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Locally equivalent Floer complexes and unoriented link cobordisms

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We show that the local equivalence class of the collapsed link Floer complex $cCFL^\infty(L)$, together with many Υ–type invariants extracted from this group, is a concordance invariant of links. In particular, we define a version of the invariants $\Upsilon_L(t)$ and $\nu^+(L)$ when L is a link and we prove that they give a lower bound for the slice genus $g_4(L)$.

Furthermore, in the last section of the paper we study the homology group $HFL'(L)$ and its behavior under unoriented cobordisms. We obtain that a normalized version of the υ–set, introduced by Ozsváth, Stipsicz and Szabó, produces a lower bound for the 4–dimensional smooth crosscap number $\gamma_4(L)$.

57K10, 57K18

1 Introduction

Hom [2017] introduced an equivalence relation on the knot Floer complex $CFK^\infty(K)$ called stable equivalence. Namely, we say that two knots are *stably equivalent* if and only if their chain complexes become filtered chain homotopy equivalent after adding some acyclic complexes. A very important result of [Hom 2017] is that if K_1 is concordant to K_2 then the complex $CFK^\infty(K_1)$ is stably equivalent to $CFK^\infty(K_2)$, which made it possible to prove that many knot invariants coming from $CFK^\infty(K)$ are indeed concordance invariants; see [Alfieri 2019; Allen 2020; Hom and Wu 2016; Kim and Livingston 2018] for some examples.

Another relation on knot Floer chain complexes was given by Zemke [2019a]: two knots K_1 and K_2 are called *locally equivalent* if there exist two maps

$$f\colon CFK^\infty(K_1)\to CFK^\infty(K_2) \quad\text{and}\quad g\colon CFK^\infty(K_2)\to CFK^\infty(K_1)$$

which preserve the filtrations (both the Alexander and algebraic filtration) and induce filtered isomorphisms in homology. Even though those two relations appear to be very different from their definition, we can actually show that they coincide. We recall that this theorem was proved in the involutive setting by Hendricks and Hom [2019].

Theorem 1.1 *Let K_1 and K_2 be two knots in S^3. Then $CFK^\infty(K_1)$ is locally equivalent to $CFK^\infty(K_2)$ if and only if they are stably equivalent.*

For the purpose of this paper, the local equivalence relation has the advantage that it can be used in the same way for links. Let us consider the chain complex $cCFL^\infty(L)$, defined from $CFL^-(L)$ by collapsing the variables $U_1, \dots, U_n$ to U and taking the tensor product with $\mathbb{F}[U, U^{-1}]$, where here $\mathbb{F}$ always denotes the field with two elements; see [Ozsváth and Szabó 2008; Ozsváth et al. 2015]. We equip $cCFL^{\infty}(L)$ with a filtration $\mathscr{F}$, obtained from the algebraic filtration and the (collapsed) Alexander filtration; such an $\mathscr{F}$ descends to homology, so we can define the filtered group $\mathscr{F}c\mathcal{HFL}^{\infty}(L)$. Based on an intuition of Alfieri [2019], we consider $\mathscr{F}$ as indexed by some particular subsets S of the plane, which he calls southwest regions, satisfying the property that if $(\bar{x}, \bar{y}) \in S$ then each (x, y) such that $x \leqslant \bar{x}$ and $y \leqslant \bar{y}$ is in S; a more precise definition is given later in Section 2.2.

We recall that two n–component links are *concordant* if there is a cobordism between them consisting of n disjoint annuli. Then the local equivalence class of $cCFL^{\infty}(L)$ and the filtered homology group $c\mathcal{HFL}^{\infty}(L)$ are a concordance invariant in the following sense.

Theorem 1.2 *Suppose that L_1 is concordant to L_2. Then there are chain maps*

$$cCFL^{\infty}(L_1) \rightleftarrows cCFL^{\infty}(L_2)$$

which preserve $\mathscr{F}$ and induce an $\mathscr{F}$–filtered isomorphism in homology. In particular, the restrictions of such isomorphisms give identifications

$$\mathscr{F}^S c\mathcal{HFL}^{\infty}_d(L_1) \cong_{\mathbb{F}} \mathscr{F}^S c\mathcal{HFL}^{\infty}_d(L_2)$$

for every $d \in \mathbb{Z}$ and southwest region S of $\mathbb{Z}^2$.

The strategy of the proof of this result consists in decomposing a concordance into standard pieces and then a careful usage of the maps introduced by Sarkar [2011] on grid diagrams. In fact, starting from Sarkar's work, we can construct maps induced by some specific cobordisms. Some of these maps were already used by the author in [Cavallo 2018].

Remark 1.3 Zemke [2019c], using different techniques, also defined maps induced by (decorated) link cobordisms, which conjecturally coincide with the ones presented in this paper. We can use such maps to give another proof of Theorem 1.2: namely, according to [Zemke 2019c, Theorems A and C] every link concordance induces a graded isomorphism in link Floer homology; while the fact that the $\mathscr{F}$–filtration is preserved follows from [Zemke 2019b, Theorem 1.4]. This argument is similar to the one in [Zemke 2019a], which proved a version of Theorem 1.2 for knots.

Theorem 1.2 allows us to define some numerical concordance invariants for links; including a generalization of Alfieri's Υ_S [2019], the ν^+–invariant of Hom and Wu [2016] (see also [Rasmussen 2003]) and the secondary upsilons, defined by Kim and Livingston [2018]. We briefly describe how to extract some of these invariants.

Write $cCFL^{\infty}_*(L)$ for the filtered chain homotopy type of the link Floer complex of L. Once we fix a filtered basis, we can represent such a model complex in the plane by (j, A), where j and A represent the minimal algebraic and Alexander filtration level, respectively, and $*$ is the Maslov grading of each generator. We use the fact that $\dim_{\mathbb{F}} \mathscr{F}^{\{j \leqslant 0\}} c\mathcal{HFL}^{\infty}_0(L) = 1$ — see Theorem 2.1 — and then compute how far we can shift the region S while being able to find a generator for such a homology class in $cCFL_0^\infty(L)$. In this way, given a southwest region S, we associate a real number to it that we call $\Upsilon_S(L)$; the complete definition can be found in Section 2.2.

In the case of knots, $\Upsilon_S(L)$ is a normalization of the invariant of Alfieri [2019]. This choice was made because, say A_t is the region of the plane consisting of the pairs (j, A) with $At + j(2-t) \leqslant 0$, we have

$$\Upsilon_{A_t}(K) = \Upsilon_K(t) \quad \text{for } t \in [0,2],$$

and the latter is the Υ–function of Ozsváth, Stipsicz and Szabó [Ozsváth et al. 2017a].

Moreover, since there is a unique homology class in $\mathscr{F}^{\{j\leqslant 0\}} c\mathscr{HFL}^\infty_{1-n}(L) \setminus \mathscr{F}^{\{j\leqslant -1\}} c\mathscr{HFL}^\infty_{1-n}(L)$, the same procedure allows us to define another family of invariants which we call $\Upsilon^*_S(L)$. Clearly, for knots we have $\Upsilon_S(K) = \Upsilon^*_S(K)$ for every S. The following proposition summarizes some of the main properties of $\Upsilon_S(L)$.

Proposition 1.4 *Suppose that L, L_1 and L_2 are links in S^3 and L has n components. Then $\Upsilon_S(L)$ and $\Upsilon^*_S(L)$ are concordance invariants and*

1. $\tau(L) = -\Upsilon'_L(0)$ *and* $\tau^*(L) = -(\Upsilon^*_L)'(0)$, *where the invariants $\tau(L)$ and $\tau^*(L)$ are defined in* [Cavallo 2018];
2. $\Upsilon_S(L) = \Upsilon_{-S}(L)$ *and* $\Upsilon^*_S(L) = \Upsilon^*_{-S}(L)$ *for any S, where $-S$ is the region obtained by reflecting S along the line $\{j - A = 0\}$;*
3. $\Upsilon_S(L) = \Upsilon_S(-L)$ *and* $\Upsilon^*_S(L) = \Upsilon^*_S(-L)$ *for any S, where $-L$ is the reverse of L;*
4. $\Upsilon_S(L^*) = -\Upsilon^*_{\overline{\iota S}}(L)$, *where L^* is the mirror image of L and $\overline{\iota S}$ is the topological closure of the complement of the region obtained by reflecting S using the central symmetry of $\mathbb{R}^2$ at the origin;*
5. $\Upsilon_{L_1\#L_2}(t) = \Upsilon_{L_1}(t) + \Upsilon_{L_2}(t)$ *and* $\Upsilon^*_{L_1\#L_2}(t) = \Upsilon^*_{L_1}(t) + \Upsilon^*_{L_2}(t)$ *for $t \in [0,2]$, where $L_1 \# L_2$ is a connected sum of L_1 and L_2;*
6. $\Upsilon_L(t) = \frac{1}{2}(1 - n + \sigma(L)) \cdot t$ *and* $\Upsilon^*_L(t) = \frac{1}{2}(n - 1 + \sigma(L)) \cdot t$ *for $t \in [0,1]$ whenever L is quasialternating and $\sigma(L)$ is the signature of the link as in* [Gordon and Litherland 1978].

We prove that each $\Upsilon_S(L)$ gives a lower bound for the slice genus $g_4(L)$, which as usual is defined as the minimum genus of a compact, oriented surface Σ properly embedded in D^4 such that $\partial\Sigma = L$. We recall that, since we can add tubes between surfaces in D^4 without increasing the genus, we can suppose that any such Σ is also connected. Moreover, in Section 4.4 we define the notion of distance $h_S(m)$ from the point $(0, m)$ to the centered southwest region S, where centered means that $(0,0) \in \partial S$; therefore, we have the following result.

Theorem 1.5 *If L is a link in S^3 with n components then*

$$-\Upsilon_S(L) \leqslant h_{\pm S}(g_4(L) + n - 1) \quad \textit{and} \quad -\Upsilon^*_S(L) \leqslant h_{\pm S}(g_4(L))$$

for every centered southwest region S of $\mathbb{R}^2$. In particular, for the classic Υ–functions,

$$-\Upsilon_L(t) \leqslant t(g_4(L) + n - 1) \quad \textit{and} \quad -\Upsilon^*_L(t) \leqslant t \cdot g_4(L) \quad \textit{for } t \in [0,1].$$

Now let us consider the southwest regions V_k for $k \geqslant 0$, defined as the subset of the plane consisting of the pairs (j, A) such that $j \leqslant 0$ and $A \leqslant k$. We can now define the invariant $\nu^+(L)$ as the minimum k such that $-2 \cdot V_L(k) := \Upsilon_{V_k}(L) = 0$. An equivalent definition of $\nu^+(L)$ was given in [Cavallo 2018, Section 4]; although, the invariant was denoted by $\nu(L)$ and the concordance invariance was not proven.

Proposition 1.6 *Suppose that L, L_1 and L_2 are links in S^3, and L has n components. Then $\nu^+(L)$ is a concordance invariant,*

$$0 \leqslant \tau(L) \leqslant \nu^+(L) \leqslant g_4(L) + n - 1 \quad \text{and} \quad \nu^+(L_1 \# L_2) \leqslant \nu^+(L_1) + \nu^+(L_2).$$

Ozsváth, Stipsicz and Szabó [Ozsváth et al. 2017b] introduced the homology group $HFL'(L)$ that they called the unoriented link Floer homology group. From $HFL'(L)$ they define the υ–set of L which is a set of 2^{n-1} integers and is an isotopy invariant of unoriented links after a suitable normalization.

Moreover, for knots they showed that $\upsilon(K)$, which coincides with $\Upsilon_K(1)$ and is the only element of the υ–set in this case, gives a lower bound for the 4*–dimensional smooth crosscap number* $\gamma_4(K)$, which is the minimum first Betti number of a compact surface F properly embedded in D^4 and such that $\partial F = L$. Note that this time F is not necessarily orientable (and always nonoriented).

Starting from these results, in this paper we consider a slightly different version of $HFL'(L)$ and we prove that it is an unoriented concordance invariant. Since it shares much information with the original group and we only use this new version, we denote it in the same way.

We say that a collection of n disjoint annuli Σ is an *unoriented concordance* between L_1 and L_2, which are n–component links, if Σ is a concordance between L'_1 and L'_2, obtained by changing the orientation of some components on L_1 and L_2 respectively.

Theorem 1.7 *The group $HFL'(L_1)[[\frac{1}{2}\sigma(L_1)]]$ is j–filtered isomorphic to $HFL'(L_2)[[\frac{1}{2}\sigma(L_2)]]$ whenever L_1 is unoriented concordant to L_2.*

From Theorem 1.7 we obtain that the wideness of the υ–set, $|\upsilon_{\max}(L) - \upsilon_{\min}(L)|$, and the numbers $\upsilon_{\max}(L) - \frac{1}{2}\sigma(L)$ and $\upsilon_{\min}(L) - \frac{1}{2}\sigma(L)$ are unoriented concordance invariants of L. Using the same techniques in Section 4.4, we show that such invariants give lower bounds for $\gamma_4^{(k)}(L)$, a version of the 4–dimensional smooth crosscap number for links. In fact, we say that $\gamma_4^{(k)}(L)$ is defined as the minimum first Betti number of a compact surface, properly embedded in D^4, which has k connected components and is bounded by L.

Theorem 1.8 *Say the n–component link L in S^3 bounds a compact, unoriented surface F, properly embedded in D^4, with k connected components. Then*

$$|k - 1 - \upsilon_{\max}(L) + \upsilon_{\min}(L)| \leqslant \gamma_4^{(k)}(L).$$

A corollary of this theorem is the following result, which was already proved in a different way by Donald and Owens [2012].

Corollary 1.9 *Every quasialternating link L can bound an unoriented, compact surface F, properly embedded in D^4, only when the Euler characteristic $\chi(F)$ is at most equal to one.*

Theorem 1.8 gives a bound that involves the wideness of $\upsilon(L)$. We give other inequalities in the following theorem.

Theorem 1.10 *Consider an n–component link L in S^3 which bounds a compact, unoriented surface F, properly embedded in D^4, with k connected components. Then*

$$\left|\upsilon_{\max}(L)-\tfrac{1}{2}\sigma(L)-\tfrac{1}{2}(k-n)\right|\leqslant\gamma_4^{(k)}(L)\quad\text{and}\quad\left|\upsilon_{\min}(L)-\tfrac{1}{2}\sigma(L)-\tfrac{1}{2}(2-k-n)\right|\leqslant\gamma_4^{(k)}(L).$$

In particular, when $k=n$,

$$\left|\upsilon_{\max}(L)-\tfrac{1}{2}\sigma(L)\right|\leqslant\gamma_4^{(n)}(L)\quad\text{and}\quad\left|\upsilon_{\min}(L)-\tfrac{1}{2}\sigma(L)+n-1\right|\leqslant\gamma_4^{(n)}(L),$$

and when $k=1$,

$$\left|\upsilon_i(L)-\tfrac{1}{2}\sigma(L)-\tfrac{1}{2}(1-n)\right|\leqslant\gamma_4^{(1)}(L)$$

for every $\upsilon_i(L)$ in the υ–set of L.

We apply this result to the family of links $L_n=T^*_{2,4}\,\#\,T^{\#n}_{3,4}$; namely, the connected sum of the mirror of the torus link $T_{2,4}$ and n torus knots $T_{3,4}$. In particular, we show that $\{L_n\}$ for $n\geqslant 0$ is a family of 2–component links such that $\gamma_4^{(1)}$ is arbitrarily large.

Corollary 1.11 *Given the link $L_n=T^*_{2,4}\,\#\,T^{\#n}_{3,4}$, we have $\gamma_4^{(2)}(L_n)=n+1$ and $\gamma_4^{(1)}(L_n)\geqslant n$ for $n\geqslant 0$.*

The paper is organized as follows: in Section 2 we summarize the construction of the link Floer complex $cCFL^\infty(L)$ and we describe how to define the filtered homology group $\mathcal{F}^S c\mathcal{HFL}^\infty(L)$ and the invariant $\Upsilon_S(L)$. Moreover, we prove the equivalence between stable and local equivalence of knot Floer chain complexes stated in Theorem 1.1. In Section 3 we prove the concordance invariance of $cCFL^\infty(L)$. In Section 4 we define the other Υ–type invariants and we prove some of their properties, including Proposition 1.4. We also give the proof of Theorem 1.5, which describes our bound for the slice genus. Finally, in Section 5 we introduce the group $HFL'(L)$ and the υ–set of L, showing that they give unoriented concordance invariants. Moreover, we study their behavior under unoriented cobordisms and we prove the lower bounds for $\gamma_4^{(k)}(L)$.

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2 Link Floer homology

2.1 Chain complex and homology

Throughout the paper we assume that the reader is familiar with the construction of the link Floer homology chain complexes, both when links are represented with multipointed Heegaard diagrams [Ozsváth and Szabó 2004a; 2004b; 2008] or grids [Manolescu et al. 2007; Ozsváth et al. 2015]. We only recall the main features.

Let us consider $\mathcal{D}=(\Sigma,\alpha,\beta,\boldsymbol{w},\boldsymbol{z})$ a multipointed Heegaard diagram for an oriented n–component link L in S^3. The chain complex $cCFL^\infty(\mathcal{D})$ is the free $\mathbb{F}[U,U^{-1}]$–module over the intersection points $\mathbb{T}=\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ in the symmetric power $\mathrm{Sym}(\Sigma,\alpha,\beta)$ — see [Ozsváth and Szabó 2004a; 2008] — where $\mathbb{F}$ is the field with two elements and $\boldsymbol{w}$ and $\boldsymbol{z}$ are two n–tuples of basepoints in Σ; see [Ozsváth and Szabó 2008]. The differential ∂^- is defined by counting pseudo-holomorphic curves on some special [Ozsváth and Szabó 2008] domains in $\mathrm{Sym}(\Sigma,\alpha,\beta)$ with Maslov index μ equal to one [Lipshitz 2006; Ozsváth and Szabó 2004b]; denote the set of such domains by π_2. Then for every intersection point x we can write

$$\partial^- x=\sum_{y\in\mathbb{T}}\sum_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=1}} m(\phi)\cdot U^{n_{\boldsymbol{w}}(\phi)}y,$$

where $m(\phi)\in\mathbb{F}$ depends also on the choice of an almost-complex structure on $\mathrm{Sym}(\Sigma,\alpha,\beta)$ and $0\leqslant n_{\boldsymbol{w}}(\phi)=n_{w_1}(\phi)+\cdots+n_{w_n}(\phi)$ is the multiplicity of the basepoints $\boldsymbol{w}$ in ϕ. Moreover, we say that

$$\partial^-(U^{\pm1}p)=U^{\pm1}\cdot\partial^- p$$

for any $x\in\mathbb{T}$ and $p\in cCFL^\infty(\mathcal{D})$.

For every $x\in\mathbb{T}$ we can assign an absolute $\mathbb{Z}$–grading, called *Maslov grading* [Ozsváth and Szabó 2008], which is denoted by $M(x)$ and can be extended to the whole complex by taking

$$M(U^{\pm}p)=M(p)\mp 2$$

for any p homogeneous. We then have

$$cCFL^\infty(\mathcal{D})=\bigoplus_{d\in\mathbb{Z}} cCFL^\infty_d(\mathcal{D})$$

as $\mathbb{F}$–vector spaces; moreover, there is a map

$$\partial^-_d\colon cCFL^\infty_d(\mathcal{D})\to cCFL^\infty_{d-1}(\mathcal{D})$$

for any $d\in\mathbb{Z}$.

The chain complex $cCFL^\infty(\mathcal{D})$ comes with a natural increasing filtration, usually denoted by the *algebraic filtration* j, defined as

$$j^t cCFL^\infty(\mathcal{D})=U^{-t}\cdot cCFL^-(\mathcal{D}),$$

where $cCFL^-(\mathcal{D})$ is the free $\mathbb{F}[U]$–module over $\mathbb{T}$. It is easy to check that the differential ∂^- respects j.

We define the homology group

$$c\mathcal{HFL}^\infty(L) = \bigoplus_{d\in\mathbb{Z}} c\mathcal{HFL}^\infty_d(L) = \bigoplus_{d\in\mathbb{Z}} \frac{\operatorname{Ker}\partial^-_d}{\operatorname{Im}\partial^-_{d+1}}.$$

Since the Maslov grading and the differential only depend on $\boldsymbol{w}$, such a group, together with the algebraic filtration, is isomorphic to $HF^\infty(S^3,n)\cong\mathbb{F}[U,U^{-1}]^{2^{n-1}}$, where the n denotes the number of basepoints in the Heegaard diagram. The filtration j descends to homology in the following way. Say $\pi_d:\operatorname{Ker}\partial^-_d\to c\mathcal{HFL}^\infty_d(L)$ is the quotient map; then

$$j^t c\mathcal{HFL}^\infty_d(L) = \pi_d(\operatorname{Ker}\partial^-_d \cap j^t cCFL^\infty(\mathcal{D})),$$

which is an $\mathbb{F}$–subspace of $c\mathcal{HFL}^\infty_d(L)$. More specifically, we have the following theorem.

Theorem 2.1 *Say the link L has n components. Then*

$$\frac{j^{(d+k)/2}c\mathcal{HFL}^\infty_d(L)}{j^{(d+k)/2-1}c\mathcal{HFL}^\infty_d(L)} \cong_{\mathbb{F}} \mathbb{F}^{\binom{n-1}{k}}$$

whenever $d\equiv k \bmod 2$ and $0\leqslant k\leqslant n-1$. It is zero otherwise.

Proof From [Ozsváth and Szabó 2008] we know that $HF^-(S^3,n)$ has 2^{n-1} generators such that exactly $\binom{n-1}{k}$ of them have Maslov grading $-k$. Since

$$HF^\infty_*(S^3,n)\cong_{\mathbb{F}[U,U^{-1}]} HF^-_*(S^3,n)\otimes_{\mathbb{F}[U]}\mathbb{F}[U,U^{-1}],$$

one has

$$c\mathcal{HFL}^\infty_d(L)\cong_{\mathbb{F}}\begin{cases}\mathbb{F}^{2^{n-2}} & \text{if } n\geqslant 2,\\ \mathbb{F} & \text{if } n=1 \text{ and } d \text{ is even},\\ \{0\} & \text{if } n=1 \text{ and } d \text{ is odd},\end{cases}$$

and this determines the Maslov gradings.

Now we want to compute the filtration j. We note that all the generators of $HF^-(S^3,n)$ have minimal j–filtration level zero. Hence, the statement is true for j^0; in fact if we substitute $d=-k$ in then we obtain the right distribution of the Maslov gradings. At this point, in order to prove the theorem, we only need to observe that the multiplication by $U^{\pm1}$ drops the minimal level of the algebraic filtration by ±1 and the Maslov grading by ±2. □

Figure 1 shows the distribution of the Maslov grading and the minimal j–level for two- and three-component links.

2.2 The Alexander and the $\mathcal{F}$–filtrations

In the same way as the Maslov grading, we can assign to every $x\in\mathbb{T}$ another absolute $\mathbb{Z}$–grading: the *Alexander grading* $A(x)$, which also is extended to $cCFL^\infty(\mathcal{D})$ by taking

$$A(U^{\pm1}p) = A(p)\mp 1$$

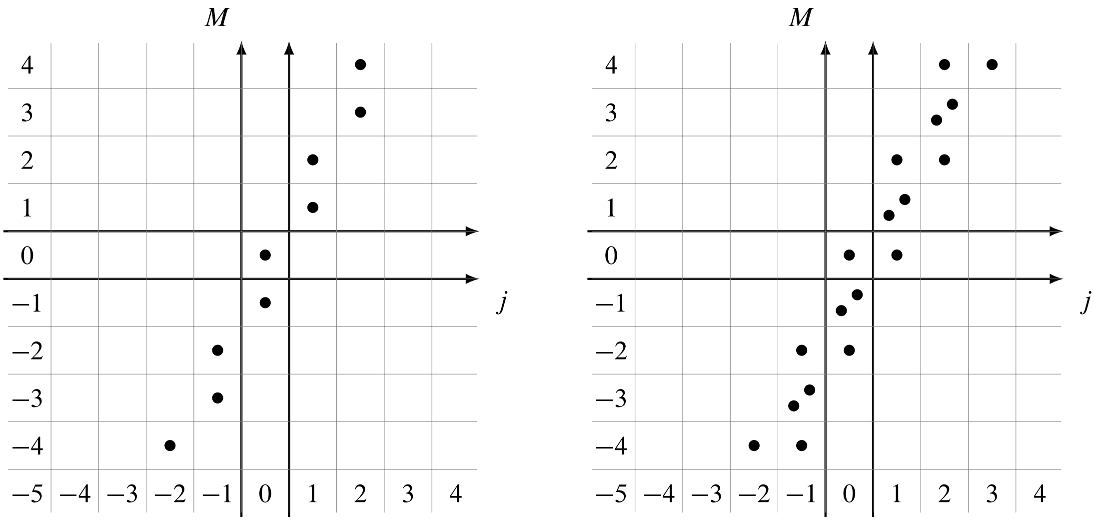

Figure 1: Maslov gradings and algebraic filtration for 2– (left) and 3–component links (right). The algebraic level j is on the x–axis and the Maslov grading is on the y–axis.

for any p homogeneous. We recall that in [Ozsváth and Szabó 2008] the Alexander grading is introduced as a multigrading $\boldsymbol{A}(x)=(A_1(x),\dots,A_n(x))$; in this paper we define $A(x):=A_1(x)+\dots+A_n(x)$.

In this case, the differential ∂^- does not preserve $A(x)$; for this reason we introduce the *Alexander filtration*. Let us consider the $\mathbb{F}[U]$–subspace $\mathscr{A}^s cCFL^\infty(\mathfrak{D})$ generated by all the elements p in $cCFL^\infty(\mathfrak{D})$ such that $A(p)\leqslant s$. The $\mathscr{A}$–filtration is an increasing filtration like j and it is such that

$$\{0\}\cong \mathscr{A}^{\underline{s}} cCFL^\infty_d(\mathfrak{D})\subset\cdots\subset \mathscr{A}^{\bar{s}} cCFL^\infty_d(\mathfrak{D})=cCFL^\infty_d(\mathfrak{D}), \tag{2-1}$$

which follows from [Ozsváth and Szabó 2008]; moreover, it is again easy to show that it is preserved by ∂^-. Note that $\bar{s}$ and $\underline{s}$ depend on d.

We define $\mathscr{F}$ for now as a double-increasing filtration. More specifically, we say that

$$\mathscr{F}^{t,s} cCFL^\infty(\mathfrak{D})=j^t cCFL^\infty(\mathfrak{D})\cap\mathscr{A}^s cCFL^\infty(\mathfrak{D})$$

and clearly ∂^- also respects $\mathscr{F}$. We now extend the $\mathscr{F}$–filtration on the homology group, in the way that it is indexed by southwest regions of the lattice $\mathbb{Z}^2$ (resp. the plane $\mathbb{R}^2$), using an idea of Alfieri [2019]. A *southwest region* $S\subset\mathbb{Z}^2$ (resp. $\mathbb{R}^2$) is a subset of $\mathbb{Z}^2$ (resp. a topological submanifold of $\mathbb{R}^2$) such that if $(\bar{t},\bar{s})\in S$ then $s\leqslant\bar{s}$ and $t\leqslant\bar{t}$ imply $(t,s)\in S$. Moreover, we require S to differ from $\varnothing$ and $\mathbb{Z}^2$ (resp. $\mathbb{R}^2$).

Consider again the map $\pi_d:\operatorname{Ker}\partial^-_d\to c\mathcal{HFL}^\infty_d(L)$. Define

$$\operatorname{Ker}\partial^-_{d,S}=\operatorname{Ker}\partial^-_d\cap\operatorname{Span}\{\mathscr{F}^{t,s}cCFL^\infty_d(\mathfrak{D})\mid(t,s)\in S\}:=\operatorname{Ker}\partial^-_d\cap\mathscr{F}^S cCFL^\infty_d(\mathfrak{D}).$$

Then we say that

$$\mathscr{F}^S c\mathcal{HFL}^\infty_d(L)=\pi_d(\operatorname{Ker}\partial^-_{d,S})\subset c\mathcal{HFL}^\infty_d(L)$$

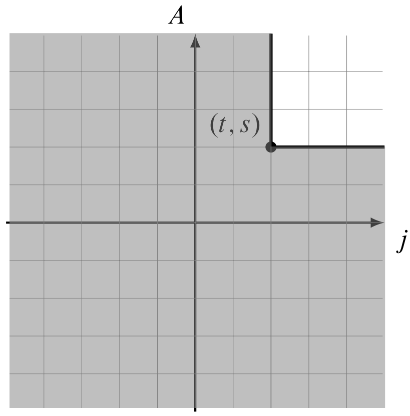

Figure 2: The southwest region $W_{t,s}$ is the subset $\{(j,A) \mid j \leqslant t \text{ or } A \leqslant s\}$ of $\mathbb{R}^2$.

for any $d \in \mathbb{Z}$. Note that the level $\mathcal{F}^{t,s}$ corresponds to the southwest region $V_{t,s} = \{(j,A) \mid j \leqslant t, A \leqslant s\}$, while $j^t := \mathcal{F}^{\{j \leqslant t\}}$ and $\mathcal{A}^s := \mathcal{F}^{\{A \leqslant s\}}$ correspond to $\{(j,A) \mid j \leqslant t\}$ and $\{(j,A) \mid A \leqslant s\}$ respectively.

The filtration $\mathcal{F}$ is increasing in the sense that if $S_1 \subset S_2$ are two southwest regions then

$$\mathcal{F}^{S_1} c\mathcal{HFL}^{\infty}_{*}(L) \subset \mathcal{F}^{S_2} c\mathcal{HFL}^{\infty}_{*}(L).$$

Moreover, it has the following property.

Proposition 2.2 *Fix an integer d, denote by $W_{t,s}$ the southwest region in Figure 2 and take $V_{t,s}$ as before. Then there exists a pair (t,s) such that $\mathcal{F}^{S} c\mathcal{HFL}^{\infty}_{d}(L) \cong \{0\}$ for every southwest region $S \subset W_{t,s}$.*

Furthermore, there is another pair (t',s') such that $\mathcal{F}^{T} c\mathcal{HFL}^{\infty}_{d}(L) \cong c\mathcal{HFL}^{\infty}_{d}(L)$ for every southwest region $T \supset V_{t',s'}$.

Proof Since $cCFL^{\infty}(\mathcal{D})$ is finitely generated as an $\mathbb{F}[U,U^{-1}]$–module, $cCFL^{\infty}_{d}(\mathcal{D})$ is a finite-dimensional $\mathbb{F}$–vector space. Then there are integers A, the minimal Alexander level containing a generator of $cCFL^{\infty}_{d}(\mathcal{D})$, and B, the same considering algebraic levels, by (2-1). If we choose $t < B$ and $s < A$ then $\mathcal{F}^{W_{t,s}} cCFL^{\infty}_{d}(\mathcal{D}) \cong \{0\}$ and so $\mathcal{F}^{W_{t,s}} c\mathcal{HFL}^{\infty}_{d}(L)$ is also zero. The first claim now follows from the fact that $\mathcal{F}$ is an increasing filtration; for the second one we reason exactly in the same way. □

From [Ozsváth and Szabó 2008] we have the following important theorem.

Theorem 2.3 (Ozsváth and Szabó) *The $\mathcal{F}$–filtered chain homotopy type of $cCFL^{\infty}(\mathcal{D})$, together with the Maslov grading, is a link invariant of L, where $\mathcal{D}$ is a Heegaard diagram for L.*

For simplicity, from now on we may denote our chain complex by $cCFL^{\infty}_{*}(L)$, implicitly referring to any of the representatives of the filtered chain homotopy type. This result guarantees that also the $\mathcal{F}$–filtration on $c\mathcal{HFL}^{\infty}_{*}$ is a link invariant, justifying our notation.

We call a graded isomorphism F between the homology groups of two links L_1 and L_2 a *filtered isomorphism* if F and its inverse F^{-1} both preserve the filtration $\mathscr{F}$. This is equivalent to say that F restricts to isomorphisms

$$\mathscr{F}^S c\mathcal{HFL}^\infty_d(L_1) \cong_{\mathbb{F}} \mathscr{F}^S c\mathcal{HFL}^\infty_d(L_2)$$

for every $d \in \mathbb{Z}$ and southwest region S of $\mathbb{Z}^2$.

When L_1 and L_2 are isotopic links, Theorem 2.3 implies that $cCFL^\infty(L_1)$ is locally equivalent to $cCFL^\infty(L_2)$, following the notation of Zemke [2019a]. This means we can find chain maps $f: cCFL^\infty(L_1) \to cCFL^\infty(L_2)$ and $g: cCFL^\infty(L_2) \to cCFL^\infty(L_1)$ which both preserve $\mathscr{F}$ and induce $\mathscr{F}$–filtered isomorphisms between $c\mathcal{HFL}^\infty(L_1)$ and $c\mathcal{HFL}^\infty(L_2)$.

Corollary 2.4 *Suppose that the link L_1 is isotopic to the link L_2 in S^3. Then there is a local equivalence between $cCFL^\infty(L_1)$ and $cCFL^\infty(L_2)$.*

Note that we can assume f to be a chain homotopy equivalence, but in general a local equivalence is not necessarily an $\mathscr{F}$–filtered chain homotopy equivalence. This would happen if the chain homotopies between f and its homotopy inverse also preserve $\mathscr{F}$.

We call a southwest region S of $\mathbb{R}^2$ *centered* if $(0,0)$ belongs to the boundary ∂S of S. Consider

$$S_k = \left\{(t,s) \in \mathbb{R}^2 \mid \left(t + \tfrac{1}{2}k, s + \tfrac{1}{2}k\right) \in S\right\},$$

where $k \in \mathbb{R}$. We define the invariant $\Upsilon_S(L)$ as follows. Given a centered southwest region S of $\mathbb{R}^2$, we say that

$$\Upsilon_S(L) := \max\{k \in \mathbb{R} \mid \mathscr{F}^{S_k} c\mathcal{HFL}^\infty_0(L) \supset \mathscr{F}^{\{j \leqslant 0\}} c\mathcal{HFL}^\infty_0(L)\}.$$

We recall that the $\mathscr{F}$–level $\{j \leqslant 0\}$ coincides with the level j^0 of the algebraic filtration. Note also that Theorem 2.1 implies $\dim_{\mathbb{F}} c\mathcal{HFL}^\infty_0(L) > 1$ for links with three or more components, but $\dim_{\mathbb{F}} \mathscr{F}^{\{j \leqslant 0\}} c\mathcal{HFL}^\infty_0(L)$ is always equal to one. For this reason, in the definition of Υ_S, we need the region S_k not only to contain a generator of the total homology in Maslov grading zero, but also that such an element is homologous to one which lives in the algebraic level j^0.

Corollary 2.5 *The real number $\Upsilon_S(L)$ is a link invariant for every southwest region S of $\mathbb{R}^2$.*

Proof This follows immediately from Proposition 2.2 and Corollary 2.4. □

2.3 Duality and mirror images

Let us start this subsection with a Heegaard diagram $\mathcal{D}$ for an oriented link L in S^3. As we recalled in Section 2.1, from $\mathcal{D}$ we obtain the chain complex $(cCFL^\infty(\mathcal{D}), \partial^-)$. We now define the corresponding dual complex $(cCFL^\infty(\mathcal{D})^*, \partial^-_*)$ as follows.

The space $cCFL^\infty(\mathcal{D})^*$ as an $\mathbb{F}[U,U^{-1}]$–module is isomorphic to

$$\operatorname{Hom}_{\mathbb{F}[U,U^{-1}]}(cCFL^\infty(\mathcal{D}), \mathbb{F}[U,U^{-1}]). \tag{2-2}$$

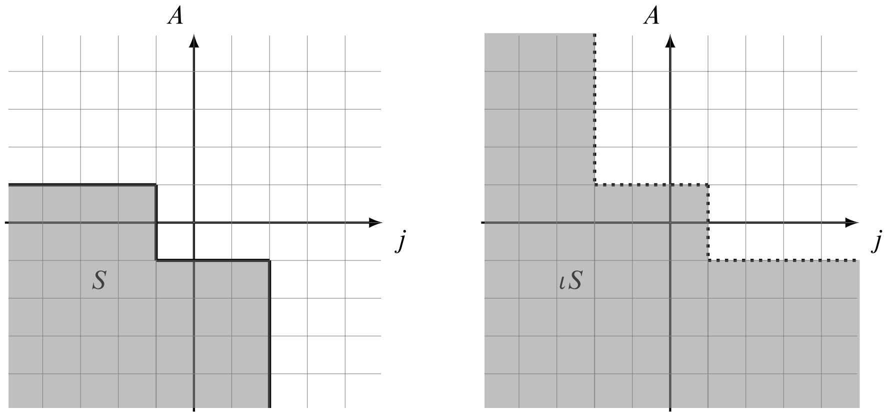

Figure 3: The dotted boundary in the picture on the right is not part of ιS.

If x is an intersection point then its dual x^* is the functional which sends x into 1 and the other intersection points into 0; and this implies $p^* \in cCFL^\infty(\mathcal{D})^*$ can be defined by $\mathbb{F}[U, U^{-1}]$–linearization of the dual of the intersection points. More specifically, we say that

$$cCFL^\infty_d(\mathcal{D})^* := (cCFL^\infty_{-d}(\mathcal{D}))^* = \{p^* \in cCFL^\infty(\mathcal{D})^* \mid 1 \in p^*(cCFL^\infty_m(\mathcal{D})) \text{ implies } m = -d\}.$$

Notice that

$$cCFL^\infty(\mathcal{D})^* \cong_\mathbb{F} \bigoplus_{d\in\mathbb{Z}} cCFL^\infty_d(\mathcal{D})^*,$$

but $cCFL^\infty(\mathcal{D})^* \not\cong_\mathbb{F} \operatorname{Hom}_\mathbb{F}(cCFL^\infty(\mathcal{D}), \mathbb{F})$. In particular, $U^{\pm 1}p^* := (U^{\mp 1}p)^*$ and thus

$$M(U^{\pm 1}p^*) = M((U^{\mp 1}p)^*) = -M(U^{\mp 1}p) = -M(p) \mp 2 = M(p^*) \mp 2$$

as expected.

We can also define the dual filtration $\mathcal{F}^*$. In order to do this, we introduce the concept of inverse ιS of a southwest region S in $\mathbb{Z}^2$ (resp. $\mathbb{R}^2$). We take ιS as the complement of the image of S under the symmetry centered in the origin of the plane; see Figure 3 for an example.

Lemma 2.6 *If S is a southwest region then ιS is also a southwest region.*

Proof The mirror image of S is a northeast region. The complement of a northeast region is a southwest region; in fact, if $(x, y) \in \iota S$ and $(\bar{x}, \bar{y}) \notin \iota S$ with $\bar{x} \leqslant x$ and $\bar{y} \leqslant y$ then $(\bar{x}, \bar{y})$ belongs to the northeast region $(\iota S)^c$, which means that (x, y) is also in $(\iota S)^c$. This is a contradiction. □

The dual filtration is defined as

$$(\mathcal{F}^*)^S cCFL^\infty_d(\mathcal{D})^* := \operatorname{Ann} \mathcal{F}^{\iota S} cCFL^\infty_{-d}(\mathcal{D}) = \{p^* \in cCFL^\infty_d(\mathcal{D})^* \mid p^*(\mathcal{F}^{\iota S} cCFL^\infty_{-d}(\mathcal{D})) = 0\}$$

for any southwest region S. We observe that if $S' \subset S$ then $\iota S \subset \iota S'$ and so $\operatorname{Ann} \mathcal{F}^{\iota S'} \subset \operatorname{Ann} \mathcal{F}^{\iota S}$. This means that $\mathcal{F}^*$ is still an increasing filtration.

The only missing part in the dual complex is the differential. We introduce ∂_*^- as follows. For every $x^* \in cCFL^\infty(\mathcal{D})^*$ and $y \in cCFL^\infty(\mathcal{D})$,

$$(\partial_*^- x^*)(y) = x^*(\partial^- y).$$

Moreover, we take $\partial_*^-(Up^*) = U \cdot \partial_*^- p^*$.

Lemma 2.7 *The map ∂_*^- is a differential, drops the Maslov grading by 1 and preserves the filtration $\mathcal{F}^*$.*

Proof First,

$$\partial_*^-(\partial_*^- x^*(y)) = (\partial_*^- x^*)(\partial^- y) = x^*(\partial^- \circ \partial^- y) = 0 = 0(y)$$

for any $y \in cCFL^\infty(\mathcal{D})$. For the second claim, suppose $p^* \in cCFL_d^\infty(\mathcal{D})^*$. Then $\partial_*^- p^*(q) = p^*(\partial^- q)$, so if $q \in cCFL_{-d+1}^\infty(\mathcal{D})$ then $\partial^- q \in cCFL_{-d}^\infty(\mathcal{D})$. In addition, if $r \notin cCFL_{-d+1}^\infty(\mathcal{D})$ is homogeneous then $\partial_*^- p^*(r) = 0$ and this implies

$$\partial_*^- p^* = \partial_*^- p^*|_{cCFL_{-d+1}^\infty(\mathcal{D})} \in (cCFL_{-d+1}^\infty(\mathcal{D}))^* = cCFL_{d-1}^\infty(\mathcal{D})^*.$$

Finally, suppose that $p^* \in (\mathcal{F}^*)^S cCFL_d^\infty(\mathcal{D})^*$ for a southwest region S. Then $p^*(\mathcal{F}^{\iota S} cCFL_{-d}^\infty(\mathcal{D})) = 0$. If $q \in \mathcal{F}^{\iota S} cCFL_{-d+1}^\infty(\mathcal{D})$ then $(\partial_*^- p^*)(q) = 0$, since $\partial^- q \in \mathcal{F}^{\iota S} cCFL_{-d}^\infty(\mathcal{D})$, which implies that

$$\partial_*^- p^* \in \operatorname{Ann} \mathcal{F}^{\iota S} cCFL_{-d+1}^\infty(\mathcal{D}) = (\mathcal{F}^*)^S cCFL_{d-1}^\infty(\mathcal{D})^*. \qquad \square$$

We can now prove that the dual complex we have just defined is related to the complex obtained from a Heegaard diagram of the mirror image of L. We denote by $\mathcal{C}_d[\![a]\!]$ the graded complex given by $\mathcal{C}_{d-a}$.

Theorem 2.8 *If $(cCFL^\infty(\mathcal{D}), \partial^-)$ is the chain complex associated to a Heegaard diagram $\mathcal{D}$ for L then there is a diagram $\mathcal{D}^*$, representing the mirror image L^* of L, such that*

$$(cCFL^\infty(\mathcal{D}^*), \partial_{\mathcal{D}^*}^-) = (cCFL^\infty(\mathcal{D})^*, \partial_*^-)[\![1-n]\!]$$

as $\mathcal{F}$–filtered, graded chain complexes.

Proof If $\mathcal{D} = (\Sigma, \alpha, \beta, \boldsymbol{w}, \boldsymbol{z})$ then $\mathcal{D}^* = (-\Sigma, \alpha, \beta, \boldsymbol{w}, \boldsymbol{z})$. This identifies the domain $\phi \in \pi_2(x, y)$ with $-\phi \in \pi_2(y, x)$; see [Ozsváth and Szabó 2008; Ozsváth et al. 2017a]. Moreover, using the formula in [Lipshitz 2006] it is easy to check that ϕ and $-\phi$ have the same Maslov index. The identification that proves the theorem is $x \to x^*$, extended U–equivariantly to the whole complexes, where x^* denotes the dual of x as before. We first show that such a map is indeed a chain map:

$$(\partial_{\mathcal{D}^*}^-(x))^*(t) = \sum_{y \in \mathbb{T}} \sum_{\substack{\phi \in \pi_2(y,x) \\ \mu(\phi)=1}} m(\phi) \cdot U^{n_{\boldsymbol{w}}(\phi)} y^*(t) = \sum_{\substack{\phi \in \pi_2(t,x) \\ \mu(\phi)=1}} m(\phi) \cdot U^{n_{\boldsymbol{w}}(\phi)},$$

$$\partial_*^-(x^*(t)) = x^*(\partial^- t) = x^*\Bigg(\sum_{y \in \mathbb{T}} \sum_{\substack{\phi \in \pi_2(t,y) \\ \mu(\phi)=1}} m(\phi) \cdot U^{n_{\boldsymbol{w}}(\phi)} y\Bigg) = \sum_{\substack{\phi \in \pi_2(t,x) \\ \mu(\phi)=1}} m(\phi) \cdot U^{n_{\boldsymbol{w}}(\phi)},$$

which holds for every generator t of $cCFL^\infty(\mathcal{D})$, and so the claim is proved.

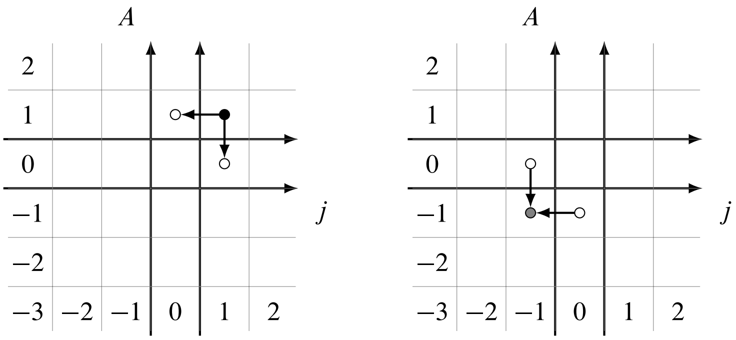

Figure 4: The complex $cCFL^\infty(T_{2,3})$ is on the left and $cCFL^\infty(T^*_{2,3})$ on the right. Both chain complexes are pictured ignoring the U–action. Black, white and gray dots represent Maslov gradings 1, 0 and -1 respectively.

Now we argue that our identification correctly shifts the Maslov and Alexander gradings. Suppose that $M(x) = d$. Then by definition it is $M_*(x^*) = -d$. We observe that by [Ozsváth and Szabó 2008],

$$M(x) - M(y) = \mu(\phi) - 2n_{\boldsymbol{w}}(\phi) = \mu(-\phi) - 2n_{\boldsymbol{w}}(-\phi) = M_{\mathcal{D}*}(y^*) - M_{\mathcal{D}*}(x^*)$$

with $\phi \in \pi_2(x, y)$, and then $M_{\mathcal{D}*}$ is reversed as a relative grading, which means $M_{\mathcal{D}*}(x) = -d + c$ with $c \in \mathbb{Z}$. Now we use the fact that the Maslov grading is always normalized in the way that the top grading, where the total homology is nontrivial, is zero [Ozsváth and Szabó 2008]. This gives $c = 1 - n$ as wanted.

Finally, consider x such that $A(x) = s$. As before, using the relation

$$A(x) - A(y) = n_{\boldsymbol{z}}(\phi) - n_{\boldsymbol{w}}(\phi)$$

whenever $\phi \in \pi_2(x, y)$ and the fact that the Alexander grading is always symmetric, we find $A_{\mathcal{D}^*}(x) = -s$. We use the definition of $\mathcal{F}^*$ to recover

$$\begin{aligned} A_*(x^*) &= \min\{a \in \mathbb{Z} \mid x^* \in (\mathcal{A}^*)^a cCFL^\infty(\mathcal{D})^*\} \\ &= \min\{a \in \mathbb{Z} \mid x^*(\mathcal{A}^{-a-1} cCFL^\infty(\mathcal{D})) = 0\} \\ &= -\max\{a \in \mathbb{Z} \mid x \notin \mathcal{A}^{a-1} cCFL^\infty(\mathcal{D})\} \\ &= -\min\{a \in \mathbb{Z} \mid x \in \mathcal{A}^{a} cCFL^\infty(\mathcal{D})\} \\ &= -s. \end{aligned}$$

□

Note that the identification in Theorem 2.8 also gives that the homology group of the mirror image of L is the dual of the homology group of L, where the latter group is defined exactly as in (2-2). Furthermore, as an example in Figure 4 we show the filtered chain complexes for the positive and the negative trefoil.

2.4 Local and stable equivalences of knot Floer chain complexes

Hom [2017] introduced a different equivalence relation for the complexes $CFK^\infty(K) = cCFL^\infty(K)$, when K is a knot. More specifically, we say that the Floer complexes associated to the knots K_1 and K_2

are stably equivalent if we have an $\mathcal{F}$–filtered chain homotopy equivalence

$$CFK^\infty(K_1)\oplus A\simeq CFK^\infty(K_2)\oplus B,$$

where A and B are acyclic chain complexes; in other words, $H_*(A)=H_*(B)=\{0\}$. Here we recall [Hendricks and Hom 2019, Corollary 3.2], which shows that the notion of stable equivalence coincides with the one of local equivalence, in the case of knots.

Lemma 2.9 (Hendricks and Hom) *If $CFK^\infty(K)$ is locally equivalent to $CFK^\infty(\bigcirc)=\mathbb{F}[U,U^{-1}]_{(0)}$ then*

$$CFK^\infty(K)\simeq \mathbb{F}[U,U^{-1}]_{(0)}\oplus A,$$

where A is acyclic.

Thanks to the following result, we can see the local equivalence relation that we defined for link Floer complexes in Section 2.2 has a natural generalization to links of the stable equivalences introduced by Hom.

Proof of Theorem 1.1 If our chain complexes are stably equivalent then, in order to define the local equivalence, we just have to take the restriction of the filtered chain homotopy equivalence and its inverse. Conversely, let us suppose that $f\colon CFK^\infty(K_1)\to CFK^\infty(K_2)$ and $g\colon CFK^\infty(K_2)\to CFK^\infty(K_1)$ define a local equivalence. Then

$$CFK^\infty(K_1)\otimes CFK^\infty(K_2)^* \underset{g\otimes\mathrm{Id}}{\overset{f\otimes\mathrm{Id}}{\rightleftarrows}} CFK^\infty(K_2)\otimes CFK^\infty(K_2)^* \underset{g'}{\overset{f'}{\rightleftarrows}} \mathbb{F}[U,U^{-1}]_{(0)}$$

where both the pairs of chain maps give local equivalences. The existence of f' and g' can be proved in the same way as in [Zemke 2019a, Lemma 2.18].

By Lemma 2.9,

$$CFK^\infty(K_2)\otimes CFK^\infty(K_2)^*\simeq \mathbb{F}[U,U^{-1}]_{(0)}\oplus A,$$
$$CFK^\infty(K_1)\otimes CFK^\infty(K_2)^*\simeq \mathbb{F}[U,U^{-1}]_{(0)}\oplus B,$$

where A and B are acyclic. Therefore,

$$\begin{aligned}CFK^\infty(K_1)\oplus A&\simeq CFK^\infty(K_1)\otimes(CFK^\infty(K_2)\otimes CFK^\infty(K_2)^*)\\&\simeq CFK^\infty(K_2)\otimes(CFK^\infty(K_1)\otimes CFK^\infty(K_2)^*)\simeq CFK^\infty(K_2)\oplus B.\end{aligned}$$ □

It is important to observe that when L is a link with n components and n is at least two, the chain complex $cCFL^\infty(L)\otimes cCFL^\infty(L)^*$ is not locally equivalent to the complex $cCFL^\infty(\bigcirc_n)$ representing the unlink. In fact, these groups have different dimensions as $\mathbb{F}[U,U^{-1}]$–modules. Furthermore, in Section 4.3 we give an example of a link L for which such a chain complex is not locally equivalent to any $cCFL^\infty(\bigcirc_m)$ for $m\in\mathbb{N}$.

3 Concordance

3.1 Canonical form of oriented link cobordisms

The definition of link cobordism is standard in literature; in particular, for this paper the reader might find helpful to look at [Cavallo 2018; Sarkar 2011]. We only recall that we always assume the connected components of a smooth cobordism $\Sigma \hookrightarrow S^3 \times I$, from L_1 to L_2, to have boundary on both the links.

Given a surface Σ as before, we assume for now that Σ is oriented; we study unoriented cobordisms only in the last section of the paper. Then Σ consists of four elementary pieces, three of them corresponding to a critical point in the cobordism: birth, band and death moves; while the fourth is a link isotopy, which represents a piece with no critical point. Band moves come in two types: *split*, if the move turns one component into two, and *merge moves*, when two components are joint into one.

For the purpose of this paper, it is more useful to consider what we call *extended birth* and *death moves*. These are the composition of a birth and a merge move and of a split and a death move respectively; see Figures 8 and 11. In addition, we call a *torus move* the composition of a split move with a merge move which rejoins together the newly created components; see Figure 6. Hence, if L_i has n_i for $i = 1, 2$ components, while Σ has k connected components and genus $g(\Sigma)$, then the *canonical form* of Σ is the composition (up to isotopies) of b extended birth moves, $n_1 - k$ merge moves, $g(\Sigma)$ torus moves, $n_2 - k$ split moves and d death moves in this specific order. This implies that Σ can be arranged as shown in Figure 5; see [Cavallo 2018].

When L_1 and L_2 both have n components, Σ is a concordance if it is the union of n disjoint annuli Σ_i, which means that each Σ_i is a knot concordance between the i^{th} components of the two links. From Figure 5 we immediately see that in the case of a concordance there are no torus moves ($g(\Sigma) = 0$). This means that the canonical form of a concordance can be decomposed into three standard pieces: extended birth moves, isotopies and extended death moves.

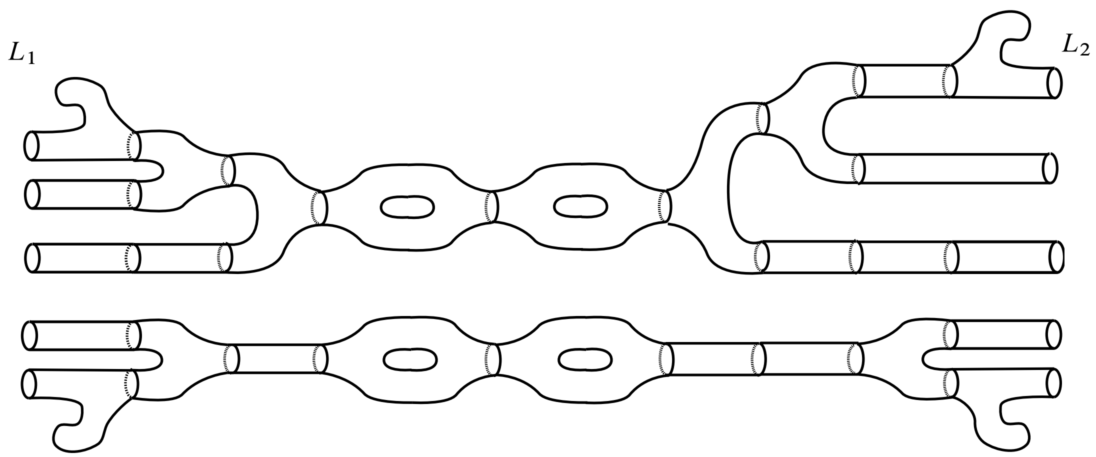

Figure 5: Canonical form of oriented cobordisms between two links.

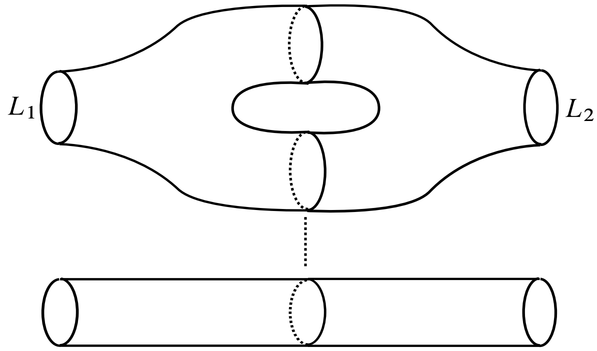

Figure 6: A torus move corresponds to two consecutive band moves on the same component.

In this section we define maps which relate the chain complexes of the links, this time constructed using grid diagrams, before and after each of these moves. Of course we do not need to study the isotopy cobordism; in fact, in this case Theorem 2.3 and Corollary 2.4 already tell us that the complexes are filtered chain homotopy equivalent and that in particular there exists a local equivalence. The strategy we follow is the same as [Cavallo 2018].

3.2 Overview on grid diagrams

A grid diagram D of an oriented n–component link L in S^3 is a grid of $\mathrm{grd}(D)\times\mathrm{grd}(D)$ squares, representing the fundamental domain of a torus, together with a set of $\mathbb{O}$–markings $\mathbb{O}=\{O_1,\dots,O_{\mathrm{grd}(D)}\}$ and one of $\mathbb{X}$–markings $\mathbb{X}=\{X_1,\dots,X_{\mathrm{grd}(D)}\}$, such that there are exactly one O and one X in every column and every row. Moreover, we choose a nonempty subset $s\mathbb{O}$ which consists of at most one $\mathbb{O}$–marking for each component of L. We call these $\mathbb{O}$–markings *special* and the others *normal*.

The link L can be drawn in D by connecting the O's with the X's on a row and the X's with the O's on a column, specifying an orientation on L. Vertical lines always overpass the horizontal lines.

The chain complex $cCFL^\infty(D)$ is an $\mathbb{F}[V_1,V_1^{-1},\dots,V_m,V_m^{-1},U,U^{-1}]$–module, where

$$\mathrm{grd}(D)-1\geqslant m\geqslant \mathrm{grd}(D)-n$$

is the number of normal $\mathbb{O}$–markings, and is freely generated by the grid states $S(D)$. The differential is given by

$$\partial^- x=\sum_{y\in S(D)}\sum_{r\in\mathrm{Rect}^\circ(x,y)} V_1^{O_1(r)}\cdots V_m^{O_m(r)}\cdot U^{O(r)}y$$

for any $y\in S(D)$, where $O_i(r)$ is equal to one if the normal $\mathbb{O}$–marking $O_i\in r$ and zero otherwise, and $O(r)$ is the number of special $\mathbb{O}$–markings in r. The set Rect° denotes some special rectangles in D; see [Ozsváth et al. 2015] for details. As in Section 2.1 we extend the differential to $cCFL^\infty(D)$ by taking $\partial^-(V_i^{\pm1}p)=V_i^{\pm1}\cdot\partial^- p$ for every $i=1,\dots,m$ and p in the complex. The variables V_i are all homotopic to U so our homology group still has a natural structure of an $\mathbb{F}[U,U^{-1}]$–module.

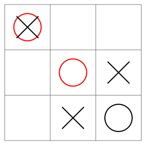

Figure 7: A grid diagram representing the unlink $\bigcirc_2$. The red circles denote the special $\mathbb{O}$–markings.

The Maslov and Alexander gradings are also combinatorially defined from D [Ozsváth et al. 2015] and each variable drops them by 2 and 1 respectively; while to define the j–filtration we need to specify that the level t is generated by the elements in

$$V_1^{i_1} \cdots V_m^{i_m} U^i \cdot cCFL^-(D),$$

where $i_1 + \cdots + i_m + i = -t$ and $cCFL^-(D)$ is the free $\mathbb{F}[V_1, \dots, V_m, U]$–module over $S(D)$. With this definitions in place we have the following theorem of [Manolescu et al. 2007]; see also [Sarkar 2011].

Theorem 3.1 (Manolescu, Ozsváth, Szabó and Thurston) *The $\mathcal{A}$–filtered chain homotopy type as an $\mathbb{F}[U, U^{-1}]$–module of $cCFL^\infty_*(D)$ coincides with the one of $cCFL^\infty_*(\mathcal{D})$ together with the Maslov grading and the algebraic filtration, where D is a grid and $\mathcal{D}$ is a Heegaard diagram for L. In particular, if D_1 and D_2 represent isotopic links then $cCFL^\infty_*(D_1)$ is locally equivalent to $cCFL^\infty_*(D_2)$.*

The way the filtered homology group $c\mathcal{HFL}^\infty(L)$ is defined and how the filtration $\mathcal{F}$ descends into homology are the same as in the previous section.

Remark 3.2 More precisely, Theorem 3.1 tells us that $cCFL^\infty_*(D_1)$ is $\mathcal{A}$–filtered, but not necessarily $\mathcal{F}$–filtered chain homotopy equivalent to $cCFL^\infty_*(D_2)$, while all the maps induced by link isotopies preserve the algebraic filtration; in the sense that, say m_1 and m_2 are the numbers of normal $\mathbb{O}$–markings in D_1 and D_2, the image of $V_1^{i_1} \cdots V_{m_1}^{i_{m_1}} U^i \cdot cCFL^\infty_*(D_1)$ is contained in $V_1^{i_1} \cdots V_{m_1}^{i_{m_1}} U^i \cdot cCFL^\infty_*(D_2)$ when $m_1 \leqslant m_2$ or $V_1^{i_1} \cdots V_{m_2}^{i_{m_2}} U^{i+i_{m_2+1}+\cdots+i_{m_1}} \cdot cCFL^\infty_*(D_2)$ when $m_1 > m_2$ for every (m_1+1)–tuple of integers $(i_1, \dots, i_{m_1}, i)$. We call this relation between $cCFL^\infty_*(D_1)$ and $cCFL^\infty_*(D_2)$ (or between the complexes given by a grid and a Heegaard diagram for the same link) an *almost filtered chain homotopy equivalence* and it implies that the complexes are locally equivalent, as stated in Theorem 3.1.

Figure 7 shows a grid diagram for the two-component unlink $\bigcirc_2$. We conclude this subsection with a lemma that we need for later.

Lemma 3.3 *Given a grid diagram D for a link, we can always change the $\mathbb{X}$–markings to obtain another diagram D' which represents an unlink.*

Proof We apply the following algorithm. Let us shift the rows of D until there is a special $\mathbb{O}$–marking in the top row (remember that D is the fundamental domain of a torus); then, starting from this $\mathbb{O}$–marking denoted by O_1, we put an $\mathbb{X}$–marking just below O_1. We keep doing this procedure with the $\mathbb{O}$–markings

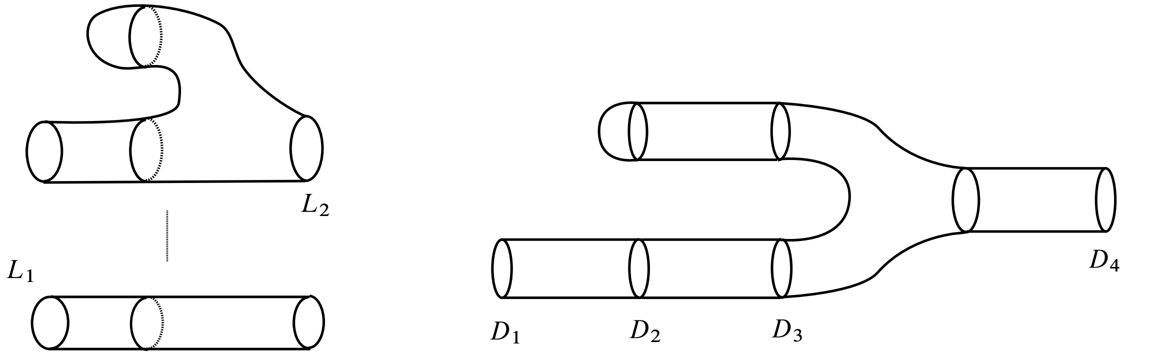

Figure 8: An extended birth move, corresponding to a 0–handle attachment followed by a 1–handle (left). The picture on the right shows only the component of Σ where the 0–handle appears.

in the row below, unless we reach an O_i such that O_{i+1} (in the row below) is special. Note that this can happen also when $i = 1$. In this case we put the new $\mathbb{X}$–marking in the same column of O_i, but not in the row below while in the row where the previous special $\mathbb{O}$–marking appeared.

When it happens that two consecutive rows j and $j+1$ both have special $\mathbb{O}$–markings on them, we put the $\mathbb{X}$–marking in the same square of O_j and we continue the algorithm from O_{j+1}. At some point we reach the lowest row; in this case, we assume the next row is the very top row (which contains a special $\mathbb{O}$–marking) and we put X accordingly.

The reader may shift the rows back to the original ordering; in any case, it is easy to check that the new diagram D' represents an unlink and the number of components coincides with the number of the special $\mathbb{O}$–markings in D. □

3.3 Extended birth moves

Let us study the concordance Σ given as in Figure 8: we first need a suitable choice of grid diagrams D_i for $i = 1, \dots, 4$, representing the links that appear in the extended birth move at the times shown in the picture. Second, we define maps $b_1 \colon D_1 \to D_2$ and $b_2 \colon D_3 \to D_4$; the first map represents the disjoint union of L_1 with an unknot, while the second one the merge move that we need in order to join the new component to L_1. Note that the diagrams D_2 and D_3 present isotopic links; then the corresponding chain complexes are related by an almost filtered chain homotopy equivalence, as in Theorem 3.1, and thus they are locally equivalent.

Let us start with b_2: the merge move is described by the diagram fragments in Figure 9, where we assumed that no special $\mathbb{O}$–markings were on the new unknotted component. More explicitly, we are picking the diagram D_3 in the way that it contains the fragment on the left in Figure 9, while D_4 is the resulting diagram after applying the move. At this point we define D_3' and D_4' as the grid diagrams obtained by applying the algorithm in Lemma 3.3 to D_3 and D_4. This means that they are both diagrams

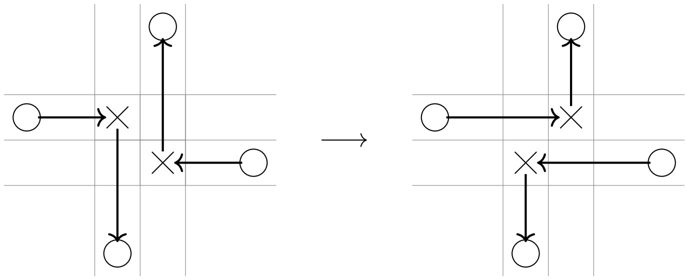

Figure 9: Band move in a grid diagram.

(the same ones!) for $\bigcirc_n$, where n is the number of components of L_1 and L_2, with the $\mathbb{O}$–markings in the same position as in D_3 and D_4.

Since the differential and the j–filtration do not depend on the position of the $\mathbb{X}$–markings as we see from their definition in Section 3.2, and this holds also for the Maslov grading [Ozsváth and Szabó 2008; Ozsváth et al. 2015], the identity map

$$\mathrm{Id}\colon cCFL^\infty(D_3') \to cCFL^\infty(D_4')$$

is a chain map, which clearly induces a graded isomorphism in homology and preserves the algebraic filtration.

Proposition 3.4 *The map $b_2 := \mathrm{Id}\colon cCFL^\infty(D_3) \to cCFL^\infty(D_4)$ preserves the Maslov grading and the $\mathcal{F}$–filtration and induces an isomorphism in homology.*

Proof In order to prove the claim we have to show that the map induces a graded isomorphism in homology and that preserves the two filtrations j and $\mathcal{A}$. The first two properties only depend on the $\mathbb{O}$–markings so they hold because b_2 is defined as the identity map; we only need to show that

$$b_2(\mathcal{A}^s cCFL^\infty(D_3)) \subset \mathcal{A}^s cCFL^\infty(D_4).$$

This can be checked by proving that $A_{D_4}(x) \leqslant A_{D_3}(x)$ for every $x \in S(D_3)$. Note that this is not obvious, even if b_2 is the identity; in fact, this time we need to consider the $\mathbb{X}$–markings in their original position, not like in D_3' and D_4', and the Alexander grading depend on the X's. Hence, we need to use a result of Sarkar [2011, Section 3.4], which gives us exactly what we need. □

We now want to define b_1. We suppose D_1 has an X in the top-right corner; then we use the move in Figure 10. Of course the new doubly marked square is not a special $\mathbb{O}$–marking. We consider the filtered NE–stabilization map

$$s^{\mathrm{NE}}\colon cCFL^\infty(D_1) \to cCFL^\infty(\widetilde{D}_2)$$

defined in [Manolescu et al. 2007; Ozsváth et al. 2015; Sarkar 2011]. Stabilizations relate isotopic links; therefore, such a map is an almost filtered chain homotopy equivalence for Theorem 3.1 and thus a local equivalence.

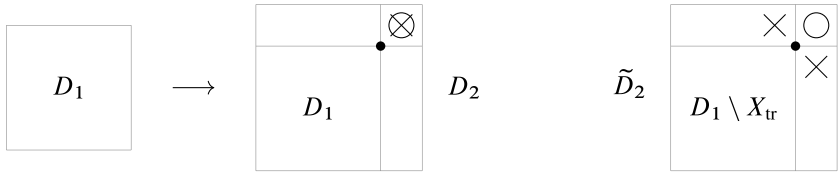

Figure 10: Birth move in a grid diagram. In the diagram $\widetilde{D}_2$, the top right $\mathbb{X}$–marking X_{tr} in D_1 does not appear.

We say that $b_1 := s^{\mathrm{NE}}: cCFL^\infty(D_1) \to cCFL^\infty(D_2)$. This makes sense because the stabilization maps, in the filtered setting, are independent of the position of the $\mathbb{X}$–markings. Hence, we have the following proposition.

Proposition 3.5 *The map $b_1: cCFL^\infty(D_1) \to cCFL^\infty(D_2)$ preserves the Maslov grading and the $\mathcal{F}$–filtration and induces an isomorphism in homology.*

Proof We cannot argue that b_1 is an $\mathcal{A}$–filtered chain homotopy equivalence, because the $\mathbb{X}$–markings in D_2 are different with respect to the ones in $\widetilde{D}_2$. On the other hand, we still have that it is a chain homotopy equivalence and preserves the j–filtration; in fact, as in Proposition 3.4 these two properties ignore the X's. Therefore, we just need to show that $A_{D_2}(s^{\mathrm{NE}}(x)) \leqslant A_{D_1}(x)$ for every $x \in S(D_1)$. This follows from another result of Sarkar [2011, Section 3.4]. □

Going back to the concordance Σ, we obtain the following theorem.

Theorem 3.6 *There is a map $b_\Sigma: cCFL^\infty(D_1) \to cCFL^\infty(D_4)$, which preserves the Maslov grading and the $\mathcal{F}$–filtration and induces an isomorphism*

$$b_\Sigma^*: c\mathcal{HFL}^\infty(L_1) \to c\mathcal{HFL}^\infty(L_2).$$

In particular, this means that

$$b_\Sigma^*(\mathcal{F}^S c\mathcal{HFL}_d^\infty(L_1)) \subset \mathcal{F}^S c\mathcal{HFL}_d^\infty(L_2)$$

for every $d \in \mathbb{Z}$ and southwest region S of $\mathbb{Z}^2$.

Proof We have that $b_\Sigma = b_2 \circ b \circ b_1$, where b is the almost filtered chain homotopy equivalence between the complexes given by D_2 and D_3. Then the statement follows from Theorem 3.1 and Propositions 3.4 and 3.5, because each piece of the map induces a graded isomorphism in homology and preserves the filtration $\mathcal{F}$. □

3.4 Extended death moves and invariance

An extended death cobordism is described in Figure 11. If $\Sigma \hookrightarrow S^3 \times I$ is such a cobordism between two n–component links L_1 and L_2 then Σ^*, the same cobordism seen in the ambient manifold $S^3 \times I$ with reversed orientation, can be considered an extended birth cobordism from L_2^* to L_1^*. Then we can prove the following proposition.

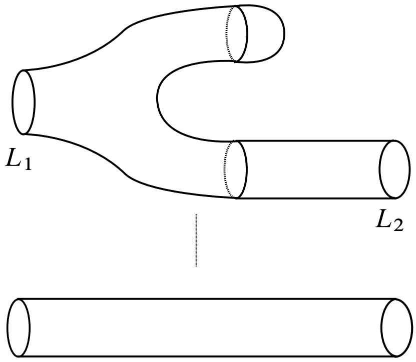

Figure 11: An extended death cobordism, corresponding to a 2–handle attachment together with a 1–handle.

Theorem 3.7 *There is a map $d_\Sigma\colon cCFL^\infty(L_1) \to cCFL^\infty(L_2)$ which preserves the Maslov grading and the $\mathcal{F}$–filtration and induces an isomorphism in homology.*

Note that, since $cCFL_d^\infty(D)$ is usually not a finite dimensional $\mathbb{F}$–vector space when D is a grid diagram, we cannot directly apply Theorem 2.8 in this case, although this can be done after more work.

Proof Denote by $cCFL^\infty(L_i)$ the filtered chain homotopy type of the complexes associated to L_i. By Theorems 2.8 and 3.1, the dual complex $cCFL^\infty(L_i)^*$ represents the almost filtered chain homotopy type of $cCFL^\infty(D_i^*)$.

We use Theorem 3.6 to say that, up to composing with some j–filtration preserving $\mathcal{A}$–filtered chain homotopy equivalences, we can suppose the existence of a map $b_{\Sigma^*}\colon cCFL^\infty(L_2)^* \to cCFL^\infty(L_1)^*$ which has all the property we want. If we take $b_{\Sigma^*,*}$ as the dual of this map then

$$b_{\Sigma^*,*}\colon cCFL^\infty(L_1) \to cCFL^\infty(L_2)$$

preserves the filtration $\mathcal{F}$ and induces precisely a graded isomorphism in homology; this is because the definition of the dual complex in Section 2.3 implies that $cCFL^\infty(L)^{**}$ has a natural identification with $cCFL^\infty(L)$ for every link L.

We conclude by saying that $d_\Sigma := b_{\Sigma^*,*}$ again up to composition with some j–filtration preserving $\mathcal{A}$–filtered chain homotopy equivalences. □

Now with this theorem set we can prove one of the main results of the paper.

Proof of Theorem 1.2 After applying Theorems 3.6 and 3.7, by considering the maps induced by a concordance Σ from L_1 to L_2, we obtain a graded isomorphism F, between the homology groups, such that $F(\mathcal{F}^S c\mathcal{HFL}_d^\infty(L_1)) \subset \mathcal{F}^S c\mathcal{HFL}_d^\infty(L_2)$, which gives

$$\dim_{\mathbb{F}} \mathcal{F}^S c\mathcal{HFL}_d^\infty(L_1) \leqslant \dim_{\mathbb{F}} \mathcal{F}^S c\mathcal{HFL}_d^\infty(L_2). \tag{3-1}$$

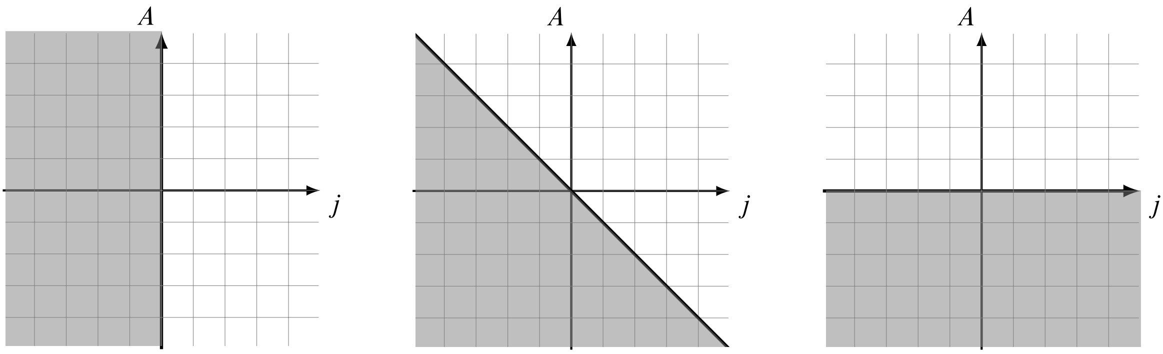

Figure 12: The centered southwest regions A_0 (left), A_1 (middle) and A_2 (right) of $\mathbb{R}^2$.

In order for F to be a filtered isomorphism we also need that it restricts to an isomorphism on each level of the $\mathcal{F}$–filtration. To see this we take another concordance Σ' from L_2 to L_1 and, in the same way as before, we get the opposite inequality with respect to (3-1). □

We now show that the Υ–type invariants are indeed concordance invariants. In order to prove this fact, we only need that the $\mathcal{F}$–filtered isomorphism type of the homology group is a concordance invariant.

Theorem 3.8 *The real number $\Upsilon_S(L)$ is a concordance invariant for every centered southwest region S in $\mathbb{R}^2$.*

Proof By Theorem 1.2,

$$\mathcal{F}^{S_k} c\mathcal{HFL}_0^\infty(L_1) \supset \mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}_0^\infty(L_1) \iff \mathcal{F}^{S_k} c\mathcal{HFL}_0^\infty(L_2) \supset \mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}_0^\infty(L_2)$$

for every $k \in \mathbb{R}$, since L_1 is concordant to L_2. By definition, this immediately implies $\Upsilon_S(L_1) = \Upsilon_S(L_2)$ for every southwest region S. □

4 Upsilon-type invariants

4.1 Definition of $\Upsilon_S^*(L)$ and the Υ–function for links

In Section 2.2 we saw that $\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}_0^\infty(L)$ is isomorphic to $\mathbb{F}$ for every link. Using Theorem 2.1 we can also argue that

$$\frac{\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}_{1-n}^\infty(L)}{\mathcal{F}^{\{j\leqslant -1\}} c\mathcal{HFL}_{1-n}^\infty(L)} \cong_{\mathbb{F}} \mathbb{F},$$

where n is the number of components of L. Then, for a given centered southwest region $S \subset \mathbb{R}^2$, we can define

$$\Upsilon_S^*(L) := \max\{k \in \mathbb{R} \mid \mathcal{F}^{S_k} c\mathcal{HFL}_{1-n}^\infty(L) \not\subset \mathcal{F}^{\{j\leqslant -1\}} c\mathcal{HFL}_{1-n}^\infty(L)\}.$$

Theorem 1.2 implies that $\Upsilon_S^*(L)$ is also a concordance invariant. Moreover, we observe that, for knots, Υ^* coincides with Υ.

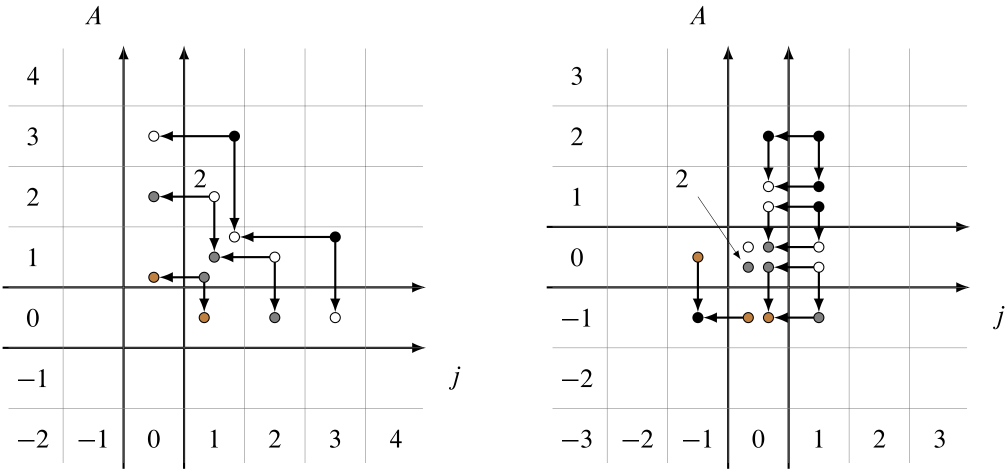

Figure 13: The complex $cCFL^\infty(T_{3,3})$ on the left and $cCFL^\infty(T'_{3,3})$ on the right. The 2 on the central staircase is the multiplicity of the subcomplex. White, gray and brown dots represent Maslov gradings 0, -1 and -2 respectively, while black dots represent the others.

In [Ozsváth et al. 2017a] the Υ–invariant is described as a piecewise linear function $\Upsilon_K(t)\colon [0,2] \to \mathbb{R}$ such that $\Upsilon_K(2-t) = \Upsilon_K(t)$ for every knot K and t. We call this function the *classical* Υ*–invariant*. In the case of links we give a similar definition, which can be seen as a particular case of Υ_S.

Consider the centered southwest region

$$A_t := \left\{(j,A) \in \mathbb{R}^2 \mid A \cdot \tfrac{1}{2}t + j\left(1 - \tfrac{1}{2}t\right) \leqslant 0\right\}$$

for $t \in [0,2]$; see Figure 12. It can be shown — see [Alfieri 2019] — that $\Upsilon_{A_t}(K) = \Upsilon_K(t)$ for every knot K. Moreover, we define

$$\Upsilon_L(t) := \Upsilon_{A_t}(L) \quad \text{and} \quad \Upsilon_L^*(t) := \Upsilon_{A_t}^*(L)$$

for every link. The reader can easily check that these $\mathbb{R}$–valued functions are piecewise linear and $\Upsilon_L(0) = \Upsilon_L^*(0) = 0$.

Example 4.1 In Figure 13 we show the chain complexes for the $(3,3)$–torus link, which can be computed from the Heegaard diagram in Figure 14. We write $T_{3,3}$ when we orient the three components in the same direction, while $T'_{3,3}$ denotes the same link with the orientation reversed on one component. From this picture we can easily compute the Υ–functions:

$$\Upsilon_{T'_{3,3}}(t) = 0 \quad \text{if } t \in [0,2], \qquad \Upsilon^*_{T'_{3,3}}(t) = \begin{cases} t & \text{if } t \in [0,1], \\ 2-t & \text{if } t \in [1,2], \end{cases}$$

$$\Upsilon_{T_{3,3}}(t) = \begin{cases} -3t & \text{if } t \in \left[0, \tfrac{2}{3}\right], \\ -2 & \text{if } t \in \left[\tfrac{2}{3}, \tfrac{4}{3}\right], \\ -6+3t & \text{if } t \in \left[\tfrac{4}{3}, 2\right], \end{cases} \qquad \Upsilon^*_{T_{3,3}}(t) = \begin{cases} -t & \text{if } t \in [0,1], \\ -2+t & \text{if } t \in [1,2]. \end{cases}$$

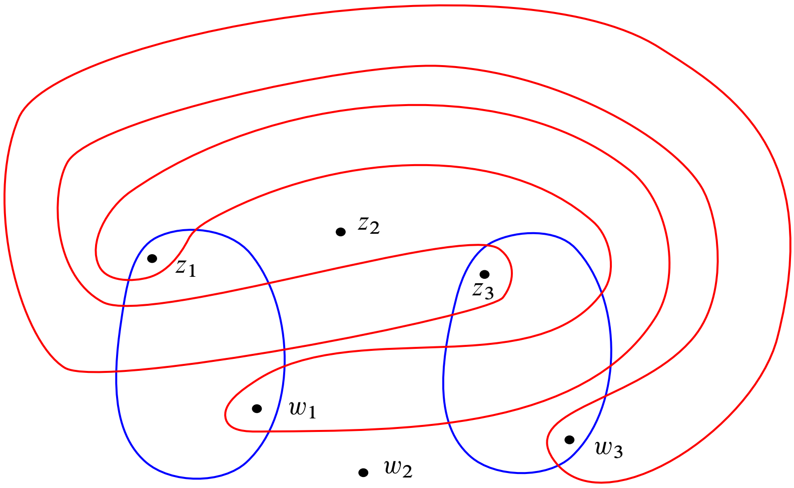

Figure 14: A Heegaard diagram for the link $T_{3,3}$. The α–curves are red, while the β–curves are blue.

Finally, we show that the classical Υ–invariants do not determine the $\mathcal{F}$–filtered isomorphism type of $c\mathcal{HFL}^\infty(L)$. In fact, take the knot $K = T_{4,5} \# T^*_{2,3;2,5} \# T^*_{2,5}$ whose homology is shown in Figure 15, where $T_{2,3;2,5}$ is the $(2,3)$–cable of $T_{2,5}$. Kim and Livingston [2018] proved that $\Upsilon_K(t) = \Upsilon^*_K(t) = 0$ for every $t \in [0,2]$. On the other hand, it is easy to check that $\Upsilon_{V_0}(K) = -2$, where $V_0 = \{(a,b) \mid a \leqslant 0,\, b \leqslant 0\}$, while $\Upsilon_{V_0}(\bigcirc) = 0$.

4.2 Symmetries

In this subsection we study some of the main properties of the Υ–invariants. We start from this proposition from [Ozsváth and Szabó 2008].

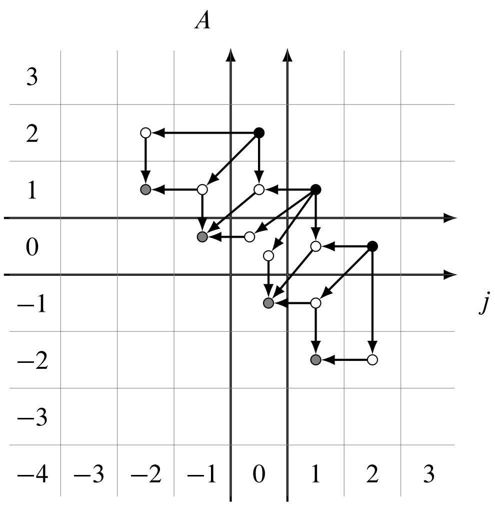

Figure 15: The acyclic summand of the chain complex $CFK^\infty(K)$, with $K = T_{4,5} \# T^*_{2,3;2,5} \# T^*_{2,5}$.

Proposition 4.2 (Ozsváth and Szabó) *The $\mathcal{F}$–filtered chain homotopy type $cCFL^\infty(L)$ of a link Floer complex is independent of the (global) orientation of L.*

In particular, we can identify the homology group of a link L and its reverse.

Corollary 4.3 *There is an $\mathcal{F}$–filtered isomorphism $c\mathcal{HFL}^\infty(L) \leftrightarrow c\mathcal{HFL}^\infty(-L)$. In particular, $\Upsilon_S(L) = \Upsilon_S(-L)$ and $\Upsilon^*_S(L) = \Upsilon^*_S(-L)$ for every centered southwest region S of $\mathbb{R}^2$.*

We remind the reader that this result is not true if we reverse the orientation only on some of the components of L, as we saw in the previous subsection with the link $T_{3,3}$.

Say $-S$ is the southwest region obtained from S after applying the reflection r of the plane with respect to the line $\{A - j = 0\}$. We prove the following property.

Theorem 4.4 *We have $\Upsilon_S(L) = \Upsilon_{-S}(L)$ and $\Upsilon^*_S(L) = \Upsilon^*_{-S}(L)$ for every centered southwest region S of $\mathbb{R}^2$. In particular, one has $\Upsilon_L(t) = \Upsilon_L(2-t)$ and $\Upsilon^*_L(t) = \Upsilon^*_L(2-t)$ for every $t \in [0,2]$.*

Proof Since a chain complex for $-L$ is obtained by switching the role of $\boldsymbol{w}$ and $\boldsymbol{z}$ in a Heegaard diagram for L, and then of the filtrations $\mathcal{A}$ and j, Corollary 4.3 tells us that $cCFL^\infty(L)$ is symmetric under r up to homotopy. Moreover, this symmetry is chain homotopic to the identity by [Sarkar 2015, Lemma 4.6] and the claim follows.

For the second part of the statement, we just need to observe that the reflected southwest region $-A_t$ corresponds to A_{2-t}. □

With this theorem set, from now on we consider the Υ–functions as defined on $[0,1]$, since their values on $[1,2]$ are then determined automatically.

Now we want to study the relation between the Υ's of L and its mirror image. We recall that, in Section 2.3, we defined ιS as the complement of the region obtained from S by applying a central symmetry. Then we say that $\overline{\iota S}$ is the topological closure of ιS.

Proposition 4.5 *For an n–component link L,*

$$\Upsilon_S(L^*) = -\Upsilon^*_{\overline{\iota S}}(L)$$

for every centered southwest region S of $\mathbb{R}^2$. In particular, we obtain $\Upsilon_{L^*}(t) = -\Upsilon^*_L(t)$ for every $t \in [0,1]$ and for a knot K one has $\Upsilon_S(K^*) = -\Upsilon_{\overline{\iota S}}(K)$.

Proof We apply Theorem 2.8 to argue that there is an identification

$$\mathcal{F}^S c\mathcal{HFL}^\infty_0(L^*) \leftrightarrow (\mathcal{F}^*)^S c\mathcal{HFL}^\infty_{n-1}(L)^* = \mathrm{Ann}\, \mathcal{F}^{\iota S} c\mathcal{HFL}^\infty_{1-n}(L)$$

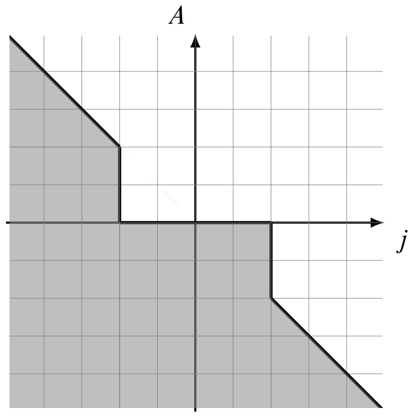

Figure 16: The centered southwest region T in the picture is such that $T = \overline{\iota T}$.

that preserves the containment relations. Hence, we only need to use the definition of Υ:

$$\begin{aligned}\Upsilon_S(L^*) &= \max\{k \in \mathbb{R} \mid \mathcal{F}^{S_k} c\mathcal{HFL}_0^\infty(L^*) \supset \mathcal{F}^{\{j \leqslant 0\}} c\mathcal{HFL}_0^\infty(L^*)\} \\ &= \max\{k \in \mathbb{R} \mid \operatorname{Ann} \mathcal{F}^{\iota S_k} c\mathcal{HFL}_{1-n}^\infty(L) \supset \operatorname{Ann} \mathcal{F}^{\{j \leqslant -1\}} c\mathcal{HFL}_{1-n}^\infty(L)\} \\ &= \max\{k \in \mathbb{R} \mid \mathcal{F}^{(\iota S)_{-k}} c\mathcal{HFL}_{1-n}^\infty(L) \subset \mathcal{F}^{\{j \leqslant -1\}} c\mathcal{HFL}_{1-n}^\infty(L)\} \\ &= -\min\{k \in \mathbb{R} \mid \mathcal{F}^{(\iota S)_k} c\mathcal{HFL}_{1-n}^\infty(L) \subset \mathcal{F}^{\{j \leqslant -1\}} c\mathcal{HFL}_{1-n}^\infty(L)\} \\ &= -\max\{k \in \mathbb{R} \mid \mathcal{F}^{\overline{\iota S}_k} c\mathcal{HFL}_{1-n}^\infty(L) \not\subset \mathcal{F}^{\{j \leqslant -1\}} c\mathcal{HFL}_{1-n}^\infty(L)\} = -\Upsilon^*_{\overline{\iota S}}(L)\end{aligned}$$

for every centered southwest region S in $\mathbb{R}^2$.

The third claim is trivial, while for the second one we note that $\overline{\iota A}_t = A_t$ for every $t \in [0, 1]$. □

We observe that the southwest regions A_t are not the only S such that $\overline{\iota S} = S$ as we see from Figure 16.

Let us recall that the homology group $\widehat{HFL}(L)$ (resp. $\widehat{\mathcal{HFL}}(L)$) is defined as the bigraded homology of the associated graded object (resp. the $\mathcal{A}$–filtered graded homology) of the complex $\widehat{CFL}(L)$, given by setting $U = 0$ in $cCFL^\infty(L)$; see [Cavallo 2018; Ozsváth and Szabó 2008] for details.

Lemma 4.6 *If a cycle in $\mathcal{F}^{\{j \leqslant 0\}} cCFL^\infty(L)$ is a generator of the homology group $c\mathcal{HFL}^\infty(L)$, and its homology class has minimal j–level zero, then its projection to $\widehat{CFL}(L)$ is a generator of $\widehat{\mathcal{HFL}}(L)$.*

Proof By [Rasmussen 2003, Lemma 4.5] we know that, up to changing basis, the complex $cCFL^\infty(L)$ is such that the differential of the bigraded object associated to $\widehat{CFL}(L)$ is zero. Therefore, if we pick a generator with minimal j–level zero then its projection cannot be zero in $\widehat{\mathcal{HFL}}(L)$, because clearly it would be homologous to an element of $U \cdot \mathcal{F}^{\{j \leqslant 0\}} cCFL^\infty(L) = \mathcal{F}^{\{j \leqslant -1\}} cCFL^\infty(L)$. □

We use the mirror image symmetry to prove the following proposition. We assume the reader to be familiar with the definition of the concordance invariants $\tau(L)$ and $\tau^*(L)$, given in [Cavallo 2018].

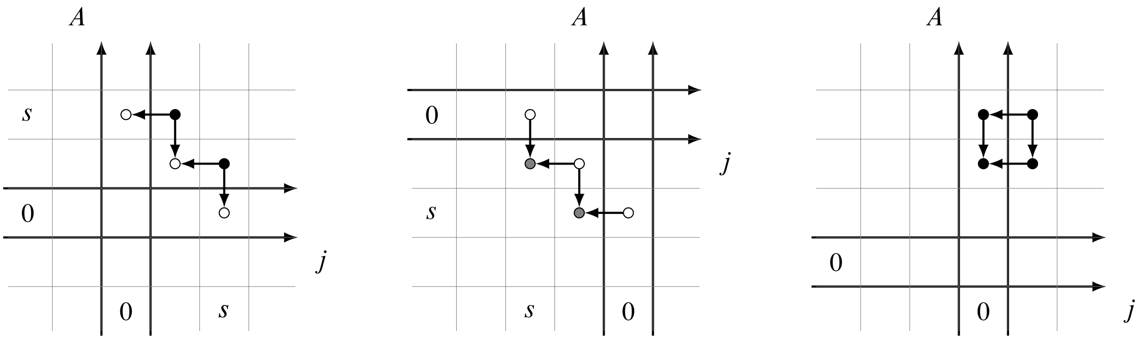

Figure 17: A positive staircase (left), a negative staircase (middle) and an acyclic square (right). The acyclic subcomplex of $cCFL^\infty(L)$, when L is as in Theorem 4.8, is the direct sum of acyclic squares.

Proposition 4.7 *For a link L,*

$$\tau(L) = -\Upsilon_L'(0) \quad \text{and} \quad \tau^*(L) = -(\Upsilon_L^*)'(0).$$

Furthermore, each slope of $\Upsilon_L(t)$ and $\Upsilon_L^*(t)$ is an integer s such that the Alexander grading subgroup $\widehat{HFL}_{*,s}(L)$ is nonzero, and if $t_0 \in (0,1)$ is a point where the slope changes from s_1 to s_2 then $t_0 \in \mathbb{Z}/|s_2 - s_1|$ and $|s_2 - s_1| > 2$.

Proof We prove the first part of the statement. We take $t \in [0, \varepsilon)$ with ε very small and we show that for such t's one has $\Upsilon_L(t) \leqslant -t \cdot \tau(L)$. Suppose that the homology class of x is a generator of $\mathcal{F}^{\{j \leqslant 0\}} c\mathcal{HFL}_0^\infty(L)$. By Lemma 4.6, $\bar{x}$, the projection of x to $\widehat{CFL}_{0,*}(L)$, is a generator of $\widehat{\mathcal{HFL}}_0(L)$. Hence, assuming $\Upsilon_L(t) > -t \cdot \tau(L)$ contradicts the fact that $\tau(L)$ is the minimum $\mathcal{A}$–level s such that $\mathcal{A}^s\widehat{\mathcal{HFL}}_0(L)$ has dimension one; see [Cavallo 2018].

We now show that $\Upsilon_L(t) \geqslant -t \cdot \tau(L)$. In fact, the same argument we used before also shows that $\Upsilon_{L^*}^*(t) \leqslant -t \cdot \tau^*(L^*)$ for $t \in [0, \varepsilon)$ and then $-\Upsilon_L(t) \leqslant -t(-\tau(L))$ from Proposition 4.5 and the symmetry properties of τ^*; see [Cavallo 2018]. This proves the claim; in fact, the version for the Υ^*–function can be proved applying Proposition 4.5.

The second part of the proposition follows from the same proof of [Ozsváth et al. 2017a, Proposition 1.4] and [Feller et al. 2019, Observation 2.2]. □

Using this result we immediately compute the Υ–functions for the Hopf links $H_\pm$. In fact $H_\pm$ is a nonsplit 2–component link that bounds an annulus in S^3. Since $\widehat{HFL}$ detects the Thurston norm [Ni 2009, Theorem 1.1], this implies that $\widehat{HFL}_{*,s}(H_\pm)$ is nonzero only when $s = -1, 0, 1$ and then $\Upsilon_{H_\pm}$ and $\Upsilon_{H_\pm}^*$ are determined by the τ–invariants, which are computed in [Cavallo 2018, Corollary 3.7]. Therefore,

$$\Upsilon_{H_+}(t) = -t, \quad \Upsilon_{H_-}^*(t) = t \quad \text{and} \quad \Upsilon_{H_+}^*(t) = \Upsilon_{H_-}(t) = 0$$

for every $t \in [0,1]$.

We conclude this subsection by stating a result of Petkova [2013] that allows us to determine $cCFL^\infty(L)$ for every nonsplit alternating link. We recall that an n–component link L is $\widehat{HFL}$*–thin* if its homology group $\widehat{HFL}_{d,s}(L)$ is supported on the line $s = d + \frac{1}{2}(n-1-\sigma(L))$, where $\sigma(L)$ is the signature of L.

Theorem 4.8 (Petkova) *Suppose that the link L has n components and it is $\widehat{HFL}$–thin. Then the chain complex $cCFL^\infty(L)$ is given as the direct sum of some $\mathbb{F}[U,U^{-1}]$–subcomplexes as in Figure 17. More specifically, for every*

$$s \in \left\{\tfrac{1}{2}(n-1-\sigma(L)) - k\right\} \quad \text{with } k = 0, \dots, n-1,$$

we have $\binom{n-1}{k}$ positive (resp. negative) staircases when s is positive (resp. negative). Moreover, the acyclic subcomplex is determined by

$$\chi(\widehat{HFL}(L))(t,t^{-1}) = \sum_{d\in\mathbb{Z}} (-1)^d \dim_{\mathbb{F}} \widehat{HFL}_{d,s}(L)\cdot t^s = (t^{1/2}-t^{-1/2})^{n-1}\cdot \nabla_L(t^{1/2}-t^{-1/2}),$$

where $\nabla_L(z)$ is the Conway normalization of the Alexander polynomial of L.

Note that quasialternating links (and then nonsplit alternating links) are $\widehat{HFL}$–thin; see [Cavallo 2018; Ozsváth et al. 2015]. In Figure 18 we show a Whitehead link and its corresponding complex.

4.3 Connected sums and disjoint unions

It follows from the work of Ozsváth and Szabó that the chain complex for a connected sum of the links L_1 and L_2 is given by the tensor product between the ones of L_1 and L_2.

Theorem 4.9 (Ozsváth and Szabó) *Given two links L_1 and L_2, denote by $L_1 \#_{i,j} L_2$ the connected sum performed on the i– and j–component of L_1 and L_2, respectively. Then*

$$cCFL^\infty(L_1 \#_{i,j} L_2) \cong cCFL^\infty(L_1) \otimes_{\mathbb{F}[U,U^{-1}]} cCFL^\infty(L_2).$$

In particular, the complex $cCFL^\infty(L_1 \# L_2)$ does not depend on i and j.

Since $\mathbb{F}[U,U^{-1}]$ is a principal ideal domain, using the Künneth formula and Theorem 2.1 on the identification in Theorem 4.9 gives

$$\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}^\infty_0(L_1 \# L_2) \cong_{\mathbb{F}} \mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}^\infty_0(L_1) \otimes_{\mathbb{F}} \mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}^\infty_0(L_2)$$

and

$$\frac{\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}^\infty_{2-n_1-n_2}(L_1 \# L_2)}{\mathcal{F}^{\{j\leqslant -1\}} c\mathcal{HFL}^\infty_{2-n_1-n_2}(L_1 \# L_2)} \cong_{\mathbb{F}} \frac{\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}^\infty_{1-n_1}(L_1)}{\mathcal{F}^{\{j\leqslant -1\}} c\mathcal{HFL}^\infty_{1-n_1}(L_1)} \otimes_{\mathbb{F}} \frac{\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}^\infty_{1-n_2}(L_2)}{\mathcal{F}^{\{j\leqslant -1\}} c\mathcal{HFL}^\infty_{1-n_2}(L_2)},$$

where n_i is the number of components of L_i and we recall that n_1+n_2-1 is the one of $L_1 \# L_2$. Furthermore, if the homology classes of x_i are generators for $\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}^\infty_0(L_i)$ then $[x_1 \otimes x_2]$ is a generator of the homology group $\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}^\infty_0(L_1 \# L_2)$. In the same way, if y_i is such that $[y_i]$ is a generator of $c\mathcal{HFL}^\infty_{1-n_i}(L_i)$, with minimal j–level zero, then $[y_1 \otimes y_2]$ is a generator of $c\mathcal{HFL}^\infty_{2-n_1-n_2}(L_1 \# L_2)$ and its minimal algebraic level is again zero.

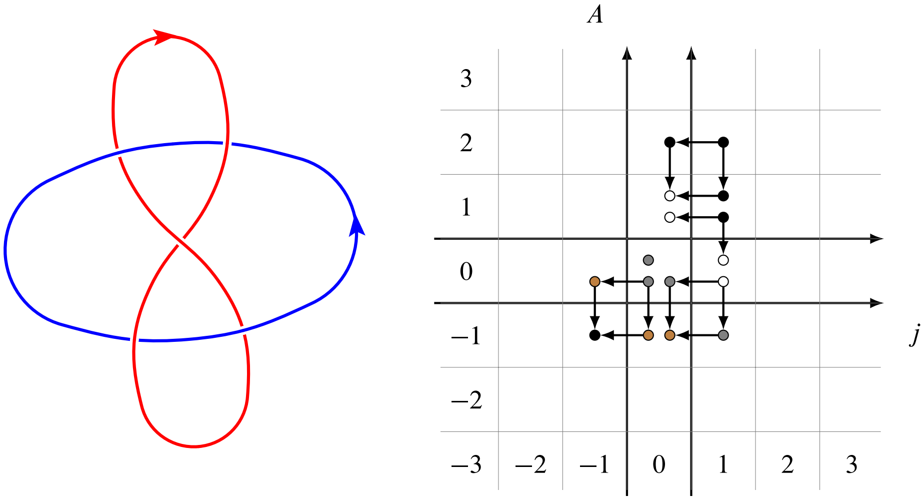

Figure 18: The complex $cCFL^{\infty}(W)$ (right) of the Whitehead link W (left).

We can now study how the Υ–invariants behave under connected sums. For every centered southwest region S of $\mathbb{R}^2$ we define

$$\mathrm{env}(S) = \{(j, A) \in \mathbb{R}^2 \mid j = a_1 + a_2 \text{ and } A = b_1 + b_2 \text{ where } (a_i, b_i) \in S \text{ for } i = 1, 2\}.$$

Clearly, the region $\mathrm{env}(S)$ is still a southwest region (unless it coincides with the whole $\mathbb{R}^2$) and $S \subset \mathrm{env}(S)$. Moreover, we take $h(S) \in \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$ as

$$\inf\{k \in \mathbb{N} \mid S_{-k} \supset \mathrm{env}(S)\}$$

and we state the following proposition.

Proposition 4.10 *Let us consider a link L_i with n_i components for $i = 1, 2$ and S a centered southwest region of $\mathbb{R}^2$. Then*

$$\Upsilon_S(L_1 \# L_2) \geqslant \Upsilon_S(L_1) + \Upsilon_S(L_2) - h(S),$$
$$\Upsilon^*_S(L_1 \# L_2) \geqslant \Upsilon^*_S(L_1) + \Upsilon^*_S(L_2) - h(S).$$

In particular, if $S = \mathrm{env}(S)$ then the Υ's and Υ^*'s are superadditive under connected sums.

Proof The proof of the two inequalities is exactly the same; hence, we only do the first one. From what we said at the beginning of the subsection we can take x and y, such that their homology classes are generators of $\mathcal{F}^{\{j \leqslant 0\}} c\mathcal{HFL}^{\infty}_0(L_i)$ for $i = 1, 2$, in the region $S_{\Upsilon_S(L_i)} = S_{\gamma_i}$, and we obtain that $[x \otimes y]$ is a generator of $\mathcal{F}^{\{j \leqslant 0\}} c\mathcal{HFL}^{\infty}_0(L_1 \# L_2)$ and $x \otimes y \in \mathcal{F}^{\mathrm{env}(S)_{\gamma_1+\gamma_2}} cCFL^{\infty}_0(L_1 \# L_2)$. Therefore, from the definition of $h(S)$ it follows that

$$\mathrm{env}(S)_{\gamma_1+\gamma_2} \subset S_{\gamma_1+\gamma_2-h(S)}$$

and $x \otimes y \in \mathcal{F}^{S_{\gamma_1+\gamma_2-h(S)}} cCFL^{\infty}_0(L_1 \# L_2)$, proving the inequality. □

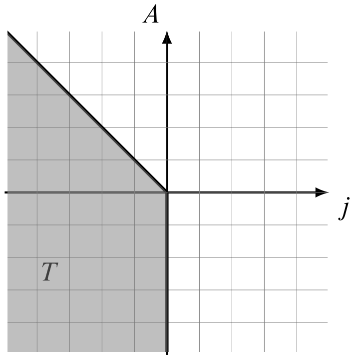

Figure 19: The centered southwest region T is such that $h(T)=0$.

There are examples of southwest regions S with $h(S)\neq 0$ and Υ is not superadditive. Take the region $V_1=\{(j,A)\in\mathbb{R}^2 \mid j\leqslant 0, A\leqslant 1\}$, then

$$-4=\Upsilon_{V_1}(T_{2,3}\,\#\,T_{2,7})<\Upsilon_{V_1}(T_{2,3})+\Upsilon_{V_1}(T_{2,7})=0+(-2)=-2.$$

Corollary 4.11 *If a centered southwest region S is such that $\overline{\iota S}=S$ and $h(S)=0$ then*

$$\Upsilon_S(L_1\,\#\,L_2)=\Upsilon_S(L_1)+\Upsilon_S(L_2) \quad \textit{and} \quad \Upsilon^*_S(L_1\,\#\,L_2)=\Upsilon^*_S(L_1)+\Upsilon^*_S(L_2)$$

for every pair of links L_1 and L_2. In particular, this holds for the classical Υ functions.

Proof By Propositions 4.5 and 4.10,

$$\Upsilon_S(L_1)+\Upsilon_S(L_2)-h(S)\leqslant\Upsilon_S(L_1\,\#\,L_2)\leqslant\Upsilon_S(L_1)+\Upsilon_S(L_2)+h(\overline{\iota S}).$$

The claim follows by using the assumption that $h(S)=h(\overline{\iota S})=0$. The same proof works for Υ^*. □

We observe that there are centered southwest regions, different from the A_t's, for which $h(S)=0$ and their Υ–invariants are superadditive; see Figure 19.

Example 4.12 By Corollary 4.11, for the positive and negative Hopf link one has $\Upsilon_{H_+\#H_-}(t)=-t$ and $\Upsilon^*_{H_+\#H_-}(t)=t$ for every $t\in[0,1]$. In particular, we have an example when $cCFL^\infty(L)\otimes cCFL^\infty(L^*)$ is not locally equivalent to the chain complex of an unlink; in fact, it is $\Upsilon_{\bigcirc_m}=0$ for every $m\in\mathbb{N}$.

The disjoint union of two links can be seen as a special connected sum. In fact, the link $L_1\sqcup L_2$ is isotopic to $L_1\,\#\bigcirc_2\#\,L_2$, where the two connected sums are performed on different components of the unlink $\bigcirc_2$.

Proposition 4.13 *The chain complex of the link $L_1\sqcup L_2$ is given by*

$$\begin{aligned} cCFL^\infty(L_1\sqcup L_2)&\cong cCFL^\infty(L_1\,\#\,L_2)\otimes_{\mathbb{F}[U,U^{-1}]}cCFL^\infty(\bigcirc_2)\\ &\cong cCFL^\infty(L_1\,\#\,L_2)\oplus cCFL^\infty(L_1\,\#\,L_2)[\![1]\!], \end{aligned}$$

where $[\![\cdot]\!]$ denotes a shift in the Maslov grading.

Proof It is easy to compute that $cCFL_*^\infty(\bigcirc_2) \cong \mathbb{F}[U,U^{-1}]_{(0)} \oplus \mathbb{F}[U,U^{-1}]_{(-1)}$. Hence, the claim follows from Theorem 4.9. □

Note that, since the chain complex for the connected sum is independent of the choice of the components, $cCFL^\infty(L_1 \sqcup L_2) = cCFL^\infty((L_1 \# L_2) \sqcup \bigcirc)$; in other words, there is an identification between the chain complexes for the disjoint union and the link obtained by adding an unknot to any connected sum of L_1 and L_2.

Corollary 4.14 *Given two links L_1 and L_2,*

$$\Upsilon_S(L_1 \sqcup L_2) = \Upsilon_S(L_1 \# L_2) \quad \textit{and} \quad \Upsilon_S^*(L_1 \sqcup L_2) = \Upsilon_S^*(L_1 \# L_2)$$

for every southwest region S of $\mathbb{R}^2$.

Proof This follows immediately from Theorem 2.1 and Proposition 4.13. □

4.4 Slice genus

Suppose that a link L has n components and bounds a smooth, compact, oriented surface $\Sigma \hookrightarrow D^4$ with genus $g(\Sigma)$ and k connected components. Then, after removing k open disks from it, we can see Σ as a smooth cobordism between the k–component unlink $\bigcirc_k$ and L. If we look at the canonical form of link cobordisms described in Section 3.1 then Σ is such that, from left to right, there are no merge moves, the torus moves are $g(\Sigma)$ in total and there are exactly $n-k$ split moves. Other than these, the cobordism Σ might have pieces representing concordances, which induce local equivalences as shown in Section 3.

The goal of this subsection is to study how much the Υ–invariants of L differ from zero ($\Upsilon_S(\bigcirc_n) = 0$ for every S) when L bounds a surface Σ as before. We use grid diagrams like in Section 3.

Let us start from the torus move; see Figure 6. We define a map t as the identity between the grid diagram representing the link before the move and the one obtained by applying Figure 9 twice. Such a map is a chain map, induces a graded isomorphism in homology and preserves the j–filtration by the same argument in Section 3.3: since the links before and after the moves have both k components, the corresponding diagrams have the same $\mathbb{O}$–markings (both normal and special). Previously we used a result of Sarkar [2011] to show that b_2 is $\mathscr{A}$–filtered of degree zero. Since now we are composing the same map twice, but the first time the number of components is increasing, this is no longer true. In fact, the map t is $\mathscr{A}$–filtered of degree 1; see [Sarkar 2011, Section 3.4].

Now we study the split moves as in the left side of Figure 20. We may want to define a map s in a similar way as what we do for t: using the same procedure for the map b_2, but this is not possible. In fact, the link L_2 has one more component than L_1, so the number of special $\mathbb{O}$–markings is different and s would not be a chain map. To avoid this problem, before applying the split move we add a disjoint unknot to L_1

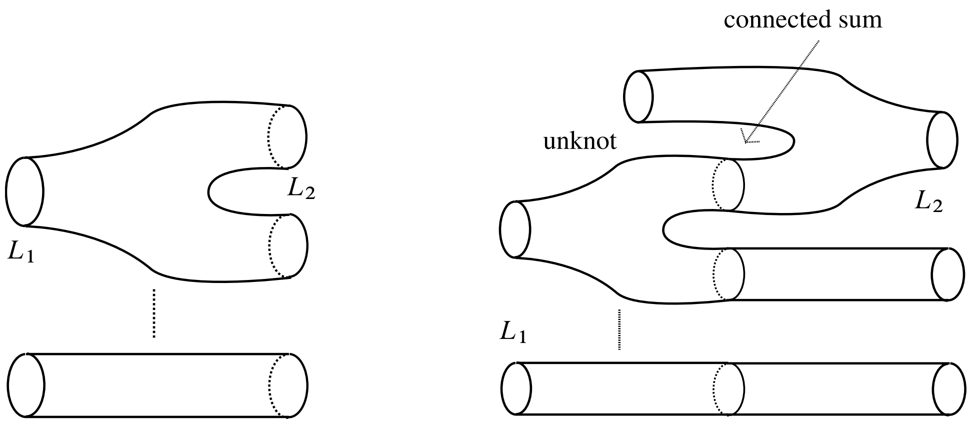

Figure 20: A split move. The two cobordisms in the picture are isotopic in $S^3 \times I$ after capping the unknot.

and after the split move we connect sum the unknot to the component without special $\mathbb{O}$–markings. This is pictured on the right side of Figure 20. In this way, we can define a map

$$s_2 \colon cCFL_0^\infty(D_1 \sqcup \bigcirc) \to cCFL_0^\infty(D_2),$$

where D_i is a grid diagram for L_i, exactly in the same way as t. Now from Proposition 4.13, the map

$$s_1 := cCFL_0^\infty(D_1') \to cCFL_0^\infty(D_1' \sqcup \bigcirc) = cCFL_0^\infty(D_1') \oplus cCFL_1^\infty(D_1')$$

is the inclusion of $cCFL_0^\infty(D_1')$ as the first summand of $cCFL_0^\infty(D_1' \sqcup \bigcirc)$; and we recall that D_i' is the grid diagram obtained from D_i by applying the algorithm in Section 3.2. Hence, the map s_1 preserves the Maslov grading and the filtration $\mathscr{F}$. We conclude that the composition

$$s := s_2 \circ s_1 \colon cCFL_0^\infty(D_1) \to cCFL_0^\infty(D_2)$$

induces a graded injective homomorphism in homology, preserves j and it is $\mathscr{A}$–filtered of degree 1.

Given a centered southwest region S of $\mathbb{R}^2$, we say that

$$S + m := \{(j, A) \in \mathbb{R}^2 \mid (j, A - m) \in S\}$$

for every $m \in \mathbb{N}$, an example is given in Figure 21. We define the nonnegative integer $h_S(m)$ as

$$\min\{k \in \mathbb{N} \mid (0, m) \in S_{-k}\}$$

and we recall that the reversed region $-S$ is defined in Section 4.2 by applying to S the reflection of $\mathbb{R}^2$ with respect to the line $\{A - j = 0\}$. Then we can prove that each Υ gives a lower bound for the genus of Σ.

Proposition 4.15 *If L is an n–component link in S^3, which bounds a surface Σ as before, then*

$$-\Upsilon_S(L) \leqslant h_{\pm S}(g(\Sigma) + n - k)$$

for every centered southwest region S of $\mathbb{R}^2$.

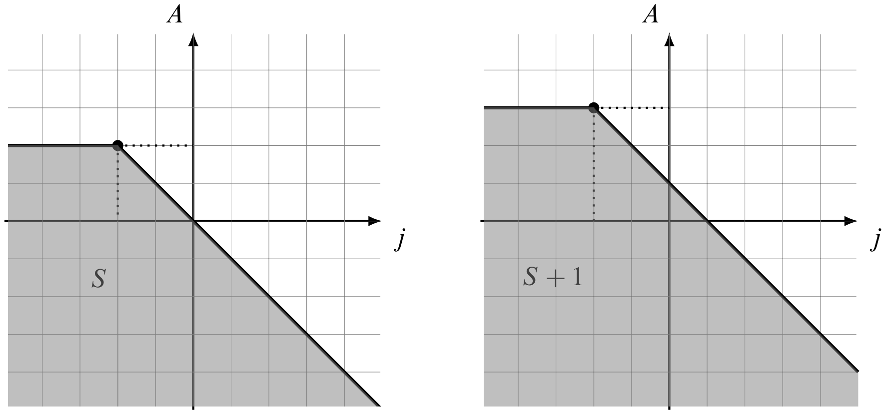

Figure 21: A centered southwest region S on the left and the southwest region $S+1$ on the right.

Proof We construct a map f_Σ by composing the maps t and s defined in this subsection, together with the concordance maps in Section 3. We obtain that

$$f_\Sigma(\mathcal{F}^S cCFL_0^\infty(\bigcirc_k)) \subset \mathcal{F}^{S+g(\Sigma)+n-k} cCFL_0^\infty(L)$$

for every S. In particular, if the homology class of x is a generator of $\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}_0^\infty(\bigcirc_k)$ then $f_\Sigma^*[x]$ is a generator of $\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}_0^\infty(L)$. This immediately implies that $\Upsilon_S(L) \geqslant -h_S(g(\Sigma)+n-k)$ and we complete the proof by observing that $\Upsilon_{-S}(L) = \Upsilon_S(L)$ from Theorem 4.4. □

A similar lower bound holds with Υ^* in place of Υ, but it is clear that the proof cannot work as the one of Proposition 4.15. In fact, we used the map s that preserves the Maslov grading, while the Υ^*–invariants of L_i are computed by finding generators in $c\mathcal{HFL}_{1-m}^\infty(L_1)$ and $c\mathcal{HFL}_{-m}^\infty(L_2)$ respectively, where m is the number of components of L_1. To jump this hurdle, in the following lemma we introduce another map s' induced by the split move.

Lemma 4.16 *Suppose that L_1 and L_2 are as in the left side of Figure 20 and D_1 and D_2 are corresponding grid diagrams. Then we can find a chain map*

$$s' \colon cCFL_d^\infty(D_1) \to cCFL_{d-1}^\infty(D_2)$$

for every $d \in \mathbb{Z}$, which preserves the $\mathcal{F}$–filtration and induces an isomorphism

$$\frac{\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}_{1-m}^\infty(L_1)}{\mathcal{F}^{\{j\leqslant -1\}} c\mathcal{HFL}_{1-m}^\infty(L_1)} \cong_{\mathbb{F}} \frac{\mathcal{F}^{\{j\leqslant 0\}} c\mathcal{HFL}_{-m}^\infty(L_2)}{\mathcal{F}^{\{j\leqslant -1\}} c\mathcal{HFL}_{-m}^\infty(L_2)},$$

where m is the number of components of L_1.

Proof We represent the split move using the fragments of D_1 and D_2 as in Figure 22, where this time the number of special $\mathbb{O}$–markings on each component is the same both before and after the move. We define s' as

$$s'(x) = \begin{cases} x & \text{if } c \in x, \\ Ux & \text{otherwise,} \end{cases} \qquad \text{and} \qquad s'(V_1 p) = U \cdot s'(p),$$

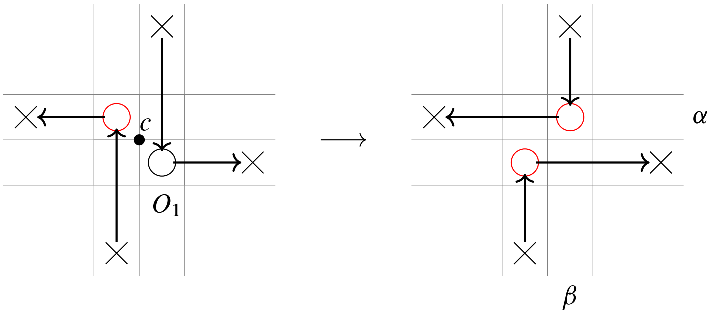

Figure 22: Another split move in a grid diagram. We recall that special $\mathbb{O}$–markings are colored in red.

for every grid state $x \in S(D_1)$, every $p \in cCFL^\infty(D_1)$ and V_i–equivariant for $i > 1$, where V_1 is the variable associated to the normal $\mathbb{O}$–marking O_1; see Figure 22.

Consider the diagrams D'_1 and D'_2, obtained by applying the algorithm in Section 3.2 to D_1 and D_2; hence, the diagrams D_i and D'_i have same $\mathbb{O}$–markings. Denote by D'_3 the diagram obtained by removing the row α and the column β from D'_2; then $cCFL^\infty(D'_2)$ is isomorphic to $cCFL^\infty(D'_3) \oplus cCFL^\infty(D'_3)[\![1]\!]$ by Proposition 4.13. Finally, let us call $\pi : cCFL^\infty(D'_2) \to cCFL^\infty(D'_3)$ the map given by adapting to our setting the homotopy inverse of the map $i \circ H$ used in [Cavallo 2018, Proposition 3.1]. This map coincide with the special destabilization in [Sarkar 2011, Section 3.3].

The map s' was also studied by Sarkar [2011, Section 3.3], and he proves that $\pi \circ s' = d^{\mathrm{NO}}$ is one of the destabilization maps in [Manolescu et al. 2007; Ozsváth et al. 2015]. Such a map is induced by the link isotopy relating D'_1 and D'_3 which means it is an almost filtered chain homotopy equivalence, and a local equivalence by Remark 3.2. Therefore, the map $s' : cCFL^\infty(D_1) \to cCFL^\infty(D_2)$ induces an injective homomorphism in homology and drops the Maslov grading by one, while the fact that s' preserves the Alexander filtration $\mathscr{A}$ is shown in [Sarkar 2011, Section 3.4].

In order to conclude the proof we just need to observe that s' does not change the minimal j–level of a generator of the homology in $\mathscr{F}^{\{j \leqslant 0\}} cCFL^\infty(D_1)$; and note that this only depends on the $\mathbb{O}$–markings. Since the identification in Proposition 4.13 is an isomorphism of chain complexes, we would have that if s' would drop the minimal j–level then the same should be true for $\pi \circ s' = d^{\mathrm{NO}}$, but this is impossible because the latter is a local equivalence. □

Now we can prove the main result of this subsection.

Proposition 4.17 *Suppose that L is an n–component link in S^3 which bounds a smooth, compact, oriented surface $\Sigma \hookrightarrow D^4$, with k connected components. Then*

$$-h_{\pm\overline{\iota S}}(g(\Sigma)) \leqslant -\Upsilon_S(L) \leqslant h_{\pm S}(g(\Sigma)+n-k),$$
$$-h_{\pm\overline{\iota S}}(g(\Sigma)+n-k) \leqslant -\Upsilon^*_S(L) \leqslant h_{\pm S}(g(\Sigma))$$

for every centered southwest region S of $\mathbb{R}^2$.

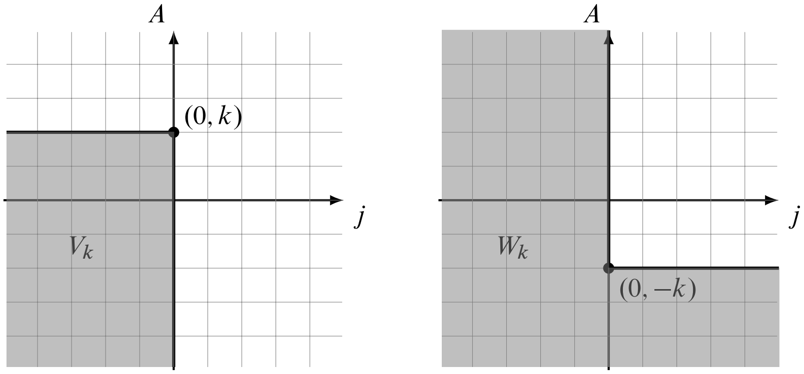

Figure 23: The centered southwest regions V_k (left) and W_k (right) of $\mathbb{R}^2$ for any integer $k \geqslant 0$.

Proof The fact that $-\Upsilon^*_S(L) \leqslant h_{\pm S}(g(\Sigma))$ follows in the same way as in Proposition 4.15 by using Lemma 4.16. Then we apply Propositions 4.5 and 4.15. □

This theorem immediately gives the lower bound in Theorem 1.5 for the *smooth slice genus* $g_4(L)$ of a link, which is defined as the minimum genus of a smooth, oriented, compact surface properly embedded in D^4 and that bounds L. For knots such lower bounds agree with the ones of Alfieri [2019] and Ozsváth, Stipsicz and Szabó [Ozsváth et al. 2017a].

Example 4.18 We observe that, when L bounds a planar (genus zero) surface in D^4, we have $\Upsilon_S(L) \leqslant 0$ and $\Upsilon^*_S(L) \geqslant 0$ for every S centered.

4.5 Other concordance invariants from the link Floer complex

4.5.1 The invariant ν^+ Let us consider the centered southwest regions

$$V_k := \{(j, A) \in \mathbb{R}^2 \mid j \leqslant 0, A \leqslant k\}$$

with $k \in \mathbb{N}$; see Figure 23. We denote the Υ–invariants associated to these regions by $-2 \cdot V_L(k) = \Upsilon_{V_k}(L)$. It follows from [Alfieri 2019] that the invariants $V_K(k)$ determine some of the invariants h_k of a knot K, which were introduced by Rasmussen [2003].

Proposition 4.19 (Alfieri) *Suppose that K is a knot in S^3. Then $V_K(k) = h_k(K)$ for every $k \in \mathbb{N}$.*

Applying Proposition 4.17 we obtain that $h_k(K) = V_K(k) \leqslant g_4(K) - k$ for a knot K and $0 \leqslant k \leqslant g_4(K)$, which coincides with [Rasmussen 2003, Corollary 7.4]. Furthermore,

$$\begin{aligned} 0 \leqslant V_L(k) &\leqslant g_4(L) + n - k - 1 && \text{if } k < g_4(L) + n - 1, \\ V_L(k) &= 0 && \text{if } k \geqslant g_4(L) + n - 1, \end{aligned}$$

and

$$V_L(k) \geqslant V_L(k+1)$$

for every link L. Finally, Theorem 3.8 tells us that $V_L(k)$ is a concordance invariant for every $k \in \mathbb{N}$.

Hom and Wu [2016] define the knot concordance invariant ν^+ and they prove that such invariant gives a lower bound for the slice genus g_4. Using our results we can easily extend ν^+ to links: we say that

$$\nu^+(L) := \min\{k \in \mathbb{N} \mid V_L(k) = 0\}.$$

It is easy to check [Hom 2017] that for knots such a definition coincides with the one in [Hom and Wu 2016] and it generalizes its well-known properties.

Proposition 4.20 *The nonnegative integer $\nu^+(L)$ is a concordance invariant of links.*

Proof If L_1 is concordant to L_2 then $V_{L_1}(k) = V_{L_2}(k)$ for every $k \in \mathbb{N}$ as we saw before. Hence, one has $V_{L_1}(k) = 0$ if and only if $V_{L_2}(k) = 0$. □

Consider the southwest regions W_k in Figure 23; we see immediately that one has $W_k = \overline{\iota V}_k$ and then $\Upsilon^*_{W_k}(L) = 2 \cdot V_{L^*}(k)$ for every $k \in \mathbb{N}$ by Proposition 4.5. We say that

$$\hat{\nu}(L) = \max\{\nu^+(L), \nu^+(L^*)\},$$

where

$$\nu^+(L^*) = \min\{k \in \mathbb{N} \mid V_{L^*}(k) = 0\} = \min\{k \in \mathbb{N} \mid \Upsilon^*_{W_k}(L) = 0\}$$

which is also a concordance invariant.

Theorem 4.21 *Suppose that L is an n–component link in S^3. Then*

$$0 \leqslant \nu^+(L) \leqslant \hat{\nu}(L) \leqslant g_4(L) + n - 1 \quad \text{and} \quad \tau(L) \leqslant \nu^+(L).$$

Furthermore, the invariants $\nu^+(L)$ and $\hat{\nu}(L)$ are subadditive:

$$\nu^+(L_1 \# L_2) \leqslant \nu^+(L_1) + \nu^+(L_2) \quad \text{and} \quad \hat{\nu}(L_1 \# L_2) \leqslant \hat{\nu}(L_1) + \hat{\nu}(L_2)$$

for every pair of links L_1 and L_2.

Proof We saw before that if $V_L(k) \neq 0$ then $k < g_4(L) + n - 1$. Since $\nu^+(L)$ is the minimal k such that $V_L(k) = 0$ and $g_4(L) = g_4(L^*)$, we conclude that $\hat{\nu}(L) \leqslant g_4(L) + n - 1$. We now show that $\tau(L) \leqslant \nu^+(L)$. Suppose that s is the minimal integer such that $V_L(s) = 0$; then there is an element x in $\mathscr{F}^{V_s} cCFL^\infty_0(L)$ whose homology class is a generator of the homology with minimal j–level zero. The claim follows from Lemma 4.6.

For the last part of the theorem, take elements x_1 and x_2 as before for L_1 and L_2 respectively. From Section 4.3 we know that

$$x_1 \otimes x_2 \in \mathscr{F}^{V_{\nu^+(L_1)+\nu^+(L_2)}} cCFL^\infty_0(L_1 \# L_2)$$

has the same properties. Since $\Upsilon_{V_k}(L_1 \# L_2) \leqslant 0$ for every k, this implies $V_{L_1 \# L_2}(\nu^+(L_1) + \nu^+(L_2)) = 0$. Now, denote by J_1 and J_2 either the links L_1 and L_2 or the links L_1^* and L_2^*, depending on which ones give the maximal $\nu^+(J_1 \# J_2)$. Then

$$\hat{\nu}(L_1 \# L_2) = \nu^+(J_1 \# J_2) \leqslant \nu^+(J_1) + \nu^+(J_2) \leqslant \hat{\nu}(L_1) + \hat{\nu}(L_2). \qquad \square$$

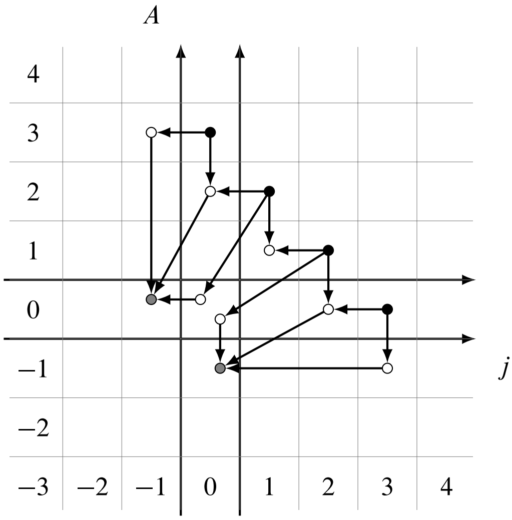

Figure 24: The relevant acyclic summand of $cCFL^{\infty}(L)$, where $L = T_{2,9} \# T^*_{2,3;2,5} \# H_-^{\# n-1}$. Here, by relevant we mean the summand which contains the generators of $\mathcal{F}^{\{j \leqslant 0\}} c\mathcal{HFL}_0^{\infty}(L)$, in the decomposition induced by the connected sum, according to Theorem 4.9. We have that $\tau(L) = 0$ and $\nu^+(L) = 2$.

Theorem 4.21 tells us that ν^+ gives a lower bound to the slice genus at least as good as the one given by τ. An example where this happens is shown in Figure 24.

Let us write $2 \cdot W_L(k) = \Upsilon_{W_k}(L)$. In the same way as before,

$$\begin{aligned} 0 \leqslant W_L(k) \leqslant g_4(L) - k \quad &\text{if } k < g_4(L), \\ W_L(k) = 0 \quad &\text{if } k \geqslant g_4(L), \end{aligned}$$

and

$$W_L(k) \geqslant W_L(k+1)$$

for every link L.

We call $\check{\nu}(L)$ the nonnegative integer

$$\max\bigl\{\min\{k \in \mathbb{N} \mid W_L(k) = 0\}, \min\{k \in \mathbb{N} \mid W_{L^*}(k) = 0\}\bigr\},$$

which shares similar properties with $\hat{\nu}(L)$. In particular, $\check{\nu}(L) = \hat{\nu}(L)$ for knots.

Theorem 4.22 *Suppose that L is an n–component link in S^3. Then $\check{\nu}$ is a concordance invariant; moreover,*

$$0 \leqslant \check{\nu}(L) \leqslant g_4(L) \quad \text{and} \quad \tau^*(L) \leqslant \check{\nu}(L).$$

Furthermore, the invariant $\check{\nu}(L)$ is also subadditive:

$$\check{\nu}(L_1 \# L_2) \leqslant \check{\nu}(L_1) + \check{\nu}(L_2)$$

for every pair of links L_1 and L_2.

Proof It follows in the same way as in the proof of Proposition 4.20 and Theorem 4.21, by applying Lemma 4.6 and Proposition 4.17. We just need to observe that

$$\min\{k \in \mathbb{N} \mid W_{L^*}(k) = 0\} = \min\{k \in \mathbb{N} \mid \Upsilon^*_{V_k}(L) = 0\}. \qquad \square$$

This result implies that if L bounds a compact planar surface properly embedded in D^4 then $\check{\nu}(L) = 0$.

4.5.2 Secondary upsilon invariants In a paper of Allen [2020] we find an example of two nonconcordant knots with the same Υ–invariants. These knots are the torus knot $T_{5,7}$ and the connected sum $T_{2,5} \# T_{5,6}$. Their chain complexes are pictured in [Allen 2020, Figures 4 and 6].

Starting from this example, we build the links $J_1 = T_{5,7} \# H_+^{\#n-1}$ and $J_2 = T_{2,5} \# T_{5,6} \# H_+^{\#n-1}$. Since we can compute the complex of the positive Hopf link,

$$cCFL^\infty(H_+) = CFK^\infty(T_{2,3}) \oplus \mathbb{F}[U, U^{-1}]_{(-1)},$$

we easily obtain that the homology groups of J_1 and J_2 are $\mathscr{F}$–filtered isomorphic. On the other hand, it is still possible to show that $cCFL^\infty(J_1)$ is not locally equivalent to $cCFL^\infty(J_2)$, which means that the filtered isomorphism (or its inverse) is not induced by a chain map that preserves the filtration $\mathscr{F}$.

In order to find an obstruction for the existence of such a map, we need to use another family of invariants, which was introduced by Kim and Livingston [2018] and by Alfieri [2019] for knots. We define the *secondary* Υ*–invariants* $\Upsilon^{(2)}_{S^+,S^-,S}(L)$ of an n–component link L as $-\Upsilon_S(L)$ plus the supremum of $k \in \mathbb{Z}$ such that

$$\mathscr{F}^{S_k \cup (S^+_{\gamma^+}) \cup (S^-_{\gamma^-})} cCFL^\infty_1(L)$$

contains a 1–chain a with $\partial^- a = x_1 + x_2$; the cycles

$$x_1 \in \mathscr{F}^{(S^+_{\gamma^+})} cCFL^\infty_0(L) \quad \text{and} \quad x_2 \in \mathscr{F}^{(S^-_{\gamma^-})} cCFL^\infty_0(L)$$

have the property that their homology classes are generators of $\mathscr{F}^{\{j \leqslant 0\}} c\mathcal{HFL}^\infty_0(L)$, where $\gamma^\pm = \Upsilon_{S^\pm}(L)$ and S^+, S^- and S are three centered southwest regions of $\mathbb{R}^2$. Note that $\Upsilon^{(2)}_{S^+,S^-,S}(L)$ can be $+\infty$, as it happens for the unknot.

We can define a *secondary* Υ^**–invariant* exactly in the same way, only this time we consider elements in Maslov gradings $1-n$ and $2-n$. For the sake of simplicity, in this subsection we only write proofs for $\Upsilon^{(2)}_{S^+,S^-,S}(L)$, but all the results also hold for this version of the invariant.

Proposition 4.23 *Let us consider a link L. Then the invariant $\Upsilon^{(2)}_{S^+,S^-,S}(L)$ is a concordance invariant for every triple of centered southwest regions S^+, S^- and S of $\mathbb{R}^2$.*

Proof Suppose that L_1 is concordant to L_2 and $\Upsilon^{(2)}_{S^+,S^-,S}(L_1) < \Upsilon^{(2)}_{S^+,S^-,S}(L_2)$. Then there is an integer $k > \Upsilon^{(2)}_{S^+,S^-,S}(L_1) + \Upsilon_S(L_1)$ such that

$$z^\pm \in \mathscr{F}^{S^\pm_{\gamma^\pm}} cCFL^\infty_0(L_2),$$

the homology class $[z^+]=[z^-]$ is the generator of $\mathcal{F}^{\{j\leqslant 0\}}c\mathcal{HFL}_0^\infty(L_2)$ and there exists

$$\beta\in\mathcal{F}^{S_k\cup(S^+_{\gamma^+})\cup(S^-_{\gamma^-})}cCFL_1^\infty(L_2)$$

with $\partial^-\beta=z^++z^-$. We recall that $\gamma^\pm=\Upsilon_{S^\pm}(L_1)=\Upsilon_{S^\pm}(L_2)$, since Υ is a concordance invariant.

From Theorem 1.2 we know that the corresponding chain complexes of two links are locally equivalent. Then we find a chain map $g\colon cCFL_0^\infty(L_2)\to cCFL_0^\infty(L_1)$, which preserves $\mathcal{F}$ and induces an $\mathcal{F}$–filtered isomorphism between $c\mathcal{HFL}_0^\infty(L_2)$ and $c\mathcal{HFL}_0^\infty(L_1)$. Therefore, we can take

$$g(z^+)+g(z^-)=g(\partial^-\beta)=\partial^-g(\beta)$$

and we have

$$g(z^\pm)\in\mathcal{F}^{S^\pm_{\gamma^\pm}}cCFL_0^\infty(L_1),$$

the homology class $[g(z^+)]=[g(z^-)]$ is the generator of $\mathcal{F}^{\{j\leqslant 0\}}c\mathcal{HFL}_0^\infty(L_1)$ and

$$g(\beta)\in\mathcal{F}^{S_k\cup(S^+_{\gamma^+})\cup(S^-_{\gamma^-})}cCFL_1^\infty(L_1).$$

This is a contradiction, because it implies $k\leqslant\Upsilon^{(2)}_{S^+,S^-,S}(L_1)+\Upsilon_S(L_1)$. □

Now we can show that the links J_1 and J_2 are not concordant. We have that

$$cCFL_{1-n}^\infty(J_1)=cCFL_0^\infty(T_{5,7})\quad\text{and}\quad cCFL_{1-n}^\infty(J_2)=cCFL_0^\infty(T_{2,5}\#T_{5,6})$$

up to acyclics; hence, if J_1 and J_2 were concordant then Proposition 4.23 should imply

$$\Upsilon^{(2)}_{S^+,S^-,S}(T_{5,7})=(\Upsilon^*)^{(2)}_{S^+,S^-,S}(J_1)=(\Upsilon^*)^{(2)}_{S^+,S^-,S}(J_2)=\Upsilon^{(2)}_{S^+,S^-,S}(T_{2,5}\#T_{5,6})$$

for every southwest regions $S^\pm$ and S. This is not true, as shown by Allen [2020].

We showed that the secondary Υ–invariants can give more information than the $\mathcal{F}$–filtered isomorphism type of $c\mathcal{HFL}^\infty(L)$; nonetheless, the following proposition holds. Here we recall that the invariants $V_L(0)$ and $W_L(0)$, corresponding to the southwest regions V_0 and W_0, are defined before in Section 4.5.

Proposition 4.24 *If $V_L(0)=W_L(0)=0$ then all of the Υ's of L are zero and all of the $\Upsilon^{(2)}$'s of L are $+\infty$. In the same way, if $V_{L^*}(0)=W_{L^*}(0)=0$ then all of the Υ^*'s of L are zero and all of the $(\Upsilon^*)^{(2)}$'s of L are $+\infty$.*

Proof Suppose that S is a centered southwest region of $\mathbb{R}^2$. Then $V_0\subset S$ and $0=\Upsilon_{V_0}(L)\leqslant\Upsilon_S(L)$. In the same way, $S\subset W_0$ and $\Upsilon_S(L)\leqslant\Upsilon_{W_0}(L)=0$. This implies $\Upsilon_S(L)=0$.

Consider two centered southwest regions $S^\pm$ of $\mathbb{R}^2$. Then $V_0\subset S^+\cap S^-$, so there is a cycle, which represents the generator of the algebraic level zero of $c\mathcal{HFL}_0^\infty(L)$, in

$$\mathcal{F}^{V_0}cCFL_0^\infty(L)\subset\mathcal{F}^{S^+}cCFL_0^\infty(L)\cap\mathcal{F}^{S^-}cCFL_0^\infty(L).$$

Since $\Upsilon_S(L)=0$ for every S from before, we obtain $\Upsilon^{(2)}_{S^+,S^-,S}(L)=+\infty$. The proof for Υ^* is exactly the same by Proposition 4.5. □

In particular, for knots we have the following corollary.

Corollary 4.25 *For a knot K, if $V_K(0) = V_{K^*}(0) = 0$ then $\Upsilon_S(K) = 0$ and $\Upsilon^{(2)}_{S^+,S^-,S}(L) = +\infty$ for all southwest regions $S^\pm$ and S of $\mathbb{R}^2$.*

Proof This follows immediately from Propositions 4.5 and 4.24. □

In fact, it is possible to prove that $V_K(0) = V_{K^*}(0) = 0$ forces $CFK^\infty(K)$ to be stably equivalent to $\mathbb{F}[U,U^{-1}]_{(0)}$, the filtered chain homotopy type of the unknot; see [Hom 2017].

5 Unoriented Heegaard Floer homology

5.1 The homology group $HFL'(L)$

Let us take a Heegaard diagram $\mathcal{D}$ for a link L in S^3. The chain complex $CFL'(L)$ is the filtered chain homotopy type of $CFL'(\mathcal{D})$, the free $\mathbb{F}[U,U^{-1}]$–module over $\mathbb{T} = \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with differential given by

$$\partial' x = \sum_{y\in\mathbb{T}} \sum_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=1}} m(\phi)\cdot U^{n_w(\phi)+n_z(\phi)} y,$$

where ϕ, $n_*(\phi)$ and $m(\phi)$ are as in Section 2.1, and

$$\partial'(U^{\pm 1} p) = U^{\pm 1}\cdot \partial' p$$

for any $x \in \mathbb{T}$ and $p \in CFL'(\mathcal{D})$.

Foe every $x \in \mathbb{T}$ we define the *δ–grading* as

$$\delta(x) = M(x) - A(x).$$

It is easy to check that, with this definition, the variable $U^{\pm 1}$ drops the δ–grading by ± 1. Moreover, there is a map

$$\partial'_d : CFL'_d(\mathcal{D}) \to CFL'_{d-1}(\mathcal{D})$$

for any $d \in \mathbb{Z}$.

The chain complex $CFL'(L)$ also has the algebraic filtration j, defined as in Section 2.1 by

$$j^t CFL'(L) = U^{-t}\cdot CFL''(L),$$

where $CFL''(L)$ is the free $\mathbb{F}[U]$–module over $\mathbb{T}$ and $t \in \mathbb{Z}$. Note that the latter group was the original unoriented chain complex defined in [Ozsváth et al. 2017b]. It is easy to check that the differential ∂' preserves j.

We define the homology group as usual:

$$HFL'(L) = \bigoplus_{d\in\mathbb{Z}} HFL'_d(L) \quad \text{and} \quad \mathcal{F}^t HFL'_d(L) = \pi_d(\operatorname{Ker}\partial'_{d,t}) := \pi_d(\operatorname{Ker}\partial'_d \cap \mathcal{F}^t CFL'(L)),$$

where $\pi_d : \operatorname{Ker}\partial'_d \to HFL'(L)$ is the quotient map.

Proposition 5.1 *For every n–component link L,*

$$HFL'(L) \cong_{\mathbb{F}[U,U^{-1}]} \mathbb{F}[U,U^{-1}]^{2^{n-1}},$$

with δ–homogeneous generators, and

$$\frac{\mathcal{F}^0 HFL'(L)}{\mathcal{F}^{-1} HFL'(L)} \cong_{\mathbb{F}} \mathbb{F}^{2^{n-1}}.$$

Proof The first claim follows from [Ozsváth et al. 2017b], while the second one from the fact that the U–action drops the δ–grading by one: each homology class in $\mathcal{F}^0 HFL'(L) \setminus \mathcal{F}^{-1} HFL'(L)$ corresponds exactly to an $\mathbb{F}[U,U^{-1}]$–summand of $HFL'(L)$. □

In [Ozsváth et al. 2017b] it is proved that $HFL'(L)$ is an isotopy link invariant. This is also implied by the following theorem.

Theorem 5.2 *There exists a chain map*

$$i : cCFL^\infty(L) \oplus cCFL^\infty(L)[\![-1]\!] \to CFL'(L),$$

which is an isomorphism of $\mathbb{F}$–vector spaces that identifies the Maslov grading with the δ–grading.

Proof Let us consider all the intersection points $x_1, \dots, x_l$ whose Maslov grading has the same parity of d. We define

$$i_d^0 : cCFL_d^\infty(L) \to CFL'_d(L),$$
$$U^{k_1}x_1 + \cdots + U^{k_l}x_l \mapsto U^{2k_1 - A(x_1)}x_1 + \cdots + U^{2k_l - A(x_l)}x_l,$$

and

$$i_d^1 : cCFL_d^\infty(L) \to CFL'_{d-1}(L),$$
$$U^{k_1}x_1 + \cdots + U^{k_l}x_l \mapsto U^{1+2k_1 - A(x_1)}x_1 + \cdots + U^{1+2k_l - A(x_l)}x_l.$$

These maps are linear by definition; let us prove that they are also injective. We observe that

$$i_d^\varepsilon(U^{k_1}x_1 + \cdots + U^{k_l}x_l) \neq 0,$$

where ε is 0 or 1, because the monomials $U^{\varepsilon + 2k_i - A(x_i)}x_i$ for $i = 1, \dots, l$ are linearly independent in $CFL'_{d-\varepsilon}(L)$; hence, the kernel of i_d^ε is trivial.

We now show that $i_d = i_d^0 + i_{d+1}^1$ is surjective. Suppose that $q = U^{h_1}x_1 + \cdots + U^{h_l}x_l \in CFL'_d(L)$. If $h_i \equiv A(x_i) \bmod 2$ then there exists a k_i such that $2k_i - A(x_i) = h_i$; otherwise, if $h_j \equiv A(x_j) + 1 \bmod 2$ then there exists a k_j such that $1 + 2k_j - A(x_j) = h_j$. Therefore, say $q = q_1 + q_2$ and q_i consists of monomials of these two kinds respectively; we find p_1 and p_2 such that

$$i_d(p_1, p_2) = i_d^0(p_1) + i_{d+1}^1(p_2) = q_1 + q_2 = q$$

and the claim follows.

Since i_d^0 and i_{d+1}^1 are both injective and their images have trivial intersection, and then give a direct sum of $CFL'_d(L)$, we obtain that each i_d is a linear isomorphism between $cCFL^\infty(L) \oplus cCFL^\infty(L)[\![-1]\!]$ in δ–grading d and $CFL'_d(L)$.

In order to complete the proof we now have to show that i is a chain map, which means $i\circ(\partial^-,\partial^-)=\partial'\circ i$. Since i is linear we can just check monomials. We have

$$\begin{aligned}(i_{d-1}\circ(\partial^-,\partial^-))(U^kx,0)&=i_{d-1}(\partial^-(U^kx),0))\\&=i^0_{d-1}(U^k\partial^-x)\\&=U^{2k}\cdot i^0_{d-1}\Bigg(\sum_{y\in\mathbb{T}}\sum_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=1}}m(\phi)\cdot U^{n_{\boldsymbol{w}}(\phi)}y\Bigg)\\&=\sum_{y\in\mathbb{T}}\sum_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=1}}m(\phi)\cdot U^{2k+2n_{\boldsymbol{w}}(\phi)-A(y)}y\end{aligned}$$

and

$$(\partial'\circ i_d)(U^kx,0)=\partial'(i^0_d(U^kx))=\partial'(U^{2k-A(x)}x)=\sum_{y\in\mathbb{T}}\sum_{\substack{\phi\in\pi_2(x,y)\\ \mu(\phi)=1}}m(\phi)\cdot U^{2k-A(x)+n_{\boldsymbol{w}}(\phi)+n_{\boldsymbol{z}}(\phi)}y.$$

To conclude we need to see that $n_{\boldsymbol{w}}(\phi)-A(y)=n_{\boldsymbol{z}}(\phi)-A(x)$ and this holds for every $\phi\in\pi_2(x,y)$; see [Ozsváth and Szabó 2008]. The proof for the monomials $(0,U^hy)$ is the same. □

The graded object associated to $CFL'(L)$ is $\widehat{CFL}'(L)$, which is the version of $\widehat{CFL}$ obtained by collapsing the bigrading accordingly. Hence, if L_1 and L_2 are isotopic links then

$$\widehat{HFL}_d{}'(L_1)\cong_{\mathbb{F}}\widehat{HFL}_d{}'(L_2)$$

for every $d\in\mathbb{Z}$. This means that both $HFL'(L)$ and $\widehat{HFL}'$ are link invariants.

5.2 The υ–set and unoriented concordance

We start this subsection with some properties of $HFL'(L)$.

Lemma 5.3 *For every link L,*

1. *if there is a chain map $F\colon cCFL^\infty(L_1)\to cCFL^\infty(L_2)$ which preserves the $\mathcal{F}$–filtration then the map $F'\colon CFL'(L_1)\to CFL'(L_2)$, defined as $i_2\circ(F\oplus F[\![-1]\!])\circ i_1^{-1}$, preserves j;*
2. *if $cCFL^\infty(L_1)$ is locally equivalent to $cCFL^\infty(L_2)$ then there is a j–filtered and δ–graded isomorphism between $HFL'(L_1)$ and $HFL'(L_2)$.*

Proof Let us prove (1). We have to show that F' is j–filtered of degree zero. We do this by proving that if $U^kx\in\mathcal{F}^tCFL'(L_1)$ then $F'(U^kx)\in\mathcal{F}^tCFL'(L_2)$ for every monomial.

We assume that $k\geqslant -t$. Then one has

$$i_1^{-1}(U^kx)=\begin{cases}(U^{(k+A(x))/2},0)&\text{if } k+A(x)\text{ is even,}\\(0,U^{(-1+k+A(x))/2})&\text{if } k+A(x)\text{ is odd.}\end{cases}$$

Now, when $k + A(x)$ is even, we can write

$$(F \oplus F[\![-1]\!])(i_1^{-1}(U^k x)) = \Big(\sum_{y \in \mathbb{T}} a(x,y) \cdot U^{(k+A(x))/2+\Delta(x,y)} y, 0 \Big),$$

with $a(x,y) \in \mathbb{F}$. This yields

$$F'(U^k x) = \sum_{y \in \mathbb{T}} a(x,y) \cdot U^{k+A(x)-A(y)+2\Delta(x,y)} y$$

and it is easy to check that we get the same result when $k + A(x)$ is odd. To conclude we need to argue that $A(y) \leqslant A(x) + 2\Delta(x,y)$. Since F preserves $\mathscr{F}$, it is both j and $\mathscr{A}$–filtered of degree zero. Therefore, it is $\Delta(x,y) \geqslant 0$ and $A(y) \leqslant A(x) + \Delta(x,y)$ whenever $a(x,y) = 1$ and the claim follows.

To prove (2) take the maps $f : cCFL^\infty(L_1) \to cCFL^\infty(L_2)$ and $g : cCFL^\infty(L_2) \to cCFL^\infty(L_1)$, which both preserve the $\mathscr{F}$–filtration. Theorem 5.2 implies that f', defined as $i_2 \circ (f \oplus f[\![-1]\!]) \circ i_1^{-1}$, and g', defined in the same way from g, induce δ–graded isomorphisms in homology. Moreover, Lemma 5.3(1) also gives that they preserve j. Hence, $HFL'(L_1)$ is j–filtered isomorphic to $HFL'(L_2)$. □

The first consequence of this lemma is that the group HFL' is also a concordance invariant.

Corollary 5.4 *If the link L_1 is concordant to the link L_2 then the unoriented link Floer homology group $HFL'(L_1)$ is j–filtered isomorphic to $HFL'(L_2)$, which means that*

$$\mathscr{F}^t HFL'_d(L_1) \cong_{\mathbb{F}} \mathscr{F}^t HFL'_d(L_2)$$

for every $t, d \in \mathbb{Z}$.

Proof From Theorem 1.2 we know that $cCFL^\infty(L_1)$ is locally equivalent to $cCFL^\infty(L_2)$. The claim then follows from Lemma 5.3(2). □

By Theorem 2.1 we know that for an n–component link L,

$$\frac{\mathscr{F}^{\{j \leqslant 0\}} c\mathcal{HFL}^\infty_d(L)}{\mathscr{F}^{\{j \leqslant -1\}} c\mathcal{HFL}^\infty_d(L)} \cong_{\mathbb{F}} \mathbb{F}^{\binom{n-1}{-d}}$$

for $d = 0, \dots, 1-n$. Let us denote by $\{h_1, \dots, h_{2^{n-1}}\}$ a basis for the direct sum of such groups, where the homology classes h_i satisfy that for each i, there is an integer k and a Maslov grading $d \in [0, 1-n]$ such that $h_i \in \mathscr{F}^{(A_1)_k} c\mathcal{HFL}^\infty_d(L) / \mathscr{F}^{(A_1)_{k+1}} c\mathcal{HFL}^\infty_d(L)$, where A_1 is the centered southwest region

$$\{(j, A) \in \mathbb{R}^2 \mid j + A \leqslant 0\}$$

that we used in Section 2.2 to define $\Upsilon_L(1)$, and, for any fixed k and d, the number of h_i with those k and d is exactly

$$\dim_{\mathbb{F}} \frac{\mathscr{F}^{(A_1)_k} c\mathcal{HFL}^\infty_d(L)}{\mathscr{F}^{(A_1)_{k+1}} c\mathcal{HFL}^\infty_d(L)}.$$

We also take h_1 to be the only homology class as above in Maslov grading 0 and $h_{2^{n-1}}$ the same, but in Maslov grading $1-n$.

We define $u_i(L)$ for $i = 1, \dots, 2^{n-1}$ as the maximum $k \in \mathbb{R}$ such that $\mathcal{F}^{(A_1)_k} c\mathcal{HFL}^{\infty}_d(L)$ contains the homology class h_i. Note that the unordered set $\{u_1(L), \dots, u_{2^{n-1}}(L)\}$ does not depend on the choice of the h_i, but only on the $\mathcal{F}$–filtered isomorphism type of $c\mathcal{HFL}^{\infty}(L)$. Moreover, $u_1(L) = \Upsilon_L(1)$ and $u_{2^{n-1}}(L) = \Upsilon^*_L(1)$.

Now let $\upsilon(L) = \{\upsilon_1(L), \dots, \upsilon_{2^{n-1}}(L)\}$ be the set of δ–gradings of a homogeneous $\mathbb{F}$–basis of

$$\frac{\mathcal{F}^0 HFL'(L)}{\mathcal{F}^{-1} HFL'(L)}.$$

This set exists by Proposition 5.1 and it does not depend on the choice of the basis, but only on the j–filtered isomorphism type of $HFL'(L)$. We have the following lemma.

Lemma 5.5 *A homogeneous $\mathbb{F}$–basis as before is obtained by taking the homology classes of elements $\{U^{k_1} q_1, \dots, U^{k_{2^{n-1}}} q_{2^{n-1}}\}$, where $q_i = i^0_{d_i}(p_i)$ and p_i represents the homology class h_i in Maslov grading d_i for every $i = 1, \dots, 2^{n-1}$.*

Proof Since i is an isomorphism from Theorem 5.2, there is an injective map $c\mathcal{HFL}^{\infty}(L) \to HFL'(L)$ identifying the Maslov grading with the δ–grading. This means that if p is a representative for h, with Maslov grading d, then $i^0_d(p)$ represents a nonzero homology class in $HFL'(L)$; moreover, representatives of distinct homology classes are sent into representatives of distinct homology classes, by Theorem 5.2.

The element $q = i^0_d(p)$ is in δ–grading d, but the minimal j–level of $[q]$ is not necessarily zero; although, since the δ–grading is an absolute $\mathbb{Z}$–grading and the U–action drops it by one, there is an integer k such that $U^k[q]$ has indeed minimal j–level equal to 0.

The fact that the set of all the $U^k q$ obtained in this way gives a basis as wanted is assured by the condition we put on the choice of the h_i. □

We use this lemma to show that the υ–set of L is closely related to the set $\{u_1(L), \dots, u_{2^{n-1}}(L)\}$.

Proposition 5.6 *Let $\upsilon(L)$ and $u_i(L)$ for $i = 1, \dots, 2^{n-1}$ be as before. Then $\upsilon_i(L) = u_i(L) + d_i$, where $u_i(L)$ is associated to the homology class h_i with Maslov grading d_i. In particular, $\upsilon_1(L) = \Upsilon_L(1)$ and $\upsilon_{2^{n-1}}(L) = \Upsilon^*_L(1) + 1 - n$.*

Proof Suppose that $p_i = U^{k_1} x_1 + \dots + U^{k_\ell} x_\ell \in cCFL^{\infty}_{d_i}(L)$ represents the homology class h_i; moreover, we assume that

$$k_j - A(U^{k_j} x_j) = 2k_j - A(x_j) \geqslant u_i(L) \quad \text{for any } j = 1, \dots, \ell$$

and $2k_1 - A(x_1) = u_i(L)$.

By Lemma 5.5, $q_i = i(p_i) = U^{2k_1 - A(x_1)} x_1 + \cdots + U^{2k_\ell - A(x_\ell)} x_\ell \in CFL'_{d_i}(L)$ represents a nonzero homology class in $HFL'_{d_i}(L)$ and $U^{-2k_1 + A(x_1)} \cdot q_i$ is in minimal algebraic level zero. Moreover, we saw that we get a homogeneous basis by considering all the h_i and then, by definition of $\upsilon(L)$,

$$\upsilon_i(L) = \delta(U^{-2k_1 + A(x_1)} \cdot q_i) = \delta(q_i) + 2k_1 - A(x_1) = d_i + u_i(L)$$

for every $i = 1, \ldots, 2^{n-1}$. □

We can shift $HFL'(L)$ in order to turn it into an unoriented link invariant.

Theorem 5.7 *$CFL'(L_1)[\![\frac{1}{2}\sigma(L_1)]\!]$ is j–filtered chain homotopy equivalent to $CFL'(L_2)[\![\frac{1}{2}\sigma(L_2)]\!]$ whenever L_1 is isotopic to L_2 as unoriented links, where σ is the signature of a link as in [Gordon and Litherland 1978]. In particular, the set*

$$\upsilon(L) - \tfrac{1}{2}\sigma(L) = \bigl\{\Upsilon_L(1) - \tfrac{1}{2}\sigma(L), \ldots, \Upsilon^*_L(1) + 1 - n - \tfrac{1}{2}\sigma(L)\bigr\}$$

is an unoriented link invariant for every link L.

Proof Changing the orientation of a link L from $\vec{L}_1$ to $\vec{L}_2$, by reversing the orientation on the i^{th} component, results in a grid diagram G where the $\mathbb{O}_i$–markings and the $\mathbb{X}_i$–markings are swapped. Then everything stays the same except for the δ–grading, which is renormalized. Using [Ozsváth et al. 2017b, Proposition 7.1] we conclude that

$$\delta_1(x) - \delta_2(x) = \tfrac{1}{2}\sigma(\vec{L}_1) - \tfrac{1}{2}\sigma(\vec{L}_2)$$

for every grid state x of G. □

It is important to note that, if we only compute the group $HFL'(L)$, we do not know how to identify $\Upsilon_L(1)$ and $\Upsilon^*_L(1) + 1 - n$ in the υ–set of L. This means that the latter is an unoriented link invariant only if considered as an unordered set of 2^{n-1} integers, up to an overall shift that can be determined from a diagram representing L. Furthermore, an analogue of the last result holds for unoriented concordant links.

Proof of Theorem 1.7 This follows in the same way as the last theorem, using Corollary 5.4. □

5.3 Unoriented cobordisms

5.3.1 Normal form and Euler number Let us denote by $\upsilon_{\max}$ (resp. $\upsilon_{\min}$) the maximal (resp. minimal) value in the υ–set of a link. From [Ozsváth et al. 2017b, Theorem 5.2] if there is an oriented saddle between L and L', where L' has one more component with respect to L, then

(5-1) $$\upsilon_{\max}(L') \leqslant \upsilon_{\max}(L) \leqslant \upsilon_{\max}(L') + 1$$

and

(5-2) $$\upsilon_{\min}(L') \leqslant \upsilon_{\min}(L) \leqslant \upsilon_{\min}(L') + 1.$$

The following inequalities agree with Proposition 4.17.

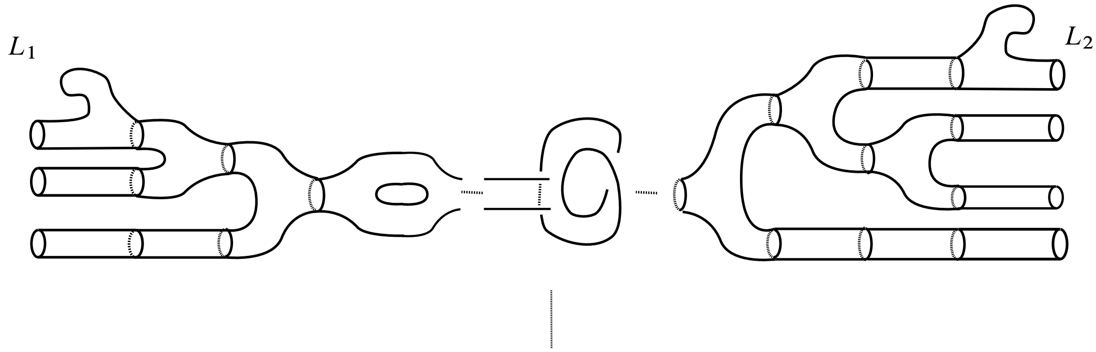

Figure 25: Canonical form of unoriented cobordisms between two links: only one connected component of F is shown. The nonorientable saddles are Möbius strips with a small open disk removed.

Proposition 5.8 *Suppose that a link L bounds a compact oriented surface Σ, properly embedded in D^4, with genus $g(\Sigma)$ and k connected components. Then*

$$-g(\Sigma)+k-n \leqslant \upsilon_{\max}(L) \leqslant g(\Sigma) \quad \textit{and} \quad -g(\Sigma)+1-n \leqslant \upsilon_{\min}(L) \leqslant g(\Sigma)+1-k.$$

Proof From Corollary 5.4 we know that $\upsilon_{\max}$ and $\upsilon_{\min}$ are concordance invariants. Hence, since every oriented cobordism Σ between $\bigcirc_k$ and an n–component link L can be decomposed as explained at the beginning of Section 4.4, and the values of $\upsilon_{\max}(\bigcirc_k)$ and $\upsilon_{\min}(\bigcirc_k)$ are 0 and $1-k$ respectively, the claim follows from (5-1) and (5-2). □

We now want to study how these invariants behave when we consider unoriented cobordisms. First, we note that there still exists a normal form; in fact, comparing the oriented case with the results of Kamada [1989] applied to cobordisms, we obtain that every unoriented cobordism F between L_1 and L_2 can be written as in Figure 25. Hence, we just need to check what happens to the υ–set when two links are related by many nonorientable saddles. Of course, we can just study the case where there is only one such move, since the general case is obtained by composing the cobordism in Figure 26.

We recall that, if F is an unoriented cobordism, there is a well-defined integer $e(F)$, called the *Euler number*, defined as

$$e(F) := \sum_{p \in F \cap F'} \varepsilon_p$$

where ε_p is the sign of a oriented basis of $T_pF \oplus T_pF'$, induced by a local orientation system of F, compared with the one given by the orientation of $T_p(S^3 \times I)$; and where F' denotes a push-off of F along the trivialization of $\nu(L_1)$ (resp. $\nu(L_2)$) in $S^3 \times \{0\}$ (resp. $S^3 \times \{1\}$) given by the Seifert framing; see [Gordon and Litherland 1978; Ozsváth et al. 2017b]. Clearly, $e(F)=0$ if F is an orientable knot cobordism.

The integer $e(F)$ can also be interpreted in the following way. Suppose that L_1 has n components, while L_2 has m; since F is homotopy equivalent to a 1–dimensional CW-complex, its normal 1–sphere

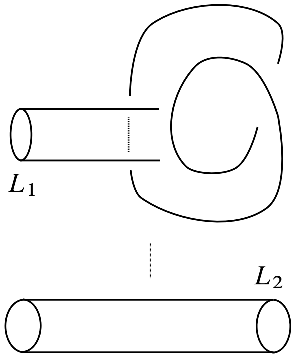

Figure 26: A nonorientable saddle corresponds to a nonoriented band move on a single component.

bundle admits a section F'. The boundary of F' consists of the links L_1' and L_2', which can be oriented accordingly to L_1 and L_2. Then

$$e(F) = \sum_{i=1}^{n} \ell\mathrm{k}(L_1^i, (L_1^i)') - \sum_{j=1}^{m} \ell\mathrm{k}(L_2^j, (L_2^j)').$$

The reader can check that this definition is independent of the choice of the section; see [Gordon and Litherland 1978].

From the previous statement we obtain that if F is the union of disjoint surfaces $F_1, \dots, F_k$ then $e(F) = e(F_1) + \cdots + e(F_k)$. In particular, a nonorientable saddle as in Figure 26 has Euler number equal to that of the unique nonorientable component.

Lemma 5.9 *Suppose that L_1 and L_2 are related by a nonorientable saddle F. Say D_1 and D_2 are planar diagrams for them such that the saddle is represented as in Figure 27. Denote by D_i' the corresponding diagram obtained from D_i by deleting all the components that do not appear in the saddle. Then*

$$e(F) = \mathrm{wr}(D_1') - \mathrm{wr}(D_2') + \varepsilon,$$

where ε is equal to 1 *if the crossing is positive and* -1 *if is negative.*

Proof From what we said before, $e(F) = e(F')$, where F' is a nonorientable saddle between K_1 and K_2, the components of the links represented by D_1' and D_2'. Since $e(F')$ is computed from a tubular

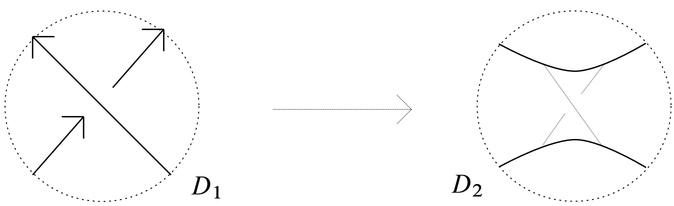

Figure 27: The nonorientable saddle is represented in the diagrams as an unoriented resolution of a crossing, where both arcs belong to the same component of L_1.

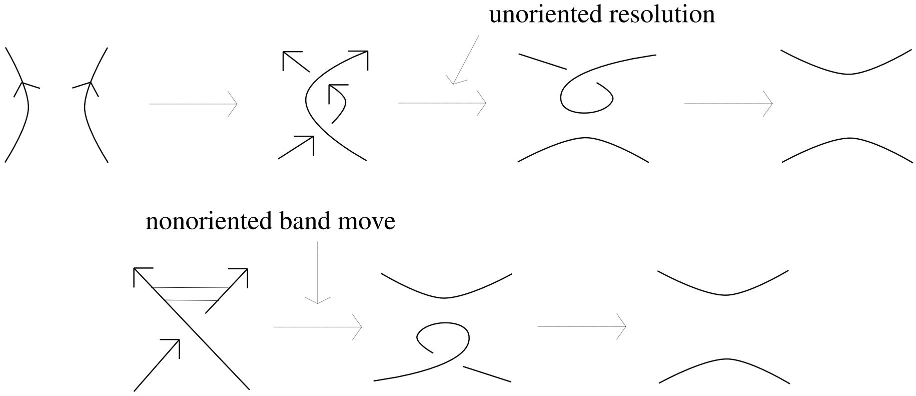

Figure 28: Each of two rows shows a direction of the equivalence of the two representations of a nonorientable saddle.

neighborhood of F' and F' is disjoint from the other annuli of F, we have that $e(F')$ can be computed using [Ozsváth et al. 2017b, Lemma 4.3]:

$$e(F') = \mathrm{wr}(D_1') - \mathrm{wr}(D_2') + \varepsilon.$$

The fact that every nonorientable saddle can be seen as an unoriented resolution of a crossing (and vice versa) follows easily from Figure 28. □

5.3.2 Unorientable saddle move We use the grid diagrams and maps defined in [Ozsváth et al. 2017b, Section 5]. Say G_1 and G_2 are grid diagrams for L_1 and L_2, which are related by a nonorientable saddle as in Figure 29. Then we have chain maps $\nu\colon CFL'(G_1) \to CFL'(G_2)$ and $\nu'\colon CFL'(G_2) \to CFL'(G_1)$, such that $\nu' \circ \nu = \nu \circ \nu' = U$, defined as

$$\nu(x) = \begin{cases} Ux & \text{if } x \cap A \neq \varnothing, \\ x & \text{if } x \cap A = \varnothing, \end{cases} \qquad \text{and} \qquad \nu'(x) = \begin{cases} x & \text{if } x \cap A \neq \varnothing, \\ Ux & \text{if } x \cap A = \varnothing, \end{cases}$$

for every grid state x.

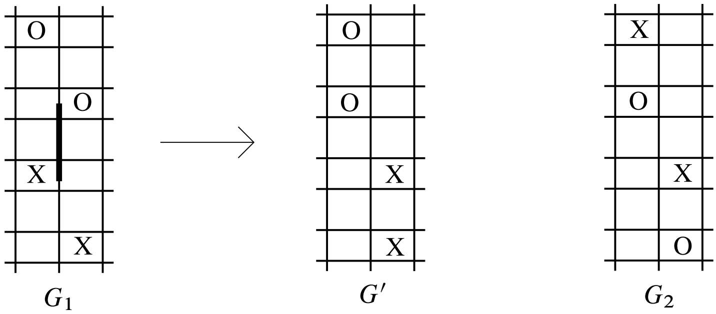

Figure 29: Nonorientable saddle in a grid diagram. Assume the markings in the first two columns of G_1 belong to the same component of L_1; we switch the $\mathbb{X}$–marking in the first column with the $\mathbb{O}$–marking in the second one to get G'. Then starting from the $\mathbb{X}$ at the bottom, we reverse all the markings on this component of the link until we obtain the diagram G_2.

Lemma 5.10 *The maps ν and ν' as before drop the δ–grading by*

$$\tfrac{1}{4}(2-e(F))-\tfrac{1}{2}(\ell\mathrm{k}(K_1,L_1\setminus K_1)-\ell\mathrm{k}(K_2,L_2\setminus K_2)),$$
$$\tfrac{1}{4}(2+e(F))+\tfrac{1}{2}(\ell\mathrm{k}(K_1,L_1\setminus K_1)-\ell\mathrm{k}(K_2,L_2\setminus K_2)),$$

respectively. Here, K_i is the component of L_i where we perform the nonorientable saddle move.

Proof Say G_1, G' and G_2 are as in Figure 29, with orientations on G_1 and G_2 given as in Section 3.2. We prove the claim for the map ν. From [Ozsváth et al. 2017b, Proposition 5.7] and its proof we have that $\delta_{G_1}(x)=\delta_{G'}(\nu(x))$ and

$$\begin{aligned}\delta_{G'}(\nu(x))-\delta_{G_2}(\nu(x)) &= -\tfrac{1}{4}[\mathrm{wr}(G_1)-\mathrm{wr}(G_2)+1-2]\\ &= -\tfrac{1}{4}[\mathrm{wr}(G_1^1)-\mathrm{wr}(G_2^1)+1-2]-\tfrac{1}{4}[2\cdot\ell\mathrm{k}(K_1,L_1\setminus K_1)-2\cdot\ell\mathrm{k}(K_2,L_2\setminus K_2)],\end{aligned}$$

where G_i^1 is the subdiagram representing K_i. Then

$$\delta_{G_2}(\nu(x))=\delta_{G_1}(x)-\tfrac{1}{4}(2-e(F))+\tfrac{1}{2}(\ell\mathrm{k}(K_1,L_1\setminus K_1)-\ell\mathrm{k}(K_2,L_2\setminus K_2))$$

by Lemma 5.9. The case of ν' is done in the same way. □

This lemma implies the following result.

Proposition 5.11 *Suppose that L_i and K_i are as before and F is the corresponding nonorientable saddle. Then*

$$\begin{aligned}\upsilon_{\max}(L_1)-\tfrac{1}{4}(2-e(F))+\tfrac{1}{2}[\ell\mathrm{k}(K_1,L_1\setminus K_1)-\ell\mathrm{k}(K_2,L_2\setminus K_2)]&\\ \leqslant \upsilon_{\max}(L_2)\leqslant \upsilon_{\max}(L_1)+\tfrac{1}{4}(2+e(F))+\tfrac{1}{2}[\ell\mathrm{k}(K_1,L_1\setminus K_1)-\ell\mathrm{k}(K_2,L_2\setminus K_2)],&\end{aligned}$$

where $L_1\setminus K_1$ and $L_2\setminus K_2$ are oriented in the same way. The same is true for $\upsilon_{\min}$.

Proof Since $\nu'\circ\nu=\nu\circ\nu'=U$ we have that ν and ν' induce isomorphisms in homology. Therefore, the claim follows from Lemma 5.10 and the definition of $\upsilon_{\max}$ and $\upsilon_{\min}$. □

These inequalities do not depend on the orientation of the components of L_1 and L_2 where the saddle appears. The proof of this statement is given in Lemma 5.12.

5.4 Bounds for the unoriented slice genus of a link

Suppose that the n–component (unoriented) link L bounds a compact, unoriented surface F, with k connected components and Euler number $e(F)$, properly embedded in D^4. Define $\boldsymbol{v}=\boldsymbol{v}_1+\cdots+\boldsymbol{v}_k$ as in Figure 30. Using the notation in [Gordon and Litherland 1978], we write

$$\lambda(\vec{L}):=\sum_{1\leqslant i<j\leqslant n}\ell\mathrm{k}(\vec{L}_i,\vec{L}_j)$$

for the *total linking number* of $\vec{L}$ and we take $\bar{e}_{\vec{L}}(F):=e(F)-2\lambda(\vec{L})$, where $\vec{L}$ means that we pick an orientation of L. Then $\bar{e}_{\vec{L}}(F)=0$ when F is oriented and $\vec{L}$ inherits its orientation from F; see [Gordon and Litherland 1978, Section 5].

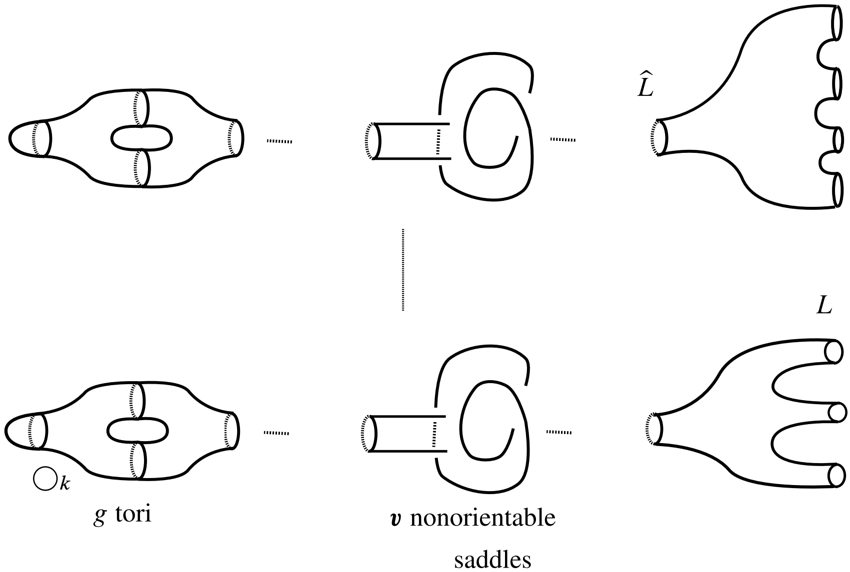

Figure 30: The number $\boldsymbol{v}_i$ denotes how many nonorientable saddles there are on each of the k components of F. In the picture we omitted the attachment of the extended birth and death moves.

Lemma 5.12 *Suppose that a link $L = \widehat{L} = \partial F$ as in Figure 30 is such that $n = k$, which means that F is the union of n disjoint unoriented surfaces F_i, each one bounding a knot. Then*

$$-g - \tfrac{1}{2}\boldsymbol{v} + \tfrac{1}{4}\bar{e}_{\vec{L}}(F) \leqslant \upsilon_{\max}(\vec{L}) \leqslant g + \tfrac{1}{2}\boldsymbol{v} + \tfrac{1}{4}\bar{e}_{\vec{L}}(F) \tag{5-3}$$

and

$$-g - \tfrac{1}{2}\boldsymbol{v} + 1 - n + \tfrac{1}{4}\bar{e}_{\vec{L}}(F) \leqslant \upsilon_{\min}(\vec{L}) \leqslant g + \tfrac{1}{2}\boldsymbol{v} + 1 - n + \tfrac{1}{4}\bar{e}_{\vec{L}}(F) \tag{5-4}$$

for every possible orientation we put on L.

Proof If $\boldsymbol{v} = 0$ then the claims are true since in this case $\bar{e}_{\vec{L}}(F) = e(F) = \lambda(\vec{L}) = 0$ (every orientation on L is compatible with one on F) and by Proposition 5.8. Suppose that $\boldsymbol{v} \geqslant 1$; we prove the last statement first. We assume (5-3) and (5-4) are satisfied for one orientation $\vec{L}$ and we prove them for another one, that we call $\vec{L}'$. Obviously, we can also suppose that $\vec{L}'$ is obtained from $\vec{L}$ by just reversing the orientation on one component of L, that we denote by K.

By [Ozsváth et al. 2015, Corollary 2.7.10] and Theorem 5.7,

$$\upsilon(\vec{L}') = \upsilon(\vec{L}) + \ell\mathrm{k}(\vec{K}, \vec{L} \setminus \vec{K}),$$

where here υ denotes either $\upsilon_{\max}$ or $\upsilon_{\min}$. Hence, since

$$\lambda(\vec{L}') = \lambda(\vec{L} \setminus \vec{K}) - \ell\mathrm{k}(\vec{K}, \vec{L} \setminus \vec{K}) = \lambda(\vec{L}) - 2\,\ell\mathrm{k}(\vec{K}, \vec{L} \setminus \vec{K}),$$

we obtain

$$\tfrac{1}{4}\bar{e}_{\vec{L}'}(F) = \tfrac{1}{4}\bar{e}_{\vec{L}}(F) + \ell\mathrm{k}(\vec{K}, \vec{L} \setminus \vec{K}).$$

This means that if we add $\ell\mathrm{k}(\vec{K}, \vec{L} \setminus \vec{K})$ to each term in the inequalities in (5-3) and (5-4) then we obtain precisely the corresponding equations for $\vec{L}'$; and this part of the proof is complete.

We now prove that the inequalities hold for at least one orientation of L. We proceed by induction on $\boldsymbol{v}$, where the initial step has been done at the beginning of the proof. Therefore, we assume that (5-3) and (5-4) hold for $\vec{L}$ and we prove them for $\vec{L}'$, where this time L' is related to L by a nonorientable saddle move as in Figure 27. Denote by K and K' the components of L and L' where the move is performed. We orient them as in the proof of Lemma 5.10 and Proposition 5.11.

We show the case of (5-3); the argument for (5-4) is exactly the same. We start by writing

$$-g - \tfrac{1}{2}\boldsymbol{v} + \tfrac{1}{4}\bar{e}_{\vec{L}}(F) \leqslant \upsilon_{\max}(\vec{L}) \quad \text{and} \quad \upsilon_{\max}(\vec{L}) \leqslant g + \tfrac{1}{2}\boldsymbol{v} + \tfrac{1}{4}\bar{e}_{\vec{L}}(F)$$

from the inductive step; we call S the saddle move and F' the surface obtained by gluing S to F, which means $\partial F' = L'$. Then the first inequality becomes

$$\begin{aligned} -g-\tfrac{1}{2}(\boldsymbol{v}+1)+\tfrac{1}{4}\bar{e}_{\vec{L}'}(F') &= -g-\tfrac{1}{2}\boldsymbol{v}+\tfrac{1}{4}\bar{e}_{\vec{L}}(F)+\big(-\tfrac{1}{2}+\tfrac{1}{4}e(S)+\tfrac{1}{2}\,\ell\mathrm{k}(\vec{K},\vec{L}\setminus\vec{K})-\tfrac{1}{2}\,\ell\mathrm{k}(\vec{K}',\vec{L}'\setminus\vec{K}')\big) \\ &\leqslant \upsilon_{\max}(\vec{L})+\big(-\tfrac{1}{2}+\tfrac{1}{4}e(S)+\tfrac{1}{2}\,\ell\mathrm{k}(\vec{K},\vec{L}\setminus\vec{K})-\tfrac{1}{2}\,\ell\mathrm{k}(\vec{K}',\vec{L}'\setminus\vec{K}')\big) \leqslant \upsilon_{\max}(\vec{L}'), \end{aligned}$$

where the first equality can be easily computed from the definition of $\bar{e}$ and the last inequality follows from Proposition 5.11. In the same way,

$$\begin{aligned} \upsilon_{\max}(\vec{L}') &\leqslant \upsilon_{\max}(\vec{L}) + \big(\tfrac{1}{2} + \tfrac{1}{4}e(S) + \tfrac{1}{2}\,\ell\mathrm{k}(\vec{K},\vec{L} \setminus \vec{K}) - \tfrac{1}{2}\,\ell\mathrm{k}(\vec{K}',\vec{L}' \setminus \vec{K}')\big) \\ &\leqslant g + \tfrac{1}{2}\boldsymbol{v} + \tfrac{1}{4}\bar{e}_{\vec{L}}(F) + \big(\tfrac{1}{2} + \tfrac{1}{4}e(S) + \tfrac{1}{2}\,\ell\mathrm{k}(\vec{K},\vec{L} \setminus \vec{K}) - \tfrac{1}{2}\,\ell\mathrm{k}(\vec{K}',\vec{L}' \setminus \vec{K}')\big) \\ &\leqslant g + \tfrac{1}{2}(\boldsymbol{v}+1) + \tfrac{1}{4}\bar{e}_{\vec{L}'}(F'). \end{aligned}$$

This concludes the proof because all the terms in (5-3) and (5-4) are preserved under concordance; hence, we can ignore extended births and deaths in F. □

This lemma allows us to prove Proposition 5.13. Suppose that L is a link which bounds an unoriented surface F in D^4, with $F_1, \dots, F_k$ as connected components, as in Figure 30. Fix an orientation on L; we need to define the integer $\lambda(\vec{L}, F) := \lambda(\boldsymbol{L}_1) + \cdots + \lambda(\boldsymbol{L}_k)$, where $\boldsymbol{L}_i$ is the oriented sublink of $\vec{L}$ such that $\boldsymbol{L}_i = \partial F_i$. Note that the orientation on L_i has nothing to do with F_i which may be nonorientable as well. We say that $\lambda(\boldsymbol{L}_i) = 0$ when $\boldsymbol{L}_i$ is a knot.

We also write $\widehat{L}$ for the k–component link which appears before the split moves in the decomposition of F in Figure 30. Hence, if we denote by $\widehat{F} \subset F$ the subsurface such that $\widehat{L} = \partial\widehat{F}$ then $\widehat{L}$ and $\widehat{F}$ satisfy the hypothesis of Lemma 5.12.

Proposition 5.13 *With the notation established above, the following inequalities are satisfied for the 2^k orientations of L which are determined by the ones on $\widehat{L}$:*

$$\begin{aligned} -g - \tfrac{1}{2}\,\boldsymbol{v} + k - n + \tfrac{1}{4}\bar{e}_{\vec{L}}(F) \leqslant \upsilon_{\max}(L) \leqslant g + \tfrac{1}{2}\,\boldsymbol{v} + \tfrac{1}{4}\bar{e}_{\vec{L}}(F), \\ -g - \tfrac{1}{2}\,\boldsymbol{v} + 1 - n + \tfrac{1}{4}\bar{e}_{\vec{L}}(F) \leqslant \upsilon_{\min}(L) \leqslant g + \tfrac{1}{2}\,\boldsymbol{v} + 1 - k + \tfrac{1}{4}\bar{e}_{\vec{L}}(F). \end{aligned}$$

Proof We have that

$$\ell k(\widehat{L}_i, \widehat{L}_j) = \sum_{t \in I_i, l \in I_j} \ell k(\vec{L}_t, \vec{L}_l)$$

for every $i, j = 1, \dots, k$, where I_a is the set of the components of L in $\boldsymbol{L}_a$ for $a = 1, \dots, k$. Therefore, one has $\lambda(\widehat{L}) + \lambda(\vec{L}, F) = \lambda(\vec{L})$. We name $F' \subset F$ the cobordism between $\widehat{L}$ and L and we obtain

$$\bar{e}_{\vec{L}}(F) = e(F) - 2\lambda(\vec{L}) = e(\widehat{F}) + e(F') - 2(\lambda(\widehat{L}) + \lambda(\vec{L}, F)) = \bar{e}_{\widehat{L}}(\widehat{F}) + (e(F') - 2\lambda(\vec{L}, F)),$$

and from this, say $F_i' = F' \cap F_i$ is a connected component of F', we argue that

$$e(F') - 2\lambda(\vec{L}, F) = \sum_{i=1}^{k} (e(F_i') - 2\lambda(\boldsymbol{L}_i))$$

by definition of the Euler number. Since each F_i' is oriented and it is a cobordism from $\widehat{L}_i = \widehat{L} \cap F_i$ to $\boldsymbol{L}_i$, we can cap F_i' off in D^4 by gluing a compact oriented surface with boundary $\widehat{L}_i$. In this way, we obtain an oriented surface G_i such that $\partial G_i = \boldsymbol{L}_i$ and $e(G_i) = e(F_i')$ for every $i = 1, \dots, k$ and then

$$\sum_{i=1}^{k} (e(F_i') - 2\lambda(\boldsymbol{L}_i)) = \sum_{i=1}^{k} (e(G_i) - 2\lambda(\boldsymbol{L}_i)) = \sum_{i=1}^{k} \bar{e}_{\boldsymbol{L}_i}(G_i) = 0$$

because the orientation on $\boldsymbol{L}_i$ is induced by the one on G_i (which is the same induced by F_i').

We have proved that $\bar{e}_{\vec{L}}(F) = \bar{e}_{\widehat{L}}(\widehat{F})$ and now we can apply Lemma 5.12 to show that

$$-g - \tfrac{1}{2}\boldsymbol{v} + \tfrac{1}{4}\bar{e}_{\vec{L}}(F) \leqslant \upsilon_{\max}(\widehat{L}) \leqslant g + \tfrac{1}{2}\boldsymbol{v} + \tfrac{1}{4}\bar{e}_{\vec{L}}(F)$$

and

$$-g - \tfrac{1}{2}\boldsymbol{v} + 1 - k + \tfrac{1}{4}\bar{e}_{\vec{L}}(F) \leqslant \upsilon_{\min}(\widehat{L}) \leqslant g + \tfrac{1}{2}\boldsymbol{v} + 1 - k + \tfrac{1}{4}\bar{e}_{\vec{L}}(F).$$

In order to conclude the proof, we apply (5-1) and (5-2) which tell us that

$$\upsilon_{\max}(\vec{L}) \leqslant \upsilon_{\max}(\widehat{L}) \leqslant \upsilon_{\max}(\vec{L}) + n - k \quad \text{and} \quad \upsilon_{\min}(\vec{L}) \leqslant \upsilon_{\min}(\widehat{L}) \leqslant \upsilon_{\min}(\vec{L}) + n - k,$$

provided that the orientation on L belongs to the 2^k ones induced by an orientation of F'. □

We can use this result to prove that the wideness of the υ–set of L gives a lower bound for the *unoriented slice genus* $\gamma_4^{(k)}(L)$, which is defined as the smallest first Betti number of a surface F as in Figure 30 with k connected components.

Proof of Theorem 1.8 This follows from Proposition 5.13 because $2g + \boldsymbol{v} + n - k$ is exactly the first Betti number of F. □

Note that Theorem 1.7 tells us that $\upsilon_{\max}(L) - \upsilon_{\min}(L)$ is an unoriented concordance invariant of L. As a consequence of Theorem 1.8 we obtain Corollary 1.9; see also [Donald and Owens 2012, Section 5] for another proof of this result.

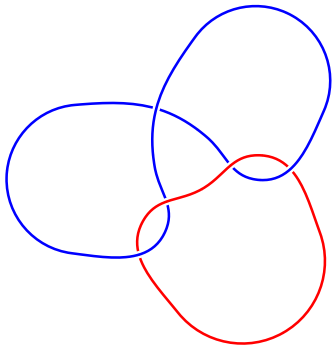

Figure 31: The link $T_{2,4}$; this link becomes the 2–component unlink after the unoriented resolution of the crossing on the blue component.

Proof of Corollary 1.9 Suppose that F is the unoriented surface with maximal value of $\chi(F)$ and say it appears like in Figure 30. As we saw in the proof of Theorem 1.8, the first Betti number of F is $2g+\boldsymbol{v}+n-k$ and then the same theorem implies

$$k-1 \leqslant 2g+\boldsymbol{v}+n-k$$

because, for a quasialternating link L, $\upsilon_{\max}(L)=\upsilon_{\min}(L)$ by Theorem 4.8.

The latter inequality can be rewritten as

$$2k-n-2g-\boldsymbol{v} \leqslant 1$$

and it is easy to check that the left-most side is precisely $\chi(F)$. □

In particular, suppose that the quasialternating link L has n components and F is the disjoint union of a disks and $n-a$ Möbius strips. Then a can be at most equal to one.

We saw in Theorem 1.7 that we can shift $HFL'(\vec{L})$ to obtain an unoriented concordance invariant of links. This suggests that we can modify the bounds in Proposition 5.13 in a way that only unoriented invariants appear. The main tool to achieve this goal is the Gordon–Litherland formula [1978, Corollary 5″],

$$\left|\sigma(\vec{L})-\tfrac{1}{2}\bar{e}_{\vec{L}}(F)\right| \leqslant \gamma_4^{(k)}(L), \tag{5-5}$$

where $L=\partial F$ and $F=F_1\sqcup\cdots\sqcup F_k$.

Proof of Theorem 1.10 We just need to apply (5-5) to Proposition 5.13. □

Note that the quantities that appear in the left-most side of all the inequalities in Theorem 1.10 are unoriented concordance invariants; in particular, they are independent of the choice of the orientation on L.

We conclude the paper with a couple of applications, which imply Corollary 1.11. First, we compute $\gamma_4^{(2)}(L_n)$ when L_n is the 2–component link $T^*_{2,4}\,\#\,T^{\#n}_{3,4}$.

Corollary 5.14 $\gamma_4^{(2)}(L_n) = n + 1$ *for every* $n \geqslant 0$.

Proof Since $T^*_{2,4}$ is nonsplit alternating we can easily compute $\upsilon_{\min}(T^*_{2,4}) = 1$ using Theorem 4.8, while the fact that $\upsilon(T^{\#n}_{3,4}) = -2n$ is known from [Ozsváth et al. 2017b, Corollary 1.4]. Moreover, applying Corollary 4.11 we obtain that

$$\upsilon_{\min}(L_n) = \upsilon_{\min}(T^*_{2,4}) + \upsilon(T^{\#n}_{3,4}) = 1 - 2n.$$

Now we just use Theorem 1.10 and remember that $\sigma(T^*_{2,4}) = 3$ and $\sigma(T_{3,4}) = -6$, so

$$\left|1 - 2n - \tfrac{1}{2}(3 - 6n) + 1\right| = \left|n + \tfrac{1}{2}\right| \leqslant n + 1 \leqslant \gamma_4^{(2)}(L_n).$$

In order to complete the proof we observe that there is a sequence of $n + 1$ nonorientable saddles that change L_n into the unlink $\bigcirc_2$: there is one from $T_{3,4}$ to the unknot and we perform one on each summand, while we can go from $T_{2,4}$ to $\bigcirc_2$ by an unoriented resolution of a crossing; see Figure 31. □

Finally, we show that $\gamma_4^{(1)}(L_n)$ can be arbitrarily large.

Corollary 5.15 $\gamma_4^{(1)}(L_n) \geqslant n$ *for every* $n \geqslant 0$.

Proof We use the last inequality in Theorem 1.10 with $\upsilon_{\min}(L_n)$ and we immediately obtain

$$\left|1 - 2n - \tfrac{1}{2}(3 - 6n - 1)\right| = n \leqslant \gamma_4^{(1)}(L_n).$$ □

We point out that these two results cannot be obtained by using Theorem 1.8 alone.

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