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Agustin Moreno

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# Algebraic and Giroux torsion in higher-dimensional contact manifolds

AGUSTIN MORENO

We construct examples in any odd dimension of contact manifolds with finite and nonzero algebraic torsion (in the sense of Latschev and Wendl (2011)), which are therefore tight and do not admit strong symplectic fillings. We prove that Giroux torsion implies algebraic 1–torsion in any odd dimension, which proves a conjecture of Massot, Niederkrüger and Wendl (2013). These results are part of the author's PhD thesis.

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## 1 Introduction

In this paper, and its followup [Moreno 2022], we address the general problem of constructing "interesting" examples of higher-dimensional contact manifolds, and developing techniques in order to compute SFT-type holomorphic curve invariants.

We will construct examples of contact manifolds in every odd dimension, presenting a geometric structure which is a higher-dimensional version of that of a *spinal open book decomposition* or SOBD, as defined in [Lisi et al. 2018] in dimension 3. The type of SOBD present in our examples, which one could call *partially planar*, mimics the notion of *planar m–torsion* domains as defined in [Wendl 2013]. We exhibit a detailed construction of an isotopy class of contact forms, which is "supported" by the SOBD, so that one may view these contact forms as "Giroux" forms. We will estimate the algebraic torsion of these examples, which we show is finite, and, in certain cases, nonzero. In those cases, the contact manifolds are tight and admit no strong symplectic fillings.

We will also relate algebraic torsion with a geometric condition, *Giroux torsion*. While this is a classical notion in dimension 3, the higher-dimensional version was introduced by Massot, Niederkrüger and Wendl [Massot et al. 2013]. We will show that the geometric presence of certain torsion domains inside a contact manifold can be detected algebraically by SFT. More concretely, Giroux torsion implies algebraic 1–torsion in any odd dimension. This proves a conjecture in [Massot et al. 2013].

The proof of this result is carried out by interpreting the Giroux torsion domains as being supported by a suitable SOBD, which we call a *Giroux SOBD*, for which we give a notion of a "Giroux form". The result follows by adapting our computations for the above partially planar model contact manifolds.

**On the invariant** The invariant we will use, algebraic torsion, was defined in [Latschev and Wendl 2011], and is a contact invariant taking values in $\mathbb{Z}^{\geq 0} \cup \{\infty\}$. It was introduced, using the machinery of *symplectic field theory*, as a quantitative way of measuring nonfillability.

The key fact about this invariant is that it behaves well under exact symplectic cobordisms, which implies that the concave end inherits any order of algebraic torsion that the convex end has. Thus, algebraic torsion may be also thought of as an obstruction to the existence of exact symplectic cobordisms. In particular, it serves as an obstruction to symplectic fillability. Moreover, there are connections to dynamics: any contact manifold with finite torsion satisfies the Weinstein conjecture (ie there exist closed Reeb orbits for every contact form).

One should mention that there are other notions of algebraic torsion in the literature which do not use SFT, but which are only 3–dimensional (see [Kutluhan et al. 2015] for the version using Heegaard Floer homology, or the appendix in [Latschev and Wendl 2011] by Hutchings, using ECH).

**Statement of results** For the SFT setup, we follow [Latschev and Wendl 2011], where we refer the reader for more details. We will take the SFT of a contact manifold $(M, \xi)$ (with coefficients) to be the homology $H_*^{\mathrm{SFT}}(M, \xi; \mathcal{R})$ of a $\mathbb{Z}_2$–graded unital $BV_\infty$–algebra $(\mathcal{A}[\![\hbar]\!], \boldsymbol{D}_{\mathrm{SFT}})$ over the group ring $R_{\mathcal{R}} := \mathbb{R}[H_2(M;\mathbb{R})/\mathcal{R}]$, for some linear subspace $\mathcal{R} \subseteq H_2(M;\mathbb{R})$. Here, $\mathcal{A} = \mathcal{A}(\lambda)$ has generators $q_\gamma$ for each good closed Reeb orbit $\gamma$ with respect to some nondegenerate contact form $\lambda$ for $\xi$, $\hbar$ is an even variable, and the operator

$$\boldsymbol{D}_{\mathrm{SFT}} \colon \mathcal{A}[\![\hbar]\!] \to \mathcal{A}[\![\hbar]\!]$$

is defined by counting rigid solutions to a suitable abstract perturbation of a $J$–holomorphic curve equation in the symplectization of $(M, \xi)$. It satisfies

- $\boldsymbol{D}_{\mathrm{SFT}}$ is odd and squares to zero,
- $\boldsymbol{D}_{\mathrm{SFT}}(1) = 0$, and
- $\boldsymbol{D}_{\mathrm{SFT}} = \sum_{k \geq 1} D_k \hbar^{k-1}$,

where $D_k \colon \mathcal{A} \to \mathcal{A}$ is a differential operator of order $\leq k$, given by

$$D_k = \sum_{\substack{\Gamma^+, \Gamma^-, g, d \\ |\Gamma^+| + g = k}} \frac{n_g(\Gamma^+, \Gamma^-, d)}{C(\Gamma^-, \Gamma^+)} q_{\gamma_1^-} \cdots q_{\gamma_{s^-}^-} z^d \frac{\partial}{\partial q_{\gamma_1^+}} \cdots \frac{\partial}{\partial q_{\gamma_{s^+}^+}}.$$

The sum ranges over all nonnegative integers $g \geq 0$, homology classes $d \in H_2(M;\mathbb{R})/\mathcal{R}$ and ordered (possibly empty) collections of good closed Reeb orbits $\Gamma^\pm = (\gamma_1^\pm, \dots, \gamma_{s^\pm}^\pm)$ such that $s^+ + g = k$. After a choice of spanning surfaces as in [Eliashberg et al. 2000, page 566; see also page 651], the projection to $M$ of each finite-energy holomorphic curve $u$ can be capped off to a 2–cycle in $M$, and so it gives rise to a homology class $[u] \in H_2(M)$, which we project to define $\overline{[u]} \in H_2(M;\mathbb{R})/\mathcal{R}$. The number $n_g(\Gamma^+, \Gamma^-, d) \in \mathbb{Q}$ denotes the count of (suitably perturbed) holomorphic curves of genus $g$ with positive asymptotics $\Gamma^+$ and negative asymptotics $\Gamma^-$ in the homology class $d$, including asymptotic markers as explained in [Eliashberg et al. 2000], or [Wendl 2016], and including rational weights arising from automorphisms. $C(\Gamma^-, \Gamma^+) \in \mathbb{N}$ is a combinatorial factor defined as $C(\Gamma^-, \Gamma^+) = s^-! s^+! \kappa_{\gamma_1^-} \cdots \kappa_{\gamma_{s^-}^-}$, where $\kappa_\gamma$ denotes the covering multiplicity of the Reeb orbit $\gamma$.

The most important special cases for our choice of linear subspace $\mathcal{R}$ are $\mathcal{R} = H_2(M; \mathbb{R})$ and $\mathcal{R} = \{0\}$, called the *untwisted* and *fully twisted* cases respectively, and $\mathcal{R} = \ker \Omega$ with $\Omega$ a closed 2–form on $M$. We shall abbreviate the latter case as $H_*^{\mathrm{SFT}}(M, \xi; \Omega) := H_*^{\mathrm{SFT}}(M, \xi; \ker \Omega)$, and the untwisted case simply by $H_*^{\mathrm{SFT}}(M, \xi) := H_*^{\mathrm{SFT}}(M, \xi; H_2(M; \mathbb{R}))$.

**Definition 1.1** Let $(M, \xi)$ be a closed manifold of dimension $2n+1$ with a positive, cooriented contact structure. For any integer $k \geq 0$, we say that $(M, \xi)$ has $\Omega$–twisted algebraic torsion of order $k$ (or $\Omega$–twisted $k$–torsion) if $[\hbar^k] = 0$ in $H_*^{\mathrm{SFT}}(M, \xi; \Omega)$. If this is true for all $\Omega$, or equivalently, if $[\hbar^k] = 0$ in $H_*^{\mathrm{SFT}}(M, \xi; \{0\})$, then we say that $(M, \xi)$ has fully twisted algebraic $k$–torsion.

We will refer to *untwisted* $k$–torsion to the case $\Omega = 0$, in which case $R_{\mathcal{R}} = \mathbb{R}$ and we do not keep track of homology classes. Whenever we refer to torsion without mention to coefficients we will mean the untwisted version. We will say that, if a contact manifold has algebraic 0–torsion for every choice of coefficient ring, then it is *algebraically overtwisted*, which is equivalent to the vanishing of the SFT, or its contact homology. By definition, $k$–torsion implies $(k+1)$–torsion, so we may define its *algebraic torsion* to be

$$\mathrm{AT}(M, \xi; \mathcal{R}) := \min\{k \geq 0 : [\hbar^k] = 0\} \in \mathbb{Z}^{\geq 0} \cup \{\infty\},$$

where we set $\min \varnothing = \infty$. We denote it by $\mathrm{AT}(M, \xi)$, in the untwisted case.

This construction is well behaved under symplectic cobordisms. Any exact symplectic cobordism $(X, \omega = d\alpha)$ with positive end $(M_+, \xi_+)$ and negative end $(M_-, \xi_-)$ gives rise to a natural $\mathbb{R}[\![\hbar]\!]$–module morphism on the untwisted SFT,

$$\Phi_X : H_*^{\mathrm{SFT}}(M_+, \xi_+) \to H_*^{\mathrm{SFT}}(M_-, \xi_-),$$

a *cobordism map*. This implies that if $(M_+, \xi_+)$ has $k$–torsion, then so does $(M_-, \xi_-)$. There is also a version with coefficients for the case of nonexact cobordisms and fillings [Latschev and Wendl 2011, Proposition 2.4].

Examples of 3–dimensional contact manifolds with any given order of torsion $k-1$, but not $k-2$, were constructed in [Latschev and Wendl 2011]. The underlying manifold is the product manifold $M_g := S^1 \times \Sigma$, for $\Sigma$ a surface of genus $g$ which is divided into two pieces $\Sigma_+$ and $\Sigma_-$ along some dividing set of simple closed curves $\Gamma$ of cardinality $k$, where the latter has genus 0, and the former has genus $g-k+1$. The contact structure $\xi_k$ is $S^1$–invariant and may be obtained, for instance, by a construction originally due to Lutz [1977]. Its isotopy class is characterized by the fact that every section $\{\mathrm{pt}\} \times \Sigma$ is a convex surface with dividing set $\Gamma$. The behavior of algebraic torsion under cobordisms then implies that there is no exact symplectic cobordisms having $(M_g, \xi_k)$ and $(M_{g'}, \xi_{k'})$ as convex and concave ends, respectively, if $k < k'$.

The existence of the analogue higher-dimensional contact manifolds was conjectured in [Latschev and Wendl 2011]. We will consider a modified version of their examples. The modification we do here consists in taking the $S^1$–factor and replacing it by a *closed* $(2n-1)$–manifold $Y$, having the special property that $Y \times I$ admits the structure of a Liouville domain (here, $I$ denotes the interval $[-1,1]$). This means that it comes with an exact symplectic form $d\alpha$, and has disconnected contact-type boundary $\partial(Y \times I, d\alpha) = (Y_-, \xi_- = \ker\alpha_-) \sqcup (Y_+, \xi_+ = \ker\alpha_+)$, where $Y_\pm$ coincide with $Y$ as manifolds, but $(Y_\pm, \xi_\pm)$ are *not necessarily contactomorphic* to each other. In fact, $Y_\pm$ have different orientations, and so they might not even be *homeomorphic* to each other (not every manifold admits an orientation-reversing homeomorphism). A Liouville domain of the form $(Y \times I, d\alpha)$ is what we will call a *cylindrical Liouville semifilling* (or simply a cylindrical semifilling). Their existence in every odd dimension was established in [Massot et al. 2013]. We immediately see that this generalizes the previous 3–dimensional example, since $S^1$ admits the *Liouville pair* $\alpha_\pm = \pm d\theta$, which means that the 1–form $e^{-s}\alpha_- + e^s\alpha_+$ is Liouville in $S^1 \times \mathbb{R}$. We prove that the manifold $Y \times \Sigma$ indeed achieves $(k-1)$–torsion (Theorem 1.2), for a suitable contact structure which we now describe.

First, for once and for good, we will fix the following notation:

**Notation** Throughout this paper, the symbol $I$ will be reserved for the interval $[-1,1]$.

We can adapt the construction of the contact structures in [Latschev and Wendl 2011] to our models. The starting idea is to decompose the manifold $M = M_g = Y \times \Sigma$ into three pieces

$$M_g = M_Y \cup M_P^\pm,$$

where $M_Y = \bigsqcup^k Y \times I \times S^1$, and $M_P^\pm = Y \times \Sigma_\pm$ (see Figure 1). We have natural fibrations

$$\pi_Y : M_Y \to Y \times I, \quad \pi_P^\pm : M_P^\pm \to Y_\pm,$$

with fibers $S^1$ and $\Sigma_\pm$, respectively, and they are compatible in the sense that

$$\partial((\pi_P^\pm)^{-1}(\mathrm{pt})) = \bigsqcup^k \pi_Y^{-1}(\mathrm{pt}).$$

While $\pi_Y$ has a Liouville domain as base, and a contact manifold as fiber, the situation is reversed for $\pi_P^\pm$, which has contact base, and Liouville fibers. This is a prototypical example of a *spinal open book decomposition*, or SOBD. While we will not give a general definition of such a notion, we refer the reader to [Moreno 2018] for a tentative one.

Using this decomposition, we can construct a contact structure $\xi_k$ which is a small perturbation of the stable Hamiltonian structure (SHS) $\xi_\pm \oplus T\Sigma_\pm$ along $M_P^\pm$, and is a *contactization* for the Liouville domain $(Y \times I, \epsilon\, d\alpha)$ along $M_Y$, for some small $\epsilon > 0$. This means that it coincides with $\ker(\epsilon\alpha + d\theta)$, where $\theta$ is the $S^1$–coordinate. We will do this in detail in Section 2.

For the contact manifolds $(M_g, \xi_k)$, we can bound their algebraic torsion. First, recall that a contact structure is hypertight if it admits a contact form without contractible Reeb orbits (which we call a

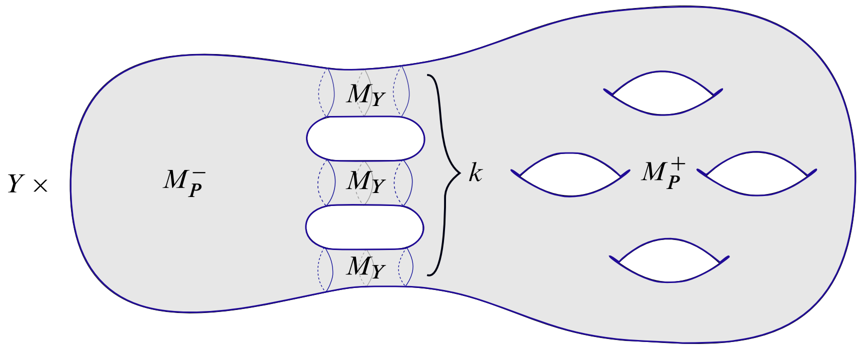


Figure 1: The SOBD structure in $M$.

hypertight contact form). In particular, there are no holomorphic disks in their symplectization, which implies that there is no 0–torsion. By a well-known theorem of Hofer and its generalization to higher dimensions by Albers and Hofer (in combination with [Borman et al. 2015]), hypertight contact manifolds are tight.

**Theorem 1.2** *For any $k \geq 1$, and $g \geq k$, the $(2n+1)$–dimensional contact manifolds $(M_g = Y \times \Sigma, \xi_k)$ satisfy $\mathrm{AT}(M_g, \xi_k) \leq k-1$. Moreover, if $(Y, \alpha_\pm)$ are hypertight, and $k \geq 2$, the corresponding contact manifold $(M_g, \xi_k)$ is also hypertight. In particular, $\mathrm{AT}(M_g, \xi_k) > 0$, and it is tight.*

In fact, the examples of Theorem 1.2 admit $\Omega$–twisted $(k-1)$–torsion, for $\Omega$ defining a cohomology class in $\mathbb{O} := \mathrm{Ann}\big(\bigoplus_k H_1(Y;\mathbb{R}) \otimes H_1(S^1;\mathbb{R})\big)$, the annihilator of $\bigoplus_k H_1(Y;\mathbb{R}) \otimes H_1(S^1;\mathbb{R}) \subseteq H_2(M_g;\mathbb{R})$. Here, we take the homology of the subregion $\bigsqcup^k Y \times \{0\} \times S^1$, lying along the region $M_Y$ where $\Sigma_\pm$ glue together. Using [Latschev and Wendl 2011, Proposition 2.4], we obtain:

**Corollary 1.3** *The examples of Theorem 1.2 do not admit weak fillings $(W, \omega)$ for which $[\omega|_{M_g}]$ is rational and lies in $\mathbb{O}$. In particular, they are not strongly fillable.*

**Remark 1.4**

- By a result of Mitsumatsu [1995], any 3–manifold $Y$ which admits a smooth Anosov flow preserving a smooth volume form satisfies that $Y \times I$ can be enriched with a cylindrical Liouville semifilling structure. Therefore any of these 3–manifolds can be used in the construction of 5–dimensional contact models with $\mathrm{AT} \leq k-1$, for any $k \geq 1$.
- The examples of Liouville cylindrical semifillings of [Massot et al. 2013] satisfy the hypertightness condition. Then we have a doubly infinite family of contact manifolds with $0 < \mathrm{AT}(M_g, \xi_k) \leq k-1$, in any dimension. These are then an instance of higher-dimensional tight but not strongly fillable contact manifolds, since they have nonzero and finite algebraic torsion. For $k=2$, this precisely computes the algebraic torsion.

Massot et al. [2013] defined a generalized higher-dimensional version of the notion of Giroux torsion. This notion is defined as follows. Consider $(Y, \alpha_+, \alpha_-)$ a *Liouville pair* on a closed manifold $Y^{2n-1}$, which means that the 1–form $\beta = \frac{1}{2}(e^s\alpha_+ + e^{-s}\alpha_-)$ is Liouville in $\mathbb{R}\times Y$. Consider also the *Giroux* $2\pi$*–torsion domain* modeled on $(Y,\alpha_+,\alpha_-)$ given by the contact manifold $(\mathrm{GT},\xi_{\mathrm{GT}}) := (Y\times[0,2\pi]\times S^1, \ker\lambda_{\mathrm{GT}})$, where

$$\lambda_{\mathrm{GT}} = \tfrac{1}{2}(1+\cos(r))\alpha_+ + \tfrac{1}{2}(1-\cos(r))\alpha_- + \sin(r)\,d\theta \tag{1}$$

and the coordinates are $(r,\theta)\in[0,2\pi]\times S^1$. Say that a contact manifold $(M^{2n+1},\xi)$ has *Giroux torsion* whenever it admits a contact embedding of $(\mathrm{GT},\xi_{\mathrm{GT}})$. In this situation, denote by $\mathbb{O}(\mathrm{GT})\subseteq H^2(M;\mathbb{R})$ the annihilator of $\mathcal{R}_{\mathrm{GT}} := H_1(Y;\mathbb{R})\otimes H_1(S^1;\mathbb{R})$, viewed as a subspace of $H_2(M;\mathbb{R})$. The following was conjectured in [Massot et al. 2013] (cf [Juhász and Kang 2018]):

**Theorem 1.5** *If a contact manifold $(M^{2n+1},\xi)$ has Giroux torsion, then it has $\Omega$–twisted algebraic 1–torsion, for every $[\Omega]\in\mathbb{O}(\mathrm{GT})$, where* GT *is a Giroux $2\pi$–torsion domain embedded in $M$.*

The proof uses the same techniques as Theorem 1.2, and the main idea is to interpret Giroux torsion domains in terms of a specially simple kind of SOBD, which we call *Giroux SOBD*.

A natural corollary is the following:

**Corollary 1.6** *If a contact manifold $(M^{2n+1},\xi)$ has Giroux torsion, then it does not admit weak fillings $(W,\omega)$ with $[\omega|_M]\in\mathbb{O}(\mathrm{GT})$ and rational, where* GT *is a Giroux $2\pi$–torsion domain embedded in $M$. In particular, it is not strongly fillable.*

This is essentially [Massot et al. 2013, Corollary 8.2], which was obtained with different methods. Observe that if $\mathcal{R}_{\mathrm{GT}} = 0$ then $(M,\xi)$ does not admit weak fillings at all. This is in fact the condition used in [Massot et al. 2013] to obstruct weak fillability.

**Disclaimer 1.7** Since the statements of our results make use of machinery from symplectic field theory, they come with the standard disclaimer that they assume that its analytic foundations are in place. They depend on the abstract perturbation scheme promised by the polyfold theory of Hofer, Wysocki and Zehnder. We shall assume that it is possible to achieve transversality by introducing an arbitrarily small abstract perturbation to the Cauchy–Riemann equation, and that the analogue of the SFT compactness theorem still holds as the perturbation is turned off. However, we have taken special care in that the approach taken not only provides results that will be fully rigorous after the polyfold machinery is complete, but also gives several direct results that are *already* rigorous.

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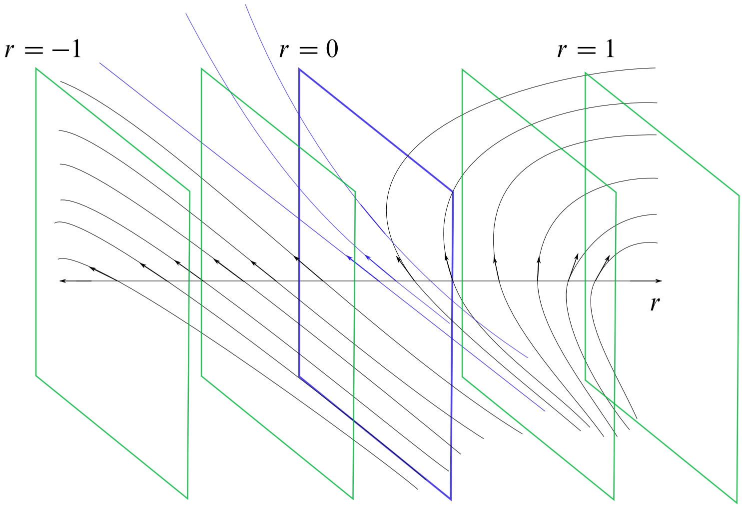


Figure 2: The qualitative behavior of the flow of the Liouville vector field $V$ on any cylindrical Liouville semifilling, for which the central slice ($r=0$) is invariant. One may informally think of such a Liouville domain as being obtained by gluing two negative symplectizations along a “noncontact hypersurface”.

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### Guide to the document

The main construction is dealt with in Section 2. We show uniqueness (Theorem 2.10) in Section 2.6. Theorem 1.2 is proved in Section 2.8.

The proof of Theorem 1.5 is dealt with in Section 3, which is basically a reformulation of the previous sections, with the key input being an adaptation of the uniqueness Theorem 2.10.

# 2 Algebraic torsion computations

## 2.1 Construction of the model contact manifolds

In this section, we construct the contact manifolds $M$ of Theorem 1.2, making use of the “double completion” construction, originally appearing in [Lisi et al. 2018].

Let $Y$ be a closed $(2n-1)$–manifold such that $(Y\times I, d\alpha)$ is a Liouville domain, for some exact symplectic form $d\alpha \in \Omega^2(Y\times I)$. See Figure 2. We assume that $\alpha=\{\alpha_r\}_{r\in I}$ is a 1–parameter family of 1–forms in $Y$. We write the symplectic form as

$$d\alpha = d\alpha_r + dr \wedge \frac{\partial \alpha_r}{\partial r}.$$

The Liouville vector field $V$, defined to be $d\alpha$–dual to $\alpha$, points outwards at each boundary component, and hence, using its flow, we can choose our coordinate $r \in I$ so that $V$ agrees with $\pm\partial_r$ near the boundary $\partial(Y \times I) = Y \times \{\pm 1\} =: Y_\pm$. Therefore, $\alpha = e^{\pm r-1}\alpha_\pm$ on $Y \times [-1, -1+\delta)$ and $Y \times (1-\delta, 1]$, respectively, for some small $\delta > 0$. Then $Y_\pm$ carries a contact structure $\xi_\pm = \ker \alpha_\pm$, where $\alpha_\pm = i_V\, d\alpha|_{T(Y_\pm)} = \alpha|_{T(Y_\pm)}$. Let $Y_r := Y \times \{r\}$, $\xi_r = \ker \alpha_r$, and we assume throughout that $\alpha_r$ is a contact form for every $r \neq 0$. Let $R_r$ be the associated Reeb vector field, which satisfies $R_r = e^{1 \mp r} R_\pm$ in the respective components of $\{|r| > 1-\delta\}$, where $R_\pm$ is the Reeb vector field of $\alpha_\pm = \alpha_{\pm 1}$. We shall make the convention that whenever we deal with equations involving $\pm$'s and $\mp$'s, one has to interpret them as to having a different sign according to the region (the "upper" sign denotes the "plus" region, and the "lower", the "minus" region).

Let $M^{2n+1} = Y \times \Sigma$, where $\Sigma$ is the orientable genus $g$ surface obtained by gluing a connected genus 0 surface with $k$ boundary components $\Sigma_-$, to a connected genus $g-k+1 > 0$ surface with $k$ boundary components $\Sigma_+$ along the boundary, by an orientation-preserving map. The surface $\Sigma$ then inherits the orientation of $\Sigma_-$, which is opposite to the one in $\Sigma_+$. On each boundary component of $\partial\Sigma_\pm$, choose collar neighborhoods $\mathcal{N}(\partial\Sigma_\pm) = (-\delta, 0] \times S^1$ (for the same $\delta$ as before), and coordinates $(t_\pm, \theta_\pm) \in \mathcal{N}(\partial\Sigma_\pm)$ such that $\partial\Sigma_\pm = \{t_\pm = 0\}$.

We will consider $\Sigma_-$ and $\Sigma_+$ to be attached at each of the $k$ boundary components by a cylinder $I \times S^1$, so that $M$ at this region is the disjoint union of $k$ copies of $Y \times I \times S^1$, with the $Y \times I$ identified with the Liouville domain above. We write the points of $M$ here as $(y, r, \theta)$, where the $\theta \in S^1$ coordinate can be chosen to coincide with $\theta_\pm$ where the gluing takes place. We shall therefore drop the subscript $\pm$ when talking about the $\theta$ coordinate. Denote also

$$\mathcal{N}(-Y) := Y \times [-1, -1+\delta) \times S^1, \quad \mathcal{N}(Y) := Y \times (1-\delta, 1] \times S^1,$$

in the above identification.

We have

$$M = M_Y \cup M_P^\pm,$$

where $M_Y = \bigsqcup^k Y \times I \times S^1$ is a region gluing $M_P^\pm = Y_\pm \times \Sigma_\pm$ together (recall Figure 1). We shall refer to them as the *spine* or *cylindrical region*, and the *positive/negative paper*, respectively. We have fibrations

$$\pi_Y \colon M_Y \to Y \times I, \quad \pi_P^\pm \colon M_P^\pm \to Y_\pm,$$

with fibers $S^1$ and $\Sigma_\pm$, respectively, and hence can be given the structure of an SOBD (see [Moreno 2018] for a definition).

We now construct an open manifold containing $M$ as a contact-type hypersurface. Denote by $\Sigma_\pm^\infty$ the open manifolds obtained from $\Sigma_\pm$ by attaching cylindrical ends of the form $(-\delta, +\infty) \times S^1$ at each boundary component, where the subset $(-\delta, 0] \times S^1$ coincides with the collar neighborhoods chosen above. The coordinates $t_\pm$ and $\theta$ extend to these ends in the obvious way, and we shall refer to the cylindrical ends as $\mathcal{N}(\partial\Sigma_\pm^\infty)$. We also consider the cylinder $\mathbb{R}\times S^1$ obtained by enlarging the cylindrical region $I\times S^1$ we had above. Denote then

$$M_P^{\pm,\infty}=Y\times\Sigma_\pm^\infty,\qquad \mathcal{N}^\infty(-Y)=Y\times(-\infty,-1+\delta)\times S^1,$$
$$M_Y^\infty=Y\times\mathbb{R}\times S^1,\qquad \mathcal{N}^\infty(Y)=Y\times(1-\delta,+\infty)\times S^1,$$

and define the *double completion* of $E$ to be

$$E^{\infty,\infty}=((-\infty,-1+\delta)\times M_P^{-,\infty})\sqcup((1-\delta,+\infty)\times M_P^{+,\infty})\sqcup((-\delta,+\infty)\times M_Y^\infty/\sim),$$

where we identify $(r,y,t_-,\theta)\in(-\infty,-1+\delta)\times Y\times\mathcal{N}(\partial\Sigma_-^\infty)$ with $(t,y,r,\theta)\in(-\delta,+\infty)\times\mathcal{N}^\infty(-Y)$ if and only if $t=t_-$, and $(r,y,t_+,\theta)\in(1-\delta,+\infty)\times Y\times\mathcal{N}(\partial\Sigma_+^\infty)$ with $(t,y,r,\theta)\in(-\delta,+\infty)\times\mathcal{N}^\infty(Y)$ if and only if $t=t_+$ (see Figure 4). By definition, the $t$ coordinate coincides with the $t_\pm$ coordinates, where these are defined, so we shall again drop the $\pm$ subscripts from the variables $t_\pm$. Note also that the $r$ coordinate is globally defined, whereas $t$ is not. Denote them by $E^{\infty,\infty}(t)$ the region of $E^{\infty,\infty}$ where the coordinate $t$ is defined.

Choose now $\lambda_\pm$ to be Liouville forms on the Liouville domains $\Sigma_\pm^\infty$, such that $\lambda_\pm=e^t\,d\theta$ on $\mathcal{N}(\partial\Sigma_\pm^\infty)$. This last expression makes sense in the region of $E^{\infty,\infty}$ where both $\theta$ and $t$ are defined, and where they are not, the form $\lambda_\pm$ makes sense. So this yields a globally defined 1–form $\lambda\in\Omega^1(E^{\infty,\infty})$ which coincides with $\lambda_\pm$ where these are defined. Also, the same argument works for $\alpha$, so we get a global $\alpha\in\Omega^1(E^{\infty,\infty})$.

For $K\gg 0$ a big constant, $\epsilon>0$ a small one, and $L\geq 1$, choose a smooth function

$$\sigma=\sigma_{\epsilon,K}^L:\mathbb{R}\to\mathbb{R}^+$$

satisfying

- $\sigma\equiv K$ on $\mathbb{R}\setminus[-L,L]$;
- $\sigma\equiv\epsilon$ on $[-L+\delta,L-\delta]$;
- $\sigma'(r)<0$, for $r\in(-L,-L+\delta)$ and $\sigma'(r)>0$ for $r\in(L-\delta,L)$.

The 1–form $\sigma\alpha$ is Liouville on $Y\times\mathbb{R}$. Indeed, if $d\mathrm{vol}$ is a positive volume form in $Y$ with respect to the $\alpha_-$–orientation, we may write

$$d\alpha^n=d\mathrm{vol}\wedge dr,\qquad \alpha_r\wedge d\alpha_r^{n-1}=\mathrm{link}(\alpha_r)\,d\mathrm{vol},$$

where the last equation defines a *self-linking* function $r\mapsto\mathrm{link}(\alpha_r)$, whose sign is opposite to that of $r\in\mathbb{R}$. Then

$$d(\sigma\alpha)^n=\sigma^{n-1}(\sigma-n\sigma'\,\mathrm{link}(\alpha_r))\,d\mathrm{vol}\wedge dr.$$

Tracking the signs, one checks that the above expression is positive.

The associated Liouville vector field is

$$(2)\qquad V_\sigma := \frac{\sigma}{\sigma + \sigma'\, dr(V)} V = \begin{cases} V & \text{on } (\mathbb{R} \setminus (-L, L)) \cup [-L+\delta, L-\delta], \\ (\sigma/(\sigma+\sigma'))\partial_r & \text{on } (L-\delta, L), \\ -(\sigma/(\sigma-\sigma'))\partial_r & \text{on } (-L, -L+\delta). \end{cases}$$

Observe that $V_\sigma$ is everywhere *positively* colinear with $V$.

After extending the form $\sigma\alpha$ to $E^{\infty,\infty}$ in the natural way, one checks that

$$\lambda_\sigma := \lambda_\sigma^L := \sigma\alpha + \lambda$$

is a Liouville form on $E^{\infty,\infty}$. Denote

$$\omega_\sigma := \omega_\sigma^L := d\lambda_\sigma = \sigma'\, dr \wedge \alpha + \sigma\, d\alpha + d\lambda,$$

which is symplectic. Denote by $X_\sigma$ the associated Liouville vector field.

If $X_\pm$ denotes the Liouville vector field on $\Sigma_\pm^\infty$ which is $d\lambda_\pm$–dual to $\lambda_\pm$, coinciding with $\partial_t$ in $\mathcal{N}(\partial\Sigma_\pm^\infty)$, we can define a smooth vector field on $E^{\infty,\infty}$ by

$$(3)\qquad X = \begin{cases} X_+ & \text{on } \{r > 1-\delta\}, \\ X_- & \text{on } \{r < -1+\delta\}, \\ \partial_t & \text{on } \{|r| < 1\}. \end{cases}$$

Then

$$X_\sigma = X + V_\sigma.$$

Denote

$$E^{L,Q} = E^{\infty,\infty}/(\{|r| > L\} \cup \{t > Q\})$$

for $Q \geq 0$ and $L \geq 1$. We have its "horizontal" and "vertical" boundaries

$$\widetilde{M}_Y^{L,Q} := \partial_h E^{L,Q} := \{t = Q\} \cap \{r \in [-L, L]\}, \quad \widetilde{M}_P^{L,Q} := \partial_v E^{L,Q} := \widetilde{M}_P^{-,L,Q} \sqcup \widetilde{M}_P^{+,L,Q},$$

where

$$\widetilde{M}_P^{-,L,Q} := \{r = -L\} \cap \{t \leq Q\}, \quad \widetilde{M}_P^{+,L,Q} := \{r = L\} \cap \{t \leq Q\}.$$

The manifold

$$\widetilde{M}^{L,Q} := \partial E^{L,Q} := \partial_h E^{L,Q} \cup \partial_v E^{L,Q}$$

is then a manifold with corners

$$\partial_h E^{L,Q} \cap \partial_v E^{L,Q} = \{|r| = L\} \cap \{t = Q\}.$$

One has

$$X_\sigma = \pm\frac{\sigma}{\sigma \pm \sigma'}\partial_r + \partial_t$$

in the corresponding components of the region $\{|r| > L-\delta\} \cap \{t \geq -\delta\}$. This means that $X_\sigma$ will be transverse to the smoothening of $\partial E^{L,Q}$ that we shall now construct.

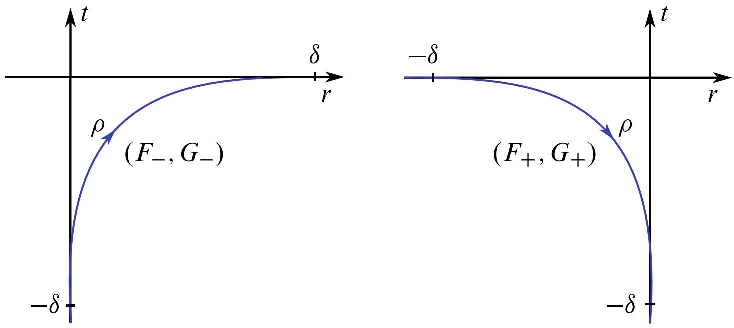


Figure 3: The paths $\rho \mapsto (F_\pm(\rho), G_\pm(\rho))$.

Choose smooth functions $F_\pm, G_\pm \colon (-\delta,\delta) \to (-\delta, 0]$ such that

$$\begin{aligned}
(F_+(\rho), G_+(\rho)) &= (\rho, 0) &&\text{and} && (F_-(\rho), G_-(\rho)) = (0, \rho) &&\text{for } \rho \le -\delta/3,\\
G_+'(\rho) &< 0 &&\text{and} && F_-'(\rho) > 0 &&\text{for } \rho > -\delta/3,\\
G_-'(\rho) &> 0 &&\text{and} && F_+'(\rho) > 0 &&\text{for } \rho < \delta/3,\\
(F_+(\rho), G_+(\rho)) &= (0, -\rho) &&\text{and} && (F_-(\rho), G_-(\rho)) = (\rho, 0) &&\text{for } \rho \ge \delta/3.
\end{aligned}$$

See Figure 3.

We now smoothen the corner $\partial_h E^{L,Q} \cap \partial_v E^{L,Q}$ by substituting the region

$$\partial E^{L,Q} \cap \big(\{t \in (Q-\delta, Q]\} \cup \{|r| \in (L-\delta, L]\}\big),$$

which contains the corners, with the smooth manifold

$$M_C^{\pm,L,Q} := \{(r = \pm L + F_\pm(\rho), t = Q + G_\pm(\rho), y, \theta) : (\rho, y, \theta) \in (-\delta,\delta) \times Y \times S^1\}.$$

The smoothened boundary can then be written as

$$M^{L,Q} := M_P^{\pm,L,Q} \cup M_C^{\pm,L,Q} \cup M_Y^{L,Q},$$

where

$$M_P^{\pm,L,Q} = \widetilde{M}_P^{\pm,L,Q} \cap \{t < Q-\delta/3\}, \quad M_Y^{L,Q} = \widetilde{M}_Y^{L,Q} \cap \{|r| < L-\delta/3\}.$$

The Liouville vector field $X_\sigma$ is transverse to this manifold, so we get a contact structure on $M^{L,Q}$ given by

$$\xi^{L,Q} = \ker(\lambda_\sigma^L|_{TM^{L,Q}}).$$

Observe that $M^{L,Q}$ is canonically diffeomorphic to $M$. So, this actually yields a contact structure on $M$. By construction, we have nonempty intersections

$$\begin{aligned}
M_P^{\pm,L,Q} \cap M_C^{\pm,L,Q} &= \{r = \pm L\} \cap \{t \in (Q-\delta, Q-\delta/3)\},\\
M_Y^{L,Q} \cap M_C^{\pm,L,Q} &= \{t = Q\} \cap \{|r| \in (L-\delta, L-\delta/3)\}.
\end{aligned}$$

We shall construct a stable Hamiltonian structure on $M^{L,Q}$ which arises as a deformation of the above contact structure, such that both coincide on $M_Y^{L,Q}$, as follows.

Choose a smooth function $\beta\colon \mathbb{R} \to [0,1]$ such that $\beta(t)=0$ for $t \le -\delta+\delta/9$, $\beta(t)=1$ for $t \ge -2\delta/3-\delta/9$, and $\beta' \ge 0$. Set

$$Z = \begin{cases} V_\sigma + \beta(t)X & \text{in the region } E^{\infty,\infty}(t) \text{ where } t \text{ is defined,} \\ (\sigma/(\sigma+\sigma'))\partial_r & \text{in } E^{\infty,\infty}(t)^c \cap \{r > 1-\delta\}, \\ -(\sigma/(\sigma-\sigma'))\partial_r & \text{in } E^{\infty,\infty}(t)^c \cap \{r < -1+\delta\}, \end{cases} \tag{4}$$

which yields a smooth vector field on $E^{\infty,\infty}$, a deformation of $X_\sigma$. Then $Z$ is still transverse to $M^{L,Q}$ and is stabilizing, so that the pair

$$\mathcal{H} := \mathcal{H}^{L,Q} := (\Lambda^{L,Q} = i_Z\omega_\sigma|_{TM^{L,Q}}, \Omega^{L,Q} = \omega_\sigma|_{TM^{L,Q}})$$

yields a stable Hamiltonian structure on $M^{L,Q}$. For $Q=0$, $(M_Y^{L,0}, \Lambda^{L,0})$ can be seen as the *contactization* of the Liouville domain $(Y \times I, \epsilon\, d\alpha)$.

Along $M_Y^{L,Q}$ the Reeb vector field is given by $R^{L,Q} = \partial_\theta/e^Q$, which is degenerate, and the space of Reeb orbits is identified with $Y \times [-L, L]$. We consider two perturbation approaches: Morse, and Morse–Bott. In the first approach we choose $H_L\colon Y \times [-L, L] \to \mathbb{R}^{\ge 0}$ to be Morse, depending only in $r$ near $r = \pm L$, satisfying $\partial_r H_L \le 0$ near $r = L$, $\partial_r H_L \ge 0$ near $r = -L$, and vanishing as one approaches $r = \pm L$. In the second approach, we choose $H_L$ to depend only on $r$ *globally*, with respect to which it is a Morse function.

If $t \mapsto \Phi_Z^t$ denotes the flow of $Z$, choose $\epsilon > 0$ sufficiently small so that the manifold

$$M^{\epsilon;L,Q} := \{\Phi_Z^{\epsilon H_L(x)}(x) \in E^{\infty,\infty} : x \in M^{L,Q}\}$$

is still transverse to $Z$. We have a stable Hamiltonian structure

$$\mathcal{H}_\epsilon := \mathcal{H}_\epsilon^{L,Q} := (\Lambda_\epsilon^{L,Q}, \Omega_\epsilon^{L,Q}) := (i_Z\omega_\sigma|_{TM^{\epsilon;L,Q}}, \omega_\sigma|_{TM^{\epsilon;L,Q}}),$$

and a decomposition

$$M^{\epsilon;L,Q} = M_Y^{\epsilon;L,Q} \cup M_C^{\epsilon;\pm,L,Q} \cup M_P^{\epsilon;\pm,L,Q},$$

where each component is the perturbation of the corresponding component of $M^{L,Q}$.

Along the region $M_C^{\epsilon;\pm,L,Q}$ the new coordinates are

$$\begin{aligned} r &= F_\pm^{\epsilon;L}(\rho) = \Phi_{V_\sigma}(\epsilon H_L(\cdot, \pm L + F_\pm(\rho)), \pm L + F_\pm(\rho)), \\ t &= G_\pm^{\epsilon;L,Q}(\rho) = Q + G_\pm(\rho) + \epsilon H_L(\cdot, \pm L + F_\pm(\rho)), \end{aligned}$$

where $\Phi_{V_\sigma}(s, \cdot)$ is the time $s$ flow of $V_\sigma$.

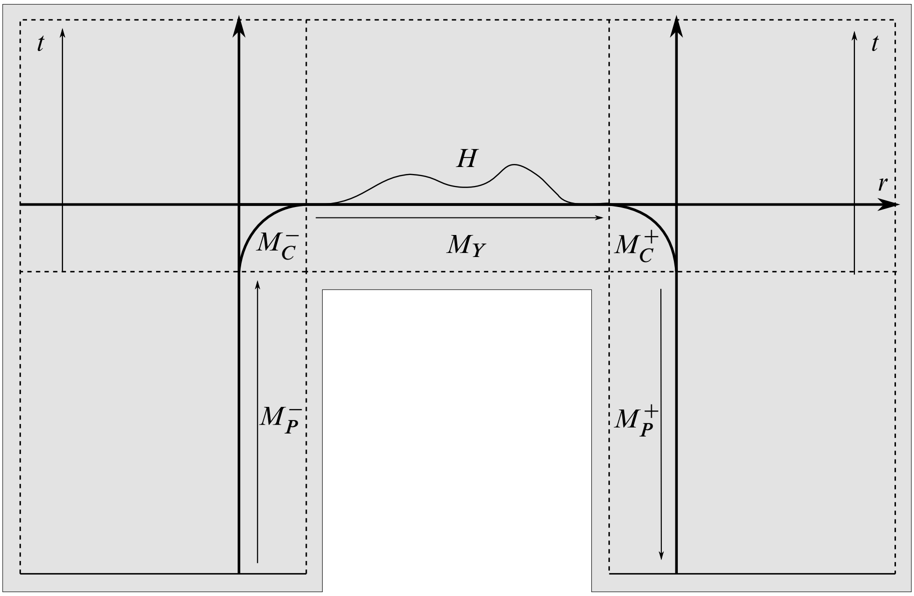


Figure 4: The double completion, and a Morse function $H$ along the spine.

We then have

$$\Lambda_\epsilon^{L,Q} = \begin{cases} \Lambda^{L,Q} = K\alpha_\pm & \text{in } M_P^{\epsilon;\pm,L,Q} = M_P^{\pm,L,Q}, \\ e^{\epsilon H_L}(\epsilon\alpha + e^Q\, d\theta) & \text{in } M_Y^{\epsilon;L,Q}, \\ \sigma(F_\pm^{\epsilon;L}(\rho))e^{\pm F_\pm^{\epsilon;L} - L}\alpha_\pm + \beta(G_\pm^{\epsilon;L,Q}(\rho))e^{G_\pm^{\epsilon;L,Q}(\rho)}\, d\theta & \text{in } M_C^{\epsilon;\pm,L,Q}. \end{cases}$$

One can similarly write down $\Omega_\epsilon^{L,Q}$ explicitly.

The Reeb vector field $R_\epsilon^{L,Q}$ associated to this stable Hamiltonian structure is

$$R_\epsilon^{L,Q} = \begin{cases} R^{L,Q} = R_\pm/K & \text{in } M_P^{\epsilon;\pm,L,Q}, \\ e^{-\epsilon H_L - Q}\big((1+\epsilon\alpha(X_{H_L}))\partial_\theta - e^Q X_{H_L}\big) & \text{in } M_Y^{\epsilon;L,Q}, \\ (1/\Phi_\pm^{\epsilon;L,Q})\big(e^{\mp F_\pm^{\epsilon;L}+L}(G_\pm^{\epsilon;L,Q})' R_\pm - e^{-G_\pm^{\epsilon;L,Q}}(\sigma' \pm \sigma)(F_\pm^{\epsilon;L})(F_\pm^{\epsilon;L})'\partial_\theta\big) & \text{in } M_C^{\epsilon;\pm,L,Q}, \end{cases}$$

where $X_{H_L}$ is the Hamiltonian vector field on $Y \times I$ associated to $H_L$, defined by $i_{X_{H_L}}\, d\alpha = -dH_L$, and

$$\Phi_\pm^{\epsilon;L,Q}(\rho) = \sigma(F_\pm^{\epsilon;L}(\rho))(G_\pm^{\epsilon;L,Q})'(\rho) - \beta(G_\pm^{\epsilon;L,Q}(\rho))(\sigma' \pm \sigma)(F_\pm^{\epsilon;L}(\rho))(F_\pm^{\epsilon;L})'(\rho). \tag{5}$$

One can check that $\Phi_\pm^{\epsilon;L,Q}$ has sign which is *opposite* to its subscript. Observe that critical points $(y,r)$ of $H$ give rise to closed Reeb orbits of the form $\gamma_p := \{p\} \times S^1 \subseteq \operatorname{crit}(H_L) \times S^1$. If we are taking the Morse approach, we have only a finite number of such orbits, and they are nondegenerate. Choosing $H_L$ to be $C^1$–small has the effect of making the vector field $X_{H_L}$ also small, so that the closed orbits which do not arise from critical points of $H_L$ have large period, including the ones not contained in $M_Y^{L,Q}$. So, taking any large (but fixed) $T \gg 0$, we can choose $H_L$ small enough so that all the periodic Reeb orbits up to period $T$ are of the form $\gamma_p^l$, for $p \in \operatorname{crit}(H_L)$, and $l \le N$, for some covering threshold $N$

depending on $T$. For the Morse–Bott case, we obtain $Y$–families of Morse–Bott orbits for each critical point of $H_L$.

**Remark 2.1** (1) One can check that $\lambda_\sigma(R_\epsilon)=1$ (recall $\lambda_\sigma|_{M^{\epsilon;L,Q}}$ is the primitive of $\Omega_\epsilon=\omega_\sigma|_{M^{\epsilon;L,Q}}$). Therefore, for compatible almost complex structure $J$ and an asymptotically cylindrical $J$–holomorphic curve $u$ with positive/negative punctures $\Gamma^\pm$, the $\Omega_\epsilon$–energy of $u$ is

$$\int u^*\Omega_\epsilon = \int u^*\,d\lambda_\sigma = \sum_{z\in\Gamma^+} T_z - \sum_{z\in\Gamma^-} T_z, \tag{6}$$

where $T_z$ is the action of the Reeb orbit corresponding to the puncture $z$. In particular, if the positive punctures correspond to critical points of $H_L$, then so will the negative ones.

(2) By inspecting the expressions of the Reeb vector field we see that there are no contractible closed Reeb orbits for the SHS, if we assume this same condition for $R_\pm$. Moreover, the direction of the Reeb vector field does not change after perturbing back to sufficiently close contact data (see Section 2.7 below), and so this also holds for the latter data. It follows that the isotopy class defined by the resulting contact structure is hypertight, and this shows the hypertightness condition of Theorem 1.2.

## 2.2 Compatible almost complex structure

**Construction** We set $L=1$ and $Q=0$, and drop the superscripts $L$ and $Q$ from all of the notation. We now define a suitable, though nongeneric, almost complex structure $J=J_\epsilon$ on the symplectization $W^\epsilon=\mathbb{R}\times M^\epsilon$, where

$$M^\epsilon = M^{\epsilon;1,0} = M_Y^\epsilon \cup M_P^{\epsilon;\pm} \cup M_C^{\epsilon;\pm}.$$

It will be compatible with the stable Hamiltonian structure $\mathcal{H}_\epsilon$, and the fibers $\Sigma_\pm$ of our fibration $\pi_P^\pm$, the "pages", will lift as holomorphic curves. We will blur the distinction between $M=M^{0;1,0}$ and its diffeomorphic perturbed copy $M^\epsilon$ (as well as for $W$ and $W^\epsilon$), so that we are actually working on a fixed $M$ with an SHS which depends on $\epsilon$.

Let $\xi_\epsilon := \ker\Lambda_\epsilon$. We will define $J$ on $\xi_\epsilon$, in an $\mathbb{R}$–invariant way, and then simply set $J(\partial_a) := R_\epsilon$.

Choose a $d\alpha$–compatible almost complex structure $J_0$ on $Y\times I$, which is cylindrical in the cylindrical ends of $Y\times I$, such that, along these, it coincides with a $d\alpha_\pm$–compatible almost complex structure $J^\pm$ on $\xi_\pm$, and maps the Liouville vector field $V$ to $R_\pm$. Observe that the vector field $\partial_\theta$ is transverse to $\xi_\epsilon$ along $M_Y$. Therefore, we may then define $J_Y=\pi_Y^*J_0$ on $\xi_\epsilon|_{M_Y}$.

Along the regions $\Sigma_\pm/\mathcal{N}(\partial\Sigma_\pm)\times Y\subseteq M_P^\pm$, and $\{t\in(-\delta,-\delta/3)\}\times Y\subseteq M_P^\pm$, the restriction of the projection $\pi_P^\pm\colon M_P^\pm\to\Sigma_\pm$ induces an isomorphism $d\pi_P^\pm\colon \xi_\epsilon/\xi_\pm \xrightarrow{\simeq} T\Sigma_\pm$. Choose $j_\pm$ to be a $d\lambda$–compatible almost complex structures on $\Sigma_\pm$, so that $j_\pm(\partial_t)=K\partial_\theta$ in $\mathcal{N}(\partial\Sigma_\pm)$. Define

$$J = J^\pm \oplus (\pi_P^\pm)^* j_\pm$$

on $\xi_\epsilon|_{M_P^\pm}$.

In $M_C^{\pm}$,

$$\xi_\epsilon = \xi_\pm \oplus \langle v_1, v_2^{\pm} \rangle,$$

where

$$v_1 = \partial_\rho, \quad v_2^{\pm} = a_\pm(\rho) R_\pm + b_\pm(\rho)\partial_\theta. \tag{7}$$

Here,

$$a_\pm = -\frac{\beta(G_\pm^\epsilon)}{e^{\pm F_\pm^\epsilon - 1}\Phi_\pm^\epsilon}, \quad b_\pm = \frac{\sigma(F_\pm^\epsilon)}{e^{G_\pm^\epsilon}\Phi_\pm^\epsilon},$$

where $\Phi_\pm^\epsilon$ is defined in (5). In the overlaps $M_C^{\pm} \cap M_Y$, one computes that $J(v_1) = g_\pm^Y(\rho) v_2^{\pm}$ where

$$g_\pm^Y := \pm \frac{e^{\mp F^\epsilon \pm(\rho)+1}}{a_\pm(\rho)} = \mp \frac{\Phi_\pm^\epsilon}{\beta(G_\pm^\epsilon)},$$

which is always positive. Similarly, in $M_P^{\pm} \cap M_C^{\pm}$, we have $J(v_1) = e^{\mp\rho} v_2^{\pm}$. Since $g_\pm^Y$ and $e^{\mp\rho}$ are both positive, we can now take any smooth positive functions $h_\pm \colon (-\delta, \delta) \to \mathbb{R}^+$ which coincide with $e^{\mp\rho}$ near $\rho = \pm\delta$ and with $g_\pm^Y$ near $\rho = \mp\delta$. We glue the two definitions by setting

$$J(v_1) = h_\pm(\rho) v_2^{\pm} := w^{\pm}, \tag{8}$$

and we make $J$ agree with $J^{\pm}$ on $\xi_\pm$.

This gives a well-defined cylindrical $J$ in $\mathbb{R} \times M$.

**Compatibility** One can check that $J$ is $\mathcal{H}_\epsilon$–compatible by straightforward computations [Moreno 2018].

**Remark 2.2** Over $\mathbb{R} \times M_P^{\pm}$, where $\Lambda_\epsilon = K\alpha_\pm$, we have $d\Lambda_\epsilon(v, Jv) \geq 0$, with equality if and only if $v^{\xi_\pm} = 0$, so the projection to $TY$ of $v$ lies in the span of $R_\pm$.

## 2.3 Finite-energy foliation

We will now consider the symplectization of our stable Hamiltonian manifold $(M, \mathcal{H}_\epsilon)$, given by

$$(W = \mathbb{R} \times M, \omega_\epsilon^\varphi = d(\varphi(a)\Lambda_\epsilon) + \Omega_\epsilon),$$

where $\varphi \in \mathcal{P}$. We will construct a finite-energy foliation of $W$ by $J$–holomorphic curves, consisting of three distinct types, which we describe in the next theorem. This is an adaptation of the construction in [Wendl 2010].

**Theorem 2.3** *There exists a finite-energy foliation of the symplectization $(W, \omega_\epsilon^\varphi)$ by simple $J$–holomorphic curves of the following types:*

- ***Trivial cylinders $C_p$**, corresponding to Reeb orbits of the form $\gamma_p = \{p\} \times S^1$ for $p \in \mathrm{crit}(H)$, and which may be parametrized by*

$$C_p \colon \mathbb{R} \times S^1 \to \mathbb{R} \times M_Y = \mathbb{R} \times (Y \times I) \times S^1, \quad C_p(s,t) = (s, p, e^{-\epsilon H(p)} t).$$

- ***Flow-line cylinders $u^a_\gamma$***, *parametrized by*

$$u^a_\gamma\colon \mathbb{R}\times S^1 \to \mathbb{R}\times M_Y = \mathbb{R}\times (Y\times I)\times S^1, \quad u^a_\gamma(s,t) = (a(s), \gamma(s), \theta(s)+t) \tag{9}$$

*for a proper function $a\colon \mathbb{R}\to\mathbb{R}$, a function $\theta\colon\mathbb{R}\to S^1$, and a map $\gamma\colon\mathbb{R}\to Y\times I$ satisfying*

$$\begin{aligned} \dot\gamma &= \nabla H(\gamma(s)), & & \\ \dot a(s) &= e^{\epsilon H(\gamma(s))}, & a(0)&=a, \\ \dot\theta(s) &= -\epsilon\alpha(\nabla H(\gamma(s))), & \theta(0)&=0. \end{aligned} \tag{10}$$

*Here, the gradient is computed with respect to the metric $g_{d\alpha,J_0} := d\alpha(\cdot\,, J_0\,\cdot)$. They have for positive/negative asymptotics the Reeb orbits corresponding to*

$$p_\pm := \lim_{s\to\pm\infty}\gamma(s)\in \mathrm{crit}(H).$$

- ***Positive/negative page-like holomorphic curves $u^\pm_{y,a}$***, *which consist of a trivial lift at symplectization level $a$ of a page $P^\pm_y := (\Sigma_\pm \setminus \mathcal{N}(\partial\Sigma_\pm))\times\{y\}$, for $y\in Y$, glued to $k$ cylindrical ends which lift the smoothened corners and then enter the symplectization of $M_Y$, asymptotically becoming a flow-line cylinder. They have $k$ positive asymptotics at Reeb orbits of the form $\gamma_p$, exactly one for each component of $M_Y$. The positive curves have genus $g-k+1>0$ and $k$ punctures, whereas the negative curves have genus $0$, and also $k$ punctures.*

**Remark 2.4** In the Morse–Bott case, one can show that $\alpha(\nabla H)=0$ for a suitable metric on $Y\times I$ [Moreno 2018, Remark 2.8], so the function $\theta$ vanishes identically.

Figure 5 summarizes the situation. We shall not distinguish the curves $u^\pm_{y,a}$ and $u^a_\gamma$ from the simple holomorphic curves that they parametrize, and we will drop the $a$ from the notation whenever we wish to refer to the equivalence class of the curves under $\mathbb{R}$–translation. A short computation shows that the flow-line cylinders are indeed holomorphic (see eg [Moreno 2018; Siefring 2017]). We now construct the page-like curves.

The pages $P^\pm_y := (\Sigma_\pm\setminus\mathcal{N}(\partial\Sigma_\pm))\times\{y\}$, for $y\in Y$, clearly lift to a holomorphic foliation of the region $\mathbb{R}\times M^\pm_P\subseteq W$, which takes the form $\{\{a\}\times P^\pm_y : a\in\mathbb{R}, y\in Y\}$. We now glue cylindrical ends to these lifts.

We have that $Jv_1 = h_\pm v^\pm_2$ and $R_\epsilon$ are both linear combinations of the vector fields $R_\pm$ and $\partial_\theta$ along $M^\pm_C$, with the coefficients only depending on $\rho$. Since these are not colinear, we have smooth functions $B, C\colon(-\delta,\delta)\to\mathbb{R}$ such that

$$\partial_\theta = BR_\epsilon + CJv_1.$$

One can in fact compute that

$$B = e^{G^\epsilon_\pm}\beta(G^\epsilon_\pm), \quad C = \frac{e^{G^\epsilon_\pm}(G^\epsilon_\pm)'}{h_\pm}. \tag{11}$$

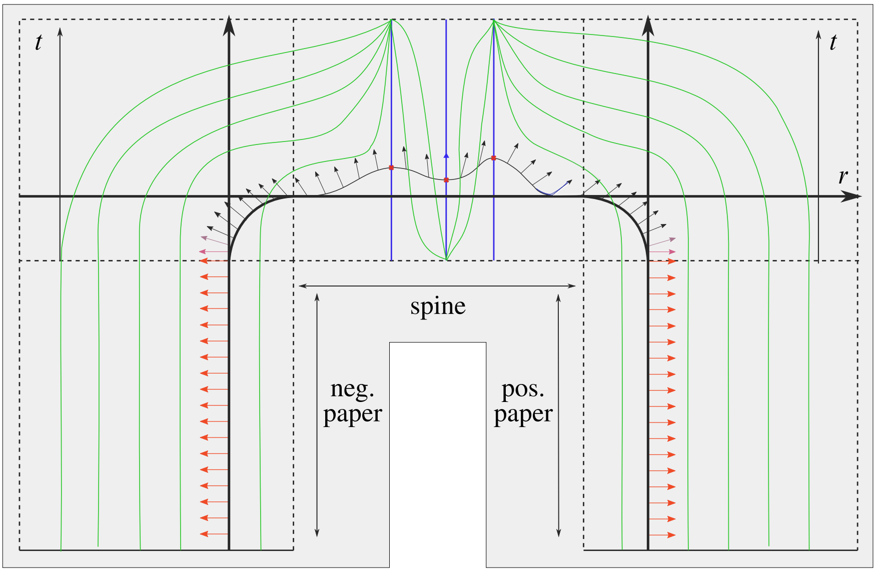


Figure 5: The double completion $E^{\infty,\infty}$, containing $M$ and its perturbed version $M^\epsilon$ as contact type hypersurfaces. The foliation by holomorphic curves is shown in green (for the nontrivial curves) and blue (for the trivial cylinders).

We have that

$$J\partial_\theta = -B\partial_a - Cv_1 = -B\partial_a - C\partial_\rho. \tag{12}$$

We conclude that

$$\langle \partial_\theta, B\partial_a + C\partial_\rho \rangle = \langle \partial_\theta, J\partial_\theta \rangle.$$

It follows that the distribution above has integral submanifolds which are unparametrized holomorphic curves. We can actually find holomorphic parametrizations given by

$$w^\pm_{y,a} : (-\delta,\delta)\times S^1 \to \mathbb{R}\times M^\pm_C = \mathbb{R}\times Y\times(-\delta,\delta)\times S^1, \quad (s,t)\mapsto (b(s), y, \rho(s), t),$$

for some fixed $y \in Y$, and functions $b, \rho : (-\delta,\delta)\to\mathbb{R}$ satisfying

$$\dot\rho(s) = C(\rho(s)), \quad \rho(-\delta) = -\delta,$$
$$\dot b(s) = B(\rho(s)), \quad b(-\delta) = a.$$

The curve $w^\pm_{y,a}$ is indeed holomorphic.

The curves $w^\pm_{y,a}$ glue with curves $u^a_\gamma$, which look like $u^a_\gamma(s,t) = (a(s), y(s), r(s), t)$ for some $y(s)\in Y$ such that $y(s)\equiv y$ near $r=\pm 1$ and $\lim_{s\to+\infty} y(s) = y^+$, and some $r:\mathbb{R}\to I$ with $\lim_{s\to+\infty} r(s) = r^+$, so that $(y^+, r^+)\in \mathrm{crit}(H)$. We may then define a $J$–holomorphic curve

$$u^\pm_{y,a} := P^\pm_{y,a}\cup w^\pm_{y,a}\cup u^a_\gamma$$

which asymptotes $k$ Reeb orbits $\gamma^\pm_{y,a;i} = \{p^\pm_{y,a;i}\}\times S^1$, where $p^\pm_{y,a;i}\in\mathrm{crit}(H)$ for $i = 1,\dots,k$, and which have genus $g(u^-_{y,a}) = 0$ and $g(u^+_{y,a}) = g-k+1$.

## 2.4 Index computations and Fredholm regularity

**Theorem 2.5** (1) *After a sufficiently small Morse perturbation making Reeb orbits along $M_Y$ nondegenerate, we can find a natural trivialization $\tau$ of the contact structure along $\gamma_p$ (inducing a trivialization $\tau^l$ along all of its covers $\gamma_p^l$), and $N \in \mathbb{N}$, which depends on $H$ and grows as $H$ gets smaller, such that the Conley–Zehnder index of $\gamma_p^l$ is given by*

$$\mu_{\mathrm{CZ}}^{\tau^l}(\gamma_p^l) = \mathrm{ind}_p(H) - n,$$

*for $l \leq N$.*

(2) *In the Morse approach, the Fredholm indexes of the curves in our finite-energy foliation are given by*

$$\begin{aligned} \mathrm{ind}(u^-_{y,a}) &= 2n(1-k) + \sum_{i=1}^{k} \mathrm{ind}_{p^-_{y,a;i}}(H), \\ \mathrm{ind}(u^+_{y,a}) &= 2n(1-g-k) + \sum_{i=1}^{k} \mathrm{ind}_{p^+_{y,a;i}}(H), \\ \mathrm{ind}(u^a_\gamma) &= \mathrm{ind}_{p_+}(H) - \mathrm{ind}_{p_-}(H). \end{aligned} \tag{13}$$

**Proof** See [Moreno 2018]. □

**Remark 2.6** Since $\mathrm{ind}_{p^+_{y,a;i}}(H) \leq 2n$ for every $i$, $\mathrm{ind}(u^+_{y,a}) \leq 2n(1-g) \leq 0$, since $g \geq 1$. This means these curves cannot possibly achieve transversality, and, after a perturbation making $J$ generic, they will disappear. For the other curves, we have the following:

**Lemma 2.7** *In the Morse–Bott case, the flow-line cylinders $u^a_\gamma$, as well as the* **negative** *page-like curves $u^-_{y,a}$, are Fredholm regular.*

The proof of this lemma will be omitted (see [Moreno 2018, Section 3.5] for details).

## 2.5 From Morse–Bott to Morse

In this section, we fix a Morse perturbation scheme. First, choose $H$ to be given by $H(y,r) = f(r)$ where $f$ is a sufficiently small and positive Morse function on $I$ and has a unique critical point at 0 with Morse index 1 (which yields the Morse–Bott situation). And then, choose a sufficiently small positive Morse function $g$ on $Y$ and extend it to a neighborhood of $Y \times \{0\}$ to make a further perturbation (obtaining the Morse case). Therefore,

$$H(y,r) = f(r) + \gamma(r)g(y),$$

where $\gamma : I \to [0,1]$ is a smooth bump function satisfying $\gamma = 0$ in the region $\{|r| > 2\delta\}$ and $\gamma = 1$ on $\{|r| \leq \delta\}$.

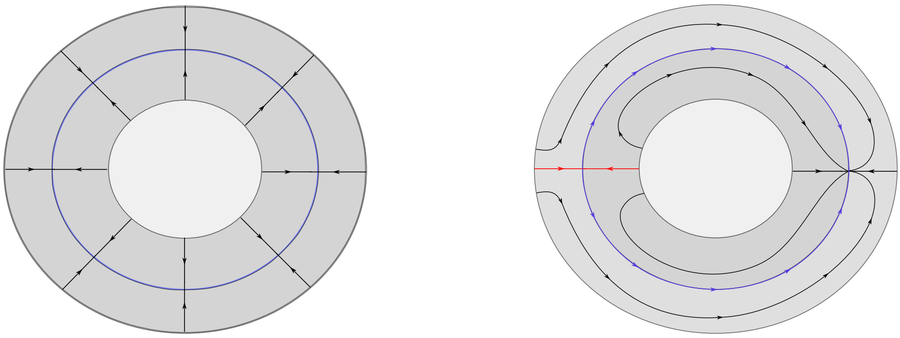

Figure 6: The Morse–Bott and the Morse scenarios, respectively, in the case where $Y = S^1$. The "evil twins" are shown in blue (see Section 2.8).

We view the Morse case as a deformation of the Morse–Bott one, via $H_t(y,r) = f(r) + t\gamma(r)g(y)$ for $t \in [0,1]$. We obtain a corresponding 1–parameter families of SHSs and compatible almost complex structures $J^t$. In the case where $g$ is chosen small, from the implicit function theorem we obtain:

**Theorem 2.8** *For Morse data sufficiently close to Morse–Bott data, all the genus zero curves in the finite-energy foliation are Fredholm regular.*

In order to simplify the torsion computation of Section 2.8, we will choose $g$ to have a unique maximum and minimum. Both scenarios are depicted in Figure 6 in the case of $Y = S^1$.

## 2.6 Uniqueness of the curves in the Morse/Morse–Bott case

In this section, we prove that the family of curves we constructed above are the unique curves (up to reparametrization and multiple covers) that asymptote Reeb orbits of the family $\{\gamma_p : p \in \mathrm{crit}(H)\}$, and with positive asymptotics in different components of $M_Y$. We do this in both the Morse and Morse–Bott situations. We assume that $H$ has a unique critical point in the interval direction, at $r = 0$ (and perhaps other critical points contained in $Y \times \{0\}$, in the Morse case). In the Morse–Bott case, we denote by $\gamma_y := \gamma_{(y,0)} = \{(y,0)\} \times S^1$ the simply covered Reeb orbit corresponding to $y \in Y$.

**Lemma 2.9** *Assume either the Morse or Morse–Bott cases. Let $N \in \mathbb{N}^{>0}$, and let $u$ be a $J$–holomorphic curve with positive asymptotics of the form $\gamma_p = \{p\} \times S^1$, for which the number of positive punctures is bounded above by $N$. Then we can find sufficiently small $\epsilon > 0$ (depending only on $N$), such that*

$$\#\Gamma^-(u) \le \#\Gamma^+(u),$$

*where $\Gamma^\pm(u)$ denotes the set of positive/negative punctures of $u$. Here, we count punctures with the covering multiplicity of their corresponding asymptotic.*

**Proof** This follows easily from Remark 2.1, and by computing the $\Omega_\epsilon$–energy [Moreno 2018]. □

**Theorem 2.10** *Let $u\colon \dot{\Sigma} \to \mathbb{R} \times M$ be a (not necessarily regular) $J$–holomorphic curve defined on some punctured Riemann surface $\dot{\Sigma}$ which is asymptotically cylindrical, and asymptotes simply covered Reeb orbits of the form $\gamma_p = \{p\} \times S^1$ at its positive ends. Assume that any two of the positive ends of $u$ lie in distinct components of $M_Y$.*

*Then there exists a sufficiently small and uniform $\epsilon > 0$, and sufficiently small Morse perturbation of the Morse–Bott case, such that, if $u$ has genus less than $g_0$ and if it is not a trivial cylinder over one of the $\gamma_p$, we have the following possibilities:*

- ***Morse–Bott case*** *$u$ has the same asymptotics and relative homology class as a curve of the form $u^{\pm}_{y,a}$ for some $y \in Y$ and $a \in \mathbb{R}$. In the case where it does not coincide with a curve $u^{-}_{y,a}$, then it has strictly positive genus; or*
- ***Morse case*** *$u$ is a flow-line cylinder $u_\gamma$.*

**Proof** We will consider two cases: either $u$ is completely contained in the region $\mathbb{R} \times M_Y$ (Case A), or it is not (Case B).

**Case A** This case is easily dealt with in the Morse–Bott scenario. By assumption, we have that $u$ has a unique positive end. Since the $\gamma_p$'s are not contractible/nullhomologous inside $M_Y$, Lemma 2.9 implies that $u$ is has one positive and one negative end, both simply covered, corresponding to Reeb orbits $\gamma_{p_\pm}$. But then the $\Omega_\epsilon$–energy of $u$ vanishes, and $u$ is necessarily is a cover of a trivial cylinder.

In the Morse case, we show that $u$ is a flow line cylinder. Again, Lemma 2.9 implies that $u$ is has one positive and one negative end, both simply covered, corresponding to Reeb orbits $\gamma_{p_\pm}$. Observe that, a priori, $u$ is not even necessarily a cylinder, since it may have positive genus.

In the degenerate case when $H \equiv 0$, the projection $\pi_Y\colon \mathbb{R} \times M_Y \to Y \times I$ is holomorphic. Then if $u$ is a holomorphic curve for this data, so is $v := \pi_Y \circ u$. Since the asymptotics of $u$ are covers of the $S^1$–fibers of $\pi_Y$, the map $v$ extends to a holomorphic curve in the closed surface $\Sigma$. But $Y \times I$ is exact, so $v$ has to be constant by Stokes' theorem. This means that $u$ is necessarily a multiple cover of a trivial cylinder.

We then see that the space of *stable holomorphic cascades* [Bourgeois 2003] in $\mathbb{R} \times Y \times I \times S^1$, the objects one obtains as limits of honest curves when one turns off the function $H$, consists of finite collections of flow-line segments and covers of trivial cylinders. If we take $H_t = tH$, and we assume we have a sequence $\{u_n\}$ of $J_{t_n}$–holomorphic maps with $t_n \to 0$ (where $J^t$ is the almost complex structure corresponding to $H_t$), with one positive and negative simply covered orbits corresponding to critical points $p_\pm$, then we obtain a *stable* holomorphic cascade $\boldsymbol{u}^H_\infty$ as a limiting object. Since the positive end of $u_n$ is simply covered, Lemma 2.9 applied to $H = 0$ implies that *every* Reeb orbit appearing in $\boldsymbol{u}^H_\infty$ is simply covered, and therefore all of its holomorphic map components cannot be multiply covered. These can then only be trivial cylinders, but stability of the cascade means that it does not have trivial cylinder components. We conclude that the space of holomorphic cascades which glue to curves as in our hypothesis consists solely of flow-lines, which are regular by the Morse–Smale condition, and come in an $(\mathrm{ind}_{p_+}(H) - \mathrm{ind}_{p_-}(H) - 1)$–dimensional family. Moreover, note that such flow-lines cylinders can *always* be glued to another flow-line cylinder. The implicit function theorem then implies that $u$ is a flow-line cylinder if $H$ is sufficiently small. Indeed, we can argue as follows. Consider the parametric moduli space

$$\bar{\mathcal{M}} := \bar{\mathcal{M}}(\mathbb{R} \times M_Y, \{J^t\}_{t\in[0,1]}; \gamma_{p_+}, \gamma_{p_-}) = \{(t,u) : t \in [0,1],\, u \in \bar{\mathcal{M}}(\mathbb{R} \times M_Y, J^t; \gamma_{p_+}, \gamma_{p_-})\},$$

where $\bar{\mathcal{M}}(\mathbb{R} \times M_Y, J^t; \gamma_{p_+}, \gamma_{p_-})$ denotes the Gromov-compactified moduli space of $J^t$–holomorphic curves in $\mathbb{R} \times M_Y$ (of any genus) which have simply covered positive asymptotic $\gamma_{p_+}$, and negative asymptotic $\gamma_{p_-}$, and where the compactification at $t = 0$ corresponds to stable holomorphic cascades. Since the Reeb orbits are simply covered, curves in the parametric moduli are somewhere injective. Then, by the Morse–Smale condition and the implicit function theorem, the parametric moduli space is, for sufficiently small $H$, an $(\mathrm{ind}_{p_+}(H) - \mathrm{ind}_{p_-}(H))$–dimensional compact manifold whose boundary contains $\bar{\mathcal{M}}(\mathbb{R} \times M_Y, J^0; \gamma_{p_+}, \gamma_{p_-})$, only consisting of flow line cylinders. The flow-line parametric moduli space

$$\begin{aligned} \bar{\mathcal{M}}_{\text{flow-line}} &:= \bar{\mathcal{M}}_{\text{flow-line}}(\mathbb{R} \times M_Y, \{J^t\}_{t\in[0,1]}; \gamma_{p_+}, \gamma_{p_-}) \\ &:= \{(t,u) : t \in [0,1],\, u \text{ corresponds to a flow-line in } \bar{\mathcal{M}}(\mathbb{R} \times M_Y, J^t; \gamma_{p_+}, \gamma_{p_-})\} \end{aligned}$$

is, a priori, a submanifold of $\bar{\mathcal{M}}(\mathbb{R} \times M_Y, \{J^t\}_{t\in[0,1]}; \gamma_{p_+}, \gamma_{p_-})$, which shares the boundary component $\bar{\mathcal{M}}(\mathbb{R} \times M_Y, J^0; \gamma_{p_+}, \gamma_{p_-})$, and has its same dimension. By thinking of $\bar{\mathcal{M}}$ as a collar neighborhood of this boundary component, we get that

$$\bar{\mathcal{M}} = \bar{\mathcal{M}}_{\text{flow-line}},$$

from which our uniqueness follows. Observe that since there are finitely many critical points, we can take $H$ (or $\epsilon > 0$) uniformly small.

**Case B** We first consider the Morse–Bott case, and deal with the Morse case later. The approach in this situation is to estimate the $\Omega_\epsilon$–energy of $u$, and to use a suitable branched cover argument. The details are as follows.

Assume the Morse–Bott case. Since every positive puncture of $u$ corresponds to a critical point, Remark 2.1 implies that so does every negative one. Let us denote by $\Gamma^\pm$ the set of positive/negative punctures of $u$, and for $z \in \Gamma^\pm$, let $\gamma_{p_z}^{\kappa_z}$ be the Reeb orbit corresponding to $z$, where $p_z \in \mathrm{crit}(H)$, $\kappa_z \geq 1$, and $\kappa_z = 1$ for $z \in \Gamma^+$. By assumption, we have that $\#\Gamma^+ \leq k$, the number of components of $M_Y$.

The $\Omega_\epsilon$–energy then has the upper bound

$$\begin{aligned} \text{(14)} \qquad \boldsymbol{E}(u) = \int_{\dot\Sigma} u^*\Omega_\epsilon &= 2\pi e^{\epsilon H(0)} \Big( \#\Gamma^+ - \sum_{z\in\Gamma^-} \kappa_z \Big) \\ &= 2\pi e^{\epsilon H(0)} (\#\Gamma^+ - \#\Gamma^-) \leq 2\pi\, \#\,\Gamma^+ \|e^{\epsilon H}\|_{C^0} \leq 2\pi k \|e^{\epsilon H}\|_{C^0}. \end{aligned}$$

By construction, we have that over the region $\Sigma_\pm \setminus \mathcal{N}(\partial\Sigma_\pm)$, the almost complex structure $J$ splits. This implies that the projection

$$\pi_P^\pm : \mathbb{R} \times Y \times \Sigma_\pm \setminus \mathcal{N}(\partial\Sigma_\pm) \to \Sigma_\pm \setminus \mathcal{N}(\partial\Sigma_\pm)$$

is holomorphic. Moreover, this is still true if we extend this region by adding a small collar

$$\{t = \mp\rho \in \left(-\delta, -\delta + \tfrac{1}{9}\delta\right]\},$$

as one can check. Denote by $B_\pm^\delta := \Sigma_\pm \setminus \mathcal{N}(\partial\Sigma_\pm) \cup \{t \in (-\delta, -\delta + \delta/9]\}$.

For each $w \in B_\pm^\delta$, the hypersurface $E_w := \mathbb{R} \times Y \times \{w\}$ is $J$–holomorphic, and $\pi_P^\pm$ has $E_w$ as fiber over $w$. Since the asymptotics of $u$ are away from $B_\pm^\delta$, the intersection of $u$ with any of the $E_w$ is necessarily a finite set of points, since they are restricted to lie in a compact part of the domain of $u$.

Assuming, without loss of generality, that $u$ indeed has a nonempty portion lying over the "plus" region $\Sigma_+ \setminus \mathcal{N}(\partial\Sigma_+)$, by positivity of intersections we have that $u$ necessarily intersects every $E_w$ for $w \in B_\pm^\delta$. By Sard's theorem, we may then find a $t_0 \geq -\delta$ such that $\pi_P^+(u)$ is transverse to the circle $\{t_0\} \times S^1 \subseteq B_+^\delta$ (over all $k$ components of the collar). If we denote $B_+^{\delta;t_0} := B_+^\delta \setminus \{t \in (t_0, -\delta + \delta/9]\}$, we have that $S_+ := (\pi_P^+ \circ u)^{-1}(B_+^{\delta;t_0}) \subseteq \dot{\Sigma}$ is a surface with boundary $\partial S_+ = (\pi_P^+ \circ u)^{-1}(\partial B_+^{\delta;t_0})$. The upshot of the discussion above is that the map

$$F_u := (\pi_P^+ \circ u)|_{S_+} : S_+ \to B_+^{\delta;t_0}$$

is a *holomorphic branched cover*, having as degree the (positive) algebraic intersection number of $u$ with any of the $E_w$ (which is independent of $w \in B_+^{\delta;t_0}$). Call this degree $\deg^+(u) := \deg(F_u)$. We wish to show that $\deg^+(u) = 1$, and so this map will be actually a biholomorphism.

Let us write $\partial S_+ = \bigcup_{i=1}^{l} C_i$, where $C_i$ is a simple closed curve whose image under $F_u$ wraps around one of the circles $\{t_0\} \times S^1$, with winding number $n_i$. By holomorphicity of $F_u$, we have $n_i > 0$. Observe that necessarily one has that $l \geq k$, since $u$ intersects every $E_w$ at least once, and in particular for every $w$ on the $k$ circles $\{t_0\} \times S^1$.

By counting preimages under this projection of a point in each the $k$ circles $\{t_0\} \times S^1$, we obtain that

$$\int_{\partial S_+} u^* \, d\theta = 2\pi \sum_{i=1}^{l} n_i = 2\pi k \deg^+(u). \tag{15}$$

Using the expression $\Omega_\epsilon = K\, d\alpha_\pm + d\lambda_\pm$, equation (15), the fact that $t_0 \geq -\delta$ and $u^* \, d\alpha_+ \geq 0$, and Stokes' theorem, we have the energy estimate

$$\begin{aligned} \boldsymbol{E}(u) &\geq \int_{S_+} u^* \Omega_\epsilon = \int_{S_+} u^*(K\, d\alpha_+ + d\lambda_+) \\ &\geq \int_{S_+} u^* \, d\lambda_+ = \int_{\partial S_+} u^*(e^t \, d\theta) = 2\pi k e^{t_0} \deg^+(u) \geq 2\pi k e^{-\delta} \deg^+(u). \end{aligned} \tag{16}$$

If we combine this with the inequality (14), we obtain

(17) $$2\pi k\|e^{\epsilon H}\|_{C^0} \geq 2\pi e^{\epsilon H(0)}(\#\Gamma^+ - \#\Gamma^-) \geq 2\pi k e^{-\delta}\deg^+(u) \geq 2\pi k e^{-\delta}.$$

Now, since we have the freedom to choose $-\delta$ and $\|\epsilon H\|_{C^0}$ as close to zero as wished, one can easily see that

$$\#\Gamma^+ = k, \quad \#\Gamma^- = 0, \quad \deg^+(u) = 1.$$

This proves that $u$ has no negative ends and precisely $k$ positive ends, and that $F_u$ gives a biholomorphism between $S_+$ and $B_+^{\delta;t_0}$. It also follows from our energy estimates that

$$\int_{S_+} u^*\, d\alpha_+ = 0.$$

Since the integrand is nonnegative, we get that $u^*\, d\alpha_+ = 0$, so $u(S_+)$ lies entirely in the almost complex 4–manifold

$$U := \mathbb{R} \times \gamma \times \Sigma_+ \subseteq \mathbb{R} \times Y \times \Sigma,$$

where $\gamma$ is a (not necessarily closed) $R_+$–Reeb orbit.

Choose now a point $y$ in the projection of $u(S_+)$ to $\gamma$, in this region. We shall prove that the projection of $u(S_+)$ to $\gamma$ consists of just the point $y$, using the Morse–Bott assumption.

Using the Reeb flow along $\gamma$ we have a local coordinate $y = y(s)$, such that $y(0) = y$. Assume by contradiction that there is some $s \neq 0$ such that $y(s)$ belongs to the projection of $u(S_+)$ to $\gamma$, and consider the family of curves $\{u^+_{y(s),a} : a \in \mathbb{R}\}$. By choosing a suitable $a$, we obtain an intersection of $u^+_{y(s),a}$ and $u$, and, by positivity of intersections, we may assume that $y(s)$ is such that $\gamma_{y(s)}$ is not an asymptotic orbit of $u$. Then the total intersection of these curves in $U$ is at least 1.

On the other hand, the set $\mathbb{R} \times (u_M \cap (u^+_{y(s),a})_M)$, where $u_M$ is the projection to $M$ of $u$, must be bounded in the $\mathbb{R}$–direction, since the corresponding Reeb orbits are bounded away from each other. This means that the standard intersection pairing is homotopy invariant (there are no contributions coming from infinity), but a priori the intersection points might move along with a given homotopy. If we choose to homotope by translating in the $\mathbb{R}$–direction, this can only happen if the projection of both curves to $M$ intersect the asymptotics of each other. Since the projection of the curve $u_{y(s'),a'}$ to $M$ does not intersect the asymptotics of $u$, we can homotope the intersections away, a contradiction. This proves the claim that $u$ is constant in $Y$.

We have obtained that the portion of $u$ which lies in the over $\Sigma_+ \setminus \mathcal{N}(\partial\Sigma_+)$ is actually contained in the 3–manifold $M_y := \mathbb{R} \times \{y\} \times (\Sigma_+ \setminus \mathcal{N}(\partial\Sigma_+))$, as is the corresponding portion of the curve $u^+_{y,a}$. But for dimensional reasons, this manifold has a unique 2–dimensional $J$–invariant distribution, given by $TM_y \cap JTM_y$. Therefore the tangent space to $u$ must coincide with the tangent space of some $u^+_{y,a}$ at every point in this region. The unique continuation theorem finally yields $u = u^+_{y,a}$.

This proves the theorem in the Morse–Bott situation. The proof in the Morse case follows from uniqueness in the Morse–Bott one, and Morse–Bott gluing. Indeed, if $u_n$ is a sequence of curves as in the statement, but for the Morse case corresponding to $H_n = f + t_n \gamma g$ with $t_n \to 0$ (as in Section 2.5), the limit configuration $\boldsymbol{u}_\infty$ is a stable holomorphic cascade; here we use the genus bounds that we imposed, in order to have compactness. By uniqueness in the Morse–Bott case, each nonconstant component of this building is a curve of the form $u^\pm_{y,a}$ or a flow-line cylinder $u_\gamma$. A priori, $\boldsymbol{u}_\infty$ may be a nodal curve with constant ghost components attached to curves of the form $u^\pm_{y,a}$; but note that in this case, stability dictates that its arithmetic genus would then have to be strictly positive (and the relative homology class and asymptotics coincide with those of $u^\pm_{y,a}$). If there is no $u^+_{y,a}$ component in $\boldsymbol{u}_\infty$, and no ghost components, all the components of $\boldsymbol{u}_\infty$ are Fredholm regular. It follows from uniqueness of Morse–Bott gluing (for the case of Fredholm regular components; see Remark 2.11 below however) that $u_n$ is of the form $u^-_{y,a}$ or a flow-line cylinder for $n$ large enough. If there is a $u^+_{y,a}$ component, then as this curve is nonregular, uniqueness of gluing might fail, but $u_n$ then has the same relative homology class and asymptotics as a curve of the form $u^+_{y,a}$. □

**Remark 2.11** (Morse–Bott gluing: existence and uniqueness) It has been generally acknowledged that the proof of Morse–Bott gluing (ie compactness, existence and uniqueness) as outlined in [Bourgeois 2003] is incomplete. This gap in the literature has been partially filled in a recent paper by Yuan Yao [2022], for the three-dimensional case, for transversely cut out components, and rigid cascades of height one (see also [Bourgeois and Oancea 2009] in arbitrary dimensions, also for height one cascades, and autonomous Hamiltonians; and the detailed analysis carried out in [Colin et al. 2010], for the three-dimensional case, written in coauthorship with Yuan Yao, which carries out a different pregluing as that of [Bourgeois 2003] with a smaller error term). It seems that we currently do not have the technology to glue together all cascades, as there are issues concerning transversality in the presence of multiple covers. Note that in the above setup gluing of the relevant curves always exists, as the curves $u^\pm_{y,a}$ in the Morse case can be viewed as the gluing of the corresponding curves $u^\pm_{y,a}$ of the Morse–Bott case, where the gluing is simply Morse gluing of their cylindrical ends. However, what is more subtle is *uniqueness* of gluing, which we assumed for the case where all components of the cascade are regular. This assumption on the foundations of gluing analysis seems to be uncontroversial, and we will stick to it. We would like to thank the referee for careful comments concerning this subtle point.

## 2.7 From the SHS to a sufficiently nondegenerate contact structure

For computations in SFT we need nondegenerate Reeb orbits and contact data. Therefore, we need to perturb the SHS $\mathcal{H}_\epsilon = (\Lambda_\epsilon, \Omega_\epsilon)$ to a nearby contact structure.

**Perturbation to contact data** Recall that we have defined an exact symplectic form $\omega_\sigma$ on the double completion, and we denote by $V_\sigma$ its associated Liouville vector field. We also have the "vertical" Liouville vector field $X$ associated to $d\lambda$, defined by expression (3), and the stabilizing vector field $Z$ for the SHS $\mathcal{H}_\epsilon$, defined by (4).

For $s \in [0,1]$, define

$$Z_s = \begin{cases} V_\sigma + ((1-s)\beta(t)+s)X & \text{in the region } E^{\infty,\infty}(t) \text{ where } t \text{ is defined,} \\ (\sigma/(\sigma+\sigma'))\partial_r + sX_+ & \text{in } E^{\infty,\infty}(t)^c \cap \{r > 1-\delta\}, \\ -(\sigma/(\sigma-\sigma'))\partial_r + sX_- & \text{in } E^{\infty,\infty}(t)^c \cap \{r < -1+\delta\}. \end{cases}$$

We have that $Z_1 = X_\sigma$ is the Liouville vector field associated to $\omega_\sigma$ and $Z_0 = Z$. This yields a family of SHSs given by

$$\mathcal{H}_{\epsilon,s} = (\Lambda_{\epsilon,s}, \Omega_\epsilon)_{s\in[0,1]} = (i_{Z_s}\omega_\sigma|_{TM^\epsilon}, \omega_\sigma|_{TM^\epsilon})_{s\in[0,1]}.$$

One can see that $\Lambda_{\epsilon,s}$ is a contact form for $s>0$. By Gray's stability, as long as $\epsilon$ and $s$ are positive and sufficiently close to zero, the isotopy class of $\xi_{\epsilon,s}$ is independent on parameters.

**Holomorphic curves for the contact data** Since we have shown that the genus zero holomorphic curves are regular for the SHS data, the implicit function theorem implies that they will survive a small perturbation to contact data, and will still be regular.

## 2.8 Proof of Theorem 1.2

Once all the technical tools are in place, we will prove Theorem 1.2. We fix the parameters $\epsilon$ and $s$, so that we work in the SFT algebra $\mathcal{A}(\Lambda_{\epsilon,s})[\![\hbar]\!]$ whose homology is $H_*^{\mathrm{SFT}}(M,\xi_{\epsilon,s})$ (which is independent of parameters). We take coefficients in $R_\Omega = \mathbb{R}[H_2(M;\mathbb{R})/\ker\Omega]$, where $\Omega \in \Omega^2(M)$ defines an element in the annihilator $\mathbb{O} := \mathrm{Ann}\big(\bigoplus^k H_1(Y;\mathbb{R}) \otimes H_1(S^1;\mathbb{R})\big)$. Here, we view $\bigsqcup^k Y \times S^1 = \bigsqcup^k Y \times S^1 \times \{0\}$ as sitting in $M_Y \subseteq M$. We will show that $(M,\xi_{\epsilon,s})$ has $\Omega$–twisted $(k-1)$–torsion for every $[\Omega] \in \mathbb{O}$, so that in particular it has $(k-1)$–torsion for the *untwisted* version of the SFT algebra.

Let us recall that for SFT to be defined, we need to introduce an *abstract perturbation* of the Cauchy–Riemann equation. We shall be doing our computation *prior* to introducing this perturbation, and prove by the end the section that this is justified. See the end of this section for more details.

**Computation of algebraic torsion** The index formula (13) implies that curves $u^-_{y,a}$ which asymptote index $2n$ critical points (maxima) at $k-1$ of the $k$ positive ends, and one index 1 critical point at the remaining one, have index 1.

Given $e_1,\dots,e_{k-1}$ maxima and $h$ an index 1 critical point, denote by $\gamma_{e_i}$ and $\gamma_h$ the corresponding nondegenerate Reeb orbits. Consider the moduli space

$$\mathcal{M} = \mathcal{M}_0(W, J_{\epsilon,s}; (\gamma_{e_1},\dots,\gamma_{e_{k-1}},\gamma_h), \varnothing)$$

of $\mathbb{R}$–translation classes of $k$–punctured genus zero $J_{\epsilon,s}$–holomorphic curves in $W = \mathbb{R}\times M$, which have no negative asymptotics, and have $\gamma_{e_1},\dots,\gamma_{e_{k-1}},\gamma_h$ as positive asymptotics. The uniqueness Theorem 2.10 implies that every element in $\mathcal{M}$ is a genus zero curve in our foliation. Since every such curve is regular, after a choice of coherent orientations as in [Bourgeois and Mohnke 2004], this moduli space is an oriented zero-dimensional manifold, which can therefore be counted with appropriate signs.

Now, a choice of coherent orientations for the moduli space of Morse flow lines induces a coherent orientation for the moduli space containing the curves $u^a_\gamma$, and we fix such a choice here onwards. We choose our function

$$H\colon Y \times I \to \mathbb{R}^{\geq 0}$$

as made explicit in Section 2.5. In particular, there are no index zero critical points, and the only index 1 critical point is given by $(\min, 0)$, where min is the unique minimum of $g$. We shall denote by $\mathcal{M}(H; p_-, p_+)$ the moduli space of *positive* unparametrized flow lines $\gamma$ connecting $p_-$ to $p_+$. Assuming the Morse–Smale condition for $H$, we have then that the zero-dimensional moduli spaces correspond to critical points satisfying $\mathrm{ind}_{p_+}(H) = \mathrm{ind}_{p_-}(H) + 1$.

We then fix $e_1, \dots, e_{k-1}, h = (\min, 0)$ as above, and we let $q_{e_i}$ and $q_h$ be the generators in SFT corresponding to the Reeb orbits associated to these critical points. Let

$$Q = q_{e_1} \cdots q_{e_{k-1}} q_h,$$

which is an element of $\mathcal{A}(\Lambda_{\epsilon,s})[\![\hbar]\!]$. In order to compute its differential, we need to count all of the rigid holomorphic curves which asymptote at its positive ends any of the Reeb orbits appearing in $Q$.

**Claim** *The holomorphic curves contributing to the differential of $Q$ are of either the following types*:

- *A holomorphic sphere $u^-_{y,a} \in \mathcal{M}$, which is in fact unique.*
- *A holomorphic cylinder $u_\gamma$ inside $r = 0$, connecting an index $2n-1$ critical point to one of the maxima $e_i$.*
- *Hypothetical nodal curves with strictly positive arithmetic genus, and constant nodal components attached to curves of the form $u^\pm_{y,a}$.*
- *Hypothetical curves of arbitrary large genus.*

Indeed, with Theorem 2.10 in mind, note first that nonnodal curves with the same asymptotics and relative homology class as curves $u^+_{y,a}$ have the same Fredholm index as such curves, which is negative, and hence do not contribute to the differential. If the curve has moreover constant ghost components attached to a curve of the form $u^\pm_{y,a}$, then its stability implies that it has strictly positive arithmetic genus $g' > g(u^\pm_{y,a}) \geq 0$ (these are listed in the third item). For each genus $g_0$, there may exist hypothetical curves of genus higher than $g_0$ not covered in the statement of Theorem 2.10; such curves are listed in the last item.

Using Theorem 2.10, it follows that only the somewhere injective (nonnodal) curves in our foliation are involved in the computation of this differential (see Proposition 2.12 below), with the potential exception of the nodal curves already described, which sometimes can have Fredholm index 1, and curves of large genus. Using that there are no index zero critical points, we see that there are only two possible ways to approach the critical point $(\min, 0)$, entering through the two different boundary components of $Y \times I$, and that there is a unique $\mathbb{R}$–class of the form $u^-_y$ which has $\gamma_h$ as a positive asymptotic (see Figure 6).

Moreover, by observing that the generic behavior is hitting a maxima, by a generic and small perturbation of the Morse function $H$ along different components of the spine, we can arrange that this curve actually defines an element of $\mathcal{M}$. The uniqueness Theorem 2.10 above shows that in fact this is the only element in $\mathcal{M}$. Finally, ruling out the curves coming from the positive side (which have the wrong index and hence are not counted by SFT), we are left only with the curves listed in our claim.

In order to count the curves of type $u_\gamma$, we observe that positive flow lines going from an index $2n-1$ critical point $p$ to an index $2n$ critical point $q$ come in "evil twins" pairs $\gamma \leftrightarrow \bar{\gamma}$: by definition of the Morse index, we have only one positive eigendirection for the Hessian of the Morse function at $p$, and the flow lines approach this point on either side of this direction. Since $H_{2n}(Y \times I) = H_{2n}(Y) = 0$ ($Y$ being a $(2n-1)$–manifold), we only have one generator of the top Morse homology chain group, which is necessarily closed under the Morse differential. Therefore, after choosing a coherent orientation of the moduli spaces of curves, the evil twins cancel each other out, and hence $\#\mathcal{M}(H; p, q) = 0$.

Fix $\Omega$ a closed 2–form such that $[\Omega] \in \mathcal{O}$. For $d \in H_2(M;\mathbb{R})$, we denote by $\bar{d} \in H_2(M;\mathbb{R})/\ker\Omega$ the class it induces, by $u_0$ the unique $\mathbb{R}$–translation class in $\mathcal{M}$, and, for a rigid holomorphic curve $v$, we denote by $\epsilon(v)$ the sign of $v$ assigned by our choice of coherent orientations. In particular, we know that $\epsilon(u_\gamma) = -\epsilon(u_{\bar{\gamma}})$.

Observe that, for the $i^{\text{th}}$ component of $M_Y$, Reeb orbits corresponding to critical points all define the same homology class $[S^1]_i \in H_1(M)$. We take as canonical representative of this class the 1–cycle given by the Reeb orbit $\gamma_{e_i}$ over the unique maxima $e_i$. For every index $2n-1$ critical point $p$ lying in the $i^{\text{th}}$ component, fix $\gamma$ an index 1 flow line joining $p$ to the maxima $e_i$. Choose the spanning surface of $\gamma_p$ to be $F_p := -\gamma \times S^1$, satisfying $\partial F_p = \gamma_p - \gamma_{e_i}$. Then, for this choice of spanning surfaces, the homology class associated to the holomorphic cylinder $u_\gamma$ is $[u_\gamma] = [F_p \cup u_\gamma]$. One thinks of $F_p$ as being attached to $u_\gamma$ at the negative end, corresponding to $\gamma_p$. Let $T_{\gamma,\bar{\gamma}}$ denote the 2–torus $\overline{\gamma \cup \bar{\gamma}} \times S^1$. Observe that

$$[u_\gamma] - [u_{\bar{\gamma}}] = [T_{\gamma,\bar{\gamma}}] \in H_1(Y) \otimes H_1(S^1) \subseteq \ker\Omega.$$

Therefore, $\overline{[u_\gamma]} = \overline{[u_{\bar{\gamma}}]}$.

According to Proposition 2.12 below, the image of $Q$ under the differential

$$\boldsymbol{D}_{\epsilon,s} \colon \mathcal{A}(\Lambda_{\epsilon,s})[\![\hbar]\!] \to \mathcal{A}(\Lambda_{\epsilon,s})[\![\hbar]\!]$$

is then given by

$$\begin{aligned}
(18)\quad \boldsymbol{D}_{\epsilon,s}(Q) &= \epsilon(u_0) z^{\overline{[u_0]}} \hbar^{k-1} + \sum_{\substack{i=1,\dots,k-1 \\ \mathrm{ind}_p(H)=2n-1}} \sum_{\gamma \in \mathcal{M}(H;p,e_i)} \epsilon(u_\gamma) z^{\overline{[u_\gamma]}} q_p \frac{\partial Q}{\partial q_{e_i}} + \hbar^k P \\
&= \epsilon(u_0) z^{\overline{[u_0]}} \hbar^{k-1} + \frac{1}{2} \sum_{\substack{i=1,\dots,k-1 \\ \mathrm{ind}_p(H)=2n-1}} \sum_{\gamma \in \mathcal{M}(H;p,e_i)} +(\epsilon(u_\gamma) + \epsilon(u_{\bar{\gamma}})) z^{\overline{[u_\gamma]}} q_p \frac{\partial Q}{\partial q_{e_i}} + \hbar^k P \\
&= \hbar^{k-1}(\epsilon(u_0) z^{\overline{[u_0]}} + \hbar P).
\end{aligned}$$

Here, $P$ contains the potential contributions of arbitrary large genus curves, and the nodal curves of genus $g' > 0$, which satisfy $|\Gamma^+| + g' = k + g' > k$, where $\Gamma^+ = \{q_{e_1}, \dots, q_{e_{k-1}}, q_h\}$ is the set of their positive asymptotics. Noting that $\epsilon(u_0) z^{\overline{[u_0]}} + \hbar P$ is invertible as a power series in $\hbar$, we conclude that our model has $(k-1)$–torsion with coefficients in $R_\Omega$. We have used that all orbits are simply covered, so no combinatorial factors appear, and we need not worry about asymptotic markers.

**Why the computation works** We now justify the computation above. Let us recall first the fact that the abstract polyfold machinery for SFT requires the introduction of an abstract perturbation to the Cauchy–Riemann equation making every holomorphic curve of positive index regular. The basic facts about this perturbation scheme, which comes from the polyfold theory of Hofer–Wysocki–Zehnder, are that:

- Every Fredholm regular index 1 holomorphic curve gives rise to a unique solution to the perturbed problem, if the perturbation is sufficiently small.
- If solutions to the perturbed problem with given asymptotic behavior exist for small perturbations, then as the perturbation is switched off they give rise to a subsequence of curves which converge to a holomorphic building with the same asymptotic behavior.

Therefore, the index 1 curves in our foliation survive and are counted, but we need to make sure there are no extra curves which need to be taken into the count. In what follows, $J$ will denote the original $J_{\epsilon,0}$ we constructed in the Morse case.

**Proposition 2.12** *The space of connected $J$–holomorphic stable buildings of index 1 which may become curves contributing to the differential of $Q = q_{e_1} \cdots q_{e_{k-1}} q_h$ after introducing an abstract perturbation, or after perturbing the SHS to a sufficiently nearby contact structure, actually consist exactly of either an index 1 curve $u_\gamma$ for some positive flow line $\gamma$, a curve $u^-_{y,a}$ for some $y \in Y$ and $a \in \mathbb{R}$, potential nodal curves of strict positive genus with constant ghost components attached to curves of the form $u^\pm_{y,a}$, or curves with arbitrary large genus.*

**Proof** It follows by inductively applying Theorem 2.10 [Moreno 2018]. □

# 3 Giroux torsion implies algebraic 1–torsion in higher dimensions

In this section, we address [Massot et al. 2013, Conjecture 4.14].

**Definition 3.1** (Giroux) Let $\Sigma$ be a compact $2n$–manifold with boundary, $\omega$ a symplectic form on the interior $\mathring{\Sigma}$ of $\Sigma$, and $\xi_0$ a contact structure on $\partial\Sigma$. The triple $(\Sigma, \omega, \xi_0)$ is an *ideal Liouville domain* if there exists an auxiliary 1–form $\beta$ on $\mathring{\Sigma}$ such that

- $d\beta = \omega$ on $\mathring{\Sigma}$;

- for any smooth function $f\colon \Sigma \to [0,\infty)$ with regular level set $\partial\Sigma = f^{-1}(0)$, the 1–form $\beta_0 = f\beta$ extends smoothly to $\partial\Sigma$ such that its restriction to $\partial\Sigma$ is a contact form for $\xi_0$.

The 1–form $\beta$ is called a *Liouville form* for $(\Sigma,\omega,\xi_0)$.

In the terminology of [Massot et al. 2013], we say that an oriented hypersurface $H$ in a contact manifold $(M,\xi)$ is a *$\xi$–round hypersurface* modeled on some closed contact manifold $(Y,\xi_0)$ if it is transverse to $\xi$ and admits an orientation-preserving identification with $S^1\times Y$ such that $\xi\cap TH = TS^1\oplus\xi_0$.

Given an ideal Liouville domain $(\Sigma,\omega,\xi_0)$, the *Giroux domain* associated to it is the contact manifold $\Sigma\times S^1$ endowed with the contact structure $\xi = \ker(f(\beta+d\theta))$, where $f$ and $\beta$ are as before, and $\theta$ is the $S^1$–coordinate. Away from $V(f)=\partial\Sigma$, the vanishing locus of $f$, this contact structure coincides with the *contactization* $\ker(\beta+d\theta)$. Over $V(f)$ it is just given by $\xi_0\oplus TS^1$, so that $V(f)\times S^1$ is a $\xi_0$–round hypersurface modeled on $(\partial\Sigma,\xi_0)$.

One may find a collar neighborhood of the form $[0,1)\times H\subseteq \Sigma\times S^1$, on which $\xi$ is given by the kernel of a contact form $\beta_0 + s\,d\theta$, where $s$ is the coordinate on the interval, $H\subseteq\partial M$ corresponds to $s=0$, $\theta$ is the coordinate in $S^1$, and $\beta_0$ a contact form for $\xi_0$ [Massot et al. 2013, Lemma 4.1]. Using these collar neighborhoods, one has a well-defined notion of gluing of two Giroux domains along boundary components modeled on isomorphic contact manifolds (see Section 3.1 below).

We also have a blow-down operation for round hypersurfaces lying in the boundary. If $H$ is a $\xi$–round boundary component of $(M,\xi)$, with orientation opposite the boundary orientation, consider the collar neighborhood $[0,1)\times H$ as before. Let $\mathbb{D}$ be the disk of radius $\sqrt{\epsilon}$ in $\mathbb{R}^2$. The map $\Psi\colon (re^{i\theta},y)\mapsto (r^2,\theta,y)$ is a diffeomorphism from $(\mathbb{D}\setminus\{0\})\times Y$ to $(0,\epsilon)\times S^1\times Y$ which pulls back $\beta_0+s\,dt$ to the contact form $\beta_0+r^2\,d\theta$. Thus we can glue $\mathbb{D}\times Y$ to $M\setminus H$ to get a new contact manifold in which $H$ has been replaced by $Y$, and the $S^1$–component of $H$ has been capped off.

Given a contact embedding of the interior of a Giroux domain $G_\Sigma := \Sigma\times S^1$ inside a contact manifold $(M,\xi)$, we shall denote by $\mathbb{O}(G_\Sigma)\subseteq H^2(M;\mathbb{R})$ the annihilator of $H_1(\Sigma;\mathbb{R})\otimes H_1(S^1;\mathbb{R})$, when the latter is viewed as a subspace of $H_2(M;\mathbb{R})$. If $N\subseteq (M,\xi)$ is a subdomain resulting from gluing together a collection of Giroux domains $G_{\Sigma_1},\dots,G_{\Sigma_k}$, we shall denote

$$\mathbb{O}(N) = \mathbb{O}(G_{\Sigma_1})\cap\cdots\cap\mathbb{O}(G_{\Sigma_k})\subseteq H_2(M;\mathbb{R}).$$

The following theorem implies Theorem 1.5 (see Example 3.3 below):

**Theorem 3.2** *If a contact manifold $(M^{2n+1},\xi)$ admits a contact embedding of a subdomain $N$ obtained by gluing two Giroux domains $G_{\Sigma_-}$ and $G_{\Sigma_+}$, such that $\Sigma_+$ has a boundary component not touching $\Sigma_-$, then $(M^{2n+1},\xi)$ has $\Omega$–twisted algebraic $1$–torsion, for every $\Omega\in\mathbb{O}(N)$. Moreover, it is also algebraically overtwisted if $N$ contains any blown down boundary components.*

The motivating example of an explicit model of such subdomain is the following:

**Example 3.3** [Massot et al. 2013] Consider $(Y, \alpha_+, \alpha_-)$ a Liouville pair on a closed manifold $Y^{2n-1}$. As in the introduction, consider the *Giroux* $2\pi$*–torsion domain* modeled on $(Y, \alpha_+, \alpha_-)$, given by the contact manifold $(\mathrm{GT}, \xi_{\mathrm{GT}}) := (Y \times [0, 2\pi] \times S^1, \ker \lambda_{\mathrm{GT}})$, where

$$\lambda_{\mathrm{GT}} = \tfrac{1}{2}(1+\cos(r))\alpha_+ + \tfrac{1}{2}(1-\cos(r))\alpha_- + \sin(r)\, d\theta \tag{19}$$

and the coordinates are $(r, \theta) \in [0, 2\pi] \times S^1$.

We may write $\lambda_{\mathrm{GT}} = f(\alpha + d\theta)$, where

$$f = f(r) = \sin(r), \quad \alpha = \tfrac{1}{2}(e^{u(r)}\alpha_+ + e^{-u(r)}\alpha_-), \quad u(r) = \log \frac{1+\cos(r)}{\sin(r)}.$$

Here, $u\colon (0,\pi) \to \mathbb{R}$ is an orientation-reversing diffeomorphism, whereas $u\colon (\pi, 2\pi) \to \mathbb{R}$ is an orientation-preserving one, which is used to pull-back the Liouville form $\frac{1}{2}(e^u\alpha + + e^{-u}\alpha_+)$ defined on $Y \times \mathbb{R}$ via a map of the form $v = \mathrm{id} \times u$.

This means that we may view $(\mathrm{GT}, \xi_{\mathrm{GT}})$ as being obtained by gluing two Giroux domains of the form $\mathrm{GT}_- = Y \times [0,\pi] \times S^1$ and $\mathrm{GT}_+ = Y \times [\pi, 2\pi] \times S^1$, along their common boundary, an $\xi_{\mathrm{GT}}$–round hypersurface modeled on $(Y, \alpha_-)$.

Therefore, from Theorem 3.2 we obtain Theorem 1.5 as a corollary.

## 3.1 Giroux SOBDs

We consider a specially simple kind of spinal open book decompositions (SOBDs), which arise on manifolds which have been obtained by gluing a family of Giroux domains along a collection of common boundary components, each a round hypersurface modeled on some contact manifold. Such is the case of the Giroux $2\pi$–torsion domains GT. Basically, these SOBDs are obtained by declaring suitable collar neighborhoods of each gluing hypersurface to be *paper* components, whereas the *spine* components are the complement of these neighborhoods.

**Construction of the SOBD** Let $(G_\pm = \Sigma_\pm \times S^1, \xi_\pm = \ker(f_\pm(\beta_\pm + d\theta)))$ be two Giroux domains which one wishes to glue along a round-hypersurface $H = Y \times S^1$ modeled on some contact manifold $(Y, \xi_0)$ and lying in their common boundaries. Fix choices of collar neighborhoods $\mathcal{N}_\pm(H)$ of $H$ inside $G_\pm$, of the form $Y \times S^1 \times [0,1]$. Take coordinates $s_\pm \in [0,1]$ such that $H = \{s_\pm = 0\}$, and $\theta \in S^1$ such that the contact structures $\xi_\pm$ are given by the kernel of the contact forms $\beta_0 + s_\pm\, d\theta$. Here, $\beta_0$ is a contact form for $\xi_0$. In these coordinates, $\beta_\pm = \beta_0/s_\pm$ and $f = f(s_\pm) = s_\pm$, and the corresponding *ideal* Liouville vector fields are $V_\pm = -s_\pm \partial_{s_\pm}$. We glue $\mathcal{N}_\pm(H)$ together in the natural way, by taking a coordinate $s \in [-1,1]$ such that $s = -s_-$, $s = s_+$, and $s = 0$ corresponds to $H$. We thus obtain a collar neighborhood $\mathcal{N}(H) = \mathcal{N}_+(H) \bigcup_\Phi \mathcal{N}_-(H) \simeq H \times [-1,1]$, where we denote by $\Phi$ the resulting gluing map. Doing this for each of the boundary components that we glue together, we obtain a decomposition for

$$M := G_+ \bigcup_\Phi G_- = \Bigl(\Sigma_+ \bigcup_\Phi \Sigma_-\Bigr) \times S^1,$$

given by $M = M_P \cup M_\Sigma$, where $M_P$ (the *paper*) is the disjoint union of the collar neighborhoods of the form $\mathcal{N}(H)$ for each of the gluing round-hypersurfaces $H$, and $M_\Sigma$ (the *spine*) is the closure of its complement in $M$. We have a natural fibration structure $\pi_P$ on $M_P$, which is the trivial fibration over the disjoint union of the hypersurfaces $H$, such that the pages (the fibers of $\pi_P$) are identified with the annuli $P := S^1 \times [-1, 1]$. We also have an $S^1$–fibration $\pi_\Sigma$ on $M_\Sigma$, which is also trivial. It has as base the disjoint union of $\Sigma_+$ and $\Sigma_-$ minus the collar neighborhoods, which we denote by $\Sigma$. Let us assign a *sign* $\epsilon(G_\pm) := \pm 1$ to each of the Giroux subdomains $G_\pm \subseteq M$.

We can also blow-down boundary components in the Giroux subdomains, and in this case we may extend our SOBD by declaring the $\mathbb{D} \times Y$ we glued in the blow-down operation to be part of the paper $M_P$ (such that its pages are disks). We declare the blown-down components to be part of the paper. We will also fix collar neighborhoods of the components of $\partial M$, which correspond to the nonglued hypersurfaces, in the very same way as we did before for the glued ones. Their components look like $\mathcal{N}_\Sigma(H) := H \times [0,1] = Y \times S^1 \times [0,1]$ for some nonglued hypersurface $H$. There is a coordinate $s \in [0,1]$ such that $\partial M = \{s = 0\}$, and such that the contact structure on the corresponding Giroux domain is given by $\ker(\beta_0 + s\, d\theta)$ for some contact form $\beta_0$ for $(Y, \xi_0)$. Denote by $\mathcal{N}_\Sigma$ the disjoint union, over all unglued $H$'s, of all of the $\mathcal{N}_\Sigma(H)$'s.

We shall refer to the SOBDs obtained by the procedure described above as *Giroux* SOBDs, where are also allowing the number of *Giroux subdomains* involved (which is the same as the number of spine components) to be arbitrary.

**The Giroux form** If we denote by $\lambda_\pm = f_\pm(\beta_\pm + d\theta)$, we can extend the expression $\lambda_+ = \beta_0 + s\, d\theta$, a priori valid on $\mathcal{N}_+(H)$, to $\mathcal{N}(H)$, by the same formula. Observe that, by choice of our coordinates, the resulting 1–form glues smoothly to $\bar{\lambda}_- := f_-(\beta_- - d\theta)$. Therefore, we may still think of the contact structure $\bar{\xi}_- = \ker \bar{\lambda}_-$ as a *contactization* contact structure over the region $(\Sigma_- \times S^1) \setminus \mathcal{N}_-(H)$, with the caveat that we need to switch the orientation in the $S^1$–direction.

Denote the resulting contact form by $\lambda := \lambda_+ \cup_\Phi \bar{\lambda}_-$. We may globally write it as $\lambda = f(\beta + d\theta)$. Here, $f$ is a function which is either strictly positive or strictly negative over the interior of the Giroux domains, and vanishes precisely at the (glued and unglued) boundary components. The 1–form $\beta$ coincides with the Liouville forms $\pm\beta_\pm$ where these are defined, and is undefined along said boundary components. In the case of blown-down components, the contact form $\lambda$ also extends in a natural way.

From this construction, in $M$, each of the subdomains $G \subset M$ used in the gluing procedure carries a *sign* $\epsilon(G)$, which we define as the sign of the function $f|_{\mathring{G}}$. Observe that $f$ coincides with $\epsilon(G)$ along every $H \times \{s_\pm = 1\}$, the inner boundary components of the collar neighborhoods. Therefore, we may isotope it relative every collar neighborhood to a smooth function which is constant equal to $\epsilon(G)$ along $G \setminus ((M_P^\delta \cap G) \cup \mathcal{N}_\partial^\delta)$ (see Figure 7). Here, $M_P^\delta$ and $\mathcal{N}_\partial^\delta$ denote small $\delta$–extensions in the interval direction of $M_P$ and $\mathcal{N}_\Sigma$, respectively, so that now $|s| \in [0, 1+\delta]$. The regions $M_P^\delta \setminus M_P$ play the role of smoothened corners. Observe that isotoping $f$ as we just did does not change the isotopy class of $\lambda$,

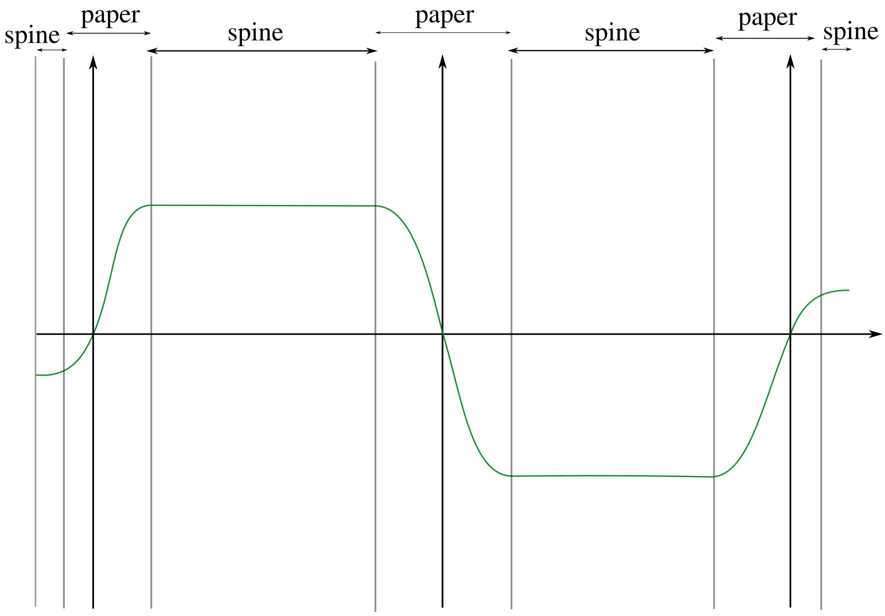


Figure 7: The isotoped function $f$ in the extended Giroux torsion domain $\mathrm{GT}^\epsilon$.

by Gray stability (version with boundary), and has the effect of transforming the Reeb vector field of $\lambda$ into $\epsilon(G)\partial_\theta$ along $M_Y$. Observe also that along the paper, we have $\lambda = \beta_0 + \gamma$, where $\gamma = s\,d\theta$ is a Liouville form for $S^1 \times [-1,1]$. Since this manifold is trivially a cylindrical Liouville semifilling, we may view $\lambda$ as a Giroux form, as we did with contact form constructed in Section 2.1.

**$\epsilon$–extension** It will be convenient to consider an $\epsilon$–extension of our SOBD, which we call $M^\epsilon$, by gluing small collars to the boundary. To each component $\mathcal{N}_\Sigma(H)$ for $H$ lying in $\partial G$ for a subdomain $G$ of $M$, we glue a collar neighborhood of the form $H \times [-\epsilon, 0]$ for some small $\epsilon > 0$. We extend our function $f$ so that the extended version coincides with $\epsilon(G)\mathrm{id}$ near $s = 0$, $f'(-\epsilon) = 0$, and $\operatorname{sgn}(f'(s)) = \epsilon(G) \neq 0$ for $s > -\epsilon$ (see Figure 7). We will define the paper $M_P^\epsilon$ to be the union of $M_P$ and the region $H \times [-\epsilon/2, 1+\delta]$, and the spine $M_\Sigma^\epsilon$ to be the union of $M_\Sigma$ with $H \times [-\epsilon, -\epsilon/2]$. The point of this extension is that now the Reeb vector field of the extended $\lambda$ coincides with $-\epsilon(G)\partial_\theta$ along the boundary.

**Remark 3.4** Because of the above discussion on orientations, from which we gathered that the $S^1$–orientation that we need depends on the sign of the Giroux domain, we rule out the case where we have a sequence $(G_0, \dots, G_{N-1})$ of Giroux subdomains of $M$, where $G_i$ has been glued to $G_{i+1}$ (modulo $N$) along some collection of boundary components, and $N > 1$ is odd. This condition is to be taken as part of the definition of a Giroux SOBD.

## 3.2 Proof of Theorem 3.2

The proof of this theorem is a reinterpretation of what we did in the construction of our contact manifold models of Section 2.1.

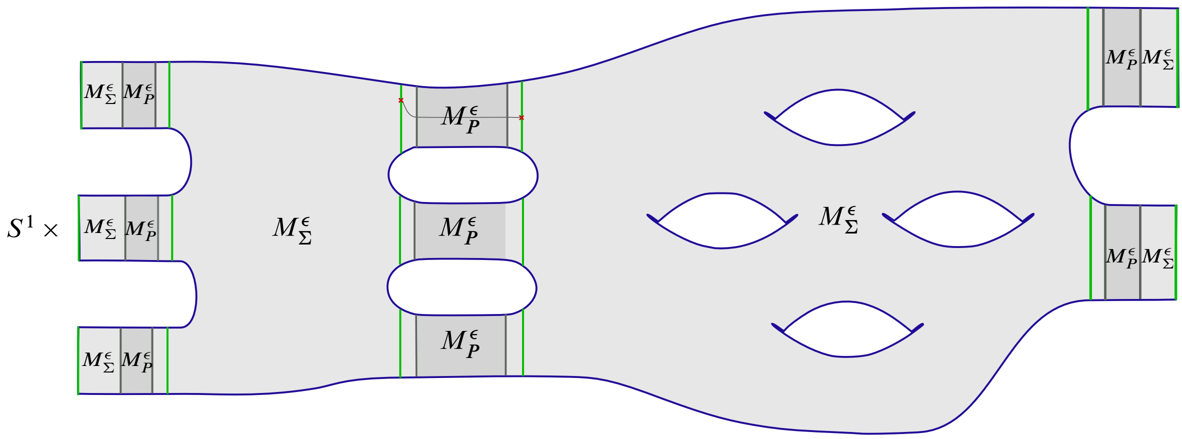


Figure 8: An $\epsilon$–extension of a domain $N$ consisting of two Giroux domains glued along three boundary components. We draw the Morse–Bott submanifolds of $H$ in green. We also draw an index 1 holomorphic cylinder (of the first kind).

**Proof of Theorem 3.2** Let $N$ be a subdomain as in the hypothesis, carrying a Giroux form $\lambda = f(\beta + d\theta)$ obtained by gluing. Since the contact embedding condition is open, and we are assuming that there are boundary components of $\Sigma_+$ not touching $\Sigma_-$, we can find a small $\epsilon > 0$ such that $M$ admits a contact embedding of the $\epsilon$–extension $N^\epsilon$. Endow this extension with a Giroux SOBD $N^\epsilon = M_P^\epsilon \cup M_\Sigma^\epsilon$ as in the previous section. On this decomposition, add corners where spine and paper glue together as we explained before. Also choose a small Morse/Morse–Bott function $H$ in $M_\Sigma^\epsilon$, which lies in the isotopy class of $f$, and vanishes as we get close to the paper. With this data, we then may construct a Morse/Morse–Bott contact form $\Lambda$ on $N$ which lies in the isotopy class of $\lambda$, along with an SHS deforming it, in the analogous way as done in Section 2.1.

Since in our situation we are allowing $\Sigma_\pm$ to be more general than a semifilling $Y \times I$, we need to specify what we mean by the Morse–Bott situation. We will take our Morse–Bott function $H$ so that it depends only on the interval parameter along the collar neighborhoods $\mathcal{N}_\Sigma$ close to the boundary and along a slightly bigger copy of $M_P^\epsilon$, and matches the function $f$ close to $\partial N$. In particular, $\partial N$ is a Morse–Bott submanifold. We also impose that $H$ is Morse in the interior of the components of $M_\Sigma$ which are away from the boundary. For simplicity, we will assume that, besides the boundary, $H$ only has exactly one Morse–Bott submanifold close to each boundary component of $M_\Sigma$ which is glued to a paper component, of the form $Y \times \{t\}$ for some $t$ (see Figure 8). The Morse situation is then obtained by a perturbation of this situation obtained by choosing Morse functions along the Morse–Bott submanifolds.

We have an almost complex structure $J'$ compatible with the SHS, for which we get a stable finite-energy foliation by holomorphic cylinders, which come in two types: either they are obtained by gluing constant lifts of the cylindrical pages and flow-line cylinders over $M_\Sigma^\epsilon$ along the corners (first kind); or they are flow-line cylinders completely contained in $M_\Sigma^\epsilon$ (second kind). Both kinds have as asymptotics simply covered Reeb orbits $\gamma_p$ corresponding to critical points $p$ of $H$, either along the boundary, or at interior points of $\Sigma$ (see Figure 9). These cylinders have two positive ends if its corresponding critical points lie in Giroux subdomains with different sign, and one positive and one negative end if these signs agree. The Fredholm index formula is exactly as before.

Cylinders of both kinds are Fredholm regular (see [Moreno 2018]). We also have a version of the uniqueness Theorem 2.10, adapted to this situation.

Make all the necessary choices to have an SFT differential

$$\boldsymbol{D}_{\mathrm{SFT}}\colon \mathcal{A}(\Lambda)[\![\hbar]\!] \to \mathcal{A}(\Lambda)[\![\hbar]\!],$$

which computes $H_*^{\mathrm{SFT}}(N, \ker\Lambda; \Omega|_N)$, where $\Omega \in \mathcal{O}(N)$. Extend these choices to $M$ so as to be able to compute $H_*^{\mathrm{SFT}}(M, \xi; \Omega)$.

Let $e$ be a maximum (index $2n$) of $H$ in $M_\Sigma \cap (\Sigma_- \times S^1)$, and let $h$ be an index 1 critical point of $H$ in $M_\Sigma \cap (\Sigma_+ \times S^1)$. We take both to lie in the Morse–Bott manifolds of the Morse–Bott case, before the Morse perturbation. Denote by $q_e$ and $q_h$ the corresponding SFT generators. Define

$$Q = q_h q_e,$$

an element of $\mathcal{A}(\Lambda)$. There is a unique (perturbed) cylinder $u$ of the first kind which has $\gamma_e$ and $\gamma_h$ as positive asymptotics, with $\mathrm{ind}(u) = 1$. If we choose $H$ so that it does not have any minimum, by uniqueness any other holomorphic curve over $M$ which may contribute to the differential of $Q$ is a flow-line cylinder completely contained in $M_\Sigma$, connecting an index $2n-1$ critical point $p$ with $e$, or potential nodal curves of strictly positive arithmetic genus.

Denote by $\bar{d}$ the element in $H_2(M;\mathbb{R})/\ker\Omega$ defined by any $d \in H_2(M;\mathbb{R})$, by $\mathcal{M}(H; p, e)$ the space of positive flow-lines connecting $p$ with $e$, $u_\gamma$ the flow-line cylinder corresponding to a flow-line $\gamma$, and $\epsilon(u)$ the sign of the holomorphic curve $u$ assigned by a choice of coherent orientations. Since $H_{2n}(\Sigma_\pm) = 0$, elements of $\mathcal{M}(H; p, e)$ come in evil twins pairs $\gamma \leftrightarrow \bar{\gamma}$ such that $\epsilon(u_\gamma) = -\epsilon(u_{\bar{\gamma}})$. As in Section 2.8, one can choose suitable spanning surfaces such that $\overline{[u_\gamma]} = \overline{[u_{\bar{\gamma}}]}$.

Then,

$$\begin{aligned}
\boldsymbol{D}_{\mathrm{SFT}} Q &= z^{\overline{[u]}}\hbar + \sum_{\substack{\gamma \in \mathcal{M}(H;p,e)\\ \mathrm{ind}_p(H) = 2n-1}} \epsilon(u_\gamma) z^{\overline{[u_\gamma]}} q_p q_h + \hbar^2 P \\
&= z^{\overline{[u]}}\hbar + \frac{1}{2} \sum_{\substack{\gamma \in \mathcal{M}(H;p,e)\\ \mathrm{ind}_p(H) = 2n-1}} (\epsilon(u_\gamma) + \epsilon(u_{\bar{\gamma}})) z^{\overline{[u_\gamma]}} q_p q_h + \hbar^2 P = \hbar(z^{\overline{[u]}} + \hbar P),
\end{aligned} \tag{20}$$

where $P$ contains the hypothetical contributions of nodal curves and hypothetical high genus curves. Inverting $z^{\overline{[u]}} + \hbar P$, this proves that $(N, \ker\Lambda)$ has $\Omega|_N$–twisted 1–torsion. This implies that $(M, \xi)$ has $\Omega$–twisted 1–torsion, since our uniqueness theorem gives that there are no holomorphic curves

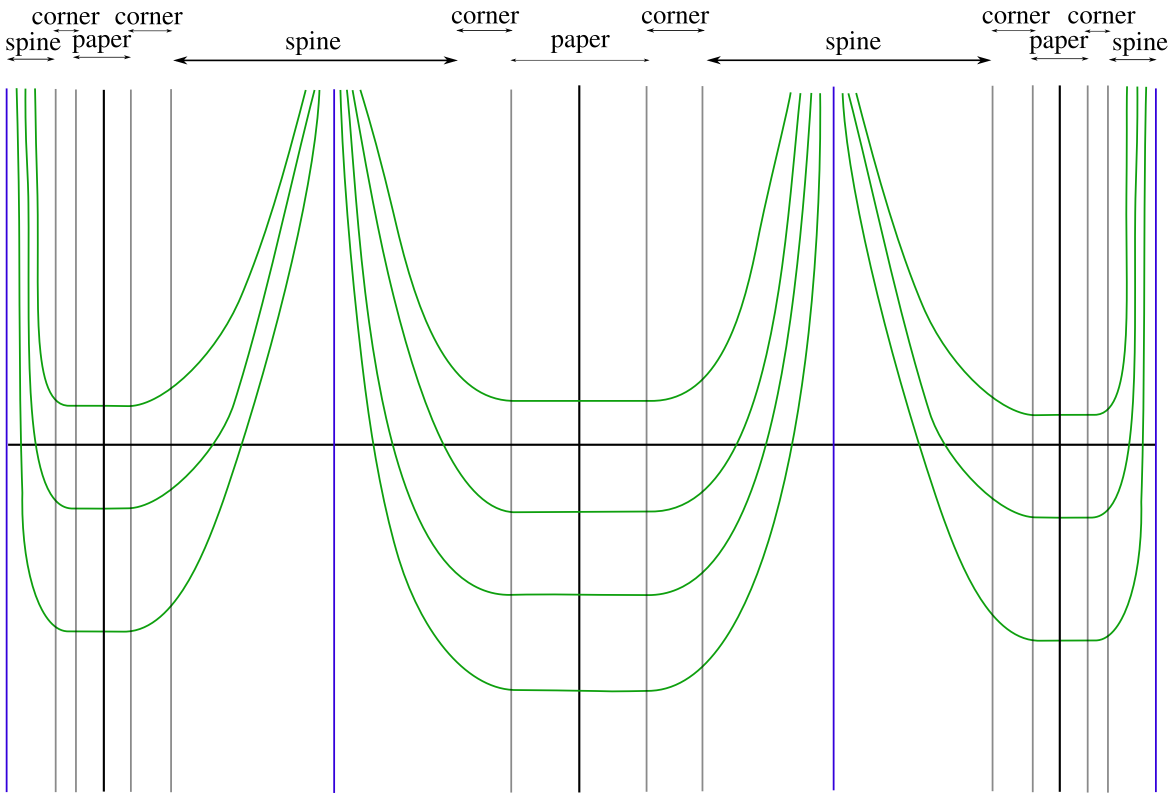


Figure 9: The foliation by holomorphic cylinders (with respect to SHS data) of the symplectization of $\mathrm{GT}^\epsilon$.

with asymptotics in $N$ which venture into $M \setminus N$, and therefore $H_*^{\mathrm{SFT}}(N, \ker \Lambda; \Omega|_N)$ embeds into $H_*^{\mathrm{SFT}}(M, \xi; \Omega)$. Here, we use that $\ker \Lambda$ is isotopic to $\ker \lambda = \xi|_N$.

For the second statement, assume that we have a blown-down boundary component, so that the corresponding $\mathbb{D} \times Y$ is a paper component with disk pages. As in Section 2.3, we have that the disk pages lift as finite-energy holomorphic planes with a single positive asymptotic $\gamma_p$, corresponding to a critical point $p$ in $M_\Sigma$. If $u$ is such a plane, its index is $\mathrm{ind}(u) = \mathrm{ind}_p(H)$. Take $p$ so that $\mathrm{ind}_p(H) = 1$, and let $P = q_{\gamma_p}$. Since there is no minima for $H$, by our uniqueness theorem we know that

$$\boldsymbol{D}_{\mathrm{SFT}}(z^{-\overline{[u]}} P) = 1 + \hbar U$$

for some $U$, and this expression is invertible. Since the choice of coefficients is arbitrary, $(N, \ker \Lambda)$ is algebraically overtwisted, which implies that $(M, \xi)$ is. $\square$

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*Institute for Advanced Study*
*Princeton, NJ, United States*

agustin.moreno2191@gmail.com

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