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We define the strong shortcut property for rough geodesic metric spaces, generalizing the notion of strongly shortcut graphs. We show that the strong shortcut property is a rough similarity invariant. We give several new characterizations of the strong shortcut property, including an asymptotic cone characterization. We use this characterization to prove that asymptotically CAT(0) spaces are strongly shortcut. We prove that if a group acts metrically properly and coboundedly on a strongly shortcut rough geodesic metric space then it has a strongly shortcut Cayley graph and so is a strongly shortcut group. Thus we show that CAT(0) groups are strongly shortcut.

To prove these results, we use several intermediate results which we believe may be of independent interest, including what we call the circle tightening lemma and the fine Milnor–Schwarz lemma. The circle tightening lemma describes how one may obtain a quasi-isometric embedding of a circle by performing surgery on a rough Lipschitz map from a circle that sends antipodal pairs of points far enough apart. The fine Milnor–Schwarz lemma is a refinement of the Milnor–Schwarz lemma that gives finer control on the multiplicative constant of the quasi-isometry from a group to a space it acts on.

20F65, 20F67, 51F30

1 Introduction

The study of interactions between nonpositive curvature and infinite group theory have a long history dating back to the work of Max Dehn on fundamental groups of surfaces in the early 20$^{\text{th}}$ century. These ideas have been developed in a variety of directions since that time and have become particularly relevant in recent decades with the emergence of geometric group theory. Various theories of nonpositively curved groups have been developed: small cancellation groups, CAT(0) groups, cubulated groups, systolic groups, quadric groups, etc. However, while the case of *negatively* curved groups has been satisfactorily unified by the seminal work of Gromov [1987] on hyperbolic groups, to this date there is no satisfactory general notion of a nonpositively curved group.

Strongly shortcut graphs were introduced in earlier work [Hoda 2022] as graphs satisfying a weak notion of nonpositive curvature. They were shown to unify a broad family of graphs of interest in geometric group theory and metric graph theory including hyperbolic graphs, standard Cayley graphs of finitely generated Coxeter groups and 1–skeletons of finite-dimensional CAT(0) cube complexes, systolic complexes and quadric complexes [Haettel et al. 2023; Hoda 2022]. They are finitely presented and have polynomial isoperimetric functions and so have decidable word problem [Hoda 2022]. Strongly shortcut groups are defined as those groups admitting a proper and cocompact action on a strongly shortcut graph [Hoda 2022]. They include a wide family of groups satisfying various nonpositive curvature conditions, including hyperbolic groups, Coxeter groups, cocompactly cubulated groups, systolic groups, quadric groups, finitely presented small cancellation groups, Helly groups, hierarchically hyperbolic groups and even the discrete Heisenberg groups [Haettel et al. 2023; Hoda 2022; Le Donne and Paddeu 2023].

A graph Γ is *strongly shortcut* if, for some $K > 1$, there is a bound on the lengths of cycles $\alpha\colon S \to \Gamma$ for which $d_X(\alpha(p), \alpha(\bar{p})) \geq (1/K) \cdot (|S|/2)$ for every antipodal pair of points $p, \bar{p} \in S$. By [Hoda 2022], a graph Γ is *strongly shortcut* if and only if, for some $K > 1$, there is a bound on the lengths of the K–bi-Lipschitz embedded cycles of Γ. A result of Papasoglu [1996, page 793] implies that strongly shortcut groups have simply connected asymptotic cones. By another result of Papasoglu [1996, page 805], this implies that strongly shortcut groups have linear isodiametric functions and, by a result of Riley [2003, Theorem C], this implies that strongly shortcut groups have linear filling length functions.

In this paper, we introduce a generalization of this notion to rough geodesic metric spaces. A metric space X is *R–rough geodesic* if, for every $x_1, x_2 \in X$, there exists a function $f\colon [0, \ell] \to X$ such that $f(0) = x_1$, $f(\ell) = x_2$, $\ell = d(x_1, x_2)$ and

$$|s - t| - R \leq d(f(s), f(t)) \leq |s - t| + R$$

for any s and t in the interval $[0, \ell]$. This is the same as X being $(1, R)$–quasigeodesic. The special case $R = 0$ is that of geodesic metric spaces. An R–rough geodesic metric space X is *strongly shortcut* if, for some $K > 1$, there is a bound on the lengths of Riemannian circles S for which there exists an R–rough 1–Lipschitz map $\alpha\colon S \to X$ that satisfies $d_X(\alpha(p), \alpha(\bar{p})) \geq (1/K) \cdot (|S|/2)$ for every antipodal pair of points $p, \bar{p} \in S$. Such a map α is called a *$1/K$–almost isometric R–circle*. We give several characterizations of the strong shortcut property, we show that a group acting metrically properly and coboundedly on a strongly shortcut rough geodesic metric space is a strongly shortcut group and we prove a few other results that may be of independent interest in metric geometry and geometric group theory. The results of this paper are applied in several upcoming papers [Haettel et al. 2023; Hoda and Krishna M S 2023; Le Donne and Paddeu 2023].

Below is a summary of our main results.

Theorem A gives several conditions that are equivalent to the strong shortcut property for rough geodesic metric spaces. Of particular note is condition (5) which expresses the strong shortcut property in terms of asymptotic cones. Conditions (2) and (3) generalize [Hoda 2022, Proposition 3.5], which expresses the strong shortcut property for graphs in terms of bi-Lipschitz cycles. These two generalizations have their own advantages: for geodesic metric spaces (ie when $R=0$) condition (2) expresses the strong shortcut property purely in terms of bi-Lipschitz maps from circles, whereas condition (3) avoids the dependence on R. Condition (4) expresses the strong shortcut property in terms of nonapproximability of certain finite metric spaces at large scale. Conditions (4) and (5) also make sense for general metric spaces and we prove that they are equivalent for general metric spaces (see Theorem 3.6). Thus one may consider condition (4) of Theorem A as a definition of the strong shortcut property for general metric spaces.

Theorem A (Theorem 3.8) *Let X be an R–rough geodesic metric space. The following conditions are equivalent*:

(1) *X is strongly shortcut.*

(2) *There exists an $L>1$ such that there is a bound on the lengths of the $(L,4R)$–quasi-isometric embeddings of Riemannian circles in X.*

(3) *There exists an $L>1$ such that for every $C\geq 0$ there is a bound on the lengths of the (L,C)–quasi-isometric embeddings of Riemannian circles in X.*

(4) *For some $L>1$ and some $n\in\mathbb{N}$, there is a bound on the $\lambda>0$ for which there exists an L–bi-Lipschitz embedding of λS_n^0 in X, where S_n^0 is the vertex set of the cycle graph S_n of length n and λS_n^0 is S_n^0 with the metric scaled by λ.*

(5) *No asymptotic cone of X contains an isometric copy of the Riemannian circle of unit length.*

The main difficulty in proving Theorem A is in the implication (2) $\implies$ (1). This is because a $1/K$–almost isometric R–circle in X does not need to be an $(L,4R)$–quasi-isometric embedding for any $L>1$; while the almost isometric condition only concerns pairs of antipodal points, the quasi-isometry condition concerns all pairs of points. The idea of the proof is that given a $1/K$–almost isometric R–circle α with $K>1$ sufficiently close to 1, we can perform surgery on α in order to obtain an $(L,4R)$–quasi-isometric embedding where L depends on K in such a way that if $K\to 1$ then $L\to 1$ also. The contrapositive $\neg$(1) $\implies$ $\neg$(2) then readily follows since any family of arbitrarily long R–circles with almost isometric constant K approaching 1 could then be surgered to produce a family of quasi-isometric embeddings with the multiplicative constant L tending to 1. The circle surgery result, which we call the circle tightening lemma, is stated in slightly simplified form in Theorem G below and expressed more formally in Lemma 4.5.

Theorem B (Theorems 3.5 and 3.8) *Let Γ be a graph. Then Γ is strongly shortcut as a graph if and only if Γ is strongly shortcut as a geodesic metric space.*

Theorem C gives several conditions that are equivalent to the strong shortcut property for groups. Condition (3) reduces the property to the existence of a strongly shortcut Cayley graph. The proof is a direct application of the fine Milnor–Schwarz lemma (Theorem H) and stability of the strong shortcut property under scaling and quasi-isometric perturbation of the metric (Theorem F).

Theorem C (Corollary 5.5) *Let G be a group. The following conditions are equivalent.*

(1) *G is strongly shortcut.*

(2) *G acts metrically properly and coboundedly on a strongly shortcut rough geodesic metric space.*

(3) *G has a finite generating set S for which the Cayley graph of (G, S) is strongly shortcut.*

Asymptotically $\mathrm{CAT}(0)$ spaces and groups were first introduced and studied by Kar [2011]. A metric space X is *asymptotically* $\mathrm{CAT}(0)$ if every asymptotic cone of X is $\mathrm{CAT}(0)$. A group is *asymptotically* $\mathrm{CAT}(0)$ if it acts properly and cocompactly on an asymptotically $\mathrm{CAT}(0)$ proper geodesic metric space. Examples of asymptotically $\mathrm{CAT}(0)$ spaces include $\mathrm{CAT}(0)$ spaces, Gromov-hyperbolic spaces and $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ with the Sasaki metric [Kar 2011].

Theorem D (Theorems 6.1 and 6.2) *Asymptotically* $\mathrm{CAT}(0)$ *rough geodesic metric spaces are strongly shortcut. Consequently*, (*asymptotically*) $\mathrm{CAT}(0)$ *groups are strongly shortcut.*

Theorem E shows that the strong shortcut condition is preserved under taking asymptotic cones. This was suggested as a desirable property for a general notion of nonpositive curvature in Gromov [1993, Section 6.E].

Theorem E (Corollary 3.9) *Let X be an R–rough geodesic metric space. If X is strongly shortcut then every asymptotic cone of X is strongly shortcut.*

Theorem F below has several consequences. In addition to showing that the strong shortcut property descends to isometric subspaces and is a rough similarity invariant, it implies that for a given strongly shortcut space, a sufficiently small bi-Lipschitz distortion of the metric preserves the strong shortcut property. This is another property which is discussed in Gromov [1993, Section 6.E].

Theorem F (Corollary 3.10) *Let X be a strongly shortcut rough geodesic metric space. Then there exists an $L_X > 1$ such that whenever Y is a rough geodesic metric space and $f : Y \to X$ is an (L_X, C)–quasi-isometric embedding up to scaling, with $C > 0$, then Y is also strongly shortcut. In particular, the strong shortcut property is a rough similarity invariant of rough geodesic metric spaces.*

In fact, Theorem F holds for general metric spaces with condition (4) of Theorem A in place of the strong shortcut property (see Proposition 3.4). It should be noted that the strong shortcut property is not

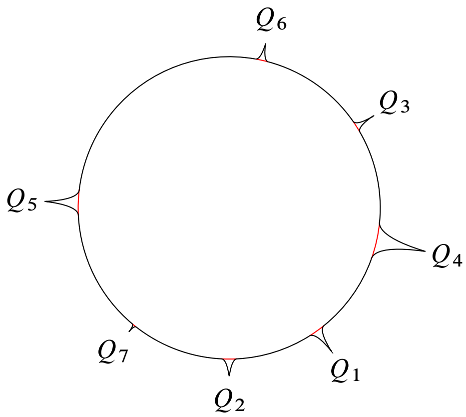

Figure 1: The closed outer path in black is a 1–Lipschitz embedding α of a Riemannian circle S. This embedding α has poor bi-Lipschitz constant but only because it badly distorts distances between relatively nearby pairs of points of S (pairs contained in the subpaths $\alpha|_{Q_i}$). If we consider only antipodal pairs of points of S then the distortion of their distances under α is much less than in these worst cases. In other words α has low distortion when viewed *globally*. The circle tightening lemma tells us that if the global distortion of α is low enough then we can perform surgery on α, replacing distorted subpaths of arbitrarily low total relative length with efficient alternatives (the red paths in the figure) in order to obtain an arbitrarily good bi-Lipschitz constant.

a quasi-isometry invariant so one cannot hope to remove the dependence on X of the quasi-isometry constant L_X. See Section 3.1 for an example.

The following result, which we call the circle tightening lemma, states that a map from a circle that satisfies a rough Lipschitz upper bound and that, on antipodes, satisfies a bi-Lipschitz lower bound can be upgraded through surgery to a rough bi-Lipschitz map. See Figure 1. The circle tightening lemma is essential in the proof of Theorem A. We believe it may be of independent interest. Here we express a slightly simplified version of the circle tightening lemma. For the formal statement, please see Lemma 4.5.

Theorem G (circle tightening lemma, Lemma 4.5) *Let X be an R–rough geodesic metric space with $R \geq 0$, let $L > 1$ and let $\varepsilon > 0$. There exists a $K > 1$ such that if $\alpha\colon S \to X$ is a sufficiently long R–rough 1–Lipschitz map from a Riemannian circle S satisfying*

$$d_X(\alpha(p), \alpha(\bar{p})) \geq \frac{1}{K} d_S(p, \bar{p})$$

for every antipodal pair $p, \bar{p} \in S$ then there exists a countable collection $\{Q_i\}_i$ of pairwise disjoint closed segments in S of total length $\sum_i |Q_i| < \varepsilon |S|$ such that shortening the Q_i and replacing the $\alpha|_{Q_i}\colon Q_i \to X$ we can obtain from α an $(L, 4R)$–quasi-isometric embedding of a circle.

Note that in the statement of the circle tightening lemma, the rough geodesicity constant R may be equal to 0 in which case the result is about 1–Lipschitz maps and L–bi-Lipschitz maps in a geodesic metric space.

We call the following refinement of the Milnor–Schwarz lemma the fine Milnor–Schwarz lemma. It is used in the proof of Theorem C. It essentially says that if a group G acts metrically properly and coboundedly on a rough geodesic space X then, up to scaling, the group G has word metrics that are quasi-isometric to X with multiplicative constant arbitrarily close to 1. We believe it may be of independent interest.

Theorem H (fine Milnor–Schwarz lemma, Lemma 5.2, Remark 5.1) *Let (X,d) be a rough geodesic metric space. Let G be a group acting metrically properly and coboundedly on X by isometries. Fix $x_0 \in X$. For $t > 0$ let S_t be the finite set defined by*

$$S_t = \{g \in G \mid d(x_0, gx_0) \le t\}$$

and consider the word metric d_{S_t} defined by S_t. (For those t where S_t does not generate G, we allow d_{S_t} to take the value ∞). Let K_t be the infimum of all $K > 1$ for which

$$(G, t d_{S_t}) \to X, \quad g \mapsto g \cdot x_0,$$

is a (K, C_K)–quasi-isometry for some $C_K \ge 0$. Then $K_t \to 1$ as $t \to \infty$.

Structure of the paper

In Section 2 we introduce basic notions that will be used throughout the paper. In Section 3 prove various characterizations of the strong shortcut property and prove that it is a rough similarity invariant. In Section 4 we state and prove the circle tightening lemma. In Section 5 we state and prove the fine Milnor–Schwarz lemma. In Section 6, we apply the results of the previous sections to prove that asymptotically CAT(0) groups are strongly shortcut.

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2 Basic notions and definitions

Let X and Y be metric spaces, let S be a set and let R be a nonnegative real. A function $f : S \to Y$ is R–roughly onto if every $y \in Y$ is at distance at most R from some point in $f(X) \subset Y$. An *R–rough isometric embedding* from X to Y is a function $f : X \to Y$ such that

$$d(x_1, x_2) - R \le d(f(x_1), f(x_2)) \le d(x_1, x_2) + R$$

for all $x_1, x_2 \in X$. An R–rough isometric embedding is the same as a $(1, R)$–quasi-isometric embedding. An R–rough isometric embedding $f : X \to Y$ is an *R–rough isometry* if it is roughly onto. An R–rough isometry is the same as a $(1, R)$–quasi-isometry.

An *R–rough geodesic* in X from x_1 to x_2 is an R–rough isometric embedding f from the interval $[0, \ell] \subset \mathbb{R}$ to X with $\ell = d(x_1, x_2)$ such that $f(0) = x_1$ and $f(\ell) = x_2$. An R–rough geodesic is the same as a $(1, R)$–quasigeodesic. A metric space (X, d) is *R–rough geodesic* if every pair of points in X is joined by an R–rough geodesic.[1] Note that rough geodesicity implies weak geodesicity, as used by Kasparov and Skandalis [1994; 2003] and others [Lafforgue 2002; Mineyev and Yu 2002]. A natural question is whether or not every rough geodesic space can be thickened, in the sense of Gromov [1993, Section 1.B] to a geodesic metric space.

Let X and Y be metric spaces, let $R \geq 0$ and let $K \geq 1$. An *R–rough K–Lipschitz map* from Y to X is a function $\alpha \colon Y \to X$ such that

$$d(\alpha(p), \alpha(q)) \leq K d(p, q) + R$$

for all $p, q \in Y$. An *R–path* in X is an R–rough 1–Lipschitz map $\alpha \colon P \to X$ from an interval $P \subset \mathbb{R}$. An *R–circle* in X is an R–rough 1–Lipschitz map $\alpha \colon S \to X$ from a Riemannian circle S. We use the notation $|F|$ to denote the length of F, where F is an interval, a Riemannian circle or a finite union of closed segments in an interval or in a Riemannian circle.

Remark 2.1 The concatenation of two R–paths need not be an R–path. However, if $\alpha_1 \colon P_1 \to X$ and $\alpha_2 \colon P_2 \to X$ are a pair of concatenable R–paths and $\gamma \colon [0, R] \to X$ is the constant path at the point of concatenation then the concatenation $\alpha_1 \gamma \alpha_2$ is an R–path.

An R–circle $\alpha \colon S \to X$ is *$1/K$–almost isometric*, for some $K > 1$, if

$$d(\alpha(p), \alpha(\bar{p})) \geq \frac{1}{K} \cdot \frac{|S|}{2}$$

for every antipodal pair of points $p, \bar{p} \in S$.

Definition 2.2 An R–rough geodesic metric space X is *strongly shortcut* if, for some $K > 1$, there is a bound on the lengths of the $1/K$–almost isometric R–cycles of X.

Remark 2.3 By Theorem 3.8, the apparent dependence on R in Definition 2.2 is not essential. That is, if X is an R–rough geodesic metric space and $R' > R$ then X is strongly shortcut if and only if it is strongly shortcut when viewed as an R'–rough geodesic metric space.

We view graphs as geodesic metric spaces with each edge isometric to a unit interval. For a graph Γ, we use the notation Γ^0 to denote the vertex set of Γ with its subspace metric. The *cycle graph S_n* of length n is the graph isometric to a Riemannian circle of length n. A *cycle* in a graph Γ is a combinatorial

[1]We include the condition $\ell = d(x_1, x_2)$ in the definition of an R–rough geodesic f only for convenience. If we do not assume it, then we can recover it up to slightly increasing the rough geodesicity constant to $R' = (1 + \sqrt{2})R$. Indeed, the remaining conditions on f imply that $|\ell - d(x_1, x_2)| \leq R$ and it can be shown that either $d(x_1, x_2) \leq R'$ (in which case any function $[0, d(x_1, x_2)] \to \{x_1, x_2\}$ is an R'–rough isometric embedding) or the composition of f with the orientation preserving linear bijection $[0, d(x_1, x_2)] \to [0, \ell]$ results in an R'–rough isometric embedding.

map $S_n \to \Gamma$ from some cycle graph S_n to Γ. A *path graph* is a graph isometric to a real interval. A *combinatorial path* in a graph Γ is a combinatorial map $P \to \Gamma$ from a path graph P.

Note that if $\alpha\colon S_n \to \Gamma$ is a cycle in a graph Γ then α is a 1–Lipschitz map from a Riemannian circle to a geodesic metric space or, in the language we have established above, an R–circle in an R–rough geodesic metric space where $R = 0$.

Definition 2.4 A graph Γ is *strongly shortcut as a graph* if, for some $K > 1$, there is a bound on the lengths of the $1/K$–almost isometric cycles of Γ.

Remark 2.5 By Theorems 3.5 and 3.8 and Corollary 3.10, the following conditions are equivalent for a graph Γ.

(1) Γ is strongly shortcut as a graph.

(2) Γ is strongly shortcut as a geodesic metric space.

(3) Γ^0 is strongly shortcut as a rough geodesic metric space.

If X is a metric space and $\lambda > 0$ then we write λX to denote the metric space obtained from X by scaling the metric by λ.

3 Characterizing the strong shortcut property

In this section we will give various characterizations of the strong shortcut property.

Lemma 3.1 *Let $\alpha\colon S \to X$ be a $1/K$–almost isometric R–circle in a metric space X. Then*

$$d(\alpha(p), \alpha(q)) \geq d(p,q) - \frac{K-1}{K} \cdot \frac{|S|}{2} - 2R$$

for all $p, q \in S$.

Proof Let $p', q' \in S \setminus \{p, q\}$ be antipodal and suppose that a geodesic segment of S visits p', p, q and q', in that order. Then

$$\begin{aligned}
\frac{1}{K} \cdot \frac{|S|}{2} &\leq d(\alpha(p'), \alpha(q')) \\
&\leq d(\alpha(p'), \alpha(p)) + d(\alpha(p), \alpha(q)) + d(\alpha(q), \alpha(q')) \\
&\leq d(p', p) + R + d(\alpha(p), \alpha(q)) + d(q, q') + R \\
&= d(p', p) + d(q, q') + d(\alpha(p), \alpha(q)) + 2R \\
&= \frac{|S|}{2} - d(p,q) + d(\alpha(p), \alpha(q)) + 2R,
\end{aligned}$$

from which we can obtain the desired inequality. □

The following definition is very useful because it applies to general metric spaces.

Definition 3.2 A metric space X *approximates n–gons* if for every $K > 1$ and every $n \in \mathbb{N}$ there exist K–bi-Lipschitz embeddings of λS_n^0 in X for arbitrarily large $\lambda > 0$.

We will see that in the case of a graph or a rough geodesic metric space *non*approximation of n–gons is equivalent to the strong shortcut property. Thus it would make sense to define the strong shortcut property for general metric spaces as nonapproximability of n–gons.

Definition 3.3 Let X and Y be metric spaces. A function $f: Y \to X$ is a (K, C)–quasi-isometry *up to scaling* if there exists a $\lambda > 0$ such that f is a (K, C)–quasi-isometry when viewed as a function from λY to X. A function $f: Y \to X$ is a *rough similarity* if, for some $C > 0$, the function f is a $(1, C)$–quasi-isometry up to scaling. A property $\mathcal{P}$ of metric spaces is a *rough similarity invariant* if whenever X satisfies $\mathcal{P}$ and $f: Y \to X$ is a rough similarity then Y also satisfies $\mathcal{P}$. A property $\mathcal{P}$ of metric spaces is a *rough approximability invariant* if, for any metric space X satisfying $\mathcal{P}$, there exists an $L_X > 1$ such that whenever $C > 0$ and $f: Y \to X$ is an (L_X, C)–quasi-isometric embedding up to scaling, then Y also satisfies $\mathcal{P}$.

Proposition 3.4 *Nonapproximability of n–gons is a rough approximability invariant of metric spaces. In particular, nonapproximability of n–gons is a rough similarity invariant of metric spaces.*

Proof Let X be a metric space that does not approximate n–gons. Then there is a $K > 1$, an $n \in \mathbb{N}$ and a $\Lambda > 0$ such that any K–bi-Lipschitz embedding of λS_n^0 in X satisfies $\lambda < \Lambda$.

Let Y be a metric space, let $t > 0$, let $L \in (1, K)$ and let $f: tY \to X$ be an (L, C)–quasi-isometric embedding. Let $K' \in (1, K/L)$ and let $\lambda' > CLK'/t$. We will show that there is a bound on the λ' for which there exists a K'–bi-Lipschitz embedding $\alpha: \lambda' S_n^0 \to Y$. Viewing such an α as map from $t\lambda' S_n^0$ to tY the composition $f \circ \alpha: t\lambda' S_n^0 \to X$ is an (LK', C)–quasi-isometric embedding. But the minimum distance between distinct points in $t\lambda' S_n^0$ is $t\lambda'$ and so one can show that $f \circ \alpha$ is a $(t\lambda' LK' + C)/(t\lambda' - LK'C)$–bi-Lipschitz embedding from $t\lambda' S_n^0$. But

$$\frac{t\lambda' LK' + C}{t\lambda' - LK'C} \to LK' < K$$

as $\lambda' \to \infty$ so there is a Λ_0 such that if $\lambda' \geq \Lambda_0$ then $(t\lambda' LK' + C)/(t\lambda' - LK'C) < K$. So if we had $\lambda' \geq \Lambda' = \max\{\Lambda_0, \Lambda/t\}$ then $f \circ \alpha$ would be a K–bi-Lipschitz embedding of λS_n^0 in X with $\lambda = t\lambda' \geq \Lambda$, which would be a contradiction. Thus Λ' bounds the λ' for which there exists a K'–bi-Lipschitz embedding $\alpha: \lambda' S_n^0 \to Y$, as required. □

Theorem 3.5 *Let Γ be a graph. Then the following conditions are equivalent.*

(1) Γ *is not strongly shortcut as a graph.*

(2) Γ *approximates n–gons.*

Proof (1) $\Longrightarrow$ (2) Let $K' > 1$ and let $\alpha: S_{n'} \to \Gamma$ be a $1/K'$–almost isometric cycle. Let $n \in \mathbb{N}$ and subdivide $S_{n'}$ into n segments of equal length, ignoring the original graph structure on $S_{n'}$. Let $Y \subset S_{n'}$ be the set of endpoints of the segments. Then Y is isometric to λS_n for $\lambda = n'/n$. Let α' be the composition of the inclusion $Y \hookrightarrow S_{n'}$ with α. Let $p, q \in Y$ be distinct. Then $d(p,q) \geq |S_{n'}|/n$ and, by Lemma 3.1,

$$d(\alpha'(p), \alpha'(q)) \geq d(p,q) - \frac{K'-1}{K'} \cdot \frac{|S_{n'}|}{2} \geq d(p,q) - \frac{K'-1}{K'} \cdot \frac{n\,d(p,q)}{2} = \left(1 - \frac{n(K'-1)}{2K'}\right) d(p,q),$$

but $d(\alpha'(p), \alpha'(q)) \leq d(p,q)$ so, when K' is small enough that $n(K'-1)/2K' < 1$, the map α' is K–bi-Lipschitz for

$$K = \left(1 - \frac{n(K'-1)}{2K'}\right)^{-1}.$$

Thus, given an arbitrary $n \in \mathbb{N}$ and an α' as above with K' small enough, we can obtain a K–bi-Lipschitz embedding of λS_n in Γ with

$$K = \left(1 - \frac{n(K'-1)}{2K'}\right)^{-1}, \quad \lambda = \frac{n'}{n}.$$

Since Γ is not strongly shortcut, there exist α as above with $K' > 1$ arbitrarily close to 1 and with n' arbitrarily large. But $K \to 1$ as $K' \to 1$ and $\lambda \to \infty$ as $n' \to \infty$ so we have K–bi-Lipschitz embeddings of λS_n with K arbitrarily close to 1 and λ arbitrarily large.

(2) $\Longrightarrow$ (1) Let $n \in \mathbb{N}$, let $K > 1$, let $\lambda > K$ and let $\alpha\colon \lambda S_n^0 \to \Gamma$ be a K–bi-Lipschitz embedding. There is a retraction $r\colon \Gamma \to \Gamma^0$ such that r is a $(1,1)$–quasi-isometry. Then the composition $r \circ \alpha$ is a $(K,1)$–quasi-isometric embedding. But distinct points in λS_n^0 are at distance at least λ and, since $K < \lambda$, this implies that $r \circ \alpha$ is L–bi-Lipschitz, where $L = (K\lambda + 1)/(\lambda - K)$. View S_n as the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with generating set $\{1\}$ and, for $i \in \mathbb{Z}/n\mathbb{Z}$, let v_i be the vertex of S_n corresponding to i. Then, for each i, we have $d(r \circ \alpha(v_i), r \circ \alpha(v_{i+1})) \leq \lfloor L\lambda \rfloor$ so there is a combinatorial path $\gamma_i\colon P_i \to \Gamma$ of length $m_i \in \{\lfloor L\lambda \rfloor - 1, \lfloor L\lambda \rfloor\}$ from $r \circ \alpha(v_i)$ to $r \circ \alpha(v_{i+1})$. For each i, identify the endpoint of P_i with the initial point of P_{i+1} to obtain a cycle $\gamma\colon S_m \to \Gamma$ with $m = \sum_{i=1}^n m_i$. Then α factors through γ via the embedding that sends v_i to the initial point of $P_i \subset S_m$. So, viewing S_n^0 as a subset of S_m^0 via this embedding, we have $r \circ \alpha(v_i) = \gamma(v_i)$, for each i. Let $x \in S_m$ and let v_i minimize $d(x, v_i)$. Then $d(x, v_i) \leq \frac{1}{2}\lfloor L\lambda \rfloor$ and if $\bar{x}$ is the antipode of x and $\bar{i} = i + \left\lfloor \frac{n}{2} \right\rfloor$ then $d(\bar{x}, v_{\bar{i}}) \leq \lfloor L\lambda \rfloor + \frac{n}{2}$, so

$$\begin{aligned}
d(\gamma(x), \gamma(\bar{x})) &\geq d(\gamma(v_i), \gamma(v_{\bar{i}})) - d(\gamma(x), \gamma(v_i)) - d(\gamma(\bar{x}), \gamma(v_{\bar{i}})) \\
&\geq d(\gamma(v_i), \gamma(v_{\bar{i}})) - d(x, v_i) - d(\bar{x}, v_{\bar{i}}) \\
&\geq d(\gamma(v_i), \gamma(v_{\bar{i}})) - \tfrac{1}{2}\lfloor L\lambda \rfloor - \lfloor L\lambda \rfloor - \tfrac{1}{2}n \\
&= d(r \circ \alpha(v_i), r \circ \alpha(v_{\bar{i}})) - \tfrac{3}{2}\lfloor L\lambda \rfloor - \tfrac{1}{2}n \\
&\geq \frac{1}{L} d_{\lambda S_n^0}(v_i, v_{\bar{i}}) - \tfrac{3}{2}\lfloor L\lambda \rfloor - \tfrac{1}{2}n \\
&= \frac{\lambda}{L}\left\lfloor \tfrac{1}{2}n \right\rfloor - \tfrac{3}{2}\lfloor L\lambda \rfloor - \tfrac{1}{2}n \\
&\geq \frac{\lambda}{L}\left(\tfrac{1}{2}(n-1)\right) - \tfrac{3}{2}L\lambda - \tfrac{1}{2}n = \frac{1}{m}\left(\frac{\lambda(n-1)}{L} - 3L\lambda - n\right) \cdot \tfrac{1}{2}|S_m|.
\end{aligned}$$

Since $m \le nL\lambda$, the above computation implies that

$$d(\gamma(x),\gamma(\bar{x})) \ge \left(\frac{n-1}{nL^2}-\frac{3}{n}-\frac{1}{L\lambda}\right)\frac{|S_m|}{2}$$

so, given α as above, we can obtain a $1/K'$–almost isometric cycle in Γ of length m, where

$$\frac{1}{K'}=\left(\frac{n-1}{nL^2}-\frac{3}{n}-\frac{1}{L\lambda}\right).$$

We need only show there exist α for which $1/K'$ is arbitrarily close to 1 and m is arbitrarily large. By hypothesis, there exist α for which $K=(n+1)/n$ and $\lambda > n$, for arbitrary $n \in \mathbb{N}$. But then, as $n \to \infty$, we have $m \ge n(\lfloor L\lambda \rfloor - 1) \to \infty$ and $L=(K\lambda+1)/(\lambda-K) \to K$ so

$$\frac{1}{K'}=\left(\frac{n-1}{nL^2}-\frac{3}{n}-\frac{1}{L\lambda}\right) \to 1. \qquad \square$$

Let X be a metric space. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Let $(b^{(m)})_{m\in\mathbb{N}}$ be a sequence in X. Let $(s^{(m)})_{m\in\mathbb{N}}$ be a sequence of positive reals such that $s^{(m)} \to \infty$ as $m \to \infty$. Consider the set

$$\mathfrak{X}'=\left\{(x_m)_{m\in\mathbb{N}} \;\middle|\; \left(\frac{d(x_m,b^{(m)})}{s^{(m)}}\right)_{m\in\mathbb{N}} \text{ is bounded}\right\}$$

of *sequences in X that are bounded with respect to* the *basepoint sequence* $(b^{(m)})_m$ and the *scaling sequence* $(s^{(m)})_m$. For $(x_m)_m, (x'_m)_m \in \mathfrak{X}'$,

$$\bar{d}((x_m)_m,(x'_m)_m)=\lim_{\mathcal{U}} \frac{d(x_m,x'_m)}{s^{(m)}}$$

defines a pseudometric on $\mathfrak{X}'$. The *asymptotic cone* $\mathfrak{X}$ of X with respect to the nonprincipal ultrafilter $\mathcal{U}$, the *basepoint sequence* $(b^{(m)})_m$ and the *scaling sequence* $(s^{(m)})_m$ is the metric space obtained from $\mathfrak{X}'$ and $\bar{d}$ by identifying $(x_m)_m$ and $(x'_m)_m$ whenever $\bar{d}((x_m)_m,(x'_m)_m)=0$.

Note that Theorem 3.6 and Corollary 3.7 apply to general metric spaces and not just rough geodesic metric spaces.

Theorem 3.6 *Let X be a metric space. Then the following conditions are equivalent.*

1. *There is an asymptotic cone of X that contains an isometric copy of the Riemannian circle of unit length.*
2. *X approximates n–gons.*

Proof (1) $\Longrightarrow$ (2) Suppose $S \subset \mathfrak{X}$ is a subspace isometric to the Riemannian circle of unit length in the *asymptotic cone* $\mathfrak{X}$ of X with respect to a nonprincipal ultrafilter $\mathcal{U}$, a *basepoint sequence* $(b^{(m)})_m$ and a *scaling sequence* $(s^{(m)})_m$. Take any $n \in \mathbb{N}$, any $K > 1$ and any $\Lambda > 0$. We will construct a K–bi-Lipschitz map $\alpha\colon \lambda S^0_n \to X$ with $\lambda \ge \Lambda$. Subdivide S into n segments of equal length and let S^0 denote the set of endpoints of the segments. For each $\varepsilon > 0$ and each $p,q \in S^0$ represented by $(p_m)_m$ and $(q_m)_m$, there is an $A^{p,q}_\varepsilon \in \mathcal{U}$ such that

$$d(p,q)-\varepsilon \le \frac{d(p_m,q_m)}{s^{(m)}} \le d(p,q)+\varepsilon$$

for all $m \in A_\varepsilon^{p,q}$. There are finitely many pairs $p,q \in S^0$ so $A_\varepsilon = \bigcap_{p,q} A_\varepsilon^{p,q} \in \mathcal{U}$. Then, for any distinct $p,q \in S^0$,

$$\Bigl(1-\frac{\varepsilon}{d(p,q)}\Bigr)s^{(m)}d(p,q) \le d(p_m,q_m) \le \Bigl(1+\frac{\varepsilon}{d(p,q)}\Bigr)s^{(m)}d(p,q)$$

for all $m \in A_\varepsilon$. But $d(p,q) \ge \frac{1}{n}$ and so

$$(1-n\varepsilon)s^{(m)}d(p,q) \le d(p_m,q_m) \le (1+n\varepsilon)s^{(m)}d(p,q)$$

for all $m \in A_\varepsilon$. So if $n\varepsilon < 1$ then, for $m \in A_\varepsilon$, the map

$$\alpha_m\colon s^{(m)}S^0 \to X, \quad p \mapsto p_m,$$

is bi-Lipschitz with bi-Lipschitz constant

$$\max\Bigl\{1+n\varepsilon, \frac{1}{1-n\varepsilon}\Bigr\} = \frac{1}{1-n\varepsilon}.$$

The space $s^{(m)}S^0$ is isometric to $(s^{(m)}/n)S^0_n$ so if we chose ε small enough that $n\varepsilon < 1$ and $1/(1-n\varepsilon) < K$ and we take $m \in A_\varepsilon$ large enough that $s^{(m)}/n \ge \Lambda$ then we can take $\alpha = \alpha_m$.

(2) $\Longrightarrow$ (1) For $m \in \mathbb{N}$, there exists a $(m+1)/m$–bi-Lipschitz map $\alpha_m\colon \lambda_m S^0_{2^m} \to X$ with $\lambda_m \ge m$. Metrize the group $(1/2^m)\mathbb{Z} = \{k/2^m \mid k \in \mathbb{Z}\} \subset \mathbb{R}$ with the subspace metric and metrize the quotient group $(1/2^m)\mathbb{Z}/\mathbb{Z}$ with the quotient metric. Then $(1/2^m)S^0_{2^m}$ is isometric to $(1/2^m)\mathbb{Z}/\mathbb{Z}$. Via this isometry we identify the vertex set $S^0_{2^m}$ with the elements of $(1/2^m)\mathbb{Z}/\mathbb{Z}$. Thus we view $(1/2^m)S^0_{2^m}$ as a metric subspace of the Riemannian circle of unit length $S = \mathbb{R}/\mathbb{Z}$. By this identification, the union $S_{\mathbb{D}} = \bigcup_{m\in\mathbb{N}} S^0_{2^m} \subset S$ is the *dyadic circle* $\mathbb{Z}\bigl[\frac{1}{2}\bigr]/\mathbb{Z}$. The dyadic circle $S_{\mathbb{D}}$ is dense in S. Thus, since asymptotic cones are complete metric spaces [Druţu and Kapovich 2018, Proposition 10.70], it will suffice to isometrically embed $S_{\mathbb{D}}$ into an asymptotic cone of X.

View α_m as an $(m+1)/m$–bi-Lipschitz map from $(1/2^m)S^0_{2^m}$ to $(1/\lambda_m 2^m)X$. Set $b^{(m)} = \alpha_m(0)$ and set $s^{(m)} = \lambda_m 2^m$. Every nonzero element of $S_{\mathbb{D}}$ can be uniquely represented as $k/2^\ell$ with k odd and satisfying $0 \le k < 2^\ell$. For any such representation $k/2^\ell$ and any $m \in \mathbb{N}$, set

$$x^{(m)}_{k/2^\ell} = \begin{cases} b^{(m)} & \text{if } m < \ell, \\ \alpha_m(k/2^\ell) & \text{if } m \ge \ell, \end{cases}$$

and set $x^{(m)}_0 = b^{(m)}$. Then, for any nonprincipal ultrafilter $\mathcal{U}$, the expression $p \mapsto (x^{(m)}_p)_m$ defines an isometric embedding of $S_{\mathbb{D}}$ into the asymptotic cone $\mathcal{X}$ of X with respect to $\mathcal{U}$, the basepoint sequence $(b^{(m)})_m$ and the scaling sequence $S^{(m)}$. Indeed, for every $p,q \in S_{\mathbb{D}}$,

$$\frac{d(x^{(m)}_p, x^{(m)}_q)}{s^{(m)}} = \frac{d\bigl(\alpha_m(p),\alpha_m(q)\bigr)}{\lambda_m 2^m} \le \frac{(m+1)/m\cdot\lambda_m d_{S_{2^m}}(p,q)}{\lambda_m 2^m} = \frac{m+1}{m}\cdot d_{S_{\mathbb{D}}}(p,q)$$

and

$$\frac{d(x^{(m)}_p, x^{(m)}_q)}{s^{(m)}} = \frac{d\bigl(\alpha_m(p),\alpha_m(q)\bigr)}{\lambda_m 2^m} \ge \frac{m/(m+1)\cdot\lambda_m d_{S_{2^m}}(p,q)}{\lambda_m 2^m} = \frac{m}{m+1}\cdot d_{S_{\mathbb{D}}}(p,q)$$

whenever m is large enough. □

Corollary 3.7 *Let X be a metric space and let $\mathfrak{X}$ be an asymptotic cone of X. Suppose that X does not approximate n–gons. Then $\mathfrak{X}$ does not approximate n–gons.*

Proof By Theorem 3.6, it suffices to show that any asymptotic cone $\mathfrak{X}'$ of $\mathfrak{X}$ does not contain an isometric copy of a Riemannian circle of unit length. But $\mathfrak{X}'$ is isometric to an asymptotic cone of X [Druţu and Kapovich 2018, Corollary 10.80] so does not contain an isometric copy of a Riemannian circle of unit length by Theorem 3.6. □

Theorem 3.8 *Let X be an R–rough geodesic metric space. The following conditions are equivalent.*

(1) *X is not strongly shortcut.*

(2) *For every $L > 1$ there exist $(L, 4R)$–quasi-isometric embeddings of arbitrarily long Riemannian circles in X.*

(3) *For every $L > 1$ there is a $C \geq 0$ such that there exist (L, C)–quasi-isometric embeddings of arbitrarily long Riemannian circles in X.*

(4) *X approximates n–gons.*

(5) *There is an asymptotic cone of X that contains an isometric copy of the Riemannian circle of unit length.*

Proof Conditions (4) and (5) are equivalent for general metric spaces, by Theorem 3.6. So it will suffice to prove the equivalence of conditions (1), (2), (3) and (4).

(1) $\Longrightarrow$ (2) Let $N = 2$, let $L > 1$ be arbitrary, let $K > 1$ be small enough to satisfy Lemma 4.5 and let $\alpha\colon S \to X$ be a $1/K$–almost isometric R–circle in X with $|S|$ arbitrarily larger than the M from Lemma 4.5. Then the limit R–circle $\alpha_\infty\colon S_\infty \to X$ given by Lemma 4.5 is an $(L, 4R)$–quasi-isometric embedding of a Riemannian circle of length at least $\frac{1}{2}|S|$.

(2) $\Longrightarrow$ (3) This is immediate.

(3) $\Longrightarrow$ (4) Let $\alpha\colon S \to X$ be an (L, C)–quasi-isometric embedding of a Riemannian circle. Let $n \in \mathbb{N}$, subdivide S into n segments of equal length and let Y be the set of endpoints of the segments. Then Y is isometric to $(|S|/n)S_n$ and

$$\Big(\frac{1}{L} - \frac{nC}{|S|}\Big) d(p,q) \leq d(\alpha(p), \alpha(q)) \leq \Big(L + \frac{nC}{|S|}\Big) d(p,q)$$

for distinct $p, q \in Y$. By hypothesis, there exist arbitrarily long α with L arbitrarily close to 1 and so $\alpha|_Y$ is a K–bi-Lipschitz embedding of λS_n for $\lambda = |S|/n$ arbitrarily large and

$$K = \max\Big\{\Big(\frac{1}{L} - \frac{nC}{|S|}\Big)^{-1}, L + \frac{nC}{|S|}\Big\}$$

arbitrarily close to 1.

(4) $\Longrightarrow$ (1) Let $n \in \mathbb{N}$, let $L > 1$, let $\lambda > 0$ and let $\alpha\colon \lambda S_n^0 \to X$ be an L–bi-Lipschitz embedding. View S_n as the Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ with generating set $\{1\}$ and, for $i \in \mathbb{Z}/n\mathbb{Z}$, let v_i be the vertex of S_n corresponding to i. Then, for each i, we have $d(\alpha(v_i), \alpha(v_{i+1})) \leq L\lambda$ so, by scaling an R–rough

geodesic, there is an R–path $\gamma'_i\colon P_i \to X$ of length $|P_i| = L\lambda$ from $\alpha(v_i)$ to $\alpha(v_{i+1})$. Let γ_i be the concatenation $c_i\gamma'_i$ where $c_i : [0, R] \to X$ is the constant path of length R at $\alpha(v_i)$. For each i, identify the endpoint of P_i with the initial point of P_{i+1} to obtain an R–circle $\gamma\colon S \to X$ with $|S| = n(L\lambda + R)$. Then α factors through γ via the embedding that sends v_i to the initial point of $P_i \subset S$. So, viewing S_n^0 as a subset of S via this embedding, we have $\alpha(v_i) = \gamma(v_i)$ for each i. Let $x \in S$ and let v_i minimize $d(x, v_i)$. Then $d(x, v_i) \le \frac{1}{2}(L\lambda + R)$ and if $\bar{x}$ is the antipode of x and $\bar{i} = i + \lfloor \frac{n}{2} \rfloor$ then $d(\bar{x}, v_{\bar{i}}) \le L\lambda + R$, so

$$\begin{aligned}
d(\gamma(x), \gamma(\bar{x})) &\ge d(\gamma(v_i), \gamma(v_{\bar{i}})) - d(\gamma(x), \gamma(v_i)) - d(\gamma(\bar{x}), \gamma(v_{\bar{i}})) \\
&\ge d(\gamma(v_i), \gamma(v_{\bar{i}})) - d(x, v_i) - R - d(\bar{x}, v_{\bar{i}}) - R \\
&\ge d(\gamma(v_i), \gamma(v_{\bar{i}})) - \tfrac{1}{2}(L\lambda + R) - (L\lambda + R) - 2R \\
&= d(\alpha(v_i), \alpha(v_{\bar{i}})) - \tfrac{1}{2}(3L\lambda + 7R) \\
&\ge \frac{1}{L} d_{\lambda S_n^0}(v_i, v_{\bar{i}}) - \tfrac{1}{2}(3L\lambda + 7R) \\
&= \frac{\lambda}{L}\left\lfloor \tfrac{1}{2}n \right\rfloor - \tfrac{1}{2}(3L\lambda + 7R) \\
&\ge \frac{\lambda}{L}\left(\tfrac{1}{2}(n-1)\right) - \tfrac{1}{2}(3L\lambda + 7R) \\
&= \frac{1}{|S|}\left(\frac{\lambda(n-1)}{L} - 3L\lambda - 7R\right) \cdot \tfrac{1}{2}|S| \\
&= \frac{1}{n(L\lambda + R)}\left(\frac{\lambda(n-1)}{L} - 3L\lambda - 7R\right) \cdot \tfrac{1}{2}|S| \\
&= \left(\frac{\lambda n - \lambda}{L^2\lambda n + nLR} - \frac{3L\lambda}{L\lambda n + nR} - \frac{7R}{L\lambda n + nR}\right) \cdot \tfrac{1}{2}|S|.
\end{aligned}$$

So, given α as above, we can obtain a $1/K$–almost isometric R–circle in X of length $|S| = n(L\lambda + R)$, where

$$\frac{1}{K} = \left(\frac{\lambda n - \lambda}{L^2\lambda n + nLR} - \frac{3L\lambda}{L\lambda n + nR} - \frac{7R}{L\lambda n + nR}\right).$$

We need only show there exist α for which $1/K$ is arbitrarily close to 1 and $|S|$ is arbitrarily large. By hypothesis, there exist α for which $L = (n+1)/n$ and $\lambda > n$, for arbitrary $n \in \mathbb{N}$. But then, as $n \to \infty$, we have $|S| = n(L\lambda + R) \to \infty$ and

$$\frac{1}{K} = \left(\frac{\lambda n - \lambda}{L^2\lambda n + nLR} - \frac{3L\lambda}{L\lambda n + nR} - \frac{7R}{L\lambda n + nR}\right) \to 1. \qquad \square$$

Corollary 3.9 *Let X be a metric space. If X is strongly shortcut then every asymptotic cone of X is strongly shortcut.*

Proof This follows immediately from Theorem 3.8 and Corollary 3.7. □

Corollary 3.10 *The strong shortcut property is a rough approximability invariant of rough geodesic metric spaces. In particular, the strong shortcut property is a rough similarity invariant of rough geodesic metric spaces.*

Proof This follows immediately from Theorem 3.8 and Proposition 3.4. □

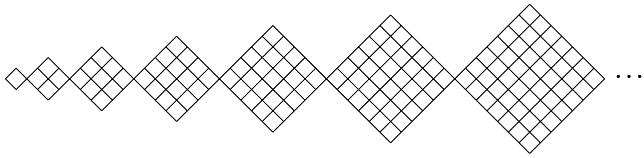

Figure 2: Continuing the pattern, one obtains an infinite graph that is strongly shortcut because it is the 1–skeleton of a finite-dimensional CAT(0) cube complex. Subdividing the *interior* edges of each $n \times n$ grid results in a quasi-isometric graph that is not strongly shortcut.

3.1 Instability under quasi-isometries

In light of Corollary 3.10, we should point out that the strong shortcut property is not a quasi-isometry invariant. The 1–skeleton of an $n \times n$ grid of squares is strongly shortcut but subdividing its *interior* edges causes its boundary cycle to become isometrically embedded. We can construct a strongly shortcut graph Γ that contains isometric copies of 1–skeletons of larger and larger $n \times n$ grids. See Figure 2. Subdividing the interior edges of each $n \times n$ grid of Γ does not change the quasi-isometry type but results in a graph that is not strongly shortcut because it contains arbitrarily long isometrically embedded cycles.

4 The circle tightening lemma

The circle tightening lemma describes how one may perform surgery on an almost isometric R–circle to obtain a quasi-isometrically embedded R–circle, assuming the various constants are chosen appropriately. A version of this lemma first appeared implicitly in the proof of a proposition in an earlier work [Hoda 2022, Proposition 3.5] where it applied only to graphs. Here we state and prove a generalization to (rough) geodesic metric spaces.

4.1 Tightening sequence for a Riemannian circle

Let S be a Riemannian circle. A *tightening sequence* for S is a sequence of intervals and Riemannian circles $(P_i)_i$, a sequence of Riemannian circles $(S_i)_i$ and sequences of maps

$$S = S_0 \hookleftarrow P_0 \to S_1 \hookleftarrow P_1 \to S_2 \hookleftarrow P_2 \to \cdots$$

such that, for each i, either

(1) (a) $P_i \hookrightarrow S_i$ and $P_i \to S_{i+1}$ are continuous paths of unit speed,
  (b) $P_i \to S_{i+1}$ is injective on the interior P_i° of P_i,
  (c) $|P_i| \geq \frac{1}{2}|S|$, and
  (d) $|S_{i+1}| < |S_i|$; or

(2) (a) $P_i^\circ = P_i = S_i = S_{i+1}$, and
  (b) $P_i \hookrightarrow S_i$ and $P_i \to S_{i+1}$ are identity maps.

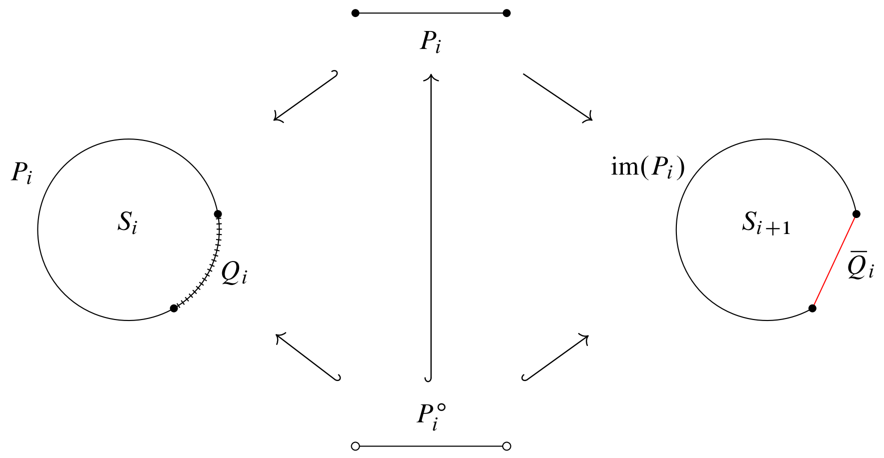

Figure 3: In a circle tightening sequence, the circle S_{i+1} is either equal to S_i or is obtained from S_i by replacing some geodesic segment Q_i of S_i with a shorter segment $\overline{Q}_i$, possibly of zero length.

See Figure 3. Then, for each i, the circle S_{i+1} is obtained from S_i by replacing $Q_i = S \setminus P_i^\circ$ with $\overline{Q}_i$, where either $Q_i = \overline{Q}_i = \varnothing$ or Q_i and $\overline{Q}_i$ are intervals with $|\overline{Q}_i| < |Q_i|$. So we also have a commutative diagram of 1–Lipschitz maps

$$\begin{tikzcd}
Q_0 \arrow[r, two heads] \arrow[d, hook] & \overline{Q}_0 \arrow[dr, hook] & Q_1 \arrow[r, two heads] \arrow[d, hook] & \overline{Q}_1 \arrow[dr, hook] & Q_2 \arrow[r, two heads] \arrow[d, hook] & \overline{Q}_2 \arrow[dr, hook] & \cdots \\
S = S_0 \arrow[rr, bend left, two heads, "\pi_0"] & P_0 \arrow[l, hook'] \arrow[r] & S_1 \arrow[rr, bend left, two heads, "\pi_1"] & P_1 \arrow[l, hook'] \arrow[r] & S_2 \arrow[rr, bend left, two heads, "\pi_2"] & P_2 \arrow[l, hook'] \arrow[r] & \cdots
\end{tikzcd}$$

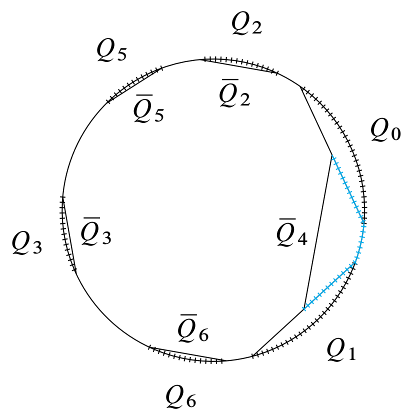

Figure 4: A circle tightening sequence that is disjoint up to 4 but not disjoint up to 5. The outer circle is the initial circle $S = S_0$. For $i \geq 0$, the circle S_{i+1} is obtained from S_i by replacing the geodesic segment $Q_i \subset S_i$ (indicated by perpendicular markings) with a shorter sequence $\overline{Q}_i$. The segment Q_4 (drawn in cyan) is the first replaced segment that cannot be viewed as a subspace of S since it is not contained in $P_{0,4}^\circ$, which can be viewed as $S \setminus \bigcup_{i=0}^{3} Q_i$.

where each $Q_i \twoheadrightarrow \overline{Q}_i$ is affine. We call the Q_i the *tightened segments* of the tightening sequence. We let $\pi^{(i)}$ denote the composition $\pi_{i-1} \circ \pi_{i-2} \circ \cdots \circ \pi_0$. A tightening sequence is *eventually constant* if $S_i = S_{i+1}$ for all large enough i.

Let $P^\circ_{0,j}$ be the limit of the diagram

$$\begin{tikzcd}
 & P_0^\circ \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & & P_1^\circ \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & \cdots & P_{j-1}^\circ \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & \\
S = S_0 & P_0 \arrow[l, hook'] \arrow[r] & S_1 & P_1 \arrow[l, hook'] \arrow[r] & \cdots & P_{j-1} \arrow[l, hook'] \arrow[r] & S_j
\end{tikzcd}$$

in the category of topological spaces and continuous maps. Concretely, we have $P^\circ_{0,0} = S_0$, $P^\circ_{0,1} = P^\circ_0$ and $P^\circ_{0,j} = P^\circ_{0,j-1} \cap P^\circ_{j-1}$ where the intersection is taken in S_{j-1}. Thus we have the commutative diagram

$$\begin{tikzcd}
 & P_{0,1}^\circ \arrow[d, equal] & & P_{0,2}^\circ \arrow[ll, hook'] \arrow[d, hook] & & P_{0,3}^\circ \arrow[ll, hook'] \arrow[d, hook] & \cdots \arrow[l, hook'] \\
 & P_0^\circ \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & & P_1^\circ \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & & P_2^\circ \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & \cdots \\
S = S_0 & P_0 \arrow[l, hook'] \arrow[r] & S_1 & P_1 \arrow[l, hook'] \arrow[r] & S_2 & P_2 \arrow[l, hook'] \arrow[r] & \cdots
\end{tikzcd}$$

We can think of $P^\circ_{0,j}$ as the original points of S that are not replaced until at least step j of the "construction" of the S_i, where the j^{th} step of the construction refers to the operation of replacing Q_j with $\overline{Q}_j$ in order to obtain S_{j+1} from S_j.

We say that a tightening sequence is *disjoint up to* j if $Q_i \subset P^\circ_{0,i}$ for all $i < j$, where $P^\circ_{0,i}$ is viewed as a subspace of S_i via the embedding $P^\circ_{0,i} \hookrightarrow S_i$. See Figure 4. We say that a tightening sequence is *completely disjoint* if it is disjoint up to j for every j.

If a tightening sequence is disjoint up to j, for $i < j$, we have $Q_i \sqcup P^\circ_{0,i+1} = P^\circ_{0,i} \hookrightarrow S$. So, for $i < j$, we may think of the Q_i as disjoint subspaces of S with $S \setminus \bigcup_{i=0}^{j-1} Q_i = P^\circ_{0,j}$ in S. Since S_j is obtained from S by replacing Q_i with $\overline{Q}_i$, for each $i < j$, we see then that the $\overline{Q}_i$, with $i < j$, embed disjointly in S_j with $P^\circ_{0,j} = S_j \setminus \bigcup_{i=0}^{j-1} \overline{Q}_i$ in S_j.

If a tightening sequence is completely disjoint then the Q_i all embed disjointly in S and the complement of their union in S is $P^\circ_{0,\infty} = \bigcap_{j=1}^{\infty} P^\circ_{0,j}$.

Lemma 4.1 *Consider a tightening sequence for a Riemannian circle S with the same notation as above. If the tightening sequence is completely disjoint and the sum $\sum_{i=0}^{\infty} |Q_i|$ of the tightened segment lengths is strictly less than $|S|$ then*

$$\delta \colon S \times S \to \mathbb{R}_{\geq 0}, \quad (x, y) \mapsto \lim_{i \to \infty} d_{S_i}(\pi^{(i)}(x), \pi^{(i)}(y)),$$

*defines a pseudometric on S such that the induced metric quotient S_∞ of (S, δ), called the **limit Riemannian circle** of the tightening sequence, is a Riemannian circle of length $\lim_{i \to \infty} |S_i|$.*

Proof The $\pi^{(i)}$ are isomorphisms on fundamental group so, for each i, we have a commuting diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{\pi}_i"] \arrow[d] & \mathbb{R} \arrow[d] \\ S_i \arrow[r, "\pi_i"] & S_{i+1} \end{tikzcd}$$

where $\mathbb{R} \to S_i$ and $\mathbb{R} \to S_{i+1}$ are the quotient maps from $(\mathbb{R}, +)$ by the subgroups $|S_i|\mathbb{Z}$ and $|S_{i+1}|\mathbb{Z}$, respectively. Then, since π_i is 1–Lipschitz, so is $\tilde{\pi}_i$. Without loss of generality, the map $\tilde{\pi}_i$ sends 0 to 0 and preserves order, in the sense that $s \le r$ implies $\tilde{\pi}_i(s) \le \tilde{\pi}_i(r)$. Let $\tilde{\pi}^{(i)}\colon \mathbb{R} \to \mathbb{R}$ be the composition $\tilde{\pi}_0 \circ \tilde{\pi}_1 \circ \cdots \circ \tilde{\pi}_{i-1}$ so that the diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{\pi}^{(i)}"] \arrow[d] & \mathbb{R} \arrow[d] \\ S \arrow[r, "\pi^{(i)}"] & S_i \end{tikzcd}$$

commutes and satisfies the same properties as the previous diagram. Then for $r \in \mathbb{R}$, the sequence $(\pi^{(i)}(r))_i$ is either nonnegative and nonincreasing or nonpositive and nondecreasing. In either case the limit exists so we can define a limit function

$$\pi^{(\infty)}\colon \mathbb{R} \to \mathbb{R}, \quad r \mapsto \lim_{i\to\infty} \pi^{(i)}(r),$$

which is also 1–Lipschitz, sends 0 to 0 and preserves order.

By assumption $\sum_{i=1}^{\infty} |Q_i| < |S|$ so $\pi^{(i)}(|S|) = |S_i| \ge |S| - \sum_{i=1}^{\infty} |Q_i| > 0$ and so

$$\pi^{(\infty)}(|S|) = \lim_{i\to\infty} |S_i| > 0.$$

For $r \in \mathbb{R}$, we have $\pi^{(i)}(r + |S|) = \pi^{(i)}(r) + |S_i|$ so

$$\pi^{(\infty)}(r + |S|) = \pi^{(\infty)}(r) + \lim_{i\to\infty} |S_i|,$$

which implies that if $\mathbb{R} \to \overline{S}$ is the quotient map of $(\mathbb{R}, +)$ with kernel $(\lim_{i\to\infty} |S_i|)\mathbb{Z}$ then the map

$$\pi^{(\infty)}\colon S \to S_\infty, \quad r + |S|\mathbb{Z} \mapsto \pi^{(\infty)}(r) + \bigl(\lim_{i\to\infty} |S_i|\bigr)\mathbb{Z},$$

is well defined and makes the diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{\pi}^{(\infty)}"] \arrow[d] & \mathbb{R} \arrow[d] \\ S \arrow[r, "\pi^{(\infty)}"] & \overline{S} \end{tikzcd}$$

commute.

Then $|\overline{S}| = \lim_{i\to\infty} |S_i|$ and, for $r + |S|\mathbb{Z}$ and $s + |S|\mathbb{Z}$ in S,

$$\begin{aligned} d_{\overline{S}}\bigl(\pi^{(\infty)}(r + |S|\mathbb{Z}), \pi^{(\infty)}(s + |S|\mathbb{Z})\bigr) &= d_{\overline{S}}\bigl(\pi^{(\infty)}(r) + |\overline{S}|\mathbb{Z}, \pi^{(\infty)}(s) + |\overline{S}|\mathbb{Z}\bigr) \\ &= \min_{k\in\mathbb{Z}} \bigl|\pi^{(\infty)}(r) - \pi^{(\infty)}(s) + |\overline{S}|k\bigr| \end{aligned}$$

$$
\begin{aligned}
&= \min_{k\in I} \left|\pi^{(\infty)}(r) - \pi^{(\infty)}(s) + |\overline{S}|k\right| \\
&= \min_{k\in I} \lim_{i\to\infty} \left|\pi^{(i)}(r) - \pi^{(i)}(s) + |\overline{S}_i|k\right| \\
&= \lim_{i\to\infty} \min_{k\in I} \left|\pi^{(i)}(r) - \pi^{(i)}(s) + |\overline{S}_i|k\right| \\
&= \lim_{i\to\infty} \min_{k\in \mathbb{Z}} \left|\pi^{(i)}(r) - \pi^{(i)}(s) + |\overline{S}_i|k\right| \\
&= \lim_{i\to\infty} d_{S_i}\bigl(\pi^{(i)}(r) + |S_i|\mathbb{Z}, \pi^{(i)}(s) + |S_i|\mathbb{Z}\bigr) \\
&= \lim_{i\to\infty} d_{S_i}\bigl(\pi^{(i)}(r + |S|\mathbb{Z}), \pi^{(i)}(s + |S|\mathbb{Z})\bigr),
\end{aligned}
$$

where $I = \{k \in \mathbb{Z} \mid |k| \le \lceil |r-s|/|\overline{S}|\rceil\}$. So the pseudometric on S pulled back from $\pi^{(\infty)}$ is δ. Then, since $\pi^{(\infty)}$ is surjective, this implies that $\overline{S}$ is S_∞, the induced metric quotient of (S,δ) and π^∞ is the quotient map. □

Remark 4.2 If a completely disjoint tightening sequence of a Riemannian circle is eventually constant then, for large enough i, the limit Riemannian circle S_∞ is isometric to the i^{th} Riemannian circle of the sequence S_i.

4.2 Tightening sequence for an R–circle

Let $R \ge 0$ and let X be an R–rough geodesic metric space. Let $\alpha\colon S \to X$ be an R–circle. A *tightening sequence* for α is a sequence of intervals and Riemannian circles $(P_i)_i$, a sequence of Riemannian circles $(S_i)_i$ and sequences of maps as in the commutative diagram

$$
\begin{tikzcd}
S \arrow[r, equal] \arrow[drrr, bend right, two heads, "\alpha"'] & S_0 \arrow[drr, bend right, two heads, "\alpha_0"] & P_0 \arrow[l, hook'] \arrow[r] & S_1 \arrow[d, "\alpha_1"'] & P_1 \arrow[l, hook'] \arrow[r] & S_2 \arrow[dll, bend left, "\alpha_2"'] & P_2 \arrow[l, hook'] \arrow[r] & \cdots \arrow[dllll, bend left, "\alpha_3"'] \\
& & & X & & & & \cdots
\end{tikzcd}
$$

such that each α_i is an R–circle and the sequence of maps

$$S = S_0 \hookleftarrow P_0 \to S_1 \hookleftarrow P_1 \to S_2 \hookleftarrow P_2 \to \cdots$$

is a tightening sequence for S. Then, by the discussion of Section 4.1, we have a diagram

$$
\begin{tikzcd}
& Q_0 \arrow[r, two heads] \arrow[d, hook] & \overline{Q}_0 \arrow[dr, bend left] & Q_1 \arrow[r, two heads] \arrow[d, hook] & \overline{Q}_1 \arrow[dr, bend left] & Q_2 \arrow[r, two heads] \arrow[d, hook] & \overline{Q}_2 \arrow[dr, bend left] & \cdots \\
S \arrow[r, equal] \arrow[drrr, bend right, two heads, "\alpha"'] & S_0 \arrow[rr, bend left, two heads, "\pi_0"] \arrow[drr, bend right, two heads, "\alpha_0"] & P_0 \arrow[l, hook'] \arrow[r, two heads] & S_1 \arrow[rr, bend left, two heads, "\pi_1"] \arrow[d, "\alpha_1"'] & P_1 \arrow[l, hook'] \arrow[r, two heads] & S_2 \arrow[rr, bend left, two heads, "\pi_2"] \arrow[dll, bend left, "\alpha_2"'] & P_2 \arrow[l, hook'] \arrow[r, two heads] & \cdots \arrow[dllll, bend left, "\alpha_3"'] \\
& & & X & & & & \cdots
\end{tikzcd}
$$

where the bounded planar regions are commuting triangles and squares.

Lemma 4.3 *Consider a tightening sequence for an R–circle $\alpha\colon S\to X$ in an R–rough geodesic metric space X, with the same notation as above. If the tightening sequence is completely disjoint and the sum $\sum_{i=0}^{\infty}|Q_i|$ of the tightened segment lengths is strictly less than $|S|$ then*

$$\alpha_\infty\colon S_\infty\to X,\quad x\mapsto \lim_{i\to\infty}\alpha_i\circ\pi^{(i)}\circ\sigma(x),$$

*defines an R–circle, called a **limit R–circle** of the tightening sequence, where S_∞ is the limit Riemannian circle of the tightening sequence and $\sigma\colon S_\infty\to S$ is a section of the quotient map $S\to S_\infty$.*

Proof Since the tightening sequence is completely disjoint, we may think of the Q_i as a collection of disjoint segments in S. For $x\in S$ either $x\notin\bigcup_{i=0}^{\infty}Q_i$ and $(\alpha_i\circ\pi^{(i)}(x))_{i=0}^{\infty}$ is a constant sequence or $x\in Q_j$ for some j and the tail sequence $(\alpha_i\circ\pi^{(i)}(x))_{i=j+1}^{\infty}$ is constant. In either case, the limit $\lim_{i\to\infty}\alpha_i\circ\pi^{(i)}(x)$ exists so we have a function $\alpha'_\infty\colon S\to X$ given by $\alpha'_\infty(x)=\lim_{i\to\infty}\alpha_i\circ\pi^{(i)}(x)$.

Let S_∞ be the limit Riemannian circle given by Lemma 4.1. So S_∞ is the induced metric quotient of (S,δ) where δ is the pseudometric give by $\delta(x,y)=\lim_{i\to\infty}d_{S_i}(\pi^{(i)}(x),\pi^{(i)}(y))$. Since each α_i is an R–circle, for $x,y\in S$,

$$d_X\bigl(\alpha_i\circ\pi^{(i)}(x),\alpha_i\circ\pi^{(i)}(y)\bigr)\le d_{S_i}(\pi^{(i)}(x),\pi^{(i)}(y))+R$$

for all i, and thus

$$d_X(\alpha'_\infty(x),\alpha'_\infty(y))\le\delta(x,y)+R$$

by taking limits as $i\to\infty$.

Then, for $x,y\in S_\infty$,

$$d_X(\alpha_\infty(x),\alpha_\infty(y))=d_X\bigl(\alpha'_\infty(\sigma(x)),\alpha'_\infty(\sigma(y))\bigr)\le\delta(\sigma(x),\sigma(y))+R=d_{S_\infty}(x,y)+R. \qquad\square$$

Remark 4.4 If a completely disjoint tightening sequence for an R–circle $\alpha\colon S\to X$ is eventually constant then, for large enough i, the limit R–circle $\alpha_\infty\colon S_\infty\to X$ is isometric over X to the i^{th} R–circle of the sequence α_i. This means that there is an isometry $S_\infty\to S$ such that diagram

$$\begin{tikzcd} S_\infty \arrow[r] \arrow[dr, "\alpha_\infty"'] & S_i \arrow[d, "\alpha_i"] \\ & X \end{tikzcd}$$

commutes.

We are ready now to state the circle tightening lemma.

Lemma 4.5 (circle tightening lemma) *Let $N>1$, let $L>1$, let $K>1$ be small enough (depending on N and L), let $R\ge 0$, let $M>0$ be large enough (depending on N, L, K and R) and let $C\ge 4R$.*

Let $\alpha\colon S\to X$ be an R–circle in an R–rough geodesic metric space. If α is $1/K$–almost isometric and $|S|>M$ then α has a completely disjoint tightening sequence such that the total length $\sum_{i=0}^{\infty}|Q_i|$ of the tightened segments is at most $|S|/N$ and the limit R–circle α_∞ is an (L,C)–quasi-isometric embedding. If, additionally, $C>0$ then such a tightening sequence exists that is eventually constant.

Lemma 4.5 is a consequence of Claims 4.7, 4.8, 4.11 and 4.14 and the strict inequalities of Claim 4.12 below, but to understand these claims we need to first define greedy tightening sequences and prove some properties about them.

4.3 Greedy tightening sequences

In order to prove Lemma 4.5 we will need to describe a tightening sequence that is constructed inductively by greedily choosing segments to tighten. Let $\alpha\colon S \to X$ be an R–circle in an R–rough geodesic metric space X, with $R \geq 0$. Let $C \geq 4R$ and let $L > 1$. We will inductively define a tightening sequence for α with the same notation as in the previous sections. Suppose we have $\alpha_i\colon S_i \to X$. If α_i is an (L, C)–quasi-isometric embedding then we extend the sequence as follows.

(1) We set $P_i^\circ = P_i = S_i = S_{i+1}$ and $Q_i = \overline{Q}_i = \varnothing$.

(2) We let $P_i \hookrightarrow S_i$ and $P_i \to S_{i+1}$ be identity maps.

(3) We let $\alpha_{i+1} = \alpha_i$.

Otherwise, the set

$$J_i = \left\{(p,q) \in S_i \times S_i \;\middle|\; d_X(\alpha_i(p), \alpha_i(q)) < \frac{1}{L} d_{S_i}(p,q) - C\right\}$$

is nonempty and $d_{S_i}(p,q) > LC \geq 0$ for any $(p,q) \in J_i$ and so $s_i = \sup\{d_{S_i}(p,q) \mid (p,q) \in J_i\} > 0$. By compactness of S, there is a sequence $(p_i^{(n)}, q_i^{(n)})_n$ in J_i that converges to some $(p_i, q_i) \in S \times S$ with $d_{S_i}(p_i, q_i) = s_i$ as $n \to \infty$. Then

$$\begin{aligned} d_X(\alpha_i(p_i), \alpha(q_i)) &\leq d_X\big(\alpha_i(p_i), \alpha(p_i^{(n)})\big) + d_X\big(\alpha_i(p_i^{(n)}), \alpha(q_i^{(n)})\big) + d_X\big(\alpha_i(q_i^{(n)}), \alpha(q_i)\big) \\ &< d_{S_i}(p_i, p_i^{(n)}) + R + \frac{1}{L} d_{S_i}(p_i^{(n)}, q_i^{(n)}) - C + d_{S_i}(q_i^{(n)}, q_i) + R \\ &\to \frac{1}{L} d_{S_i}(p_i, q_i) - C + 2R \end{aligned}$$

as $n \to \infty$. So

$$(*) \qquad d_X(\alpha_i(p_i), \alpha(q_i)) \leq \frac{1}{L} d_{S_i}(p_i, q_i) - C + 2R,$$

$$(\dagger) \qquad d_{S_i}(p_i, q_i) > 0 \quad \text{and} \quad d_{S_i}(p_i, q_i) \geq L(C - 2R)$$

hold. Let Q_i be a geodesic segment of S_i between p_i and q_i. In the case where p_i and q_i are antipodal in S_i, there are two geodesic segments between p_i and q_i; in this case we let Q_i be the geodesic segment whose intersection with $P_{0,i}^\circ$ has greatest total length. Let P_i° be the complement of Q_i and let P_i be the closure of P_i°. Let $\gamma_i'\colon \overline{Q}_i' \to X$ be an R–rough geodesic from $\alpha_i(p_i)$ to $\alpha_i(q_i)$. For $x \in X$, let $c_x\colon [0, R] \to X$ denote the constant path of length R at x. Let $\gamma_i\colon \overline{Q}_i \to X$ be the concatenation $c_{\alpha_i(p_i)}\gamma_i' c_{\alpha_i(q_i)}$. We have

$$0 \leq |\overline{Q}_i| - 2R = d_X(\alpha_i(p_i), \alpha_i(q_i)) \leq \frac{1}{L}|Q_i| - C + 2R$$

and so, since $C \geq 4R$,

$$(\ddagger) \qquad |\overline{Q}_i| \leq \frac{1}{L}|Q_i|.$$

We obtain $\alpha_{i+1}\colon S_{i+1} \to X$ from $\alpha_i|_{P_i}$ and γ_i by identifying the corresponding endpoints of $\overline{Q}_i$ with p_i and q_i in P_i. Then, by consideration of Remark 2.1, the map α_{i+1} is an R–circle.

Remark 4.6 The inequalities

$$d_X(\alpha_{i+1}(p_i), \alpha_{i+1}(x)) \leq d_{S_{i+1}}(p_i, x), \quad d_X(\alpha_{i+1}(q_i), \alpha_{i+1}(x)) \leq d_{S_{i+1}}(q_i, x)$$

hold for any $x \in \overline{Q}_i$.

This completes the description of our inductive construction. Any tightening sequence for α obtained in this way is called an (L,C)*–greedy tightening sequence* for α. The importance of this construction for us is evident from the following claim.

Claim 4.7 *Let $L > 1$, let $K > 1$, let $C \geq 4R$ and consider an (L,C)–greedy tightening sequence for a $1/K$–almost isometric R–circle $\alpha\colon S \to X$ in an R–rough geodesic metric space X. If the tightening sequence is completely disjoint and the sum $\sum_{i=0}^{\infty}|Q_i|$ of the tightened segment lengths is strictly less than $|S|$ then any limiting R–circle $\alpha_\infty\colon S_\infty \to X$ is an (L,C)–quasi-isometric embedding.*

Proof Let S_∞ be the limit Riemannian circle given by Lemma 4.1. So S_∞ is the induced metric quotient of (S,δ) where δ is the pseudometric given by $\delta(x,y) = \lim_{i\to\infty} d_{S_i}(\pi^{(i)}(x), \pi^{(i)}(y))$. Let $\alpha_\infty\colon S_\infty \to X$ be a limit R–circle as in Lemma 4.3. So α_∞ is an R–circle defined by $\alpha_\infty(x) = \lim_{i\to\infty} \alpha_i \circ \pi^{(i)} \circ \sigma(x)$, where $\sigma\colon S_\infty \to S$ is a section of the quotient map $S \to S_\infty$.

If α_∞ is not an (L,C)–quasi-isometric embedding then, since $R \leq C$,

$$d_X(\alpha_\infty(x), \alpha_\infty(y)) < \frac{1}{L} d_{S_\infty}(x,y) - C$$

for some $x, y \in S_\infty$, which then must be distinct. But

$$d_{S_\infty}(x,y) = \lim_{i\to\infty} d_{S_i}\big(\pi^{(i)}(\sigma(x)), \pi^{(i)}(\sigma(y))\big),$$
$$d_X(\alpha_\infty(x), \alpha_\infty(y)) = \lim_{i\to\infty} d_X\big(\alpha_i \circ \pi^{(i)}(\sigma(x)), \alpha_i \circ \pi^{(i)}(\sigma(y))\big),$$

where, by complete disjointness, $\big(\pi^{(i)}(\sigma(x))\big)_i$ and $\big(\pi^{(i)}(\sigma(y))\big)_i$ are eventually constant. So, for all large enough j,

$$d_X\big(\alpha_j \circ \pi^{(j)}(\sigma(x)), \alpha_j \circ \pi^{(j)}(\sigma(y))\big) = \lim_{i\to\infty} d_X\big(\alpha_i \circ \pi^{(i)}(\sigma(x)), \alpha_i \circ \pi^{(i)}(\sigma(y))\big)$$

but also, for all large enough j,

$$\lim_{i\to\infty} d_X\big(\alpha_i \circ \pi^{(i)}(\sigma(x)), \alpha_i \circ \pi^{(i)}(\sigma(y))\big) < \frac{1}{L} d_{S_j}\big(\pi^{(j)}(\sigma(x)), \pi^{(j)}(\sigma(y))\big) - C$$

so, for all large enough j,

$$d_X\big(\alpha_j(\pi^{(j)} \circ \sigma(x)), \alpha_j(\pi^{(j)} \circ \sigma(y))\big) < \frac{1}{L} d_{S_j}\big(\pi^{(j)} \circ \sigma(x), \pi^{(j)} \circ \sigma(y)\big) - C$$

which implies that $\left(\pi^{(j)} \circ \sigma(x), \pi^{(j)} \circ \sigma(y)\right) \in J_j$, for all large enough j. But then, for all large enough j,

$$d_{S_j}\left(\pi^{(j)} \circ \sigma(x), \pi^{(j)} \circ \sigma(y)\right) \leq s_j = |Q_j|$$

with $\lim_{j\to\infty} |Q_j| = 0$ so

$$d_{S_\infty}(x, y) = \lim_{j\to\infty} d_{S_j}\left(\pi^{(j)} \circ \sigma(x), \pi^{(j)} \circ \sigma(y)\right) = 0$$

which is a contradiction. □

4.4 Eventual constantness and greedy tightening sequences

Consider a greedy tightening sequence with notation as in Section 4.3. Note that if $S_i = S_{i+1}$ for some i then $S_i = S_{i+1} = S_{i+2} = \cdots$, so the tightening sequence is eventually constant. Moreover, for any i for which $S_i \neq S_{i+1}$, we have

$$\begin{aligned} |S_i| - |S_{i+1}| = |Q_i| - |\bar{Q}_i| \geq |Q_i| - \frac{1}{L}|Q_i| &= \left(1 - \frac{1}{L}\right)|Q_i| \\ &\geq \left(1 - \frac{1}{L}\right)L(C - 2R) = (L-1)(C-2R) \end{aligned}$$

by (†) and (‡). If $R > 0$ then, since $C \geq 4R$, we have $C > 2R > 0$. If $R = 0$ then $C > 2R$ is equivalent to $C > 0$. Thus, if $C > 0$ then $|S_i| - |S_{i+1}| \geq (L-1)(C-2R) > 0$. This implies the following claim.

Claim 4.8 *If $R \geq 0$, $L > 1$ and $C > 0$ then any (L, C)–greedy tightening sequence for an R–circle in an R–rough geodesic space is eventually constant.*

4.5 Disjointness and greedy tightening sequences

Consider a greedy tightening sequence with notation as in Section 4.3. Assume that α is $1/K$–almost isometric and that the tightening sequence is disjoint up to j. Recall that, by the discussion in Section 4.1, we may think of the Q_i, with $i < j$, as disjoint subspaces of S.

If $i < j$ and $Q_i \neq \varnothing$ then, by Lemma 3.1 and (∗),

$$|Q_i| - \frac{K-1}{K} \cdot \frac{|S|}{2} - 2R \leq d_X(\alpha(p_i), \alpha(q_i)) \leq \frac{1}{L}|Q_i| - C + 2R,$$

but $C \geq 4R$ so

$$|Q_i| - \frac{K-1}{K} \cdot \frac{|S|}{2} \leq \frac{1}{L}|Q_i|$$

for all $i < j$. Hence, we have established the following claim.

Claim 4.9 *If an (L, C)–greedy tightening sequence for a $1/K$–almost isometric R–circle is disjoint up to j then*

$$|Q_i| \leq \left(\frac{K-1}{K} \cdot \frac{L}{L-1}\right)\frac{|S|}{2}$$

for any $i < j$, where Q_i is the i^{th} replaced segment of the tightening sequence.

By this claim, we can find a pair of points p,q in the closure of $S \setminus \bigl(\bigcup_{i=1}^{j-1} Q_i\bigr)$ at distance

$$d_S(p,q) \geq \frac{|S|}{2} - \frac{K-1}{K} \cdot \frac{L}{L-1} \cdot \frac{|S|}{4}.$$

Let A_1 and A_2 be the two segments of S between p and q. If $I_1 = \{i < j \mid Q_i \subseteq A_1\}$ then, since α_j is an R–circle,

$$\begin{aligned} d_X(\alpha(p),\alpha(q)) &\leq d_{S_j}(p,q) + R \\ &\leq |A_1| - \sum_{i \in I_1} |Q_i| + \sum_{i \in I_1} |\bar{Q}_i| + R \\ &\leq |A_1| - \sum_{i \in I_1} |Q_i| + \frac{1}{L} \sum_{i \in I_1} |Q_i| + R \\ &= |A_1| - \frac{L-1}{L} \sum_{i \in I_1} |Q_i| + R \end{aligned}$$

where the last inequality follows by (‡). Corresponding relations for A_2 and $I_2 = \{i < j \mid Q_i \subseteq A_2\}$ also hold, and so, by Lemma 3.1,

$$\begin{aligned} |S| - \frac{L-1}{L} \sum_{i<j} |Q_i| + 2R \geq 2d_X(\alpha(p),\alpha(q)) &\geq 2d_S(p,q) - \frac{K-1}{K} \cdot |S| - 4R \\ &\geq |S| - \left(\frac{K-1}{K} \cdot \frac{L}{L-1}\right) \frac{|S|}{2} - \frac{K-1}{K} \cdot |S| - 4R, \end{aligned}$$

which establishes the following claim.

Claim 4.10 *If an (L,C)–greedy tightening sequence for a $1/K$–almost isometric R–circle is disjoint up to j then*

$$\sum_{i<j} |Q_i| \leq \left(\frac{K-1}{K} \cdot \frac{L(3L-2)}{2(L-1)^2}\right)|S| + \frac{6LR}{L-1}$$

where Q_i is the i^{th} replaced segment of the tightening sequence.

Claim 4.10 implies that

$$\sum_{i<j} |Q_i| \leq \left(\frac{K-1}{K} \cdot \frac{L(3L-2)}{2(L-1)^2} + \frac{6LR}{|S|(L-1)}\right)|S|,$$

so if

$$\frac{K-1}{K} \cdot \frac{L(3L-2)}{2(L-1)^2} < \frac{1}{N}$$

then, if $|S| > M$ for some M depending only on K, L, R and N, then

$$\frac{K-1}{K} \cdot \frac{L(3L-2)}{2(L-1)^2} + \frac{6LR}{|S|(L-1)} < \frac{1}{N}$$

and so $\sum_{i<j} |Q_i| < |S|/N$. Since

$$\frac{K-1}{K} \cdot \frac{L(3L-2)}{2(L-1)^2} < \frac{1}{N}$$

is equivalent to

$$K < \frac{NL(3L-2)}{(3N-2)L^2-(2N-4)L-2},$$

we have established the following claim.

Claim 4.11 *Let $N > 1$, let $L > 1$, let $R \geq 0$ and let $K > 1$ satisfy*

$$K < \frac{NL(3L-2)}{(3N-2)L^2-(2N-4)L-2}.$$

Then there exists an $M > 0$ such that if an (L, C)–greedy tightening sequence for a $1/K$–almost isometric R–circle $\alpha\colon S \to X$ of length $|S| > M$ is disjoint up to j then

$$\sum_{i<j} |Q_i| < \frac{|S|}{N}$$

where the Q_i are the tightened segments of the tightening sequence.

The following claim about rational functions has a short and elementary proof. We will make use of it below.

Claim 4.12 *The inequalities*

$$1 < \frac{L(9L^2-3L-4)}{7L^3+3L^2-10L+2}, \qquad \frac{L(9L^2-3L-4)}{7L^3+3L^2-10L+2} \leq \frac{L(5L-4)}{3L^2-2},$$
$$1 < \frac{NL(3L-2)}{(3N-2)L^2-(2N-4)L-2}, \quad \frac{L(9L^2-3L-4)}{7L^3+3L^2-10L+2} \leq \frac{L(7L-6)}{5L^2-2L-2}$$

hold for any $L > 1$ and $N > 1$.

The next claim is essential in proving disjointness of greedy tightening sequences.

Claim 4.13 *Let $K > 1$, $L > 1$ and $R \geq 0$. There exists an $M > 0$ such that if X is an R–rough geodesic metric space, $\alpha\colon S \to X$ is a $1/K$–almost isometric R–circle with $|S| > M$ and*

$$K < \frac{L(9L^2-3L-4)}{7L^3+3L^2-10L+2},$$

and $C \geq 4R$ then any (L, C)–greedy tightening sequence for α that is disjoint up to j satisfies the following statement. With notation as above, if $(p, q) \in J_j$ and Q is a geodesic segment from p to q in S_j, then $Q \subset P^\circ_{0,j}$, where we view $P^\circ_{0,j}$ as a subspace of S_j via the embedding $P^\circ_{0,j} \hookrightarrow S_j$.

Proof First we will show that Q is not contained in $\overline{Q}_i$ for any $i < j$. Recall that $\overline{Q}_i$ is the concatenation $A\overline{Q}'_i B$ where $\alpha_j|_{\overline{Q}'_i}\colon \overline{Q}'_i \to X$ is an R–rough geodesic and α_j is constant on A and B, each of which is isometric to $[0, R]$. By (†), we have $|Q| \geq (C-2R)L \geq (4R-2R)L \geq 2R$ and $|Q| > 0$ so we cannot have $Q \subseteq A$ or $Q \subseteq B$. We also cannot have $\overline{Q}'_i \subseteq Q \subseteq \overline{Q}_i$ since then

$$|\overline{Q}_i| - 2R = d_X(\alpha_i(p_i), \alpha_i(q_i)) = d_X(\alpha_j(p), \alpha_j(q)) < \frac{1}{L}|Q| - C \leq \frac{1}{L}|\overline{Q}_i| - C < |\overline{Q}_i| - C,$$

which contradicts $C \geq 4R$. So, if $Q \subseteq \overline{Q}_i$ then some endpoint of Q is contained in $\overline{Q}'_i$. But this implies that $\alpha_j|_Q \colon Q \to X$ is a $2R$–rough geodesic and so, by $(*)$ and (†),

$$|Q| - 2R \leq d_X(\alpha_j(p), \alpha_j(q)) < \frac{1}{L}|Q| - C < |Q| - C,$$

which, again, contradicts $C \geq 4R$. Thus we see that Q is not contained in $\overline{Q}_i$ for an $i < j$. Hence Q intersects $P^\circ_{0,j}$ nontrivially in S_j.

See Figure 5. We will define a segment $\bar{A}_p \subset S_j$ containing p and a corresponding segment $A_p \subset S$. (Note that $\bar{A}_p$ does not denote the closure of A_p here.) If p is contained in the interior of $\overline{Q}_i$ for some $i < j$ then let $\bar{A}_p = \overline{Q}_i$ and let $A_p = Q_i$. Otherwise, let $A_p = \bar{A}_p = \{p\}$. Define A_q and $\bar{A}_q$ similarly for q. A priori, it is possible that $\bar{A}_p = \bar{A}_q$. Let Q^- be obtained from Q by subtracting the interiors of $\bar{A}_p$ and $\bar{A}_q$ and let $Q^+ \to S_j$ extend $Q \hookrightarrow S_j$ so as to include a full copy of $\bar{A}_p$ and a full copy of $\bar{A}_q$. Let $Q_0^- \subset S$ be obtained from $Q^- \subset S_j$ by replacing any $\overline{Q}_i \subset Q^-$ with $Q_i \subset S$, for $i < j$. Let $Q_0^+ \to S$ be obtained from $Q^+ \to S_j$ by replacing any $\overline{Q}_i \hookrightarrow S_j$, where $i < j$, with $Q_i \hookrightarrow S$.

Let p^+ and q^+ be the images of the endpoints of Q_0^+ in S, with p^+ the endpoint corresponding to p and q^+ the endpoint corresponding to q. Let p^- and q^- be the endpoints of Q_0^- in S, with p^- the endpoint corresponding to p and q^- the endpoint corresponding to q. Then we have

$$\begin{aligned}
d_X(\alpha(p^+), \alpha(q^+)) &\leq d_X(\alpha(p^+), \alpha_j(p)) + d_X(\alpha_j(p), \alpha_j(q)) + d_X(\alpha_j(q), \alpha(q^+)) \\
&\leq d_{\bar{A}_p}(p^+, p) + d_X(\alpha_j(p), \alpha_j(q)) + d_{\bar{A}_q}(q, q^+) \\
&< d_{\bar{A}_p}(p^+, p) + \frac{1}{L}|Q| - C + d_{\bar{A}_q}(q, q^+) \\
&= d_{\bar{A}_p}(p^+, p) + \frac{1}{L}\big(d_{\bar{A}_p}(p, p^-) + |Q^-| + d_{\bar{A}_q}(q^-, q)\big) + d_{\bar{A}_q}(q, q^+) - C \\
&\leq d_{\bar{A}_p}(p^+, p) + d_{\bar{A}_p}(p, p^-) + \frac{1}{L}|Q^-| + d_{\bar{A}_q}(q^-, q) + d_{\bar{A}_q}(q, q^+) - C \\
&= |\bar{A}_p| + \frac{1}{L}|Q^-| + |\bar{A}_q| - C \\
&\leq |\bar{A}_p| + \frac{1}{L}|Q_0^-| + |\bar{A}_q| - C \\
&\leq \frac{1}{L}|A_p| + \frac{1}{L}|Q_0^-| + \frac{1}{L}|A_q| - C = \frac{1}{L}|Q_0^+| - C
\end{aligned}$$

where the second inequality follows from Remark 4.6 and the last inequality follows from (‡). By assumption, Q nontrivially intersects at least one $\overline{Q}_i$, with $i < j$. Let m be minimal such that Q nontrivially intersects $\overline{Q}_m$. Then, since Q intersects $P^\circ_{0,j}$ nontrivially, the image of $Q_0^+ \to S_m$ must strictly contain Q_m.

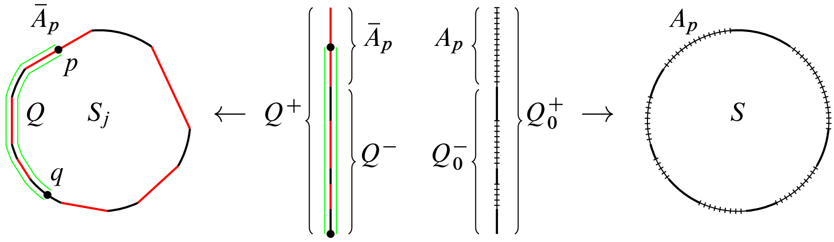

Figure 5: In the proof of Claim 4.13, from a geodesic segment Q of S_j (indicated by a green outline) we obtain Q^{-1} by removing the interiors of any $\overline{Q}_i$ that are partially contained in Q. We obtain $Q^+ \to S_j$ by extending the inclusion $Q \hookrightarrow S_j$ to include full copies of any $\overline{Q}_i$ that are partially contained in Q. From $Q^- \hookrightarrow S_j$ and $Q^+ \to S_j$ we obtain $Q_0^- \hookrightarrow S$ and $Q_0^+ \to S$ in $S = S_0$ by replacing any $\overline{Q}_i \hookrightarrow S_j$ with $Q_i \hookrightarrow S$. The $\overline{Q}_i$ with $i < j$ are drawn in red in S_j. The Q_i with $i < j$ are drawn with perpendicular markings in S.

So, if $Q_0^+ \to S_m$ was the inclusion of a geodesic segment, we would have $(p^+, q^+) \in J_m$, which would contradict $d_{S_m}(p_m, q_m) = s_m$. Thus $|Q_0^+| > \frac{1}{2}|S_m|$. But then

$$|Q_0^+| > \frac{|S_m|}{2} \geq \frac{|S|}{2} - \frac{1}{2}\sum_{i<m}|Q_i| \geq \frac{|S|}{2} - \left(\frac{K-1}{K} \cdot \frac{L(3L-2)}{2(L-1)^2}\right)\frac{|S|}{2} - \frac{3LR}{L-1}$$
$$= \left(1 - \frac{K-1}{K} \cdot \frac{L(3L-2)}{2(L-1)^2} - \frac{6LR}{|S|(L-1)}\right)\frac{|S|}{2}$$

by Claim 4.10 while

$$|Q_0| \leq \left(\frac{K-1}{K} \cdot \frac{L}{L-1}\right)\frac{|S|}{2}$$

by Claim 4.9. So $|Q_0^+| \leq |Q_0|$ would imply

$$1 - \frac{K-1}{K} \cdot \frac{L(3L-2)}{2(L-1)^2} - \frac{6LR}{|S|(L-1)} < \frac{K-1}{K} \cdot \frac{L}{L-1}$$

which is equivalent to the inequality

$$(\S) \qquad 1 < \frac{K-1}{K} \cdot \frac{L(5L-4)}{2(L-1)^2} + \frac{6LR}{|S|(L-1)}.$$

By hypothesis and Claim 4.12, we have $K < L(5L-4)/(3L^2-2)$ which is equivalent to

$$1 > \frac{K-1}{K} \cdot \frac{L(5L-4)}{2(L-1)^2}$$

so if $|S| > M'$ for some M' depending only on K, L and R then we would have

$$1 > \frac{K-1}{K} \cdot \frac{L(5L-4)}{2(L-1)^2} + \frac{6LR}{|S|(L-1)}$$

and this would contradict (§). Hence, assuming $|S|$ is greater than this M', we have $|Q_0^+| > |Q_0|$.

Then if $Q_0^+ \to S$ were the inclusion of a geodesic segment, we would have $(p^+, q^+) \in J_0$ and this would contradict $d_S(p_0, q_0) = s_0$. Thus $|Q_0^+| > \frac{1}{2}|S|$. On the other hand

$$\begin{aligned}
|Q_0^+| &= |Q_0^-| + |A_p| + |A_q| \\
&\le |Q^-| + \sum_{i<j}(|Q_i| - |\bar{Q}_i|) + |A_p| + |A_q| \\
&\le \frac{|S_j|}{2} + \sum_{i<j}(|Q_i| - |\bar{Q}_i|) + |A_p| + |A_q| \\
&= \frac{|S|}{2} + \frac{1}{2}\sum_{i<j}(|Q_i| - |\bar{Q}_i|) + |A_p| + |A_q| \\
&\le \frac{|S|}{2} + \frac{1}{2}\sum_{i<j}|Q_i| + |A_p| + |A_q| \\
&\le \frac{|S|}{2} + \left(\frac{K-1}{K} \cdot \frac{L(3L-2)}{4(L-1)^2}\right)|S| + \frac{3LR}{L-1} + \left(\frac{K-1}{K} \cdot \frac{L}{L-1}\right)|S| \\
&= \frac{|S|}{2} + \frac{K-1}{K} \cdot \frac{L}{L-1} \cdot \left(\frac{3L-2}{4(L-1)} + 1\right)|S| + \frac{3LR}{L-1} \\
&= \frac{|S|}{2} + \frac{K-1}{K} \cdot \frac{L}{L-1} \cdot \frac{7L-6}{4(L-1)} \cdot |S| + \frac{3LR}{L-1} \\
&= \left(\frac{1}{2} + \frac{K-1}{K} \cdot \frac{L(7L-6)}{4(L-1)^2} + \frac{3LR}{|S|(L-1)}\right)|S|,
\end{aligned}$$

where the last inequality follows from Claims 4.9 and 4.10. By hypothesis and Claim 4.12,

$$K < \frac{L(7L-6)}{5L^2 - 2L - 2},$$

which is equivalent to

$$\frac{1}{2} + \frac{K-1}{K} \cdot \frac{L(7L-6)}{4(L-1)^2} < 1.$$

Thus, if $|S| > M''$ for some M'' depending only on K, L and R then

$$\frac{1}{2} + \frac{K-1}{K} \cdot \frac{L(7L-6)}{4(L-1)^2} + \frac{3LR}{|S|(L-1)} < 1$$

and so $|Q_0^+| < |S|$ so that Q_0^+ embeds in S. In this case, the endpoints p^+ and q^+ of Q_0^+ in S are at distance

$$d_S(p^+, q^+) \ge \left(\frac{1}{2} - \frac{K-1}{K} \cdot \frac{L(7L-6)}{4(L-1)^2} - \frac{3LR}{|S|(L-1)}\right)|S|$$

but we also have

$$d_X(\alpha(p^+), \alpha(q^+)) < \frac{1}{L}|Q_0^+| - C \le \frac{1}{L}\left(\frac{1}{2} + \frac{K-1}{K} \cdot \frac{L(7L-6)}{4(L-1)^2} + \frac{3LR}{|S|(L-1)}\right)|S| - C$$

which, by Lemma 3.1 and $C \geq 4R$, implies

$$\frac{1}{L}\left(\frac{1}{2}+\frac{K-1}{K}\cdot\frac{L(7L-6)}{4(L-1)^2}+\frac{3LR}{|S|(L-1)}\right)|S| > \left(\frac{1}{2}-\frac{K-1}{K}\cdot\frac{L(7L-6)}{4(L-1)^2}-\frac{3LR}{|S|(L-1)}\right)|S|-\frac{K-1}{K}\cdot\frac{|S|}{2}$$

which is equivalent to the inequality

$$(\P) \qquad \frac{K-1}{K}\cdot\frac{9L^2-3L-4}{2(L-1)^2}+\frac{6R(L+1)}{|S|(L-1)} > \frac{L-1}{L}.$$

By hypothesis,

$$K < \frac{L(9L^2-3L-4)}{7L^3+3L^2-10L+2}$$

which is equivalent to

$$\frac{K-1}{K}\cdot\frac{9L^2-3L-4}{2(L-1)^2} < \frac{L-1}{L}$$

so if $|S| > M'''$ for some M''' depending only on K, L and R then we would have

$$\frac{K-1}{K}\cdot\frac{9L^2-3L-4}{2(L-1)^2}+\frac{6R(L+1)}{|S|(L-1)} < \frac{L-1}{L}$$

which contradicts (¶).

Therefore, if $|S| > M = \max\{M', M'', M'''\}$, which depends only on K, L and R then assuming the existence of a j for which $Q \not\subset P^\circ_{0,j}$ leads us to a contradiction. □

Claim 4.14 *Let $K > 1$, $L > 1$ and $R \geq 0$. There exists an $M > 0$ such that if X is an R–rough geodesic metric space, $\alpha\colon S \to X$ is a $1/K$–almost isometric R–circle with $|S| > M$ and*

$$K < \frac{L(9L^2-3L-4)}{7L^3+3L^2-10L+2},$$

and $C \geq 4R$ then any (L,C)–greedy tightening sequence for α is completely disjoint.

Proof Consider an (L,C)–greedy tightening sequence for α with notation as above. For the sake of finding a contradiction, suppose $j \geq 1$ is the least integer with $Q_j \not\subset P^\circ_{0,j}$. As above we view the $\bar{Q}_i$ with $i < j$ as disjoint segments of S_j with $S_j \setminus \bigcup_{i=0}^{j-1} \bar{Q}_m = P^\circ_{0,j}$.

Since $Q_j \not\subset P^\circ_{0,j}$, we have $\bar{Q}_m \cap Q_j \neq \varnothing$, for some $m < j$. Recall that (p_j, q_j) is the limit of a sequence $(p_j^{(n)}, q_j^{(n)})_n$ in J_j. For each n, let $Q_j^{(n)}$ be a geodesic segment between $p_j^{(n)}$ and $q_j^{(n)}$ in S_j. By Claim 4.13, $Q_j^{(n)} \subset P^\circ_{0,j}$, so $\bar{Q}_m \cap Q_j \subset \{p_m, q_m\} \cap \{p_j, q_j\}$. Without loss of generality, we may assume $q_m = p_j$. See Figure 6.

Since $Q_j^{(n)} \subset P^\circ_{0,j}$, we may think of the $Q_j^{(n)}$ as segments of S_m, by the embedding $P^\circ_{0,j} \hookrightarrow S_m$. Each $Q_j^{(n)}$ is a geodesic segment in S_m since the complementary segment of $Q_j^{(n)}$ in S_m is even longer than the complementary segment of $Q_j^{(n)}$ in S_j. Thus $(p_j^{(n)}, q_j^{(n)})_n$ is a sequence in J_m. For each n, let $Q_m^{(n)}$ be a segment between $p_m^{(n)}$ and $q_m^{(n)}$ such that $(Q_m^{(n)})_n$ converges to Q_m in Hausdorff distance. The

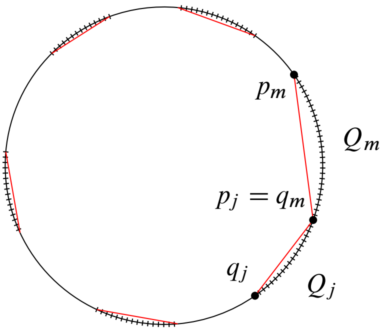

Figure 6: A greedy tightening sequence that is disjoint up to j but not up to $j+1$ with Q_j intersecting the prior Q_i only at endpoints, as in the proof of Claim 4.14.

circular orders on the triples $(p_m^{(n)}, q_m^{(n)}, q_j^{(n)})$ and $(p_m^{(n)}, p_j^{(n)}, q_j^{(n)})$ are eventually constant and equal. Let $(A^{(n)})_n$ be a sequence of segments in S_m from $p_m^{(n)}$ to $q_j^{(n)}$ such that $A^{(n)}$ eventually contains $q_m^{(n)}$ or, equivalently, eventually contains $p_j^{(n)}$.

Let $\varepsilon > 0$ satisfy $\varepsilon < \frac{1}{3}|Q_j|$ and $\varepsilon \le LR/(L+1)$. Then, for n large enough,

$$|A^{(n)}| \ge |Q_m^{(n)}| + |Q_j^{(n)}| - d_{S_m}(q_m^{(n)}, p_j^{(n)}) > |Q_m| - \varepsilon + |Q_j| - \varepsilon - \varepsilon = |Q_m| + |Q_j| - 3\varepsilon > |Q_m|$$

and

$$\begin{aligned}
d_X\big(\alpha_m(p_m^{(n)}), \alpha_m(q_j^{(n)})\big) \\
&\le d_X\big(\alpha_m(p_m^{(n)}), \alpha_m(q_m^{(n)})\big) + d_X\big(\alpha_m(q_m^{(n)}), \alpha_m(p_j^{(n)})\big) + d_X\big(\alpha_m(p_j^{(n)}), \alpha_m(q_j^{(n)})\big) \\
&< \frac{1}{L} d_{S_m}(p_m^{(n)}, q_m^{(n)}) - C + d_{S_m}(q_m^{(n)}, p_j^{(n)}) + R + \frac{1}{L} d_{S_m}(p_j^{(n)}, q_j^{(n)}) - C \\
&< \frac{1}{L}|Q_m| + \varepsilon - C + \varepsilon + R + \frac{1}{L}|Q_j| + \varepsilon - C \\
&= \frac{1}{L}(|Q_m| + |Q_j|) - 2C + R + 3\varepsilon \\
&< \frac{1}{L}(|A^{(n)}| + 3\varepsilon) - 2C + R + 3\varepsilon \\
&= \frac{1}{L}|A^{(n)}| - 2C + R + \frac{3(L+1)}{L} \cdot \varepsilon \\
&\le \frac{1}{L}|A^{(n)}| - C
\end{aligned}$$

since $C \ge 4R$. So, if $A^{(n)}$ is a geodesic segment for arbitrarily large n then, for some n, we would have $(p_m^{(n)}, q_j^{(n)}) \in J_m$ and $d_{S_m}(p_m^{(n)}, q_j^{(n)}) > |Q_m| = s_m$, which is a contradiction. Thus eventually

$$|Q_m^{(n)}| + |Q_j^{(n)}| = |A^{(n)}| > \tfrac{1}{2}|S_m|$$

and so $|Q_m| + |Q_j| \ge \frac{1}{2} S_m$.

Then, by Claim 4.9,

$$\frac{S_m}{2} \le 2\left(\frac{K-1}{K} \cdot \frac{L}{L-1}\right)\frac{|S|}{2}$$

while

$$\frac{|S_m|}{2} \ge \frac{|S|}{2} - \frac{1}{2}\sum_{i<m} |Q_i| \ge \frac{|S|}{2} - \left(\frac{K-1}{K} \cdot \frac{L(3L-2)}{2(L-1)^2}\right)\frac{|S|}{2} - \frac{3LR}{L-1}$$
$$= \left(1 - \frac{K-1}{K} \cdot \frac{L(3L-2)}{2(L-1)^2} - \frac{6LR}{|S|(L-1)}\right)\frac{|S|}{2}$$

by Claim 4.10. Combining these we obtain

$$\frac{K-1}{K} \cdot \frac{L(7L-6)}{2(L-1)^2} + \frac{6LR}{|S|(L-1)} \ge 1$$

but, by hypothesis and Claim 4.12,

$$K < \frac{L(7L-6)}{5L^2-2L-2},$$

which is equivalent to

$$\frac{K-1}{K} \cdot \frac{L(7L-6)}{2(L-1)^2} < 1,$$

so, if $|S|$ is large enough (depending only on K, L and R) then we have a contradiction. □

5 The fine Milnor–Schwarz lemma

The fine Milnor–Schwarz lemma is a refinement of the Milnor–Schwarz lemma that gives finer control on the multiplicative constant of the quasi-isometry. In this section we will state and prove this version of the Milnor–Schwarz lemma. As a consequence we will prove that every strongly shortcut group has a strongly shortcut Cayley graph. A corresponding statement should hold for any rough approximability invariant of metric spaces.

Let (X,d) be a rough geodesic metric space. Let G be a group acting coboundedly on X by isometries. Let

$$S_{x_0,t} = \{g \in G \mid d(x_0, gx_0) \le t\}$$

for $x_0 \in X$ and $t \in \mathbb{R}_{\ge 0}$.

Remark 5.1 If the action of G is metrically proper then the $S_{x_0,t}$ are finite.

Let $\Gamma_{x_0,t}$ be the graph with vertex set G and with an edge of length t between g and g' whenever $g' = gs$ for some $s \in S_{x_0,t}$. So when $S_{x_0,t}$ generates G, the graph $\Gamma_{x_0,t}$ is the Cayley graph of G for the generating set $S_{x_0,t}$ with edges scaled by t. Let $d_{x_0,t}$ be the graph metric on $\Gamma_{x_0,t}$ where we set $d_{x_0,t}(g,h) = \infty$ when g and h are in different components of $\Gamma_{x_0,t}$. Then, when $S_{x_0,t}$ generates G, the metric $d_{x_0,t}$ is the word metric on G for the generating set $S_{x_0,t}$ scaled by t. Let $f_{x_0,t}\colon (G, d_{x_0,t}) \to (X,d)$ be defined by $f_{x_0,t}(g) = gx_0$. Let $K_{x_0,t}$ be the infimum of all $K \geq 1$ for which there exists some $C_K > 0$ such that $f_{x_0,t}$ is a (K, C_K)–quasi-isometry. Note that if $f_{x_0,t}$ is not a quasi-isometry (eg if $S_{x_0,t}$ does not generate G) then $K_{x_0,t} = \infty$.

Lemma 5.2 (fine Milnor–Schwarz lemma) *Let (X, d) be a rough geodesic metric space. Let G be a group acting coboundedly on X by isometries. Let $x_0 \in X$ and let $K_{x_0,t}$ be defined as above for $t \in \mathbb{R}_{\geq 0}$. Then $K_{x_0,t} \to 1$ as $t \to \infty$.*

Proof Let $g \in G$. We will prove that $d\bigl(f_{x_0,t}(1), f_{x_0,t}(g)\bigr) \leq d_{x_0,t}(1, g)$. If $d_{x_0,t}(1, g) = \infty$ there is nothing to show and so $d_{x_0,t}(1, g) = Mt$ for some $M \in \mathbb{N}_{\geq 0}$ and there is a combinatorial path defined by

$$1 = g_0, g_1, g_2, \dots, g_M = g$$

in $\Gamma_{x_0,t}$. The $g_{i-1}^{-1} g_i$ are contained in $S_{x_0,t}$ and so, by the triangle inequality,

$$\begin{aligned} d(f_{x_0,t}(1), f_{x_0,t}(g)) &= d(x_0, gx_0) \\ &\leq d(x_0, g_1x_0) + d(g_1x_0, g_2x_0) + \cdots + d(g_{M-1}x_0, gx_0) \\ &= d(x_0, g_1x_0) + d(x_0, g_1^{-1}g_2x_0) + \cdots + d(x_0, g_{M-1}^{-1}gx_0) \\ &\leq Mt \\ &= d_{x_0,t}(1, g). \end{aligned}$$

We now establish a lower bound on $D = d(f_{x_0,t}(1), f_{x_0,t}(g))$. Since G acts coboundedly on X, the orbit Gx_0 is a quasi-onto subspace of X. Hence Gx_0 is roughly isometric to X and so Gx_0 is also a rough geodesic metric space. Let R be the rough geodesicity constant of Gx_0. Let $\alpha\colon [0, D] \to Gx_0$ be an R–rough geodesic from $f_{x_0,t}(1) = x_0$ to $f_{x_0,t}(g) = gx_0$. Assume that $t > R$ and subdivide $[0, D]$ into at most $\lceil D/(t-R) \rceil$ segments of length at most $t - R$. Let $0 = a_0 < a_1 < a_2 < \cdots < a_M = D$ be the endpoints of the segments and let $g_i x_0 = \alpha(a_i)$, for each i, with $g_0 = 1$ and $g_M = g$. Then

$$d(x_0, g_i^{-1} g_{i+1} x_0) = d(g_i x_0, g_{i+1} x_0) = d(\alpha(a_i), \alpha(a_{i+1})) \leq |a_i - a_{i+1}| + R \leq t$$

for each i. Thus $g_i^{-1} g_{i+1} \in S_{x_0,t}$, for each i, and so

$$1 = g_0, g_1, g_2, \dots, g_{M-1}, g_M = g$$

defines a combinatorial path in $\Gamma_{x_0,t}$. Hence

$$\begin{aligned} d_{x_0,t}(1, g) \leq tM &\leq t\left\lceil \frac{D}{t-R} \right\rceil \\ &= t\left\lceil \frac{d\bigl(f_{x_0,t}(1), f_{x_0,t}(g)\bigr)}{t-R} \right\rceil \\ &\leq t\left(\frac{d\bigl(f_{x_0,t}(1), f_{x_0,t}(g)\bigr)}{t-R} + 1 \right) \\ &= \frac{t}{t-R} d\bigl(f_{x_0,t}(1), f_{x_0,t}(g)\bigr) + t \end{aligned}$$

and so we have $((t-R)/t) d_{x_0,t}(1, g) - (t-R) \leq d(f_{x_0,t}(1), f_{x_0,t}(g))$.

For $g, g' \in G$,

$$d(f_{x_0,t}(g), f_{x_0,t}(g')) = d(gx_0, g'x_0) = d(x_0, g^{-1}g'x_0) = d(f_{x_0,t}(1), f_{x_0,t}(g^{-1}g'))$$

and $d_{x_0,t}(g, g') = d_{x_0,t}(1, g^{-1}g')$, so

$$\frac{t-R}{t} d_{x_0,t}(g, g') - (t-R) \leq d(f_{x_0,t}(g), f_{x_0,t}(g')) \leq d_{x_0,t}(g, g');$$

hence $1 \leq K_{x_0,t} \leq (t-R)/t$ when $t > R$. This implies that $K_{x_0,t} \to 1$ as $t \to \infty$. $\square$

Corollary 5.3 (Milnor–Schwarz lemma) *If the G–action on X is metrically proper then G is finitely generated and, for any finite generating set S, the map*

$$f_s \colon (G, d_S) \to X, \quad g \mapsto gx_0,$$

is a quasi-isometry, where d_S is the word metric for the generating set S.

Proof By Lemma 5.2, if t is large enough then $K_{x_0,t} < \infty$. Thus $S_{x_0,t}$ generates G. By Remark 5.1, the generating set $S_{x_0,t}$ is finite so G is finitely generated. Since $K_{x_0,t} < \infty$ and since scaling the metric d_S by a factor of t preserves the quasi-isometry type, the map f_S is a quasi-isometry for $S = S_{x_0,t}$. But the identity map on G is a bi-Lipschitz equivalence from $(G, d_{S'})$ to $(G, d_{S''})$ where S' and S'' are any two finite generating sets and $d_{S'}$ and $d_{S''}$ are the corresponding word metrics. Thus f_S is a quasi-isometry for any generating set S. $\square$

Corollary 5.4 *Let G be a group. If G acts metrically properly and coboundedly on a strongly shortcut rough geodesic metric space X then G has a finite generating set S for which the Cayley graph of (G, S) is strongly shortcut. In particular, the group G is strongly shortcut.*

Proof By Corollary 3.10, there exists an $L_X > 1$ such that whenever $C > 0$ and Y is a rough geodesic metric space and $f \colon Y \to X$ is an (L_X, C)–quasi-isometry up to scaling, then Y is strongly shortcut. But, by Lemma 5.2, there is a Cayley graph Γ of G, a $C > 0$ and an (L_X, C)–quasi-isometry up to scaling $f \colon \Gamma \to X$. So Γ is strongly shortcut as a rough geodesic metric space. Then, by Remark 2.5, the Cayley graph Γ is strongly shortcut as a graph. $\square$

Corollary 5.5 *Let G be a group. The following conditions are equivalent:*

(1) *G is strongly shortcut.*

(2) *G acts metrically properly and coboundedly on a strongly shortcut rough geodesic metric space.*

(3) *G has a finite generating set S for which the Cayley graph of (G, S) is strongly shortcut.*

6 Asymptotically CAT(0) spaces

In this section we will apply the characterizations of Section 3 to prove that asymptotically CAT(0) rough geodesic metric spaces are strongly shortcut. By the results of Section 5, this will imply that asymptotically CAT(0) groups are strongly shortcut.

Asymptotically CAT(0) spaces and groups were first introduced and studied by Kar [2011]. A metric space X is *asymptotically* CAT(0) if every asymptotic cone of X is CAT(0). A group is *asymptotically* CAT(0) if it acts properly and cocompactly on an asymptotically CAT(0) proper geodesic metric space. (Note that the condition that the action be on a proper metric space does not make this definition more restrictive than the definition given by Kar [2011] since the definition of proper action in Kar [2011, page 77] seems to be that of Bridson and Haefilger [1999, Section I.8.2] and any geodesic metric space admitting a cocompact action that is proper by this more restricted definition is a proper metric space.) For an introduction to CAT(0) geodesic metric spaces, see Bridson and Haefliger [1999].

Theorem 6.1 *Asymptotically* CAT(0) *rough geodesic metric spaces are strongly shortcut.*

Proof By uniqueness of geodesics in CAT(0) geodesic metric spaces, there is no isometric copy of a Riemannian circle in the asymptotic cone of an asymptotically CAT(0) metric space X. So, by Theorem 3.8, any asymptotically CAT(0) rough geodesic metric space is strongly shortcut. □

Theorem 6.2 *Asymptotically* CAT(0) *groups are strongly shortcut.*

Proof A proper and cocompact action on a proper metric space is metrically proper and cobounded. So, by Theorem 6.1, any asymptotically CAT(0) group G acts metrically properly and coboundedly on a strongly shortcut geodesic metric space. Thus, by Corollary 5.5, the group G is strongly shortcut. □

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