AGT

Algebraic & Geometric Topology

Volume 24 (2024)

Wall-crossing from Lagrangian cobordisms

JEFF HICKS

msp

Algebraic & Geometric Topology **24**:6 (2024) 3069–3138
DOI: 10.2140/agt.2024.24.3069
Published: 7 October 2024

Wall-crossing from Lagrangian cobordisms

JEFF HICKS

Biran and Cornea showed that monotone Lagrangian cobordisms give an equivalence of objects in the Fukaya category. However, there are currently no known nontrivial examples of monotone Lagrangian cobordisms with two ends. We look at an extension of their theory to the pearly model of Lagrangian Floer cohomology and unobstructed Lagrangian cobordisms. In particular, we examine the suspension cobordism of a Hamiltonian isotopy and the Haug mutation cobordism between mutant Lagrangian surfaces. In both cases we show that these Lagrangian cobordisms can be unobstructed by a bounding cochain, and additionally induce an A_∞ homomorphism between the Floer cohomology of the ends. This gives a first example of a two-ended Lagrangian cobordism giving a nontrivial equivalence of Lagrangian Floer cohomology.

A brief computation is also included which shows that the incorporation of a bounding cochain from this equivalence accounts for the "instanton-corrections" considered by Auroux (2007), Pascaleff and Tonkonog (2020) and Rizell, Ekholm and Tonkonog (2022) for the wall-crossing formula between Chekanov and product tori in $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$.

We additionally prove some auxiliary results that may be of independent interest. These include a weakly filtered version of the Whitehead theorem for A_∞ algebras and an extension of Charest and Woodward's stabilizing divisor model of Lagrangian Floer cohomology to Lagrangian cobordisms.

53D37, 53D40

1 Introduction

Wall-crossing...

The wall-crossing phenomenon for Lagrangian submanifolds is an observation that the count of holomorphic disks with boundary on a family of Lagrangian submanifolds need not be continuous over the family. The count of these disks and how this count changes over families play an important role in describing the space of Lagrangian submanifolds up to Hamiltonian isotopy.

The moduli space of Lagrangian branes can be understood locally by constructing coordinate charts. A Lagrangian brane is a Lagrangian submanifold equipped with a unitary local system. Nearby any Lagrangian, the *flux of a Lagrangian isotopy* builds a local $(\mathbb{C}^*)^k$ chart, where $k = \dim(H^1(L;\mathbb{R}))$. An expectation which has been proven in good examples (such as Auroux [3] and Palmer and Woodward [28]) is that the *open Gromov–Witten* (*OGW*) *potential*, which records an area-weighted count of the holomorphic disks with boundary on a given Lagrangian L, is a holomorphic map in these coordinates. Both the symplectic area of these disks and the flux of an isotopy are complexified by the unitary local system on the Lagrangian submanifold.

Locally, the flux-charts and the OGW potential are holomorphic. However, neither of these can consistently provide coordinates or functions globally. An inconsistency occurs when a holomorphic disk "bubbles" over a Lagrangian isotopy, causing a discontinuity in the OGW potential. To consistently construct coordinates on the moduli space of Lagrangian submanifolds, one must incorporate "instanton corrections" to the count of holomorphic disks and flux computation. Auroux [3] and Kontsevich and Soibelman [22] phrased these corrections in terms of a wall-crossing formula. In this paradigm, the moduli space of Lagrangian submanifolds is divided into chambers of Lagrangians which do not bound Maslov index 0 disks. Separating these chambers are "walls" consisting of the Lagrangian submanifolds which bound Maslov index 0 disks. To transition from coordinates on one chamber to another, one computes a wall-crossing formula given by the count of Maslov index 0 disks, which appropriately modifies the OGW potential and flux computation.

A particular example of wall-crossing occurs when a monotone Lagrangian torus L bounds a Lagrangian disk D. For such pairs there exists another Lagrangian, called the mutation $\mu_D(L)$, lying in a different chamber. Palmer and Woodward [28], Pascaleff and Tonkonog [29] and Rizell, Ekholm and Tonkonog [31] explicitly computed the wall-crossing formula between these two chambers. Notably, this computation gives a wall-crossing formula between two Lagrangians which are *not* Hamiltonian isotopic.

The geometric justification for these wall-crossing transformations comes from the homological mirror symmetry conjecture of Kontsevich [21]. Associated to a symplectic space X is the Fukaya category $\mathrm{Fuk}(X)$ whose objects are Lagrangian submanifolds. The conjecture predicts that, for a Calabi–Yau manifold X, there exists a mirror Calabi–Yau manifold $\check{X}$ such that the symplectic geometry of X as recorded by $\mathrm{Fuk}(X)$ is interchanged with the complex geometry of $\check{X}$ as recorded by $D^b\,\mathrm{Coh}(\check{X})$. One way to recover the space $\check{X}$ is to study the moduli of points on $\check{X}$. A useful perspective comes from the SYZ conjecture (see Strominger, Yau and Zaslow [32]), which presents mirror spaces as dual Lagrangian torus fibrations. From this viewpoint, a candidate mirror to the skyscraper sheaf of a point in $\check{X}$ is a Lagrangian torus fiber of the SYZ fibration $X \to Q$.

As Hamiltonian isotopic Lagrangian submanifolds give quasi-isomorphic objects of the Fukaya category, the moduli of objects in the Fukaya category is a good proxy for the Hamiltonian isotopy classes of Lagrangian submanifolds. This Fukaya-categorical interpretation has the advantage that the incorporation of "instanton corrections" naturally arises in the construction of the Fukaya category. The presence of holomorphic disks with boundary on a Lagrangian L complicates the construction of the Fukaya category considerably. To account for the presence of holomorphic disks, one must equip each Lagrangian submanifold with additional data. The objects of the Fukaya category are pairs (L, b), where b is a "bounding cochain" deforming L as an object of the category. This deformation encodes an algebraic cancellation of holomorphic disks with boundary on L. Hamiltonian isotopy still produces an equivalence of such pairs, where disk bubbling is recorded by modifications of the bounding cochain. In the examples of crossing a wall, Lagrangians $(L,0)$ in one chamber are equivalent to (L',b') in the second chamber. The perspective of Fukaya, Oh, Ohta and Ono [17] is that the nontrivial deformation b' should be considered as the correction which occurs in the wall-crossing formula.

. . . and Lagrangian cobordisms

In the previous discussion, we focused on Hamiltonian isotopy as the geometric equivalence relation on objects of the Fukaya category. Another such equivalence relation was exhibited by Biran and Cornea [8], who proved that Lagrangian submanifolds related by monotone Lagrangian cobordism are equivalent objects in the Fukaya category. As every Hamiltonian isotopy gives an example of a Lagrangian cobordism, this indeed generalizes the previously considered equivalences. However, the monotonicity condition precludes the existence of Maslov index 0 disks, so examples considered in [8] do not realize the wall-crossing phenomenon.

A natural extension is to consider Lagrangian cobordisms equipped with bounding cochains. Such a cobordism (K,b) is predicted to yield equivalences in the Fukaya category between the ends $(L^+, b|_{L^+})$ and $(L^-, b|_{L^-})$. One interesting example of a Lagrangian cobordism is the *mutation cobordism* constructed by Haug [18], which relates mutant Lagrangians. This is an example of a nonmonotone Lagrangian cobordism, as mutant Lagrangians L and $\mu_D(L)$ are generally nonisomorphic as objects of the Fukaya category.

1.1 Results

Our goal here is to extend our understanding of wall-crossing between Hamiltonian isotopic Lagrangians to wall-crossing between cobordant Lagrangians. We use the *pearly Floer complex* $\mathrm{CF}^\bullet(L)$ as a receptacle for counting holomorphic disks. This is a deformation of the Morse complex $\mathrm{CM}^\bullet(L)$ constructed by inserting holomorphic disks with boundary on L into the flow lines of the Morse function. $\mathrm{CF}^\bullet(L)$ is a filtered A_∞ algebra. We first prove that the pearly Floer complex of a Lagrangian cobordism is compatible with the pearly Floer complex of the ends:

Theorem 1.1.1 (restatement of Corollary C.5.2) *The projections of the Floer cohomology of a Lagrangian cobordism to that of its ends $\beta^\pm\colon \mathrm{CF}^\bullet(K)\to\mathrm{CF}^\bullet(L^\pm)$ are A_∞ homomorphisms.*

We employ these new Floer results to extend [8] to the nonmonotone and cylindrical setting:

Theorem 1.1.2 (paraphrasing of Proposition 3.0.2) *Suppose that $K\colon L^-\rightsquigarrow L^+$ is a Lagrangian cobordism diffeomorphic to $L^+\times\mathbb{R}$. Then there is a homotopy equivalence of filtered A_∞ algebras $\Theta_K\colon \mathrm{CF}^\bullet(L^-)\to\mathrm{CF}^\bullet(L^+)$.*

As every Hamiltonian isotopy of Lagrangians produces a cylindrical suspension Lagrangian cobordism realizing that isotopy, this generalizes invariance of Floer cohomology of Hamiltonian isotopic Lagrangians to the unobstructed setting. The method of proof differs from the Hamiltonian isotopy setting and uses that $\mathrm{CF}^\bullet(K)$ is the deformation of the mapping cylinder $\mathrm{CM}^\bullet(K)$. In the setting where L^- is tautologically unobstructed, we show that K is unobstructed by a bounding cochain whose restriction to L^- is trivial but whose restriction to L^+ may not be.

The first nonmonotone and noncylindrical example that we look at is the mutation cobordism. We similarly prove that this cobordism gives an equivalence:

Theorem 1.1.3 (paraphrasing Theorem 4.2.5) *Let $K_{\mu_D^\epsilon}: L \rightsquigarrow \mu_D^\epsilon(L)$ be a mutation cobordism satisfying Definition 4.2.6. Then there exists a deforming cochain $b \in \mathrm{CF}^\bullet(K_{\mu_D^\epsilon})$ such that $\mathrm{CF}^\bullet{}_b(K_{\mu_D^\epsilon})$ is an A_∞ mapping cocylinder, yielding a map of filtered A_∞ algebras:*

$$\Theta_{\mu_D^\epsilon}: \mathrm{CF}^\bullet{}_{b^-}(\mu_D^\epsilon(L)) \to \mathrm{CF}^\bullet{}_{b^+}(L).$$

The proof is more difficult than the cylindrical case, as $\mathrm{CF}^\bullet(K)$ is *not* the deformation of a mapping cylinder. To show that K is unobstructed, we must construct a bounding cochain for $K_{\mu_D^\epsilon}$ accounting for an interesting holomorphic disk with boundary on $K_{\mu_D^\epsilon}$. The main observation for mutation cobordisms is that the portion of the pearly differential on $\mathrm{CF}^\bullet(K)$ arising from this holomorphic disk exactly cancels out the noncylindrical topology of K.

We verify in a computation included in the appendices that this theorem can be applied to the setting of monotone Chekanov and product tori in $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$. In this example, the bounding cochain on the Chekanov and product tori inherited from the Lagrangian cobordism corrects the cohomology of these mutant Lagrangians. This correction matches the wall-crossing transformations constructed by Auroux [4]. This gives (to our knowledge) the first example of a noncylindrical two-ended Lagrangian cobordism yielding an equivalence in the Fukaya category. This also gives a Fukaya category interpretation to the wall-crossing result of Pascaleff and Tonkonog [29], and the invariance of immersed Floer cohomology in this setting (as computed by Palmer and Woodward [28]). Our result can be considered as an application of the viewpoints of Biran and Cornea [8] and Fukaya, Oh, Ohta and Ono [17] on wall-crossing to non-Hamiltonian isotopic Lagrangian submanifolds.

We include some auxiliary results: a weakly filtered version of the Whitehead theorem (Theorem A.2.1), a definition of domain and label-dependent perturbation systems for the pearly model (Appendix B), a construction of stabilizing divisors for Lagrangian cobordisms (Lemmas C.1.2 and C.1.3), and compatibility of the pearly model of a Lagrangian cobordism with its ends (Theorem C.5.1).

1.2 Structure

Section 2 discusses the pearly model of Floer cohomology in the setting of Lagrangian cobordisms. We review some existing versions of the pearly model in Section 2.2, as well as provide an outline extending the pearly model to the noncompact setting. We summarize the relevant results from the appendices which are required for the remainder of the paper. The first is a way to construct a perturbation datum for the purpose of computing specific structure constants for the pearly model. The second is the extension of the stabilizing divisor pearly model of Charest and Woodward [10] to the setting of Lagrangian cobordisms. The proofs of these results are delayed until Appendices B and C.

In Section 3, we look at a toy model where we recover invariance of $\mathrm{CF}^\bullet(L)$ under Hamiltonian isotopy using nonmonotone Lagrangian cobordisms. The core of the proof is to show that the Lagrangian cobordism gives an example of an A_∞ mapping cocylinder, from which a continuation map can be recovered by the application of Theorem A.3.2. The cobordism is our first example of a nonmonotone cobordism giving an equivalence of objects in the Fukaya category, and the method of proof provides a road map to the more general case considered later.

Section 4 contains our main theorem. We first introduce the Lagrangian mutation cobordism from Haug [18] in Section 4.1. We take some care to work out the parametrization of this cobordism explicitly in the setting of $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$. This care is necessary to later characterize the kinds of holomorphic disks which can appear with boundary on this cobordism, and to compute explicitly the flux swept between the ends of the Lagrangian cobordism. Section 4.2 looks at the mutation cobordism constructed from a mutation pair — a Lagrangian L bounding a Lagrangian disk D. The remainder of the section is spent proving Theorem 4.2.5, which shows that this mutation cobordism K_{μ_D}, when equipped with an appropriate bounding cochain, gives an equivalence in the Fukaya category. The idea of the proof is very similar to Section 3, in that we show that the mutation cobordism K_{μ_D} has the Floer cohomology of a mapping cocylinder. This is complicated by the fact that the underlying topology of K_{μ_D} is not that of a cylinder. We outline the proof here:

- In Section 4.3.1 we construct a Morse function for K_{μ_D} and point out how this cobordism fails to be a topological cylinder between its ends.
- In Section 4.3.2 we find a holomorphic disk with boundary on K_{μ_D}.
- In Section 4.4.1 we show that this holomorphic disk contributes to $m^0_{K_{\mu_D}}$ in such a way that its lowest order contributions can be canceled out by a deforming cochain d_ϵ.
- In Section 4.4.2 we show that there is a homotopy between $\mathrm{CF}^\bullet{}_{d_\epsilon}(K_{\mu_D})$ and its left end $\mathrm{CF}^\bullet{}_{d_\epsilon|_{L^-}}(L^-)$ which uses the component of the deformed differential arising from the holomorphic disk.
- In Section 4.4.3 we present K_{μ_D} as a weakly filtered A_∞ mapping cocylinder (see Definition A.3.1) using the homotopy constructed above and conclude the existence of a continuation map.

In Section 5 we look at an application of Theorem 4.2.5 to the Chekanov and product tori in $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$. In addition to showing that these objects have the same Floer cohomology (and conjecturally correspond to the same object of the Fukaya category), the bounding cochains inherited from the Lagrangian cobordism deform the local Flux coordinates on the moduli space of Chekanov and product tori. Assuming convergence of sums taken with $\mathbb{C}$ coefficients and a multiple cover formula for Maslov index 0 disks, these coordinate changes are shown to match the computations made in Auroux [3].

We additionally include three appendices with auxiliary results. Appendix A states some known facts for filtered A_∞ algebras and morphism of A_∞ algebras. It also includes the statements of Theorems A.2.1 and A.3.2. The first is an extension of the homotopy transfer principle of Kadeishvili [20] or the filtered A_∞ Whitehead theorem (see Fukaya, Oh, Ohta and Ono [17]) to the setting of weakly filtered A_∞

homotopies. The second is a proof of existence and characterization of mapping cocylinders for filtered A_∞ algebras.

Appendix B extends the construction of Charest and Woodward [10] to use perturbations that depend on the domain and the labels of the domain. The main application, Lemma B.3.2, shows that under strong conditions we can use a nonperturbed almost complex structure to compute some of the structure coefficients for the pearly model.

Appendix C adapts the construction of [10] to Lagrangian cobordisms. The main results are Lemmas C.1.2 and C.1.3 which construct Donaldson divisors of large degree for Lagrangian cobordisms. The Donaldson divisors can be made weakly stabilizing. Furthermore, the divisors and weakly stabilized almost complex structures are split (across the product $X \times \mathbb{C}$) outside of a compact set. This allows us to construct a pearly model for Lagrangian cobordisms (Corollary C.4.2) and obtain compatibility between the pearly model of a Lagrangian cobordism and that of its ends (Corollary C.5.2).

1.3 Notation

We introduce here several pieces of notation that are used throughout the paper.

- Let $A_1, \dots, A_k$ be vector spaces. The componentwise projection is the map
$$\pi^{k_1|\cdots|k_i} : (A_1 \oplus \cdots \oplus A_i)^{\otimes k_1 + \cdots + k_i} \to A_1^{\otimes k_1} \otimes \cdots \otimes A_i^{\otimes k_i}.$$
- Let $A = A^- \oplus A^0 \oplus A^+$ and let $m^k : A^{\otimes k} \to A$ be a multilinear map over field $\mathbb{F}$. When we want to restrict the domain and codomain we'll write
$$m^{\ell_1 \cdots \ell_k ; \ell_0} : \bigotimes_{i=1}^{k} A^{\ell_i} \to A^{\ell_0},$$
where the ℓ_i are chosen from the labels $\{+, -, 0\}$. In this notation $m^{;\ell} : \mathbb{F} \to A^\ell$ denotes an element in A^ℓ.
- Let $f : X^+ \to X^-$ be a map of topological spaces. The mapping cylinder is the topological space $\operatorname{cyl}(f) = (X^+ \times I) \sqcup_{(x^+ \times \{1\} \sim f(x^+))} X^-$. This comes with inclusion maps $i_{X^\pm} : X^\pm \to \operatorname{cyl}(f)$ and a pushforward map $\pi : \operatorname{cyl}(f) \to X^-$. We denote the maps on the cochains by
$$\beta^\pm := (i_{X^\pm})^* : C^*(\operatorname{cyl}(f)) \to C^*(X^\pm) \quad \text{and} \quad \alpha^- := (\pi)^* : C^*(X^-) \to C^*(\operatorname{cyl}(f)).$$
- All symplectic manifolds considered are rational and aspherical. Unless specifically noted, all Lagrangian submanifolds considered are spin, graded, rational and embedded.

Acknowledgements

First and foremost, I would like to thank my advisor Denis Auroux for his patience and explanations during my graduate studies at UC Berkeley. I would also like to thank Paul Biran and Luis Haug, with whom I had many very productive conversations furthering my understanding of the antisurgery cobordism. Additionally, I'd like to acknowledge Andrew Hanlon for the helpful comments on the introduction to this paper, as well as Ailsa Keating, Nick Sheridan and Chris Woodward for several valuable discussions. This paper further benefited from the very thoughtful comments of the referee, addressing expositional and mathematical gaps from an earlier version of this manuscript, especially with regards to adapting the stabilizing divisor approach of the pearly model to the setting of Lagrangian cobordisms. Finally, I would also like to thank ETH Zürich for their hospitality in Fall 2018 during which a portion of this work was completed.

This work was partially supported by NSF grants DMS-1406274 and DMS-1344991, Simons Foundation grant 385573, Simons Collaboration on Homological Mirror Symmetry, EPSRC grant EP/N03189X/1, *Classification*, *computation and construction*: *new methods in geometry*, and ERC grant 850713.

2 Background

2.1 Lagrangian cobordisms

Lagrangian cobordisms are Lagrangian submanifolds in $X \times \mathbb{C}$ which interpolate between Lagrangian submanifolds of X. In addition to providing an interesting relation on Lagrangian submanifolds of X, cobordant Lagrangian submanifolds are frequently equivalent objects of the Fukaya category.

Definition 2.1.1 [1] Let L^- and L^+ be Lagrangian submanifolds of X. A *two-ended Lagrangian cobordism* between L^- and L^+ is a proper Lagrangian submanifold $K \subset X \times \mathbb{C}$ which satisfies the following conditions:

- **Fibered over ends** There exist constants $t^- < t^+$ such that
  $$K \cap \{(x,z) \mid \mathrm{Re}(z) \geq t^+ - \epsilon\} = L^+ \times \mathbb{R}_{\geq t^+ - \epsilon}, \quad K \cap \{(x,z) \mid \mathrm{Re}(z) \leq t^- + \epsilon\} = L^- \times \mathbb{R}_{\leq t^- + \epsilon}.$$
- **Compactness** There exists a constant $s > 0$ such that the projection $\pi_{\iota\mathbb{R}} : K \to \iota\mathbb{R} \subset \mathbb{C}$ is contained within an interval $[-\iota s, \iota s]$.

We denote such a cobordism by $K : L^- \rightsquigarrow L^+$.

We visualize a Lagrangian cobordism by drawing its projection to the $\mathbb{C}$ parameter, as in Figure 1. There is a generalization of this definition to cobordisms with multiple ends, which requires that the Lagrangian cobordism fibers over rays of fixed argument outside of a compact subset of $\mathbb{C}$. A specialization of the result of [8] proves that the equivalence relation of monotone cobordance descends to the Fukaya category.

Theorem 2.1.2 *Suppose that $K : L^- \rightsquigarrow L^+$ is a monotone Lagrangian cobordism. Then L^- and L^+ are equivalent objects of the Fukaya category.*

In [8], it is stated that this result is expected to hold in a more general setting where the Lagrangians are equipped with the additional data of orientations, local systems, bounding cochains, etc. We look to understand this result in the context of pearly algebras of cobordant Lagrangians, where the Lagrangian cobordism may be nonmonotone but unobstructed by a bounding cochain.

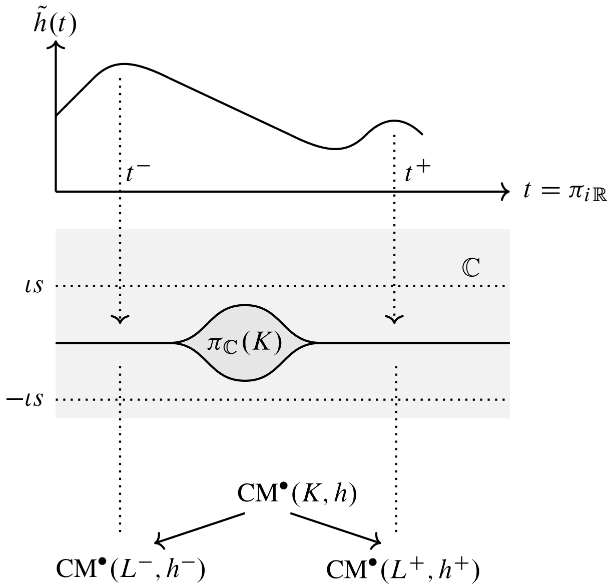

Figure 1: The profile of the Morse function for a cobordism. Bottlenecks are inserted on the ends of the cobordism.

Definition 2.1.3 Let (L^-, h^-) and (L^+, h^+) be two Lagrangians equipped with choices of Morse function, spin structure and grading. An *admissible Lagrangian cobordism* between these is the data of (K, h), where

- K is a Lagrangian cobordism between L^- and L^+,
- $h\colon K \to \mathbb{R}$ is a Morse function, and
- we have a spin structure and grading for K compatible with those of the ends.

The Morse function h is required to satisfy the following compatibility conditions:

(i) The Morse flow restricted to the fibers above real coordinates t^- and t^+ is determined by $h^\pm$:

$$\operatorname{grad} h|_{\pi_{\mathbb{R}}^{-1}(t^-)} = (\operatorname{grad} h^-, 0) \quad \text{and} \quad \operatorname{grad} h|_{\pi_{\mathbb{R}}^{-1}(t^+)} = (\operatorname{grad} h^+, 0).$$

(ii) The fibers above $t^\pm$ are perturbations of Morse–Bott maximums in the sense that, at points $q \in K$,

$$dt(-\operatorname{grad} h)|_{\pi_{\mathbb{R}}(q) < t^-} < 0, \quad dt(-\operatorname{grad} h)|_{t^- < \pi_{\mathbb{R}}(q) < t^- + \epsilon} > 0,$$
$$dt(-\operatorname{grad} h)|_{\pi_{\mathbb{R}}(q) > t^+} > 0, \quad dt(-\operatorname{grad} h)|_{t^+ - \epsilon < \pi_{\mathbb{R}}(q) < t^+} < 0.$$

One way to construct an admissible Morse function for a Lagrangian cobordism is to consider the function $\tilde{h}$ for K, as drawn in Figure 1, which is only dependent on t, and then take a Morsification which is constructed in the regions around $t^\pm$ using an $h^\pm$–perturbation. The function $\tilde{h}$ is Morse–Bott near the ends, and we call critical submanifolds $L^\pm \times \{t^\pm\}$ the bottlenecks of the cobordism.

Proposition 2.1.4 *Let K be a Lagrangian cobordism equipped with an admissible Morse function.*

- *Even though K is noncompact, the moduli space of Morse flow lines between two critical points admits a compactification by broken flow lines.*
- *The Morse flow lines γ of $\mathrm{CM}^\bullet(K,h)$ for which $\pi_{\mathbb{R}}(\gamma) = t^\pm$ are exactly the Morse flow lines of $\mathrm{CM}(L^\pm, h^\pm)$.*

As a result, we obtain projections of the Morse cochain complexes to the ends:

$$\mathrm{CM}^\bullet(L^-,h^-) \xleftarrow{\ \beta^-\ } \mathrm{CM}^\bullet(K,h) \xrightarrow{\ \beta^+\ } \mathrm{CM}^\bullet(L^+,h^+)$$

2.2 Pearly model for Floer complexes

The receptacle we choose for the count of holomorphic disks with boundary on a compact Lagrangian L is the *pearly Floer cohomology* of a Lagrangian. This is a filtered A_∞ algebra $\mathrm{CF}^\bullet(L,h)$ specified by a choice of a Lagrangian submanifold L, admissible Morse function $h\colon L \to \mathbb{R}$ and perturbation data. The algebra structure is determined by taking a count of treed disks, which are flow trees of the Morse function h deformed with insertions of holomorphic disks with boundary on L at the vertices. Several versions of this algebra have been constructed with a variety of conditions imposed on the Lagrangian L and the manifold X. Some models of the pearly Floer cohomology include [6; 10; 16; 24].

In [6], a version of the pearly quantum homology for monotone Lagrangian submanifolds was constructed. In this model, the symplectic spaces studied are symplectically convex at infinity, and the Lagrangian submanifolds considered are compact and monotone. With these hypotheses, [6] gave the definition of the pearly quantum cohomology $\mathrm{QC}_\bullet(L,h)$, which is a deformation of the Morse chain complex $\mathrm{CM}_\bullet(L,h)$. Furthermore, in [7] they extend this theory to the setting where $K \subset X \times \mathbb{C}$ is a monotone Lagrangian cobordism (Definition 2.1.1). Lagrangian cobordisms have pearly quantum homology with inclusion maps $\mathrm{QH}_\bullet(L^\pm,h) \to \mathrm{QH}_\bullet(K,h)$, where $L^\pm$ are the ends of the Lagrangian cobordism K.

In [10], the pearly model is constructed with relaxed requirements for the Lagrangian L. For compact symplectic spaces X with rational symplectic form, [10] defines a pearly algebra $\mathrm{CF}^\bullet(L,h)$. The key piece of additional input into this construction is a stabilizing divisor which intersects every holomorphic disk with boundary on L. These intersection points stabilize the domain of the holomorphic disks, allowing one to construct domain-dependent perturbations to the Cauchy Riemann equations. With this deformation, [10] cuts out a regularized moduli space of treed disks.

Theorem 2.2.1 [10] *Consider (X,ω) a rational symplectic manifold. Let $L \subset X$ be a spin and graded Lagrangian submanifold and $h\colon L \to \mathbb{R}$ a Morse function. There exist a weakly stabilizing divisor $D_X \subset X \setminus L$ and admissible domain-dependent perturbation datum $\mathcal{P}$ for L defining a filtered A_∞ algebra*

$\mathrm{CF}^\bullet(L,h,\mathcal{P})$. *The filtered* A_∞ *homotopy type of* $\mathrm{CF}^\bullet(L,h,\mathcal{P})$ *is independent of Morse function, weakly stabilizing divisor and admissible domain-dependent perturbation datum.*

For this reason, when the choice of perturbation $\mathcal{P}$ or Morse function h is unimportant, we will write $\mathrm{CF}^\bullet(L,h)$ or $\mathrm{CF}^\bullet(L)$ instead.

In Appendices B and C we make two extensions to the construction of [10]. We include short summaries of these two appendices here, and highlight the results from those sections which are necessary for the main portion of this article.

2.2.1 Domain and label-dependent perturbations The first modification, constructed in Appendix B, is to introduce a perturbation datum for treed pseudoholomorphic disks which depends not only on the domain, but also on the labeling of the domain. The labels are chosen from the critical points of the Morse function h.

The advantage to using this set of perturbations comes from examples where a geometrically chosen almost complex structure and Morse function provide us with a computable set of regularly cut out pseudoholomorphic flow trees between a specific set of Morse critical points. We'd like to keep these pseudoholomorphic flow trees, but it is unclear if our geometrically chosen almost complex structure is regularizing for pseudoholomorphic treed disks between other Morse critical points. By allowing our perturbation datum to depend additionally on the limiting labels of the pseudoholomorphic flow trees, we can keep our perturbation fixed on the areas where we know how to make our calculation, and modify it elsewhere. See Section B.2.3 for more detailed comments on where domain and label-dependent perturbations fit in with other regularizations methods.

A specific example of where this works out is given in Section B.3, the results of which we now state. Let J^0 be an almost complex structure and let $h\colon L\to\mathbb{R}$ be a Morse–Smale function. Suppose that there is a J^0–holomorphic disk $u_{\mathrm{ex}}\colon(D^2,\partial D^2)\to(X,L)$ such that

- the boundary of u_{ex} is transverse to all upwards and downward flow spaces of critical points of h, and
- the disk u_{ex} is regular and is the unique disk that intersects the stabilizing divisor at a single point.

Suppose that $\deg(x_1)=1$, $\deg(x_0)=2$, and that there are no Morse flow lines from x_1 to x_0. The hypothesis on the regularity of u_{ex} and transversality of ∂u_{ex} with the flow spaces shows that unbroken pseudoholomorphic flow-trees whose type belongs to

$$\mathbb{I}^{\leq 1}_{*;x_0}:=\{\underline{\Gamma}_P\mid \mathrm{ind}(\underline{\Gamma}_P)\leq 0,\text{input label is empty or } x_1,\text{ output is } x_0,\text{ at most 1 interior marked point}\}$$

are regularly cut for $\mathcal{P}^0$, the trivial perturbation determined by J^0 and h.

The hypothesis on [10, Theorem 4.19] is too restrictive to show that the perturbation system $\mathcal{P}^0_{\mathbb{I}^{\leq 1}_{*;x_0}}$ defined for types $\underline{\Gamma}_P\in\mathbb{I}^{\leq 1}_{*;x_0}$ can be coherently extended to a regular perturbation system for all combinatorial types. This is because $\mathbb{I}^{\leq 1}_{*;x_0}$ is not downward closed (informally: the coherence conditions for combinatorial types in $\mathbb{I}^{\leq 1}_{*;x_0}$ involve combinatorial types outside $\mathbb{I}^{\leq 1}_{*;x_0}$; see Definition B.1.4). For instance, the combinatorial type of a Morse flow tree with inputs y and x_1 and output x_0 appears in the downward closure of $\mathbb{I}^{\leq 1}_{*;x_0}$, but not in $\mathbb{I}^{\leq 1}_{*;x_0}$ — this tree could additionally break at further critical points, giving more broken types in the downward closure of our original set of combinatorial types. It is very unlikely that the trivial perturbation of Morse flow lines will be regularizing for all of these combinatorial types, as Morse flow-trees are never regular for the trivial perturbation (unless the moduli space is empty!). So, we will usually not be able to find a $\mathcal{P}$ which is a coherent regular extension of $\mathcal{P}^0_{\mathbb{I}^{\leq 1}_{*;x_0}}$.

However, we can take the following strategy: given $\mathcal{P}^0$ which regularly cuts out our moduli spaces $\mathbb{I}^{\leq 1}_{*;x_0}$ for some subset of types, we look for a regular coherent perturbation $\mathcal{P}$, *not necessarily agreeing with* $\mathcal{P}^0$, but having the same moduli spaces for the desired set of combinatorial types.

Lemma 2.2.2 (paraphrasing Lemma B.3.2) *There exists a full regular coherent perturbation system $\mathcal{P}$ such that, for every combinatorial type in $\underline{\Gamma}_P \in \mathbb{I}^{\leq 1}_{*;x_0}$, the count of $\mathcal{P}$–perturbed pseudoholomorphic curves of type $\underline{\Gamma}_P$ agrees with the count of $\mathcal{P}^0$–perturbed pseudoholomorphic curves.*

2.2.2 Pearly model for Lagrangian cobordisms The second modification of the pearly model of [10], constructed in Appendix C, concerns the extension of the results of [7] to [10]. We show that a perturbation datum and stabilizing divisors may be chosen so that projection maps from the pearly Floer cohomology of a Lagrangian cobordism to the ends exist. While we use [10] as a framework due to our familiarity with this particular version of the pearly model, we expect the techniques employed here should transfer in the most general sense to other models of Lagrangian Floer cohomology, or any abstract regularization method used to construct the moduli spaces of pseudoholomorphic treed disks.

Lemma 2.2.3 (paraphrasing of Lemmas C.1.2 and C.1.3) *Let $K\colon L^- \rightsquigarrow L^+$ be a Lagrangian cobordism and $h\colon K \to \mathbb{R}$ a compatible Morse function. There exists a weakly stabilizing divisor $D \subset (X \times \mathbb{C}) \setminus K$, and $\mathcal{P}_K$ for K satisfying the following properties*:

- *$\mathcal{P}_K$ is admissible, in that it is E–weakly stabilized by the divisor D, regular and coherent.*
- *$\mathcal{P}_K$ consists of domain-dependent choices of almost complex structure for which the projection $\pi_{\mathbb{C}}\colon X \times \mathbb{C} \to \mathbb{C}$ is holomorphic outside of a compact set.*

We use this to prove that the moduli space of pseudoholomorphic treed disks is compact (Proposition C.4.1) and compatible with the perturbation systems $\mathcal{P}_{L^\pm}$ for the ends $L^\pm$ of the Lagrangian cobordism (Theorem C.5.1). Compactness allows us to define the pearly model for Lagrangian cobordisms (Corollary C.4.2), while compatibility proves that the projections

$$\mathrm{CF}^\bullet(L^-,h^-) \xleftarrow{\ \beta^-\ } \mathrm{CF}^\bullet(K,h) \xrightarrow{\ \beta^+\ } \mathrm{CF}^\bullet(L^+,h^+)$$

are filtered A_∞ homomorphisms (Corollary C.5.2). This is the extension of Proposition 2.1.4 to Lagrangian Floer theory.

3 Toy model: continuation maps from cylindrical cobordisms

We now give an example where the results of [8] apply to the nonmonotone setting. While we allow our Lagrangian cobordisms to possibly bound holomorphic disks, we require that the cobordism's topology be cylindrical. A relevant set of examples are the suspension of a Hamiltonian isotopy.

Definition 3.0.1 (suspension of a Hamiltonian isotopy) Let $H_t\colon X\times\mathbb{R}_t\to\mathbb{R}$ be a Hamiltonian with support on $[t^-,t^+]$. Let $\phi_t\colon L\times\mathbb{R}_t\to X$ be a Lagrangian isotopy induced by the Hamiltonian flow of H_t. The *suspension of the Hamiltonian isotopy* ϕ_t is the Lagrangian cobordism K_{H_t} given by the embedding

$$L\times\mathbb{R}\hookrightarrow X\times\mathbb{C}\quad (x,t)\mapsto\big(\phi_t(x),t+\iota H_t(\phi_t(x))\big).$$

More generally, given a Lagrangian homotopy $\phi_t\colon L\times\mathbb{R}_t\to X$ which is exact (in the sense that $\iota_{(d\phi/dt)}\omega|_{L_t}$ is an exact form for all t), there exists a similar construction for the suspension of the exact homotopy. In a similar vein to the equivalences constructed in [8], the suspension Lagrangian cobordism gives a continuation map for the pearly algebra of L.

Proposition 3.0.2 *Let $L^-,L^+\subset X$ be two Lagrangian submanifolds, with Morse functions $h^\pm\colon L^\pm\to\mathbb{R}$. Suppose that $K\subset X\times\mathbb{C}$ is an embedded Lagrangian cobordism with ends on L^- and L^+. Furthermore, suppose that there is a diffeomorphism $K\cong L^-\times\mathbb{R}$. Then there exists a filtered A_∞ homomorphism $\Theta_K\colon \mathrm{CF}^\bullet(L^-;h^-)\to\mathrm{CF}(L^+;h^+)$.*

Proof We pick a particular Morse function h for K. Take a preliminary function

$$\tilde{h}(t)\colon\mathbb{R}\to\mathbb{R} \tag{1}$$

which has maximums at t^- and t^+ and a minimum at $t^+-\epsilon$; see Figure 1. This defines a Morse–Bott function on K. We Morsify this function by taking perturbations based on the functions $h^\pm$ as considered in Proposition 2.1.4. By construction, we have an identification

$$\mathrm{CF}^\bullet(K,h)=\mathrm{CF}^\bullet(L^-,h^-)\oplus\mathrm{CF}^\bullet(L^+,h^+)[1]\oplus\mathrm{CF}^\bullet(L^+,h^+)$$

as graded vector spaces. From Theorem C.5.1, the pearly differential on K_H decomposes as

$$m^1_K=\begin{pmatrix} m^{-;-} & 0 & 0\\ m^{-;0} & m^{0;0} & m^{+;0}\\ 0 & 0 & m^{+;+}\end{pmatrix}, \tag{2}$$

where $m^{\pm;\pm}$ matches $m^1_{L^\pm}$.

See Section 1.3 for notation. We additionally have the projection A_∞ homomorphisms $\beta^\pm\colon\mathrm{CF}^\bullet(K,h)\to\mathrm{CF}^\bullet(L^\pm,h^\pm)$. The map $m^{+;0}$ is a deformation of the Morse differential on K by terms of nonzero filtration. Since the Morse portion of the differential

$$\underline{m}^{+;0}\colon\mathrm{CM}^\bullet(L^+,h^+)\to\mathrm{CM}^\bullet(L^+,h^+)[1]$$

is the identity, the Floer differential $m^{+;0}$ which arises as a deformation by higher energy terms remains an isomorphism. Therefore $\mathrm{CF}^\bullet(L^-,h^-) \leftarrow \mathrm{CF}^\bullet(K,H) \to \mathrm{CF}^\bullet(L^+,h^+)$ is a filtered A_∞ mapping cocylinder (Definition A.3.1). We obtain a map $\alpha\colon \mathrm{CF}^\bullet(L^-,h^-) \to \mathrm{CF}^\bullet(K,h)$ and a pushforward–pullback map (Theorem A.3.2)

$$\Theta_K := \beta^+ \circ \alpha\colon \mathrm{CF}^\bullet(L^-,h^-) \to \mathrm{CF}(L^+,h^+). \qquad \square$$

Remark 3.0.3 The hypothesis that K is diffeomorphic to $L^- \times \mathbb{R}$ can be weakened to assuming that K is an h–cobordism, in which case the map $\underline{m}^{+;0}$ is a quasi-isomorphism. While Definition A.3.1 would no longer apply, one can use [17, Theorem 4.2.45] to obtain the continuation map.

It immediately follows that there is a continuation map for the pearly model of Lagrangian Floer theory.

Corollary 3.0.4 *Let L^- and L^+ be Hamiltonian isotopic Lagrangian submanifolds. There exists a pullback–pushforward map*

$$\Theta_{H_t}\colon \mathrm{CF}^\bullet(L^-,h^-) \to \mathrm{CF}^\bullet(L^+,h^+)$$

called the ***continuation map*** *of H_t.*

Proof Consider the suspension cobordism K_{H_t} and apply Proposition 3.0.2. $\square$

Additionally, the presence of such a map proves the existence of maps between the Maurer–Cartan spaces of these Lagrangians.

Corollary 3.0.5 *Let L^- and L^+ be Hamiltonian isotopic Lagrangian submanifolds. If $\mathrm{CF}^\bullet(L^-)$ is unobstructed by the bounding cochain b^-, then L^+ is unobstructed by the bounding cochain $b^+ = (\Theta_{H_t})_* b^-$. Furthermore, the continuation map can be made a map of uncurved A_∞ algebras $(\Theta_{H_t})_\flat\colon {\mathrm{CF}^\bullet}_{b^-}(L^-) \to {\mathrm{CF}^\bullet}_{b^+}(L^+)$.*

Proof The existence of a filtered A_∞ homomorphism implies that these maps exist; see Claim A.1.4. $\square$

4 Cobordisms and mutations

In this section we look at cobordisms arising from mutation configurations. We prove that in certain scenarios such a cobordism gives an A_∞ mapping cocylinder between the Floer cohomologies of its ends. In Section 4.1 we introduce a local model for mutation in $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$. This local description is used to construct a holomorphic disk with boundary on the Lagrangian cobordism. In Section 4.2 we use this disk to build a continuation map from a Lagrangian mutation cobordism.

4.1 Review of the Haug cobordism

This is a review of the mutation cobordism constructed in [18] between the monotone Chekanov and product tori in $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$. We assemble this cobordism from pieces built in Lefschetz fibrations. The fibration we consider is $W = z_1 z_2\colon \mathbb{C}^2 \to \mathbb{C}$, which has a single nodal fiber at the origin.

We first give a description of Lagrangian surgery in the sense of [23; 30] for Lagrangians in Lefschetz fibrations. Consider the paths

$$\ell_{\uparrow/\downarrow}\colon [0,R]\to\mathbb{C} \quad \text{given by } r\mapsto \pm\imath r^2$$

in the base of the Lefschetz fibration with ends on the critical value of the fibration. We take lifts of these paths to give Lagrangian thimbles $L_\uparrow, L_\downarrow \subset \mathbb{C}^2$. These Lagrangians have the topology of D^2 and can be parametrized in coordinates by

$$L_\uparrow = \{(x+\imath y, y+\imath x) \mid x^2+y^2\le R^2\} \quad\text{and}\quad L_\downarrow = \{(x+\imath y, -y-\imath x) \mid x^2+y^2\le R^2\}.$$

We now recall the construction of Lagrangian surgery and the associated suspension cobordism from [7]. We pick a *surgery profile curve*, which is a map $(a(t)+\imath b(t))\colon[-R,R]\to\mathbb{C}$. The functions $a(t)$ and $b(t)$ are nonstrictly increasing, and we require that $(a(t)+\imath b(t)) = t$ for $t<c$ and $(a(t)+\imath b(t))=\imath t$ for $t>c$. This neck provides a local model for the Lagrangian surgery. To insert this neck at the transverse intersection point of two Lagrangian submanifolds, we take a Darboux chart around the intersection point which identifies these Lagrangian submanifolds with the totally real and imaginary subspaces of $\mathbb{C}^n$ and interpolate between these two subspaces using the surgery profile curve. For the example we are concerned with, the surgery $L_\uparrow \# L_\downarrow$ is explicitly described in coordinates by

$$\big\{\big((a(t)+\imath b(t))(\hat{x}+\imath\hat{y}), (a(t)+\imath b(t))(\hat{y}+\imath\hat{x})\big) \mid \hat{x}^2+\hat{y}^2=1\big\}. \tag{3}$$

With this construction, we take special care with the order of the summands in the connect sum. Our convention is that the end of the neck which corresponds to $t\in[-R,-c]$ corresponds to the first summand of the Lagrangian connect sum. The *neck width* of a mutation measures the amount of flux swept out as one takes an isotopy from $L_\uparrow \# L_\downarrow$ to $L_\downarrow \# L_\uparrow$. The projection of this cobordism under W is

$$W(L_\uparrow \# L_\downarrow) = \imath(a(t)+\imath b(t))^2(\hat{x}^2+\hat{y}^2) = \imath(a(t)+\imath b(t))^2,$$

whose image under W is a curve in the complex plane:

$$\ell_\uparrow \# \ell_\downarrow\colon [-R,R]\to\mathbb{C}, \quad t\mapsto \big(-2a(t)b(t)+\imath(a(t)^2-b(t)^2)\big).$$

Notice that when $t\in[-R,-c]$ the coordinate $b(t)$ equals 0 and the curve matches the positive imaginary axis. Similarly, when $t\in[c,R]$ we have that $a(t)=0$ and the curve matches the negative imaginary axis. The real component of $\ell_\uparrow \# \ell_\downarrow$ is always nonnegative. See Figures 2 and 3 for diagrams of the pieces described above. We can similarly construct a profile for $L_\downarrow \# L_\uparrow$ and corresponding path $\ell_\downarrow \# \ell_\uparrow$.

Given a Lagrangian surgery, [8] describes *trace cobordisms* $K_l\colon L_\uparrow \# L_\downarrow \rightsquigarrow L_\downarrow \sqcup L_\uparrow$ and $K_r\colon L_\downarrow \sqcup L_\uparrow \rightsquigarrow L_\downarrow \# L_\uparrow$. There is a comparison between the neck width of the cobordism and the area of the projection of the cobordism to $\mathbb{C}$, which is that the area of projection of the cobordism is at least the neck width. By concatenating these cobordisms together, we obtain a Lagrangian cobordism $K_r\circ K_l\colon L_\uparrow \# L_\downarrow \rightsquigarrow L_\uparrow \# L_\downarrow$. The projection of $K_r\circ K_l$ to $\mathbb{C}$ is the eye-shaped Lagrangian drawn in Figure 2. We additional draw the slices of the Lagrangian cobordism, $K|_t := \pi_X(\pi_{\mathbb{R}}^{-1}(t)\cap K)$.

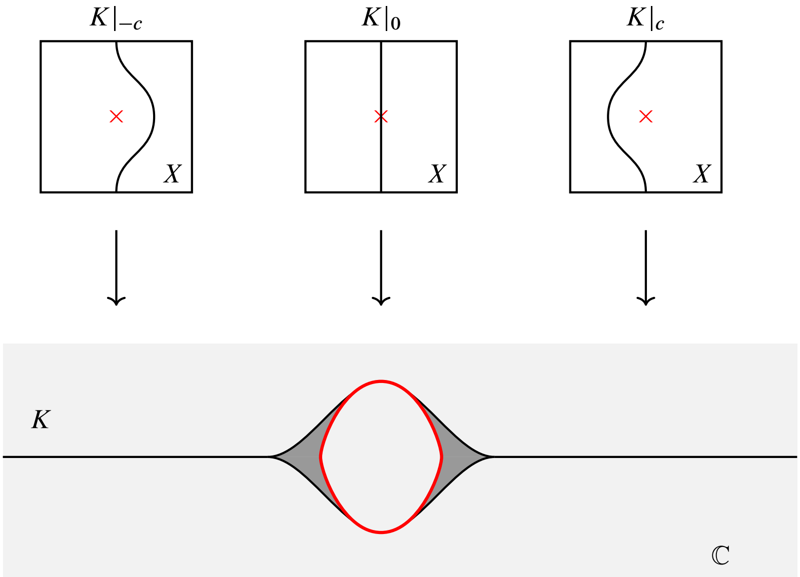

Figure 2: The projection of the eye-shaped Lagrangian cobordism, $(K_r \circ K_l \cap U) \subset X \times \mathbb{C} \to \mathbb{C}$. The cycle $\pi_U^{-1}(0,0)$ is highlighted in red.

Remark 4.1.1 Another way of stating the relation between area of the projection and the neck width is to say that the *shadow* of the cobordism $K_r \circ K_l$, as defined in [13], is an upper bound for the widths of the two surgeries.

We now wish to make comparisons between the surgery of the thimbles $L^{\pm}$, Chekanov and product tori, and the Whitney sphere. In the base of the Lefschetz fibration, one can construct the Whitney sphere by taking a c–shaped path $\ell_c\colon [-S,S] \to \mathbb{C}$ from the negative end of $\ell_\downarrow$ to the positive end of $\ell_\uparrow$ in a clockwise fashion, so that the concatenation $\ell_\downarrow \cdot \ell_c \cdot \ell_\uparrow$ is a matching path from the single critical point of the fibration to itself. Let $L_c\colon [-S,S] \times S^1 \to \mathbb{C}$ be the corresponding parallel transport of the vanishing cycle. The union $L_c \cup L_\downarrow \cup L_\uparrow$ is the Whitney sphere L_{S^2}. By attaching L_c to $L_\downarrow \# L_\uparrow$, we will obtain the monotone Chekanov torus, denoted by L^-. By attaching L_c to $L_\uparrow \# L_\downarrow$, we obtain the monotone product torus, which will be denoted by L^+. See Figure 3 for a picture of these Lagrangian submanifolds.

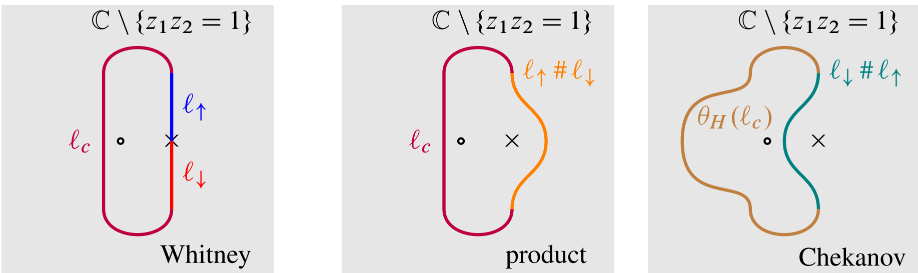

Figure 3: All of the paths in the Lefschetz fibration needed to assemble the tori and cobordism.

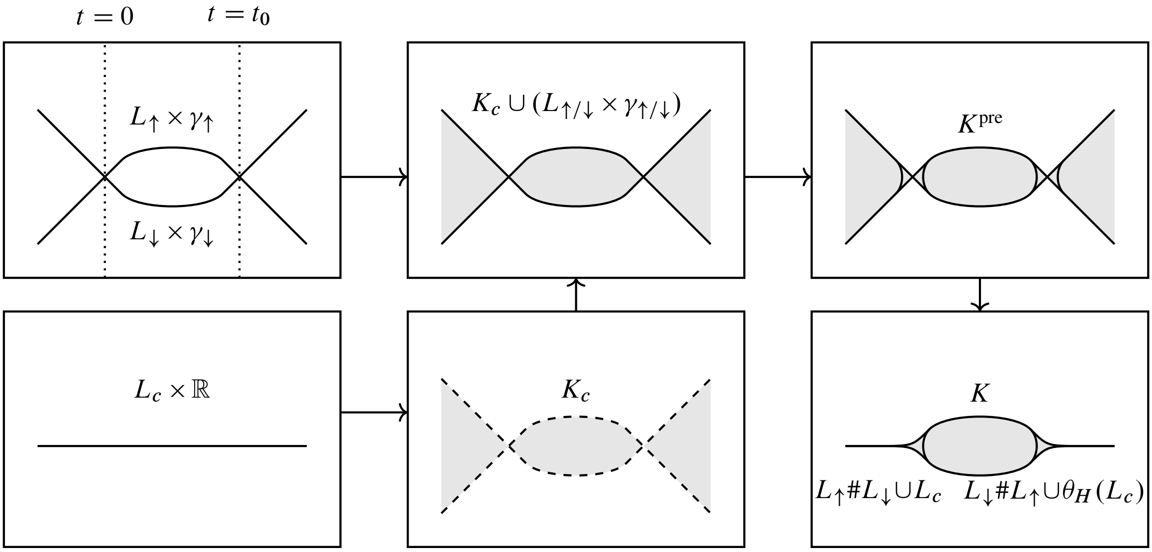

Figure 4: Steps to build our Lagrangian cobordism by projection to the cobordism $\mathbb{C}$ parameter of $X\times\mathbb{C}$.

Ideally at this point we would glue the cobordism $L_c\times\mathbb{R}$ to the concatenation $K_r\circ K_l$ to obtain a cobordism between the Chekanov and product tori. However, the Lagrangian $L_c\times\mathbb{R}$ does not glue to the cobordism $K_r\circ K_l$, as the boundary of $K_r\circ K_l$ corresponding to the ends of the thimbles $L^{\uparrow/\downarrow}$ moves around the parameter space of the cobordism along curves dictated by the gluing profile. We therefore modify $L_c\times\mathbb{R}$ by a Hamiltonian isotopy so these pieces overlap and we can glue them together.

To achieve this, we need a more precise description of the construction of the Lagrangian surgery trace cobordism. The surgery trace cobordism is obtained by performing the surgery in one dimension higher and stretching the neck of the surgery; see [7, Section 6.1]. We will build our cobordism between Chekanov and product tori by performing surgery on an immersed Lagrangian submanifold. This immersed Lagrangian submanifold is the suspension of an exact homotopy $\phi_t\colon L_{S_2}\times\mathbb{R}\to X$ which fixes the points of self-intersection. However, the primitive $H_t\colon L_{S^2}\times\mathbb{R}\to\mathbb{R}$ for $\iota_{(d\phi_t/dt)}\omega|_{L_{S_2}}$ which is used to construct the suspension cobordism will take different values at the self-intersection points of the Whitney sphere, so the Lagrangian suspension is embedded at all t_0 where $H_{t_0}\colon L\times\{t_0\}\to\mathbb{R}$ is nonconstant. As we will later want to compute the flux of this exact homotopy, we construct this suspension of the exact homotopy using the pieces drawn in Figure 4. Take paths $\gamma_{\uparrow/\downarrow}(t)\colon\mathbb{R}\to\mathbb{C}$ as drawn in Figure 4 and consider the cylindrical Lagrangian cobordisms $L_\downarrow\times\gamma_\downarrow$ and $L_\uparrow\times\gamma_\uparrow$. Choose a parametrization of these paths so that $\gamma_{\uparrow/\downarrow}(t)=t+\iota f_{\uparrow/\downarrow}(t)$; see Figure 4, top left. By taking a translation of these paths, we assume that $f_{\uparrow/\downarrow}(0)=0$ and $f^{\pm}(t_0)=0$.

We will now glue to $(L_\downarrow\times\gamma_\downarrow)\sqcup(L_\uparrow\times\gamma_\uparrow)$ a Lagrangian which is topologically $L_c\times\mathbb{R}$. This cobordism will be the suspension of a time-dependent Hamiltonian isotopy of L_c. We pick a Hamiltonian $H_t\colon L_c\times\mathbb{R}\to\mathbb{R}$ which satisfies the following properties:

- The value of H_t on $L_c \times \mathbb{R} = [-S, S] \times S^1 \times \mathbb{R}$ is pulled back from $[-S, S]$ and is increasing in s.
- There exists a constant ϵ such that

$$H_t(L_c|_{s\in[-S,-S+\epsilon)}) = f_\downarrow(t) \quad \text{and} \quad H_t(L_c|_{s\in[S,S-\epsilon)}) = f_\uparrow(t).$$

We define the cobordism K_c to be the suspension of this Hamiltonian isotopy; see Figure 4, bottom left and bottom middle. By design, the Hamiltonian flow X_{H^t} is constant on the region $[\pm S, \pm S \mp \epsilon) \times S^1$ of L_c. Therefore the portion of K_c corresponding to the $[\pm S, \pm S \mp \epsilon) \times S^1 \times \mathbb{R}$ is exactly $L_c|_{s\in[\pm S,\pm S\mp\epsilon)} \times \gamma_{\uparrow/\downarrow}$; see Figures 3 and 4.

Claim 4.1.2 *The union $K_c \cup (L_\uparrow \times \gamma_\uparrow) \cup (L_\downarrow \times \gamma_\downarrow)$ is a smooth Lagrangian submanifold with* 2 *transverse self-intersections*; *see Figure 4, top middle.*

We now construct the mutation cobordism from $K_c \cup (L_\uparrow \times \gamma_\uparrow) \cup (L_\downarrow \times \gamma_\downarrow)$. First, we apply a Lagrangian surgery at both of the self-intersection points to obtain the Lagrangian submanifold K^{pre} (Figure 4, top right), which is an embedded Lagrangian submanifold. The slices of this Lagrangian cobordism at times 0 and t_0 are

$$K^{\mathrm{pre}}|_0 = (L_\uparrow \# L_\downarrow \cup L_c) \quad \text{and} \quad K^{\mathrm{pre}}|_{t_0} = (L_\downarrow \# L_\uparrow) \cup \theta_H(L_c).$$

These are product and Chekanov tori, respectively.

We construct the *mutation cobordism* K_{μ_D} (Figure 4) by taking the portion of K^{pre} between 0 and t_0 whose projection to the real component lies in $[0, t_0]$. To obtain a Lagrangian cobordism from this space, we take this portion and stretch it out to obtain a Lagrangian cobordism with ends. This stretching process is described in [19, Section 3.1.1], where it is called the truncation of K^{pre} [19, Definition 3.1.5]. In the notation of that paper, we write

$$K_{\mu_D} := K^{\mathrm{pre}}\|_{[0,t_0]}.$$

Remark 4.1.3 The construction of this Lagrangian cobordism can also be done by locally identifying the Lagrangians $L_\uparrow \times \gamma_\uparrow$ and $L_\downarrow \times \gamma_\downarrow$ with $\mathbb{R}^2 \times \mathbb{R}, \imath\mathbb{R}^2 \times \imath\mathbb{R} \subset \mathbb{C}^2 \times \mathbb{C} = \mathbb{C}^{2+1}$, then following the construction of the Lagrangian surgery trace in [7, Lemma 6.1.1]. The cutting and cylindrical neck attachment construction there is similar to the truncation procedure from [19], and produces a Lagrangian isotopic submanifold.

4.2 General mutations

In this section we look at a generalization of the above construction.

Theorem 4.2.1 (Haug [18]) *Suppose that D is a Lagrangian disk with boundary cleanly contained within an oriented Lagrangian submanifold L. There exists an immersed Lagrangian $\alpha_D(L) \looparrowright X$, called the **Lagrangian antisurgery**[1] **of L along D**, such that*

- *$\alpha_D(L)$ is topologically obtained from L by performing surgery along ∂D,*

[1]The character α is chosen for antisurgery as it looks like an immersed Lagrangian.

- *$\alpha_D(L)$ agrees with L outside of a small neighborhood of D,*
- *if L was embedded and disjoint from the interior of D, then $\alpha_D(L)$ has a single self-intersection point.*

Antisurgery is inverse to Lagrangian surgery in the sense that if L' is obtained from L by resolving a self-intersection point, then there exists an antisurgery disk on L' such that $\alpha_D(L') = L$. However, the choices in both surgery and antisurgery mean that the process of applying antisurgery followed by surgery need not construct the same Lagrangian. By combining antisurgery with surgery, we can obtain a new embedded Lagrangian. We now restrict to complex dimension two.

Definition 4.2.2 [29] Let L be an embedded Lagrangian and D a surgery disk. Let $\alpha_D(L)$ be obtained from D by antisurgery. A *mutation of L along D* is the Lagrangian $\mu_D^\epsilon(L)$ obtained from $\alpha_D(L)$ by resolving the resulting single self-intersection point in opposite direction. The width ϵ of the mutation is the sum of the fluxes swept from L to $\alpha_D(L)$ to $\mu_D^\epsilon(L)$.

An example of Lagrangians obtained by mutation are the monotone Chekanov and product tori in $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$, where surgery occurs along the disk $z_1 = \bar{z}_2$. It is expected that Lagrangians which are related by mutation give different charts on the moduli space of Lagrangian submanifolds in the Fukaya category, and that these charts are related by a wall-crossing formula tied to cluster transformation [29].[2] Just as there exists a Lagrangian cobordism between the Chekanov and product tori in $\mathbb{C}^2$, there exist Lagrangian cobordisms between Lagrangians related by mutation.

Theorem 4.2.3 (Haug [18]) *Let L be a Lagrangian and let D be an antisurgery disk for L. There exists a Lagrangian cobordism between L and $\mu_D^\epsilon(L)$.*

Definition 4.2.4 (Pascaleff and Tonkonog [29]) A *mutation configuration* is a pair (L, D), where D is an antisurgery disk for L with boundary in a primitive homology class. The *mutation cobordism* is the cobordism K_{μ_D} with ends on L and $\mu_D^\epsilon(L)$. We call $[\partial D^2] \subset H_1(L, \mathbb{Z})$ the mutation class or mutation direction.

A key feature of Lagrangian mutation is that the Lagrangians L and $\mu_D^\epsilon(L)$ are Lagrangian isotopic.

We will show that the Lagrangian cobordism K_{μ_D} can induce a continuation map between the Lagrangians L and $\mu_D^\epsilon L$. The proof attempts to follow the toy model of the suspension cobordism considered in Section 3. However, the mutation cobordism is not topologically a cylinder so we cannot simply copy our proof from Section 3. To overcome this difference, we will prove the following replacements for cylindricity of the cobordism K_{μ_D}:

- We first describe the Morse theory of K_{μ_D} in Section 4.3.1.

[2]We note here that our definition of mutation differs slightly from that in [29], as in our setting there is some symplectic flux swept between $L, \alpha_D(L)$ and $\mu_D^\epsilon(L)$. Usually one requires that the amount of symplectic flux between these Lagrangians is zero. This would be necessary to preserve monotonicity of a Lagrangian submanifold.

- Section 4.3.2 constructs a holomorphic disk with boundary on this cobordism and Section 4.4.1 shows that the m^0 contribution of this disk can be deformed away to higher filtration.
- We show that the restriction of the differential to the right end of the cobordism can be inverted in Section 4.4.2, allowing us to apply a theorem on mapping cocylinders for A_∞ algebras in Section 4.4.3.

This will give us enough leverage to construct a mapping cocylinder from the Lagrangian cobordism K_{μ_D} and prove the following wall-crossing theorem:

Theorem 4.2.5 *Suppose that (L, D) is an isolated mutation configuration (Definition 4.2.6). Then there exists a deforming cochain d_ϵ for $\mathrm{CF}^\bullet(K_{\mu_D})$ such that ${\mathrm{CF}^\bullet}_{d_\epsilon}(K_{\mu_D})$ is an A_∞ mapping cocylinder between ${\mathrm{CF}^\bullet}_{\beta^-_*(d_\epsilon)}(L^-)$ and ${\mathrm{CF}^\bullet}_{\beta^+_*(d_\epsilon)}(L^+)$.*

Definition 4.2.6 is required to rule out the possible appearance of holomorphic disks with boundary on K_{μ_D} which may appear during the "suspension" portion of the cobordism K_{μ_D}.

Definition 4.2.6 Let $\epsilon = \omega(u_{\mathrm{ex}})$. We say that the mutation μ_D is isolated if the only holomorphic disk u with boundary on K_{μ_D}, $\langle [\partial u], x^\pm \rangle \neq 0$ and $\omega(u) \leq \omega(u_{\mathrm{ex}})$ is u_{ex}.

We anticipate that it should be possible to decompose a mutation cobordism into a concatenation of suspensions and mutation cobordisms so that the mutation is isolated. The cobordism decomposition framework studied in [19] should provide this decomposition in general. We will show that the prototypical example of a mutation cobordism between Chekanov and product type tori studied in Section 5 is isolated.

Notation 4.2.7 We will say deforming cochain to denote an element $d \in \mathrm{CF}^\bullet(K_{\mu_D})$ which we use to deform the A_∞ algebra (as in [17, Definition 3.6.9]) but which is not necessarily a bounding cochain (ie m^0_d may or may not be 0). Usually we will employ these to deform the curvature in such a way that $\nu(m^0_d)$ is larger than some specified amount.

4.3 Geometric input

We will simplify our notation substantially by restricting to the setting where $L = T^2$. The Floer cochain complex of K_{μ_D} is dependent on the data of a Morse function. By picking an appropriate Morse function, we can simplify the later discussion.

Notation 4.3.1 In this section let $L^- = L$ and $L^+ = \mu_D(L)$. K_{μ_D} is constructed from L^- by attachment of a 1–cell and a 0–cell. However, there is a slightly more symmetric treatment to K_{μ_D}. Consider an intermediate Lagrangian L^0, which is obtained by performing antisurgery on L^-. This is an immersed Lagrangian 2–sphere. Both L^- and L^+ may be obtained from L^0 via attachment of a 1–cell. By abuse of notation, we will also use L^-, L^+ and L^0 to refer to the submanifolds of K_{μ_D} which (upon inclusion into $X \times \mathbb{C}$ and projection to X) correspond to the slices L^-, L^+ and L^0.

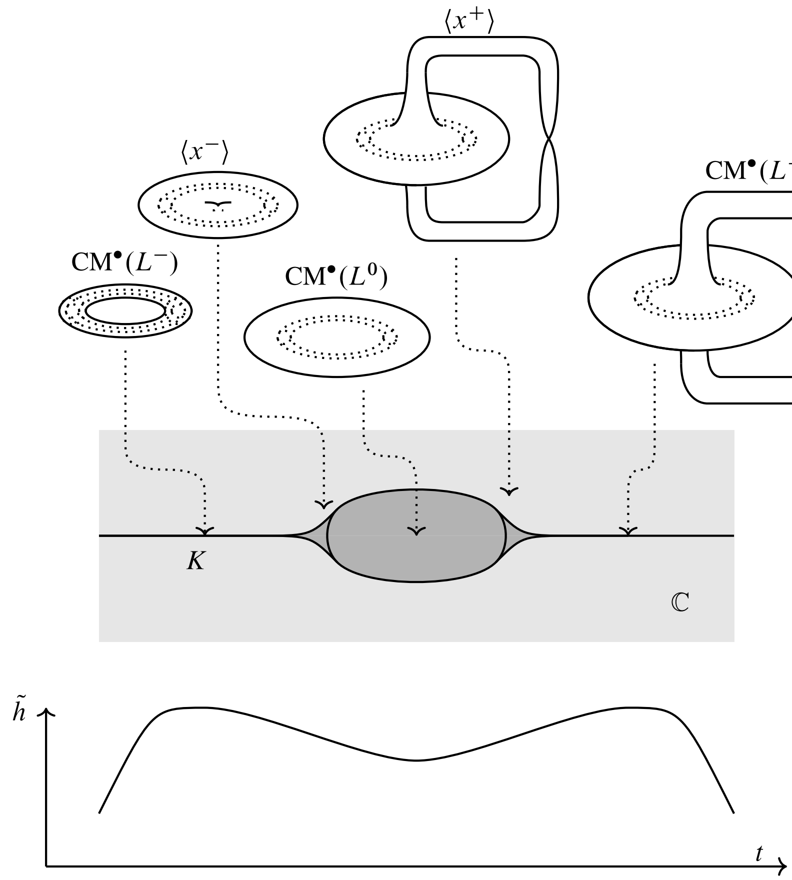

Figure 5: The sublevel sets of the Morse–Bott function for K_{μ_D} at the critical values of $\tilde{h}$. Notice that L^0 separates the cobordism into two halves.

4.3.1 Morse theory for K_{μ_D} Consider a Morse–Bott function $\tilde{h}\colon K_{\mu_D} \to \mathbb{R}$ which has a minimum along L^0, maximums at the ends $L^\pm$ and two index-2 critical points corresponding to the attached 1–cells (Figure 5).

We now give a decomposition of the Morse cochain complex, with the goal of understanding where $\mathrm{CM}^\bullet(K_{\mu_D})$ fails to be a mapping cocylinder. The Morse complex $\mathrm{CM}^\bullet(K_{\mu_D})$ has the shape

$$\mathrm{CM}^\bullet(K_{\mu_D}) = \left(\begin{array}{ccccc} \mathrm{CM}^\bullet(L^-) & & & & \mathrm{CM}^\bullet(L^+) \\ & \searrow \quad \searrow & & \swarrow \quad \swarrow & \\ & x^- \to & \mathrm{CM}^\bullet(L^0)[1] & \leftarrow x^+ & \end{array} \right),$$

where x^- and x^+ are the critical points corresponding to the handles. We will call the subspace spanned by x^- and x^+

$$E^0 := \Lambda\langle x^-, x^+\rangle.$$

Claim 4.3.2 *The spaces spanned by*

$$\mathrm{CF}^\bullet(L^-) \oplus \mathrm{CF}^\bullet(L^0) \oplus E^0 \quad \textit{and} \quad \mathrm{CF}^\bullet(L^+) \oplus \mathrm{CF}^\bullet(L^0) \oplus E^0$$

are filtered A_∞ ideals of $\mathrm{CF}^\bullet(K_{\mu_D})$.

Proof These are the kernels of $\beta^\pm \colon \mathrm{CF}^\bullet(K_{\mu_D}) \to \mathrm{CF}^\bullet(L^\pm)$, which, by Theorem C.5.1, are A_∞ homomorphisms. □

Notation 4.3.3 For this section, whenever we write an A_∞ product map, it will always mean the A_∞ product structure on $\mathrm{CF}^\bullet(K_{\mu_D})$.

Setting $A^\pm = \mathrm{CM}^\bullet(L^\pm)$ and $A^0 = \mathrm{CM}^\bullet(L^0) \oplus E^0$, we can write the differential on $\mathrm{CM}^\bullet(K_{\mu_D}) = A^- \oplus A^0 \oplus A^+$ as

$$\underline{m}^1 = \begin{pmatrix} (\underline{m})^{-;-} & 0 & 0 \\ (\underline{m})^{-;0} & (\underline{m})^{0;0} & (\underline{m})^{+;0} \\ 0 & 0 & (\underline{m})^{+;+} \end{pmatrix},$$

where $(\underline{m})^{\pm;\pm} = m^1_{\mathrm{CM}^\bullet(L^\pm)}$ are the Morse differentials. If the ideal $A^+ \to A^0$ was nullhomotopic — for example if $(\underline{m})^{+;0}$ was an isomorphism — then $\mathrm{CM}^\bullet(K_{\mu_D})$ would be a mapping cocylinder. We will further decompose this cochain complex by identifying subspaces of $\mathrm{CM}^\bullet(L^\pm)$ which correspond to the mutation direction and tease out exactly how our complex fails to be a mapping cocylinder.

The kernel of $(\underline{m})^{+;0}$ is one dimensional. We pick an element c^{u+} which generates the kernel so that we may write

$$\ker((\underline{m})^{+;0}) = \Lambda \cdot \langle c^{u+} \rangle.$$

We restrict to the 1–cochains $E^\pm := \mathrm{CM}^1(L^\pm)$. Pick elements $c^{w\pm} \in E^\pm$ such that $(\underline{m})^{\pm;0}(c^{w\pm}) = x^\pm$. Then the classes $c^{w\pm}$ and $c^{u\pm}$ now span the $E^\pm$.

Remark 4.3.4 The class c^{u+} is determined completely by the cobordism as the Morse cohomology class whose downward flow space is the surgery disk. However, the selection of c^{w+} requires a choice similar to the choices required to produce local coordinates on the moduli space of Lagrangian tori in Section 5.2.

With this description, the noncylindricity of $\mathrm{CM}^\bullet(K_{\mu_D})$ exactly corresponds to the nonsurjectivity of the map $(\underline{m})^{+;0}$ in the diagram

$$E^- \xrightarrow{(\underline{m})^{-;0}} E^0 \xleftarrow{(\underline{m})^{+;0}} E^+.$$

In the chosen basis $\{c^{u\pm}, c^{w\pm}\}$ for $E^\pm$ and $\{x^-, x^+\}$ for E_0, the maps $(\underline{m})^{\pm;0}$ are

$$(\underline{m})^{-;0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (\underline{m})^{+;0} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark 4.3.5 In the more general case where L is not a torus, identify $E^\pm$ as the preimage of the above-defined map $m^{\pm;0} \colon \mathrm{CM}^\bullet(L^\pm) \to E^0$. This will include the kernel of $m^\pm$. A choice of basis $\{c^{u\pm}, c^{w\pm}\}$ for $E^\pm$ and a splitting $H^1(L^\pm) = H^1(L)^\pm / E^\pm \oplus E^\pm$ will be required to construct a bounding cochain at a later step; see (5). The portion which is independent of the handle, $H^1(L^\pm)/E^\pm$, is isomorphic to $H^1(L^0)$. This corresponds to the "cylindrical" portion of the Morse theory of K, on which $m^{+;0}$ is invertible at filtration zero.

The construction of the deforming cochain d_ϵ in Theorem 4.2.5 depends on these choices. Taking different choices will yield different deforming cochains d_ϵ, and thus identify L^- and L^+ with a different choice of bounding cochain $\beta_*^\pm(d_\epsilon)$. We expect that looking at all possible choices in (5) identifies a locus of the Maurer–Cartan solutions $\mathcal{MC}(L^-)$ with a locus in $\mathcal{MC}(L^+)$, corresponding to the choices of bounding cochains which make L^- and L^+ isomorphic objects of the Fukaya category. From a mirror symmetry perspective, when L^- and L^+ are SYZ fibers we propose this locus is mirror to the intersection between the torus charts constructed around the mirror points to L^- and L^+.

A different viewpoint is that the A_∞ homomorphism constructed by $\mathrm{CF}^\bullet{}_{d_\epsilon}(K_{\mu_D})$ between $\mathrm{CF}^\bullet{}_{\beta_*^-(d_\epsilon)}(L^-)$ and $\mathrm{CF}^\bullet{}_{\beta_*^+(d_\epsilon)}(L^+)$ induces a map between their Maurer–Cartan solutions. By then appropriately identifying the Maurer–Cartan elements of $\mathrm{CF}^\bullet{}_{\beta_*^\pm(d_\epsilon)}(L^\pm)$ with those of $\mathrm{CF}^\bullet(L^\pm)$, we obtain a pairing between certain Maurer–Cartan elements of L^- with those of L^+. This is the approach taken in Section 5.

4.3.2 A holomorphic disk We now show that for a specific choice of almost complex structure there exists a pseudoholomorphic disk with boundary on K_{μ_D}.

Proposition 4.3.6 *There exists a choice of complex structure such that K_{μ_D} bounds a regular Maslov index-0 disk*

$$u_{\mathrm{ex}}\colon (D^2, \partial D^2) \to (X \times \mathbb{C}, K_{\mu_D}).$$

The contribution of this disk to the m^0 term is $x^+ + x^-$.

Proof We proceed by exhibiting the disk in the local model constructed in Section 4.1. Following [18, Section 2.3] we can construct a standard neighborhood U of the Lagrangian antisurgery disk D, identifying U with a subset of $\mathbb{C}^2$ (which, by abuse of notation, we shall also call U). We choose a complex structure for X which matches the standard complex structure on $U \subset \mathbb{C}^2$. We can pick this identification so that the Lagrangians L^- and L^+ when restricted to U match the local model based on the Lefschetz fibration (see [18, Section 2.3] for implanting the local model). In these coordinates $L_\downarrow \cap L_\uparrow = (0,0)$. Then

$$L^- \cap U = (L_\downarrow \# L_\uparrow) \cap U \quad \text{and} \quad L^+ \cap U = (L_\uparrow \# L_\downarrow) \cap U.$$

The restriction $K_U := K_{\mu_D} \cap (U \times \mathbb{C})$ is the eye-shaped Lagrangian cobordism drawn in Figure 2. Recall that K_U is obtained by first considering $(L_\downarrow \times \gamma_\downarrow) \cup (L_\uparrow \times \gamma_\uparrow)$, then resolving the self-intersections to obtain K_U^{pre}, and subsequently truncating to get a Lagrangian cobordism. The chart $(L_\downarrow \times \gamma_\downarrow) \cup (L_\uparrow \times \gamma_\uparrow)$ contains a holomorphic 2–gon with ends limiting the two self-intersection points. The goal is to show that, when surgery is applied at both self-intersections, the corners of this bigon are rounded to obtain a holomorphic disk with boundary on K_U^{pre}. The truncation will occur away from the boundary of this holomorphic disk, so the holomorphic disk survives to give a holomorphic disk with boundary on K_U.

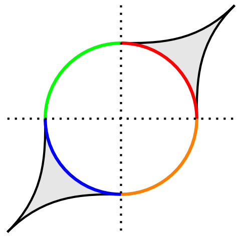

Figure 6: Dividing K_U into four quarters.

The holomorphic bigon $u_2\colon D^2 \setminus \{-1,1\} \to U \times \mathbb{C}$ with boundary on $(L_\downarrow \times \gamma_\downarrow) \cup (L_\uparrow \times \gamma_\uparrow)$ has a particularly nice description. Let $u_2'\colon \mathbb{R} \times [0,1] \to \mathbb{C}$ be the holomorphic bigon which parametrizes the area between the curves $\gamma_\downarrow$ and $\gamma_\uparrow$. Then u_2 is given by

$$u_2\colon D^2 \setminus \{-1,1\} \to U \times \mathbb{C}, \quad z \mapsto ((0,0), u_2'(z)).$$

From this observation, our candidate construction will be to show that $\pi_U^{-1}(0,0)$ has clean S^1 intersection with K_U^{pre}. The interior of the S^1 will then be a holomorphic disk with boundary on S^1.

We need to describe the intersection of $\pi_U^{-1}(0,0)$ with the chart of K_U^{pre} which corresponds to the surgery neck. Using the same surgery profile as in (3), the surgery chart for K_U^{pre} is parametrized by

$$\bigl\{\bigl((a(t)+\imath b(t))(\hat{x}+\imath\hat{y}), (a(t)+\imath b(t))(\hat{y}+\imath\hat{x}), (a(t)+\imath b(t))\hat{z}\bigr) \mid \hat{x}^2+\hat{y}^2+\hat{z}^2=1\bigr\}, \tag{4}$$

where the first two coordinates are in U and the last coordinate factors through the cobordism parameter. By setting $\hat{x}=\hat{y}=0$ and $\hat{z}=1$, we obtain a smooth curve that passes through the surgery neck parametrizing the clean intersection between the complex line $\pi_U^{-1}(0,0)$ and the surgery neck. We call the corresponding disk $u_{\mathrm{ex}}\colon (D^2, \partial D^2) \to (U \times \mathbb{C}, K_{\mu_D})$. We now show that this disk is regular using a criterion of [27]. This requires computing the partial Maslov indices of the boundary of u_{ex}. The boundary of u_{ex} can be broken up into 4 components:

- the "neck portions", which correspond to where the boundary passes through the surgery neck of K^{pre}, and
- the "fiber trivial portions", where the boundary is contained in $L_\uparrow \times \gamma_\uparrow$ and $L_\downarrow \times \gamma_\downarrow$.

For simplicity, we will assume that the profile curve $a(t)+\imath b(t)$ draws a quarter circle, even though this doesn't provide a smooth Lagrangian surgery neck. We also will assume that the curves $\gamma_{\uparrow/\downarrow}$ are quarter circles, and choose coordinates so that $K|_U$ has projection as drawn in Figure 6. We now consider a parametrization of the boundary of the holomorphic disk $\gamma(\theta)\colon S^1 \to \partial u_{\mathrm{ex}}$ such that we may parametrize $TK^{\mathrm{pre}}|_{\partial u_{\mathrm{ex}}}$ by maps $C \cdot A(\theta)\colon \mathbb{R}^3 \to \mathbb{C}^3$, Here C is a constant matrix, and $A(\theta)$ describes the rotation of the tangent space as we progress through the coordinate θ. We subdivide S^1 into four quarters corresponding to the portions of ∂u_{ex} which pass through the right surgery chart, $L_\uparrow \times \gamma_\uparrow$ chart, left surgery chart and $L_\downarrow \times \gamma_\downarrow$ chart.

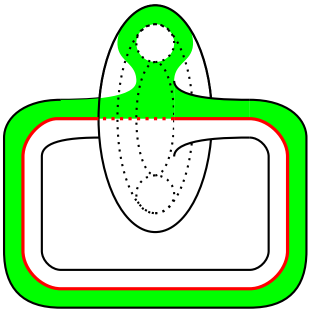

Figure 7: The red cycle in the cobordism K represents the boundary of an exceptional Maslov index-0 disk contributing to nontrivial $m^0_{K_{\mu_D}}$. The green 2–cycle represents a "deforming cochain" which cancels out the lowest-order contribution of this disk.

- For $0 \leq \theta < \frac{1}{2}\pi$, the curve $\gamma(\theta)$ passes through the right surgery chart. A basis for the tangent space of (4) along the curve $(0, 0, e^{\iota\theta})$ is given by
$$A(\theta) = \begin{pmatrix} e^{-\iota\theta} & 0 & 0 \\ 0 & e^{-\iota\theta} & 0 \\ 0 & 0 & e^{\iota\theta} \end{pmatrix}.$$
- Next, for $\theta < \frac{1}{2}\pi \leq \theta \leq \pi$, the curve $\gamma(\theta)$ passes through the chart parametrized by $L_\uparrow \times \gamma_\uparrow$. The tangent space in the fiber is constant, and we only see rotation of the tangent space in the base component of $X \times \mathbb{C}$:
$$A(\theta) = \begin{pmatrix} -\iota & 0 & 0 \\ 0 & -\iota & 0 \\ 0 & 0 & e^{\iota\theta} \end{pmatrix}$$

The rotation of the tangent space through the left surgery chart and $L_\downarrow \times \gamma_\downarrow$ charts is analogous. This shows that the tangent bundle splits as a direct sum of rank-1 pieces. The first two pieces rotate a negative half-turn, while the component corresponding to the cobordism parameter of $X \times \mathbb{C}$ makes a full positive rotation. The partial Maslov indices of the loop $TK^{\text{pre}}|_{\partial u_e x}$ are -1, -1 and 2. Therefore, this is a disk of Maslov index 0, and it is regular by the criterion of [27, Theorem II].

The homological class of this disk is the loop in the cobordism K which is introduced from the two 1–handle attachments. Therefore, the upward flow spaces of $x^\pm$ each meet the boundary of this disk once. Since our local model was chosen to have the same J–structure as X, this gives us an isolated Maslov index 0 disk $u_{\text{ex}} \colon D^2 \to X \times \mathbb{C}$ whose boundary lies on the cobordism K, and whose boundary contributes to the Morse cohomology class $x^+ + x^-$. □

The boundary of this disk is drawn in Figure 7. This is a holomorphic disk which is isolated in the following sense:

Claim 4.3.7 *For the standard holomorphic structure, the disk u_{ex} is the only disk contained in U with boundary on K_{μ_D} in this relative homology class.*

Proof Every disk with boundary on $K_{\mu_D} \cap (U \times \mathbb{C}) \subset U \times \mathbb{C}$ in the same relative homology class as u_{ex} gives a class of disk in $H_2\bigl(\mathbb{C}, \pi_{\mathbb{C}}(K_{\mu_D} \cap (U \times \mathbb{C}))\bigr)$. The boundary $\pi_{\mathbb{C}}(\partial u_{\mathrm{ex}})$ — drawn as the red loop in Figure 2 — circles around a connected component of $\mathbb{C} \setminus \pi_{\mathbb{C}}(K_{\mu_D} \cap (U \times \mathbb{C}))$ whose area is $\omega_{U\times\mathbb{C}}(u_{\mathrm{ex}})$. The red loop is the minimal loop which encircles this connected component. Therefore, any other holomorphic disk $u\colon (D^2, \partial D^2) \to (U \times \mathbb{C}, K_{\mu_D} \cap (U \times \mathbb{C}))$ with boundary in the same class as u_{ex} will have projection $\pi_{\mathbb{C}} \circ u$ with area at least $\omega_{\mathbb{C}}(u_{\mathrm{ex}})$.

As $\omega_{U\times\mathbb{C}}(u) = \omega_{\mathbb{C}}(u) + \omega_U(u)$, the projection $\pi_U \circ u$ has zero area. However, $\pi_U \circ u$ is a holomorphic map, so the zero-area condition means that $\pi_U \circ u$ is constant. The proof is completed upon checking for which $p \in U$ the preimage $\pi_U^{-1}(p) \cap K \cap U$ contains a loop in the appropriate homology class. This homology class necessarily transverses the $(L_\downarrow \times \gamma_\downarrow)$ and $(L_\uparrow \times \gamma_\uparrow)$ charts, so $p \in L_\uparrow \cap L_\downarrow = \{(0,0)\}$. Therefore u_{ex} is the only such example. □

4.4 Algebraic manipulations

Notation 4.4.1 Let ϵ be the area $\omega_{X\times\mathbb{C}}(u_{\mathrm{ex}})$.

The presence of the disk u_{ex} is the first step to proving that $\mathrm{CF}^\bullet(K_{\mu_D})$ is a mapping cocylinder. The disk deforms the Morse products in two significant ways:

- The holomorphic disk contributes to m^0 as $m^0_{[u_{\mathrm{ex}}]} = T^\epsilon(x^+ + x^-)$.
- The boundary ∂u_{ex} intersects the downward flow space of c^{u+} at a point and the upward flow space of x^- at a point. This will deform the Morse differential on $\mathrm{CM}^\bullet(K_{\mu_D})$. The nonsurjectivity of the map $\underline{m}_K^{+;0}$ in the Morse setting (in comparison to the cylindrical setting given by (2)) is an obstruction to giving $\mathrm{CM}^\bullet(K_{\mu_D})$ the structure of a mapping cylinder. In particular, the contribution of u_{ex} to the differential makes $\langle m^1_{[u_{\mathrm{ex}}]}(c^{w+}), x^-\rangle = T^\epsilon$, whereas the underlying Morse differential has $\langle \underline{m}^1_{[u_{\mathrm{ex}}]}(c^{w+}), x^-\rangle = 0$. When this deformation is considered, the map $m^{+;0}\colon E^+ \to E^0$ is surjective, giving us some hope that $\mathrm{CF}^\bullet(K_{\mu_D})$ is a mapping cylinder.

This nearly proves that $\mathrm{CF}^\bullet(K_{\mu_D})$ is a mapping cocylinder. However, Theorem A.3.2 requires that $(m^{+;0})^{-1}(m^0)$ lives in positive filtration. As a result, we cannot immediately apply Theorem A.2.1. This leaves the following steps to construct a mapping cocylinder:

(i) **Deforming to increase $\nu(m^0|_{E^0})$** In Section 4.4.1 we first equip K_{μ_D} with a deforming cochain d_ϵ which will cancel out the curvature contribution of the disk u_{ex}, so that the energy of $m^0_{d_\epsilon}$ will be greater than ϵ.

(ii) **Inverting $(m_{d_\epsilon})^{+;0}$** In Section 4.4.2 we show that the map $(m_{d_\epsilon})^{+;0}$ can now be inverted. This is complicated by the introduction of the deforming cochain d_ϵ.

(iii) **Applying Theorem A.3.2** In Section 4.4.3 we conclude that ${\mathrm{CF}^\bullet}_{d_\epsilon}(K_{\mu_D})$ is a filtered mapping cocylinder. As a corollary, we conclude that K_{μ_D} is unobstructed whenever L^- is.

We will make a simplifying assumption (Definition 4.2.6) to complete (ii).

4.4.1 Deforming to remove $m^0|_{E^0}$ When working over the Novikov field we can prove that a map is invertible by showing that it is invertible at low filtration. We will denote by $\nu\colon \Lambda \to \mathbb{R} \cup \{\infty\}$ the Novikov valuation, with the convention that $\nu(0) = \infty$. Following [33], we will use *non-Archimedean normed vector spaces*, which are Λ–vector spaces equipped with a filtration function $\ell\colon A \to \mathbb{R} \cup \{\infty\}$ satisfying[3] the axioms of [33, Definition 2.2].

We introduce some notation which will help us use this method of proof:

Definition 4.4.2 Let (A, ℓ_A) and (B, ℓ_B) be non-Archimedean normed vector spaces. The *filtration* of a map $\Theta\colon A \to B$ is the largest jump in the filtration map under Θ:

$$\ell_{A,B}(\Theta) := \sup_{v : \ell_A(v) = 0} \{\ell_B(\Theta(v))\}.$$

The *leading order* of a map $\Theta\colon A \to B$ is the largest drop in filtration map under Θ:

$$\mathrm{ord}_{A,B}(\Theta) := \inf_{v : \ell_A(v) = 0} \{\ell_B(\Theta(v))\}.$$

Our reason for using filtration of maps is to construct inverses:

Claim 4.4.3 *If $\ell_{A,B}(\Theta) < \infty$, then Θ has a left inverse. If this is also a right inverse, then*

$$\ell_{A,B}(\Theta) = -\mathrm{ord}_{B,A}(\Theta^{-1}) \quad \textit{and} \quad \ell_{B,A}(\Theta^{-1}) = -\mathrm{ord}_{A,B}(\Theta).$$

Proof The condition $\ell_{A,B}(\Theta) < \infty$ states that $\Theta(v) \neq 0$ for any $v \neq 0$, so our map has a left inverse.

Suppose that Θ^{-1} is the inverse of Θ. Let v_i be a sequence of vectors with $\ell_A(v_i) = 0$ such that $\lim_{i\to\infty} \ell_B(\Theta(v_i)) = \ell_{A,B}(\Theta)$. Since $\ell_A(\Theta^{-1} \circ \Theta(v_i)) = \ell_A(v_i) = 0$, we obtain that $\mathrm{ord}_{B,A}(\Theta^{-1}) \geq -\ell_{A,B}(\Theta)$. Similarly, let w_i be a sequence of vectors with $\ell_B(w_i) = 0$ such that $\lim_{i\to\infty} \ell_B(\Theta^{-1})(w_i) = \mathrm{ord}_{B,A}(\Theta^{-1})$. Since $\ell_B(\Theta \circ \Theta^{-1}(w_i)) = \ell_B(w_i) = 0$, we obtain that $\ell_{A,B}(\Theta) \leq -\mathrm{ord}_{B,A}(\Theta^{-1})$. □

If one possesses a bound on the filtration and order of a map, then one can obtain a bound on their sum.

Claim 4.4.4 *Suppose that $\ell_{A,B}(\Theta_1) < \mathrm{ord}_{A,B}(\Theta_2)$. Then $\ell_{A,B}(\Theta_1 + \Theta_2) = \ell_{A,B}(\Theta_1)$.*

Proof First we note that for any vector v,

$$\ell_B(\Theta_1(v)) \leq \ell_{A,B}(\Theta_1) < \mathrm{ord}_{A,B}(\Theta_2) \leq \ell_B(\Theta_2(v)),$$

and so by application of the non-Archimedean triangle inequality,

$$\ell_B(\Theta_1(v) + \Theta_2(v)) = \ell_B(\Theta_1(v)).$$

Let v_i be a sequence of vectors with $\ell_A(v_i) = 0$ realizing $\lim_{i\to\infty} \ell_B(\Theta_1(v_i)) = \ell_{A,B}(\Theta_1)$. Then

$$\ell_{A,B}(\Theta_1 + \Theta_2) \geq \lim_{i\to\infty} \ell_B(\Theta_1(v_i) + \Theta_2(v_i)) = \lim_{i\to\infty} \ell_B(\Theta_1(v_i)) = \ell_{A,B}(\Theta_1).$$

[3]To match the conventions from [10], our filtration function has the opposite sign convention of [33].

For the other direction, let w_i be a sequence of zero-filtration vectors with $\lim_{i\to\infty}\ell_B((\Theta_1+\Theta_2)(w_i)) = \ell_{A,B}(\Theta_1+\Theta_2)$. Then

$$\ell_{A,B}(\Theta_1+\Theta_2) = \lim_{i\to\infty}\ell_B(\Theta_1(w_i)+\Theta_2(w_i)) = \lim_{i\to\infty}\ell_B(\Theta_1(w_i)) \leq \ell_{A,B}(\Theta_1). \qquad \square$$

Definition 4.4.5 Let $\Theta\colon A\to B$ be a map. We say that Θ has leading terms $\Theta_{\leq\lambda}$ of filtration λ and write

$$\Theta = \Theta_{\leq\lambda} + O(\lambda)$$

if $\Theta = \Theta_{\leq\lambda}+R$, with

$$\ell_{A,B}(\Theta_{\leq\lambda}) = \lambda < \operatorname{ord}_{A,B}(R).$$

We will now assume that μ_D is an isolated mutation.

Claim 4.4.6 *Suppose that $\mu_D(L)$ is an isolated mutation. Let $\pi_{E^0}\colon \mathrm{CF}^\bullet(K_{\mu_D})\to E_0$ be the standard projection. Define $\lambda := \ell_{E_0}(\pi_{E^0}\circ m^0)$. There exists a deforming cochain $d\in \mathrm{CF}^\bullet(K_{\mu_D})$ which increases the filtration of the curvature, giving*

$$\ell_{E_0}(\pi_{E^0}\circ m^0_d) > \lambda$$

with $\ell(d)\geq\lambda$.

Proof The restriction $\pi_{E^0}\circ m^1|_{E^-\oplus E^+}\colon E^-\oplus E^+\to E^0$ surjects, and has a right inverse D. After choosing this right inverse we can construct

$$d := -D(m^0)|_{\Lambda\langle x^-,x^+\rangle}. \tag{5}$$

Because the order of D is zero, $\ell(d)\geq\lambda$. Then

$$\begin{aligned}\ell_{E_0}(\pi_{E^0}\circ m^0_d) &= \ell_{E_0}\Big(\pi_{E^0}\circ m^0 + \pi_{E^0}\circ m^1(d) + \sum_k \pi_{E^0}\circ m^k(d^{\otimes k})\Big)\\ &\geq \min\Big(\ell_{E_0}(\pi_{E^0}\circ(m^0+m^1(d))), \ell_{E_0}\Big(\sum_k \pi_{E^0}\circ m^k(d^{\otimes k})\Big)\Big).\end{aligned}$$

By Definition 4.2.6, the lowest-energy term of m^0 is $T^{\omega(u_{\mathrm{ex}})}\cdot(x^-+x^+)$, which exactly cancels $\underline{m}^1(d)$. Therefore, $\ell_{E_0}\big(\pi_{E^0}(m^0+m^1(d))\big)>\lambda$. Since $\ell(m^k(d^{\otimes k}))\geq k\lambda$, we conclude

$$\ell_{E_0}(\pi_{E^0}\circ m^0_d) > \min(\lambda, 2\lambda) = \lambda. \qquad \square$$

4.4.2 Inverting $m^{+;0}|_{E^+}$

Choose the deforming cochain from Claim 4.4.6 so that $\ell(\langle m^0_{d_\epsilon}, E^0\rangle) > \epsilon$. Because our mutation is assumed to be isolated (Definition 4.2.6), we obtain a lower bound for the filtration of the deforming cochain:

$$\nu(d_\epsilon)\geq\epsilon. \tag{6}$$

Claim 4.4.7 *The map $(m^{+;0})_{d_\epsilon}\colon E^+\to E^0$ is invertible, with $\operatorname{ord}_{E^+,E^0}\big(((m^{+;0})_{d_\epsilon})^{-1}\big)\geq -\epsilon$.*

Proof We expand $(m^{+;0})_{d_\epsilon}$ termwise:

$$(m_{d_\epsilon})^{+;0}|_{E^+} = m^{+;0} + m_0^2(d_\epsilon\otimes\mathrm{id}+\mathrm{id}\otimes d_\epsilon)|_{E^+}\pi_{E^0}\circ\Big(\sum_{k>2} m^k(d_\epsilon^{\otimes k_1}\otimes\mathrm{id}\otimes d_\epsilon^{\otimes k_2})\Big)\Big|_{E^+}.$$

We then compute the valuation of each term. We will compute this map with respect to the basis $\{c^{u+}, c^{w+}\}, \{x^-, x^+\}$.

- The first term $m^{+;0}$ can be explicitly computed, as the isolated mutation condition means that the space of pearly flow lines of energy less than or equal to ϵ are cut out regularly, and the property Lemma B.3.2 means that these are the only pearly flow lines which contribute to the differential at an order of ϵ:

$$m^{+;0} = \begin{pmatrix} T^\epsilon & 0 \\ T^\epsilon & 1 \end{pmatrix} + O(\epsilon).$$

- The terms $m^2(d_\epsilon \otimes \mathrm{id} + \mathrm{id} \otimes d_\epsilon)|_{E^+}$ can be further split by the homology class of the disks which deforms the product m^2:

$$m^2 \circ (d_\epsilon \otimes \mathrm{id})|_{E^+} = \sum_{\substack{\beta \in H^2(X,L) \\ \omega(\beta) \geq 0}} m^2_\beta \circ (d_\epsilon \otimes \mathrm{id})\Big|_{E^+}.$$

Whenever $\omega(\beta) \geq 0$, the term $m^2_\beta(d_\epsilon \otimes c^{u+})$ has filtration at least $\nu(d_\epsilon) + \nu(\beta) > \epsilon$. So we need only worry about the classical portion product, $(\underline{m})^2(d_\epsilon \otimes \mathrm{id})$. Since $\mathrm{CM}^\bullet(L^+) \oplus \langle x^+ \rangle \oplus \mathrm{CM}^\bullet(L^0)$ is an ideal of $\mathrm{CM}^\bullet(K_{\mu_D})$,

$$\langle (\underline{m})^2(d_\epsilon \otimes c^{u+}), x^- \rangle = 0.$$

So we can write

$$\underline{m}^2(d_\epsilon \otimes \mathrm{id} + \mathrm{id} \otimes d_\epsilon)|_{E^+} = \begin{pmatrix} 0 & 0 \\ A_{u^+x^+} & A_{w^+,x^+} \end{pmatrix} + O(\epsilon),$$

where either $A_{u^+x^+}$ and A_{w^+,x^+} have filtration ϵ or are zero.

- The remaining terms $\left\langle \sum_{k>2} m^k(d_\epsilon^{\otimes k_1} \otimes \mathrm{id} \otimes d_\epsilon^{\otimes k_2}), x^- \right\rangle$ necessarily have filtration at least 2ϵ because the product is filtered and because of (6).

In summary, in the basis $\{c^{u+}, c^{w+}\}, \{x^-, x^+\}$,

$$\pi_{E^0} \circ (m_{d_\epsilon})^{+;0}|_{E^+} = \begin{pmatrix} T^\epsilon & 0 \\ T^\epsilon + A_{u^+x^+} & 1 + A_{w^+x^+} \end{pmatrix} + O(\epsilon).$$

Since the order-ϵ portion of this map is invertible by Claim 4.4.3, the map $\pi_{E^0} \circ (m^{+;0})_{d_\epsilon}|_{E^+}$ is invertible. The order of the inverse is at least $-\epsilon$. □

Note that a similar argument shows that $(m_{d_\epsilon})^{+;0} \colon \mathrm{CF}^\bullet{}_{d_\epsilon}(L^+) \to \mathrm{CF}^\bullet{}_{d_\epsilon}(L^0) \oplus E^0$ is invertible.

4.4.3 Checking conditions of Theorem A.3.2 To show that $\mathrm{CF}^\bullet(K_{\mu_D})$ is an A_∞ mapping cocylinder (Definition A.3.1), set $A^\pm = \mathrm{CF}^\bullet{}_{d_\epsilon}(L^\pm)$ and $A^0 = \mathrm{CM}^\bullet{}_{d_\epsilon}(L^0) \oplus E^0$. We've proven that $\mathrm{CF}^\bullet{}_{d_\epsilon}(K_{\mu_D}) = A^- \oplus A^0 \oplus A^+$ as a vector space. By Theorem C.5.1, the differential on this complex is of the form

$$\begin{pmatrix} (m_{d_\epsilon})^1_{A^-} & 0 & 0 \\ (m_{d_\epsilon})^{-;0} & (m_{d_\epsilon})^{0;0} & (m_{d_\epsilon})^{+;0} \\ 0 & 0 & (m_{d_\epsilon})^1_{A^+} \end{pmatrix}.$$

By Claim 4.4.7, the map $(m_{d_\epsilon})^{+;0}$ is invertible with order greater than $-\epsilon$. By Definition 4.2.6, $\nu((m_{d_\epsilon})^{+;0} \circ m^0_{d_\epsilon}) \geq 0$. We therefore satisfy the conditions for a mapping cocylinder (Definition A.3.1) and may apply Theorem A.3.2. This concludes the proof of Theorem 4.2.5.

5 Examples: wall-crossings and mutations

We now explore some applications of Proposition 3.0.2 and Theorem 4.2.5. We review an example from [3], computing wall-crossings for Chekanov and product tori in the complement of an anticanonical divisor. Consider the Lefschetz fibration with total space $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$

$$W : \mathbb{C}^2 \setminus \{z_1 z_2 = 1\} \to (\mathbb{C} \setminus \{1\}) \quad \text{given by} \quad (z_1, z_2) \mapsto z_1 z_2$$

(see Figure 8). We symplectically inflate this manifold at the removed divisor by taking the completion along the removed hypersurface. This resulting manifold has the same topology as $(\mathbb{C}^*)^2 \setminus \{z_1 z_2 = 1\}$. The new symplectic form can be chosen so that the projection $W : \mathbb{C}^2 \setminus \{xy = 1\} \to \mathbb{C} \setminus \{1\}$ remains a symplectic fibration. The regular fibers of this map have the topology of $\mathbb{C}^*$ and can be given an SYZ fibration

$$\text{val}_{W^{-1}(z)} : W^{-1}(z) \to \mathbb{R} \quad \text{given by} \quad (z_1, z_2) \mapsto |z_1|^2 - |z_2|^2$$

which is a restriction of the global Hamiltonian to a fiber. The Lefschetz fibration has a single degenerate fiber $z_1 z_2 = 0$. The base of the fibration can also be equipped with an SYZ fibration. We will take the fibration of the base given by loops $\gamma_r(\theta) := 1 + re^{2\pi\iota\theta}$. The symplectic parallel transport map given by the Lefschetz fibration preserves the isotopy class of SYZ fibers of $W^{-1}(z)$; as a result, one can build an SYZ fibration for the total space $\mathbb{C}^2 \setminus \{z_1 z_2 = 1\}$ by taking the circles $\text{val}^{-1}_{W^{-1}(z)}(s)$ and parallel transporting them along circles $\gamma_r(\theta)$ of the second fibration to obtain Lagrangian tori

$$L_{\gamma_r, |w|} = \{(z_1, z_2) \in \mathbb{C}^2 \setminus \{z_1 z_2 = 1\} \mid z_1 z_2 \in \gamma_r \text{ and } |z_1|^2 - |z_2|^2 = \log |w|\}.$$

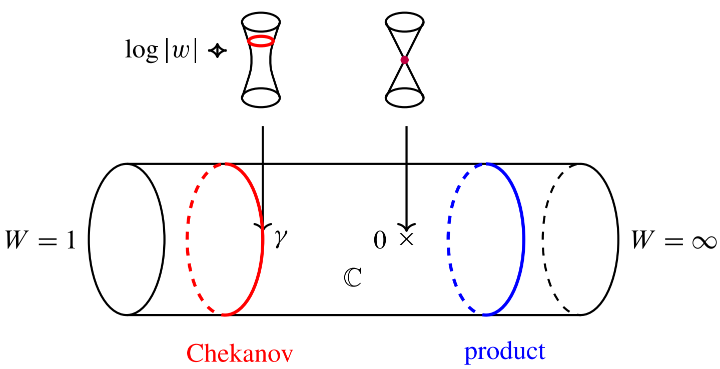

Figure 8: Symplectic fibration over $W : \mathbb{C}^2 \setminus \{z_1 z_2 = 1\} \to \mathbb{C} \setminus \{1\}$. Lagrangian tori are created by parallel transporting cycles in the fibers by loops in the base. Loops on the left of the critical value are of Chekanov type, while those on the right are product type.

The resulting SYZ fibration has one degenerate fiber, which occurs when $\log|w|=0$ and r approaches 1. The degenerate fiber $L_{\gamma_1,1}$ is the Whitney sphere, an immersed Lagrangian sphere with a single double point. We may generalize this construction to Lagrangians $L_{\gamma,|w|}$ for curves $\gamma\colon S^1\to\mathbb{C}\setminus\{1\}$ which wind around the removed point a single time. Such curves are divided into three types: the *Chekanov* type curves which additionally wind around the origin, the *product* type curves which do not, and those curves γ which contain the origin. If the curve γ contains the origin, we say the Lagrangian $L_{\gamma,|w|}$ is on the wall between Chekanov and product type. By lifting these curves to Lagrangian submanifolds via parallel transport of cycles in the fibers, we obtain the Chekanov and product type Lagrangian tori in $\mathbb{C}^*\setminus z_1z_2=1$. We will denote the Chekanov type Lagrangian tori by $L^-_{\gamma,|w|}$ and the product type tori by $L^+_{\gamma,|w|}$.

5.1 Uncorrected charts on the moduli space

Wall-crossing for Lagrangian submanifolds is a phenomenon which occurs when we try to parametrize the space of these Lagrangian tori with coordinates. These coordinates should be constructed over the Novikov ring, although for the purposes of this exposition we will use complex coordinates and unitary local systems. We consider Lagrangian branes, which are tuples $(L_{\gamma,|w|},b)$ where $b\in H^1(L,\iota\mathbb{R})$ gives us a unitary local system on L_γ via deformation of the Floer cohomology following [3, Lemma 4.1]. The space of Hamiltonian isotopy classes of Chekanov (resp. product) Lagrangian branes comes with local coordinates from measuring the flux of an isotopy, which we now describe.

Let (L_0,b_0) and (L_1,b_1) be two Lagrangian submanifolds equipped with local systems and let L_t be a Lagrangian isotopy between these two Lagrangians. Fix $c\in C_1(L_0)$. The Lagrangian isotopy gives a cylinder $c\times I\subset X$ with boundary $(c\times\{0\})\sqcup(c\times\{1\})\subset L_0\sqcup L_1$. The *flux* of this isotopy along c_0 is the quantity

$$\mathrm{Flux}_{L_t}(c):=-\left(\int_{c\times\{0\}}b_0\right)+\left(\int_{c\times\{1\}}b_1\right)+\left(\int_{c_0\times I}\omega\right).$$

This defines a complex-valued cohomological class $\mathrm{Flux}_{L_t}\in H^1(L_0,\mathbb{C})$. After picking a basepoint for our moduli space and a basis for homology, the flux cohomology class gives us local coordinates on the moduli space of Lagrangians, up to Hamiltonian isotopy.

In our example of $X=\mathbb{C}^2\setminus\{z_1z_2=0\}$, one choice of basis comes from compactifying X to $\mathbb{C}^2$ and noting that $L^\pm_{\gamma,|w|}\subset(\mathbb{C}^2)$ now bound holomorphic disks whose boundary classes pick out a basis for homology.

- On the Chekanov family, we call these two classes c_w and c_u. The class $c_w\in H_1(L^-_{\gamma,|w|})$ is the class of the circle in a fiber of the moment map which is parallel transported around to obtain the Lagrangian $L^-_{\gamma,|w|}$. When $|w|=1$, this is the vanishing cycle of the Lefschetz fibration. The second class is obtained by compactifying the total space to $\mathbb{C}^2$. In $\mathbb{C}^2$, there is a family of Maslov-2 disks holomorphic disks with boundary on $L^-_{\gamma,|w|}\subset\mathbb{C}^2$. The homology class of the boundary of such a class is called c_u.
- For the product family, we call these two classes c_r and c_s. They are both obtained from considering $L^+_{\gamma,|w|}$ inside the compactification $\mathbb{C}^2$, where the Lagrangian torus bounds two families of Maslov-2

holomorphic disks. If $L^+_{\gamma,|w|}$ is the standard product torus $(r_1 e^{\iota\theta_r}, r_2 e^{\iota\theta_s})$, then the classes c_r and c_s correspond to the meridional and longitudinal classes of the product torus. We call the corresponding classes of disks $c_r, c_s \subset H_1(L^+_{\gamma,|w|})$.

Once we have fixed these homology classes, we can construct coordinates on the space of Chekanov (resp. product) Lagrangian branes by measuring flux against a fixed Lagrangian. Fix the loop $\gamma_0 = 1 + e^{\iota\theta}$. This loop is neither of Chekanov nor product type. However, it still makes sense to measure the flux against the Whitney sphere $L_{\gamma_0,1}$ for both the Chekanov and product Lagrangians.

Claim 5.1.1 *Let L_t be a Lagrangian isotopy with $L_0 = L_{\gamma_0,1}$ and L_t of Chekanov (resp. product) type for all $t \in (0,1]$. Then the flux class $\mathrm{Flux}_{L_t} \in H^1(L_1, \mathbb{C})$ depends only on the Lagrangian L_1.*

The flux constructs local coordinates on the space of Chekanov (resp. product) type tori.

Definition 5.1.2 For $(u, w) \in (\mathbb{C}^*)^2$, define the class of Lagrangians

$$[L^-_{u,w}]_{\mathrm{Flux}} := \{(L^-_{\gamma,|w|}, b) \mid \exp(\mathrm{Flux}_{L_t}(c_u)) = u, \exp(\mathrm{Flux}_{L_t})(c_w) = w\},$$

where L_t is an isotopy of Chekanov type tori starting at $L_{\gamma_0,1}$ and ending at $(L^-_{\gamma,|w|}, b)$. For $(r,s) \in (\mathbb{C}^*)^2$, define the class of Lagrangians

$$[L^+_{r,s}]_{\mathrm{Flux}} := \{(L^+_{\gamma,|w|}, b) \mid \exp(\mathrm{Flux}_{L_t}(c_r)) = r, \exp(\mathrm{Flux}_{L_t}(c_s)) = s\},$$

where L_t is an isotopy of product type tori starting at $L_{\gamma_0,1}$ and ending at $(L^+_{\gamma,|w|}, b)$.

These classes are subsets of the equivalence classes of Lagrangians under the relation of Hamiltonian isotopy.

Claim 5.1.3 *The classes $[L^-_{u,w}]_{\mathrm{Flux}}$ (resp. $[L^+_{r,s}]_{\mathrm{Flux}}$) are the equivalence classes of Chekanov (resp. product) tori under the equivalence relation of Hamiltonian isotopy through Chekanov (resp. product) tori. Furthermore, with the standard complex structure, no member of such an isotopy will bound a holomorphic disk in $(\mathbb{C}^*)^2 \setminus \{z_1 z_2 = 1\}$.*

The classes $[L^-_{u,w}]_{\mathrm{Flux}}$ and $[L^+_{r,s}]_{\mathrm{Flux}}$ allow us to use the coordinates (u, w) and (r, s) to parametrize charts on the moduli space of Lagrangian tori. We will frequently refer to a specific representative of each class as $L^-_{u,w}$ or $L^+_{r,s}$. Notice that each Lagrangian torus $L_{\gamma_r,|w|}$ from Figure 8 is either of Chekanov type, product type or lies on a "wall" between these two types. If L_{γ_r} is not on the wall, then $L_{\gamma_r,|w|}$ belongs to a distinct $[L^{\pm}]_{\mathrm{Flux}}$ class.

For purposes of exposition, we will assume for the remainder of this discussion that we can work safely with complex coefficients and avoid convergence issues. Consider the identification between bounding cochains and local systems

$$(L^+_{r,s}, ac^r + bc^s) \mapsto (L^+_{r\exp(a),s\exp(b)}, 0). \tag{7}$$

When the underlying chain model for the Lagrangian is the de Rham complex, this identification is known. We first learned of this in [4, Lemma 4.1], and [15] constructs a version of Lagrangian Floer theory based on the de Rham complex for which this holds. We will assume that the same identification holds in the pearly model.

Assumption 5.1.4 The identification given in (7) holds for the pearly model.

To justify the appearance of the exponential in the pearly model, one needs to compute the equivalence between the flow-tree model for Morse theory and the de Rham model; an equivalence between these two chain models is exhibited in [22].

5.2 Correction from Lagrangian cobordism

The main computation for this example is the following sharpening of the minimal-energy disk requirement of Definition 4.2.6, which was necessary to apply our wall-crossing computation, to the specific example of K_{μ_D}. We note that there is a relation between the flux of the surgeries used to construct the cobordism, the flux relative boundary of the isotopy θ_{H_t} used to construct K_c from Claim 4.1.2, and the area of the holomorphic disk from Proposition 4.3.6. Let $\operatorname{width}(L_\downarrow \# L_\uparrow)$ be the flux of the Lagrangian isotopy between the surgery and the immersed Lagrangian $L_\downarrow \cup L_\uparrow$. We arrange for $L_\downarrow \cup L_\uparrow \cup L_c$ to be exactly homotopic to $L_{\gamma_0,1}$, the standard Whitney sphere from which we compute our flux coordinates. Define the quantities

$$u_1 = \exp(-\operatorname{width}_{L_\downarrow \# L_\uparrow}), \quad u_2 = \exp(\operatorname{Flux}_{\theta_{H_t}(L_c)}(\ell_c)),$$
$$s = \exp(\operatorname{width}_{L_\uparrow \# L_\downarrow}), \qquad z = \exp\left(\int_{\partial u_{\text{ex}}} b + \int_{u_{\text{ex}}} \omega\right),$$

where ℓ_c is the list of the curve ℓ_c from Figure 3 to the L_c chart. From the definition of flux and surgery width, we see that the product side of the mutation cobordism K_{μ_D} is the Lagrangian $L^+_{s,s}$. The flux between the Chekanov end of the mutation cobordism and the Whitney sphere is the amount swept out by the surgery neck, along with the flux relative boundary of the component L_c. Write $u = u_1 u_2$ so that the Chekanov end of the mutation cobordism is $L^-_{u,1}$ From relations in the homology of K_{μ_D}, we observe that $u_2 = (u_1)^{-1} s z$, and so

$$s = u/z. \tag{8}$$

Proposition 5.2.1 *For the standard choice of complex structure, the only pseudoholomorphic disk with boundary on K_{μ_D} is the one described in Proposition 4.3.6 and its multiple covers.*

Proof We show that this disk is unique. We look at the Lagrangian cobordism $K_{\mu_D} \subset X \times \mathbb{C}$ under the projections $\pi_X : X \times \mathbb{C} \to X$, $\pi_{\mathbb{C}} : X \times \mathbb{C} \to \mathbb{C}$ and $W : X \to \mathbb{C}$. The first projection that we look at is $W \circ \pi_X : K_{\mu_D} \to \mathbb{C}$. The regions where we have performed Hamiltonian isotopy and surgery sweep out flux corresponding to shaded regions in the projection; see Figure 9. We let U be the neighborhood drawn in Figure 9. Every disk with boundary on K_{μ_D} must either have boundary contained within the red region U or have boundary completely disjoint from the red region U, by the open mapping principle.

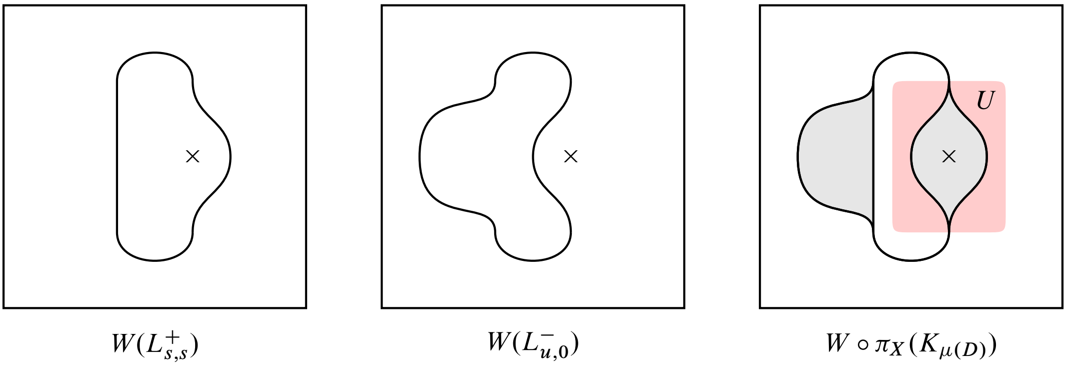

Figure 9: Disks in the Lagrangian cobordism K_{μ_D} must either have boundary contained inside of the red region, or outside of the red region.

We now look at $K_{\mu_D} \cap (W \circ \pi_X)^{-1}(U)$. This is the eye-shaped cobordism from Figure 2. By Claim 4.3.7 the only disks which appear here are u_{ex} and its multiple covers. The complement of the region given by U cannot bound holomorphic disks for topological reasons. This characterizes the disks which may appear on K_{μ_D}. □

Corollary 5.2.2 *The mutation cobordism $K_{\mu_D} : L^-_{u,1} \rightsquigarrow L^+_{s,s}$ is an isolated mutation.*

5.2.1 Wall-crossing computation We now compute $\pi^{\pm}_*(d_\epsilon)$ from Theorem 4.2.5. From our discussion on orientations, we have that the Lagrangian cobordism K_{μ_D} does not identify $L^-_{u,1}$ with $L^+_{s,s}$ with the standard choices of spin structures. We therefore will denote the ends of the cobordism by $\mathring{L}^-_{u,1}$ and $\mathring{L}^+_{s,s}$ to signify that these Lagrangians have a different spin structure than the Lagrangians previously considered. This is important, as Lagrangians equipped with different spin structures can represent distinct objects in the Fukaya category. We choose the Morse functions for the $\mathring{L}^-_{u,1}$ and $\mathring{L}^+_{s,s}$ matching the one chosen in Section 5.1 so that the coordinates for the moduli spaces can be identified with our previous computation.

The vanishing cycle on the Chekanov side is c^u, while the vanishing cycle on the product side is $c^r + c^s$. To construct the continuation map from a mutation cobordism, we must pick a splitting of the vector spaces $E^{\pm}$ from Section 4.3.1. This involves picking classes $c^{w\pm}$ as in Remark 4.3.4 On the Chekanov side we take the class $c^{w-} := c^w$ and on the product side we choose the class $c^{w+} := c^r$.

With this choice of Morse function, we show, with Assumption 5.2.3, that the curvature term is

$$m^0_{K_{\mu_D}} = \log(1+z)(x^+ + x^-). \tag{9}$$

Assumption 5.2.3 For each k, there exists a domain-dependent perturbation $J^k_z : D^2 \to \text{End}(TX \times \mathbb{C})$ such that there is a unique J^k_z pseudoholomorphic disk $u_k : (D^2, \partial D^2) \to (X \times \mathbb{C}, K_{\mu_D})$ with the property that $[u_k] = k[u]$.

Proof of (9) given Assumption 5.2.3 The A_∞ product structure in [10] is defined by the count of adapted treed holomorphic disks weighted by the number of interior leaves. The curvature term, up to sign, is given by

$$m^0 = \sum_{x_0, u \in \mathcal{M}(X,L,D,x_0)_0} \pm(\sigma(u)!)^{-1} T^{\omega(u)} x_0.$$

where the $\pm$ sign is determined by the orientation of the moduli space and $\sigma(u)$ is the number of interior leaves.

Consider the preimages of the stabilizing divisor $\{z_1, \dots, z_k\} = u_k^{-1}(D^2) \subset D^2$. We now define a domain-dependent perturbation for the combinatorial type $\underline{\Gamma}^k$ which has one disk with k interior marked points and one output flow line. Each $C \in \mathcal{M}(\underline{\Gamma}^k)$ with disk component D^2_C is labeled with interior marked points $p_1, \dots, p_k$ and attachment point $s_0 \in \partial D^2_C$ for the flow line. For each disk, there exists a unique automorphism $\phi_C : D^2_C \to D^2$ with

$$\phi_C(p_1) = z_1 \quad \text{and} \quad \phi_C(s_0) = 1.$$

We define the perturbation of the almost complex structure $\mathcal{P}(\underline{\Gamma}^k) : \mathcal{U} \to TX$ by pullback of the almost complex structure from Assumption 5.2.3,

$$\mathcal{P}(\underline{\Gamma}^k)(C, z) = J^k_{\phi_C(z)}.$$

By Assumption 5.2.3, for this choice of perturbation, there exists a unique pseudoholomorphic treed disk $u_C : C \to X$ with boundary on L and flow line limiting to x_0. However, most of these disks are not adapted to the divisor D, as there is no reason that $u_C^{-1}(D) = \phi^{-1}(\{z_1, \dots, z_k\})$ should be the same as the set of marked points $p_1, \dots, p_k$. In fact, we are only guaranteed that $\phi^{-1}(z_1) = p_1$ by construction.

The only difference between the domains C are the positions of the marked points, and so the moduli spaces of $\mathcal{P}$–perturbed D–adapted pseudoholomorphic treed disks are in bijection with the domains for which $\phi^{-1}(\{z_1, \dots, z_k\}) = \{p_1, \dots, p_k\}$. There are $(k-1)!$ such domains (as p_1 must be mapped to z_1, and the remaining $k-1$ marked points can be chosen freely).

This means that (given Assumption 5.2.3) there exist choices of the perturbation datum such that $\#\mathcal{M}_{\mathcal{P}}(X, L, D, \underline{\Gamma}^k) = (k-1)!$. This gives us (modulo 2) the formula for multiple covers, and (up to a sign)

$$\langle m^0, x_0 \rangle = \sum \frac{(k-1)!}{k!} T^{k\omega[u]} = \log(1+z). \qquad \square$$

The deformation from Claim 4.4.6 can be extended to a bounding cochain:

$$b = \log(1+z)c^w + \log(1+z)c^s.$$

By restricting b_ϵ to the ends of the cobordism, we obtain a continuation map $\mathrm{CF}^\bullet{}_{b^-}(\mathring{L}^-_{u,1}) \to \mathrm{CF}^\bullet{}_{b^+}(\mathring{L}^+_{s,s})$.

If we assume convergence over complex coefficients and identify the bounding cochain with a local system, we obtain a correspondence between Lagrangians

$$(\mathring{L}^-_{u,1+z}) \sim (L^-_{u,1}, \log(1+z)c^w)) \sim (\mathring{L}^+_{s,s}, \log(1+z)c^s) \sim (\mathring{L}^+_{s,s(1+z)}).$$

Setting $w = 1+z$ and noting that $s = u/z$ from (8),

$$(\mathring{L}^-_{u,w}) \sim \mathring{L}^+_{u/(w-1),uw/(w-1)}.$$

This closely matches the identification from [3], with the only difference being the signs. This discrepancy can be explained by the spin structures on the Lagrangian cobordism K_{μ_D}. As the Lagrangian cobordism K_{μ_D} admits an embedding into $\mathbb{R}^3$, it has a trivial tangent bundle and is therefore spin. However, there is no spin structure on K_{μ_D} which restricts to the standard spin structures [11] on both $L^\pm$.

Appendix A Mapping cylinders for curved A_∞ algebras

A.1 A brief review of filtered A_∞ algebras and homomorphisms

Our notation for filtered A_∞ algebras and homomorphisms follows [17]. We make some modifications to extend to the setting of weakly filtered A_∞ algebras and homomorphisms.

Notation A.1.1 We will sometimes write the composition $f \circ g$ as $\left[\begin{smallmatrix} g \\ f \end{smallmatrix}\right]$. This both saves space, and makes some of the algebraic manipulations easier to follow.

A *filtered A_∞ algebra* $(A^\bullet, m^k)$ is a free graded Λ–vector space $A^\bullet$ with Λ–linear cohomologically graded higher products for each $k \geq 0$

$$m^k\colon (A^\bullet)^{\otimes k} \to (A^{\bullet+2-k}),$$

with extra data and satisfying the filtered A_∞ algebra axioms given in [17, Definition 3.2.20]. The key relation is the quadratic A_∞ relation, which we will denote by

$$\sum_{k_1+k'+k_2=k} (-1)^{\clubsuit(\underline{x},k_1)} \begin{bmatrix} \mathrm{id}^{\otimes k_1} \otimes m^{k'} \otimes id^{\otimes k_2} \\ m^{k_1+1+k_2} \end{bmatrix} (x_1,\dots,x_k),$$

where the sign is determined by the cohomological grading of the domain

$$\clubsuit(\underline{x},k_1) := k_1 + \sum_{j=1}^{k_1} \deg(x_j).$$

Notation A.1.2 We will suppress $\underline{x} = (x_1,\dots,x_k)$ in future computations, but continue to use the notation $\clubsuit(\underline{x},i)$ for the purposes of determining signs.

If $m^0 = 0$, then $(A^\bullet, m^1)$ is a chain complex and we say that $A^\bullet$ is *uncurved* or *tautologically unobstructed*. There is a filtration map $\ell\colon A^\bullet \to \mathbb{R}$ defined by the smallest λ such that $v \in F^\lambda A$. Up to a change of sign convention, this makes A a non-Archimedean normed vector space in the sense of [33, Definition 1.2.3]. The filtration zero portion of products $\underline{m}^k : A^{\otimes k} \to A$ are a tautologically unobstructed A_∞ algebra.

We from now on suppress the cohomological index and, when the product structure is clear, we will simply notate an A_∞ algebra as A. An *ideal* of A is a subspace $I \subset A$ such that, for every $b \in I$ and $a_1, \dots, a_{k-1} \in A$,

$$m^k(a_1 \otimes \cdots \otimes a_j \otimes b \otimes a_{j+1} \otimes \cdots \otimes a_{k-1}) \in I.$$

Note that we *do not require* $m^0 \in I$. The set $A_{>0}$ of elements with positive filtration is an A_∞ ideal and $A/A_{\geq 0}$ is a tautologically unobstructed A_∞ algebra.

We take a variation of [17, Definition 5.2.1] and adapt it to [17, Definition 3.2.7]:

Definition A.1.3 Let (A, m_A^k) and (B, m_B^k) be A_∞ algebras. A *weakly filtered A_∞ homomorphism* from A to B is a sequence of graded maps

$$f^k : A^{\otimes k} \to B$$

satisfying the following conditions:

- **Weakly filtered** The maps nearly preserve energy, that is,

$$f^k(F^{\lambda_1}A, \dots, F^{\lambda_k}A) \subset F^{-ck+\sum_{i=1}^k \lambda_i} B$$

  for some fixed constant c called the *energy loss of f* with $c < \ell(m_A^0)$.
- **Quadratic A_∞ relations** The f^k, m_A^k and m_B^k mutually satisfy the quadratic filtered A_∞ homomorphism relations

$$\sum_{k_1+k'+k_2=k} (-1)^{\clubsuit(\underline{x},k_1)} \begin{bmatrix} \mathrm{id}^{\otimes k_1} \otimes m_A^{k'} \otimes \mathrm{id}^{k_2} \\ f^{k_1+1+k_2} \end{bmatrix} = \sum_{i_1+\cdots+i_j=k} \begin{bmatrix} f^{i_1} \otimes \cdots \otimes f^{i_j} \\ m_B^j \end{bmatrix}.$$

The composition of weakly filtered A_∞ homomorphisms is again a weakly filtered A_∞ homomorphism.

We follow [17, Definition 3.6.9] and write $m_a^k : A^{\otimes k} \to A$ for the deformation of the A_∞ structure at a element $a \in A$ for positive valuation. We say that b is a bounding cochain if $(A^\bullet, m_b)$ is tautologically unobstructed, and write

$$\mathcal{MC}_{>c}(A) := \{b \in A \mid m_b^0 = 0 \text{ and } \ell(b) > c\}$$

for the space of Maurer–Cartan elements of filtration at least c [17, Definition 3.6.4]. Given $f : A \to B$ a weakly filtered A_∞ morphism of energy loss c, there exists a pushforward map $f_* : \mathcal{MC}_{>c}(A) \to \mathcal{MC}_{>0}(B)$.

It will be convenient for us to use the following notation:

Claim A.1.4 *Let $f : A \to B$ be a filtered A_∞ algebra morphism. Then there exists an A_∞ homomorphism*

$$f_\flat : (A, m_A) \to (B, (m_B)_{f_*(0)})$$

where $f_\flat$ is defined by

$$f_\flat^k = \begin{cases} f^k & \text{for } k > 0, \\ 0 & \text{if } k = 0. \end{cases}$$

Claim A.1.5 *Let $f: A \to B$ be a filtered A_∞ algebra morphism. Let $a \in A$ be a deforming element. Then there is a map*

$$f_a: (A, (m_A^k)_a) \to (B, m_B^k).$$

A.2 Curved homotopy transfer theorem

Theorem A.2.1 (weakly filtered Whitehead theorem) *Let A and B be filtered A_∞ algebras. Suppose that we have maps*

$$A \underset{\beta^k}{\overset{\alpha}{\rightleftarrows}} B \circlearrowleft h$$

such that β^k is an A_∞ homomorphism, with $\beta^k = 0$ for all $k \neq 1$, and the following conditions hold:

(10) $$\beta \circ m_B^1 - m_A^1 \circ \beta = \beta \circ m_B^1 \circ m_B^1 \circ h,$$

(11) $$\alpha \circ m_A^1 - m_B^1 \circ \alpha = h \circ m_B^1 \circ m_B^1 \circ \alpha,$$

(12) $$h \circ m_B^1 + m_B^1 \circ h = \alpha \circ \beta - \mathrm{id} - h \circ m_B^1 \circ m_B^1 \circ h,$$

$$\beta \circ h = 0, \quad h \circ \alpha = 0, \quad \beta \circ \alpha = \mathrm{id}_A, \quad \ell(h \circ m_B^0) > 0.$$

Then we can extend

- *α to a weakly filtered A_∞ morphism $\alpha^k: A^{\otimes k} \to B$, and*
- *h to a weakly filtered A_∞ homotopy $h^k: B^{\otimes k} \to B$.*

Note that this relates to the homotopy transfer theorem of [20] when $m_B^0 = 0$, as (10)–(12) become the chain map and homotopy relations. The additional relations are analogues of the strong deformation retract conditions. In the filtered setting they tell us that β and α intertwine the curvatures of m_B^1 and m_A^1:

$$\beta \circ m_B^1 \circ m_B^1 \circ \alpha = m_A^1 \circ m_A^1.$$

Theorem A.2.1 is also related to [17, Theorem 4.2.45], which proves that if α and β are filtered A_∞ homomorphisms and h is filtered, then h extends to an A_∞ homotopy between α and β whenever (10)–(12) hold at zero valuation. We now examine the possibility of adapting [17, Theorem 4.2.45] with the weaker condition that h is weakly filtered.

The proof of [17, Theorem 4.2.45] first proves the theorem for unfiltered A_∞ algebras. The proof for the unfiltered A_∞ algebra proceeds by constructing the maps $m_\alpha^k: A^{\otimes k} \to A$ inductively over the index of k. This proof necessarily uses the fact that f^0 and m_A^0 are trivial, as otherwise higher terms f^{k+i} show up in the k–input quadratic A_∞ relations.

The filtered case is proven by induction on the filtration, with the unfiltered setting serving as a base case. Two difficulties occur when extending to weakly filtered homotopies. In [17, Theorem 4.2.45] the definition of homotopy is given in terms of filtered A_∞ homomorphisms $h\colon A \to \mathfrak{A}$, where $\mathfrak{A}$ is a model of $[0,1] \times A$. One can generalize these to weakly filtered A_∞ homotopies by considering weakly filtered homomorphisms $h\colon A \to \mathfrak{A}$ and weakly filtered A_∞ structures. Difficulties with these approaches are commented on in the discussion surrounding [17, Theorem 5.2.35], where a similar statement is considered for weakly filtered maps of A_∞ bimodules.

Further, when h is only weakly filtered, we can no longer use the proof strategy for [17, Theorem 4.2.45] as h no longer provides a weak homotopy between $\alpha \circ \beta$ and id_B on the tautologically unobstructed A_∞ algebra $B/B_{>0}$. A possible approach to the weak-homotopy case is given in [17, Section 7.2.6], which examines G–filtered A_∞ algebras. A G–filtered A_∞ structure [17, Definition 7.2.67] is a collection of $m^{k,\beta}\colon A^{\otimes k} \to A$ indexed by $k \in \mathbb{N}$ and β in a monoid G. The monoid G should have a unique invertible element. We also require that every element of G possess only finite factorization. That is, for all $\beta \in G$,

$$\|\beta\| := \sup\{k \mid \beta = \beta_1 + \cdots + \beta_k = \beta \text{ and } \beta_i \neq e\} < \infty,$$

The A_∞ products are given by

$$m^k = \sum_{\beta \in G} T^{\lambda(\beta)} m^{k,\beta}.$$

satisfying the A_∞ which respects the structure of G. Roughly, the proof of [17, Theorem 7.2.72] inducts on k and $\|\beta\|$, and uses this finite factorization property. In spirit, our approach is much like this one.

Instead of strengthening the condition on the filtration, we choose instead to strengthen the condition on the hypothesis of the homotopy equivalence (10)–(12). The upshot is that we can reexpress m_A^K in terms of the maps β, α, m_B^K and h, allowing us to decompose the map based on the number of m_B^0 terms which occur in the expansion. This is used as a replacement for the monoid structure, and we induct on this quantity, rather than the filtration, in our proof. We now give the construction of the maps in Theorem A.2.1.

Proof of Theorem A.2.1

Our proof is a modification of the inductive proof given in Markl's Christmas carp paper [25], using the sign conventions from [17]. We include the steps in the induction proof which differ from the proof in the unfiltered setting. Notably, our construction inducts over two variables, which correspond degree filtration and the number of B–curvature terms that appear in the A_∞ relations. Outside of these changes, the only difference between the weakly filtered and unfiltered proof is the additional care that must be spent with the ranges of indices due to the presence of curvature terms. Place a partial order on pairs (n,k) so that $(k',n') \leq (k,n)$ if $n' \leq n$ and $k' \leq k$. Define the maps

$$\alpha^{1,0} := \alpha, \quad \alpha^{0,1} = h \circ m_B^0, \quad m_A^{1,0} := m_A^1, \quad m_A^{0,1} = \beta \circ m_B^0, \quad \chi^{0,1} = m_B^0, \quad \chi^{0,0} = 0 \quad \text{and} \quad \chi^{1,0} = 0.$$

Suppose that, for all $(k', n') \leq (k, n)$, we've defined the maps $\alpha^{k',n'}$ and $\chi^{k',n'}$. Then inductively define the kernel

$$\rho^{k,n} = -\sum_{r\geq 2} \sum_{\substack{k_1+\cdots+k_r=k \\ n_1+\cdots+n_r=n \\ (0,0)\leq(n_i,k_i)}} \begin{bmatrix} \alpha^{k_1,n_1} \otimes \cdots \otimes \alpha^{k_r,n_r} \\ m_B^r \end{bmatrix},$$

error terms

$$\chi^{k,n} = \sum_{r\geq 2} \sum_{\substack{k_1+\cdots+k_r=k \\ n_1+\cdots+n_r=n \\ (0,0)\leq(n_i,k_i)}} (-1)^{\clubsuit(\underline{x}, \sum_{j=1}^{i-1} k_j)} \begin{bmatrix} \alpha^{k_1,n_1} \otimes \cdots \otimes \chi^{k_i,n_i} \otimes \cdots \otimes \alpha^{k_r,n_r} \\ h \circ m_B^r \end{bmatrix},$$

and maps

$$\alpha^{k,n} = -h \circ \rho^{k,n} \quad \text{and} \quad m_A^{k,n} = -\beta \circ \rho^{k,n}.$$

The first two error terms are

(13) $$\chi^{1,1} = h \circ m^B (m_B^0 \otimes \alpha^{1,0} + (-1)^{\clubsuit} \alpha^{1,0} \otimes m^0) = h \circ m_B^1 \circ m_B^1 \circ \alpha^{1,0},$$

(14) $$\chi^{0,2} = h \circ m^2 (\alpha^{0,1} \otimes \chi^{0,1} + (-1)^{\clubsuit} \alpha^{0,1} \otimes \chi^{0,1}) = h \circ m_B^1 \circ m_B^1 \circ h \circ m_B^0.$$

The filtrations of these maps are increasing in n, as we can prove inductively that

$$\ell(\rho^{k,n})(a_1, \ldots, a_k) \geq n\ell(m^0) - (n+k)\ell(h) + \sum_{i=1}^{k} \ell(a_i),$$

and by the condition that $\ell(h \circ m_B^0) > 0$ we see that $\sum_{n=0}^{\infty} \rho^{n,k}$ converges over Novikov coefficients. Most of the $m_A^{k,n}$ terms end up being trivial, as by applying the A_∞ homomorphism relations for β we obtain

$$m^{k,n} = \begin{bmatrix} h \circ \rho^{k_1,n_1} \otimes \cdots \otimes h \circ \rho^{k_r,n_r} \\ \beta \circ m_B^r \end{bmatrix} = \begin{bmatrix} h \circ \rho^{k_1,n_1} \otimes \cdots \otimes h \circ \rho^{k_r,n_r} \\ m_A^r \circ (\beta^{\otimes r}) \end{bmatrix},$$

which vanishes whenever the composition $\beta \circ h$ appears. So this term is zero unless $(k, n) = (k, 0)$. We can therefore write

$$m_A^k = m_A^k \circ (\beta \circ \alpha)^{\otimes k} = \beta \circ m_B^{k,0} \circ (\alpha)^{\otimes k} = m_A^{k,0} = \sum_n m_A^{k,n}.$$

We will prove by induction on the product order for (k, n) that the following relation $I(k, n)$ holds:

(15) $$\sum_{\substack{k_1+\cdots+k_r=k \\ n_1+\ldots+n_r=n}} \begin{bmatrix} \alpha^{k_1,n_1} \otimes \cdots \otimes \alpha^{k_r,n_r} \\ m_B^r \end{bmatrix} = \sum_{\substack{j_1+j'+j_2=k \\ n_1+n'=n}} (-1)^{\clubsuit(\underline{x}, j_1)} \begin{bmatrix} \mathrm{id}^{\otimes j_1} \otimes m_A^{j',n'} \otimes \mathrm{id}^{\otimes j_2} \\ \alpha^{j_1+j_2+1,n_1} \end{bmatrix} - \chi^{k,n} + \chi^{k,n+1}.$$

We note that summing all of the $I(k, n)$ relations for fixed k yields the A_∞ homomorphism relation for $\alpha^k := \sum_n \alpha^{k,n}$.

Proof The base cases for our induction are when $(k,n)\in\{(0,0),(0,1),(1,0)\}$.

- The relation $I(0,0)$ states
$$m_B^0=\chi^{0,1}.$$
- A computation using the relation (13) shows that $I(1,0)$ follows from (11).
- A computation using (14) shows that $I(0,1)$ follows from (12).

Now assume that we've shown that $I(k',n')$ holds for all $(k',n')<(k,n)$. As $(k,n)\neq(0,0),(1,0),(0,1)$, we know that $r\geq 2$ in the left-hand side of $I(k,n)$. We can therefore reexpress it as

$$\sum_{\substack{k_1+\cdots+k_r=k\\ n_1+\cdots+n_r=n}}\begin{bmatrix}\alpha^{k_1,n_1}\otimes\cdots\otimes\alpha^{k_r,n_r}\\ m_B^r\end{bmatrix}=-\rho^{k,n}+m_B^1\circ\alpha^{k,n}=-\rho^{k,n}-m_B^1\circ h\circ\rho^{k,n}$$

By applying $m_B^1\circ h\circ\rho^{k,n}=-\alpha\circ m_A^{k,n}-\rho^{k,n}-h\circ m_B^1\circ m_B^1\circ\alpha^{k,n}-h\circ m_B^1\circ\rho^{k,n}$ from (12) and canceling the highlighted terms, this equals

$$\alpha\circ m_A^{k,n}+h\circ m_B^1\circ m_B^1\circ\alpha^{k,n}+\sum_{r\geq 2}\sum_{\substack{k_1+\cdots+k_r=k\\ n_1+\cdots+n_r=n\\ (0,0)\leq(n_i,k_i)}}\begin{bmatrix}\alpha^{k_1,n_1}\otimes\cdots\otimes\alpha^{k_r,n_r}\\ h\circ m_B^1\circ m_B^r\end{bmatrix}$$
$$=\alpha\circ m_A^{k,n}+\sum_{r\geq 1}\sum_{\substack{k_1+\cdots+k_r=k\\ n_1+\cdots+n_r=n\\ (0,0)\leq(n_i,k_i)}}\begin{bmatrix}\alpha^{k_1,n_1}\otimes\cdots\otimes\alpha^{k_r,n_r}\\ h\circ m_B^1\circ m_B^r\end{bmatrix}.$$

Applying the A_∞ relations for $m^1\circ m_B^r$, we get

$$\alpha\circ m_A^{k,n}+\sum_{\substack{r_1+r'+r_2=r\geq 2\\ r_1+r_2>0}}\sum_{\substack{k_1+\cdots+k_r=k\\ n_1+\cdots+n_r=n}}(-1)^{\heartsuit(\underline{x},\underline{k},r_1)}\begin{bmatrix}\alpha^{k_1,n_1}\otimes\cdots\otimes\alpha^{k_r,n_r}\\ \mathrm{id}^{\otimes r_1}\otimes m_B^{r'}\otimes\mathrm{id}^{\otimes r_2}\\ h\circ m_B^{r_1+r_2+1}\end{bmatrix}.$$

Here $\heartsuit(\underline{x},\underline{k},r_1):=r_1+\sum_{i=1}^{r_1}\bigl(1-k_i+\sum_{j=1}^{k_i}\deg(x_j)\bigr)$. We regroup the terms which lie above $m_B^{r'}$ in the composition to obtain

$$\alpha^{1,0}\circ m_A^{k,n}+\sum_{\substack{r_1+r'+r_2=r\geq 1\\ r_1+r_2\geq 0\\ j_1+j'+j_2=k\\ m_1+m'+m_2=n}}\sum_{\substack{k_1+\cdots+k_{r_1}=j_1\\ k_{r_1+1}+\cdots+k_{r_2+r'}=j'\\ k_{r-r_1}+\cdots+k_r=j_2\\ n_1+\cdots+n_{r_1}=m_1\\ n_{r_1+1}+\cdots+n_{r_2+r'}=m'\\ n_{r-r_1}+\cdots+n_r=m_2}}(-1)^{\heartsuit(\underline{x},\underline{k},r_1)}\begin{bmatrix}\mathrm{id}^{\otimes j_1}\otimes\begin{bmatrix}\bigotimes_{i=r_1+1}^{r_1+r'}(\alpha^{k_i,n_i})\\ m_B^{r'}\end{bmatrix}\otimes\mathrm{id}^{\otimes j_2}\\ \bigotimes_{i=1}^{r_1}(\alpha^{k_i,n_i})\otimes\mathrm{id}\otimes\bigotimes_{i=r-r_2}^{r}(\alpha^{k_i,n_i})\\ h\circ m_B^{r_1+r_2+1}\end{bmatrix}.$$

When taken modulo 2, $\heartsuit(\underline{x},\underline{k},r_1)\equiv\sum_{i=1}^{r_1}k_i+\sum_{j=1}^{j_1}\deg(x_i)\equiv\clubsuit(\underline{x},j_1)$. When grouped over the index (j',m'), the shaded block is the left-hand side of the relation $I(j',m')$. If $m'=n$ then $j_1,j_2\neq 0$, and therefore $j' < k$. Hence $(j', m') < (j, m)$. Let $\underline{x}_{j_1,j_2}$ be the tuple $(x_{j_1+1}, \dots, x_{j_1+j'})$. We can apply the inductive hypothesis to get

$$\alpha^{1,0} \circ m_A^{k,n} + \sum_{\substack{r_1+r'+r_2=r\geq 1\\ r_1+r_2>0\\ j_1+j'+j_2=k\\ m_1+m'+m_2=n}} (-1)^{\clubsuit(\underline{x},j_1)} \sum_{\substack{k_1+\cdots+k_{r_1}=j_1\\ k_{r-r_1}+\cdots+k_r=j_2\\ n_1+\cdots+n_{r_1}=m_1\\ n_{r-r_1}+\cdots+n_r=m_2\\ l_1+l'+l_2=j'\\ o_1+o_2=m'}} (-1)^{\clubsuit(\underline{x}_{j_1,j_2},l_1)} \begin{bmatrix} \mathrm{id}^{\otimes j_1} \otimes \begin{bmatrix} \mathrm{id}^{\otimes l_1} \otimes m_A^{l',o_1} \otimes \mathrm{id}^{\otimes l_2} \\ \alpha^{l_1+l_2+1,o_2} \end{bmatrix} \otimes \mathrm{id}^{\otimes j_2} \\ \bigotimes_{i=1}^{r_1}(\alpha^{k_i,n_i}) \otimes \mathrm{id} \otimes \bigotimes_{i=r-r_2}^{r}(\alpha^{k_i,n_i}) \\ h \circ m_B^{r_1+r_2+1} \end{bmatrix}$$

$$+ \sum_{\substack{r_1+r'+r_2=r\geq 1\\ r_1+r_2>0\\ j_1+j'+j_2=k}} (-1)^{\clubsuit(\underline{x},j_1)} \sum_{\substack{k_1+\cdots+k_{r_1}=j_1\\ k_{r_1+1}+\cdots+k_{r_2+r'}=j'\\ k_{r-r_1}+\cdots+k_r=j_2\\ n_1+\cdots+n_r=n}} \begin{bmatrix} \mathrm{id}^{\otimes j_1} \otimes (-\chi^{r',n'} + \chi^{r',n'+1}) \otimes \mathrm{id}^{\otimes j_2} \\ \bigotimes_{i=1}^{r_1}(\alpha^{k_i,n_i}) \otimes \mathrm{id} \otimes \bigotimes_{i=r-r_2}^{r}(\alpha^{k_i,n_i}) \\ h \circ m_B^{r_1+r_2+1} \end{bmatrix}.$$

The remainder of the proof proceeds as in the tautologically unobstructed setting by applying the recursive definition of $\chi^{n,k}$ and $\rho^{n,k}$, and using the relation $(-1)^{\clubsuit(\underline{x},j_1)}(-1)^{\clubsuit(\underline{x}_{j_1,j_2},l_1)} = (-1)^{\clubsuit(\underline{x},j+l_1)}$.

We now define the homotopy, again following [25]. Inductively define the maps

$$\phi^k := \sum_{\substack{k_1+k'+k_2=k\\ j_1+\cdots+j_r=k_1\\ r+k_2\geq 1, k'\geq 1}} (-1)^{\clubsuit(\underline{x},k_1)} \begin{bmatrix} \beta^{\otimes k_1} \otimes h^{k'} \otimes \mathrm{id}^{\otimes k_2} \\ \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_r} \otimes \mathrm{id}^{\otimes k_2+1} \\ m^{r+1+k_2} \end{bmatrix},$$

where $h^k := h \circ \phi^k$. The A_∞ homotopy relations

$$\sum_{\substack{k_1+k'+k_2=k\\ j_1+\cdots+j_r=k_1\\ r+k_2\geq 0, k'\geq 1}} (-1)^{\clubsuit(\underline{x},k_1)} \begin{bmatrix} (\alpha\circ\beta)^{j_1} \otimes \cdots \otimes (\alpha\circ\beta)^{j_r} \otimes h^{k'} \otimes \mathrm{id}^{\otimes k_2} \\ m^{r+1+k_2} \end{bmatrix}$$
$$= (\alpha\circ\beta)^k - \mathrm{id}^k - \sum_{k_1+k'+k_2=k, k_1+k_2\geq 0} (-1)^{\clubsuit(\underline{x},k_1)} \begin{bmatrix} \mathrm{id}^{\otimes k_1} \otimes m_B^{k'} \otimes \mathrm{id}^{\otimes k_2} \\ h^{k_1+k_2+1} \end{bmatrix}$$

are proved inductively. In the above expression, $\mathrm{id}^k = \mathrm{id}$ if $k = 1$, and is zero otherwise. On the right-hand side k' may be equal to 0. Then

$$\sum_{\substack{k_1+k'+k_2=k\\ j_1+\cdots+j_r=k_1\\ r+k_2\geq 0, k'\geq 1}} (-1)^{\clubsuit(\underline{x},k_1)} \begin{bmatrix} \beta^{\otimes k_1} \otimes h^{k'} \otimes \mathrm{id}^{\otimes k_2} \\ \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_r} \otimes \mathrm{id}^{\otimes k_2+1} \\ m^{r+1+k_2} \end{bmatrix}$$
$$= m_B^1 \circ h \circ \phi^k + \phi^k = \alpha\circ\beta\circ\phi^k - h\circ m_B^1 \circ \phi^k - h \circ m_B^1 \circ m_B^1 \circ h \circ \phi^k$$

$$= \alpha \circ \beta \circ \phi^k + \sum_{\substack{k_1+k'+k_2=k \\ j_1+\cdots+j_r=k_1 \\ r+k_2\geq 0, k'\geq 1}} (-1)^{\clubsuit(\underline{x},k_1)} \begin{bmatrix} \beta^{\otimes k_1} \otimes h^{k'} \otimes \mathrm{id}^{\otimes k_2} \\ \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_r} \otimes \mathrm{id}^{\otimes k_2+1} \\ h \circ m_B^1 \circ m^{r+1+k_2} \end{bmatrix}$$

$$= \alpha \circ \beta \circ \phi^k + \sum_{\substack{k_1+k'+k_2=k \\ j_1+\cdots+j_r=k_1 \\ i_1+i'+i_2=r+k_2+1\geq 1 \\ k'\geq 1}} (-1)^{\clubsuit(\underline{x},k_1)+\clubsuit(\underline{y},j_1)} \begin{bmatrix} \beta^{\otimes k_1} \otimes h^{k'} \otimes \mathrm{id}^{\otimes k_2} \\ \alpha^{j_1} \otimes \cdots \otimes \alpha^{j_r} \otimes \mathrm{id}^{\otimes k_2+1} \\ \mathrm{id}^{\otimes i_1} \otimes m_B^{i'} \otimes \mathrm{id}^{\otimes i_2} \\ h \circ m^{i_1+i_2+1} \end{bmatrix},$$

where $\underline{y} = y_1 \otimes \cdots \otimes y_{r+k_2+1}$ is the partial composition

$$(\alpha^{j_1} \otimes \cdots \otimes \alpha^{j_r} \otimes \mathrm{id}^{\otimes k_2+1}) \circ (\beta^{\otimes k_1} \otimes h^{k'} \otimes \mathrm{id}^{\otimes k_2}) \circ (\underline{x}).$$

Collecting into terms for which $j_1 + j' < r$, $j_1 < r < j_1 + j'$ and $r < j_1 + 1$ allows us to either apply the A_∞ homomorphism for α^k or the induction hypothesis to push the $m_B^{j'}$ term through to the top of the composition. The remainder of the proof is standard. □

Remark A.2.2 The relations (10)–(12) require some motivation. The A_∞ relations can be proved noninductively by unpacking the definition of ρ into the sum over compositions defined over stable trees, as in [22]. The A_∞ and homotopy relations correspond to contractions and expansions of these trees at internal vertices and edges; the argument in some crucial way uses the fact that the contractions of stable trees at internal edges are again stable trees.

To generalize to the filtered setting, one must consider stable trees with k internal and n external leaves, where the internal leaves are labeled with curvature terms coming from m^0. The A_∞ and homotopy relations again correspond to contractions and expansions. However, the contraction of a stable tree at an internal leaf is not necessarily stable. The terms added to (10)–(12) cancel out these contributions coming from nonstable trees.

We give the noninductive proof of Theorem A.2.1 in the preprint version of this paper.

A.3 Morphisms are mapping cocylinders

In the category of chain complexes, there is a dictionary between morphisms and mapping cocylinders. We now extend this dictionary to filtered A_∞ algebras.

Definition A.3.1 Let A^+ and A^- be two filtered A_∞ algebras. A *cocylinder* from A^+ to A^- is a filtered A_∞ algebra B which

- as a vector space is isomorphic to $A^- \oplus A^0 \oplus A^+$,

- has differential of the form
$$\begin{pmatrix} m^{-;-} & 0 & 0 \\ m^{-;0} & m^{0;0} & m^{+;0} \\ 0 & 0 & m^{+;+} \end{pmatrix},$$
where $m^{+;0}$ is an isomorphism with inverse satisfying $\ell((m^{+;0})^{-1} \circ m^0) > 0$, and
- has projections of chain complexes
$$\begin{array}{ccc} & B & \\ {}^{\beta^-}\swarrow & & \searrow^{\beta^+} \\ A^- & & A^+ \end{array}$$
which can be extended to A_∞ homomorphisms $\beta^k_\pm$, with $\beta^k_\pm = 0$ for all $k \neq 1$.

We denote such a mapping cocylinder by
$$A^- \leftrightarrow B \to A^+.$$

The cylinders from A^- to A^+ are in correspondence with morphisms $f : A^- \to A^+$.

Theorem A.3.2 (cylinders are mapping cocylinders) *Let A^- and A^+ be two filtered A_∞ algebras.*

(i) *To every cocylinder $A^- \leftrightarrow B \to A^+$, we can associate a morphism $\Theta_B : A^- \to A^+$.*

(ii) *To every morphism $f : A^- \to A^+$, we can associate a cocylinder*
$$A^- \leftrightarrow B_f \to A^+.$$

(iii) *These constructions are compatible in the sense that $\Theta_{B_f} = f$.*

Outline of proof The construction of a morphism associated to a mapping cocylinder proceeds by showing that the projection β_-, and the map $\alpha : A^- \to B$ and homotopy $h : B \to B$ given by
$$\alpha(x) = (x, 0, (m^{+;0})^{-1} \circ m^{-;0}(x)) \quad \text{and} \quad h(x) = (0, 0, (m^{+;0})^{-1}(x)),$$
satisfy the conditions of Theorem A.2.1, giving us an A_∞ extension of α. The morphism associated to the cocylinder is the "pullback–pushforward" composition $\Theta_B := \beta_+ \circ \alpha$.

The construction of a mapping cocylinder associated to a morphism follows from a more general construction: Let C be any filtered A_∞ algebra, and let $f : A^- \to C$ and $g : A^+ \to C$ be filtered A_∞ homomorphisms. Then C can be equipped with the structure of a filtered (A^+, A^-) bimodule whose bimodule product
$$m^{k_1|1|k_1}_{A^-|C|A^+} : (A^-)^{\otimes k_1} \otimes C \otimes (A^+)^{\otimes k_2} \to C$$
is given (up to sign) by
$$m^{k_1|1|k_2}_{A^-|C|A^+} = \sum_{\substack{h_1+\cdots+h_{\alpha_1}=k_1 \\ i_1+\cdots+i_{\alpha_2}=k_2}} m_C^{\alpha_1+1+\alpha_2} \circ (f^{h_1} \otimes \cdots \otimes f^{h_{\alpha_1}} \otimes \mathrm{id}_C \otimes g^{i_1} \otimes \cdots \otimes g^{i_{\alpha_2}}).$$

The proof of the A_∞ bimodule relations uses the A_∞ homomorphism relations of f and g. Given an A_∞ bimodule C, we define (up to sign) an A_∞ product on $A^- \oplus C[1] \oplus A^+$ by

$$m^k_{A^- \times_C A^+} = (m^k_{A^+} \circ \pi^{k|0|0}_{A^-|C|A^+}) \oplus \left(f^k \circ \pi^{k|0|0}_{A^-|C|A^+} + \left(\sum_{k_1+1+k_2=k} m^{k_1|1|k_2}_{A^-|C|A^+} \circ \pi^{k_1|1|k_2} \right) + g^k \circ \pi^{0|0|k} \right) \oplus (m^k_{A^+} \circ \pi^{0|0|k}),$$

for which checking the A_∞ relations becomes a check of the bimodule structure. Furthermore, the natural projections $\beta_\pm \colon A^- \oplus C[1] \oplus A^+ \to A^\pm$ are A_∞ homomorphisms.

To obtain the mapping cocylinder, we let $C = A^+$ and $g = \mathrm{id}$. □

We include the following remark, which we could not find in the literature:

Remark A.3.3 We expect that the A_∞ algebra $A^- \oplus C[1] \oplus A^+$ is the homotopy fiber product, $A^- \times_C A^+$ in the category of filtered A_∞ algebras. Some specializations are:

- If $A = A^- = A^+ = C$ and $f = g = \mathrm{id}_A$, then $A \times_A A$ is a model of $A \times [0,1]$, similar to that constructed in [17, Lemma 4.2.25].
- For filtered A_∞ algebras, a morphism $f \colon A \to C$ does not make C a left A module, but rather only an (A, A) bimodule. We note that if there exists a morphism $0 \to C$, then we can make C an $(A, 0)$ bimodule, and so it becomes a left A module. This observation is related to the existence of mapping cones and the existence of bounding cochains.

By existence of a pushforward map on bounding cochains, C is unobstructed if and only if there exists a filtered A_∞ homomorphism $0 \colon 0 \to C$. We may therefore restate the above observation as $f \colon A \to C$ has a mapping cone if and only if C is unobstructed.

Appendix B Domain and label-dependent perturbation systems

We adopt language from [10, Section 4.2]. Pick $h \colon L \to \mathbb{R}$ a Morse function. We denote by Γ a combinatorial type of treed disk. We use C to denote a treed disk, $\mathcal{M}(\Gamma)$ for the moduli space of treed disks of fixed combinatorial type, $\overline{\mathcal{M}}(\Gamma)$ to denote its closure, and $\overline{\mathcal{U}}(\Gamma)$ for the universal treed disk composed of pairs (C, x), where $x \in C$ is a point. We will denote the forgetful map by $C \colon \overline{\mathcal{U}}(\Gamma) \to \overline{\mathcal{M}}(\Gamma)$. The universal treed disk is covered with two sets, $\overline{\mathcal{U}}(\Gamma) = \overline{\mathcal{S}}(\Gamma) \cup \overline{\mathcal{T}}(\Gamma)$, which consist of pairs (C, x) where x lies in either the surface or tree component of C [10, Section 4.2]. There are morphisms $\mathcal{M}(\Gamma) \to \mathcal{M}(\Gamma')$ arising from morphisms of underlying combinatorial types [10, Definition 4.12]; they are called collapsing edges/making an edge finite or nonzero, cutting edges, locality and forgetting forgettable edges. For this section, we will focus on the morphisms of making an edge finite and cutting edges. Let $V(\Gamma)$ denote the number of external leaves of Γ and $N(\Gamma)$ denote the number of interior marked points of Γ. We use $\ell(e)$ for the length of an edge e.

A *labeled* combinatorial type $\underline{\Gamma}$ is a combinatorial type Γ, along with labels

- $\underline{x} = \{x_0; x_1, \dots, x_{V(\Gamma)-1} \mid x_i \in \mathrm{Crit}(h)\}$ a sequence of critical points of a Morse function h, indexed by the leaves of Γ (we reserve x_0 for the label of the root of the tree), and
- at each k–broken edge e, a sequence of breaking labels $B_e := \{b_1, \dots, b_k\}$.

The moduli spaces of domains of fixed labeled type cannot be compactified by simply considering labeled types, as the resulting compactification will not be Hausdorff. When an edge length goes to infinity, the domain will not know what label to assign the breaking in the compactification stratum. Therefore, we help the domains remember how they are supposed to break. A *prebroken* combinatorial type $\underline{\Gamma}_P$ is a labeled combinatorial type $\underline{\Gamma}$, along with the data of prebreaking labels P, which consists of

- an assignment to each edge e of a prebreaking number $k'(e)$,
- a labeling at each k' prebroken edge or vertex a sequence of prebreaking labels $P_e := \{x_1, \dots, x_{k'}\}$, and
- an ordering at each edge e of $B_e \sqcup P_e$.

We will sometimes treat the data of P as a set, so that $P' \subset P$ means that the prebreaking labels of $\underline{\Gamma}_{P'}$ are a subset of the labels of $\underline{\Gamma}_P$, or $\underline{\Gamma}_{P \setminus P'}$ is the prebroken combinatorial type where we've removed the P' prebreakings.

A *prebroken* disk domain $\underline{C}_P$ of combinatorial type $\underline{\Gamma}_P$ is a treed disk with prebroken marks, which is of a sequence of labeled points $(t_1, x_1), \dots, (t_{k'(e)}, x_{k'(e)})$ at each edge e. Here $0 < t_1 < \cdots < t_{k'(e)} < \ell(e)$ is a sequence of points on the edge e and the prebreaking labels are given by $P_e = \{x_i\}_{i=1}^{k'(e)}$. There exists a moduli space of prebroken combinatorial types $\mathcal{M}(\underline{\Gamma}_P)$, as well as a universal curve of prebroken combinatorial types $\mathcal{U}(\underline{\Gamma}_P)$.

B.0.1 Morphisms of prebroken combinatorial types We now look at morphisms of prebroken combinatorial types. We include the morphisms from [10, Definition 4.6]: cutting edges, collapsing edges, making an edge length finite, making an edge length nonzero, forgetting tails and making an edge weight finite or nonzero. We add in a new morphism: forgetting a prebreaking. The main changes between morphisms of prebroken combinatorial types and combinatorial types are in cutting edges, making an edge length finite and forgetting a prebreaking. We discuss these changes, as well as the corresponding maps on the moduli space of prebroken treed disks and their universal curves.

(i) **Cutting an edge** If $\underline{\Gamma}_P$ has a breaking b_i occurring at edge e then:

- $\underline{\Gamma}'_P \to \underline{\Gamma}_P$, the morphism of cutting an edge e at b_i, replaces Γ with a disconnected graph Γ'. Let v_1 and v_2 be the edges of e. The graph Γ' removes the edge e and adds in two vertices w_1 and w_2 connected to edges $e_1 = v_1 w_1$ and $e_2 = v_2 w_2$.
- The labels of $\underline{\Gamma}'$ are the same as on $\underline{\Gamma}$, except at the new vertices w_1 and w_2, which both obtain the label b_i.
- Labels for prebreakings and breakings of $\underline{\Gamma}'_P$ are otherwise inherited from $\underline{\Gamma}_P$.

There are induced homeomorphisms $\mathcal{M}(\underline{\Gamma}'_P)\to\mathcal{M}(\underline{\Gamma}_P)$ and $\mathcal{U}(\underline{\Gamma}'_P)\to\mathcal{U}(\underline{\Gamma}_P)$ as in [10]. See the arrows labeled (i) in Figure 10 for a visualization of this identification.

(ii) **Making an edge finite** We now discuss the morphism of making an infinite broken edge finite. We say that Γ is obtained from Γ' by making an edge finite if the edges of Γ and Γ' agree away from a single edge e, where $\rho(e)=\infty$ and $\rho'(e)\in(0,\infty)$. In this case, $\mathcal{M}(\Gamma')$ embeds in $\mathcal{M}(\Gamma)$ as a codimension-1 boundary $\mathcal{M}(\Gamma')\times[0,\epsilon)$.

Given prebroken combinatorial types $\underline{\Gamma}_P$ and $\underline{\Gamma}'_{P'}$, we say that $\underline{\Gamma}_P$ is obtained from $\underline{\Gamma}'_{P'}$ by making an edge finite at breaking b_i if

- the underlying unlabeled combinatorial types are related by such a morphism, and
- the edge $e\in\underline{\Gamma}'_{P'}$ has one additional prebroken marking x_i which matches the marking of the breaking $b_i\in B_e$ for $\underline{\Gamma}_P$.

Unfortunately, it is no longer the case that $\mathcal{M}(\underline{\Gamma}'_{P'})$ embeds as a codimension-1 boundary into $\mathcal{M}(\underline{\Gamma}_P)$, due to the extra choice in determining where to place the prebroken point. Therefore we cannot construct a compactification of $\mathcal{M}(\underline{\Gamma}_P)$ by simply including broken configurations. However, there remains enough structure for us to later define what a coherent perturbation is in this setting, and thus obtain moduli spaces of pseudoholomorphic treed disks which have the appropriate broken configurations in Section B.2. For $P'\subset P$ a set of markings, let $\mathcal{M}^{P'>1/2}(\underline{\Gamma}_P)$ denote the portion of the moduli space of prebroken treed disks where the distance between the prebreakings $t_i\in e$ of P' and the boundary of e is greater than $\frac{1}{2}$. When P' consists of a single marking (x_i,t_i), this space is homeomorphic to

$$\mathcal{M}^{\{x_i\}>1/2}(\underline{\Gamma}_P)=\left\{(\underline{C}'_{P'},r,t_s)\mid \underline{C}'_{P'}\in\mathcal{M}(\underline{\Gamma}'_{P'})\text{ for }r\in[0,\epsilon)\text{ and }t_i\in\left(\tfrac{1}{2},\ell(e)-\tfrac{1}{2}\right)\subset e\right\}.$$

By abuse of notation, we denote this set by $\mathcal{M}(\underline{\Gamma}')\times[0,\epsilon)\times\left(\frac{1}{2},\ell(e)-\frac{1}{2}\right)$. There is similarly an identification of the portion of the universal curve living over the region $\mathcal{U}^{\{x_i\}>1/2}(\underline{\Gamma})$ with $\mathcal{U}(\underline{\Gamma}')\times[0,\epsilon)\times\left(\frac{1}{2},\ell(e)-\frac{1}{2}\right)$. See the arrows labeled (ii) Figure 10 for a visualization of this identification.

(iii) **Forgetting prebreakings** Let $\underline{\Gamma}_P$ be a prebroken combinatorial type, and let $P'\subset P$ be a subset of the labels. Then we have forgetful maps on the moduli space of treed disks $\mathcal{M}(\underline{\Gamma}_P)\to\mathcal{M}(\underline{\Gamma}_{P'})$ as well as on the universal curve

$$\operatorname{res}_{P'\subset P}:\mathcal{U}(\underline{\Gamma}_P)\to\mathcal{U}(\underline{\Gamma}_{P'}).$$

Let $P'\subset P$ denote a subset of the prebreakings. We say that a prebroken treed disk $\underline{C}_P$ belongs to $\mathcal{M}^{P'<1/4}(\underline{\Gamma}_P)$ if the markings $(t,p)\in P'$ all have a distance less than $\frac{1}{4}$ from the boundary of the edges. Similarly, let $\mathcal{U}^{P'<1/4}(\underline{\Gamma}_P)$ denote the portion of the universal curve whose prebroken domain belongs to $\mathcal{M}^{P'<1/4}(\underline{\Gamma}_P)$. When we look at a single prebreaking labeled x_i, the neighborhood $\mathcal{U}^{\{x_i\}<1/4}(\underline{\Gamma}_P)$ can be identified with

$$\left\{(C,x,t_i)\mid (C,x)\in\mathcal{U}(\underline{\Gamma}_{P\setminus\{t_i\}})\text{ for }t_i\in\left[0,\min\left(\tfrac{1}{4},\ell(e)\right)\sqcup\left(\max\left(\ell(e)-\tfrac{1}{4},0\right),\ell(e)\right)\right]\right\}.$$

See the arrows labeled (iii) in Figure 10 for a diagram of the forgetting prebreakings identification.

B.0.2 Boundary regions on prebroken disks The moduli space of prebroken curves of fixed combinatorial type is noncompact. In addition to the usual sequences of treed disks which do not converge in the moduli space of treed disks, the prebreakings lead to new kinds of sequences that do not have convergent subsequences. There are two new kinds of phenomena we must account for:

(i) the distance between two prebreakings goes to zero, or

(ii) one of the prebreakings moves towards the boundary of an edge.

We will later force the prebreakings to remain disjoint from each other, so the first kind of noncompactness will not affect the noncompactness of the moduli spaces of pseudoholomorphic treed disks we construct.

The second kind of noncompactness we handle only when constructing the moduli space of pseudoholomorphic treed disks. We will consider a perturbation datum which is only domain and label (but not prebreaking) dependent over $\mathcal{M}^{P'<1/4}(\underline{\Gamma}_P)$ for every $P' \subset P$ (and thus independent of the position of prebreakings in P'). Such a choice of perturbation datum has the property that, whenever we look at a sequence of pseudoholomorphic treed disks with P' prebreakings moving towards the boundary of the edge, we arrive in a portion of the moduli space of pseudoholomorphic treed disks which is cut out by the $P \setminus P'$–dependent J–holomorphic curve. In this chart, it is no longer a problem that the P' prebreakings move towards the edge.

B.1 The perturbation datum and coherence

A perturbation datum for type $\underline{\Gamma}_P$ [10, Definition 4.10] is an assignment of domain and prebreaking dependent Morse function (ie a map $h_\Gamma\colon \overline{\mathcal{T}}(\underline{\Gamma}_P) \times L \to \mathbb{R}$) and domain and prebreaking dependent tame almost complex structure ($J_\Gamma\colon \bar{\mathcal{S}}(\underline{\Gamma}_P) \times X \to \mathrm{End}(TX)$) to each prebroken treed disk $\underline{C}_P$. These are required to agree with the prescribed Morse function $h\colon L \to \mathbb{R}$ and tame almost complex structure $J\colon TX \to TX$ near the boundaries of the edges and disks. We will denote perturbation datum for type $\underline{\Gamma}_P$ by $\mathcal{P}(\underline{\Gamma}_P)$, so that

$$\mathcal{P}(\underline{\Gamma}_P)(\underline{C}_P, x) = \begin{cases} J(\underline{\Gamma}_P)(\underline{C}_P, x) \in \mathrm{End}(TX) & \text{if } (\underline{C}_P, x) \in \bar{\mathcal{S}}(\underline{\Gamma}_P), \\ h(\underline{\Gamma}_P)(\underline{C}_P, x) \in C^\infty(L, \mathbb{R}) & \text{if } (\underline{C}_P, x) \in \overline{\mathcal{T}}(\underline{\Gamma}_P). \end{cases}$$

Definition B.1.1 A *stabilizing Morse function* is a Morse function $h\colon L \to \mathbb{R}$ whose critical values are distinct; for $x_i \in \mathrm{Crit}(h)$, denote the level set of h containing the critical point x_i by $H_{x_i} := \{h^{-1}(h(x_i))\}$.

A perturbation datum $\mathcal{P}(\underline{\Gamma}_P)$ is *stabilizing with respect to D* if it satisfies [10, Definition 4.26]; this roughly states that every solution to the $\mathcal{P}(\underline{\Gamma}_P)$–perturbed Floer equation will meet the stabilizing divisor in the appropriate number of intersection points (see Section C.1). The perturbations of almost complex structure must be chosen in such a way that D remains an almost complex hypersurface.

A perturbation datum $\mathcal{P}(\underline{\Gamma}_P)$ is stabilizing with respect to h if, away from the critical points, the gradient $\nabla h(\underline{\Gamma}_P)(C_P, x)$ is transverse to $h^{-1}(\mathrm{Crit}(h))$. We say that $\mathcal{P}(\underline{\Gamma}_P)$ is stabilizing if it is stabilizing for both D and h.

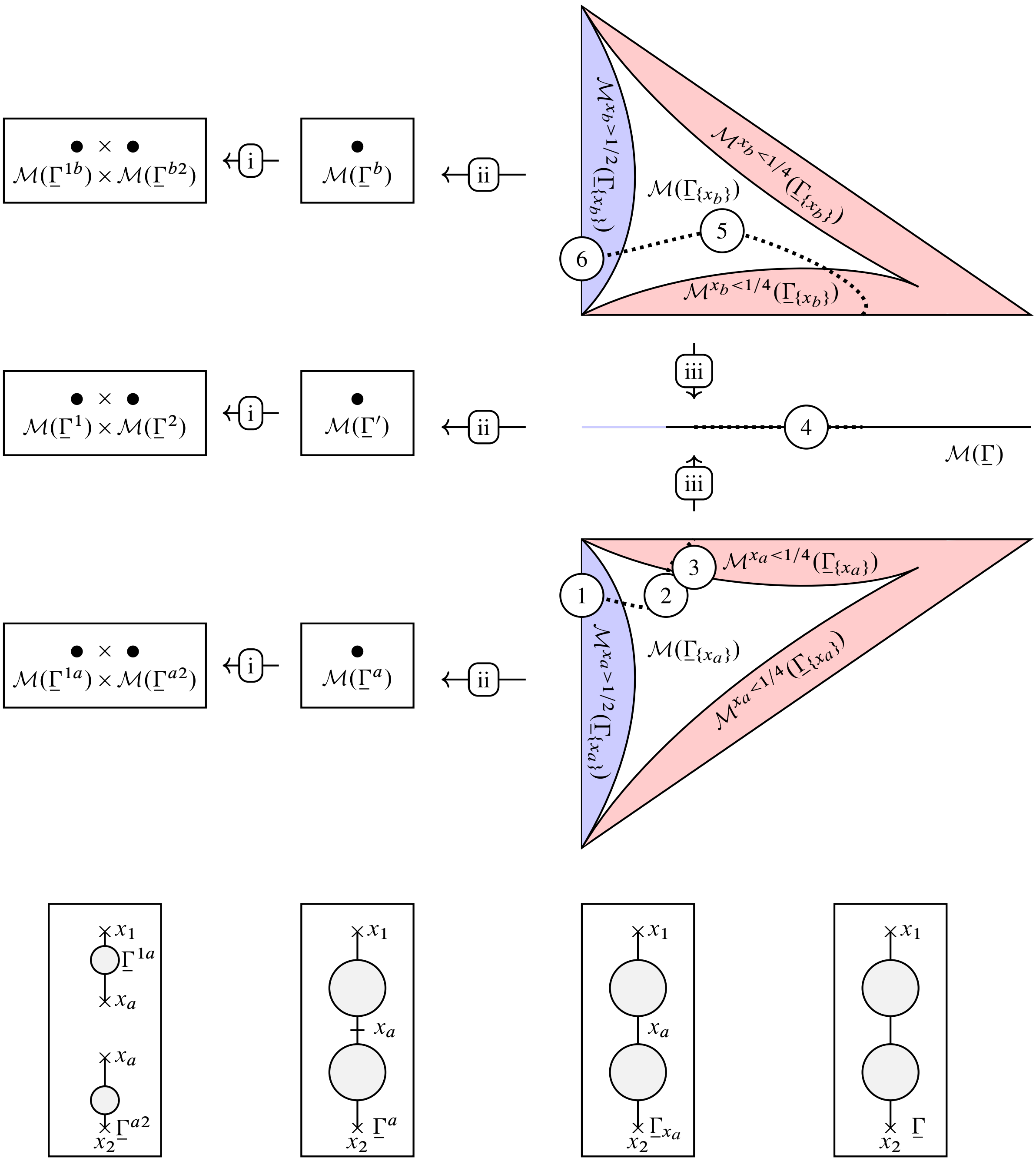

Figure 10: Describing the moduli space of prebroken regular treed disks. Here $\underline{\Gamma}$ is the combinatorial labeled type between two critical points x_1 and x_2 with two interior vertices. $\underline{\Gamma}_{\{t_a\}}$ and $\underline{\Gamma}_{\{t_b\}}$ have the same underlying combinatorial type, but additionally have prebreakings at t_a and t_b, respectively, labeled with classes $x_a, x_b \in \mathrm{Crit}(h)$. $\underline{\Gamma}'$, $\underline{\Gamma}^a$ and $\underline{\Gamma}^b$ are broken types, whereas $\underline{\Gamma}_{1a}$, $\underline{\Gamma}_{a2}$, $\underline{\Gamma}_{1b}$ and $\underline{\Gamma}_{b2}$ are combinatorial types between x_1, x_2, x_a and x_b with indices indicating the labels. The points marked 1–6 refer to the pseudoholomorphic treed disks which appear in Figure 11.

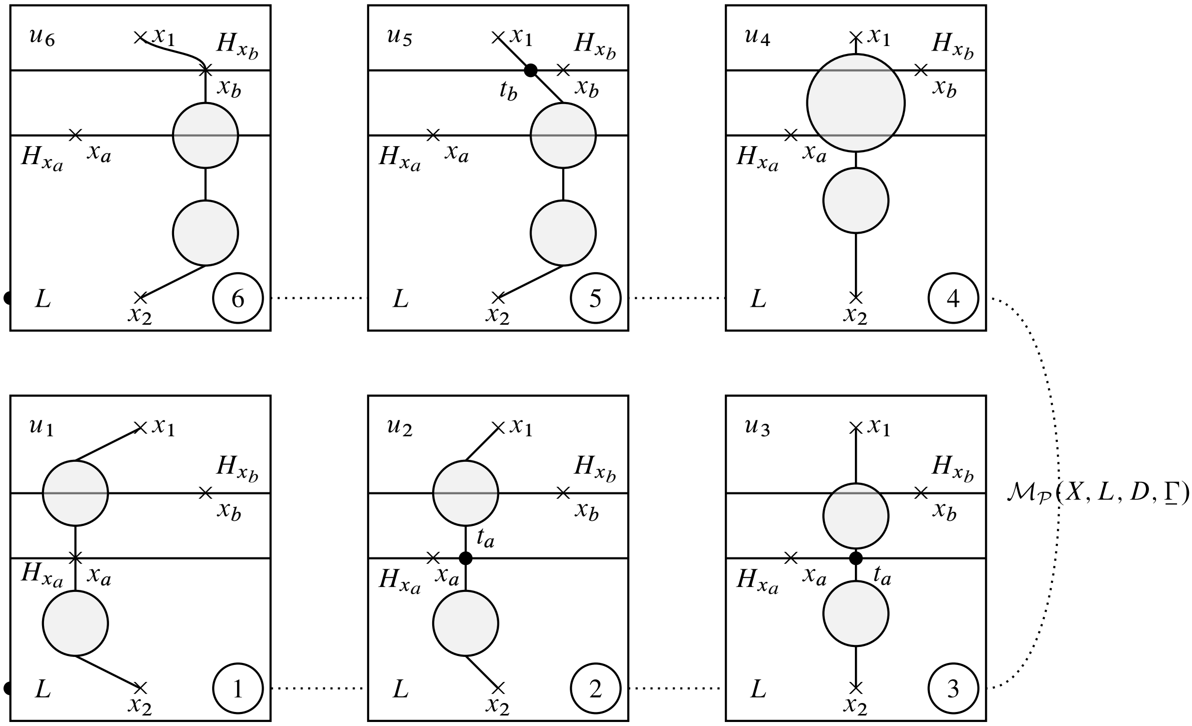

Figure 11: An example of the moduli space of labeled pseudoholomorphic pearly flow lines of type $\underline{\Gamma}$. The moduli space is represented by the dotted line, and the diagrams represent the image of $u_i(T)$ and $u_i(\partial S)$ inside of L for various maps u_i. We also mark critical points x_1, x_2, x_a and x_b of a Morse function $h\colon L \to \mathbb{R}$, as well as level sets H_{x_i}. The projection $\underline{C}_P(\mathcal{M}^{\mathrm{un}}_{\mathcal{P}}(X, L, D, \underline{\Gamma}))$ to the moduli space of types (drawn in Figure 10) is represented by the dotted line in that figure, with the circled numerals 1–6 identifying the combinatorial type of each of these pseudoholomorphic treed disks.

Given a regularizing perturbation datum, one can define $\mathcal{P}(\underline{\Gamma}_P)$–perturbed pseudoholomorphic treed disks, which are maps $u\colon \underline{C}_P \to X$ satisfying the conditions of [10, Definition 4.13].

Definition B.1.2 (modification of [10, Definition 4.17]) A stable prebroken pseudoholomorphic treed disk $u\colon \underline{C}_P \to X$ with boundary on L is adapted to D and h if it satisfies the (a) stable surface and (b) leaf axioms from [10, Definition 4.17], and additionally:

(c) **Prebroken axiom** Each prebroken point $t_i \in \underline{C}_P$ with label x_i is mapped to the H_{x_i}. Furthermore, each connected component of $u^{-1}(H_{x_i})$ contains a prebroken point t_i with label x_i.

One upshot of using labeled treed disks is that, as $\underline{\Gamma}$ contains the data of which critical points of h its flow lines limit to, we may now define $\operatorname{ind}(\underline{\Gamma})$, the expected dimension of the moduli space of labeled treed holomorphic disks. We denote by $\mathcal{M}_{\mathcal{P}}(X, L, D, h, \underline{\Gamma}_P)$ the moduli space of $\mathcal{P}$–perturbed pseudoholomorphic prebroken labeled treed disks which are adapted to D and h. Because the perturbations that we choose for Morse functions are still required to be transverse to the level sets H_{x_i} for $x_i \in \mathrm{Crit}(h)$, transversality for the conditions imposed by the prebroken axiom are automatically satisfied on the interior of $\mathcal{M}_{\mathcal{P}}(X, L, D, h, \underline{\Gamma}_P)$. As before, we write the forgetful map $\underline{C}_P : \mathcal{M}_{\mathcal{P}}(X, L, D, h, \underline{\Gamma}_P) \to \mathcal{M}(\underline{\Gamma}_P)$. We say that the perturbation datum $\mathcal{P}(\underline{\Gamma}_P)$ is *regular* if the moduli space $\mathcal{M}_{\mathcal{P}}(X, L, D, h, \underline{\Gamma}_P)$ is cut out transversely.

Remark B.1.3 Suppose that $\underline{\Gamma}_P$ is a combinatorial type with prebreakings. Let $\mathcal{M}^{\mathrm{un}}_{\mathcal{P}}(X, L, D, \underline{\Gamma}_P)$ be the moduli space of pseudoholomorphic treed disks which are adapted to D, but only satisfy the following weakening of the prebroken axiom:

(c′) **Weak prebroken axiom** Each prebroken point $t_i \in \underline{C}_P$ with label x_i is mapped to the H_{x_i}.

We note that if $P' \subset P$ is a subset of labels, and $\mathcal{P}(\underline{\Gamma}_P)$ arises from $\mathcal{P}(\underline{\Gamma}_{P'})$ by pullback under forgetting prebreakings, then every adapted $\mathcal{P}(\underline{\Gamma}_P)$ pseudoholomorphic disk is an example of a $\mathcal{P}(\underline{\Gamma}_{P'})$ weakly adapted pseudoholomorphic disk, simply given by forgetting the position of the marked points in $P \setminus P'$. Since the prebroken axiom determines the positions of these marked points uniquely, we obtain an inclusion

$$\mathcal{M}_{\mathcal{P}}(X, L, D, h, \underline{\Gamma}_P) \subset \mathcal{M}^{\mathrm{un}}_{\mathcal{P}}(X, L, D, \underline{\Gamma}_{P'}).$$

which is a diffeomorphism onto its image.

We will want perturbations that are chosen consistently between different labeled combinatorial types. Let $\mathbb{T}$ denote the set of all prebroken combinatorial types for (L, h), and use $\mathbb{I} \subset \mathbb{T}$ to denote a subset of the prebroken combinatorial types. A *perturbation system* $\mathcal{P}_{\mathbb{I}}$ is a choice of perturbation datum $\mathcal{P}_{\mathbb{I}}(\underline{\Gamma}_P)$ for every prebroken combinatorial type $\underline{\Gamma}_P \in \mathbb{I}$. If $\mathcal{P}_{\mathbb{T}}$ is a perturbation system for all prebroken combinatorial types in $\mathbb{T}$, we will call $\mathcal{P}_{\mathbb{T}}$ a *full perturbation system*. For a fixed perturbation system $\mathcal{P}_{\mathbb{I}}$ and prebroken combinatorial type $\underline{\Gamma}_P \in \mathbb{I}$, we write $\mathcal{M}_{\mathcal{P}_{\mathbb{I}}}(X, L, D, h, \underline{\Gamma}_P)$ for the moduli space of $\mathcal{P}_{\mathbb{I}}(\underline{\Gamma}_P)$–perturbed pseudoholomorphic treed disks.

Adopting notation from the discussion preceding [10, Theorem 4.19], we write $\underline{\Gamma}'_{P'} \prec \underline{\Gamma}_P$ if

- there exists a morphism of combinatorial types $\underline{\Gamma}'_{P'} \to \underline{\Gamma}_P$ corresponding to collapsing edges/making an edge or weight finite or nonzero, or
- there exists a morphism of combinatorial types $\underline{\Gamma}_P \to \underline{\Gamma}'_{P'}$ given by cutting edges, or
- $\underline{\Gamma}_P = \underline{\Gamma}^1_{P_1} \cup \underline{\Gamma}^2_{P2}$ is a disconnected type and $\underline{\Gamma}'_{P'} = \underline{\Gamma}^1_{P_1}$ or $\underline{\Gamma}^2_{P_2}$, or
- there exists a morphism of combinatorial types $\underline{\Gamma}_P \to \underline{\Gamma}'_{P'}$ given by forgetting prebroken labels.

Definition B.1.4 If $\mathbb{I}$ satisfies the property that $\underline{\Gamma}_P \in \mathbb{I}$ and $\underline{\Gamma}'_P \prec \underline{\Gamma}_P$ implies $\underline{\Gamma}'_P \in \mathbb{I}$, then we say that $\mathbb{I}$ is *downward closed.*

Definition B.1.5 (following [10, Definition 4.12]) We say that $\mathcal{P}(\underline{\Gamma})$ and $\mathcal{P}(\underline{\Gamma}')$ are coherent over $\underline{\Gamma}'_{P'} \prec \underline{\Gamma}_P$ if:

(a) if there is a morphism of type "collapsing edges/making an edge nonzero" $\underline{\Gamma}'_{P'} \to \underline{\Gamma}_P$ with $\underline{\Gamma}_P, \underline{\Gamma}'_{P'} \in \mathbb{I}$, then $\mathcal{P}(\underline{\Gamma}'_P)$ is the pullback of the perturbation $\mathcal{P}(\underline{\Gamma}_P)$, or

(b) if there exists a morphism of cutting edges $\underline{\Gamma}_P \to \underline{\Gamma}'_{P'}$ we require that $\mathcal{P}(\underline{\Gamma}'_P)$ is the pushforward of the perturbation $\mathcal{P}(\underline{\Gamma}_P)$, or if $\underline{\Gamma}'_{P'}$ is a disjoint union of $\underline{\Gamma}^1_{P^1}$ and $\underline{\Gamma}^2_{P^2}$ then we require that the perturbation arises as pullback,

(c)–(d) they satisfy the locality and forgettable edges axiom from [10, Definition 4.12].

In addition to these coherence conditions, we modify the condition for breaking (so that it now incorporates the prebreaking point) and add a new condition for forgetting labels:

(e) **Coherence under making an infinite edge finite** Suppose that $\underline{\Gamma}'_{P'} \to \underline{\Gamma}_P$ is a morphism of making an edge finite. Recall that we have an identification $\mathcal{M}(\underline{\Gamma}'_{P'}) \times [0,\epsilon) \times \left(\frac{1}{2}, \ell(e) - \frac{1}{2}\right) \subset \mathcal{M}(\underline{\Gamma}_P)$, where P' has one more prebreaking than P. Given $t_s \in \left(\frac{1}{2}, \infty\right) \sqcup \left(-\infty, -\frac{1}{2}\right)$, consider the inclusion

$$i_{t_s} : \mathcal{M}(\underline{\Gamma}'_{P'}) \hookrightarrow \mathcal{M}(\underline{\Gamma}_P) \quad \text{given by } \; C \mapsto (C, 0, t_s).$$

Denote by $i_{t_s} : \mathcal{U}(\underline{\Gamma}'_{P'}) \hookrightarrow \mathcal{U}(\underline{\Gamma}_P)$ the corresponding inclusion of the universal curve. We require that $\mathcal{P}(\underline{\Gamma}'_{P'}) = i^*_{t_s} \mathcal{P}(\underline{\Gamma}_P)$ for all t_s, so that near the boundary the perturbation is not dependent on the position of t_s.

(f) **Coherence under forgetting labels** When $P' \subset P$, so that we have a forgetful map $\mathcal{U}^{P'<1/4}(\underline{\Gamma}_P) \to \mathcal{U}^{P \setminus P'}(\underline{\Gamma}_{P \setminus P'})$, we require the perturbation datum to arise as the pullback over the forgetting premarkings map, that is

$$\mathcal{P}|_{\mathcal{U}^{P'<1/4}(\underline{\Gamma}_P)} = \mathrm{res}^*_{P' \subset P} \, \mathcal{P}(\underline{\Gamma}_{P \setminus P'}).$$

We say that $\mathcal{P}_{\mathbb{I}}$ is coherent if $\mathbb{I}$ is downward closed, and $\mathcal{P}_{\mathbb{I}}$ is coherent over all relations $\underline{\Gamma}'_{P'} \prec \underline{\Gamma}_P \in \mathbb{I}$.

Remark B.1.6 The revised condition (e) states that if t_i belongs to an edge that is about to break, then perturbations do not depend on the position of t_i.

Note that (f) means that near the region where a prebreaking t_i approaches the boundary of an edge, the perturbation system is not allowed to depend on the position of the prebreaking t_i.

B.2 From coherent perturbations to Floer cohomology

We say that a stabilizing perturbation system $\mathcal{P}_{\mathbb{I}}$ is *regular* if it is regular for all labeled types $\underline{\Gamma}_P \in \mathbb{I}$ with $\mathrm{ind}(\underline{\Gamma}_P) \leq 1$. Suppose we have a regular coherent perturbation system. Then by (f) and Remark B.1.3,

$$\mathcal{M}_{\mathcal{P}_{\mathbb{I}}}(X, L, D, h, \underline{\Gamma}_P)|_{\mathcal{M}^{P'<1/4}(\underline{\Gamma}_P)} \subset \mathcal{M}^{\mathrm{un}}_{\mathcal{P}_{\mathbb{I}}}(X, L, D, \underline{\Gamma}_{P \setminus P'}).$$

Furthermore, when these spaces are regularly cut out, this is a diffeomorphism onto its image. We say, for prebreakings P_1 and P_2 and curves $u_i \in \mathcal{M}_{\mathcal{P}_{\mathbb{I}}}(X, L, D, h, \underline{\Gamma}_{P_i})|_{\mathcal{M}^{P<1/4}(\underline{\Gamma})}$, that $u_1 \sim u_2$ if they have the same image in $\mathcal{M}^{\mathrm{un}}_{\mathcal{P}_{\mathbb{I}}}(X, L, D, \underline{\Gamma})$. Property (f) allows us to construct a moduli space of $\mathcal{P}_{\mathbb{I}}$–perturbed pseudoholomorphic disks which do not have boundary components arising from prebreakings moving to the boundary of flow lines; see Section B.0.2(ii). In summary:

Claim B.2.1 *Suppose that $\mathcal{P}_{\mathbb{I}}$ is a regular coherent perturbation system. For a labeled type $\underline{\Gamma}$, we can define a moduli space*

$$\mathcal{M}_{\mathcal{P}_{\mathbb{I}}}(X, L, D, \underline{\Gamma}) := \bigcup_{\text{labelings } P \text{ for } \underline{\Gamma}} \mathcal{M}_{\mathcal{P}_{\mathbb{I}}}(X, L, D, h, \underline{\Gamma}_P)/\sim$$

which is a smooth manifold.

See Figure 11 in conjunction with Figure 10 for how $\mathcal{M}_{\mathcal{P}_{\mathbb{I}}}(X, L, D, \underline{\Gamma})$ is assembled from charts consisting of prebroken types.

B.2.1 Extending perturbation systems When proving the existence of a full perturbation system, it is desirable to construct a regular coherent perturbation system for some small collection of types $\mathbb{I}$, and then extend this regular coherent perturbation system to a full perturbation system.

Definition B.2.2 Let $\mathbb{I}$ be a collection of types, and $\mathcal{P}_{\mathbb{I}}$ be a perturbation system. We say that $\mathcal{P}_{\mathbb{J}}$ extends $\mathcal{P}_{\mathbb{I}}$ if $\mathbb{I} \subset \mathbb{J}$ and, for every $\underline{\Gamma}_P \in \mathbb{I}$,

$$\mathcal{P}_{\mathbb{I}}(\underline{\Gamma}_P) = \mathcal{P}_{\mathbb{J}}(\underline{\Gamma}_P).$$

We will assume that [10, Theorem 4.19] extends to prebroken treed disks.

Assumption B.2.3 (existence of pearly model for labeled types) Let $L \subset X$ be a Lagrangian submanifold and $D \subset X$ be a weakly stabilizing divisor. Suppose we have already picked $\mathcal{P}_{\mathbb{I}}$ a regular coherent perturbation system for a downward closed set of combinatorial types $\mathbb{I}$. Then there exists a regular full coherent perturbation system $\mathcal{P}_{\mathbb{T}}$ extending $\mathcal{P}_{\mathbb{I}}$. Additionally, $\mathcal{P}_{\mathbb{T}}$ can be chosen so that whenever we have Gromov compactness for choices of perturbations and pseudoholomorphic treed disks, and $\operatorname{ind}(\underline{\Gamma}) \leq 1$, the moduli spaces of a regularly coherent and weakly stabilized perturbation system have appropriate tubular neighborhoods providing a compactification. The orientations of these neighborhoods agree with the boundary stratification (following [10, Theorem 4.19]):

$$\partial\overline{\mathcal{M}}_{\mathcal{P}_{\mathbb{T}}}(X, L, D, \underline{\Gamma}) = \bigcup_{\underline{\Gamma}^1 \circ \underline{\Gamma}^2 = \underline{\Gamma}} \overline{\mathcal{M}}_{\mathcal{P}_{\mathbb{T}}}(X, L, D, \underline{\Gamma}^1) \times \overline{\mathcal{M}}_{\mathcal{P}_{\mathbb{T}}}(X, L, D, \underline{\Gamma}^2). \tag{16}$$

Here $\underline{\Gamma}^2 \circ \underline{\Gamma}^1$ is the concatenation of labeled trees where the outgoing label of $\underline{\Gamma}^1$ matches an incoming label for $\underline{\Gamma}^2$.

The differences between [10, Theorem 4.19] and Assumption B.2.3 come from prebreakings. With regards to coherence over the morphism of making an infinite edge finite (which has been modified), we note that the new coherence condition (Definition B.1.5(e)) forces the perturbation to be independent of the position of prebreaking in a neighborhood of where breaking occurs. As a consequence, the argument for producing the boundary stratification is no different than in the domain (and label) independent setting.

B.2.2 From coherence to A_∞ relations Let $\mathcal{P}_{\mathbb{T}}$ be a full coherent regular perturbation system for (X, L, D). For $\beta \in H_2(X, L)$, let $\mathcal{M}_{\mathcal{P}_{\mathbb{T}}}(X, L, D, \underline{\Gamma}, \beta)$ be the set of pseudoholomorphic treed disks where the sum of the homology classes of the disk components is β. For fixed $\underline{x}$ a set of labels, and $\beta \in H_2(X, L)$, we denote by

$$\overline{\mathcal{M}}_{\mathcal{P}_{\mathbb{T}}}(X, L, D, \underline{x}, \beta) = \bigcup_{\underline{\Gamma} \text{with labels } \underline{x}} \overline{\mathcal{M}}_{\mathcal{P}_{\mathbb{T}}}(X, L, D, \underline{\Gamma}, \beta)/\sim$$

where these spaces are glued along their matching boundary components (corresponding to making an edge length nonzero). Whenever these moduli spaces have expected dimension less than or equal to one, we have a boundary stratification

$$\partial \overline{\mathcal{M}}_{\mathcal{P}}(X, L, D, \underline{y}, \beta) = \bigcup_{\substack{\underline{x}_1 \circ_k \underline{x}_2 = \underline{y} \\ \beta_1 + \beta_2 = \beta}} \mathcal{M}_{\mathcal{P}_{\mathbb{T}}}(X, L, D, \underline{x}_1, \beta_1) \times \mathcal{M}_{\mathcal{P}_{\mathbb{I}}}(X, L, D, \underline{x}_2, \beta_2), \tag{17}$$

where $\underline{x}_1 \circ_k \underline{x}_2 = \underline{y}$ is the concatenation of the outgoing label of $\underline{x}_2$ with the k^{th} incoming label of $\underline{x}_1$. The count of treed disks in these moduli spaces allows us to construct the pearly model of L.

Definition B.2.4 (following [10, Definition 4.28]) Suppose that L is a Lagrangian submanifold as in Assumption B.2.3, and let $\mathcal{P}_{\mathbb{T}}$ be a full coherent regular perturbation system. The *pearly Floer cohomology* of L is the A_∞ algebra $\mathrm{CF}^\bullet(L) := \Lambda\langle \mathrm{Crit}(h)\rangle$, whose A_∞ structure coefficients are given by [10, (4.34)]

$$\langle m^k(x_1, \dots, x_k), x_0\rangle := \sum_{\beta \in H_2(X,L)} (-1)^{\heartsuit} (\sigma(u)!)^{-1} T^{\omega(\beta)} \cdot \#\mathcal{M}_{\mathcal{P}}(X, L, D, \underline{x}, \beta),$$

where $\underline{x} = \{x_0; x_1, \dots, x_k\}$, # is the count of 0–dimensional components of the moduli space with orientation, $\sigma(u)$ denotes the number of interior marked points and $\heartsuit = \sum_{i=1}^k i|x_i|$.

B.2.3 Remarks on relation to abstract perturbation techniques Given the large number of different approaches to regularizing Floer theory, we provide some context for where this framework fits in with respect to other constructions. Regularization techniques can be characterized by how "uniform" or "dependent" they are. Generally, the uniform regularizations retain much of the underlying geometric data of the symplectic manifold and almost complex structure chosen, and so they can be useful for computations. However, this must be balanced against the additional flexibility that comes with using dependent regularizations, which is necessary for achieving regularity in some settings and can be advantageous for proving abstract properties of Floer cohomology. We give a list of different regularization techniques below, ordered from "most uniform" to "most dependent".

- Global perturbations of almost complex structures place us in the most geometric setting and therefore provide us with the best tools for explicit computations of moduli spaces of pseudoholomorphic treed disks. One can hope with the right choice of global perturbations that techniques like the open-mapping principle still hold, and thereby use geometric principles to characterize the moduli spaces contributing to the structure coefficients of the A_∞ structure. Problematically, global perturbations are not flexible enough to give us regularization in examples we consider (in particular, the nonmonotone setting).

- Domain-dependent perturbations give us all the flexibility we need to construct perturbations for defining the pearly Floer cochains in the nonmonotone setting. However, it is difficult to explicitly construct a domain-dependent perturbation.
- Domain and label-dependent perturbations (what we use here) give us slightly more flexibility. The upshot is that one may be able to use geometric perturbations for particular labeled combinatorial types contributing to particular structure constants in the Floer cohomology.
- Abstract perturbation techniques (from polyfolds and Kuranishi structures) give us the greatest flexibility. The gain is that it is possible to be extremely precise about what kinds of perturbation you want to take. However, the perturbations will generally not come from any kind of geometric source, making some portions of the theory more difficult to work with.

It is a general expectation that one should be able to use a regularizing perturbation datum from one of these "uniform" regularization techniques inside the framework of a "dependent" regularization technique. For instance, one could construct a "domain-dependent" perturbation by just defining it everywhere to be given by a global perturbation of almost complex structure.

The trick, which we now exploit, is to use uniform perturbations where you need to use geometric properties for computations, then later use dependent perturbations to achieve regularity and coherence more broadly.[4] For our applications, domain and label-dependent perturbations prove to be flexible enough that we can choose the perturbations we want where we want them.

B.3 Application of domain and label-dependent perturbations: explicit computation of structure coefficients

We now give an application of using domain and label-dependent perturbations.

Consider the situation from Section 2.2.1. Recall that the hypothesis stated that there was a unique J^0–regular holomorphic disk u_{ex} with one intersection with the stabilizing divisor. We assumed that u_{ex} intersected all upward and downward flow spaces of critical points transversely. We also looked at two Morse critical points x_1 and x_0 in degrees 1 and 2, respectively, without a flow line between them. From this we concluded that treed disks with combinatorial type in

$$\mathbb{I}^{\leq 1}_{*;x_0} := \{\underline{\Gamma}_P \mid \operatorname{ind}(\underline{\Gamma}_P) \leq 0, \text{input is empty or } x_1, \text{ output is } x_0, \text{ at most 1 interior marked point}\}$$

are regularly cut for $\mathcal{P}^0$, the trivial perturbation determined by J^0 and h. We can additionally prove from the hypothesis that there are no broken flow trees with output on x_0 and one interior marked point.

Case 1 $\boldsymbol{V(\underline{\Gamma}) = 0}$ Let $\underline{\Gamma}$ be the labeled combinatorial type of treed disks with one vertex labeled with the class of u_{ex} and root labeled with x_0, as in Figure 12, top. As the disk u has the minimal number of stabilizing interior marked points, the only kinds of broken configurations which may occur are those where the semi-infinite edge breaks (as indicated by $\underline{\Gamma}'$ in the right-hand side of Figure 12, top). As the

[4] In polyfold theory, Katrin Wehrheim explained this once as *the Obamacare principle*, based on the idea that "if you like the plan you have, you can keep it" [26].

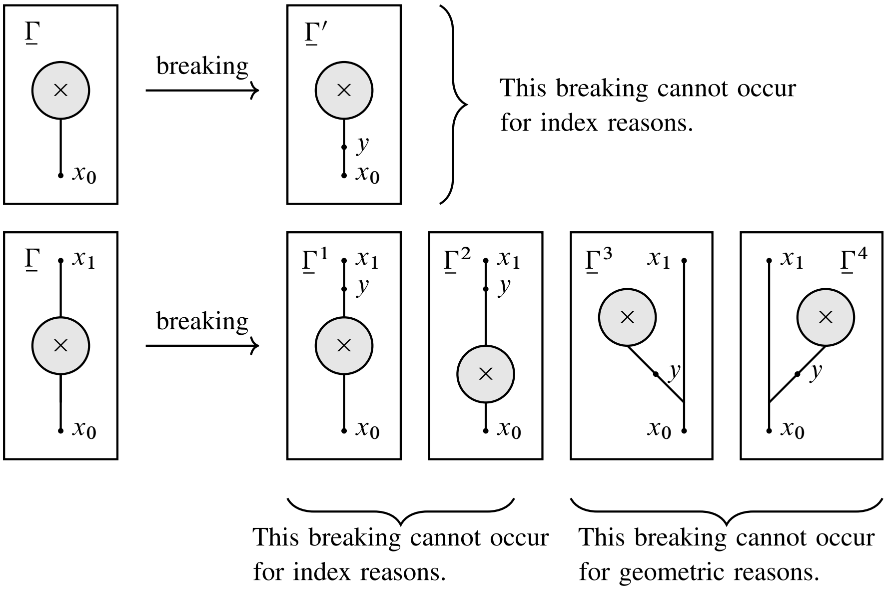

Figure 12: Possible breakings of flow trees between x_1 and x_0 with a single marked point. Top: possible breakings of a pearly flow line between x_0 and x_1 which has a minimal energy disk. Bottom: possible breakings of type with output on x_0.

disk u_{ex} is transverse to all upwards and downward flow spaces and is regular, the configuration with output on y is a regular configuration; this implies that y has degree 2. This means that the flow line between x_0 and y_0 has expected dimension -1; since the function h is Morse–Smale, there are no such flow lines. Therefore, there are no broken configurations of the type $\underline{\Gamma}'$.

Case 2 $\boldsymbol{V(\underline{\Gamma}) = 1}$ In the case that we are looking at pearly trajectories between x_0 and x_1, the combinatorial types of breaking which can occur (due to the minimal interior marked point conditions) are listed in Figure 12, bottom. By the same reasoning as the case with one output, the first two types of breaking (of combinatorial type) $\underline{\Gamma}^1$ and $\underline{\Gamma}^2$ cannot occur. Without additional conditions, the combinatorial type $\underline{\Gamma}^3$ or $\underline{\Gamma}^4$ may occur (for instance, if there are nonregular Morse flow trees with inputs x_1 and y and output x_0). As we assumed that there are no Morse flow lines from x_1 to x_0, there are no unperturbed Morse flow trees that contain x_1 as input and x_0 as an output. Therefore the moduli space of unperturbed pearly flow trees of type $\underline{\Gamma}^3$ or $\underline{\Gamma}^4$ is empty.

From this we conclude that $\mathcal{P}^{\mathbf{0}}_{\mathbb{I}^{\leq 1}_{*;x_0}}$ is regular for broken types in $\mathbb{I}^{\leq 1}_{*;x_0}$ as well.

Definition B.3.1 Let $\mathbb{I}$ be a collection of types, and $\mathcal{P}^{\mathbf{0}}_{\mathbb{I}}$ be a perturbation system defined for combinatorial types in $\mathbb{I}$. We say that $\mathcal{P}_{\mathbb{J}}$ weakly agrees with $\mathcal{P}^{\mathbf{0}}_{\mathbb{I}}$ if, for every $\underline{\Gamma} \in \mathbb{I} \cap \mathbb{J}$ with $\operatorname{ind}(\underline{\Gamma}_P) \leq 0$, the perturbation data agree in a small neighborhood of the labeled combinatorial types contained in these moduli spaces. That is, we require that

$$\underline{C}_P(\mathcal{M}_{\mathcal{P}^0_{\mathbb{I}}}(X, L, D, h, \underline{\Gamma}_P)) = \underline{C}_P(\mathcal{M}_{\mathcal{P}_{\mathbb{J}}}(X, L, D, h, \underline{\Gamma}_P))$$

and there exists an open neighborhood $U \supset \underline{C}_p(\mathcal{M}_{\mathcal{P}_{\mathbb{I}}}(X, L, D, h, \underline{\Gamma}_P))$ such that for all $\underline{C}_P \in U$ and $x \in \underline{C}_P$,

$$\mathcal{P}_{\mathbb{J}}(\underline{\Gamma}_P)(\underline{C}_P, x) = \mathcal{P}^0_{\mathbb{I}}(\underline{\Gamma}_P)(\underline{C}_P, x).$$

We say that $\mathcal{P}_{\mathbb{J}}$ weakly extends $\mathcal{P}^0_{\mathbb{I}}$ if they weakly agree and $\mathbb{I} \subset \mathbb{J}$.

Lemma B.3.2 *Let $\mathcal{P}^0_{\mathbb{I}^{\leq 1}_{*;x_0}}$ be the trivial perturbation on the set of combinatorial types from the above discussion. There exists a full regular coherent perturbation system $\mathcal{P}$ which is a weak extension of $\mathcal{P}^0_{\mathbb{I}^{\leq 1}_{*;x_0}}$.*

The remainder of the section will construct $\mathcal{P}$. We describe the construction of the perturbation datum in steps. Let $\mathbb{I}^0_{y;z}$ denote the combinatorial types of flow lines with no disks from y to z. For any $\underline{\Gamma}_P \in \bigcup_{y,z \in \mathrm{Crit}(h)} \mathbb{I}^0_{y;z}$, take $\mathcal{P}(\underline{\Gamma}_P)$ to be the trivial perturbation datum. The trivial perturbation is coherent. Since h is Morse, $\mathcal{P}(\underline{\Gamma}_P)$ is regular. The choices made for $\mathcal{P}$ so far agree weakly with $\mathcal{P}^0_{\mathbb{I}^{\leq 1}_{*,x_1}}$.

We now pick a perturbation datum $\mathcal{P}(\underline{\Gamma}_P)$ for $\underline{\Gamma}_P \in \bigcup_{z \in \mathrm{Crit}(h)} \mathbb{I}^1_z$, the types of index 0 with one internal marked point and one output.

Claim B.3.3 *The perturbation datum $\mathcal{P}(\underline{\Gamma}_P)$ can be picked for prebroken combinatorial types $\underline{\Gamma}_P \in \mathbb{I}^1_z$ in such a way that it is regular, weakly agrees with $\mathcal{P}^0_{\mathbb{I}^{\leq 1}_{*,x_1}}$ on the combinatorial types $\underline{\Gamma} \in \mathbb{I}^1_{x_0}$, and is coherent with previously made choices over $\bigcup_{y,z \in \mathrm{Crit}(h)} \mathbb{I}_{y;z}$.*

Proof The definition of the perturbation depends on the number of prebreaking points (ie critical values of h) on the flow line between the boundary of the disk u_{ex} and critical point x_0. We discuss the case where there are 0 or 1 prebreaking points; the remaining cases are similar.

- Suppose that the single $\mathcal{P}^0$–pseudoholomorphic curve of type $\underline{\Gamma}_P \in \mathbb{I}^1_{x_0}$ from Figure 12, top, has no prebreaking. Then $\{\mathbb{I}^1_{x_0}\}$ is downward closed, and we can choose a perturbation datum for this type freely. Then choose perturbation data for the rest of $\bigcup_z \mathbb{I}^1_z$.
- Suppose that the single $\mathcal{P}^0$–pseudoholomorphic curve of type $\underline{\Gamma}_P \in \mathbb{I}^1_{x_0}$ has a single prebreaking, and let $\underline{C}_P$ be the prebroken disk domain. We assume that this prebreaking has distance of at least $\frac{1}{4}$ from the disk (so that the coherence condition for forgetting prebreakings does not hold near this prebroken combinatorial type). Choose $\mathcal{P}(\underline{\Gamma}'_{P'})$ for all $\underline{\Gamma}'_{P'} \in \bigcup_{z \in \mathrm{Crit}(h)} \mathbb{I}^1_z$ with no prebreakings. We can choose this perturbation to be as close to $\mathcal{P}^0$ as desired. Having chosen these $\mathcal{P}(\underline{\Gamma}'_{P'})$, we now define $\mathcal{P}(\underline{\Gamma}_P)$. The moduli space $\mathcal{M}(\underline{\Gamma}_P)$ is a 1–dimensional ray that measures the distance of the prebroken label to the marked point on the disk u_{ex}.

Let $U \subset \mathcal{M}(\underline{\Gamma}_P)$ be a small neighborhood of $\underline{C}_P$. Choose $\mathcal{P}(\underline{\Gamma}_P)$ which interpolates between $\mathcal{P}^0(\underline{\Gamma}_P)$ over the neighborhood U and the previously chosen perturbations $\mathcal{P}_{\underline{\Gamma}'_{P'}}$ as the distance of the prebreaking goes to infinity. The moduli of $\mathcal{P}(\underline{\Gamma}_P)$–curves with domain belonging to the complement of U is empty, since

- $\mathcal{P}(\underline{\Gamma}'_{P'})$ was chosen close to $\mathcal{P}^0$,
- the moduli spaces of $\mathcal{P}^0$–curves with domain in the complement of U is empty, and
- emptiness is preserved under small changes of perturbation datum.

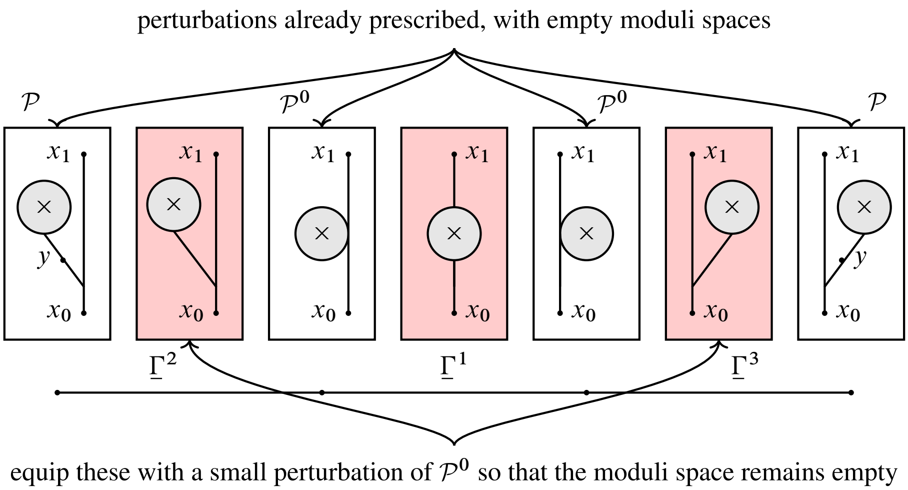

Figure 13: The three remaining tree types.

So $\mathcal{P}(\underline{\Gamma}_P)$ agrees with $\mathcal{P}^0(\underline{\Gamma}_P)$ in a neighborhood of the domains which are represented by pseudoholomorphic treed disks, which is to say that $\mathcal{P}(\underline{\Gamma})$ weakly agrees with $\mathcal{P}^0(\underline{\Gamma}_P)$.

The proof is similar for examples where there are several prebreakings on the domain $\underline{C}_P$ representing the single $\mathcal{P}^0$–pseudoholomorphic curve of type $\mathbb{I}^1_{x_0}$. □

We now handle the types of Morse flow trees. Denote by $\mathbb{I}^0_{x,y;z}$ the moduli space of Morse flow trees with inputs x and y and output z. We pick the perturbation datum $\mathcal{P}$ for types in $\bigcup_{x,y,z} \mathbb{I}^0_{x,y;z}$. The only coherence conditions we need to worry about are coherence conditions from Morse flow lines; in particular, regular and coherent $\mathcal{P}$ can be chosen to weakly extend $\mathcal{P}^0$ on $\bigcup_y \mathbb{I}_{x_1,y;x_0}$. Here, obtaining a weak agreement is easy: any sufficiently small choice of perturbation $\mathcal{P}$ will do the trick, as $\mathcal{M}_{\mathcal{P}^0}(X, L, D, \underline{\Gamma}) = \varnothing$ for $\underline{\Gamma} \in \bigcup_y \mathbb{I}_{x_1,y;x_0}$. The same argument holds for types $\underline{\Gamma} \in \bigcup_y \mathbb{I}_{y,x_1;x_0}$.

We now extend $\mathcal{P}$ to $\mathbb{I}^1_{x_1;x_0}$, the set of treed disks with input on x_1, output on x_0 and one interior marked point. This contains three combinatorial types which have expected dimension 0, which are highlighted in Figure 13. On the moduli spaces of types $\underline{\Gamma}^2$ and $\underline{\Gamma}^3$, we choose the perturbation datum which

- satisfies the coherence conditions for $\mathcal{P}$ as we go to broken configurations, and
- agrees with $\mathcal{P}^0$ as we go towards the ghost-bubbled configuration which forms a common boundary with $\underline{\Gamma}^1$ (which is also forced by coherence).

We note that if we previously chose $\mathcal{P}$ to be sufficiently small, $\mathcal{M}_{\mathcal{P}}(X, L, D, \underline{\Gamma}^i)$ will remain empty for $i \in 2, 3$, as the emptiness of moduli spaces is an open condition in the space of perturbations. On $\underline{\Gamma}^1$, we take $\mathcal{P}(\underline{\Gamma}^1) = \mathcal{P}^0(\underline{\Gamma}^1)$.

We have now defined regular and coherent $\mathcal{P}$ on a downward closed set of types containing $\mathbb{I}^{\leq 1}_{*;x_0}$. By construction, $\mathcal{P}$ is a weak extension of $\mathcal{P}^0_{\mathbb{I}^{\leq 1}_{*;x_0}}$. By [9, Theorem 4.19], we can extend this to a regular and coherent perturbation on all types.

Appendix C Stabilizing divisors and pearly model for Lagrangian cobordisms

In this appendix, we drop the requirement that $\omega(\pi_2(X)) < 0$. For simplicity of exposition, we assume that X is compact, $H_1(L)$ is torsion free and that $[\omega] \in H^2(X, \mathbb{Z})$.

C.1 Stabilizing divisors: background and summary

We use $J_\tau(X, \omega)$ to denote the space of ω–tame almost complex structures. A symplectic divisor is a symplectic hypersurface $D \subset X$. If $[D]$ is Poincaré dual to $k[\omega]$, we say that the degree of D is k. We say that J is adapted to D if $J(TD) = TD$.

A *weakly stabilizing divisor* [10, Definition 3.8] for a Lagrangian $L \subset X$ is a symplectic divisor $D \subset X$ disjoint from L and for which there exists a $J_D \in \mathcal{J}_\tau(X, \omega)$ adapted to D so that every J_D–holomorphic disk or sphere intersects D in at least one point.

A divisor is *of sufficiently large degree* for an almost complex structure J_D and a Lagrangian L if, for all J_D–holomorphic spheres u_{S^2} and disks u_{D^2} with boundary on L,

$$PD([D])([u_{S^2}]) \geq 2c_1(X)([u_{S^2}]) + \dim(X) + 1 \quad \text{and} \quad PD([D])([u_{D^2}]) \geq 1.$$

Lemma [9, Lemma 3.9] *Let L be a Lagrangian submanifold and J be an almost complex structure. There exists a constant k_m such that, for every $0 < \theta < 1$, there exists a $k_\theta > 0$ such that for all $k > k_\theta$ we can find a θ–approximately J–holomorphic divisor D of degree $k_m k$ which is of sufficiently large degree for L.*

Note that the complex structure J_D for which D is weakly stabilizing will likely not be the structure J we start with. To use weakly stabilizing divisors for the purposes of constructing open Gromov–Witten invariants, one needs the divisor to transversely intersect J_D–holomorphic curves; additionally, one would like an open set worth of J_Ds to use as domain-dependent perturbations.

Definition C.1.1 [10, Definition 4.24] For $E > 0$, an almost complex structure $J_D \in \mathcal{J}_\tau(X)$ is E–stabilized by D if and only if whenever $u\colon (\Sigma, \partial\Sigma) \to (X, L)$ is a nonconstant J_D–holomorphic curve (where $\Sigma \in \{D^2, S^2\}$) of energy less than E, we have:

- **Nonconstant spheres** There are no spheres $u\colon \Sigma \to X$ whose images are contained in D.
- **Sufficient intersections** Each sphere (resp. disk) has at least three (resp. one) intersection points with the divisor D.

Lemma [10, Lemma 4.25] *For θ sufficiently small, suppose that D has sufficiently large degree for every J_D which is θ–close to J. For each energy $E > 0$, the set of E–stabilized tame almost complex structures which are D–adapted and θ–close to J is open and dense (in the set of tame almost complex structures which are D–adapted and θ–close to J).*

C.1.1 Comparison to previous work We adapt the following lemmas to the setting of Lagrangian cobordisms:

- [9, Lemma 3.9] and [12, Lemma 8.11], which show that there exist weakly stabilizing divisors for pseudoholomorphic disks and spheres, and
- [10, Lemma 4.25], itself an extension of [12, Proposition 8.14 and Corollary 8.20], which shows that for a fixed energy bound we can find a dense set of stabilized almost complex structures.

Let $K: L^+ \rightsquigarrow L^-$ be a Lagrangian cobordism. Denote by $V \subset \mathbb{C}$ the open set with compact closure such that $K|_{\pi_{\mathbb{C}}^{-1}(\mathbb{C}\setminus V)}$ contains only the ends of the Lagrangian cobordism.

Lemma C.1.2 (weakly stabilizing divisors for Lagrangian cobordisms) *Let $K: L^+ \rightsquigarrow L^-$ be a Lagrangian cobordism. Pick an almost complex structure $J_X \times \iota$ for $X \times \mathbb{C}$. There exists a constant k_m such that, for every $0 < \theta < 1$, there exists a $k_\theta > 0$ such that for all $k > k_\theta$ we can find a θ–approximately J–holomorphic divisor D of degree $k_m k$ which is of sufficiently large degree for K. Furthermore, this divisor can be chosen so that $D|_{\pi_{\mathbb{C}}^{-1}(\mathbb{C}\setminus V)} = D_X \times (\mathbb{C} \setminus V)$ for some divisor $D_X \subset X$.*

For this divisor, we can find J_D which are weakly stabilized by D and belong to

$$\mathcal{J}_{\tau,V}(X \times \mathbb{C}, \omega + \omega_{\mathbb{C}}) := \{J \in J_\tau(X \times \mathbb{C}, \omega) \mid \exists J_z \in J_\tau(X, \omega) \text{ with } J = J_z \times \iota \text{ outside } \pi_{\mathbb{C}}^{-1}(V)\},$$

the set of almost complex structures on $X \times \mathbb{C}$ which are split and constant outside of a given compact subset V. There exists a restriction map $\operatorname{res}: \mathcal{J}_{\tau,V}(X \times \mathbb{C}, \omega + \omega_{\mathbb{C}}) \to \mathcal{J}_\tau(X, \omega)$ given by restricting to a fiber $X \times \{z\}$ with $z \notin V$.

Lemma C.1.3 (density of E–stabilized almost complex structures) *Given $K: L^+ \rightsquigarrow L^-$, choose θ and D as above. Then for $E > 0$ there exists an open and dense subset of almost complex structures*

$$\mathcal{J}_V^*(X \times \mathbb{C}, D, J, \theta, E) \subset \mathcal{J}_{\tau,V}(X \times \mathbb{C}, D, J, \theta)$$

of almost complex structures which are E–stabilized by D.

C.2 Background: divisors in the complement of a Lagrangian

We first recall the construction of weakly stabilizing divisors for Lagrangian submanifolds $L \subset X$ where X is compact. Portions of this algorithm will come into play when we construct weakly stabilizing divisors for Lagrangian cobordisms $K \subset X \times \mathbb{C}$.

The primary method (employed in [2; 14]) for constructing symplectic divisors is to present them as the zero sets of some line bundle. For $J \in J_\tau(X, \omega)$, let $E_X \to X$ be a Hermitian line bundle with connection ∇^E whose curvature is $i\omega$. The zero sets of transverse sections of E_X^k will be Poincaré dual to $k[\omega]$. When we have a J–holomorphic section $s: X \to E$ which is transverse to the zero section, the zero set $s^{-1}(0)$ is a symplectic divisor. The condition of being J–holomorphic can be weakened substantially while still preserving the symplecticity of the divisor. A common way to weaken this construction is to consider a sequence of sections $s_k\colon X \to E^k$ which are *asymptotically holomorphic* and uniformly transverse to 0; then for sufficiently large k, the sections $s_k^{-1}(0)$ will give symplectic divisors of degree k. Theorem 3 of [2] provides a tool for constructing asymptotically holomorphic sections.

Definition C.2.1 Let $V \subset X$ be a subset. We say that sections $s_k\colon X \to E^k$ *fall off from* V if there exist constants $C, \lambda > 0$ such that $|s_k(x)| < C\exp(-\lambda d(y, V)^2)$, where d is the metric induced by ω and J. A section is *concentrated at* V if there exists a constant c such that $|s_k(x)| > c$ for all $x \in V$, and the sections s_k fall off from V.

Sections which are concentrated along a subset V can be perturbed to make them transverse to the zero section without losing transversality over V. This is because if a section is concentrated at V, it is "highly transverse" to the zero section (in the sense that it is very nonzero!). This idea can be extended to sections which intersect the zero section.

Definition C.2.2 [14, Definition 17] We say that sections $s_k\colon X \to E^k$ are η–transverse to zero over V if, for all $x \in V$, whenever $|s_k(x)| < \eta$ the covariant derivative $\nabla s_k(x)\colon T_xX \to E_x^k$ is surjective with bound $|\nabla s_k(x)| > k^{1/2}\eta$. The metric we use here is the metric induced by ω and J.

Sections $s_k\colon X \to E^k$ have a *neighborhood of* η*–transversality* over V' if s_k is η–transverse over all x with $d(x, V') < 4k^{-1/3}$. We say that a section is *transversely extendible over* U if U has compact closure and s_k has a neighborhood of η–transversality over ∂U.

Given U a set, let $U^{=} := \{x \in U \mid d(x, \partial U) > 4k^{-1/3}\}$.

From the definition, it is immediate that if $s_k\colon X \to E^k$ is concentrated at V then it is transversely extendible over $X = U \setminus V$. The notion of transversely extendible is based on the following theorem:

Theorem C.2.3 (adapted from [2, Theorem 3]) *Fix $\epsilon > 0$. Given $s_k\colon X \to E$ asymptotically holomorphic sections and any open subset $U \subset X$ with compact closure, there exist asymptotically holomorphic sections $\tilde{s}_k\colon X \to E^k$ and $\tilde{\eta} > 0$ such that*

- *the sections are $\tilde{\eta}$–transverse over $U^{=}$,*
- *$|\tilde{s}_k - s_k| < \epsilon$ and $|\nabla\tilde{s}_k - \nabla s_k| < k^{1/2}\epsilon$, and*
- *$s_k(x) = \tilde{s}_k(x)$ on the complement of U.*

An immediate corollary employed in [5] is that whenever X is compact and $s_{k,V}\colon X \to E^k$ are asymptotically holomorphic sections concentrated on V, there exist η–transverse holomorphic sections which are nonvanishing on V. A small modification allows us to replace "concentrated" with "transversely extendible in the complement".

Corollary C.2.4 *Suppose that $s_{k,\mathrm{out}}\colon X \to E^k$ is asymptotically holomorphic, transverse to zero over $X \setminus U$ and transversely extendible over U. Additionally assume that $s_{k,\mathrm{out}}^{-1}(0)|_{X\setminus U}$ is a symplectic divisor* *over $X \setminus U$ for $k \gg 0$. There exists $\tilde{\eta} > 0$ and asymptotically holomorphic sections $\tilde{s}_k : X \to \mathbb{C}$, agreeing with s_k over $X \setminus U$ and $\tilde{\tilde{\eta}}$ transverse to zero over U. In particular, $\tilde{s}_k^{-1}(0)$ is a symplectic divisor for $k \gg 0$.*

Proof By definition of transverse extendibility, $s_{k,\mathrm{out}}$ has a neighborhood of η–transversality along $\partial(U)$. This implies that $s_{k,\mathrm{out}}$ is η–transverse in $U \setminus U^{=}$.

Pick ϵ small enough that $\epsilon < \frac{1}{2}\eta$. By Theorem C.2.3, we can construct sections $\tilde{s}_k$ with ϵ specified which are $\tilde{\eta}$–transverse over all points in $U^{=} \subset U$. As $s_{k,\mathrm{out}}$ is η–transverse in $U \setminus U^{=}$, $|\tilde{s}_k - s_k| < \epsilon$ and $|\nabla \tilde{s}_k - \nabla s_k| < k^{1/2}\epsilon$, we obtain that $\tilde{s}_k$ is at least $(\eta - \epsilon)$–transverse over $U \setminus U^{=}$. Therefore the $\tilde{s}_k$ are $\tilde{\tilde{\eta}} = \min\left(\tilde{\eta}, \frac{1}{2}\eta\right)$–transverse over U.

It follows that, for $k \gg 0$, $\tilde{s}_k^{-1}(0)|_U$ is a symplectic hypersurface. Since $\tilde{s}^{-1}(0)|_{X\setminus U} = s^{-1}(0)|_{X\setminus U}$ (which is symplectic for $k \gg 0$ by assumption), we obtain that, for $k \gg 0$, $\tilde{s}^{-1}(0)$ is a symplectic hypersurface. □

We now return to the problem of finding D disjoint from L; this amounts to constructing asymptotically J–holomorphic and η–transverse sections s_k such that $s_k(x) \neq 0$ for all $x \in L$.

Lemma 3.9 of [10] observes that these sections must satisfy an additional requirement if we would like $D_k = s_k^{-1}(0)$ to be weakly stabilizing. Suppose that D_k is given as the zero set of sections $s_k : X \to E^k$, which are nonvanishing on L. The sections determine trivializations τ_k of $E^k \to L$; the connection 1–form written in this trivialization determines a class $\alpha_{\tau_k} \in H^1(L, \mathbb{R})$. We can compute the intersection number of D_k with a class of disk $u : (D^2, \partial D^2) \to (X, L)$ by

$$[u] \cdot [D_k] = k \int_{D^2} u^*\omega - \int_{[\partial D^2]} u^*\alpha_{\tau_k}. \tag{18}$$

The only way we can hope for this to be positive is if the connection 1–forms α_{τ_k} are bounded; we call such a selection of trivializations *bounded*. Lemma 3.9 of [10] proves that for fixed L, boundedness of α_{τ_k} is a sufficient condition for the divisors D_k to be of sufficiently large degree for large enough k.

The bounded curvature form requirement following (18) can be accommodated in the construction of [5], which we now recall. The construction of $s_{k,L}$ starts by finding a constant C_L and picking trivializations $\tau_k : L \to E^k$ with

$$|\tau_k(x)| = 1 \quad \text{and} \quad |\nabla \tau_k(x)|_g < C_L.$$

These trivializations can additionally be chosen so that their connection 1–forms α_{τ_k} are bounded. Associated to $p \in L$ a point, [5] constructs asymptotically holomorphic and uniformly bounded sections

$$s_{k,p,L}(x) := \frac{\tau_k(p)}{|s_{k,p}(p)|} s_{k,p}(x),$$

where $s_{k,p}(x)$ is the asymptotically holomorphic section concentrated at a point p.

Consider a *finite* set of points $P(k) \subset K$ with the property that the radius $1/\sqrt{k}$ balls centered at $x \in P(k)$ cover $K \cap \pi_{\mathbb{C}}^{-1}(V)$, and the radius $1/(3\sqrt{k})$ balls around the $x \in P(k)$ are disjoint from each other and $K \cap \pi_{\mathbb{C}}^{-1}(\mathbb{C} \setminus V)$. By construction, the arguments of $s_{k,p,L}(x)$ do not differ by much; see (19). Therefore

$$s_{k,L} := \sum_{p \in P(k)} s_{k,p,L}(x)$$

is an asymptotically holomorphic section concentrated along L. To obtain a section which is η–transverse everywhere (giving us symplectic divisors), we apply Corollary C.2.4 to perturb by a small amount in the complement of L.

C.3 Construction of weakly stabilizing divisors for Lagrangian cobordisms

We adapt the constructions above for Lagrangian cobordisms $K \subset X \times \mathbb{C}$. Again, we assume that X is compact, $H_1(K)$ is torsion free and that $[\omega] \in H^2(X, \mathbb{Z})$.

C.3.1 Toy case: cobordism with empty ends We start with a toy case: let $K \subset X \times \mathbb{C}$ be a compact Lagrangian submanifold, and suppose that X is compact. We construct a weakly stabilizing divisor for K.

Take $V := \{z \mid |z| < R\} \subset \mathbb{C}$ a compact subset such that the ends of K are disjoint from V. Additionally, choose $J = J_X \times \iota$ a split almost complex structure. We now sketch how to construct a symplectic divisor in the complement of K. The approach is similar to the one before. Consider the bundle $E = (E_X \boxtimes E_{\mathbb{C}})$, with connection $\nabla^{E_X} \boxtimes \nabla^{E_{\mathbb{C}}}$ whose curvature form is $\omega_X + \omega_{\mathbb{C}}$. We take trivializations $\tau_k \colon K \to E^k$ which satisfy the connection 1–form bound following (18). By application of [5] we can construct an η–transverse asymptotically J–holomorphic section $s_{k,K} \colon X \times \mathbb{C} \to E$ which is concentrated along K.

We cannot from here immediately proceed using the construction of [5], as we need to perturb over a noncompact set to achieve transversality. However, we can find a perturbation $s_{k,\mathrm{out}} \colon X \times \mathbb{C} \to E^k$ which is transverse (but not η–transverse!) and asymptotically holomorphic outside of the region V. Consider the section $s_{k,R} \colon \mathbb{C} \to E_{\mathbb{C}}$ modeled after the section which is asymptotically holomorphic and concentrated on S^1_R from [5, page 746]; this can be explicitly written in coordinates as

$$s_{k,R}(z) = \rho_k \exp\bigl(\tfrac{1}{2}k(R - |z|)^2\bigr),$$

where $\rho_k \colon \mathbb{C} \to \mathbb{R}$ is a bump function equal to 1 outside a small neighborhood of the origin. The $s_{k,R}(z)$ define asymptotically holomorphic sections which are nonvanishing outside of $|z| > R$. Furthermore, $s_{k,R}(z)$ is concentrated along the boundary of S^1_R.

Now pick $s_{k,X} \colon X \to E$ a section which is asymptotically holomorphic and η–transverse. This gives us the section $s_{k,\mathrm{out}} := s_{k,X} \boxtimes s_{k,R} \colon X \times \mathbb{C} \to E^k$. For $(x, z) \in X \times \mathbb{C}$ with $||z| - R| < k^{-1/3}$, we have

$$\begin{aligned} |s_{k,\mathrm{out}}| + |\nabla^E s_{k,\mathrm{out}}| &= |s_{k,X}| \cdot |s_{k,R}| + |\nabla^{E_X} s_{k,X}| \cdot |s_{k,R}| + |s_{k,X}| \cdot |\nabla^{E_{\mathbb{C}}} s_{k,R}| \\ &> (|s_{k,X}| + |\nabla^{E_X} s_{k,X}|)|s_{k,R}| > \tfrac{1}{3}\eta. \end{aligned}$$

So, $s_{k,\mathrm{out}}$ is $\frac{1}{3}\eta$–transverse in a neighborhood of $|z| = R$.

This means $s_{k,\mathrm{out}}+s_{k,K}$ is $\frac{1}{3}\eta$ transverse in a neighborhood of the boundary of $\partial(X\setminus(K\cup\pi_{\mathbb{C}}^{-1}(\mathbb{C}\setminus V))$, which means that it is transversely extendible over the complement of $K\cup\pi_{\mathbb{C}}^{-1}(\mathbb{C}\setminus V)$. Since the zero locus of $s_{k,\mathrm{out}}+s_{k,K}$ restricted to $K\cup\pi_{\mathbb{C}}^{-1}(\mathbb{C}\setminus V)$ is symplectic for $k\gg 0$, by Corollary C.2.4 we obtain a section $\tilde{s}_k\colon X\times\mathbb{C}\to E^k$ whose zero sets $s_k^{-1}(0)$ are symplectic for $k\gg 0$ and agree with the zero set of $s_{k,\mathrm{out}}+s_{k,K}$ over $K\cup\pi_{\mathbb{C}}^{-1}(\mathbb{C}\setminus V)$. That portion is

$$s_k^{-1}(0)|_{\pi_{\mathbb{C}}^{-1}(\mathbb{C}\setminus V)}=s_{k,\mathrm{out}}^{-1}(0)|_{\pi_{\mathbb{C}}^{-1}(\mathbb{C}\setminus V)}=s_{k,X}^{-1}(0)\times(\mathbb{C}\setminus V).$$

C.3.2 Full case: construction of Donaldson divisors for Lagrangian cobordisms We first note that there is a flat trivialization of $\tau_{k,\mathbb{R}}\colon\mathbb{R}\to E_{\mathbb{C}}^k$ for the Lagrangian $\mathbb{R}\subset\mathbb{C}$. Pick an almost complex structure $J_X\times\iota$ for $X\times\mathbb{C}$. As in the construction from [5], we take trivializations $\tau_{k,K}\colon K\to E^k$; we ask that they satisfy the following additional properties:

- The sections satisfy the bounds from [5, Lemma 2].
- There exist choices of trivializations $\tau_k^{\pm}\colon L^{\pm}\to E_X^k$ such that τ_k splits as $\tau_k^{\pm}\times\tau_{\mathbb{R}}$ when restricted to the ends of the cobordism, $L^+\times[t^+,\infty)$ and $L^-\times(-\infty,t^-]$.
- Additionally, the trivializations τ_+ and τ_- chosen above satisfy $\tau_+(x)=\tau_-(x)$ at each $x\in L^+\cap L^-$.

The last condition can be achieved with the argument of [10, Lemma 3.11(b)].

The divisor we construct will be the zero set of a transverse θ–approximately holomorphic section $s_k\colon X\times\mathbb{C}\to E^k$ which is nonvanishing on K. The restriction $s_k|_K\colon K\to E^k$ provides a trivialization homotopic to the sections $\tau_{k,K}$. Provided that these have bounded connection forms (following the discussion of (18)), the resulting divisors we construct will become weakly stabilizing when $k\gg 0$; see [9, Lemma 3.9].

Using the construction of [5] there exist constants $C,\eta>0$ and asymptotically J_X–holomorphic η–transverse sections $s_{k,X}\colon X\to E^k$ which have the property that $s_{k,X}(x)>C$ for all $x\in L^+\cup L^-$; the section arises as a perturbation of a section concentrated on $L^+\cup L^-$. For any given $\theta>0$, we can choose the perturbation small enough that for large k and any $x\in L^+\cup L^-$ the argument of the section $s_{k,X}$ is within θ of the argument of $\tau_k^{\pm}$ [5, page 746], ie

$$\sup_{x\in L^{\pm}}\left|\arg\left(\frac{s_{k,X}(x)}{\tau_k^{\pm}(x)}\right)\right|<\theta. \tag{19}$$

Take $\theta<\frac{1}{8}\pi$ in the above construction, and define the section $s_{k,X\times\mathbb{C},\mathrm{out}}:=s_{k,X}\boxtimes s_{k,R}\colon X\times\mathbb{C}\to E^k$. The section is asymptotically $(J_X\times\iota)$–holomorphic, transverse to the zero section outside a noncompact set and nonvanishing over the ends of the Lagrangian cobordism. Additionally, $s_{k,X\times\mathbb{C},\mathrm{out}}$ has the property that, for sufficiently large k, the argument of this section approximates (up to θ) the argument of $\tau_{k,K}$ over $X\times(\mathbb{C}\setminus V)$.

It remains to modify this section so that it is nonvanishing over the entirety of K and add further perturbations to obtain $\tilde{\eta}$–transversality over $\pi_{\mathbb{C}}^{-1}(V)$. Consider locally concentrated perturbations whose arguments are approximately determined by τ_k:

$$s_{k,p,K}(x,z) := \frac{\tau_k(p)}{|s_{k,p}(p)|} s_{k,p}(x,z).$$

For each k, consider a subset of points $P(k) \subset K$ whose radius $1/\sqrt{k}$ balls cover $K \cap \pi_{\mathbb{C}}^{-1}(V)$ and are distance at least $1/(3\sqrt{k})$ from each other. By [5], we have that

$$s_{k,K,\mathrm{int}}(x) := \tfrac{1}{6}\eta \sum_{p \in P(k)} s_{k,p,K}(x)$$

is concentrated at $K \setminus \pi^{-1}(U)$. Furthermore, for any θ and k sufficiently large,

$$\sup_{x \in K \setminus \pi_{\mathbb{C}}^{-1}(V)} \left| \arg\left(\frac{s_{k,K,\mathrm{int}}(x)}{\tau_k(x)} \right) \right| < \theta.$$

Therefore, for sufficiently large k, the sections $s_{k,K,\mathrm{int}}(x)$ and $s_{k,X\times\mathbb{C},\mathrm{out}}(x)$ have arguments agreeing up to error of $2\theta = \frac{1}{4}\pi$ over all $x \in K \cap \pi_{\mathbb{C}}^{-1}(V)$; therefore the sum $|s_{k,K,\mathrm{int}} + s_{k,X\times\mathbb{C},\mathrm{out}}|$ is at least $\frac{1}{6}\eta$ over $K \cap \pi^{-1}(V)$.

In summary, we have constructed a section $s_k := s_{k,K,\mathrm{int}} + s_{k,X\times\mathbb{C},\mathrm{out}}$ which is η–transverse over a neighborhood of $\partial(X \times V \setminus K)$, and $s_k^{-1}(0)$ is a symplectic divisor over a neighborhood of $K \cup \pi_{\mathbb{C}}^{-1}(\mathbb{C} \setminus V)$ for $k \gg 0$. From here we apply Corollary C.2.4 to construct asymptotically holomorphic sections $\tilde{s}_k \colon X \times \mathbb{C} \to E^k$ whose zero set is symplectic for $k \gg 0$; as in the toy case,

$$\tilde{s}_k^{-1}(0)|_{X\times(\mathbb{C}\setminus V)} = s_k^{-1}(0)|_{X\times(\mathbb{C}\setminus V)} = s_{k,X}^{-1}(0) \times (\mathbb{C} \setminus V).$$

It follows from [9, Lemma 3.9] and [12, Lemma 8.11] that for k sufficiently large, $\tilde{s}_k^{-1}(0)$ is a divisor of sufficiently large degree for K. This completes the proof of Lemma C.1.2.

C.3.3 Weak stability We now show that for k sufficiently large, the Donaldson divisors D constructed as the zero sets of $\tilde{s}_k \colon X \times \mathbb{C} \to E^k$ described in Section C.3.2 are weakly stabilizing. The argument follows [9, Section 4.5].

Given $J \in \mathcal{J}_{\tau,V}(X \times \mathbb{C}, \omega + \omega_{\mathbb{C}})$, we use Section C.3.2 to construct $D \subset X \times \mathbb{C}$ which splits in the complement of $\pi_{\mathbb{C}}^{-1}(V)$, is θ–approximately holomorphic and has sufficiently large degree k. This restricts to fibers $X \times \{z\}$ for $z \notin K$ to give us divisors D_X which are θ–approximately holomorphic with respect to the almost complex structure J_X.

Denote by

$$\mathcal{J}_\tau(X, D_X, J_X, \theta) := \{J_{X,D_X} \in \mathcal{J}_\tau(X,\omega) \mid \|J_X - J_{X,D_X}\| < \theta \text{ and } J_{X,D_X}(TD_X) = TD_X\},$$
$$\mathcal{J}_{\tau,V}(X \times \mathbb{C}, D, J, \theta) := \{J_D \in \mathcal{J}_{\tau,V}(X \times \mathbb{C}, \omega + \omega_{\mathbb{C}}) \mid \|J - J_D\| < \theta \text{ and } J_D(TD) = TD\},$$

the almost complex structures which are θ–close and adapted to the divisor D. There is a restriction map $\mathrm{res} \colon \mathcal{J}_{\tau,V}(X \times \mathbb{C}, D, J, \theta) \to \mathcal{J}_\tau(X, D_X, J_X, \theta)$. The argument of [9, Lemma 3.9] carries over here, so for

$\theta > 0$ there exists $d_0(\theta)$ such that if D has degree $d_0(\theta)$ then D is sufficiently large [9, Definition 4.24] for all almost complex structures which are θ–close to J.

Proof of Lemma C.1.3 As in [9, Lemma 4.9] and [10, Lemma 4.25], the proof is similar to that of [12, Proposition 8.14]. The argument in [9, Lemma 4.9] first shows that $\mathcal{J}^*_V(X\times\mathbb{C}, D, J, \theta, E)$ is open.

Consider a convergent sequence J^i in the complement of $\mathcal{J}_{\tau,V}(X\times\mathbb{C}, D, J, \theta, E)$ where there are J^i–holomorphic spheres u^i which are contained in D. Then at least one of the following two holds:

- There is an infinite subsequence of J^i–holomorphic u^i with $\mathrm{Im}(\pi_{\mathbb{C}}\circ u^i)\subset V$. Since the images of these spheres are contained in a compact set, we may apply Gromov compactness to show that a limiting subsequence converges to u with $\mathrm{Im}(u)\subset D$. Therefore, the limit of the J^i is in the complement of $\mathcal{J}^*_V(X\times\mathbb{C}, D, J, \theta, E)$.
- There is an infinite subsequence of J^i–holomorphic u^i with $\mathrm{Im}(\pi_{\mathbb{C}}\circ u^i)\not\subset V$. By the open mapping principle, the $\pi_{\mathbb{C}}\circ u^i$ are constant, and so we obtain a subsequence of J^i_X–holomorphic maps with image $\pi_X\circ u^i$ in D_X. The same Gromov compactness argument shows that the limit of the J^i_X is in the complement of $\mathcal{J}^*(X, D_X, J_X, \theta, E)$. Therefore the limit of the J^i is in the complement of $\mathcal{J}^*_V(X\times\mathbb{C}, D, J, \theta, E)$.

Next, we show that $\mathcal{J}^*_V(X\times\mathbb{C}, D, J, \theta, E)$ is dense. This is done by showing that $\mathcal{J}^*_V(X\times\mathbb{C}, D, J, \theta, E)$ contains $\mathcal{J}^{\mathrm{reg}}_V(X\times\mathbb{C}, J, \theta, E)$, the set of almost complex structures such that all simple holomorphic curves up to energy E are regular. The argument that these regularizing complex structures are E–stabilized is identical to the proof given in [9, Lemma 4.9], which follows the ideas of [12, Proposition 8.11]. The portion which differs between our setting and the one considered in [9] is proving that the regularizing almost complex structures $\mathcal{J}^{\mathrm{reg}}_V(X\times\mathbb{C}, J, \theta, E)$ are comeager in $J_{\tau,V}(X\times\mathbb{C}, D, J, \theta)$.

The standard proof, which we use here, is to show that the universal Cauchy–Riemann operator

$$\bar\partial^{X\times\mathbb{C}}\colon \mathcal{B}(K)\times J_{\tau,V}(X\times\mathbb{C}, D, J, \theta)\to\mathcal{E}_{X\times\mathbb{C}}\quad\text{given by }\ (u, J)\mapsto\bar\partial_J^{X\times\mathbb{C}}(u)$$

is transverse to 0. Here $\mathcal{B}(K)$ is the Banach manifold of disks with boundary on K, and $\mathcal{E}_{X\times\mathbb{C}}$ is the Banach bundle whose fiber at $u\in\mathcal{B}$ is sections of $\Omega^{0,1}(u^*T(X\times\mathbb{C}))$. Given a pair (u, J), we break into two cases: if $\mathrm{Im}(u)$ is disjoint from $\pi_{\mathbb{C}}^{-1}(V)$, or if $u\subset\pi_{\mathbb{C}}^{-1}(V)$.

(i) $\mathbf{Im}(\pi\circ u)\cap V=\varnothing$ The open mapping principle implies that $\bar\partial_J^{X\times\mathbb{C}}(u)=0$ only when $\pi_{\mathbb{C}}(u)$ is constant. From this setting, we apply [10, Lemma 4.25] to show that $\bar\partial^X\colon\mathcal{B}(L^\pm)\times\mathcal{J}_\tau(X, D_X, J_X, \theta)\to\mathcal{E}_X$ is a submersion. It follows that the map $\bar\partial^{X\times\mathbb{C}}$ is a submersion as well.

(ii) $\mathbf{Im}(\pi\circ u)\cap V\neq\varnothing$ We fall into the setting described by [12, Definition 5.5]. Then [12, Lemma 5.6] states[5] that the universal Cauchy–Riemann operator is transverse to 0 when restricted to variations of J fixed over the complement of $U=\pi^{-1}(V)$ as long as the image of u is not contained within U.

[5] In [12] there is different notation: their set V is our set U.

Thus, the universal Cauchy–Riemann operator is transverse to 0. By Sard–Smale the set of regular J, $\mathcal{J}_V^{\text{reg}}(X \times \mathbb{C}, J, \theta, E)$, is comeager. □

C.4 Compactness

The above constructions allow us to define regular moduli spaces of $\mathcal{P}$–pseudoholomorphic treed disks with boundary on a Lagrangian cobordism K. Let X be a rational symplectic manifold, $K \subset X \times \mathbb{C}$ a Lagrangian cobordism and $D \subset X \times \mathbb{C}$ chosen so that Lemma C.1.3 holds. We say a that perturbation datum is *stabilizing and cobordism admissible* if it is chosen from the neighborhoods $\mathcal{J}_V^*(X \times \mathbb{C}, D, J, \theta, E)$. The perturbation system is called *admissible* if it is coherent, regular, stabilizing and cobordism admissible. We now need the cobordism analogue of [9, Theorem 4.27].

Proposition C.4.1 *Let $\mathcal{P}_{\mathbb{I}}$ be an admissible perturbation system. For any $\underline{\Gamma}$ of $\operatorname{ind}(\underline{\Gamma}) \leq 1$, the moduli space $\mathcal{M}_{\mathcal{P}_{\mathbb{I}}}(X \times \mathbb{C}, K, D, \underline{\Gamma})$ is compact, and its boundary components are given by Morse flow line breaking in K.*

Proof The only modification needed from [9, Theorem 4.27] is to address the use of Gromov compactness in the setting of $X \times \mathbb{C}$. We give a brief recap of the argument used by [6]. Consider a domain-dependent pseudoholomorphic map $u \colon \Sigma \to X$. Since we choose domain-dependent almost complex structures from $\mathcal{J}_{\tau,V}(X \times \mathbb{C}, \omega + \omega_{\mathbb{C}})$, the map $\pi_{\mathbb{C}} \circ u$ is holomorphic for points whose image lies outside of V. As a result, we may apply the maximum principle to show that holomorphic disks with boundary on a Lagrangian cobordism K either

- have image contained in $X \times V$, or
- live in a fiber of the projection so that $\pi_{\mathbb{C}}(u)$ is constant.

By choosing a Morse function whose gradient flow points outwards along the ends of the Lagrangian cobordism, we can show that the image of a holomorphic treed disk is contained within $X \times V$ (see Lemma C.5.3 for a full argument). We can therefore apply Gromov compactness. □

By [10, Theorem 4.19] there exists an admissible perturbation datum.

Corollary C.4.2 *Let X be a compact rational symplectic manifold. Let $K \colon L^+ \rightsquigarrow L^-$ be a spin and graded Lagrangian cobordism and $h \colon K \to \mathbb{R}$ an admissible Morse function. There exists a comeager set of perturbation data such that $\operatorname{CF}^\bullet(K, h, \mathcal{P})$ defines a filtered A_∞ algebra.*

C.5 Pearly model for Lagrangian cobordisms

The main result of this section is the matching of moduli spaces of pseudoholomorphic disks for L^+ with those of K. Given a prebroken type $\underline{\Gamma}_P$ for $L^+ \subset X$, denote by $(i_*^+)\underline{\Gamma}_P$ the type for $K \subset X$ whose underlying tree has combinatorial type Γ and whose prebreakings and labels are determined by the identification $i_*^+ \operatorname{Crit}(h^+) \subset \operatorname{Crit}(h)$. Denote by $\mathbb{I}_+^+ := i_*^+ \mathbb{T}_{L^+}$ the set of prebroken combinatorial types of (K,h) whose labels come from the pushforward of labels on (L,h^+). Let $\mathbb{I}_+$ denote the trees which have an outgoing edge labeled by a critical point in $i^+_* \operatorname{Crit}(h^+) \subset \operatorname{Crit}(h)$.

Theorem C.5.1 (compatibility of pearly model of cobordisms) *Let $K\colon L^- \rightsquigarrow L^+$ be a Lagrangian cobordism, and $h\colon K \to \mathbb{R}$ be a compatible Morse function. There exist admissible perturbations systems $\mathcal{P}$ for K and $(i^+)^*\mathcal{P}$ for L^+ such that, for any labeled combinatorial type $\underline{\Gamma}_P \in \mathbb{I}_+$,*

$$\mathcal{M}_{\mathcal{P}}(X\times\mathbb{C},K,D,\underline{\Gamma}_P) = \begin{cases} \mathcal{M}_{(i^+)^*\mathcal{P}}(X,L^+,D_X,\underline{\Gamma}'_P) & \text{if } \mathbb{I}^+_+ \ni \underline{\Gamma}_P = i^+_*\underline{\Gamma}'_P, \\ \varnothing & \text{if } \underline{\Gamma}_P \in \mathbb{I}_+ \setminus \mathbb{I}^+_+. \end{cases}$$

Proof The admissibility of the pullback perturbation $(i^+)^*\mathcal{P}$ is proven in Corollary C.5.4. The characterization of the moduli spaces is (20) and Claim C.5.5. □

An algebraic corollary of this statement is the pearly model equivalent of Proposition 2.1.4:

Corollary C.5.2 *Let $K\colon L^+ \rightsquigarrow L^-$ be a Lagrangian cobordism. Then the projections*

$$\begin{array}{ccccc} & & CF^\bullet(K,h) & & \\ & \swarrow^{\beta^-} & & \searrow^{\beta^+} & \\ CF^\bullet(L^-,h^-) & & & & CF^\bullet(L^+,h^+) \end{array}$$

are filtered A_∞ homomorphisms.

Proof The conditions on the moduli spaces imposed by Theorem C.5.1 show, whenever $\{x_i\}$ is a sequence of Floer cochains with at least one x_i corresponding to a critical point in $\operatorname{Crit}(h) \setminus i^+_* \operatorname{Crit}(h^+)$, that

$$m^k(x_1 \otimes \cdots \otimes x_k) \in \Lambda\langle \operatorname{Crit}(h) \setminus i^+_* \operatorname{Crit}(h^+)\rangle \subset CF^\bullet(K,h).$$

This proves that $\Lambda\langle \operatorname{Crit}(h) \setminus i^+_* \operatorname{Crit}(h^+)\rangle$ is an A_∞ ideal and that $CF^\bullet(K,h)/\langle \operatorname{Crit}(h) \setminus i^+_* \operatorname{Crit}(h^+)\rangle$ is an A_∞ homomorphism. Also, by Theorem C.5.1, the A_∞ product on $CF^\bullet(K,h)/\langle \operatorname{Crit}(h) \setminus i^+_* \operatorname{Crit}(h^+)\rangle$ matches the A_∞ structure on $CF^\bullet(L^+,h^+)$.

The same argument holds on the negative end as well. □

C.5.1 Proof of compatibility of moduli spaces

Lemma C.5.3 *Let $\mathcal{P}$ be an admissible perturbation datum for K. Consider a combinatorial type $\underline{\Gamma} \in \mathbb{I}^+_+$ and $u \in \mathcal{M}_{\mathcal{P}_{\mathbb{I}^+_+}}(X\times\mathbb{C},K,D,\underline{\Gamma})$. We claim that $\pi_{\mathbb{C}}(u) = t^+$.*

Proof Let $\gamma_e\colon (s^-_e, s^+_e) \to K$ be the Morse flow lines of u, and let $u_v\colon (D^2_v, \partial D^2_v) \to (X\times\mathbb{C}, K)$ be the $\mathcal{P}$–regular holomorphic disks of u, where v and e denote vertices and edges of $\underline{\Gamma}$, respectively. We make two observations:

- If $\pi_{\mathbb{C}} \circ \gamma(s_+) = t^+$, then $\pi_{\mathbb{C}} \circ \gamma(s_e^-) = t^+$ because $\operatorname{grad} h^+$ points away from t^+.
- As the composition $u \circ \pi_{\mathbb{C}} \colon D^2 \to X \times \mathbb{C} \to \mathbb{C}$ is holomorphic outside of V, if there exists a point $z_0 \in D^2$ such that $\pi_{\mathbb{C}} \circ u_v(z_0) = t^+$, then $\pi_{\mathbb{C}} \circ u_v(z) = t^+$ by the maximum principle.

At the root of the tree $\pi_{\mathbb{C}}(\gamma_{e_0}(s_e^+)) = \pi_{\mathbb{C}}(x_0) = t^+$. Recursively working upwards through the tree from the root and applying the observations shows that $\pi_{\mathbb{C}}(u) = t^+$. □

From this, we can draw two conclusions:

Firstly, given $\underline{\Gamma}_P$ for $L^+ \subset X$ and admissible perturbation $\mathcal{P}(i_*^+ \underline{\Gamma}_P)$ for the corresponding type on K, we obtain a domain-dependent perturbation of almost complex structure $(i^+)^*\mathcal{P}(\underline{\Gamma}_P)$ by using the restriction map $\operatorname{res} \colon \mathcal{J}_{\tau,V}(X \times \mathbb{C}, D, J, \theta) \to \mathcal{J}_\tau(X, D_X, J_X, \theta)$. Every $\mathcal{P}$–pseudoholomorphic disk u of type $i_*^+ \underline{\Gamma}_P$ lies completely inside $X \times \{t^+\}$, and (by compatibility of our two perturbation systems) gives rise to an $(i^+)^*\mathcal{P}(\underline{\Gamma}_P)$–pseudoholomorphic disk $\pi_X(u)$ with boundary on $L^+ \subset X$. Therefore (as a set)

$$\mathcal{M}_{\mathcal{P}}(X \times \mathbb{C}, K, D, i_*^+ \underline{\Gamma}_P)) \subset \mathcal{M}_{(i^+)^*\mathcal{P}}(X, L, D_X, \underline{\Gamma}_P).$$

Because the gradient flow of an admissible Morse function points outwards at t^+, and $\mathcal{P}$ is regular and split in a neighborhood of $X \times \{t^+\}$, we have that $\pi_X(u)$ is a regular disk for $(i^+)^*\mathcal{P}$.

Similarly, the lift of any $u \in \mathcal{M}_{(i^+)^*\mathcal{P}}(X, L, D_X, \underline{\Gamma}_P)$ to $u \times \{t^+\}$ is a solution for the $\mathcal{P}$–perturbed $\bar{\partial}$ equation. This shows that

$$\mathcal{M}_{\mathcal{P}}(X \times \mathbb{C}, K, D, i_*^+ \underline{\Gamma}_P) = \mathcal{M}_{(i^+)^*\mathcal{P}}(X, L, D_X, \underline{\Gamma}_P). \tag{20}$$

Since $\mathcal{P}$ is regular, $u \times \{t^+\}$ is a regularly cut out treed disk, so $\pi_X(u \times \{t^+\}) = u$ is regular. We conclude that $(i^+)^*\mathcal{P}$ is a regular perturbation system.

Corollary C.5.4 *Let $\mathcal{P}_{\mathbb{T}_K}$ be an admissible perturbation system for K. Then $(i^+)^*\mathcal{P}_{\mathbb{T}_L}$ is an admissible perturbation system for L^+.*

Secondly, let $\mathbb{I}_+ \setminus \mathbb{I}_+^+$ denote the set of labeled trees with outgoing label in $i_*^+ \operatorname{Crit}(h^+)$, but at least one incoming label in $\operatorname{Crit}(h) \setminus i_*^+ \operatorname{Crit}(h^+)$.

Claim C.5.5 *If $\underline{\Gamma}_P \in \mathbb{I}_+ \setminus \mathbb{I}_+^+$, then $\mathcal{M}(X \times \mathbb{C}, K, D, h, \underline{\Gamma}_P) = \varnothing$.*

Proof For all $\underline{\Gamma}_P \in \mathbb{I}_+ \setminus \mathbb{I}_+^+$, $\underline{\Gamma}_P$ has an input label which does not belong to $i_*^+ \operatorname{Crit}(h^+)$. Suppose for contradiction that we have a pseudoholomorphic treed disk u of type $\underline{\Gamma}_P$. Since the output label of $\underline{\Gamma}_P$ does still belong to $i_*^+ \operatorname{Crit}(h^+)$, we may apply Lemma C.5.3 and conclude that $\pi_{\mathbb{C}}(u) = t^+$. However, this implies that all the labels of $\underline{\Gamma}_P$ belong to $i_*^+ \operatorname{Crit}(h^+)$, a contradiction. Therefore $\mathcal{M}_{\mathcal{P}}(K, X \times \mathbb{C}, D, \underline{\Gamma}_P) = \varnothing$. □

References

[1] **V I Arnold**, *Lagrange and Legendre cobordisms, I*, Funktsional. Anal. i Prilozhen. 14 (1980) 1–13 MR Zbl In Russian; translated in Funct. Anal. Appl. 14 (1980) 167–177

[2] **D Auroux**, *Asymptotically holomorphic families of symplectic submanifolds*, Geom. Funct. Anal. 7 (1997) 971–995 MR Zbl

[3] **D Auroux**, *Mirror symmetry and T–duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. 1 (2007) 51–91 MR Zbl

[4] **D Auroux**, *Special Lagrangian fibrations, wall-crossing, and mirror symmetry*, from "Geometry, analysis, and algebraic geometry", Surv. Differ. Geom. 13, International, Somerville, MA (2009) 1–47 MR Zbl

[5] **D Auroux**, **D Gayet**, **J-P Mohsen**, *Symplectic hypersurfaces in the complement of an isotropic submanifold*, Math. Ann. 321 (2001) 739–754 MR Zbl

[6] **P Biran**, **O Cornea**, *A Lagrangian quantum homology*, from "New perspectives and challenges in symplectic field theory" (M Abreu, F Lalonde, L Polterovich, editors), CRM Proc. Lect. Notes 49, Amer. Math. Soc., Providence, RI (2009) 1–44 MR Zbl

[7] **P Biran**, **O Cornea**, *Lagrangian cobordism, I*, J. Amer. Math. Soc. 26 (2013) 295–340 MR Zbl

[8] **P Biran**, **O Cornea**, *Lagrangian cobordism and Fukaya categories*, Geom. Funct. Anal. 24 (2014) 1731–1830 MR Zbl

[9] **F Charest**, **C Woodward**, *Floer trajectories and stabilizing divisors*, J. Fixed Point Theory Appl. 19 (2017) 1165–1236 MR Zbl

[10] **F Charest**, **C T Woodward**, *Floer cohomology and flips*, Mem. Amer. Math. Soc. 1372, Amer. Math. Soc., Providence, RI (2022) MR Zbl

[11] **C-H Cho**, *Holomorphic discs, spin structures, and Floer cohomology of the Clifford torus*, Int. Math. Res. Not. 2004 (2004) 1803–1843 MR Zbl

[12] **K Cieliebak**, **K Mohnke**, *Symplectic hypersurfaces and transversality in Gromov–Witten theory*, J. Symplectic Geom. 5 (2007) 281–356 MR Zbl

[13] **O Cornea**, **E Shelukhin**, *Lagrangian cobordism and metric invariants*, J. Differential Geom. 112 (2019) 1–45 MR Zbl

[14] **S K Donaldson**, *Symplectic submanifolds and almost-complex geometry*, J. Differential Geom. 44 (1996) 666–705 MR Zbl

[15] **K Fukaya**, *Cyclic symmetry and adic convergence in Lagrangian Floer theory*, Kyoto J. Math. 50 (2010) 521–590 MR Zbl

[16] **K Fukaya**, **Y-G Oh**, *Zero-loop open strings in the cotangent bundle and Morse homotopy*, Asian J. Math. 1 (1997) 96–180 MR Zbl

[17] **K Fukaya**, **Y-G Oh**, **H Ohta**, **K Ono**, *Lagrangian intersection Floer theory: anomaly and obstruction, I*, AMS/IP Stud. Adv. Math. 46.1, Amer. Math. Soc., Providence, RI (2009) MR Zbl

[18] **L Haug**, *Lagrangian antisurgery*, Math. Res. Lett. 27 (2020) 1423–1464 MR Zbl

[19] **J Hicks**, *Lagrangian cobordisms and Lagrangian surgery*, Comment. Math. Helv. 98 (2023) 509–595 MR Zbl

[20] **T V Kadeishvili**, *On the theory of homology of fiber spaces*, Uspekhi Mat. Nauk 35 (1980) 183–188 MR Zbl In Russian; translated in Russian Math. Surv. 35 (1980) 231–238

[21] **M Kontsevich**, *Homological algebra of mirror symmetry*, from "Proceedings of the International Congress of Mathematicians, I" (S D Chatterji, editor), Birkhäuser, Basel (1995) 120–139 MR Zbl

[22] **M Kontsevich**, **Y Soibelman**, *Homological mirror symmetry and torus fibrations*, from "Symplectic geometry and mirror symmetry" (K Fukaya, Y-G Oh, K Ono, G Tian, editors), World Sci., River Edge, NJ (2001) 203–263 MR Zbl

[23] **F Lalonde**, **J-C Sikorav**, *Sous-variétés lagrangiennes et lagrangiennes exactes des fibrés cotangents*, Comment. Math. Helv. 66 (1991) 18–33 MR Zbl

[24] **J Li**, **K Wehrheim**, *A_∞ structures from Morse trees with pseudoholomorphic disks*, preliminary draft (2014) https://math.berkeley.edu/~katrin/papers/disktrees.pdf

[25] **M Markl**, *Transferring A_∞ (strongly homotopy associative) structures*, Rend. Circ. Mat. Palermo Suppl. 79 (2006) 139–151 MR Zbl

[26] **B Obama**, *The President's weekly address*, 6 June 2009 https://www.presidency.ucsb.edu/node/286656

[27] **Y-G Oh**, *Riemann–Hilbert problem and application to the perturbation theory of analytic discs*, Kyungpook Math. J. 35 (1995) 39–75 MR Zbl

[28] **J Palmer**, **C Woodward**, *Invariance of immersed Floer cohomology under Lagrangian surgery*, preprint (2019) arXiv 1903.01943

[29] **J Pascaleff**, **D Tonkonog**, *The wall-crossing formula and Lagrangian mutations*, Adv. Math. 361 (2020) art. id. 106850 MR Zbl

[30] **L Polterovich**, *The surgery of Lagrange submanifolds*, Geom. Funct. Anal. 1 (1991) 198–210 MR Zbl

[31] **G D Rizell**, **T Ekholm**, **D Tonkonog**, *Refined disk potentials for immersed Lagrangian surfaces*, J. Differential Geom. 121 (2022) 459–539 MR Zbl

[32] **A Strominger**, **S-T Yau**, **E Zaslow**, *Mirror symmetry is T–duality*, Nuclear Phys. B 479 (1996) 243–259 MR Zbl

[33] **M Usher**, **J Zhang**, *Persistent homology and Floer–Novikov theory*, Geom. Topol. 20 (2016) 3333–3430 MR Zbl

School of Mathematics, University of Edinburgh
Edinburgh, United Kingdom

jeff.hicks@ed.ac.uk

https://www.maths.ed.ac.uk/~jhicks2/

Received: 23 March 2020 Revised: 1 April 2022

Algebraic & Geometric Topology

msp.org/agt

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US $705/year for the electronic version, and $1040/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

Algebraic & Geometric Topology

Volume 24 Issue 6 (pages 2971–3570) 2024