the set $\mathrm{B}_{\delta_q}(q)$.

In total we get the finite set $\Delta = \{\delta_q, \delta_{q_1}, \dots, \delta_{q_{m_S}}, \delta_{q_1^1}, \dots, \delta_{q^1_{m_S}}, \delta_{q_1^2}, \dots, \delta_{q^2_{m_S}}, \dots\}$. Let $\delta := \min \Delta$ and $V := \mathrm{B}_\delta(q)$. Then we have $\widehat{D}(x) = \widehat{D}(q)$ for all $x \in V$ because of the properties (3-1) and (3-2) and the construction of δ. □

3.2 Proof of Theorem 1.1

Now we can prove the main result of this paper.

Proof of Theorem 1.1 We consider a generic smooth isotopy $(K_r)_{r\in[0,1]}$ of knots where K_0 and K_1 are generic, and want to show that $\mathrm{Cord}(K_0) \cong \mathrm{Cord}(K_1)$. Therefore, we will look at all the cases listed in Lemma 3.1. Since there are, according to Lemma 3.1, only finitely many nongeneric knots during this isotopy, we can simplify notation by considering a generic isotopy $(K_r)_{r\in[-\varepsilon,\varepsilon]}$, $\varepsilon > 0$, such that

- in case (i) of Lemma 3.1 K_r is generic for all $r \in [-\varepsilon, \varepsilon]$;
- in case (ii) of Lemma 3.1 K_r is generic for all $r \in [-\varepsilon, \varepsilon] \setminus \{0\}$ and K_0 satisfies exactly one of the cases (ii)(1)–(ii)(14).

First, we construct isomorphisms which we will need in cases (i) and (ii)(1)–(ii)(13). Let

$$n_0 := |\mathrm{Crit}_0(K_0)| \quad \text{and} \quad n_1 := |\mathrm{Crit}_1(K_0)|.$$

Since no critical points appear or disappear (in the cases (i) and (ii)(1) to (ii)(13)), for all $r \in [-\varepsilon, \varepsilon]$,

$$|\mathrm{Crit}_0(K_r)| = n_0 \quad \text{and} \quad |\mathrm{Crit}_1(K_r)| = n_1.$$

So for $r \in [-\varepsilon, \varepsilon]$ let

$$\mathrm{Crit}_0(K_r) = \{g_r^1, \dots, g_r^{n_0}\} \quad \text{and} \quad \mathrm{Crit}_1(K_r) = \{k_r^1, \dots, k_r^{n_1}\}.$$

Now we define the sets

$$\mathrm{Crit}_0^{[-\varepsilon,\varepsilon]} := \bigcup_{r \in [-\varepsilon,\varepsilon]} \{r\} \times \{g_r^1, \dots, g_r^{n_0}\} \subset [-\varepsilon, \varepsilon] \times T^2,$$
$$\mathrm{Crit}_1^{[-\varepsilon,\varepsilon]} := \bigcup_{r \in [-\varepsilon,\varepsilon]} \{r\} \times \{k_r^1, \dots, k_r^{n_1}\} \subset [-\varepsilon, \varepsilon] \times T^2.$$

Since the isotopy is smooth, we can number the critical points such that the maps

$$\Psi_0 \colon [-\varepsilon, \varepsilon] \times \mathrm{Crit}_0(K_{-\varepsilon}) \to \mathrm{Crit}_0^{[-\varepsilon,\varepsilon]}, \qquad (r, g^i_{-\varepsilon}) \mapsto g^i_r \quad \text{for } i = 1, \dots, n_0,$$
$$\Psi_1 \colon [-\varepsilon, \varepsilon] \times \mathrm{Crit}_1(K_{-\varepsilon}) \to \mathrm{Crit}_1^{[-\varepsilon,\varepsilon]}, \qquad (r, k^i_{-\varepsilon}) \mapsto k^i_r \quad \text{for } i = 1, \dots, n_1,$$

are continuous with respect to the first component. For all $r \in [-\varepsilon, \varepsilon]$ we define the linear maps $\Phi_{0,r}$ and $\Phi_{1,r}$ on generators by

$$\begin{aligned}
\Phi_{0,r} \colon C_0(K_{-\varepsilon}) &\to C_0(K_r),\\
g^i_{-\varepsilon} &\mapsto \Psi_0(r, g^i_{-\varepsilon}) = g^i_r, \quad i = 1, \dots, n_0,\\
\lambda^{\pm 1} &\mapsto \lambda^{\pm 1},\\
\mu^{\pm 1} &\mapsto \mu^{\pm 1},\\
\Phi_{1,r} \colon C_1(K_{-\varepsilon}) &\to C_1(K_r),\\
k^i_{-\varepsilon} &\mapsto \Psi_1(r, k^i_{-\varepsilon}) = k^i_r, \quad i = 1, \dots, n_1.
\end{aligned}$$

For all $r \in [-\varepsilon, \varepsilon]$ we extend $\Phi_{0,r}$ to an algebra homomorphism. Obviously, $\Phi_{0,r}$ and $\Phi_{1,r}$ are isomorphisms for all $r \in [-\varepsilon, \varepsilon]$.

For each k^i_r, $r \in [-\varepsilon, \varepsilon], i = 1, \dots, n_1$, choose $k^i_{r,+}, k^i_{r,-} \in W^u_r(k^i_r)$ to determine $D_r(k^i_r)$. Choose $T < \infty$ such that for all $i = 1, \dots, n_1$,

$$\varphi_0^T(k^i_{0,+}) \in \mathrm{B}_{\varepsilon_0^{j_i}}(g_0^{j_i}), \quad \varphi_0^T(k^i_{0,-}) \in \mathrm{B}_{\varepsilon_0^{l_i}}(g_0^{l_i}),$$

where the $\varepsilon_0^{j_i}$ and $\varepsilon_0^{l_i}$ are chosen such that the properties (3-1) and (3-2) are satisfied. This must also hold for all chords created by splitting a chord according to relation (iv) during its movement along the unstable manifold of k^i_r. Since, according to Lemma 2.22, these are only finitely many chords and all chords which are moved along the gradient flow reach a $\mathrm{B}_{\varepsilon_0^i}(g_0^i)$ after finite time, such a T exists. After possibly increasing T it can be achieved that for all $r \in [-\varepsilon, \varepsilon]$, $i = 1, \dots, n_1$, and matching $\varepsilon_r^{j_i}$ and $\varepsilon_r^{l_i}$ (since $[-\varepsilon, \varepsilon] \times T^2$ is compact),

$$\varphi_r^T(k^i_{r,+}) \in \mathrm{B}_{\varepsilon_r^{j_i}}(g_r^{j_i}), \quad \varphi_r^T(k^i_{r,-}) \in \mathrm{B}_{\varepsilon_r^{l_i}}(g_r^{l_i}).$$

Lemma 3.3 implies that there exists a $\bar{\delta} > 0$ such that for all $i = 1, \dots, n_1$ and all $x \in \mathrm{B}_{\bar{\delta}}(k^i_{0,\pm})$,

$$\widehat{D}_0(x) = \widehat{D}_0(k^i_{0,\pm}).$$

Let $\hat{\varepsilon} > 0$ be small enough that for all $r \in [-\varepsilon, \varepsilon]$ the properties (3-5) to (3-10) required in the proof of Lemma 3.3 are satisfied (which are formulated there for ε_F, ε_B and ε_S). Such an $\hat{\varepsilon}$ exists since the required properties are open and $[-\varepsilon, \varepsilon]$ is compact. The starting times are always $s_0 = 0$. Since the solution of a differential equation depends continuously on the initial conditions and the function, there exists a $\delta > 0$ with $\delta < \bar{\delta}$ such that for all $i = 1, \dots, n_1$ and all $x \in T^2$ with $|x - k^i_{0,\pm}| < \delta$ and $|X_r(x) - X_0(x)| < \delta$ for all $s \in [0, T]$,

$$|\varphi_r^s(x) - \varphi_0^s(k^i_{0,\pm})| < \hat{\varepsilon}.$$

If ε is chosen small enough, it follows that for all $i = 1, \dots, n_1$ and all $r \in [-\varepsilon, \varepsilon]$,

$$\widehat{D}_r(k^i_{r,\pm}) = \Phi_{0,r} \circ \widehat{D}_{-\varepsilon}(k^i_{-\varepsilon,\pm}).$$

So we get, for all $i = 1, \dots, n_1$ and all $r \in [-\varepsilon, \varepsilon]$,

$$D_r(k^i_r) = \Phi_{0,r} \circ D_{-\varepsilon}(k^i_{-\varepsilon}) = \Phi_{0,r} \circ D_{-\varepsilon} \circ \Phi^{-1}_{1,r}(k^i_r).$$

Therefore, we have in case (i)

$$D_r = \Phi_{0,r} \circ D_{-\varepsilon} \circ \Phi^{-1}_{1,r}.$$

Thus, where $\langle M \rangle$ means the ideal generated by the set M,

$$\begin{aligned}
\mathrm{Cord}(K_r) &= C_0(K_r)/I_r \\
&= C_0(K_r)/\langle D_r(C_1(K_r))\rangle \\
&= \Phi_{0,r}(C_0(K_{-\varepsilon}))/\langle \Phi_{0,r} \circ D_{-\varepsilon} \circ \Phi^{-1}_{1,r}(C_1(K_r))\rangle \\
&= \Phi_{0,r}(C_0(K_{-\varepsilon}))/\langle \Phi_{0,r} \circ D_{-\varepsilon}(C_1(K_{-\varepsilon}))\rangle \\
&\overset{(1)}{=} \Phi_{0,r}\big(C_0(K_{-\varepsilon})/\langle D_{-\varepsilon}(C_1(K_{-\varepsilon}))\rangle\big) \\
&= \Phi_{0,r}(C_0(K_{-\varepsilon})/I_{-\varepsilon}) \\
&= \Phi_{0,r}(\mathrm{Cord}(K_{-\varepsilon})) \\
&\cong \mathrm{Cord}(K_{-\varepsilon}).
\end{aligned}$$

Equality (1) holds since $\Phi_{0,r}$ is an algebra isomorphism for all $r \in [-\varepsilon, \varepsilon]$.

So in case (i) the cord algebra remains the same up to a canonical isomorphism.

Now we assume that K_r is generic for all $r \in [-\varepsilon, \varepsilon] \setminus \{0\}$ and K_0 satisfies exactly one of the cases (ii)(1) to (ii)(14) of Lemma 3.1. We want to show that $\mathrm{Cord}(K_{-\varepsilon}) \cong \mathrm{Cord}(K_\varepsilon)$ in each of these cases. Without loss of generality, we can number the critical points such that in the cases (ii)(1) to (ii)(10) k^1_0 is the critical point of index 1 that is mentioned in these statements. The above consideration concerning $D_r(k^i_r)$ only holds for $i = 2, \dots, n_1$ and we have to look at $D_r(k^1_r)$. Furthermore, in the cases (ii)(8) to (ii)(10) we number the critical points such that g^1_0 is the critical point of index 0 that is mentioned in these statements.

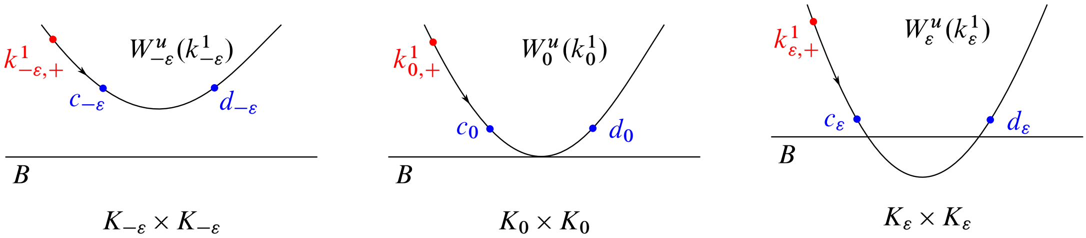

Figure 27: The unstable manifold of the chord k^1_0 is tangent to B.

Case (ii)(1) $(W^u_0(k^1_0) \not\pitchfork B)$ We consider the situation as in Figure 27. The following consideration holds analogously for the other possible situations: direction of the flow from the right to the left, $k^1_{r,-}$, and $W^u_0(k^1_0)$ tangent to $(\{0\} \times K_0) \subset B$.

For $r \in [-\varepsilon, \varepsilon]$ choose c_r and d_r as in Figure 27. Then for ε small enough we get on the one hand $\widehat{D}_\varepsilon(d_\varepsilon) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(d_{-\varepsilon})$ and on the other hand $\widehat{D}_r(c_r) = \widehat{D}_r(d_r)$ for all $r \in [-\varepsilon, 0)$ and

$$\begin{aligned}\widehat{D}_r(c_r) &= \widehat{D}_r(d_r)\lambda\lambda^{-1} \quad \text{(rel. (iii))}\\ &= \widehat{D}_r(d_r)\end{aligned}$$

for all $r \in (0, \varepsilon]$, since the endpoint of the chord intersects the basepoint once in the reversed direction of the orientation of the knot and once in the direction of the orientation of the knot. Thus, we get

$$D_\varepsilon(k^1_\varepsilon) = \Phi_{0,\varepsilon} \circ D_{-\varepsilon} \circ \Phi^{-1}_{1,\varepsilon}(k^1_\varepsilon).$$

It follows that

$$D_\varepsilon = \Phi_{0,\varepsilon} \circ D_{-\varepsilon} \circ \Phi^{-1}_{1,\varepsilon}.$$

By the analogous computation as in case (i) with $r = \varepsilon$ we get $\mathrm{Cord}(K_\varepsilon) \cong \mathrm{Cord}(K_{-\varepsilon})$.

Case (ii)(2) $(W^u_0(k^1_0) \not\pitchfork S_0)$ We consider the situation as in Figure 28. The chords labeled h^i_r, for $i = 1, 2, 3$, lie on the unstable manifold of k^1_r such that $\varphi^{t_i}_r(k^1_{r,+}) = h^i_r, t_i > 0$, and $t_i > t_j$ if $i > j$. (We will use this notation in the following cases, too.) For $r \in (0, \varepsilon]$ the chord intersects the knot in its interior at two different times when flowing along the negative gradient, and we get c^1_r, c^2_r, d^1_r and d^2_r by the application of relation (iv). According to Lemma 3.1, we can assume that there is no critical point between c^1_r and d^1_r and between c^2_r and d^2_r, otherwise we would have also the case (ii)(13), but this would be a

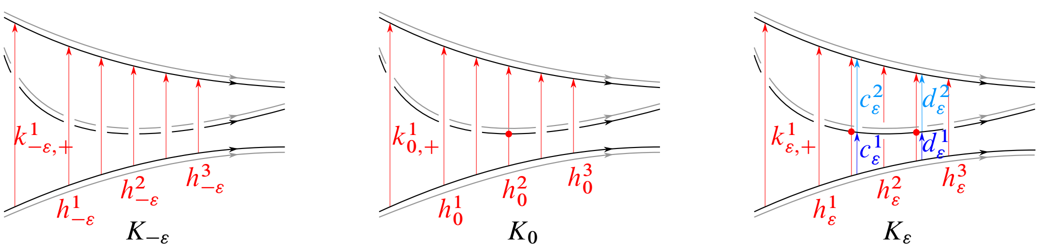

Figure 28: The unstable manifold of the chord k^1_0 is tangent to S_0.

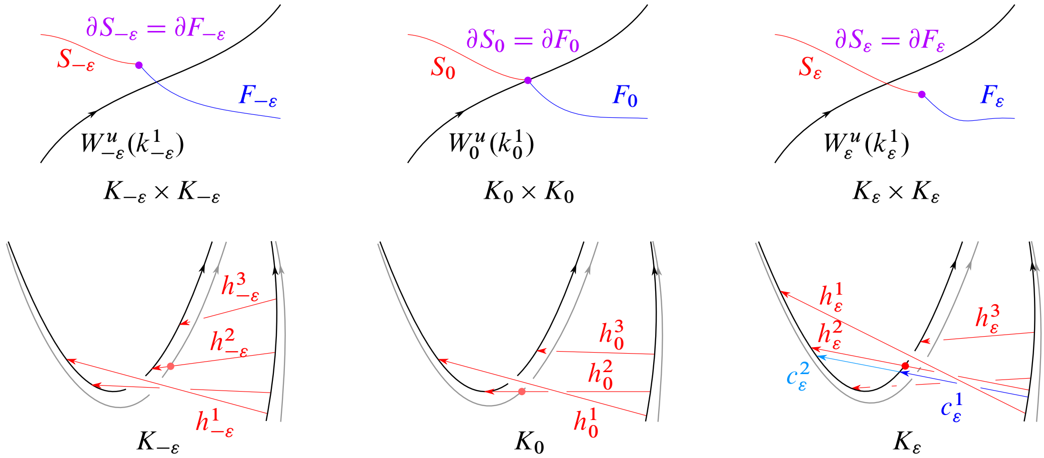

Figure 29: The unstable manifold of the chord k^1_0 intersects ∂S_0 and ∂F_0.

contradiction to this lemma since only one of these cases occurs at any one time. So if ε is chosen small enough, we have $\widehat{D}_r(c^1_r) = \widehat{D}_r(d^1_r)$ and $\widehat{D}_r(c^2_r) = \widehat{D}_r(d^2_r)$ for $r > 0$. Thus, we get

$$\begin{aligned} \widehat{D}_r(h^1_r) &= \widehat{D}_r(h^2_r) + \widehat{D}_r(c^1_r)\widehat{D}_r(c^2_r) && \text{(rel. (iv))} \\ &= \widehat{D}_r(h^3_r) - \widehat{D}_r(d^1_r)\widehat{D}_r(d^2_r) + \widehat{D}_r(c^1_r)\widehat{D}_r(c^2_r) && \text{(rel. (iv))} \\ &= \widehat{D}_r(h^3_r). \end{aligned}$$

As in case (ii)(1) we get $D_\varepsilon(k^1_\varepsilon) = \Phi_{0,\varepsilon} \circ D_{-\varepsilon} \circ \Phi^{-1}_{1,\varepsilon}(k^1_\varepsilon)$, $D_\varepsilon = \Phi_{0,\varepsilon} \circ D_{-\varepsilon} \circ \Phi^{-1}_{1,\varepsilon}$ and thus by the analogous computation as above $\mathrm{Cord}(K_\varepsilon) \cong \mathrm{Cord}(K_{-\varepsilon})$.

The other possible situations are to be treated analogously.

Case (ii)(3) ($W^u_0(k^1_0) \not\pitchfork F_0$) Analogous to case (ii)(1) with relation (ii) instead of relation (iii), ie $\mu^{\pm 1}$ instead of $\lambda^{\pm 1}$.

Case (ii)(4) ($W^u_0(k^1_0) \cap \partial S_0 \neq \varnothing$ and $W^u_0(k^1_0) \cap \partial F_0 \neq \varnothing$) We consider the situation as in Figure 29. As we can see from the figure, we get $\widehat{D}_r(h^1_r) \overset{\text{rel. (ii)}}{=} \widehat{D}_r(h^3_r)\mu^{-1}$ for $r < 0$ and $\widehat{D}_r(h^3_r) \overset{\text{rel. (ii)}}{=} \widehat{D}_r(c^1_r)\mu$ for $r > 0$. If ε is chosen small enough, we have $\widehat{D}_r(h^3_r) = \Phi_{0,r} \circ \widehat{D}_{-\varepsilon}(h^3_{-\varepsilon})$. Now we can compute for $r > 0$:

$$\begin{aligned} \widehat{D}_r(h^1_r) &= \widehat{D}_r(h^3_r) + \widehat{D}_r(c^1_r)\widehat{D}_r(c^2_r) && \text{(rel. (iv))} \\ &= \widehat{D}_r(h^3_r) + \widehat{D}_r(h^3_r)\mu^{-1}(1-\mu) && \text{(rel. (i))} \\ &= \widehat{D}_r(h^3_r)\mu^{-1} \\ &= \Phi_{0,r} \circ \widehat{D}_{-\varepsilon}(h^3_{-\varepsilon})\mu^{-1} \\ &= \Phi_{0,r} \circ \widehat{D}_{-\varepsilon}(h^1_{-\varepsilon}). \end{aligned}$$

As in case (ii)(1) we get $D_\varepsilon(k^1_\varepsilon) = \Phi_{0,\varepsilon} \circ D_{-\varepsilon} \circ \Phi^{-1}_{1,\varepsilon}(k^1_\varepsilon)$, $D_\varepsilon = \Phi_{0,\varepsilon} \circ D_{-\varepsilon} \circ \Phi^{-1}_{1,\varepsilon}$ and thus by the analogous computation as above $\mathrm{Cord}(K_\varepsilon) \cong \mathrm{Cord}(K_{-\varepsilon})$.

The other possible situations are to be treated analogously.

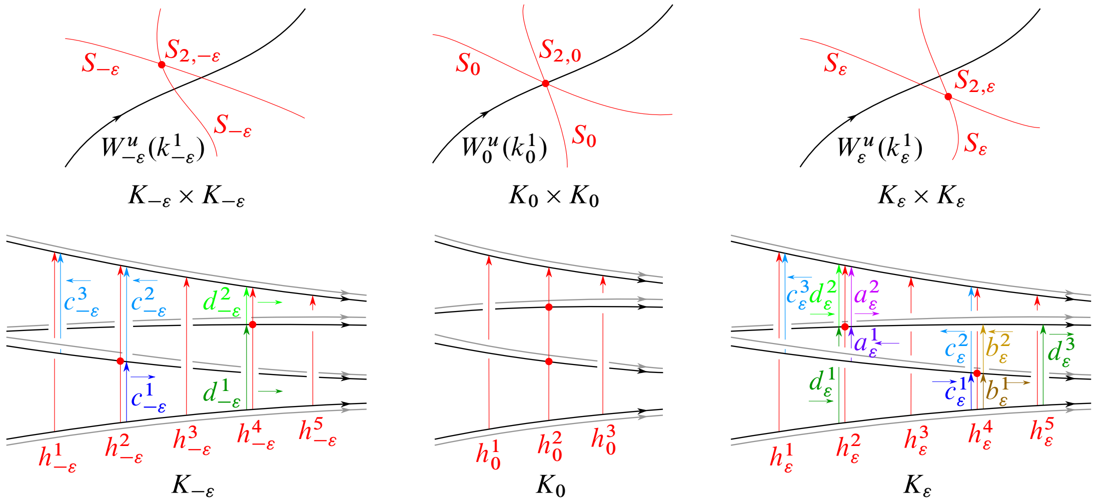

Figure 30: The unstable manifold of the chord k^1_0 intersects $S_{2,0}$.

Case (ii)(5) $(W^u_0(k^1_0)\cap S_{2,0}\neq\varnothing)$ We consider the situation as in Figure 30. As in case (ii)(2) we can assume that there are no critical points in the shown region. The small horizontal arrows indicate the direction in which the chords move along the gradient vector field. From the figure we get

$$\hat{D}_{-\varepsilon}(c^2_{-\varepsilon}) \overset{\text{rel. (ii)}}{=} \mu^{-1}\hat{D}_{-\varepsilon}(c^3_{-\varepsilon}), \quad \hat{D}_\varepsilon(a^2_\varepsilon)=\hat{D}_\varepsilon(d^2_\varepsilon),$$
$$\hat{D}_\varepsilon(b^2_\varepsilon) \overset{\text{rel. (ii)}}{=} \mu^{-1}\hat{D}_\varepsilon(a^1_\varepsilon), \quad \hat{D}_\varepsilon(b^1_\varepsilon)=\hat{D}_\varepsilon(c^1_\varepsilon).$$

If ε is chosen small enough, then

$$\hat{D}_\varepsilon(c^1_\varepsilon)=\Phi_{0,\varepsilon}\circ\hat{D}_{-\varepsilon}(c^1_{-\varepsilon}), \quad \hat{D}_\varepsilon(d^2_\varepsilon)=\Phi_{0,\varepsilon}\circ\hat{D}_{-\varepsilon}(d^2_{-\varepsilon}), \quad \hat{D}_\varepsilon(d^3_\varepsilon)=\Phi_{0,\varepsilon}\circ\hat{D}_{-\varepsilon}(d^1_{-\varepsilon}),$$
$$\hat{D}_\varepsilon(c^3_\varepsilon)=\Phi_{0,\varepsilon}\circ\hat{D}_{-\varepsilon}(c^3_{-\varepsilon}), \quad \hat{D}_\varepsilon(h^5_\varepsilon)=\Phi_{0,\varepsilon}\circ\hat{D}_{-\varepsilon}(h^5_{-\varepsilon}).$$

Now we can compute

$$\begin{aligned}
\hat{D}_{-\varepsilon}(h^1_{-\varepsilon}) &\overset{\text{rel. (iv)}}{=} \hat{D}_{-\varepsilon}(h^3_{-\varepsilon})-\hat{D}_{-\varepsilon}(c^1_{-\varepsilon})\hat{D}_{-\varepsilon}(c^2_{-\varepsilon})\\
&\overset{\text{rel. (iv)}}{=} \hat{D}_{-\varepsilon}(h^5_{-\varepsilon})+\hat{D}_{-\varepsilon}(d^1_{-\varepsilon})\hat{D}_{-\varepsilon}(d^2_{-\varepsilon})-\hat{D}_{-\varepsilon}(c^1_{-\varepsilon})\mu^{-1}\hat{D}_{-\varepsilon}(c^3_{-\varepsilon}),\\
\hat{D}_\varepsilon(h^1_\varepsilon) &\overset{\text{rel. (iv)}}{=} \hat{D}_\varepsilon(h^3_\varepsilon)+\hat{D}_\varepsilon(d^1_\varepsilon)\hat{D}_\varepsilon(d^2_\varepsilon)\\
&\overset{\text{rel. (iv)}}{=} \hat{D}_\varepsilon(h^5_\varepsilon)-\hat{D}_\varepsilon(c^1_\varepsilon)\hat{D}_\varepsilon(c^2_\varepsilon)+\hat{D}_\varepsilon(d^1_\varepsilon)\hat{D}_\varepsilon(d^2_\varepsilon)\\
&\overset{\text{rel. (iv)}}{=} \hat{D}_\varepsilon(h^5_\varepsilon)-\hat{D}_\varepsilon(c^1_\varepsilon)\big(\mu^{-1}\hat{D}_\varepsilon(c^3_\varepsilon)-\mu^{-1}\hat{D}_\varepsilon(a^1_\varepsilon)\hat{D}_\varepsilon(a^2_\varepsilon)\big)+\big(\hat{D}_\varepsilon(d^3_\varepsilon)-\hat{D}_\varepsilon(b^1_\varepsilon)\hat{D}_\varepsilon(b^2_\varepsilon)\big)\hat{D}_\varepsilon(d^2_\varepsilon)\\
&= \hat{D}_\varepsilon(h^5_\varepsilon)-\hat{D}_\varepsilon(c^1_\varepsilon)\mu^{-1}\hat{D}_\varepsilon(c^3_\varepsilon)+\hat{D}_\varepsilon(b^1_\varepsilon)\hat{D}_\varepsilon(b^2_\varepsilon)\hat{D}_\varepsilon(d^2_\varepsilon)+\hat{D}_\varepsilon(d^3_\varepsilon)\hat{D}_\varepsilon(d^2_\varepsilon)-\hat{D}_\varepsilon(b^1_\varepsilon)\hat{D}_\varepsilon(b^2_\varepsilon)\hat{D}_\varepsilon(d^2_\varepsilon)\\
&= \Phi_{0,\varepsilon}\circ\big(\hat{D}_{-\varepsilon}(h^5_{-\varepsilon})-\hat{D}_{-\varepsilon}(c^1_{-\varepsilon})\mu^{-1}\hat{D}_{-\varepsilon}(c^3_{-\varepsilon})+\hat{D}_{-\varepsilon}(d^1_{-\varepsilon})\hat{D}_{-\varepsilon}(d^2_{-\varepsilon})\big)\\
&= \Phi_{0,\varepsilon}\circ\hat{D}_{-\varepsilon}(h^1_{-\varepsilon}).
\end{aligned}$$

As in case (ii)(1) we get $D_\varepsilon(k^1_\varepsilon)=\Phi_{0,\varepsilon}\circ D_{-\varepsilon}\circ\Phi^{-1}_{1,\varepsilon}(k^1_\varepsilon)$ and $D_\varepsilon=\Phi_{0,\varepsilon}\circ D_{-\varepsilon}\circ\Phi^{-1}_{1,\varepsilon}$, and thus by the analogous computation as above, $\mathrm{Cord}(K_\varepsilon)\cong\mathrm{Cord}(K_{-\varepsilon})$.

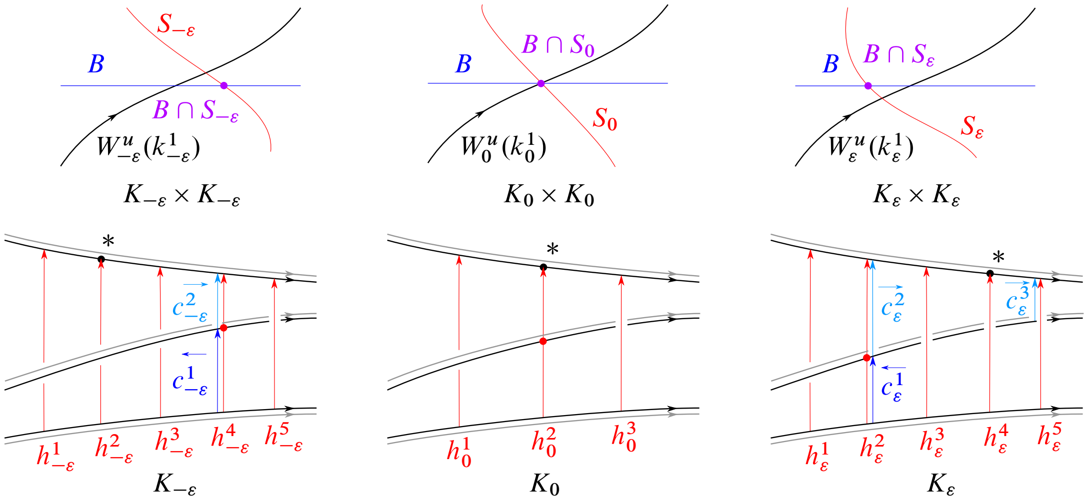

Figure 31: The unstable manifold of the chord k_0^1 intersects $B\cap S_0$.

The other possible situations are to be treated analogously.

Case (ii)(6) $(W_0^u(k_0^1)\cap B\cap S_0\neq\varnothing)$ We consider the situation as in Figure 31. As in case (ii,2) we can assume that there are no critical points in the shown region. If ε is chosen small enough, then

$$\widehat{D}_\varepsilon(c_\varepsilon^1)=\Phi_{0,\varepsilon}\circ\widehat{D}_{-\varepsilon}(c_{-\varepsilon}^1),\quad \widehat{D}_\varepsilon(c_\varepsilon^3)=\Phi_{0,\varepsilon}\circ\widehat{D}_{-\varepsilon}(c_{-\varepsilon}^2),\quad \widehat{D}_\varepsilon(h_\varepsilon^5)=\Phi_{0,\varepsilon}\circ\widehat{D}_{-\varepsilon}(h_{-\varepsilon}^5).$$

Now we can compute

$$\begin{aligned}
\widehat{D}_{-\varepsilon}(h^1_{-\varepsilon}) &= \widehat{D}_{-\varepsilon}(h^3_{-\varepsilon})\lambda^{-1} && \text{(rel. (iii))}\\
&= \widehat{D}_{-\varepsilon}(h^5_{-\varepsilon})\lambda^{-1}-\widehat{D}_{-\varepsilon}(c^1_{-\varepsilon})\widehat{D}_{-\varepsilon}(c^2_{-\varepsilon})\lambda^{-1} && \text{(rel. (iv))},\\
\widehat{D}_\varepsilon(h^1_\varepsilon) &= \widehat{D}_\varepsilon(h^3_\varepsilon)-\widehat{D}_\varepsilon(c^1_\varepsilon)\widehat{D}_\varepsilon(c^2_\varepsilon) && \text{(rel. (iv))}\\
&= \widehat{D}_\varepsilon(h^5_\varepsilon)\lambda^{-1}-\widehat{D}_\varepsilon(c^1_\varepsilon)\widehat{D}_\varepsilon(c^3_\varepsilon)\lambda^{-1} && \text{(rel. (iii))}\\
&= \Phi_{0,\varepsilon}\circ\big(\widehat{D}_{-\varepsilon}(h^5_{-\varepsilon})\lambda^{-1}-\widehat{D}_{-\varepsilon}(c^1_{-\varepsilon})\widehat{D}_{-\varepsilon}(c^2_{-\varepsilon})\lambda^{-1}\big)\\
&= \Phi_{0,\varepsilon}\circ\widehat{D}_{-\varepsilon}(h^1_{-\varepsilon}).
\end{aligned}$$

As in case (ii)(1) we get $D_\varepsilon(k_\varepsilon^1)=\Phi_{0,\varepsilon}\circ D_{-\varepsilon}\circ\Phi_{1,\varepsilon}^{-1}(k_\varepsilon^1)$ and $D_\varepsilon=\Phi_{0,\varepsilon}\circ D_{-\varepsilon}\circ\Phi_{1,\varepsilon}^{-1}$, and thus by the analogous computation as above, $\mathrm{Cord}(K_\varepsilon)\cong\mathrm{Cord}(K_{-\varepsilon})$.

The other possible situations are to be treated analogously.

Case (ii)(7) $(W_0^u(k_0^1)\cap F_0\cap S_0\neq\varnothing)$ Analogous to case (ii)(6) with relation (ii) instead of relation (iii), ie $\mu^{\pm1}$ instead of $\lambda^{\pm1}$.

Case (ii)(8) (a) $(g_0^1\in B)$ We consider the situation as in Figure 32. The red and blue trajectories are unstable manifolds of critical points of index 1 or trajectories starting at chords that are generated

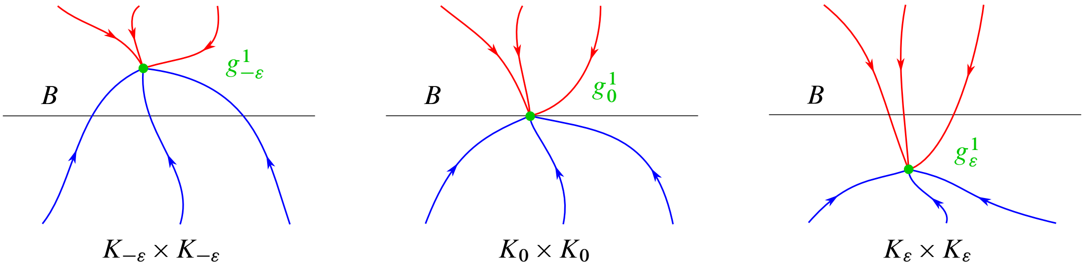

Figure 32: The critical point g_0^1 of index 0 intersects the basepoint.

by splitting according to relation (iv). According to Lemma 2.22, there are only finitely many such trajectories. In K_0 none of the red and blue trajectories is tangent to B, otherwise we would also have case (ii)(1) or (ii)(12) of Lemma 3.1, but this would be a contradiction to this lemma since only one of these cases occurs at any one time. If ε is chosen small enough, we can guarantee the following: In $K_{-\varepsilon}$ all of the blue trajectories intersect B in a small neighborhood of $g_{-\varepsilon}^1$, but none of the red ones, and in K_ε all of the red trajectories intersect B in a small neighborhood of g_ε^1, but none of the blue ones. It follows that in $K_{-\varepsilon}$ we have a contribution of $g_{-\varepsilon}^1$ along the red trajectories and of $g_{-\varepsilon}^1\lambda^{-1}$ along the blue trajectories. In K_ε we have a contribution of $g_\varepsilon^1\lambda$ along the red trajectories and of g_ε^1 along the blue trajectories. So we can construct the canonical isomorphism

$$\begin{aligned}\operatorname{Cord}(K_{-\varepsilon}) &\xrightarrow{\sim} \operatorname{Cord}(K_\varepsilon),\\ g_{-\varepsilon}^1 &\mapsto g_\varepsilon^1\lambda,\\ g_{-\varepsilon}^i &\mapsto g_\varepsilon^i, \quad i=2,\dots,n_0,\\ \lambda^{\pm 1} &\mapsto \lambda^{\pm 1},\\ \mu^{\pm 1} &\mapsto \mu^{\pm 1}.\end{aligned}$$

The other possible situations are to be treated analogously.

(b) $(k_0^1\in B)$ We consider the situation as in Figure 33. If ε is chosen small enough, then

$$\widehat{D}_\varepsilon(k_{\varepsilon,+}^1)=\Phi_{0,\varepsilon}\circ\widehat{D}_{-\varepsilon}(\tilde{k}_{-\varepsilon,+}^1),\quad \widehat{D}_\varepsilon(\tilde{k}_{\varepsilon,-}^1)=\Phi_{0,\varepsilon}\circ\widehat{D}_{-\varepsilon}(k_{-\varepsilon,-}^1).$$

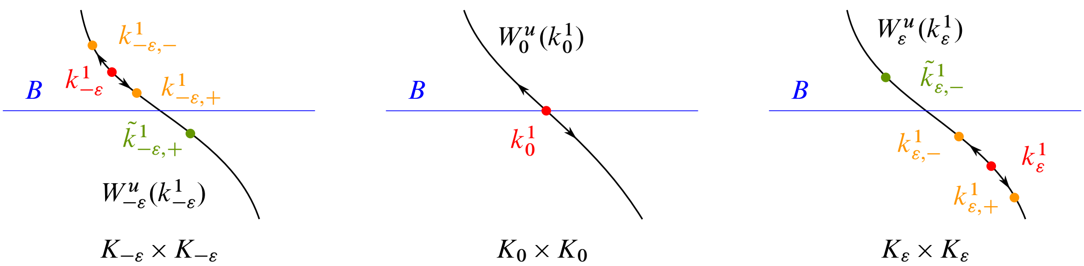

Figure 33: The critical point k_0^1 of index 1 intersects the basepoint.

Now we can compute

$$\begin{aligned}
D_\varepsilon(k^1_\varepsilon) &= \widehat{D}_\varepsilon(k^1_{\varepsilon,+}) - \widehat{D}_\varepsilon(k^1_{\varepsilon,-}) \\
&= \widehat{D}_\varepsilon(k^1_{\varepsilon,+}) - \widehat{D}_\varepsilon(\tilde{k}^1_{\varepsilon,-})\lambda^{-1} && \text{(rel. (iii))} \\
&= \Phi_{0,\varepsilon} \circ \big(\widehat{D}_{-\varepsilon}(\tilde{k}^1_{-\varepsilon,+}) - \widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,-})\lambda^{-1}\big) \\
&= \Phi_{0,\varepsilon} \circ \big((\widehat{D}_{-\varepsilon}(\tilde{k}^1_{-\varepsilon,+})\lambda - \widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,-}))\lambda^{-1}\big) \\
&= \Phi_{0,\varepsilon} \circ \big((\widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,+}) - \widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,-}))\lambda^{-1}\big) && \text{(rel. (iii))} \\
&= \Phi_{0,\varepsilon} \circ (D_{-\varepsilon}(k^1_{-\varepsilon})\lambda^{-1}) \\
&= (\Phi_{0,\varepsilon} \circ D_{-\varepsilon}(k^1_{-\varepsilon}))\lambda^{-1}.
\end{aligned}$$

With this result we get, where $R = \mathbb{Z}[\lambda^{\pm 1}, \mu^{\pm 1}]$ is the commutative ring as described above and $\langle M \rangle_R$ is the ideal generated by the set M,

$$\begin{aligned}
\operatorname{Cord}(K_\varepsilon) &= C_0(K_\varepsilon)/I_\varepsilon \\
&= C_0(K_\varepsilon)/\langle D_\varepsilon(\{k^2_\varepsilon, \dots, k^{n_1}_\varepsilon\}), D_\varepsilon(k^1_\varepsilon)\rangle_R \\
&= C_0(K_\varepsilon)/\langle \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(\{k^2_{-\varepsilon}, \dots, k^{n_1}_{-\varepsilon}\}), \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(k^1_{-\varepsilon})\lambda^{-1}\rangle_R \\
&= C_0(K_\varepsilon)/\langle \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(\{k^2_{-\varepsilon}, \dots, k^{n_1}_{-\varepsilon}\}), \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(k^1_{-\varepsilon})\rangle_R \\
&= C_0(K_\varepsilon)/\langle \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(\{k^1_{-\varepsilon}, \dots, k^{n_1}_{-\varepsilon}\})\rangle_R \\
&= \Phi_{0,\varepsilon}(C_0(K_{-\varepsilon}))/\Phi_{0,\varepsilon}(I_{-\varepsilon}) \\
&= \Phi_{0,\varepsilon}(C_0(K_{-\varepsilon})/I_{-\varepsilon}) \quad \text{(since } \Phi_{0,\varepsilon} \text{ is an algebra isomorphism)} \\
&= \Phi_{0,\varepsilon}(\operatorname{Cord}(K_{-\varepsilon})) \\
&\cong \operatorname{Cord}(K_{-\varepsilon}).
\end{aligned}$$

The other possible situations are to be treated analogously.

Case (ii)(9) (a) ($g^1_0 \in S_0$) We consider the situation as in Figure 34. The trajectories drawn in the upper figures are unstable manifolds of critical points of index 1 or trajectories starting at chords that are generated by splitting according to relation (iv). According to Lemma 2.22, there are only finitely many such trajectories. In K_0 none of these trajectories is tangent to S_0, otherwise we would also have case (ii)(2) or (ii)(12) of Lemma 3.1, but this would be a contradiction to this lemma since only one of these cases occurs at any one time. If ε is chosen small enough, we can guarantee that (in the example of Figure 34):

- In $K_{-\varepsilon}$ all of the "lower" (red and brown) trajectories intersect $S_{-\varepsilon}$ in a small neighborhood of $g^1_{-\varepsilon}$, but none of the "upper" (blue and purple) ones.
- In $K_{-\varepsilon}$ we get for the chords that are split off from the "lower" trajectories according to relation (iv): $\widehat{D}_{-\varepsilon}(c^1_{-\varepsilon}) = \widehat{D}_{-\varepsilon}(d^1_{-\varepsilon})$ and $\widehat{D}_{-\varepsilon}(c^2_{-\varepsilon}) = \widehat{D}_{-\varepsilon}(d^2_{-\varepsilon})$.
- In K_ε all of the "upper" trajectories intersect S_ε in a small neighborhood of g^1_ε, but none of the "lower" ones.

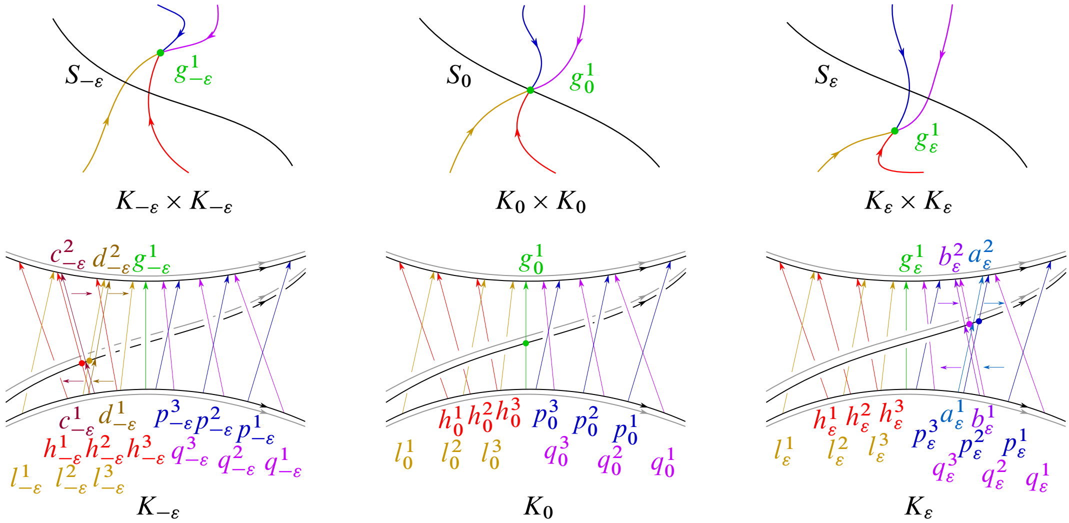

Figure 34: The critical point g^1_0 of index 0 intersects the knot in its interior.

- In K_ε we get for the chords that are split off from the "upper" trajectories according to relation (iv): $\widehat{D}_\varepsilon(a^1_\varepsilon) = \widehat{D}_\varepsilon(b^1_\varepsilon)$ and $\widehat{D}_\varepsilon(a^2_\varepsilon) = \widehat{D}_\varepsilon(b^2_\varepsilon)$.
- Furthermore, we get $\widehat{D}_\varepsilon(a^1_\varepsilon) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(c^1_{-\varepsilon})$ and $\widehat{D}_\varepsilon(a^2_\varepsilon) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(c^2_{-\varepsilon})$.

It follows that in $K_{-\varepsilon}$ we have a contribution of $g^1_{-\varepsilon}$ along the "upper" trajectories and of

$$g^1_{-\varepsilon} - \widehat{D}_{-\varepsilon}(c^1_{-\varepsilon})\widehat{D}_{-\varepsilon}(c^2_{-\varepsilon})$$

along the "lower" trajectories according to relation (iv). In K_ε we have a contribution of

$$g^1_\varepsilon + \widehat{D}_\varepsilon(a^1_\varepsilon)\widehat{D}_\varepsilon(a^2_\varepsilon) = g^1_\varepsilon + \Phi_{0,\varepsilon} \circ (\widehat{D}_{-\varepsilon}(c^1_{-\varepsilon})\widehat{D}_{-\varepsilon}(c^2_{-\varepsilon}))$$

along the "upper" trajectories and of g_ε along the "lower" trajectories. So we can construct the following canonical isomorphism:

$$\begin{aligned} \mathrm{Cord}(K_{-\varepsilon}) &\xrightarrow{\sim} \mathrm{Cord}(K_\varepsilon), \\ g^1_{-\varepsilon} &\mapsto g^1_\varepsilon + \widehat{D}_\varepsilon(a^1_\varepsilon)\widehat{D}_\varepsilon(a^2_\varepsilon), \\ g^i_{-\varepsilon} &\mapsto g^i_\varepsilon, \quad i = 2, \dots, n_0, \\ \lambda^{\pm 1} &\mapsto \lambda^{\pm 1}, \\ \mu^{\pm 1} &\mapsto \mu^{\pm 1}. \end{aligned}$$

The other possible situations are to be treated analogously.

(b) ($k^1_0 \in S_0$) We consider the situation as in Figure 35. As before, we can assume, according to Lemma 3.1, that the only critical point in the shown region is k^1_r. If ε is chosen small enough, then

$$\widehat{D}_\varepsilon(d^1_\varepsilon) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(c^1_{-\varepsilon}), \qquad \widehat{D}_\varepsilon(d^2_\varepsilon) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(c^2_{-\varepsilon}),$$
$$\widehat{D}_\varepsilon(\tilde{k}^1_{\varepsilon,+}) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,+}), \quad \widehat{D}_\varepsilon(k^1_{\varepsilon,-}) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(\tilde{k}^1_{-\varepsilon,-}).$$

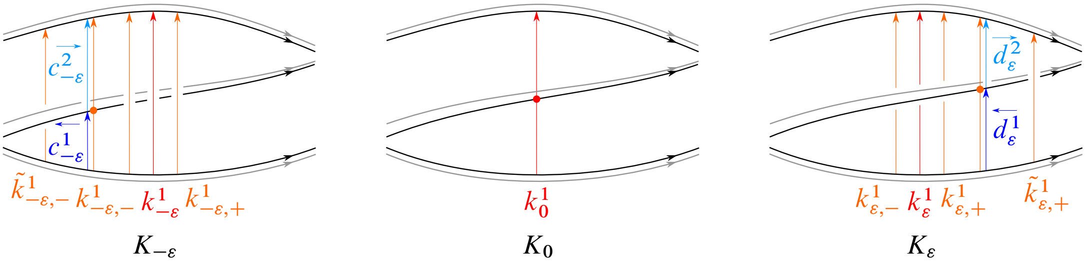

Figure 35: The critical point k^1_0 of index 1 intersects the knot in its interior.

Now we can compute

$$\begin{aligned}
D_\varepsilon(k^1_\varepsilon) &= \widehat{D}_\varepsilon(k^1_{\varepsilon,+}) - \widehat{D}_\varepsilon(k^1_{\varepsilon,-}) \\
&= \widehat{D}_\varepsilon(\tilde{k}^1_{\varepsilon,+}) - \widehat{D}_\varepsilon(d^1_\varepsilon)\widehat{D}_\varepsilon(d^2_\varepsilon) - \widehat{D}_\varepsilon(k^1_{\varepsilon,-}) && \text{(rel. (iv))} \\
&= \Phi_{0,\varepsilon}\circ\big(\widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,+}) - \widehat{D}_{-\varepsilon}(c^1_{-\varepsilon})\widehat{D}_{-\varepsilon}(c^2_{-\varepsilon}) - \widehat{D}_{-\varepsilon}(\tilde{k}^1_{-\varepsilon,-})\big) \\
&= \Phi_{0,\varepsilon}\circ\big(\widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,+}) - \widehat{D}_{-\varepsilon}(c^1_{-\varepsilon})\widehat{D}_{-\varepsilon}(c^2_{-\varepsilon}) - \big(\widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,-}) - \widehat{D}_{-\varepsilon}(c^1_{-\varepsilon})\widehat{D}_{-\varepsilon}(c^2_{-\varepsilon})\big)\big) && \text{(rel. (iv))} \\
&= \Phi_{0,\varepsilon}\circ\big(\widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,+}) - \widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,-})\big) \\
&= \Phi_{0,\varepsilon}\circ D_{-\varepsilon}(k^1_{-\varepsilon}).
\end{aligned}$$

As before we get $D_\varepsilon(k^1_\varepsilon) = \Phi_{0,\varepsilon}\circ D_{-\varepsilon}\circ\Phi^{-1}_{1,\varepsilon}(k^1_\varepsilon)$, $D_\varepsilon = \Phi_{0,\varepsilon}\circ D_{-\varepsilon}\circ\Phi^{-1}_{1,\varepsilon}$ and thus by the analogous computation as above $\mathrm{Cord}(K_\varepsilon) \cong \mathrm{Cord}(K_{-\varepsilon})$.

The other possible situations are to be treated analogously.

Case (ii)(10) ($k \in F_0$ for a critical point k of index 0 or 1) Analogous to case (ii,8) with relation (ii) instead of relation (iii).

Case (ii)(11) (there exists a trajectory between two critical points of index 1 along the vector field $-\nabla E_0$) We consider the situation as in Figure 36. According to Lemma 3.1, we have $k^2_r \notin (B \cup S_r \cup F_r)$ for all $r \in [-\varepsilon,\varepsilon]$. Since this is an open property, we can choose neighborhoods U_r of every k^2_r such that $U_r \cap (B \cup S_r \cup F_r) = \varnothing$. For all $r \in [-\varepsilon,\varepsilon]$ choose $c_r \in W^u_r(k^1_r) \cap U_r$. If ε is small enough, we get $\widehat{D}_r(c_r) = \widehat{D}_r(k^2_{r,+})$ for $r < 0$ and $\widehat{D}_r(c_r) = \widehat{D}_r(k^2_{r,-})$ for $r > 0$. If $W^u_0(k^1_0)$ intersects B, F_0, or S_0 between $k^1_{0,+}$ and c_0, we can choose ε small enough such that $W^u_r(k^1_r)$ intersects B, F_r, and S_r for all r

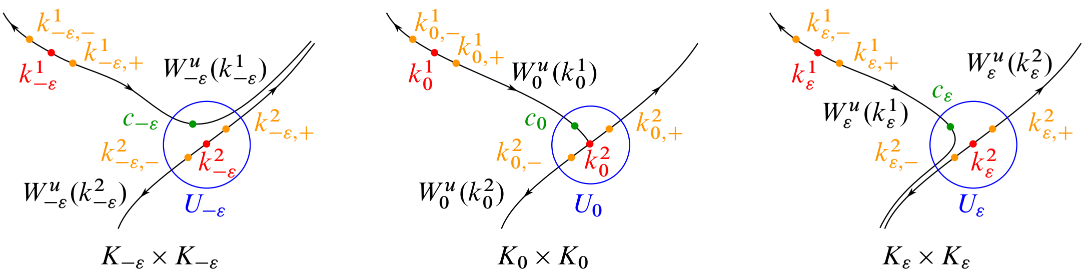

Figure 36: The unstable manifold of the chord k^1_0 ends at the critical point k^2_0 of index 1.

in the same way between $k^1_{r,+}$ and c_r. That means, while computing $\widehat{D}_r(k^1_{r,+})$ we get the same result up to the point c_r for all $r \in [-\varepsilon, \varepsilon]$. So without loss of generality we can assume $\widehat{D}_r(k^1_{r,+}) = \widehat{D}_r(c_r)$. Therefore, we get

$$\widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,+}) = \widehat{D}_{-\varepsilon}(k^2_{-\varepsilon,+}), \quad \widehat{D}_{\varepsilon}(k^1_{\varepsilon,+}) = \widehat{D}_{\varepsilon}(k^2_{\varepsilon,-}).$$

If ε is small enough, then

$$\widehat{D}_{\varepsilon}(k^2_{\varepsilon,+}) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(k^2_{-\varepsilon,+}), \quad \widehat{D}_{\varepsilon}(k^2_{\varepsilon,-}) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(k^2_{-\varepsilon,-}), \quad \widehat{D}_{\varepsilon}(k^1_{\varepsilon,-}) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,-}).$$

With this we can compute the ideal

$$\begin{aligned}
\langle D_\varepsilon(C_1(K_\varepsilon))\rangle_R &= \langle D_\varepsilon(k^1_\varepsilon), D_\varepsilon(\{k^2_\varepsilon, \dots, k^{n_1}_\varepsilon\})\rangle_R \\
&= \langle D_\varepsilon(k^1_\varepsilon) + D_\varepsilon(k^2_\varepsilon), D_\varepsilon(\{k^2_\varepsilon, \dots, k^{n_1}_\varepsilon\})\rangle_R \\
&= \langle \widehat{D}_\varepsilon(k^1_{\varepsilon,+}) - \widehat{D}_\varepsilon(k^1_{\varepsilon,-}) + \widehat{D}_\varepsilon(k^2_{\varepsilon,+}) - \widehat{D}_\varepsilon(k^2_{\varepsilon,-}), D_\varepsilon(\{k^2_\varepsilon, \dots, k^{n_1}_\varepsilon\})\rangle_R \\
&= \langle \widehat{D}_\varepsilon(k^2_{\varepsilon,+}) - \widehat{D}_\varepsilon(k^1_{\varepsilon,-}), D_\varepsilon(\{k^2_\varepsilon, \dots, k^{n_1}_\varepsilon\})\rangle_R \\
&= \langle \Phi_{0,\varepsilon} \circ (\widehat{D}_{-\varepsilon}(k^2_{-\varepsilon,+}) - \widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,-})), \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(\{k^2_{-\varepsilon}, \dots, k^{n_1}_{-\varepsilon}\})\rangle_R \\
&= \langle \Phi_{0,\varepsilon} \circ (\widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,+}) - \widehat{D}_{-\varepsilon}(k^1_{-\varepsilon,-})), \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(\{k^2_{-\varepsilon}, \dots, k^{n_1}_{-\varepsilon}\})\rangle_R \\
&= \langle \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(k^1_{-\varepsilon}), \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(\{k^2_{-\varepsilon}, \dots, k^{n_1}_{-\varepsilon}\})\rangle_R \\
&= \langle \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(C_1(K_{-\varepsilon}))\rangle_R.
\end{aligned}$$

It follows that

$$\mathrm{Cord}(K_\varepsilon) = \Phi_{0,\varepsilon}(\mathrm{Cord}(K_{-\varepsilon})) \cong \mathrm{Cord}(K_{-\varepsilon}).$$

The other possible situations are to be treated analogously.

Case (ii)(12) (one of the cases (ii)(1) to (ii)(10) holds for a chord which is generated by the application of relation (iv) (where in the cases (ii)(1) to (ii)(7) the unstable manifolds are to be replaced by trajectories along the vector field $-\nabla E_0$, starting at this chord)) Analogous to the cases (ii)(1) to (ii)(10).

Case (ii)(13) (a chord which is generated by the application of relation (iv) runs along the vector field $-\nabla E_0$ to a critical point of index 1) Analogous to case (ii)(11).

Case (ii)(14) (creation/cancellation of a pair of critical points of index 0 and 1 or of index 1 and 2) It suffices to consider the creation of a pair of critical points since the other case corresponds to the inverse isotopy.

(a) Creation of a pair of critical points of index 0 and 1. So let $K_0 \times K_0$ contain a critical point p of birth type of index 0. For the definition of a critical point of birth/death type see [3] after the proof of Lemma 9.3. Note that the index of p is defined in the same way as the Morse index of a critical point, ie the maximal dimension of the subspace on which the quadratic form $v \mapsto \mathrm{Hess}_p\, E(v, v)$ is negative definite.

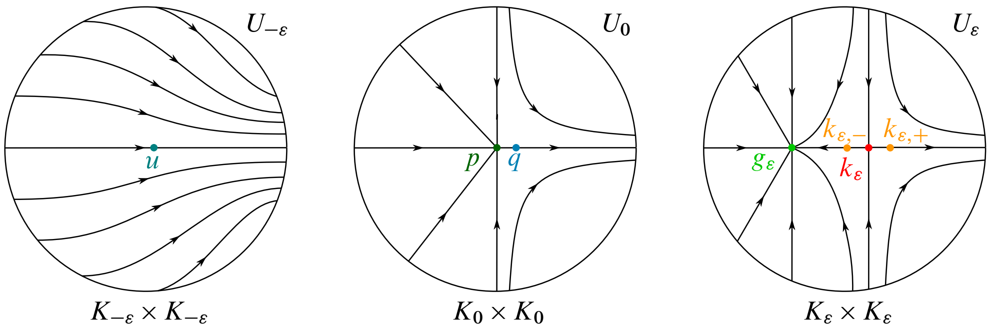

Figure 37: Creation of a pair of critical points of index 0 and 1.

Since K_0 is generic with one exception, the proofs of the Lemmas 2.17, 2.20 and 2.23 also work for p and $W_0^u(p)$. With a similar transversality argument as in these lemmata we can show that p does not lie on the unstable manifold of a critical point of index 1.

We can choose neighborhoods U_r and coordinates such that we can represent the situation as in Figure 37 (see [3, proof of Lemma 9.6]). Here, g_ε and k_ε in the right figure are nondegenerate critical points of index 0 and 1, respectively. Since we get two additional critical points during the isotopy, we have to modify the linear maps $\Phi_{0,r}$ and $\Phi_{1,r}$ a little bit:

Let

$$n_0 := |\mathrm{Crit}_0(K_{-\varepsilon})| \quad \text{and} \quad n_1 := |\mathrm{Crit}_1(K_{-\varepsilon})|.$$

Then for all $r \in [-\varepsilon, 0)$ we have

$$|\mathrm{Crit}_0(K_r)| = n_0 \quad \text{and} \quad |\mathrm{Crit}_1(K_r)| = n_1$$

and for all $r \in (0, \varepsilon]$ we have

$$|\mathrm{Crit}_0(K_r)| = n_0 + 1 \quad \text{and} \quad |\mathrm{Crit}_1(K_r)| = n_1 + 1.$$

So let

$$\mathrm{Crit}_0(K_r) = \begin{cases} \{g_r^1, \dots, g_r^{n_0}\} & \text{if } r \in [-\varepsilon, 0), \\ \{g_0^1, \dots, g_0^{n_0}, p\} & \text{if } r = 0, \\ \{g_r^1, \dots, g_r^{n_0}, g_r\} & \text{if } r \in (0, \varepsilon], \end{cases}$$
$$\mathrm{Crit}_1(K_r) = \begin{cases} \{k_r^1, \dots, k_r^{n_1}\} & \text{if } r \in [-\varepsilon, 0], \\ \{k_r^1, \dots, k_r^{n_1}, k_r\} & \text{if } r \in (0, \varepsilon]. \end{cases}$$

Now we define for $i = 0, 1$,

$$\mathrm{Crit}_i^{[-\varepsilon,\varepsilon]} := \bigcup_{r \in [-\varepsilon,\varepsilon]} \{r\} \times \mathrm{Crit}_i(K_r) \subset [-\varepsilon, \varepsilon] \times T^2.$$

Since the isotopy is smooth, we can number the critical points so that the maps

$$\Psi_0 \colon [-\varepsilon, \varepsilon] \times \mathrm{Crit}_0(K_{-\varepsilon}) \to \mathrm{Crit}_0^{[-\varepsilon,\varepsilon]}, \qquad (r, g_{-\varepsilon}^i) \mapsto g_r^i, \quad i = 1, \dots, n_0,$$
$$\Psi_1 \colon [-\varepsilon, \varepsilon] \times \mathrm{Crit}_1(K_{-\varepsilon}) \to \mathrm{Crit}_1^{[-\varepsilon,\varepsilon]}, \qquad (r, k_{-\varepsilon}^i) \mapsto k_r^i, \quad i = 1, \dots, n_1,$$

are continuous with respect to the first component. For all $r \in [-\varepsilon, \varepsilon]$ we define the linear maps $\Phi_{0,r}$ and $\Phi_{1,r}$ on generators by

$$\begin{aligned}\Phi_{0,r} : C_0(K_{-\varepsilon}) &\to \langle g_r^1, \dots, g_r^{n_0} \rangle_R, \\ g_{-\varepsilon}^i &\mapsto \Psi_0(r, g_{-\varepsilon}^i) = g_r^i, \quad i = 1, \dots, n_0, \\ \lambda^{\pm 1} &\mapsto \lambda^{\pm 1}, \\ \mu^{\pm 1} &\mapsto \mu^{\pm 1}, \\ \Phi_{1,r} : C_1(K_{-\varepsilon}) &\to \langle k_r^1, \dots, k_r^{n_1} \rangle_{\mathbb{Z}}, \\ k_{-\varepsilon}^i &\mapsto \Psi_1(r, k_{-\varepsilon}^i) = k_r^i, \quad i = 1, \dots, n_1.\end{aligned}$$

For all $r \in [-\varepsilon, \varepsilon]$ we extend $\Phi_{0,r}$ to an algebra homomorphism. Obviously, $\Phi_{0,r}$ and $\Phi_{1,r}$ are isomorphisms for all $r \in [-\varepsilon, \varepsilon]$.

By the same consideration as above, it follows that for ε small enough,

$$\widehat{D}_\varepsilon(k_{\varepsilon,\pm}^i) = \Phi_{0,\varepsilon} \circ \widehat{D}_{-\varepsilon}(k_{-\varepsilon,\pm}^i)$$

for all $i = 1, \dots, n_1$. Thus, we have for all $i = 1, \dots, n_1$,

$$D_\varepsilon(k_\varepsilon^i) = \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(k_{-\varepsilon}^i) = \Phi_{0,\varepsilon} \circ D_{-\varepsilon} \circ \Phi_{1,\varepsilon}^{-1}(k_\varepsilon^i).$$

Now we choose $q \in W_0^u(p)$ such that $\{\varphi_0^s(q) : s \leq 0\} \cap (S \cup F \cup B) = \varnothing$. Analogous to the above consideration it can be shown that there exists an open neighborhood U of $(0, q)$ in $[-\varepsilon, \varepsilon] \times T^2$ such that for all $(r, x) \in U$,

$$\Phi_{0,0} \circ \Phi_{0,r}^{-1} \circ \widehat{D}_r(x) = \widehat{D}_0(q).$$

For ε small enough we can assume that we have $U \cap (\{r\} \times T^2) \neq \varnothing$ for all $r \in [-\varepsilon, \varepsilon]$. Choose $u \in U \cap (\{-\varepsilon\} \times T^2)$. Also choose $k_{\varepsilon,+}$ and $k_{\varepsilon,-}$ to determine $D_\varepsilon(k_\varepsilon)$. Without loss of generality, $k_{\varepsilon,+}$ and $k_{\varepsilon,-}$ can be chosen as shown in Figure 37. If the orientation of the knot is reversed, $k_{\varepsilon,+}$ and $k_{\varepsilon,-}$ are swapped and the sign of $D_\varepsilon(k_\varepsilon)$ changes. However, nothing changes in the quotient. Then we get

$$\Phi_{0,0}(\widehat{D}_{-\varepsilon}(u)) = \widehat{D}_0(q) = \Phi_{0,0} \circ \Phi_{0,\varepsilon}^{-1}(\widehat{D}_\varepsilon(k_{\varepsilon,+})).$$

In addition we get

$$\widehat{D}_\varepsilon(k_{\varepsilon,-}) = g_\varepsilon,$$

since $\dim W_\varepsilon^s(g_\varepsilon) = 2$ and g_ε lies arbitrarily close to k_ε for ε sufficiently small. Therefore, it follows that

$$D_\varepsilon(k_\varepsilon) = \widehat{D}_\varepsilon(k_{\varepsilon,+}) - \widehat{D}_\varepsilon(k_{\varepsilon,-}) = \Phi_{0,\varepsilon}(\widehat{D}_{-\varepsilon}(u)) - g_\varepsilon.$$

Further, we get

$$C_0(K_\varepsilon) = \langle \Phi_{0,\varepsilon}(C_0(K_{-\varepsilon})), g_\varepsilon \rangle_R, \quad C_1(K_\varepsilon) = \langle \Phi_{1,\varepsilon}(C_1(K_{-\varepsilon})), k_\varepsilon \rangle_{\mathbb{Z}}.$$

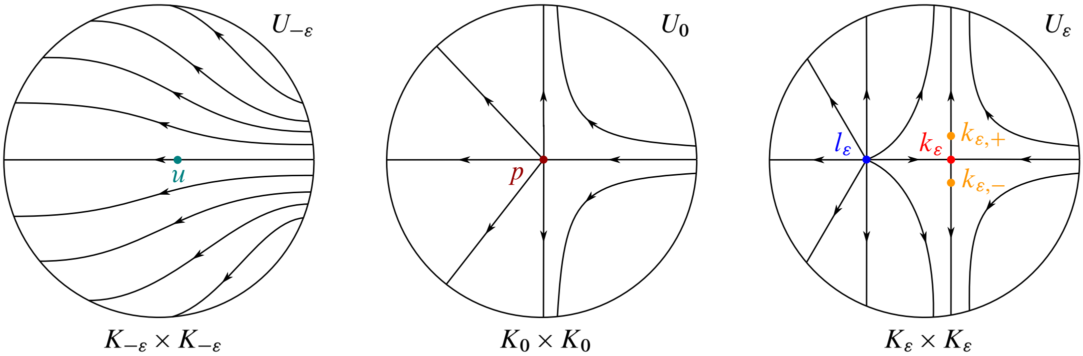

Figure 38: Creation of a pair of critical points of index 1 and 2.

Now we can compute

$$\begin{aligned}
\mathrm{Cord}(K_\varepsilon) &= C_0(K_\varepsilon)/I_\varepsilon \\
&= C_0(K_\varepsilon)/\langle D_\varepsilon(C_1(K_\varepsilon))\rangle \\
&= C_0(K_\varepsilon)/\langle D_\varepsilon(\{k_\varepsilon^1, \dots, k_\varepsilon^{n_1}\}), D_\varepsilon(k_\varepsilon)\rangle \\
&= \langle \Phi_{0,\varepsilon}(C_0(K_{-\varepsilon})), g_\varepsilon\rangle_R / \langle \Phi_{0,\varepsilon} \circ D_{-\varepsilon} \circ \Phi_{1,\varepsilon}^{-1}(\{k_\varepsilon^1, \dots, k_\varepsilon^{n_1}\}), D_\varepsilon(k_\varepsilon)\rangle \\
&= \langle \Phi_{0,\varepsilon}(C_0(K_{-\varepsilon})), g_\varepsilon\rangle_R / \langle \Phi_{0,\varepsilon} \circ D_{-\varepsilon}(\{k_{-\varepsilon}^1, \dots, k_{-\varepsilon}^{n_1}\}), \Phi_{0,\varepsilon}(\hat{D}_{-\varepsilon}(u)) - g_\varepsilon\rangle \\
&= \langle \Phi_{0,\varepsilon}(C_0(K_{-\varepsilon})), g_\varepsilon\rangle_R / \langle \Phi_{0,\varepsilon}(I_{-\varepsilon}), \Phi_{0,\varepsilon}(\hat{D}_{-\varepsilon}(u)) - g_\varepsilon\rangle \\
&\overset{(1)}{=} \Phi_{0,\varepsilon}(C_0(K_{-\varepsilon}))/\Phi_{0,\varepsilon}(I_{-\varepsilon}) \\
&\overset{(2)}{=} \Phi_{0,\varepsilon}(C_0(K_{-\varepsilon})/I_{-\varepsilon}) \\
&= \Phi_{0,\varepsilon}(\mathrm{Cord}(K_{-\varepsilon})) \\
&\cong \mathrm{Cord}(K_{-\varepsilon}).
\end{aligned}$$

Equality (1) holds because on the one hand we have $\hat{D}_{-\varepsilon}(u) \in C_0(K_{-\varepsilon})$ and on the other hand we have, in the quotient, $g_\varepsilon = \Phi_{0,\varepsilon}(\hat{D}_{-\varepsilon}(u))$. It follows that $\langle g_\varepsilon\rangle \subset \Phi_{0,\varepsilon}(C_0(K_{-\varepsilon}))$. Equality (2) holds because $\Phi_{0,\varepsilon}$ is an algebra isomorphism.

(b) Creation of a pair of critical points of index 1 and 2. So let $K_0 \times K_0$ contain a critical point p of birth type of index 1.

With a similar transversality argument as before we can show that p does not lie on the stable or unstable manifold of a critical point of index 1.

We can choose neighborhoods U_r and coordinates such that we can represent the situation as in Figure 38 (see [3, proof of Lemma 9.6]). Here, k_ε and l_ε in the right figure are nondegenerate critical points of index 1 and 2, respectively. We can divide $K_r \times K_r$ into open subsets bounded by "broken" trajectories (see [1, Figure 3.1]) such that $K_r \times K_r$ is the union of the closures of these open sets. Figure 39

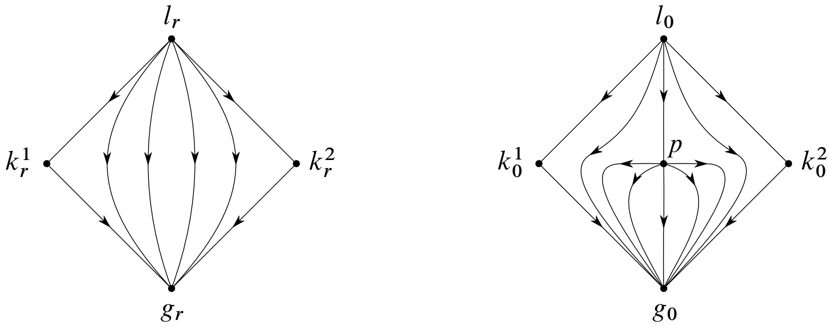

Figure 39: Broken trajectories and position of the critical point p of birth type of index 1.

shows this schematically on the left, where l_r, k_r^1, k_r^2 and g_r are critical points with $\mathrm{Ind}(l_r) = 2$, $\mathrm{Ind}(k_r^1) = \mathrm{Ind}(k_r^2) = 1$, and $\mathrm{Ind}(g_r) = 0$. These open subsets are of the form

$$\{x \in K_r \times K_r : \lim_{s\to\infty} \varphi_r^s(x) = g_r \text{ and } \lim_{s\to-\infty} \varphi_r^s(x) = l_r\}.$$

A "broken" trajectory starts at a critical point of index 2, runs along a stable manifold of a critical point of index 1 to that point, then on the unstable manifold of that point to a critical point of index 0.

So p lies in such an open subset of $K_0 \times K_0$, as shown in Figure 39 on the right. Thus, it is obvious that the trajectories starting from $k_{\varepsilon,+}$ and $k_{\varepsilon,-}$ end at the same critical point g_ε of index 0. If these trajectories intersect the sets S_ε, F_ε, and B in the same way (as well as all flow lines resulting from the application of relation (iv)) such that $\widehat{D}_\varepsilon(k_{\varepsilon,+}) = \widehat{D}_\varepsilon(k_{\varepsilon,-})$, we get $D_\varepsilon(k_\varepsilon) = 0$, and therefore $\mathrm{Cord}(K_{-\varepsilon}) \cong \mathrm{Cord}(K_\varepsilon)$.

If this is not the case, we perturb the 1–parameter family $(E_r)_{r\in[-\varepsilon,\varepsilon]}$ of energy functions with the help of Lemma 2.19 such that we get the 1–parameter family $(E'_r)_{r\in[-\varepsilon,\varepsilon]}$ with the following properties (see also Figure 40):

(i) $E'_{-\varepsilon} = E_{-\varepsilon}$ and $E'_\varepsilon = E_\varepsilon$.

(ii) The flow lines starting at p are arbitrarily close to each other, so they run inside a δ–tube for a $\delta > 0$.

(iii) The flow lines starting at $k'_{r,+}$ and $k'_{r,-}$ run also within a δ–tube for all $r \in (0, \hat{\varepsilon})$ where $0 < \hat{\varepsilon} < \varepsilon$; $\hat{\varepsilon}$ is chosen such that the energy functions E'_r are generic for all $r \in [-\hat{\varepsilon}, \hat{\varepsilon}] \setminus \{0\}$.

Now we choose δ small enough such that the flow lines starting at $k'_{\hat{\varepsilon},+}$ and $k'_{\hat{\varepsilon},-}$ intersect the sets $S_{\hat{\varepsilon}}$, $F_{\hat{\varepsilon}}$ and B in the same way (and also all flow lines that result from the application of relation (iv)). We now determine the cord algebras for the knots $K_r, r \in [-\varepsilon, \varepsilon]$ with respect to the energy functions E'_r, except in the finitely many cases where K_r is nongeneric, and get:

(i) For $r \in [-\varepsilon, 0)$ and $r \in (\hat{\varepsilon}, \varepsilon]$ the cases (ii)(1) to (ii)(13) from Lemma 3.1 may occur. We have shown already that the cord algebra does not change in these cases.

(ii) For $r \in (0, \hat{\varepsilon})$ the cord algebra does not change since only generic energy functions occur.

(iii) Since $D'_{\hat{\varepsilon}}(k'_{\hat{\varepsilon}}) = \widehat{D}'_{\hat{\varepsilon}}(k'_{\hat{\varepsilon},+}) - \widehat{D}'_{\hat{\varepsilon}}(k'_{\hat{\varepsilon},-}) = 0$, the cord algebra stays the same for all $r \in (0, \varepsilon]$.

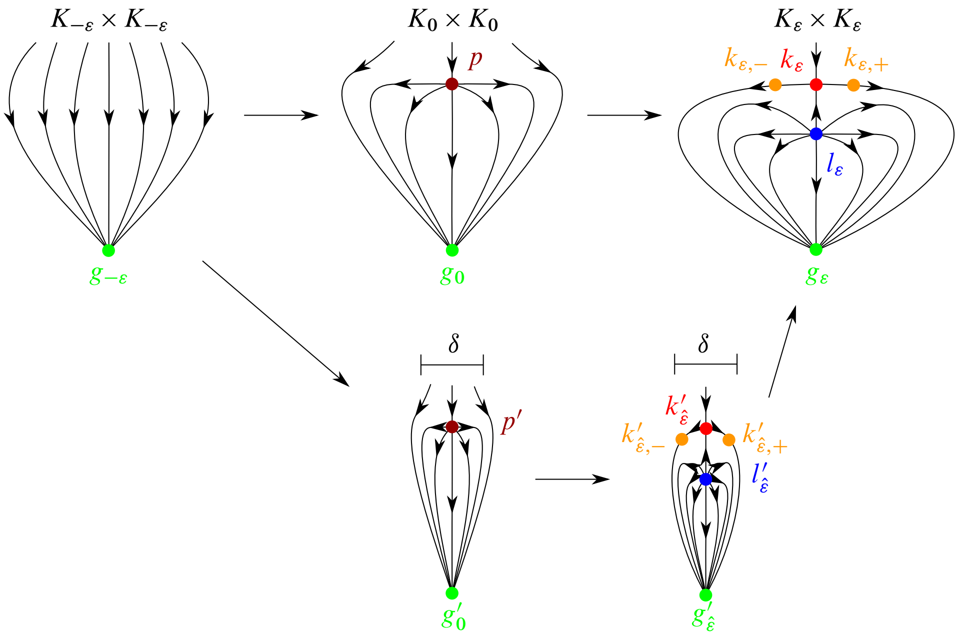

Figure 40: Perturbation of the 1–parameter family of energy functions.

It remains to be shown that the cord algebra does not change at the transition over $r = 0$. For this we use the linear maps as described above:

$$\begin{aligned}
\Phi_{0,r}\colon C_0(K_{-\hat\varepsilon}) &\to C_0(K_r),\\
g^i_{-\hat\varepsilon} &\mapsto \Psi_0(r, g^i_{-\hat\varepsilon}) = g^i_r, \quad i = 1,\dots,n_0,\\
\lambda^{\pm 1} &\mapsto \lambda^{\pm 1},\\
\mu^{\pm 1} &\mapsto \mu^{\pm 1},\\
\Phi_{1,r}\colon C_1(K_{-\hat\varepsilon}) &\to \langle k^1_r,\dots,k^{n_1}_r\rangle_{\mathbb{Z}},\\
k^i_{-\hat\varepsilon} &\mapsto \Psi_1(r, k^i_{-\hat\varepsilon}) = k^i_r, \quad i = 1,\dots,n_1,
\end{aligned}$$

where $r \in [-\hat\varepsilon, \hat\varepsilon]$. By the same consideration as above, it follows that for ε small enough,

$$D'_{\hat\varepsilon}(k^i_{\hat\varepsilon}) = \Phi_{0,\hat\varepsilon} \circ D'_{-\hat\varepsilon}(k^i_{-\hat\varepsilon}) = \Phi_{0,\hat\varepsilon} \circ D'_{-\hat\varepsilon} \circ \Phi^{-1}_{1,\hat\varepsilon}(k^i_{\hat\varepsilon})$$

for all $i = 1,\dots,n_1$. Now we can compute

$$\begin{aligned}
\mathrm{Cord}(K_{\hat\varepsilon}) &= C_0(K_{\hat\varepsilon})/I_{\hat\varepsilon}\\
&= C_0(K_{\hat\varepsilon})/\langle D'_{\hat\varepsilon}(C_1(K_{\hat\varepsilon}))\rangle\\
&= C_0(K_{\hat\varepsilon})/\langle D'_{\hat\varepsilon}(\{k^1_{\hat\varepsilon},\dots,k^{n_1}_{\hat\varepsilon}\}), \underbrace{D'_{\hat\varepsilon}(k'_{\hat\varepsilon})}_{=0}\rangle\\
&= C_0(K_{\hat\varepsilon})/\langle D'_{\hat\varepsilon}(\{k^1_{\hat\varepsilon},\dots,k^{n_1}_{\hat\varepsilon}\})\rangle\\
&= \Phi_{0,\hat\varepsilon}(C_0(K_{-\hat\varepsilon}))/\langle \Phi_{0,\hat\varepsilon} \circ D_{-\hat\varepsilon} \circ \Phi^{-1}_{1,\hat\varepsilon}(\{k^1_{\hat\varepsilon},\dots,k^{n_1}_{\hat\varepsilon}\})\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \Phi_{0,\hat{\varepsilon}}(C_0(K_{-\hat{\varepsilon}}))/\langle \Phi_{0,\hat{\varepsilon}} \circ D_{-\hat{\varepsilon}}(C_1(K_{-\hat{\varepsilon}}))\rangle \\
&= \Phi_{0,\hat{\varepsilon}}(C_0(K_{-\hat{\varepsilon}}))/\Phi_{0,\hat{\varepsilon}}(I_{-\hat{\varepsilon}}) \\
&= \Phi_{0,\hat{\varepsilon}}(C_0(K_{-\hat{\varepsilon}})/I_{-\hat{\varepsilon}}) \\
&= \Phi_{0,\hat{\varepsilon}}(\mathrm{Cord}(K_{-\hat{\varepsilon}})) \\
&\cong \mathrm{Cord}(K_{-\hat{\varepsilon}}). \qquad \square
\end{aligned}
$$

4 Isomorphism of the topological definition of the cord algebra and the definition using Morse theory

In this section we will prove that our definition of the cord algebra is isomorphic to the topological definition given in [2, Definition 2.6].

Theorem 4.1 $$\mathrm{Cord}_{\mathrm{topological}}(K) \cong \mathrm{Cord}_{\mathrm{Morse}}(K).$$

Proof That our definition of the cord algebra agrees with the usual one in the literature follows from results in [2] which we now recall. There is a chain of isomorphisms

$$\mathrm{Cord}(K) \cong H_0(\Sigma, D) \cong H_0(\Sigma_{\mathrm{pl}}, D) \cong H_0(\Sigma_{\mathrm{lin}}, D)$$

where the first isomorphism is [2, Proposition 1.3]. The proof of the second and third isomorphism will be given below. Here $\mathrm{Cord}(K)$ denotes the topological definition of the cord algebra. Σ denotes the space of broken strings, ie words of alternating Q–strings (in $\mathbb{R}^3 \setminus K$) and N–strings (in $N \setminus K$ for a tubular neighborhood N of K), and

$$\Sigma_{\mathrm{lin}} \subset \Sigma_{\mathrm{pl}} \subset \Sigma$$

denote the subspaces whose Q–strings are linear and piecewise linear, respectively. The differential is given by

$$D = \partial + \delta$$

where ∂ is the singular boundary operator, and $\delta = \delta_Q + \delta_N$ is defined by decomposing Q– and N–strings, respectively, when they intersect K in their interior.

Proposition 4.2 $$H_0(\Sigma, D) \cong H_0(\Sigma_{\mathrm{pl}}, D).$$

Proof We have

$$H_0(\Sigma, D) = C_0(\Sigma)/\mathrm{im}\, D(C_1(\Sigma)).$$

The image of D is a two-sided ideal in $C_0(\Sigma)$ (see [2, the third paragraph before Proposition 2.2]). Thus, we can write

$$H_0(\Sigma, D) = C_0(\Sigma)/\langle \mathrm{im}\ D(C_1(\Sigma))\rangle.$$

As a consequence, we have to define

$$H_0(\Sigma_{\mathrm{pl}}, D) := C_0(\Sigma_{\mathrm{pl}})/\langle \mathrm{im}\ D(C_1(\Sigma_{\mathrm{pl}}))\rangle$$

to get an isomorphism. We extend the maps i_{pl} and $\mathbb{F}_0$ to maps

$$\tilde{i}_{\text{pl}}\colon H_0(\Sigma_{\text{pl}}, D) \hookrightarrow H_0(\Sigma, D), \quad \beta \in C_0(\Sigma_{\text{pl}}) \mapsto i_{\text{pl}}\beta,$$

and

$$\widetilde{\mathbb{F}}_0\colon H_0(\Sigma, D) \to H_0(\Sigma_{\text{pl}}, D), \quad \beta \in C_0(\Sigma) \mapsto \mathbb{F}_0\beta.$$

From $\mathbb{F}_0 i_{\text{pl}} = \mathbb{1}$ [2, Proposition 7.1(i), first statement] it follows that $\widetilde{\mathbb{F}}_0 \tilde{i}_{\text{pl}} = \mathbb{1}$.

To prove $\tilde{i}_{\text{pl}}\widetilde{\mathbb{F}}_0 = \mathbb{1}$, we take a $\beta \in C_0(\Sigma)$. From [2, Proposition 7.1(i), second statement], it follows that

$$i_{\text{pl}}\mathbb{F}_0\beta - \beta = D\mathbb{H}_0\beta.$$

Since the right-hand side is an element of $D(C_1(\Sigma))$, we get

$$\tilde{i}_{\text{pl}}\widetilde{\mathbb{F}}_0\beta - \beta = 0. \qquad \square$$

Analogously, we can prove

$$H_0(\Sigma_{\text{pl}}, D) \cong H_0(\Sigma_{\text{lin}}, D).$$

In order to compute $H_0(\Sigma_{\text{lin}}, D)$, in [2, the paragraph preceding Proposition 7.14] the following operations are introduced for $i = 0, 1$ and a large $T > 0$:

$$f^T\colon C_i(\Sigma_{\text{lin}}) \to C_i(\Sigma_{\text{lin}}), \quad H^T\colon C_i(\Sigma_{\text{lin}}) \to C_{i+1}(\Sigma_{\text{lin}}).$$

Here for $\beta \in C_i(\Sigma_{\text{lin}})$ the chains $f^T\beta$ and $H^T\beta$ defined by moving the Q–strings in β for time T, respectively for times in $[0, T]$, under the negative gradient flow of the energy functional $E\colon K \times K \to \mathbb{R}$. The N–strings are dragged along without creating new intersections with K.

Applying [2, (7.6) and (7.8)] to f^T and H^T yields operations

$$\mathbb{F}^T\colon C_i(\Sigma_{\text{lin}}) \to C_i(\Sigma_{\text{lin}}), \quad \mathbb{H}^T\colon C_i(\Sigma_{\text{lin}}) \to C_{i+1}(\Sigma_{\text{lin}}),$$

defined by

$$\mathbb{F}^T = \sum_{d=0}^{\infty} f_d^T, \quad f_d^T = \sum_{i=0}^{d} (H^T\delta)^i f^T (\delta H^T)^{d-i},$$

$$\mathbb{H}^T = \sum_{d=0}^{\infty} H_d^T, \quad H_d^T = H^T(\delta H^T)^d.$$

By [2, (7.9)] they satisfy the relations

$$D\mathbb{H}_0^T = \mathbb{F}_0^T - \mathbb{1}, \quad \mathbb{H}_0^T D + D\mathbb{H}_1^T = \mathbb{F}_1^T - \mathbb{1}.$$

This implies that

$$H_0(\Sigma_{\text{lin}}) \cong \text{im } \mathbb{F}_0^T / \langle \text{im } D\mathbb{F}_1^T \rangle.$$

By definition of $\mathbb{F}_0^T$ and $\mathbb{F}_1^T$, for T sufficiently large we have

$$\text{im}\, \mathbb{F}_0^T = \widetilde{C}_0, \quad \text{im}\, D\mathbb{F}_1^T = \widetilde{D}(\widetilde{C}_1)$$

where

- $\widetilde{C}_i$ is the nc-algebra generated by index i binormal chords connected by N–strings for $i = 0, 1$;

- $\widetilde{D}\colon \widetilde{C}_1 \to \widetilde{C}_0$ is defined like our boundary operator $D\colon C_1(K) \to C_0(K)$, with the difference that the contributions from intersections with the framing and the basepoint are ignored and instead N–strings are dragged along without intersecting K.

Pulling tight the remaining N–strings, recording their intersections with the framing and the basepoint and then removing them, yields the isomorphism

$$\text{im } \mathbb{F}_0^T / \langle \text{im } D\mathbb{F}_1^T \rangle \cong C_0(K)/I_K$$

where the right-hand side is our definition of the cord algebra. Combined with the isomorphisms above this yields the desired isomorphism

$$\text{Cord}(K) \cong C_0(K)/I_K. \qquad \square$$

5 Final remark

Finally, we will briefly compare the topological definition of Ng's cord algebra with our definition using Morse theory.

The original version of the cord algebra is easy to define. However, when calculating the cord algebra for a given knot the following must be considered: if one has found several generators and relations, it is still to be shown that no further generators or relations exist; this proof must be given for each knot individually.

The definition of the cord algebra with the help of Morse theory is very complex, since first some properties of generic knots and a generic framing are to be shown, and then the boundary map is to be defined. As can be seen in the examples, the determination of the individual relations is also laborious and must be carried out very carefully. However, since there are only finitely many critical points of index 1, one has surely found all relations and generators, as soon as one has determined the boundary map for each of these critical points.

Appendix Proof of Lemma 2.14

Remark A.1 Let K be a knot of length L and $\gamma\colon S^1 \cong \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3$ be an arclength parametrization of K. Assume $\ddot{\gamma}(s) \neq 0$ for all $s \in S^1$. In the proof of the lemma it is used that the framing ν satisfies the following conditions:

(i) $\dot{\nu}(s) \neq 0$ at all points $\gamma(s)$ for which a chord exists that is tangent to K at $\gamma(s)$. According to Lemma 2.11, these are only finitely many points.

(ii) Consider the map $\bar{n}\colon S^1 \to S^2$, $s \mapsto \ddot{\gamma}(s)/|\ddot{\gamma}(s)|$. For all points $\gamma(s)$ with $\nu(s) = \bar{n}(s)$, we have $\dot{\nu}(s) \neq \dot{\bar{n}}(s)$.

(iii) For two particular finite sets $R_1, R_2 \subset K \times K$, which are described in more detail in the proof, we have $F^s \cap (R_1 \cup R_2) = \varnothing$.

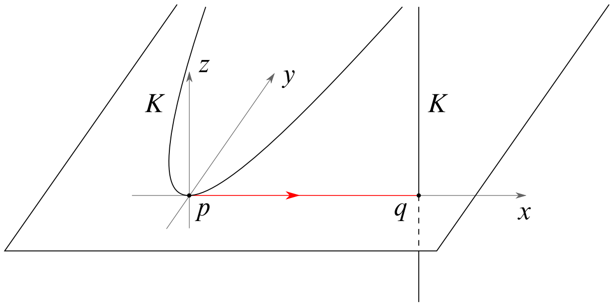

Figure 41: Local model of K in a neighborhood of a boundary point of S.

Proof of Lemma 2.14 Let K be a knot and $\gamma\colon S^1 \to \mathbb{R}^3$ be an arclength parametrization of K. Assume $\ddot{\gamma}(s) \neq 0$ for all $s \in S^1$. Let ν be a framing of K.

(1) Let $(p,q) \in \partial S$ be a chord that is tangent to K at the point $p = \gamma(s)$. We can assume that $\dot{\gamma}(s)$ points in the direction of the chord (p,q), otherwise we can reparametrize. We choose local coordinates (x,y,z) in $\mathbb{R}^3$, where the x–axis points in the direction of the chord (p,q), therefore in the direction of $\dot{\gamma}(s)$, the y–axis points in the direction of $\ddot{\gamma}(s)$, and the z–axis points in the direction of $\dot{\gamma}(s) \times \ddot{\gamma}(s)$ such that we have $p = (0,0,0)$ and $q = (1,0,0)$. Then K near p can be written as a graph over the x–axis (with $y = \kappa x^2 + O(x^3)$ and $z = O(x^3)$) and near q as a graph over the z–axis (with $x = 1 + O(z^2)$ and $y = O(z^2)$). There $2\kappa \neq 0$ is the curvature of K at p (see [2, in the proof of Lemma 7.10]). Note that κ is always positive due to our choice of the local coordinates. So we can assume a local model of K near p and q (see Figure 41):

K can be written near p as a graph over the x–axis,

$$y = \kappa x^2, \quad z = 0;$$

and near q as a graph over the z–axis,

$$x = 1, \quad y = 0.$$

Thus, chords in a neighborhood of (p,q) can be written as $(s(x), t(z)) \in K \times K$ such that

$$\gamma(s(x)) = \begin{pmatrix} x \\ \kappa x^2 \\ 0 \end{pmatrix} \quad \text{and} \quad \gamma(t(z)) = \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix}.$$

With this we have

$$p = \gamma(s(0)) \quad \text{and} \quad q = \gamma(t(0)).$$

The normal plane at the point $\gamma(s(x))$ near p is

$$N(x) = \Big\{ v \in \mathbb{R}^3 : \Big\langle v, \frac{d}{dx}\gamma(s(x)) \Big\rangle = 0 \Big\} = \operatorname{span}\left(\begin{pmatrix} -2\kappa x \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

With

$$\alpha(x) := \begin{pmatrix} -2\kappa x \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

we get

$$N(x) = \operatorname{span}(\alpha(x), \beta).$$

Thus,

$$\langle \gamma(t(z)) - \gamma(s(x)), \alpha(x) \rangle = \left\langle \begin{pmatrix} 1-x \\ -\kappa x^2 \\ z \end{pmatrix}, \begin{pmatrix} -2\kappa x \\ 1 \\ 0 \end{pmatrix} \right\rangle = -2\kappa x + 2\kappa x^2 - \kappa x^2 = \kappa x^2 - 2\kappa,$$

$$\langle \gamma(t(z)) - \gamma(s(x)), \beta \rangle = \left\langle \begin{pmatrix} 1-x \\ -\kappa x^2 \\ z \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = z.$$

We define the map

$$n \colon D^2((p,q)) \setminus \{(p,q)\} \to S^1,$$
$$(s(x), t(z)) \mapsto \frac{\big(\langle \gamma(t(z)) - \gamma(s(x)), \alpha(x) \rangle, \langle \gamma(t(z)) - \gamma(s(x)), \beta \rangle\big)}{\big|\big(\langle \gamma(t(z)) - \gamma(s(x)), \alpha(x) \rangle, \langle \gamma(t(z)) - \gamma(s(x)), \beta \rangle\big)\big|} = \frac{(\kappa x^2 - 2\kappa x, z)}{|(\kappa x^2 - 2\kappa x, z)|}.$$

Now we set $x(u) = \cos u$ and $z(u) = \sin u$ for $u \in [0, 2\pi]$ and consider the map

$$\hat{n} \colon S^1 \to S^1,$$
$$\begin{aligned} u \mapsto n\big(s(x(u)), t(z(u))\big) &= \frac{(\kappa x^2(u) - 2\kappa x(u), z(u))}{|(\kappa x^2(u) - 2\kappa x(u), z(u))|} \\ &= \frac{(\kappa \cos^2 u - 2\kappa \cos u, \sin u)}{|(\kappa \cos^2 u - 2\kappa \cos u, \sin u)|} \\ &= \frac{((\cos u - 2)\kappa \cos u, \sin u)}{|((\cos u - 2)\kappa \cos u, \sin u)|}. \end{aligned}$$

The mapping degree of $\hat{n}$ can be determined immediately:

$$\deg(\hat{n}) = 1,$$

ie (p,q) is a boundary point of F^s and hence we have shown $\partial^s S \subset \partial F^s$.

Let us now consider a framing

$$\nu(x) = a(x)\frac{\alpha(x)}{|\alpha(x)|} + b(x)\beta$$

with $a^2(x) + b^2(x) = 1$. Then

$$(s(x), t(z)) \in F^s \iff \begin{pmatrix} a(x) \\ b(x) \end{pmatrix} = n(s(x), t(z)) = \frac{1}{((\kappa x^2 - 2\kappa x)^2 + z^2)^{\frac{1}{2}}} \begin{pmatrix} \kappa x^2 - 2\kappa x \\ z \end{pmatrix}.$$

If the two equations are squared, both yield the same equation

$$b^2(x)(\kappa x^2 - 2\kappa x)^2 = a^2(x)z^2. \tag{A-1}$$

Case 1 ($a(0), b(0) \neq 0$ or, equivalently, $a(0), b(0) \neq \pm 1$) Then there exists a neighborhood of 0 with $a(x), b(x) \neq 0$ for all x in that neighborhood. In this neighborhood, (A-1) can be solved for z^2 and z can be written as a function of x:

$$z^2(x) = \frac{b^2(x)(\kappa x^2 - 2\kappa x)^2}{a^2(x)}.$$

In addition, it must hold that

$$\begin{aligned} \operatorname{sign}(z(x)) &\overset{!}{=} \operatorname{sign}(b(x)), \quad \text{where } \operatorname{sign}(b(x)) \text{ is constant in a neighborhood of } x = 0, \\ \operatorname{sign}(\kappa x^2 - 2\kappa x) &\overset{!}{=} \operatorname{sign}(a(x)), \quad \text{where } \operatorname{sign}(a(x)) \text{ is constant in a neighborhood of } x = 0. \end{aligned}$$

It follows that

$$\begin{aligned} x > 0 &\quad \text{for } a(0) < 0, \\ x < 0 &\quad \text{for } a(0) > 0. \end{aligned}$$

Thus, in a neighborhood of (p, q), F^s is a smooth curve with boundary point (p, q).

Case 2 ($b(0) = 0$ or, equivalently, $a(0) = \pm 1$) From (A-1) it follows immediately $z = 0$ and it must hold that

$$\operatorname{sign}(\kappa x^2 - 2\kappa x) \overset{!}{=} \operatorname{sign}(a(x)), \quad \text{where } \operatorname{sign}(a(x)) \text{ is constant in a neighborhood of } x = 0.$$

It follows that

$$\begin{aligned} x > 0 &\quad \text{for } a(0) = -1, \\ x < 0 &\quad \text{for } a(0) = 1. \end{aligned}$$

Thus, in a neighborhood of (p, q), F^s is also a smooth curve with boundary point (p, q).

Case 3 ($a(0) = 0$ or, equivalently, $b(0) = \pm 1$) From (A-1) it follows immediately $x = 0$ (the second solution $x = 2$ is not relevant since x is only considered in a small neighborhood of 0) and it must hold that

$$\operatorname{sign}(z) \overset{!}{=} \operatorname{sign}(b(x)), \quad \text{where } \operatorname{sign}(b(x)) \text{ is constant in a neighborhood of } x = 0.$$

It follows that

$$\begin{aligned} z > 0 &\quad \text{for } b(0) = 1, \\ z < 0 &\quad \text{for } b(0) = -1. \end{aligned}$$

Thus, in a neighborhood of (p, q), F^s is also a smooth curve with boundary point (p, q).

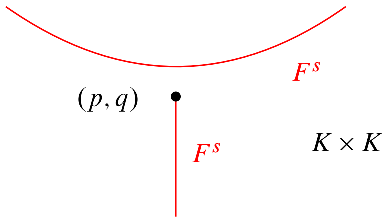

Figure 42: $F^s \subset K \times K$ in a neighborhood of the chord (p,q).

According to condition (i) from Remark A.1, we have $\dot{v}(s(0)) \neq 0$ because the chord (p,q) is tangent to K at $p = \gamma(s(0))$. Since $b(0) = \pm 1$, and thus b reaches a maximum or minimum, we get $b'(0) = 0$. So $a'(0) \neq 0$ is satisfied. Using this and L'Hospital's rule, we can compute

$$\lim_{x \to 0} z^2(x) = \frac{4\kappa^2}{(a'(0))^2}.$$

It follows that

$$\lim_{x \to 0} z(x) = \operatorname{sign}(b(0)) \frac{2\kappa}{|a'(0)|}.$$

This means that F^s can approach the chord (p,q) in $K \times K$ very close, but in this case does not pass through this chord. Together with the above, this results in a picture like the one in Figure 42.

So far we have only considered a local model of K and neglected the terms of higher order. These must now be taken into account in a further step. However, we will not carry out this consideration here in detail, but only note the following:

Let $p = \gamma(s_0)$ and $q = \gamma(t_0)$. Now we only look at chords with starting point $\gamma(s)$ for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$ and endpoint $\gamma(t)$ for $t \in (t_0 - \delta, t_0 + \delta)$, where ε and δ have to be chosen sufficiently small. The above calculations will then be carried out again, taking into account the higher-order terms. In the course of the calculations, ε and δ may have to be reduced several times in order to guarantee that the terms of higher order are small enough. Compare also the proof of Lemma 7.10 in [2].

(2) Now we consider the behavior of F near the diagonal $\Delta \subset K \times K$. Let $p = \gamma(s) \in K$. We choose coordinates (x, y, z) as in (1) such that $(\dot{\gamma}(s), \ddot{\gamma}(s), \dot{\gamma}(s) \times \ddot{\gamma}(s))$ is a basis of this coordinate system. Thus, K can be considered as a graph over the x–axis with $y = \kappa x^2$ and $z = 0$. So, the normal plane at the point $\gamma(s(x))$ is, as in (1),

$$N(x) = \operatorname{span}(\alpha(x), \beta) \quad \text{where } \alpha(x) = \begin{pmatrix} -2\kappa x \\ 1 \\ 0 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We define the following map, which is the projection onto the normal plane $N(x)$ and a normalization:

$$n_x \colon \mathbb{R}^3 \to S^1 \subset N(x), \quad v \mapsto \frac{(\langle v, \alpha(x)/|\alpha(x)| \rangle, \langle v, \beta \rangle)}{|(\langle v, \alpha(x)/|\alpha(x)| \rangle, \langle v, \beta \rangle)|}.$$

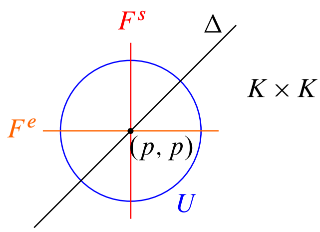

Figure 43: $F \subset K \times K$ in a neighborhood of the chord (p, p).

Let (x_1, x_2), where $x_1 \neq x_2$, be a chord in a neighborhood U of the chord $(p, p) \in \Delta$; see Figure 43. Then

$$n_{x_1}\big(\gamma(s(x_2)) - \gamma(s(x_1))\big) = n_{x_1}\left(\begin{pmatrix} x_2 - x_1 \\ \kappa x_2^2 - \kappa x_1^2 \\ 0 - 0 \end{pmatrix}\right)$$
$$= \frac{(-2\kappa x_2 x_1 + 2\kappa x_1^2 + \kappa x_2^2 - \kappa x_1^2, 0)}{|(-2\kappa x_2 x_1 + 2\kappa x_1^2 + \kappa x_2^2 - \kappa x_1^2, 0)|} = \frac{(\kappa(x_2 - x_1)^2, 0)}{|(\kappa(x_2 - x_1)^2, 0)|} = (1, 0).$$

In the local coordinates of $N(0)$ the following holds: $\ddot{\gamma}(s(0))/|\ddot{\gamma}(s(0))| = (1, 0)$. As a consequence, if $\nu(0) = (1, 0) \in N(0)$, we have $\nu(x) \neq (1, 0)$ in a neighborhood of $x = 0$ according to condition (ii) from Remark A.1, and so we get

$$F^s \cap U = \{(x_1, x_2) \in U : x_1 = 0, x_2 \neq x_1\}.$$

The set F^s is not defined on the diagonal because there the projection of a chord onto the normal plane is the zero vector. However, we can extend F^s in a canonical way on the diagonal (see Figure 43):

$$F^s \cap U = \{(x_1, x_2) \in U : x_1 = 0\}.$$

Assumption (ii) from Remark A.1 is necessary for the following reason: if $\nu(x) \equiv (1, 0)$ in a neighborhood of $x = 0$, $\dim F^s = 2$ would follow because every chord in that neighborhood would intersect the framing.

Also here we will not carry out the consideration of the terms of higher order. However, this is to be realized analogous to the above consideration.

(3) It remains to show that $F^s \setminus (\partial S \cup \Delta)$ is a one-dimensional submanifold.

We define the maps $\nu_1, \nu_2 \in C^\infty(S^1, S^2)$ by

$$\nu_1(s) = \frac{\ddot{\gamma}(s)}{|\ddot{\gamma}(s)|} \quad \text{and} \quad \nu_2(s) = \frac{\dot{\gamma}(s) \times \ddot{\gamma}(s)}{|\dot{\gamma}(s) \times \ddot{\gamma}(s)|}.$$

Thus, we have $(\dot{\gamma}(s), \nu_1(s), \nu_2(s))$ as an orthonormal basis of $\mathbb{R}^3$ for all $s \in S^1$. So the framing can be written as

$$\nu\colon S^1 \to S^2, \quad s \mapsto a_1(s)\nu_1(s) + a_2(s)\nu_2(s),$$

where $(a_1, a_2) =: \hat{a} \in C^\infty(S^1, S^1)$. We define the map

$$a\colon S^1 \to S^1 \times S^1, \quad s \mapsto (s, \hat{a}(s)).$$

We also define the following map, which is the orthogonal projection of a chord onto the normal plane at the starting point of the chord and a normalization:

$$\hat{n}\colon T^2 \setminus (\partial S \cup \Delta) \to S^1, \quad (t,u) \mapsto \frac{\begin{pmatrix} \langle \gamma(u)-\gamma(t), \nu_1(t)\rangle \\ \langle \gamma(u)-\gamma(t), \nu_2(t)\rangle \end{pmatrix}}{\left|\begin{pmatrix} \langle \gamma(u)-\gamma(t), \nu_1(t)\rangle \\ \langle \gamma(u)-\gamma(t), \nu_2(t)\rangle \end{pmatrix}\right|}.$$

This map is well defined since the boundary of S and the diagonal are excluded. According to Definition 2.8, a chord (t,u) intersects the framing at its starting point if and only if $\hat{n}(t,u)=\hat{a}(t)$. We consider the map

$$n\colon T^2 \setminus (\partial S \cup \Delta) \to S^1 \times S^1, \quad (t,u) \mapsto (t, \hat{n}(t,u)).$$

We want to show that a is transverse to n. This holds if and only if the map

$$a \times n\colon S^1 \times (T^2 \setminus (\partial S \cup \Delta)) \to (S^1 \times S^1) \times (S^1 \times S^1), \quad (s,t,u) \mapsto (a(s), n(t,u)) = (s, \hat{a}(s), t, \hat{n}(t,u)),$$

is transverse to $\Delta_{(S^1\times S^1)\times(S^1\times S^1)}$, where

$$\Delta_{(S^1\times S^1)\times(S^1\times S^1)} = \{(s,t,u,v) \in (S^1 \times S^1) \times (S^1 \times S^1) : (s,t) = (u,v)\}$$

is the diagonal in $(S^1 \times S^1) \times (S^1 \times S^1)$. The tangent space at points of this diagonal is

$$T_{(s,t,s,t)}\Delta_{(S^1\times S^1)\times(S^1\times S^1)} = \operatorname{span}\left(\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}\right).$$

In addition,

$$D_{(s,t,u)}(a \times n) = \begin{pmatrix} 1 & 0 & 0 \\ \dot{\hat{a}}(s) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{\partial \hat{n}}{\partial t}(t,u) & \frac{\partial \hat{n}}{\partial u}(t,u) \end{pmatrix}.$$

The derivatives of $\hat{a}$ and $\hat{n}$ each have two components, but can be regarded as real-valued by the identification $S^1 \cong \mathbb{R}/\mathbb{Z}$. The matrix is to be understood in this sense. Furthermore,

$$(a \times n)(s,t,u) \in \Delta_{(S^1\times S^1)\times(S^1\times S^1)} \iff s = t, \hat{a}(s) = \hat{n}(t,u).$$

Thus, $a \times n$ is transverse to the diagonal $\Delta_{(S^1\times S^1)\times(S^1\times S^1)}$ if and only if for all points (s,t,s,t) in $(S^1 \times S^1) \times (S^1 \times S^1)$ with $(s,t,s,t) = (a \times n)(s,s,u)$ the following holds:

$$\operatorname{span}\left(\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\ \dot{\hat{a}}(s)\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\ \frac{\partial \hat{n}}{\partial s}(s,u)\end{pmatrix}, \begin{pmatrix}0\\0\\0\\ \frac{\partial \hat{n}}{\partial u}(s,u)\end{pmatrix}\right) = \mathbb{R}^4.$$

This is satisfied if

$$\text{(A-2)} \qquad \dot{\hat{a}}(s) \neq \frac{\partial \hat{n}}{\partial s}(s,u) \quad \text{or} \quad \dot{\hat{a}}(s) = \frac{\partial \hat{n}}{\partial s}(s,u) \quad \text{and} \quad \frac{\partial \hat{n}}{\partial u}(s,u) \neq 0.$$

Assuming this, $(a\times n)^{-1}(\Delta_{(S^1\times S^1)\times(S^1\times S^1)})$ is a one-dimensional submanifold of $S^1\times(T^2\setminus(\partial S\cup\Delta))$ since $\operatorname{codim}(\Delta_{(S^1\times S^1)\times(S^1\times S^1)}\subset(S^1\times S^1)\times(S^1\times S^1))=2$. The following holds:

$$\begin{aligned}(a\times n)^{-1}(\Delta_{(S^1\times S^1)\times(S^1\times S^1)}) &= \{(s,t,u):a(s)=n(t,u)\}\\ &= \{(s,t,u):(s,\hat a(s))=(t,\hat n(t,u))\}\\ &= \{(s,s,u):(s,\hat a(s))=(s,\hat n(s,u))\}.\end{aligned}$$

The latter set can be considered as

$$\{(s,u)\in T^2\setminus(\partial S\cup\Delta):\hat a(s)=\hat n(s,u)\},$$

ie the set of all chords that intersect the framing at its starting point. As a consequence,

$$F^s\setminus(\partial S\cup\Delta)\subset K\times K$$

is a one-dimensional submanifold.

Together with the results from (1) and (2) the following holds: $F^s\subset K\times K$ is a one-dimensional submanifold with boundary and $\partial F^s=\partial^s S$.

So it remains to show that condition (A-2) holds generically. For this we want to show the following:

(i) $M_1:=\left\{(s,u):\frac{\partial\hat n}{\partial u}(s,u)=0\right\}\subset T^2\setminus(\partial S\cup\Delta)$ is a one-dimensional submanifold.

(ii) Let $M_2:=\left\{(s,u):\hat a(s)=\hat n(s,u),\dot{\hat a}(s)=\frac{\partial\hat n}{\partial s}(s,u)\right\}\subset T^2\setminus(\partial S\cup\Delta)$. Then $M_1\cap M_2=\varnothing$.

Proof of these two statements:

(i) In Figure 44 we can see that

$$\begin{aligned}\frac{\partial\hat n}{\partial u}(s,u)=0 &\iff \frac{\partial(\gamma(u)-\gamma(s))}{\partial u}\in\operatorname{span}(\dot\gamma(s),\hat n(s,u))\\ &\iff \dot\gamma(u)\in\operatorname{span}(\dot\gamma(s),\hat n(s,u))\\ &\iff \langle\dot\gamma(u),\hat n(s,u)\times\dot\gamma(s)\rangle=0\\ &\iff \langle\dot\gamma(u),(\gamma(u)-\gamma(s))\times\dot\gamma(s)\rangle=0\\ &\iff \langle\dot\gamma(s)\times\dot\gamma(u),\gamma(u)-\gamma(s)\rangle=0.\end{aligned}$$

We define the maps (for k big enough)

$$\begin{aligned}f:C^k(S^1,\mathbb{R}^3)\times(T^2\setminus(\partial S\cup\Delta))\to\mathbb{R},&\quad (\gamma,s,u)\mapsto\langle\dot\gamma(s)\times\dot\gamma(u),\gamma(u)-\gamma(s)\rangle,\\ f_\gamma:T^2\setminus(\partial S\cup\Delta)\to\mathbb{R},&\quad (s,u)\mapsto f(\gamma,s,u).\end{aligned}$$

We want to show that 0 is a regular value for f_γ. First, we show that the map

$$D_{(\gamma,s,u)}f:C^k(S^1,\mathbb{R}^3)\times\mathbb{R}^2\to\mathbb{R}$$

is surjective for all (γ,s,u) with $f(\gamma,s,u)=0$:

Let (γ,s,u) be such that $f(\gamma,s,u)=\langle\dot\gamma(s)\times\dot\gamma(u),\gamma(u)-\gamma(s)\rangle=0$.

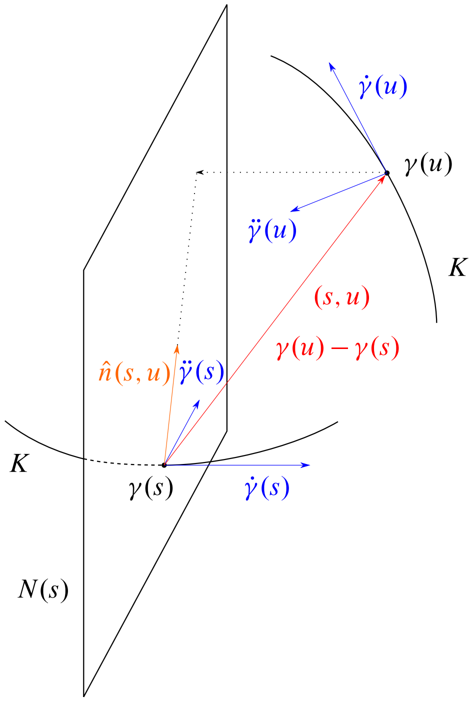

Figure 44: The chord (s,u) with the first and second derivatives of the parametrization γ at the start and endpoint of the chord and $\hat{n}(s,u)$.

We assume that $D_{(\gamma,s,u)}f$ is the zero map and determine $D_{(\gamma,s,u)}f\cdot(\hat{\gamma},\hat{s},\hat{u})$:

$$\frac{\partial f}{\partial s}(\gamma,s,u)\cdot\hat{s}=\langle\ddot{\gamma}(s)\times\dot{\gamma}(u),\gamma(u)-\gamma(s)\rangle\hat{s},$$
$$\frac{\partial f}{\partial u}(\gamma,s,u)\cdot\hat{u}=\langle\dot{\gamma}(s)\times\ddot{\gamma}(u),\gamma(u)-\gamma(s)\rangle\hat{u},$$
$$\frac{\partial f}{\partial \gamma}(\gamma,s,u)\cdot\hat{\gamma}=\langle\dot{\hat{\gamma}}(s)\times\dot{\gamma}(u),\gamma(u)-\gamma(s)\rangle+\langle\dot{\gamma}(s)\times\dot{\hat{\gamma}}(u),\gamma(u)-\gamma(s)\rangle+\langle\dot{\gamma}(s)\times\dot{\gamma}(u),\hat{\gamma}(u)-\hat{\gamma}(s)\rangle.$$

Since $\langle\dot{\gamma}(s)\times\dot{\gamma}(u),\gamma(u)-\gamma(s)\rangle=0$, we can also write

$$\frac{\partial f}{\partial s}(\gamma,s,u)=\langle(\ddot{\gamma}(s)+\lambda\dot{\gamma}(s))\times\dot{\gamma}(u),\gamma(u)-\gamma(s)\rangle,$$
$$\frac{\partial f}{\partial u}(\gamma,s,u)=\langle\dot{\gamma}(s)\times(\ddot{\gamma}(u)+\mu\dot{\gamma}(u)),\gamma(u)-\gamma(s)\rangle$$

for any $\lambda,\mu\in\mathbb{R}$.

Assuming $\gamma(u)-\gamma(s)=0$, it follows that $s=u$ and hence $(s,u)\in\Delta$. However, this is excluded by the definition of f.

Assuming $(\gamma(u)-\gamma(s))\parallel\dot{\gamma}(u)$, it follows that the chord (s,u) is tangent to K. Thus, we have $(s,u)\in\partial S$. However, this is also excluded by the definition of f. The case $(\gamma(u)-\gamma(s))\parallel\dot{\gamma}(s)$ is analogous.

Assume that $\ddot{\gamma}(s), \dot{\gamma}(s)$ and $\dot{\gamma}(u)$ are linearly independent. The set

$$A := \{(\ddot{\gamma}(s) + \lambda\dot{\gamma}(s)) \times \dot{\gamma}(u) : \lambda \in \mathbb{R}\}$$

describes a straight line with $0 \notin A$. So $\operatorname{span}(A)$ is a plane through the origin with $\operatorname{span}(A) \perp \dot{\gamma}(u)$. Thus,

$$\langle(\ddot{\gamma}(s) + \lambda\dot{\gamma}(s)) \times \dot{\gamma}(u), \gamma(u) - \gamma(s)\rangle = 0 \ \forall\ \lambda \in \mathbb{R} \iff (\gamma(u) - \gamma(s)) \parallel \dot{\gamma}(u).$$

It follows that $(s, u) \in \partial S$. This is a contradiction to the definition of f. So $\ddot{\gamma}(s), \dot{\gamma}(s)$ and $\dot{\gamma}(u)$ are linearly dependent and we get

$$\dot{\gamma}(u) = \lambda_1\ddot{\gamma}(s) + \lambda_2\dot{\gamma}(s) \text{ with } \lambda_1, \lambda_2 \in \mathbb{R}$$

since $\ddot{\gamma}(s)$ and $\dot{\gamma}(s)$ are linearly independent. It follows that the set

$$\{(\ddot{\gamma}(s) + \lambda\dot{\gamma}(s)) \times \dot{\gamma}(u) : \lambda \in \mathbb{R}\} = \{(\lambda\lambda_1 - \lambda_2)\dot{\gamma}(s) \times \ddot{\gamma}(s) : \lambda \in \mathbb{R}\}$$

is a straight line through the origin. Since

$$\frac{\partial f}{\partial s}(\gamma, s, u) = \langle(\ddot{\gamma}(s) + \lambda\dot{\gamma}(s)) \times \dot{\gamma}(u), \gamma(u) - \gamma(s)\rangle = 0$$

for all $\lambda \in \mathbb{R}$ according to the assumption, it follows that

$$\gamma(u) - \gamma(s) \in \operatorname{span}(\dot{\gamma}(s), \ddot{\gamma}(s)),$$

hence

$$\gamma(u) - \gamma(s) = \lambda_3\ddot{\gamma}(s) + \lambda_4\dot{\gamma}(s) \quad \text{with } \lambda_3, \lambda_4 \in \mathbb{R}.$$

Now we can rewrite the three summands of $\frac{\partial f}{\partial \gamma}(\gamma, s, u) \cdot \hat{\gamma}$:

$$\begin{aligned}
\langle\dot{\hat{\gamma}}(s) \times \dot{\gamma}(u), \gamma(u) - \gamma(s)\rangle &= (\lambda_2\lambda_3 - \lambda_1\lambda_4)\langle\dot{\gamma}(s) \times \ddot{\gamma}(s), \dot{\hat{\gamma}}(s)\rangle,\\
\langle\dot{\gamma}(s) \times \dot{\hat{\gamma}}(u), \gamma(u) - \gamma(s)\rangle &= -\lambda_3\langle\dot{\gamma}(s) \times \ddot{\gamma}(s), \dot{\hat{\gamma}}(u)\rangle,\\
\langle\dot{\gamma}(s) \times \dot{\gamma}(u), \hat{\gamma}(u) - \hat{\gamma}(s)\rangle &= \lambda_1\langle\dot{\gamma}(s) \times \ddot{\gamma}(s), \hat{\gamma}(u) - \hat{\gamma}(s)\rangle.
\end{aligned}$$

All in all, with $\bar{\lambda} := \lambda_2\lambda_3 - \lambda_1\lambda_4$ we get

$$\frac{\partial f}{\partial \gamma}(\gamma, s, u) \cdot \hat{\gamma} = \langle\dot{\gamma}(s) \times \ddot{\gamma}(s), \bar{\lambda}\dot{\hat{\gamma}}(s) - \lambda_3\dot{\hat{\gamma}}(u) + \lambda_1(\hat{\gamma}(u) - \hat{\gamma}(s))\rangle.$$

We have $\lambda_3 \neq 0$, otherwise $\gamma(u) - \gamma(s) = \lambda_4\dot{\gamma}(s)$, ie $(s, u) \in \partial S$. So we can choose $\hat{\gamma}$ such that $\dot{\hat{\gamma}}(u) = \dot{\gamma}(s) \times \ddot{\gamma}(s)$ and $\dot{\hat{\gamma}}(s) = \hat{\gamma}(u) - \hat{\gamma}(s) = 0$. With this choice we have

$$\frac{\partial f}{\partial \gamma}(\gamma, s, u) \cdot \hat{\gamma} \neq 0$$

and therefore the map $D_{(\gamma,s,u)} f$ is not the zero map. Hence $D_{(\gamma,s,u)} f$ is surjective.

Thus, 0 is a regular value for f and $f^{-1}(0)$ is a Banach manifold (see McDuff and Salamon [9, proof of Theorem 3.1.6] which is based on an infinite dimensional version of Sard's theorem). The projection map $\mathrm{pr}_1 \colon f^{-1}(0) \to C^k(S^1, \mathbb{R}^3)$, for k big enough, is smooth. According to the Sard–Smale theorem, almost

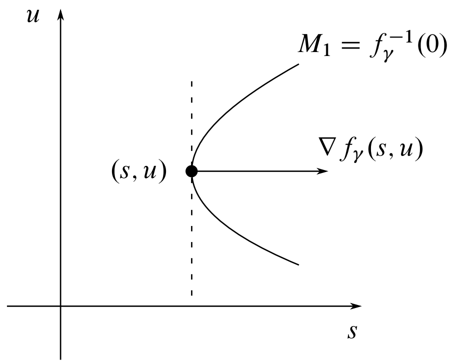

Figure 45: A point $(s,u) \in M_1$ with $\frac{\partial f_\gamma}{\partial u}(s,u) = 0$.

all points in $C^k(S^1, \mathbb{R}^3)$ are regular values for pr_1, so the parametrization γ of K is generically a regular value. Thus, $\mathrm{pr}_1^{-1}(\gamma) \subset f^{-1}(0)$ is a Banach submanifold and 0 is a regular value for f_γ (see [9, proof of Theorem 3.1.6]). So $M_1 = f_\gamma^{-1}(0)$ is a one-dimensional submanifold and the assertion (i) is shown.

(ii) Let $(s,u) \in M_1$.

(a) First, we consider the case

$$\frac{\partial f_\gamma}{\partial u}(s,u) = 0,$$

ie in a neighborhood of s the set M_1 cannot be written as a graph over the s–axis; see Figure 45. To simplify the notation we define the function $g_\gamma := \partial f_\gamma / \partial u$. So,

$$\begin{Bmatrix} f_\gamma(s,u) = \langle \dot\gamma(s) \times \dot\gamma(u), \gamma(u) - \gamma(s)\rangle = 0 \\ g_\gamma(s,u) = \langle \dot\gamma(s) \times \ddot\gamma(u), \gamma(u) - \gamma(s)\rangle = 0 \end{Bmatrix}$$

$$\iff \begin{Bmatrix} \dot\gamma(s) \in \mathrm{span}(\dot\gamma(u), \gamma(u) - \gamma(s)) \\ \ddot\gamma(u) \in \mathrm{span}(\dot\gamma(s), \gamma(u) - \gamma(s)) \overset{(s,u)\notin\partial S}{=} \mathrm{span}(\dot\gamma(u), \gamma(u) - \gamma(s)) \end{Bmatrix}$$

$$\iff \begin{Bmatrix} \dot\gamma(s) = \mu_1 \dot\gamma(u) + \mu_2(\gamma(u) - \gamma(s)),\ \mu_1, \mu_2 \in \mathbb{R},\ \mu_1 \neq 0 \\ \ddot\gamma(u) = \mu_3 \dot\gamma(u) + \mu_4(\gamma(u) - \gamma(s)),\ \mu_3, \mu_4 \in \mathbb{R},\ \mu_4 \neq 0 \end{Bmatrix}$$

where μ_1 must not vanish, otherwise we have $(s,u) \in \partial S$, and μ_4 must not vanish, since $\dot\gamma(u)$ and $\ddot\gamma(u)$ are linearly independent.

We define the function

$$h_\gamma \colon T^2 \setminus (\partial S \cup \Delta) \to \mathbb{R}^2, \quad (s,u) \mapsto (f_\gamma(s,u), g_\gamma(s,u)),$$

and want to show that $h_\gamma^{-1}((0,0))$ is a finite set for generic knots. This holds if $D_{(s,u)} h_\gamma$ is surjective for all (s,u) with $h_\gamma(s,u) = (0,0)$. Let $(s,u) \in T^2 \setminus (\partial S \cup \Delta)$ with $h_\gamma(s,u) = (0,0)$. Then

$$D_{(s,u)} h_\gamma = \begin{pmatrix} \langle \ddot\gamma(s) \times \dot\gamma(u), \gamma(u) - \gamma(s)\rangle & \mu_3 \langle \ddot\gamma(s) \times \dot\gamma(u), \gamma(u) - \gamma(s)\rangle \\ 0 & \mu_1 \langle \dot\gamma(u) \times \dddot\gamma(u), \gamma(u) - \gamma(s)\rangle \end{pmatrix}.$$

So $D_{(s,u)}h_\gamma$ is surjective, ie $\det(D_{(s,u)}h_\gamma) \neq 0$, if and only if

$$\langle \ddot{\gamma}(s) \times \dot{\gamma}(u), \gamma(u) - \gamma(s) \rangle \neq 0 \quad \text{and} \quad \langle \dot{\gamma}(u) \times \ddot{\gamma}(u), \gamma(u) - \gamma(s) \rangle \neq 0,$$

since $\mu_1 \neq 0$.

First, we look at $\langle \dot{\gamma}(u) \times \ddot{\gamma}(u), \gamma(u) - \gamma(s) \rangle$. We will need the torsion of the knot at the point $\gamma(u)$. Recall that for all $u \in S^1$ the torsion $\tau(u) \in \mathbb{R}$ is the unique real number for which $\dot{b}(u) = \tau(u)\bar{n}(u)$ holds, where $\bar{n}(u) := \ddot{\gamma}(u)/|\ddot{\gamma}(u)|$ and $b(u) := \dot{\gamma}(u) \times \bar{n}(u)$. We calculate

$$\dot{b}(u) = \dot{\gamma}(u) \times \frac{|\ddot{\gamma}(u)|^2 \dddot{\gamma}(u) - \langle \ddot{\gamma}(u), \dddot{\gamma}(u) \rangle \ddot{\gamma}(u)}{|\ddot{\gamma}(u)|^3},$$

and then we can write

$$\tau(u)\ddot{\gamma}(u) = \dot{\gamma}(u) \times \dddot{\gamma}(u) - \frac{\langle \ddot{\gamma}(u), \dddot{\gamma}(u) \rangle}{|\ddot{\gamma}(u)|^2} \dot{\gamma}(u) \times \ddot{\gamma}(u).$$

Therefore,

$$\begin{aligned} \langle \dot{\gamma}(u) \times \dddot{\gamma}(u), \gamma(u) - \gamma(s) \rangle &= \tau(u)\langle \ddot{\gamma}(u), \gamma(u) - \gamma(s) \rangle + \frac{\langle \ddot{\gamma}(u), \dddot{\gamma}(u) \rangle}{|\ddot{\gamma}(u)|^2} \underbrace{\langle \dot{\gamma}(u) \times \ddot{\gamma}(u), \gamma(u) - \gamma(s) \rangle}_{=0 \text{ since } \ddot{\gamma}(u) \in \operatorname{span}(\dot{\gamma}(u), \gamma(u) - \gamma(s))} \\ &= \tau(u)\langle \ddot{\gamma}(u), \gamma(u) - \gamma(s) \rangle. \end{aligned}$$

According to the above consideration, we have $\ddot{\gamma}(u) \in \operatorname{span}(\dot{\gamma}(u), \gamma(u) - \gamma(s))$. Besides, $\ddot{\gamma}(u)$ and $\dot{\gamma}(u)$ are linearly independent. Therefore,

$$\gamma(u) - \gamma(s) \in \operatorname{span}(\dot{\gamma}(u), \ddot{\gamma}(u)),$$

and thus

$$\gamma(u) - \gamma(s) = \mu_5 \ddot{\gamma}(u) + \mu_6 \dot{\gamma}(u), \quad \mu_5, \mu_6 \in \mathbb{R},\ \mu_5 \neq 0,$$

where μ_5 must not vanish, otherwise the chord (s,u) would be tangent to K. It follows that

$$\tau(u)\langle \ddot{\gamma}(u), \gamma(u) - \gamma(s) \rangle = \tau(u)\langle \ddot{\gamma}(u), \mu_5 \ddot{\gamma}(u) + \mu_6 \dot{\gamma}(u) \rangle = \tau(u)\mu_5 |\ddot{\gamma}(u)|^2.$$

This expression does not vanish if $\tau(u) \neq 0$. For a generic knot $\tau(u) = 0$ holds for only finitely many $u \in S^1$. So let $\{\bar{u}_i : i = 1, \ldots, N_u\} \subset S^1$, for an $N_u \in \mathbb{N}$, be the set of points for which $\tau(\bar{u}_i) = 0$ holds. Consider the set

$$\begin{aligned} \widetilde{R}_1 &:= \left\{ (s,u) \in M_1 : \frac{\partial f_\gamma}{\partial u}(s,u) = 0, \tau(u) = 0 \right\} \\ &= \{(s,u) \in T^2 \setminus (\partial^s S \cup \Delta) : h_\gamma(s,u) = 0, \tau(u) = 0\} \\ &= \{(s,\bar{u}_i) \in T^2 \setminus (\partial^s S \cup \Delta) : h_\gamma(s,\bar{u}_i) = 0, i \in \{1, \ldots, N_u\}\}. \end{aligned}$$

We can choose open neighborhoods of $\partial^s S$ and the diagonal of $K \times K$ of which we already know that F^s is a submanifold within these neighborhoods. Let U be the union of these neighborhoods. Thus, it suffices to consider the set

$$R_1 := \widetilde{R}_1 \cap (T^2 \setminus U).$$

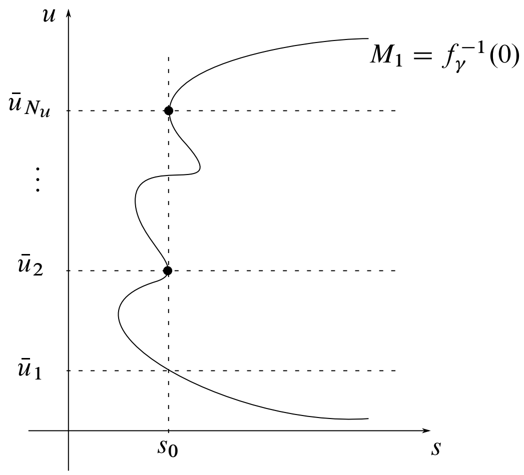

Figure 46: The set R_1 is finite.

We now argue that R_1 is a finite set. We have $M_1 = f_\gamma^{-1}(0)$. Therefore, for all $(s,u) \in f_\gamma^{-1}(0)$, we have $\nabla f_\gamma(s,u) \perp M_1$. If $(s,\bar{u}_i) \in M_1$, then

$$\frac{\partial f_\gamma}{\partial u}(s,\bar{u}_i) = 0$$

according to the assumption. So M_1 is tangent to the straight line $\{(s,u) : s = \text{const}\}$ at the point $(s,\bar{u}_i) \in R_1$; see Figure 46. Since the set $\{\bar{u}_i : i = 1,\dots,N_u\} \subset S^1$ is finite, no further point in R_1 can be contained in a sufficiently small neighborhood of $(s,\bar{u}_i)$. Therefore, R_1 is a discrete set. Since $M_1 \cap (T^2 \setminus U)$ is compact, R_1 is finite. In the example of Figure 46 we have $(s_0,\bar{u}_2), (s_0,\bar{u}_{N_u}) \in R_1$.

Now let's look at the entry

$$\frac{\partial f_\gamma}{\partial s}(s,u) = \langle \ddot{\gamma}(s) \times \dot{\gamma}(u), \gamma(u) - \gamma(s) \rangle$$

in $D_{(s,u)}h_\gamma$. According to the assumption, we have $f_\gamma(s,u) = \langle \dot{\gamma}(s) \times \dot{\gamma}(u), \gamma(u) - \gamma(s) \rangle = 0$. It follows that

$$\frac{\partial f_\gamma}{\partial s}(s,u) = \langle (\ddot{\gamma}(s) + \lambda\dot{\gamma}(s)) \times \dot{\gamma}(u), \gamma(u) - \gamma(s) \rangle$$

for all $\lambda \in \mathbb{R}$. Assume

$$\frac{\partial f_\gamma}{\partial s}(s,u) = 0.$$

Then we get, as in (i),

$$\gamma(u) - \gamma(s) = \lambda_3\ddot{\gamma}(s) + \lambda_4\dot{\gamma}(s), \quad \lambda_3,\lambda_4 \in \mathbb{R},\ \lambda_3 \neq 0.$$

Thus, we can write

$$0 = f_\gamma(s,u) = \lambda_3\langle \dot{\gamma}(s) \times \dot{\gamma}(u), \ddot{\gamma}(s) \rangle \quad \text{and} \quad 0 = g_\gamma(s,u) = \lambda_3\langle \dot{\gamma}(s) \times \ddot{\gamma}(u), \ddot{\gamma}(s) \rangle,$$

or, since $\lambda_3 \neq 0$,

$$\langle \dot{\gamma}(s) \times \dot{\gamma}(u), \ddot{\gamma}(s) \rangle = 0 \quad \text{and} \quad \langle \dot{\gamma}(s) \times \ddot{\gamma}(u), \ddot{\gamma}(s) \rangle = 0.$$

Therefore,

$$\dot\gamma(u), \ddot\gamma(u), \gamma(u)-\gamma(s) \in \operatorname{span}(\dot\gamma(s), \ddot\gamma(s)).$$

We define the maps (for k big enough)

$$\tilde f\colon C^k(S^1,\mathbb{R}^3)\times(T^2\setminus(\partial S\cup\Delta))\to\mathbb{R}^3,\quad (\gamma,s,u)\mapsto\begin{pmatrix}\langle\dot\gamma(s)\times\ddot\gamma(s),\dot\gamma(u)\rangle\\ \langle\dot\gamma(s)\times\ddot\gamma(s),\ddot\gamma(u)\rangle\\ \langle\dot\gamma(s)\times\ddot\gamma(s),\gamma(u)-\gamma(s)\rangle\end{pmatrix},$$

$$\tilde f_\gamma\colon T^2\setminus(\partial S\cup\Delta)\to\mathbb{R}^3,\qquad (s,u)\mapsto f(\gamma,s,u).$$

We want to show that 0 is a regular value for $\tilde f_\gamma$. First, we show that the map

$$D_{(\gamma,s,u)}\tilde f\colon C^k(S^1,\mathbb{R}^3)\times\mathbb{R}^2\to\mathbb{R}^3$$

is surjective for all (γ,s,u) with $\tilde f(\gamma,s,u)=0$:

Let (γ,s,u) be such that $\tilde f(\gamma,s,u)=0$. Let $(\hat\gamma,0,0)\in C^k(S^1,\mathbb{R}^3)\times\mathbb{R}^2$. We determine

$$\begin{aligned}D_{(\gamma,s,u)}\tilde f\cdot(\hat\gamma,0,0)&=\frac{\partial\tilde f}{\partial\gamma}(\gamma,s,u)\cdot\hat\gamma\\ &=\begin{pmatrix}\langle\dot{\hat\gamma}(s)\times\ddot\gamma(s)+\dot\gamma(s)\times\ddot{\hat\gamma}(s),\dot\gamma(u)\rangle+\langle\dot\gamma(s)\times\ddot\gamma(s),\dot{\hat\gamma}(u)\rangle\\ \langle\dot{\hat\gamma}(s)\times\ddot\gamma(s)+\dot\gamma(s)\times\ddot{\hat\gamma}(s),\ddot\gamma(u)\rangle+\langle\dot\gamma(s)\times\ddot\gamma(s),\ddot{\hat\gamma}(u)\rangle\\ \langle\dot{\hat\gamma}(s)\times\ddot\gamma(s)+\dot\gamma(s)\times\ddot{\hat\gamma}(s),\gamma(u)-\gamma(s)\rangle+\langle\dot\gamma(s)\times\ddot\gamma(s),\hat\gamma(u)-\hat\gamma(s)\rangle\end{pmatrix}.\end{aligned}$$

If we choose $\hat\gamma_1$, $\hat\gamma_2$, and $\hat\gamma_3$ such that $\dot{\hat\gamma}_i(s)=\ddot{\hat\gamma}_i(s)=0$, for $i=1,2,3$, and

- $\dot{\hat\gamma}_1(u)=\dot\gamma(s)\times\ddot\gamma(s)$ and $\ddot{\hat\gamma}_1(u)=\hat\gamma_1(u)-\hat\gamma_1(s)=0$, we get
$$D_{(\gamma,s,u)}\tilde f\cdot(\hat\gamma_1,0,0)=\begin{pmatrix}|\dot\gamma(s)\times\ddot\gamma(s)|^2\\0\\0\end{pmatrix};$$
- $\ddot{\hat\gamma}_2(u)=\dot\gamma(s)\times\ddot\gamma(s)$ and $\dot{\hat\gamma}_2(u)=\hat\gamma_2(u)-\hat\gamma_2(s)=0$, we get
$$D_{(\gamma,s,u)}\tilde f\cdot(\hat\gamma_2,0,0)=\begin{pmatrix}0\\|\dot\gamma(s)\times\ddot\gamma(s)|^2\\0\end{pmatrix};$$
- $\hat\gamma_3(u)-\hat\gamma_3(s)=\dot\gamma(s)\times\ddot\gamma(s)$ and $\dot{\hat\gamma}_3(u)=\ddot{\hat\gamma}_3(u)=0$, we get
$$D_{(\gamma,s,u)}\tilde f\cdot(\hat\gamma_3,0,0)=\begin{pmatrix}0\\0\\|\dot\gamma(s)\times\ddot\gamma(s)|^2\end{pmatrix}.$$

Since $|\dot\gamma(s)\times\ddot\gamma(s)|^2\neq0$, the surjectivity results from any linear combination of these three choices. Thus, 0 is a regular value for $\tilde f$ and $\tilde f^{-1}(0)$ is a Banach manifold (see [9, proof of Theorem 3.1.6]). The projection map $\mathrm{pr}_1\colon\tilde f^{-1}(0)\to C^k(S^1,\mathbb{R}^3)$, for k big enough, is smooth. According to the Sard–Smale theorem, almost all points in $C^k(S^1,\mathbb{R}^3)$ are regular values for pr_1, so the parametrization γ of K is generically a regular value. Thus, $\mathrm{pr}_1^{-1}(\gamma) \subset \tilde{f}^{-1}(0)$ is a Banach submanifold and 0 is a regular value for $\tilde{f}_\gamma$ (see [9, proof of Theorem 3.1.6]). It follows that $\tilde{f}_\gamma^{-1}(0) = \varnothing$, ie for a generic knot we have $\langle \ddot{\gamma}(s) \times \dot{\gamma}(u), \gamma(u) - \gamma(s) \rangle \neq 0$ in case (a).

Now we choose some open neighborhoods as follows:

- According to (1), around the finitely many points $\partial^s S$, F^s is a smooth curve with endpoint in $\partial^s S$. So we can choose open neighborhoods of these finitely many points of which we already know that within these neighborhoods F^s is a one-dimensional submanifold with boundary. Let $U_{\partial^s S}$ be the union of these neighborhoods.
- According to (2), there exists an open neighborhood U_Δ of the diagonal in $K \times K$ such that $F^s \cap U_\Delta$ is a one-dimensional submanifold.
- We can choose the framing such that $F^s \cap R_1 = \varnothing$ since R_1 depends only on the knot and is finite. So there exist open neighborhoods $U_{(s,u)_i}$ of all points $(s,u)_i \in R_1$ such that for all i, we have $F^s \cap U_{(s,u)_i} = \varnothing$. Let $U_{R_1} := \bigcup_i U_{(s,u)_i}$.

Let $U := U_{R_1} \cup U_{\partial^s S} \cup U_\Delta$. We define the map

$$\tilde{h}_\gamma := h_\gamma|_{T^2 \setminus U}.$$

According to the above, $D_{(s,u)}\tilde{h}_\gamma$ is surjective for all (s,u) with $\tilde{h}_\gamma(s,u) = (0,0)$. So the set

$$R_2 := \tilde{h}_\gamma^{-1}((0,0))$$

is finite since $T^2 \setminus U$ is compact.

We choose the framing so that $F^s \cap R_2 = \varnothing$.

(b) Since R_2 is finite, ie

$$\frac{\partial f_\gamma}{\partial u}(s,u) = 0$$

for only finitely many $(s,u) \in M_1$, $M_1 \setminus R_2$ can be split into finitely many disjoint subsets $M_{1,i}$, for $i = 1, \dots, N_{M_1}$, with $N_{M_1} \in \mathbb{N}$, each of which is connected, such that for all $i = 1, \dots, N_{M_1}$, $M_{1,i}$ can be written as a graph over the s–axis; ie

$$M_{1,i} = \{(s, u_i(s)) : s \in U_i\},$$

where $U_i \subset S^1$ is open, $u_i : U_i \to \mathbb{R}$ is smooth, and $\lim_{s \to s_0} h_\gamma(s_0, u_i(s)) = (0,0)$ for $s_0 \in \partial U_i$. We define the set

$$M_{U_i} := \{s \in U_i : \hat{a}(s) = g_1(s), \dot{\hat{a}}(s) = g_2(s)\} \subset S^1 \cong \mathbb{R}/\mathbb{Z}$$

with

$$g_1 : S^1 \to S^1, \quad s \mapsto \hat{n}(s, u(s)),$$
$$g_2 : S^1 \to \mathbb{R}, \quad s \mapsto \frac{\partial \hat{n}}{\partial s}(s, u(s)).$$

The set of all 1–jets from S^1 to S^1 is

$$J^1(S^1, S^1) = S^1 \times S^1 \times \mathbb{R}.$$

Furthermore,

$$L_i := \{(s, g_1(s), g_2(s)) : s \in U_i\} \subset J^1(S^1, S^1)$$

is a submanifold, since L_i is a graph over the s–axis, with $\operatorname{codim}(L_i \subset J^1(S^1, S^1)) = 2$. We can perturb $\hat{a}$ in the space $C^k(S^1, S^1)$, for k big enough, so that the map

$$h\colon S^1 \to J^1(S^1, S^1), \quad s \mapsto (s, \hat{a}(s), \dot{\hat{a}}(s)),$$

is transverse to $\bigcup_{i=1}^{N_{M_1}} L_i$ (see [7, Theorem 2.9]). So we get

$$M_U := \bigcup_{i=1}^{N_{M_1}} M_{U_i} = h^{-1}\Bigl(\bigcup_{i=1}^{N_{M_1}} L_i\Bigr) = \varnothing$$

since $\operatorname{codim}(M_U \subset S^1) = 2$.

Thus, the condition (A-2) is satisfied and this proves the lemma. □

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