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CHAO WANG

WEIBIAO WANG

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Extendable periodic automorphisms of closed surfaces over the 3–sphere

CHAO WANG
WEIBIAO WANG

A periodic automorphism of a surface Σ is said to be extendable over S^3 if it extends to a periodic automorphism of the pair (Σ, S^3) for some possible embedding $\Sigma \hookrightarrow S^3$. We classify and construct all extendable automorphisms of closed surfaces, with orientation-reversing cases included. Moreover, they can all be induced by automorphisms of S^3 on Heegaard surfaces. As a byproduct, the embeddings of surfaces into lens spaces are discussed.

57M60, 57S17, 57S25

1 Introduction

Let Σ_g be a connected closed orientable surface of genus g. Denote the automorphism group of a manifold by $\mathrm{Aut}(\cdot)$. A torsion $f \in \mathrm{Aut}(\Sigma_g)$ is called *extendable* over a 3–manifold M if there exists an embedding $e\colon \Sigma_g \hookrightarrow M$ and a periodic automorphism ϕ of M, such that $e \circ f = \phi \circ e$. In other words, f can be induced by a symmetry ϕ of M on some embedded surface.

Existing results on this topic are mainly discussed in smooth category and concerned with the case $M = S^3$ or $M = \mathbb{R}^3$. For concrete examples, see the work of Guo, Wang, Wang and Zhang [6], which determined the extendability over S^3 for all periodic maps on Σ_2. Equivalent conditions for a periodic orientation-preserving automorphism to be extendable over $\mathbb{R}^3$ were first given by Rüedy [8], while the orientation-reversing ones are classified by Costa [4]. The former work was recently generalized for S^3 in the orientation-preserving category by Ni, Wang and Wang [7], who put forward the following question.

Question 1.1 Consider orientation-reversing automorphisms — ie either f or ϕ in the definition, or both of them, can be orientation-reversing. How do we classify the periodic maps on Σ_g that are extendable over S^3? Can the corresponding embedded surfaces always be chosen as Heegaard ones?

In this paper, we are going to solve this problem and construct all periodic extendable maps. As a consequence, we give a positive answer to the latter half of the question. Also, we work in the smooth category, thus by the geometrization of finite group actions on 3–manifolds, each torsion $\phi \in \mathrm{Aut}(S^3)$ is conjugate to an orthogonal action on the standard 3–sphere $S^3 \subset \mathbb{R}^4$.

Moreover, Funayoshi and Koda [5] proved that for $g \leq 2$, if a map $f_0 \in \mathrm{Aut}(\Sigma_g)$ extends to an automorphism of S^3 with respect to some embedding $\Sigma_g \hookrightarrow S^3$, then f_0 can be realized as the restriction of some $\phi_0 \in \mathrm{Aut}(S^3)$ on a Heegaard surface. It remains open whether that still holds for higher genus cases. Here f_0 and ϕ_0 are not necessarily periodic, but according to a recent result of Ni, Wang and Wang [7], if f_0 is periodic, then ϕ_0 can also be chosen as a torsion. Thus we obtain a partial answer to that problem.

Theorem 1.2 *If an automorphism $\phi_0 \in \mathrm{Aut}(S^3)$ induces a periodic map $f \in \mathrm{Aut}(\Sigma_g)$ with respect to some embedding $\Sigma_g \hookrightarrow S^3$, then f can also be induced by a periodic automorphism ϕ of S^3 on a Heegaard surface.*

To classify periodic extendable maps, we first introduce some necessary invariants in Section 2. With them we state our main results, Theorems 2.4, 2.5 and 2.6, which provide equivalent conditions for a periodic map on Σ_g to be extendable over S^3. Basic examples will be presented in Section 3, and all periodic extendable automorphisms can be constructed from them. In fact, in Section 4 we modify the basic examples to get more extendable maps. Meanwhile, we prove that for a surface automorphism f satisfying the extension conditions listed in the main theorems, f must be conjugate to one of them. In Section 5 we finally prove the necessity of the extension conditions, case by case. Note that when a periodic automorphism ϕ of the pair (Σ_g, S^3) preserves the orientation of S^3, the quotient orbifold pair $(\Sigma_g/\phi, S^3/\phi)$ is related to an embedded surface in a lens space, so we deal with the topic in Section 6.

Extendability characterizes whether a symmetry of a surface can be induced by those of 3–manifolds. The notion can be generalized in different ways and some efforts have been made to understand it. For instance, a finite subgroup G of $\mathrm{Aut}(\Sigma_g)$ (or a G–action) is called *extendable* over S^3 with respect to an embedding $e\colon \Sigma_g \hookrightarrow S^3$ if there exists a group monomorphism φ from G to $\mathrm{Aut}(S^3)$ such that $\varphi(h) \circ e = e \circ h$ holds for each $h \in G$. The maximum order of extendable groups for fixed genus g is discussed by Wang, Wang, Zhang and Zimmermann in [10; 11; 12]. And as a contrast to Theorem 1.2, there are extendable finite group actions on Σ_{21} and Σ_{481} whose extensions cannot be realized on Heegaard surfaces [11].

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2 Classification of periodic maps and main theorems

The classification of periodic maps on closed orientable surfaces was finished by Yokoyama [13; 14; 15], but see also Costa [3]. Indeed, Yokoyama completed the classification for all compact surfaces. We introduce the involved invariants and present the results here in a convenient way for our task. The notation will be used throughout the paper.

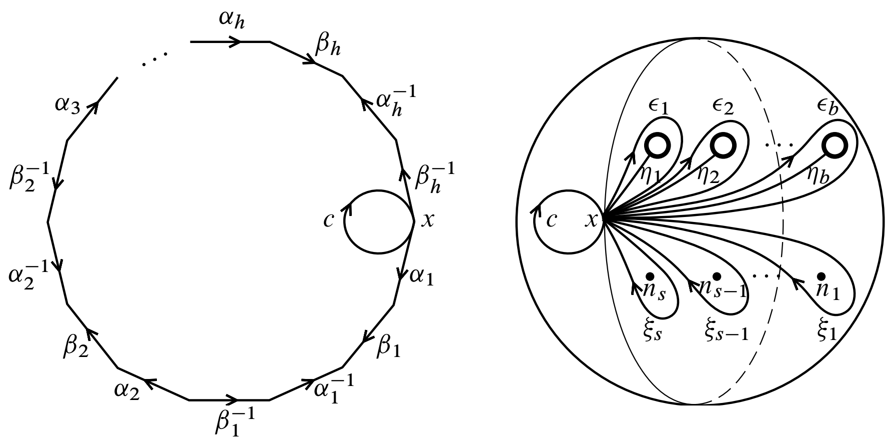

Figure 1: A model for an orientable 2–orbifold.

Given a periodic map $f \in \mathrm{Aut}(\Sigma_g)$ of order n, we obtain an orbifold Σ_g/f. Suppose Σ_g/f has s isolated singular points of indices $n_1, n_2, \dots, n_s$, respectively, and the underlying space $|\Sigma_g/f|$ is a compact surface of genus h, with b boundary components. We can cut Σ_g/f along a simple closed curve c with no singular point on it, to get a genus h surface with connected boundary and a simpler orbifold, whose underlying space is a sphere with $b+1$ open disks removed. See Figure 1 for the orientable case and Figure 2 for the nonorientable case. Take a basepoint x on c and choose oriented loops α_i, β_i or δ_i for $1 \le i \le h$ along with ϵ_j, η_j for $1 \le j \le b$ and ξ_k for $1 \le k \le s$ as in the figures. Here $\{\alpha_i, \beta_i\}_{1 \le i \le h}$ (or $\{\delta_i\}_{1 \le i \le h}$) is a canonical generator set for fundamental group of a closed surface; η_j is a round trip between the basepoint and the j^{th} boundary component of $|\Sigma_g/f|$; ϵ_j surrounds the j^{th} boundary component; and ξ_k surrounds the k^{th} singular point. Then by van Kampen's theorem for orbifolds (see Scott [9, Section 2]), there is a canonical presentation of the orbifold fundamental group $\pi_1(\Sigma_g/f)$ given by

$$\Big\langle \alpha_1, \beta_1, \dots, \alpha_h, \beta_h, \eta_1, \epsilon_1, \dots, \eta_b, \epsilon_b, \xi_1, \dots, \xi_s \;\Big|\; \prod_{i=1}^{h} [\alpha_i, \beta_i] \prod_{j=1}^{b} \epsilon_j \prod_{k=1}^{s} \xi_k = 1,\ \xi_k^{n_k} = 1 \text{ for } 1 \le k \le s,\ \eta_j = \epsilon_j^{-1} \eta_j \epsilon_j \text{ and } \eta_j^2 = 1 \text{ for } 1 \le j \le b \Big\rangle$$

if $|\Sigma_g/f|$ is orientable, or

$$\Big\langle \delta_1, \dots, \delta_h, \eta_1, \epsilon_1, \dots, \eta_b, \epsilon_b, \xi_1, \dots, \xi_s \;\Big|\; \prod_{i=1}^{h} \delta_i^2 \prod_{j=1}^{b} \epsilon_j \prod_{k=1}^{s} \xi_k = 1,\ \xi_k^{n_k} = 1 \text{ for } 1 \le k \le s,\ \eta_j = \epsilon_j^{-1} \eta_j \epsilon_j \text{ and } \eta_j^2 = 1 \text{ for } 1 \le j \le b \Big\rangle$$

if $|\Sigma_g/f|$ is nonorientable. The collection of such generators,

$$\mathcal{G} = \begin{cases} \{\alpha_1, \beta_1, \dots, \alpha_h, \beta_h, \eta_1, \epsilon_1, \dots, \eta_b, \epsilon_b, \xi_1, \dots, \xi_s\} & \text{if } |\Sigma_g/f| \text{ is orientable,} \\ \{\delta_1, \dots, \delta_h, \eta_1, \epsilon_1, \dots, \eta_b, \epsilon_b, \xi_1, \dots, \xi_s\} & \text{if } |\Sigma_g/f| \text{ is nonorientable,} \end{cases}$$

is called a *canonical generator system* of $\pi_1(\Sigma_g/f)$.

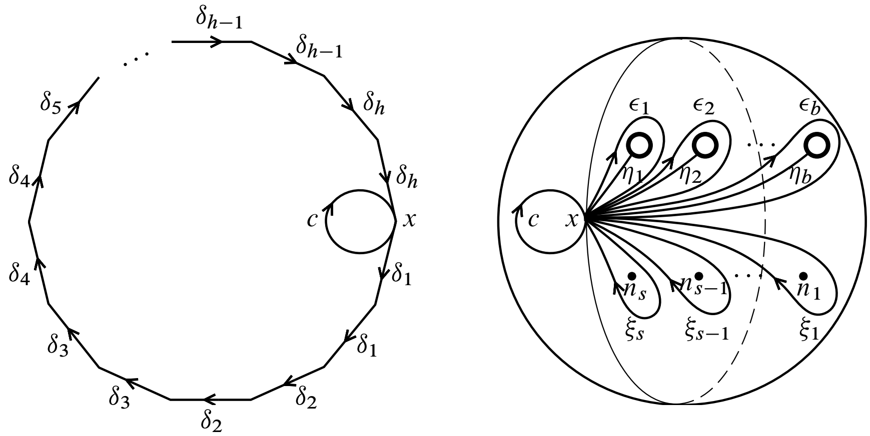

Figure 2: A model for a nonorientable 2–orbifold.

The action of the finite cyclic group $\langle f\rangle$ on Σ_g induces a short exact sequence

$$1 \to \pi_1(\Sigma_g) \xrightarrow{\rho_*} \pi_1(\Sigma_g/f) \xrightarrow{\psi} \langle f\rangle \to 1,$$

where $\rho\colon \Sigma_g \to \Sigma_g/f$ is the quotient map. The following proposition shows that ψ plays a key role in the classification of periodic maps up to conjugacy.

Proposition 2.1 [6, Theorem 2.2; 3, Lemma 4.1] *Suppose $f, f' \in \mathrm{Aut}(\Sigma_g)$ are periodic maps of the same order and with epimorphisms ψ and ψ', respectively. Then f and f' are conjugate if and only if there exists an orbifold homeomorphism $H\colon \Sigma_g/f \to \Sigma_g/f'$ such that $\iota\circ\psi = \psi'\circ H_*$, where $\iota\colon \langle f\rangle \to \langle f'\rangle$ is defined by $\iota(f) = f'$.*

Given a canonical generator system $\mathcal{G}$ of $\pi_1(\Sigma_g/f)$, we obtain a collection of elements $\psi(\gamma)(\gamma\in\mathcal{G})$ in $\langle f\rangle$. Mapping f to the generator 1 of the additive group $\mathbb{Z}_n$, we identify $\langle f\rangle$ with $\mathbb{Z}_n$. Without ambiguity, we use integers to represent their images under the modulo n homomorphism $\mathbb{Z}\to\mathbb{Z}_n$. Then the *isotropy invariant* of f is defined to be

$$\begin{cases} \pm(\psi(\epsilon_1), \psi(\epsilon_2), \dots, \psi(\epsilon_b);\ \psi(\xi_1), \psi(\xi_2), \dots, \psi(\xi_s)) & \text{if } |\Sigma_g/f| \text{ is orientable,} \\ (\pm\psi(\epsilon_1), \pm\psi(\epsilon_2), \dots, \pm\psi(\epsilon_b);\ \pm\psi(\xi_1), \pm\psi(\xi_2), \dots, \pm\psi(\xi_s)) & \text{if } |\Sigma_g/f| \text{ is nonorientable.} \end{cases}$$

Here the plus-minus symbol is added universally or individually to make the defined object an invariant of f and independent of the choice of $\mathcal{G}$. Also, the order of $\epsilon_1, \epsilon_2, \dots, \epsilon_b$ and that of $\xi_1, \xi_2, \dots, \xi_s$ can be changed arbitrarily. Thus, for example, $\pm(4, 2; -8, -6)$ and $\pm(-2, -4; 6, 8)$ are the same invariant; $(\pm4, \pm2; \pm(-8), \pm(-6))$ and $(\pm2, \pm4; \pm6, \pm8)$ are the same invariant.

To classify periodic maps on closed surfaces, we still need two more invariants for two special cases.

Case 1 $n/2$ is even, $|\Sigma_g/f|$ is nonorientable (thus closed), and there is no $\pm n/2$ among $\psi(\xi_1), \dots, \psi(\xi_s)$. Then we define $h_1(f)$ as

$$h_1(f) = \sum_{i=1}^{h} \psi(\delta_i) + \sum_{k=1}^{s} \chi_k \psi(\xi_k), \quad \text{where } \chi_k = \begin{cases} 0 & \text{if } \psi(\xi_k) \in \{2, 4, \dots, \frac{1}{2}n - 2\} \subset \mathbb{Z}_n, \\ 1 & \text{if } \psi(\xi_k) \in \{\frac{1}{2}n + 2, \frac{1}{2}n + 4, \dots, n-2\} \subset \mathbb{Z}_n. \end{cases}$$

Note that each $\psi(\delta_i)$ is odd and each $\psi(\xi_k)$ is even, so $h_1(f)$ has the same parity as h.

Case 2 $|\Sigma_g/f|$ is nonorientable with genus $h = 2$. We choose

$$m = \gcd\big(\psi(\delta_1) + \psi(\delta_2), \psi(\epsilon_1), \dots, \psi(\epsilon_b), \psi(\xi_1), \dots, \psi(\xi_s), n\big),$$

where $\gcd(\cdots)$ means the greatest common divisor. Let f_m be the order m map induced by f on $|\Sigma_g/f^m|$, and

$$\psi_m \colon \pi_1(|\Sigma_g/f^m|/f_m) \to \langle f_m \rangle = \mathbb{Z}_m, \quad \text{identifying } f_m \leftrightarrow 1,$$

be the corresponding epimorphism. Then we define $h_2(f)$ as a multiset

$$h_2(f) = \{\pm\psi_m(\delta_1), \pm\psi_m(\delta_2)\} \subset \mathbb{Z}_m.$$

The following theorem tells that $f \in \mathrm{Aut}(\Sigma_g)$ is determined, up to conjugacy, by the topology of $|\Sigma_g/f|$, the isotropy invariant, and $h_1(f)$ and $h_2(f)$ if defined. Moreover, their values are independent of the choice of canonical generator system $\mathcal{G}$ for $\pi_1(\Sigma_g/f)$. The theorem is a combination of Proposition 2 in Section 1 and Theorem 1 in Section 2 of [13], Theorems 2.2, 3.2, 4.2 and Propositions 2.4, 3.4, 4.4 of [14], and Theorems 3.2–3.5 together with Proposition 3.2 of [15].

Theorem 2.2 (classification theorem) *Let $f, f' \in \mathrm{Aut}(\Sigma_g)$ be two periodic maps of the same order n, with $|\Sigma_g/f| \cong |\Sigma_g/f'|$.*

1. *Suppose $|\Sigma_g/f|$ and $|\Sigma_g/f'|$ are orientable. Then f and f' are conjugate if and only if they have the same isotropy invariant, up to a reordering of the singular points and boundary components of the corresponding orbifolds.*
2. *Suppose $|\Sigma_g/f|$ and $|\Sigma_g/f'|$ are nonorientable. Then f and f' are conjugate if and only if they satisfy the following conditions:*
   - (i) *Up to a reordering of the singular points and boundary components of the corresponding orbifolds, f and f' have the same isotropy invariant.*
   - (ii) *If $n/2$ is even and there is no $\pm n/2$ in the isotropy invariant, then $h_1(f) = h_1(f')$.*
   - (iii) *If $|\Sigma_g/f|$ and $|\Sigma_g/f'|$ have genus 2, then $h_2(f) = h_2(f')$.*

If $f \in \mathrm{Aut}(\Sigma_g)$ extends to $\phi \in \mathrm{Aut}(S^3)$ with some embedding $\Sigma_g \hookrightarrow S^3$, there are four types of (f, ϕ):

- **type $(+,+)$** f, ϕ are both orientation-preserving,
- **type $(-,-)$** f, ϕ are both orientation-reversing,
- **type $(+,-)$** f preserves the orientation of Σ_g while ϕ reverses that of S^3,
- **type $(-,+)$** f reverses the orientation of Σ_g while ϕ preserves that of S^3.

Now we can state our main theorems with the notation listed above. The first one is a recent result from [7]. We include it here for the sake of completeness.

Theorem 2.3 *A periodic orientation-preserving map $f \in \mathrm{Aut}(\Sigma_g)$ is extendable over S^3 in type $(+,+)$ if and only if the isotropy invariant $\pm(\psi(\xi_1), \psi(\xi_2), \dots, \psi(\xi_s))$ is*

$$\pm(\underbrace{\alpha, \alpha, \dots, \alpha}_{t}, \underbrace{-\alpha, -\alpha, \dots, -\alpha}_{t}, \underbrace{\beta, \beta, \dots, \beta}_{s/2-t}, \underbrace{-\beta, -\beta, \dots, -\beta}_{s/2-t})$$

up to rearrangement, where α, β are elements of coprime orders p, q in $\mathbb{Z}_n$ respectively, and $0 \le 2t \le s$.

Moreover, in that case, the conjugacy class of $\langle f \rangle$ is uniquely determined by its period n, the genus h of the quotient orbifold Σ_g/f, and the parameters s, t, p and q.

Theorem 2.4 *A periodic orientation-reversing map $f \in \mathrm{Aut}(\Sigma_g)$ is extendable over S^3 in type $(-,-)$ if and only if there is a generator α of $\mathbb{Z}_n$, such that there are only $\pm 2\alpha$ and 0 in the isotropy invariant, and one of the following situations holds:*

(1) *$n/2$ is odd and $|\Sigma_g/f|$ is orientable. If $n > 2$, then up to rearrangement the isotropy invariant $\pm(\psi(\epsilon_1), \psi(\epsilon_2), \dots, \psi(\epsilon_b); \psi(\xi_1), \psi(\xi_2), \dots, \psi(\xi_s))$ is*

$$\pm(\underbrace{2\alpha, \dots, 2\alpha}_{t-\lceil s/2 \rceil}, \underbrace{-2\alpha, \dots, -2\alpha}_{t-\lfloor s/2 \rfloor}, \underbrace{0, \dots, 0}_{s+b-2t}; \underbrace{2\alpha, \dots, 2\alpha}_{\lceil s/2 \rceil}, \underbrace{-2\alpha, \dots, -2\alpha}_{\lfloor s/2 \rfloor}),$$

where $\lceil s/2 \rceil \le t \le \lfloor (s+b)/2 \rfloor$.

(2) *$n/2$ is odd and $|\Sigma_g/f|$ is nonorientable without boundary. If $n > 2$, then h and s have the same parity.*

(3) *$n/2$ is even, and $|\Sigma_g/f|$ is nonorientable without boundary. If $n > 4$ and s is even, then*

$$h_1(f) = \begin{cases} -s\alpha & \text{if } 2\alpha \in \{2, 4, \dots, n/2-2\} \subset \mathbb{Z}_n, \\ s\alpha & \text{if } 2\alpha \in \{n/2+2, n/2+4, \dots, n-2\} \subset \mathbb{Z}_n. \end{cases}$$

If $n = 4$ and $s = 0$, then $h_1(f) = 0 \in \mathbb{Z}_4$.

Moreover, in that case, the conjugacy class of $\langle f \rangle$ is uniquely determined by n, h, b, s and t if (1) holds, and by n, h and s if (2) or (3) holds.

Theorem 2.5 *A periodic orientation-preserving map $f \in \mathrm{Aut}(\Sigma_g)$ is extendable over S^3 in type $(+,-)$ if and only if n is even, $s \ge 2$ and there is a generator α of $\mathbb{Z}_n$ such that, up to rearrangement, the isotropy invariant $\pm(\psi(\xi_1), \psi(\xi_2), \dots, \psi(\xi_s))$ is*

$$\pm(\alpha, (-1)^{s-1}\alpha, -2\alpha, 2\alpha, -2\alpha, 2\alpha, \dots, (-1)^s \cdot 2\alpha).$$

If $n = 2$, then $s = 2$ and the isotropy invariant is $\pm(\alpha, -\alpha)$.

Moreover, in that case, the conjugacy class of $\langle f \rangle$ is uniquely determined by n, h and s.

Theorem 2.6 *A periodic orientation-reversing map $f \in \mathrm{Aut}(\Sigma_g)$ is extendable over S^3 in type $(-,+)$ if and only if one of the following situations happens*:

(1) *$n/2$ is odd, and $|\Sigma_g/f|$ is orientable with nonempty and connected boundary, ie $b=1$. If $n>2$, then s is odd and there exists a generator α of $\mathbb{Z}_n$ such that, up to rearrangement, the isotropy invariant $\pm(\psi(\epsilon_1);\ \psi(\xi_1),\psi(\xi_2),\dots,\psi(\xi_s))$ is*

$$(2\alpha;\ \underbrace{2\alpha,\dots,2\alpha}_{(s-1)/2},\underbrace{-2\alpha,-2\alpha,\dots,-2\alpha}_{(s+1)/2}).$$

(2) *$n/2$ is odd, and $|\Sigma_g/f|$ is nonorientable with nonempty and connected boundary, ie $b=1$. If $s>0$, then s is odd, and there exists a factor l of $n/2$ and a generator α of $\mathbb{Z}_n$ such that the isotropy invariant $(\pm\psi(\epsilon_1);\ \pm\psi(\xi_1),\dots,\pm\psi(\xi_s))$ is*

$$(\pm 2\alpha;\ \pm 2l\alpha,\pm 2l\alpha,\dots,\pm 2l\alpha).$$

(3) *$|\Sigma_g/f|$ is nonorientable without boundary, ie $b=0$. Then, up to rearrangement, the isotropy invariant $(\pm\psi(\xi_1),\dots,\pm\psi(\xi_s))$ is*

$$(\underbrace{\pm\beta,\dots,\pm\beta}_{t},\underbrace{\pm\gamma,\dots,\pm\gamma}_{s-t}),$$

where $0\le t\le s$, and $\beta,\gamma\in\mathbb{Z}_n$ have coprime orders p,q, respectively. If $t=0$ then set $p=1$, otherwise t should be odd; if $t=s$ then set $q=1$, otherwise $s-t$ should be odd. Let $n=pql$; then l is even and $l/2$ has the same parity as h. If $h=1$, then $l=2$. In addition, the following conditions hold:

(i) *If $h_1(f)$ is defined, ie $n/2$ is even and $2\notin\{p,q\}$, then $h_1(f)\equiv l/2 \pmod{l}$.*

(ii) *If $h=2$, without loss of generality assume p is odd. Then f is conjugate to some power of the map f_0 whose invariants are*

$$\text{isotropy invariant} = (\underbrace{\pm ql,\dots,\pm ql}_{t},\underbrace{\pm pl,\dots,\pm pl}_{s-t}),$$
$$h_1(f_0)=\tfrac{1}{2}n-\max\{t,1\}\cdot\tfrac{1}{2}ql-\max\{s-t,1\}\cdot\tfrac{1}{2}pl\in\mathbb{Z}_n \quad \text{(if defined)},$$
$$h_2(f_0)=\{\pm k,\pm(l/2-k)\}\subset\mathbb{Z}_{l/2},$$

where k is the smallest positive integer that satisfies

$$k\equiv qm_0 \pmod{p},\quad k\equiv p \pmod{2q}\quad \textit{and}\quad \gcd(k,n)=1,$$

and m_0 is the smallest positive integer that satisfies

$$m_0\equiv \tfrac{1}{2}l+1 \pmod{l}\quad \textit{and}\quad \gcd(m_0,p)=1.$$

Moreover, the conjugacy class of such an $\langle f\rangle$ is uniquely determined by the parameters n, h and s if (1) *happens*; *by n, h, s and l if* (2) *happens*; *and by n, h, s, t, p and q if* (3) *happens.*

notation	model $S^3 = \mathbb{R}^3 \cup \{\infty\}$	model $S^3 \subset \mathbb{C}^2$
0	$(0,0,0)$	$(0,1)$
∞	∞	$(0,-1)$
Z	$\{(0,0,z) : z \in \mathbb{R}\} \cup \{\infty\}$	$\{(w_1,w_2) : w_1 = 0, \lvert w_2\rvert = 1\}$
S^1	$\{(x,y,0) : x^2+y^2 = 1\}$	$\{(w_1,w_2) : \lvert w_1\rvert = 1, w_2 = 0\}$
S^2	$\{(x,y,z) : x^2+y^2+z^2 = 1\}$	$\{(w_1,w_2) : \mathrm{Re}(w_2) = 0, \lvert w_1\rvert^2+\lvert w_2\rvert^2 = 1\}$

Table 1: Notation for identified objects in two models of S^3.

3 Basic examples of extendable maps

Before proving the main theorems, we give some basic examples of extendable maps. In each of them we choose a $\phi \in \mathrm{Aut}(S^3)$ of order n and an embedded surface Σ which is ϕ–invariant; thus ϕ induces a surface automorphism $\phi|_\Sigma$. In the next section we will show that every periodic extendable map can be constructed from these examples just with some inessential modifications.

Two models for S^3 will be used. One is $\mathbb{R}^3 \cup \{\infty\}$, where ϕ acts as an element of $O(3)$. The other is the unit sphere in $\mathbb{C}^2$,

$$\{(w_1,w_2) \in \mathbb{C}^2 : |w_1|^2+|w_2|^2 = 1\}.$$

Write $w_1 = x_1+iy_1$ and $w_2 = x_2+iy_2$ with $x_1,y_1,x_2,y_2 \in \mathbb{R}$. Then ϕ acts as an element of $O(4)$ on $S^3 \subset \mathbb{C}^2 \cong \mathbb{R}^4 \cong \mathbb{R}\langle x_1,y_1,x_2,y_2\rangle$. The two models are connected by a stereographic projection in $\mathbb{C}^2 \cong \mathbb{R}^4$:

$$\{(w_1,w_2) \in \mathbb{C}^2 : |w_1|^2+|w_2|^2 = 1\} \to \mathbb{R}^3 \cup \{\infty\}, \quad (x_1+iy_1, x_2+iy_2) \mapsto \left(\frac{x_1}{1+x_2}, \frac{y_1}{1+x_2}, \frac{y_2}{1+x_2}\right).$$

It identifies some objects in the two models; see Table 1, where $\mathrm{Re}(\cdot)$ denotes the real part of a complex number. The notation will be used in this and the next two sections.

3.1 Type $(+,+)$

Example 3.1 The automorphism ϕ acts on $S^3 \subset \mathbb{C}^2$ as $\phi(w_1,w_2) = (w_1e^{2\pi i/n}, w_2e^{2\pi i/n})$, ie a $2\pi/n$–rotation on both coordinate components. Let Σ be the torus

$$T = \left\{(w_1,w_2) \in \mathbb{C}^2 : |w_1| = |w_2| = \tfrac{\sqrt{2}}{2}\right\}.$$

As long as $n > 1$ (as is always assumed), ϕ^m for $m = 1,2,\dots,n-1$ has no fixed point on S^3; thus Σ/ϕ is a closed orientable surface, and by the Riemann–Hurwitz formula (or just by topological observation) we know it is a torus. So $\phi|_\Sigma$ satisfies the conditions in Theorem 2.3 with $g = 1$, $h = 1$, $s = t = 0$ and $p = q = 1$.

3.2 Type $(-,-)$

Example 3.2 $S^3 = \mathbb{R}^3 \cup \{\infty\}$, n is even, and $\phi \in O(3)$ has the matrix

$$\begin{pmatrix} \cos(2p\pi/n) & -\sin(2p\pi/n) & 0 \\ \sin(2p\pi/n) & \cos(2p\pi/n) & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

ie ϕ is the composition of a $(2p\pi/n)$–rotation around the z–axis and a reflection across the xy–plane.

(1) Let Σ be the unit sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

  (i) If $p = 1$, then $\phi|_\Sigma$ satisfies Theorem 2.4(2) or (3) with $g = 0$, $h = 1$, $s = 1$ and $\alpha = 1$. Here we do not check $h_1(\phi|_\Sigma)$ for it must be as in the theorem, according to Section 5. In the degenerate case $n = 2$, we actually have $s = 0$, which is inessential in our discussion. Similarly, we will also omit the calculation of the invariants h_1 and h_2, and ignore the degenerate cases below.

  (ii) If $p = 2$ and $n/2$ is odd, then $\phi|_\Sigma$ satisfies Theorem 2.4(1) with $g = 0$, $h = 0$, $b = 1$, $s = 1$, $t = 1$ and $\alpha = 1$.

(2) Let Σ be the boundary of a ϕ–invariant regular neighborhood of the circle

$$S^1 = \{(x,y,z) \in \mathbb{R}^3 : z = 0, x^2 + y^2 = 1\}.$$

For instance, Σ can be chosen as the torus T in Example 3.1.

  (i) If $p = 1$, then $\phi|_\Sigma$ satisfies Theorem 2.4(2) or (3) with $g = 1$, $h = 2$, $s = 0$ and $\alpha = 1$.

  (ii) If $p = 2$ and $n/2$ is odd, then $\phi|_\Sigma$ satisfies Theorem 2.4(1) with $g = 1$, $h = 0$, $b = 2$, $s = 0$, $t = 1$ and $\alpha = 1$.

(3) Suppose $p = 2$ and $n/2$ is odd. Let β be the line segment connecting the points $(0,0,1)$ and $(0,1,0)$ in $\mathbb{R}^3$. Then the union of $\phi^m(\beta)$ for $1 \le m \le n$ is a connected graph; see Figure 3, left. Choose a ϕ–invariant regular neighborhood and let Σ be its boundary. Then $\phi|_\Sigma$ satisfies Theorem 2.4(1) with $g = n/2 - 1, h = 0, b = 1, s = 2, t = 1$ and $\alpha = 1$.

(4) Suppose $p = 2$ and $n/2$ is odd. Let C_+ be the circle

$$\{(x,y,z) \in \mathbb{R}^3 : z = 1, x^2 + y^2 = 1\},$$

and β' the line segment connecting the points $(0,1,1)$ and $(0,1,0)$ in $\mathbb{R}^3$. Then the union of $\phi^m(C_+ \cup \beta')$ for $1 \le m \le n$ is a connected graph; see Figure 3, right. Choose a ϕ–invariant regular neighborhood and let Σ be its boundary. Then $\phi|_\Sigma$ satisfies Theorem 2.4(1) with $g = n/2 + 1$, $h = 1$, $b = 1$, $s = 0$ and $t = 0$.

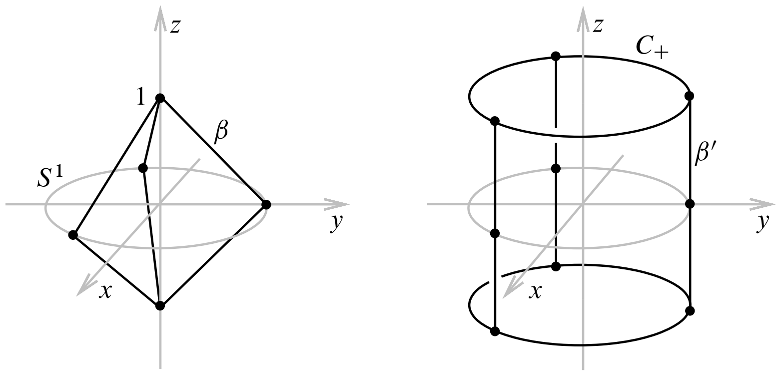

Figure 3: ϕ–invariant graphs, $n = 6$.

3.3 Type $(+,-)$

Example 3.3 $S^3 = \mathbb{R}^3 \cup \{\infty\}$, and $\phi \in O(3)$ has the same matrix as in the last example with n even and $p = 1$.

(1) Let Σ be the sphere xy–plane $\cup \{\infty\}$. Then $\phi|_\Sigma$ satisfies the conditions in Theorem 2.5 with $g = 0, h = 0, b = 0, s = 2$ and $\alpha = 1$.

(2) Choose an arc
$$\gamma = \{(x, y, z) \in \mathbb{R}^3 : y^2 + (z-1)^2 = 4, y \geq 0, x = 0\},$$
and write $\Gamma_1 = \bigcup_{m=1}^{n/2} \phi^{2m}(\gamma)$ and $\Gamma_2 = \phi(\Gamma_1)$; see Figure 4. Let Σ be defined by
$$\{X \in \mathbb{R}^3 : \mathrm{dist}(X, \Gamma_1) = \mathrm{dist}(X, \Gamma_2)\} \cup \{\infty\},$$
where dist is the Euclidean distance in $\mathbb{R}^3$. The two components of $S^3 - \Sigma$ are regular neighborhoods of Γ_1 and Γ_2, so Σ is a Heegaard surface of genus $n/2-1$. As ϕ exchanges Γ_1 and Γ_2, it preserves Σ. If $n \geq 4$, there are three singular points in the quotient orbifold Σ/ϕ, which are provided by the

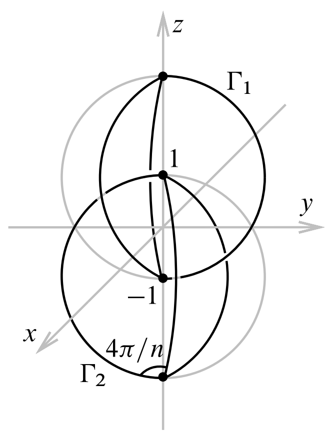

Figure 4: A ϕ–invariant graph with two components, $n = 6$.

orbits $\{(0,0,0)\}$, $\{(0,0,\pm 2)\}$, $\{\infty\}$; if $n=2$, the orbit $\{(0,0,\pm 2)\}$ gives a regular point so there are only two singular points. Now by the Riemann–Hurwitz formula we see that $\phi|_\Sigma$ satisfies the conditions in Theorem 2.5 with $g=n/2-1$, $h=0$, $b=0$, $s=3$ (if $n=2$ then $s=2$) and $\alpha=1$.

3.4 Type $(-,+)$

Example 3.4 $S^3 \subset \mathbb{C}^2$ and $\phi(w_1,w_2)=(w_1 e^{2\pi i/(n/2)}, -w_2)$, where n is even and $n/2$ is odd. Let Σ be the sphere

$$S^2=\{(w_1,w_2)\in S^3 \subset \mathbb{C}^2 : \mathrm{Re}(w_2)=0\}.$$

Then $\phi|_\Sigma$ satisfies Theorem 2.6(1) with $g=0, h=0, b=1, s=1$ (if $n=2$ then $s=0$).

Example 3.5 Σ is chosen as the union of ∞ and a plane in $\mathbb{R}^3$ passing through the origin. Let ϕ act on $S^3\subset\mathbb{C}^2$ as $\phi(w_1,w_2)=(-w_1,-w_2)$. Then $\phi|_\Sigma$ satisfies Theorem 2.6(3) with $g=0$, $h=1$, $b=0$, $s=t=0$, $p=q=1$ and $l=n=2$.

Example 3.6 The torus

$$T=\Big\{(w_1,w_2)\in\mathbb{C}^2 : |w_1|=|w_2|=\tfrac{\sqrt{2}}{2}\Big\}$$

splits $S^3\subset\mathbb{C}^2$ into two solid tori,

$$V_1=\Big\{(w_1,w_2)\in S^3\subset\mathbb{C}^2 : |w_1|\geq\tfrac{\sqrt{2}}{2}\Big\} \quad\text{and}\quad V_2=\Big\{(w_1,w_2)\in S^3\subset\mathbb{C}^2 : |w_2|\geq\tfrac{\sqrt{2}}{2}\Big\}.$$

The $(2,2)$–torus link

$$L=\Big\{\Big(\tfrac{\sqrt{2}}{2}e^{i\theta}, \pm\tfrac{\sqrt{2}}{2}e^{i\theta}\Big) : 0\leq\theta\leq 2\pi\Big\}$$

on T bounds an annulus in V_1:

$$\Big\{(re^{i\theta}, \pm\sqrt{1-r^2}e^{i\theta}) : 0\leq\theta\leq 2\pi, \tfrac{\sqrt{2}}{2}\leq r\leq 1\Big\},$$

which divides V_1 into two solid tori $V_{1,1}$ and $V_{1,2}$; see Figure 5.

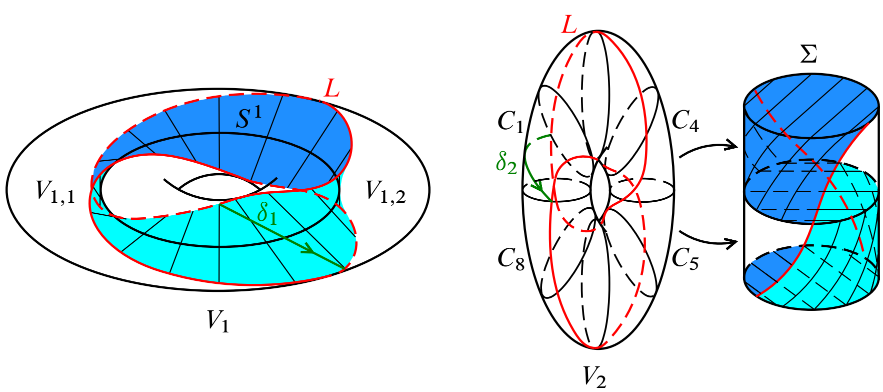

Figure 5: A genus 1 Heegaard surface in S^3, $d=4$.

Fix a positive number d, and cut V_2 into $2d$ cylinders

$$C_k = \left\{ (w_1, w_2) \in S^2 : |w_2| \geq \tfrac{\sqrt{2}}{2}, \frac{(k-1)\pi}{d} \leq \operatorname{Arg}(w_2) < \frac{k\pi}{d} \right\} \quad \text{for } 1 \leq k \leq 2d.$$

Define

$$H_1 = V_{1,1} \cup C_1 \cup C_3 \cup \cdots \cup C_{2d-1} \quad \text{and} \quad H_2 = V_{1,2} \cup C_2 \cup C_4 \cup \cdots \cup C_{2d},$$

and let Σ be $\partial H_1 = \partial H_2$. For each $k = 1, 2, \dots, 2d$, L divides the annulus $T \cap \partial C_k$ into two disks, so each of $V_{1,1}, V_{1,2}$ intersects C_k at a disk, which implies H_1, H_2 are both solid tori and Σ is a genus 1 Heegaard surface of S^3.

(1) Suppose d is odd and consider the automorphism of S^3 defined by

$$\phi(w_1, w_2) = (w_1 e^{2\pi i/d}, w_2 e^{2\pi i(d+2)/2d}).$$

Decompose it as

$$(w_1, w_2) \xrightarrow{\phi_1} (w_1 e^{2\pi i/d}, w_2 e^{2\pi i/d}) \xrightarrow{\phi_2} (w_1 e^{2\pi i/d}, w_2 e^{2\pi i(1/d+1/2)}).$$

On V_1, ϕ_1 preserves each component of L while ϕ_2 does not, thus ϕ exchanges them and also the solid tori $V_{1,1}$ and $V_{1,2}$. On V_2, ϕ sends C_k to C_{k+2+d}, thus changes the parity of the subscript. Therefore, ϕ exchanges H_1 and H_2 and induces an f on Σ, which satisfies Theorem 2.6(2) with $n = 2d$, $g = 1$, $h = 1$, $b = 1$, $s = 0$ and $l = d$.

(2) Suppose d is even and consider the automorphism ϕ given by the composition

$$(w_1, w_2) \to (w_1 e^{2\pi i/2d}, w_2 e^{2\pi i/2d}) \to (w_1 e^{2\pi i/2d}, w_2 e^{2\pi i(1/2d+1/2)}).$$

Similarly, it induces a periodic map on Σ which satisfies Theorem 2.6(3) with $n = 2d$, $g = 1$, $h = 2$, $b = 0$, $s = t = 0$, $p = q = 1$ and $l = 2d$. Then $h_1(\phi|_\Sigma)$ and $h_2(\phi|_\Sigma)$ must be as in the theorem, though we can compute them directly. Choose a canonical generator system of $\pi_1(\Sigma/\phi)$:

- δ_1, represented by the oriented geodesic segment on $\Sigma \cap V_1$ from
$$\left(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right) \quad \text{to} \quad \left(\tfrac{\sqrt{2}}{2} e^{2\pi i/2d}, \tfrac{\sqrt{2}}{2} e^{2\pi i(1+d)/2d}\right),$$
- δ_2, represented by the oriented curve on $\Sigma \cap C_1$ from
$$\left(\tfrac{\sqrt{2}}{2} e^{2\pi i(d+1)/2d}, \tfrac{\sqrt{2}}{2} e^{2\pi i/2d}\right) \quad \text{to} \quad \left(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right).$$

By the definition of the epimorphism ψ, the deck transformation $\phi^{\psi(\delta_1)}$ sends the initial $\left(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}\right)$ to the terminal

$$\left(\tfrac{\sqrt{2}}{2} e^{2\pi i/2d}, \tfrac{\sqrt{2}}{2} e^{2\pi i(1+d)/2d}\right).$$

So $\psi(\delta_1)$ is 1; similarly, $\psi(\delta_2) = d-1$. Hence, $h_1(\phi|_\Sigma) = d \in \mathbb{Z}_{2d}$ and $h_2(\phi|_\Sigma) = \{\pm 1, \pm(d-1)\} \subset \mathbb{Z}_d$.

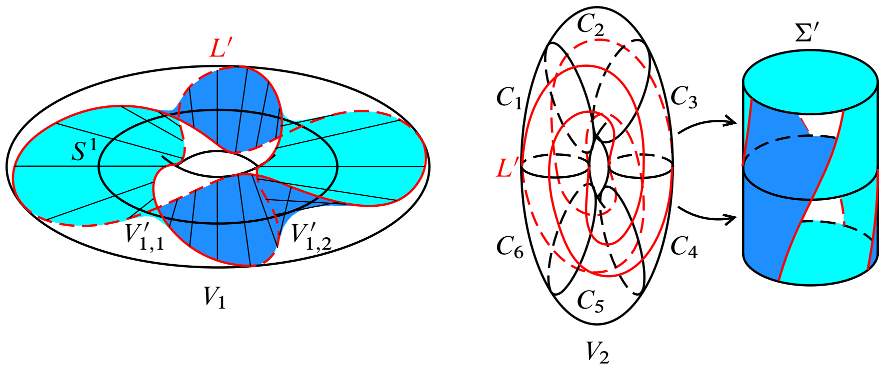

Figure 6: A genus $d+1$ Heegaard surface in S^3, $d=3$.

Example 3.7 Replace the $(2,2)$–torus link L in the last example by the $(2,4)$–torus link

$$L' = \left\{\left(\tfrac{\sqrt{2}}{2}e^{i\theta}, \pm\tfrac{\sqrt{2}}{2}e^{2i\theta}\right) : 0 \le \theta \le 2\pi\right\},$$

and define the corresponding handlebodies $V'_{1,1}$, $V'_{1,2}$, H'_1 and H'_2 similarly. Then we obtain another Heegaard splitting of S^3; see Figure 6. Note that now each of $V'_{1,1}$ and $V'_{1,2}$ intersects C_k at two disks, hence the Heegaard surface $\Sigma' = \partial H'_1 = \partial H'_2$ has genus $d+1$.

(1) Suppose d is odd and consider the automorphism of S^3 defined by the composition

$$(w_1, w_2) \to (w_1 e^{2\pi i/d}, w_2 e^{2\pi i\cdot 2/d}) \to (w_1 e^{2\pi i/d}, w_2 e^{2\pi i(2/d+1/2)}).$$

With the Riemann–Hurwitz formula, we see that it induces a periodic map on Σ' which satisfies Theorem 2.6(2) with $n=2d$, $g=d+1$, $h=2$, $b=1$, $s=0$ and $l=d$.

(2) Suppose d is odd and consider the composition

$$(w_1, w_2) \to (w_1 e^{2\pi i/2d}, w_2 e^{2\pi i\cdot 2/2d}) \to (w_1 e^{2\pi i/2d}, w_2 e^{2\pi i(2/2d+1/2)}).$$

Then the induced map on Σ' satisfies Theorem 2.6(3) with $n=2d$, $g=d+1$, $h=3$, $b=0$, $s=t=0$, $p=q=1$ and $l=2d$.

4 Realizations of the extensions

Suppose that a periodic map f on a closed surface satisfies the extension conditions as in one of Theorems 2.3–2.6. We are going to prove that it is extendable over S^3 in the corresponding type.

For each case, we will show first that, up to conjugacy, the cyclic group $\langle f\rangle$ is determined by the invariants used in the theorems. The strategy is to show that fixing a possible collection of such invariants, the following assertions hold:

(1) The isotropy invariant of f can be normalized. That is to say, there exists some integer m coprime to n (thus $\langle f^m\rangle = \langle f\rangle$) such that the isotropy invariant for f^m has a "normal" form uniquely determined by the given invariants. The "normalizations" below are the same.

(2) If $h_1(f)$ exists, ie $|\Sigma_g/f|$ is nonorientable without boundary, $n/2$ is even and there is no $n/2$ in the isotropy invariant, then $h_1(f)$ can be normalized without changing the isotropy invariant.

(3) If $h_2(f)$ exists, ie $|\Sigma_g/f|$ is nonorientable with genus $h = 2$, then $h_2(f)$ can be normalized without changing the isotropy invariant and $h_1(f)$ (if defined).

Then we have an integer m coprime to n, such that the isotropy invariant of f^m and $h_1(f^m), h_2(f^m)$ (if defined) have a "normal" form, which is uniquely determined by the given invariants. Moreover, the genus g of the surface can be figured out from the Riemann–Hurwitz formula. According to Theorem 2.2, the conjugacy class of f^m is determined. Therefore, $\langle f\rangle = \langle f^m\rangle$ is also determined.

It then suffices to realize extendable maps with any possible invariants involved. We will construct them by modifying the examples in Section 3, so that Theorem 1.2 follows.

The following lemmas will be used to normalize some cases in our discussion.

Lemma 4.1 *Suppose $f \in \mathrm{Aut}(\Sigma_g)$ has period n. Let k, m be integers with $km \equiv 1 \pmod{n}$. Denote the corresponding epimorphisms for f and f^m by*

$$\psi : \pi_1(\Sigma_g/f) \to \langle f\rangle = \mathbb{Z}_n, \quad \textit{identifying} \quad f \leftrightarrow 1,$$
$$\psi' : \pi_1(\Sigma_g/f) \to \langle f^m\rangle = \mathbb{Z}_n, \quad \textit{identifying} \quad f^m \leftrightarrow 1,$$

respectively. Then for each $\gamma \in \pi_1(\Sigma_g/f)$, we have $\psi'(\gamma) = k\psi(\gamma)$.

Proof Naturally we have $(f^m)^{\psi'(\gamma)} = f^{\psi(\gamma)}$, so the conclusion follows. □

Lemma 4.2 *Suppose $f \in \mathrm{Aut}(\Sigma_g)$ has period n, and m is a factor of n. Let f_m be the order m map induced by f on $|\Sigma_g/f^m|$, and*

$$\psi_m : \pi_1(|\Sigma_g/f^m|/f_m) \to \langle f_m\rangle = \mathbb{Z}_m, \quad \textit{identifying} \quad f_m \leftrightarrow 1,$$

be the corresponding epimorphism. Then there is a commutative diagram for ψ_m and ψ,

$$\begin{array}{ccc} \pi_1(\Sigma_g/f) & \xrightarrow{\ \psi\ } & \mathbb{Z}_n \\ {\scriptstyle F_*}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \bmod m} \\ \pi_1(|\Sigma_g/f^m|/f_m) & \xrightarrow{\ \psi_m\ } & \mathbb{Z}_m \end{array}$$

where F_* is induced by the natural forgetful map.

Proof Fix a basepoint for $\pi_1(\Sigma_g/f)$. It is a regular orbit $X \subset \Sigma_g$ under the f–action. Fix $x_0 \in X$. For $\gamma \in \pi_1(\Sigma_g/f)$, γ can be represented by the image of an oriented curve in Σ_g that connects x_0 to some $x \in X$. Then $\psi(\gamma) \in \mathbb{Z}_n$ is determined by the equation

$$f^{\psi(\gamma)}(x_0) = x.$$

On the other hand, the oriented curve descends to $|\Sigma_g/f^m|$ and connects $[x_0]$ to $[x]$. It represents $F_*\gamma$ in $\pi_1(|\Sigma_g/f^m|/f_m)$. So $\psi_m(F_*\gamma)\in\mathbb{Z}_m$ is determined by the equation

$$(f_m)^{\psi_m(F_*\gamma)}([x_0])=[x].$$

Obviously, $\psi(\gamma)$ satisfies the equation, so $\psi_m(F_*\gamma)\equiv\psi(\gamma) \pmod m$. □

Lemma 4.3 (1) *Suppose* $\gcd(a,b,c)=1$. *Then there exists an integer* d *such that* $\gcd(a+bd,c)=1$.

(2) *If* $\gcd(m,n)=p$, *then there exists an integer* k *such that* $\gcd(k,n)=1$ *and* $km\equiv p \pmod n$.

(3) *Suppose* p, q, p_0 *and* q_0 *are integers. The congruence equation system*

$$\begin{cases} x\equiv p_0 \pmod p,\\ x\equiv q_0 \pmod q,\end{cases}$$

has a solution $x\in\mathbb{Z}$ *if and only if* $p_0\equiv q_0 \pmod{\gcd(p,q)}$. *Moreover, if it has a solution, then the solution is unique modulo the least common multiple* $\operatorname{lcm}(p,q)$.

(4) *Suppose* λ *and* $\mu\in\mathbb{Z}_n$ *have orders* p *and* q, *respectively, and* $\gcd(p,q)=1$, *so* $n=pql$. *Then there exists a generator* τ *of* $\mathbb{Z}_n$ *such that* $\lambda=\tau ql$ *and* $\mu=\tau pl$.

Proof (1) Factorize c into $p_1^{s_1}\cdots p_I^{s_I}q_1^{t_1}\cdots q_J^{t_J}$, where p_i,q_j are prime and $p_i\mid a$, $q_j\nmid a$ for each i,j. Let $d=q_1^{t_1}\cdots q_J^{t_J}$, and suppose $\gcd(a+bd,c)=c_0$. We only need to show $c_0=1$. For each p_i, we have $p_i\mid a$, $p_i\mid c$, $p_i\nmid b$, $p_i\nmid d$; for each q_j, we have $q_j\nmid a$, $q_j\mid d$, $q_j\mid c$. So none of p_i,q_j divides $a+bd$, thus c_0 must equal 1.

(2) Assume $m=m_0p$ and $n=n_0p$; then $\gcd(m_0,n_0)=1$. Let k_1 satisfy $k_1m_0\equiv 1 \pmod{n_0}$. By (1), we just choose $k=k_1+n_0d\ (d\in\mathbb{Z})$ such that $\gcd(k,n)=1$.

(3) If x exists, assume $x=p_0+k_1p=q_0+k_2q$, where $k_1,k_2\in\mathbb{Z}$. Then $p_0-q_0=-k_1p+k_2q$, so $p_0\equiv q_0 \pmod{\gcd(p,q)}$.

Conversely, if $p_0\equiv q_0 \pmod{\gcd(p,q)}$, there are integers k, a and b such that $p_0-q_0=k\cdot\gcd(p,q)=k(ap+bq)$. Then $x=p_0-kap=q_0+kbq$ is a solution of the equations.

If x and x' are both solutions, then $x\equiv x' \pmod p$ and $x\equiv x' \pmod q$, so $x\equiv x' \pmod{\operatorname{lcm}(p,q)}$.

(4) Assume $\lambda=\lambda_0ql$ and $\mu=\mu_0pl$ with $1\le\lambda_0<p$, $1\le\mu_0<q$ and $\gcd(\lambda_0,p)=\gcd(\mu_0,q)=1$. By (3), there exists an integer τ_0 such that $\tau_0\equiv\lambda_0 \pmod p$ and $\tau_0\equiv\mu_0 \pmod q$. Also, $\gcd(\tau_0,pq)=1$. By (1), we just choose $\tau=\tau_0+pqd\ (d\in\mathbb{Z})$ such that $\gcd(\tau,n)=1$. □

4.1 Type (+, +)

Proposition 4.4 *Suppose* f *satisfies the conditions in Theorem 2.3. Then the conjugacy class of* $\langle f\rangle$ *is uniquely determined by* n, h, s, t, p *and* q.

Proof We fix n, h, s, t, p and q and follow the strategy at the beginning of this section.

Assume $n = pql$ for some positive integer l. By Lemma 4.3(4), the isotropy invariant is

$$(\underbrace{\tau ql, \dots, \tau ql}_{t}, \underbrace{-\tau ql, \dots, -\tau ql}_{t}, \underbrace{\tau pl, \dots, \tau pl}_{s/2-t}, \underbrace{-\tau pl, \dots, -\tau pl}_{s/2-t}),$$

where τ is a generator of $\mathbb{Z}_n$. By Lemma 4.1, the isotropy invariant for f^τ is

$$(ql, \dots, ql, -ql, \dots, -ql, pl, \dots, pl, -pl, \dots, -pl).$$

Moreover, g can be figured out from the Riemann–Hurwitz formula,

$$2-2g = n\Big(2-2h-2t\Big(1-\frac{1}{p}\Big)-(s-2t)\Big(1-\frac{1}{q}\Big)\Big).$$

According to Theorem 2.2(1), f^τ is determined up to conjugacy and thus so is $\langle f\rangle = \langle f^\tau\rangle$. □

Maps extendable in type $(+,+)$ and the corresponding embedded surfaces have been described in [7]. We introduce another approach to constructing them as follows, and the strategy will be applied later to obtain all extendable maps in the other three types from the basic examples in Section 3.

Example 4.5 Let ϕ be id_{S^3} and Σ be a trivially embedded sphere in S^3 which does not intersect the two circles S^1 and Z; see Figure 7. Then of course $\phi|_\Sigma$ satisfies the conditions in Theorem 2.3 with parameters $g = h = 0$, $s = t = 0$ and $n = p = q = 1$. We can modify Σ as in the figure to get another ϕ–invariant surface Σ', which has larger genus and more intersection points with S^1 and Z. Then we consider a branched covering $S^3 \to S^3$ with branch sets $S^1 \cup Z$ (both upstairs and downstairs). Lifting Σ' from the downstairs S^3 to the upstairs S^3, we obtain a surface $\widetilde{\Sigma}$. Let $\widetilde{\phi}$ be a periodic automorphism of S^3 such that the quotient map $S^3 \to |S^3/\widetilde{\phi}| \cong S^3$ is exactly the branched covering. By this means, the extension for a map in Theorem 2.3 with any possible parameters $\widetilde{n} = \widetilde{p}\widetilde{q}$, $\widetilde{h}$, $\widetilde{s}$, $\widetilde{t}$, $\widetilde{p}$ and $\widetilde{q}$ can be realized as $(\widetilde{\phi}|_{\widetilde{\Sigma}}, \widetilde{\phi})$.

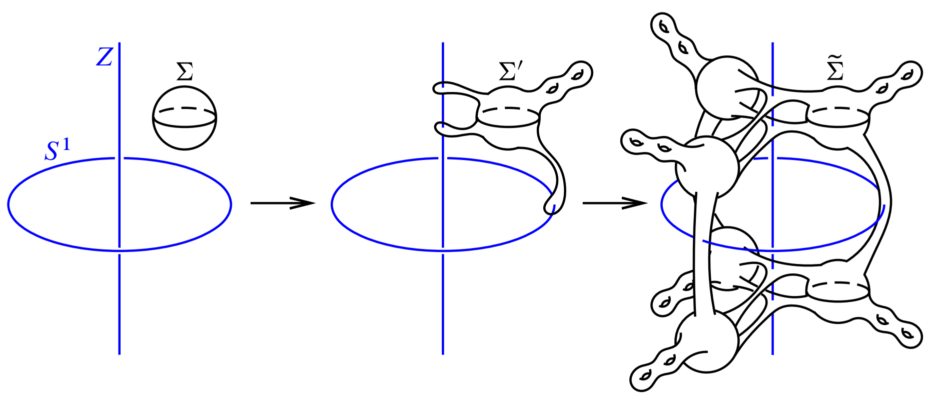

Figure 7: Constructing new extendable maps with surgeries, $\widetilde{n} = 6$, $\widetilde{h} = 2$, $\widetilde{s} = 6$, $\widetilde{t} = 1$, $\widetilde{p} = 2$, $\widetilde{q} = 3$.

Generally, suppose we have a periodic automorphism ϕ of S^3 and a ϕ–invariant Heegaard surface Σ such that the induced map $\phi|_\Sigma$ satisfies the conditions in Theorem 2.3 with parameters g, h, s, t, n, p and q. We can make modifications in a fundamental domain for the ϕ–action on (Σ, S^3), and extend them ϕ–equivariantly. In other words, we apply the following surgeries, and then lift the surface in $|S^3/\phi|$ to Σ' in S^3:

- **Genus surgery** Modifying the quotient surface $|\Sigma/\phi|$ in $|S^3/\phi|$ to increase the genus by adding a "handle".
- **Singular surgery** Modifying the quotient surface $|\Sigma/\phi|$ in $|S^3/\phi|$ to enlarge the intersection sets $|\Sigma/\phi| \cap |S^1/\phi|, |\Sigma/\phi| \cap |Z/\phi|$ by an isotopy which seems like "stretching out hands to grip the axes".

After that, take a branched covering $S^3 \to S^3$ with branch sets $S^1 \cup Z$ (both upstairs and downstairs). Lifting Σ' and ϕ from the downstairs S^3 to the upstairs S^3, we obtain a surface $\widetilde{\Sigma}$ and a periodic automorphism $\widetilde{\phi}$. In other words, we conduct the following surgery:

- **Branch surgery** Suppose ϕ acts on $S^3 \subset \mathbb{C}^2$ as
$$\phi(w_1, w_2) = (w_1 e^{2\pi i m_1/n}, w_2 e^{2\pi i m_2/n})$$
with $\gcd(n, m_1, m_2) = 1$. Let $\widetilde{\phi}\colon S^3 \to S^3$ be the map defined by
$$\widetilde{\phi}(w_1, w_2) = (w_1 e^{2\pi i m_1/nn_1}, w_2 e^{2\pi i m_2/nn_2})$$
with $\gcd(m_1, n_1) = \gcd(m_2, n_2) = \gcd(n_1, n_2) = 1$. Then the $\widetilde{\phi}^n$–action induces a branched covering map
$$\omega\colon S^3 \to |S^3/(\widetilde{\phi})^n| \cong S^3, \quad (w_1, w_2) \mapsto (|w_1| e^{i \cdot n_1 \cdot \operatorname{Arg}(w_1)}, |w_2| e^{i \cdot n_2 \cdot \operatorname{Arg}(w_2)}).$$
For a ϕ–invariant surface $\Sigma' \subset |S^3/(\widetilde{\phi})^n| \cong S^3$, write $\widetilde{\Sigma} = \omega^{-1}(\Sigma')$.

As $|\widetilde{\Sigma}/\widetilde{\phi}| \cong |\Sigma'/\phi|$, we have constructed a new extendable map $\widetilde{\phi}|_{\widetilde{\Sigma}}$, which still satisfies the conditions in Theorem 2.3. Denote the corresponding parameters by $\widetilde{g}$, $\widetilde{n}$, $\widetilde{h}$, $\widetilde{s}$, $\widetilde{t}$, $\widetilde{p}$ and $\widetilde{q}$. We see that $\widetilde{h} - h$ is the genus of the added "handle" in the genus surgery; $2\widetilde{t}, \widetilde{s} - 2\widetilde{t}$ are the cardinalities of $|\Sigma'/\phi| \cap |S^1/\phi|$ and $|\Sigma'/\phi| \cap |Z/\phi|$, respectively; $\widetilde{p} = pn_2$, $\widetilde{q} = qn_1$; $\widetilde{n} = nn_1n_2$; and $\widetilde{g}$ is determined by the Riemann–Hurwitz formula. Moreover, $\widetilde{\Sigma}$ can be chosen as a Heegaard surface in S^3.

Now from Example 3.1 we have an extendable map with parameters $n > 1$, $h = 1$, $s = t = 0$ and $p = q = 1$. Apply the three surgeries. Similarly, we can obtain each extendable map with possible parameters $\widetilde{g}$, $\widetilde{n}$, $\widetilde{h}$, $\widetilde{s}$, $\widetilde{t}$, $\widetilde{p}$ and $\widetilde{q}$ with $\widetilde{n} > \widetilde{p}\widetilde{q}$. Note that in this case $\widetilde{h}$ must be nonzero for otherwise the corresponding epimorphism $\widetilde{\psi}$ cannot be surjective. Also, during the surgeries we can always make $\widetilde{\Sigma}$ a Heegaard surface of S^3.

So far we have proved that for a surface map satisfying the conditions in Theorem 2.3, it is extendable in type $(+,+)$. Moreover, according to its parameters, we can construct such a map together with an extension as above. For a direct (and essentially the same) description, readers may refer to [7].

4.2 Type $(-,-)$

Proposition 4.6 (1) *Suppose f satisfies Theorem 2.4*(1)*. Then the conjugacy class of $\langle f\rangle$ is uniquely determined by n, h, b, s and t.*

(2) *Suppose f satisfies Theorem 2.4*(2) *or* (3)*. Then the conjugacy class of $\langle f\rangle$ is uniquely determined by n, h and s.*

Proof (1) Fix n, h, b, s and t. Just as in the proof of Proposition 4.4, we may figure out g from the Riemann–Hurwitz formula, and normalize the isotropy invariant to make $\alpha = 1 \in \mathbb{Z}_n$. Then by Theorem 2.2(1), $\langle f\rangle$ is determined up to conjugacy.

(2) Fix n, h and s. Similarly figure out g. We first normalize α to be $1 \in \mathbb{Z}_n$ and normalize $h_1(f)$ (if defined) to be $-s \in \mathbb{Z}_n$ as below. Consider f^α. Denote by ψ' the corresponding epimorphism for f^α. Fix a canonical generator system

$$\mathscr{G} = \{\delta_1, \dots, \delta_h, \xi_1, \dots, \xi_s\}$$

of $\pi_1(\Sigma_g/f)$. Without loss of generality, assume that $\psi(\xi_i) = 2\alpha$ for $i = 1, 2, \dots, s_0$ and $\psi(\xi_j) = -2\alpha$ for $j = s_0+1, \dots, s$. By Lemma 4.3 we have $\psi'(\xi_i) = 2$ for $i = 1, 2, \dots, s_0$ and $\psi'(\xi_j) = -2$ for $j = s_0+1, \dots, s$. So the isotropy invariant of f^α is $(\pm 2, \pm 2, \dots, \pm 2)$.

In the following two cases, $h_1(f)$ is not defined:

- $n/2$ is odd;
- $n = 4$ and $s \neq 0$, thus we have $2\alpha = 2 = n/2$ in the isotropy invariant.

So the normalization is straightforward, just by replacing f with f^α.

If $n/2$ is even, $n > 4$ and s is even, or $n = 4$ with $s = 0$, then $h_1(f)$ satisfies the condition (3) in the theorem. We only need to check that $h_1(f^\alpha) = -s$, so we can replace f by f^α directly. If $2\alpha \in \{2, 4, \dots, n/2-2\} \subset \mathbb{Z}_n$, by definition we have

$$h_1(f) = \sum_{i=1}^{h} \psi(\delta_i) - (s - s_0) \times 2\alpha \in \mathbb{Z}_n.$$

Therefore,

$$\sum_{i=1}^{h} \psi(\delta_i) = h_1(f) + (s - s_0) \times 2\alpha = -s\alpha + (s - s_0) \times 2\alpha = (s - 2s_0)\alpha.$$

It follows that $\sum_{i=1}^{h} \psi'(\delta_i) = s - 2s_0$ and

$$h_1(f^\alpha) = \sum_{i=1}^{h} \psi'(\delta_i) - 2(s - s_0) = -s \in \mathbb{Z}_n.$$

If $2\alpha \in \{n/2+2, n/2+4, \dots, n-2\}$, a similar check shows that $\sum_{i=1}^{h} \psi(\delta_i)$ equals $(s-2s_0)\alpha$ and $h_1(f^\alpha)$ equals $-s$ as well.

In the remaining case, where $n/2$ is even, $n > 4$ and s is odd, also we have

$$h_1(f^\alpha) = \sum_{i=1}^{h} \psi'(\delta_i) - 2(s - s_0) \in \mathbb{Z}_n.$$

Therefore,

$$2h_1(f^\alpha) + 2s = 2\sum_{i=1}^{h} \psi'(\delta_i) + \sum_{j=1}^{s} \psi'(\xi_j) = \psi'\Big(\prod_{i=1}^{h} \delta_i^2 \prod_{j=1}^{s} \xi_j\Big) = 0 \in \mathbb{Z}_n.$$

So $h_1(f^\alpha) = -s$ or $h_1(f^\alpha) = -s + n/2$. If $h_1(f^\alpha) = -s$, we just replace f by f^α; otherwise we replace f by $f^{(n/2+1)\alpha}$. In fact, if $h_1(f^\alpha) = -s + n/2$, we can check that the isotropy invariant of $f^{(n/2+1)\alpha}$ is also $(\pm 2, \pm 2, \dots, \pm 2)$, and

$$\begin{aligned} h_1(f^{(n/2+1)\alpha}) = \big(\tfrac{1}{2}n + 1\big)\sum_{i=1}^{h} \psi'(\delta_i) - 2(s - s_0) &= \big(\tfrac{1}{2}n + 1\big)\big(-s + \tfrac{1}{2}n + 2(s - s_0)\big) - 2(s - s_0) \\ &= -s \in \mathbb{Z}_n. \end{aligned}$$

Now we have finished the normalization of the isotropy invariant and $h_1(f)$. It suffices to normalize $h_2(f)$ for the case $h = 2$. If $s > 0$, $h_2(f)$ must be $\{\pm 1, \pm 1\} \subset \mathbb{Z}_2$ as $\gcd(2\alpha, n) = 2$, so $\langle f \rangle$ is determined. Assume $h = 2, s = 0$ below, thus by definition and Lemma 4.2, we have

$$h_2(f) = \{\pm\psi(\delta_1), \pm\psi(\delta_2)\} \subset \mathbb{Z}_d \quad \text{and} \quad d = \gcd(\psi(\delta_1) + \psi(\delta_2), n).$$

If $n/2$ is odd, from the relation

$$2(\psi(\delta_1) + \psi(\delta_2)) = \psi(\delta_1^2\delta_2^2) = \psi(1) = 0 \in \mathbb{Z}_n$$

we see that $\psi(\delta_1) + \psi(\delta_2) = 0 \in \mathbb{Z}_n$ since $\psi(\delta_1)$ and $\psi(\delta_2)$ are both odd. If $n/2$ is even, with the assumptions in the theorem we also have $\psi(\delta_1) + \psi(\delta_2) = h_1(f) = 0$. So $d = n$ holds for either case. Now ψ is surjective, so $\psi(\delta_1)$ and $\psi(\delta_2)$ are both coprime to n. Choose $m = \psi(\delta_1)$, then f^m normalizes the invariant h_2. In fact, let ψ'' be the corresponding epimorphism for f^m, then by Lemma 4.1 we have $\psi''(\delta_1) = 1$ and $\psi''(\delta_2) = -1$. Therefore,

$$h_2(f^m) = \{\pm\psi''(\delta_1), \pm\psi''(\delta_2)\} = \{\pm 1, \pm(n-1)\} \subset \mathbb{Z}_n.$$

Moreover, f and f^m have no isotropy invariant, and $h_1(f^m)$, if it exists, equals $\psi''(\delta_1) + \psi''(\delta_2) = 0 = h_1(f)$. As a consequence, the conjugacy class of $\langle f \rangle = \langle f^m \rangle$ is uniquely determined. □

For type $(-,-)$, extendable maps have been constructed in Costa's fine work [4]. It was not explained why his examples are all the extendable ones, and Proposition 4.6 irons out this flaw. Also, it is convenient to realize them from Example 3.2 with surgeries.

Example 4.7 Suppose f satisfies the condition (1) in Theorem 2.4 with parameters n, $h = 2$, $b = 6$, $s = 5$ and $t = 4$. Then it can be realized from Example 3.2(1)(ii). In fact, the constructed surface Σ_g can also be directly described as follows (just with a little deformation). See Figure 8.

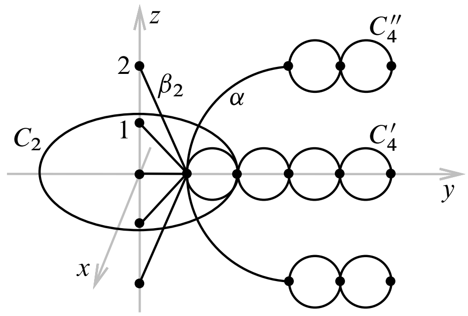

Figure 8: The graph Γ $(I=2, J=1, K=3, L=2)$.

Write

$$\begin{aligned} C_j &= \{(x,y,z)\in\mathbb{R}^3 : z=0, x^2+y^2=j^2\},\\ C_k' &= \bigl\{(x,y,z)\in\mathbb{R}^3 : x=0, \bigl(y-\bigl(k+\tfrac{1}{2}\bigr)\bigr)^2+z^2=\tfrac{1}{4}\bigr\},\\ C_l'' &= \bigl\{(x,y,z)\in\mathbb{R}^3 : x=0, \bigl(y-\bigl(l+\tfrac{1}{2}\bigr)\bigr)^2+(z-2)^2=\tfrac{1}{4}\bigr\},\\ \alpha &= \{(x,y,z)\in\mathbb{R}^3 : x=0, y\le 3, z\ge 0, (y-3)^2+z^2=4\},\end{aligned}$$

and let β_i be the line segment connecting the points $(0,0,i), (0,1,0)$. Choose $\phi\in\mathrm{Aut}(S^3)$ as in Example 3.2(1)(ii), ie the composition of a $2\pi/(n/2)$–rotation around the z–axis and a reflection across the xy–plane. For nonnegative integers I, J, K and L, choose Σ to be the boundary of a ϕ–invariant regular neighborhood of the graph

$$\Gamma_{I,J,K,L} \triangleq \bigcup_{m=1}^{n}\phi^m\biggl(\alpha\cup\biggl(\bigcup_{i=0}^{I}\beta_i\biggr)\cup\biggl(\bigcup_{j=1}^{J}C_{j+1}\biggr)\cup\biggl(\bigcup_{k=1}^{J+K}C_k'\biggr)\cup\biggl(\bigcup_{l=1}^{L}C_{l+2}''\biggr)\biggr).$$

Σ is a connected surface of genus

$$g=\tfrac{1}{2}n(2I+2J+K+2L)-2I.$$

Then $\phi|_\Sigma$ satisfies the conditions (1) in Theorem 2.4 with $h=L$, $b=2J+K+1$, $s=2I+1$ and $t=I+J+1$. In particular, the case $I=2$, $J=1$, $K=3$ and $L=2$ in the figure gives $h=2$, $b=6$, $s=5$ and $t=4$.

We give an explanation for the idea of the example. Beginning with the basic example, we have an initial surface Σ, which is isotopic to the boundary of a ϕ–invariant regular neighborhood of the initial graph $\Gamma_{0,0,0,0}$. The parameters for the surface map $\phi|_\Sigma$ are n, $h=0$, $b=1$ and $s=t=1$. We hope to enlarge the parameters by modifying the quotient space pair $(|\Sigma/\phi|, |S^3/\phi|)$. Just as in type $(+,+)$, we apply genus surgery and singular surgery to enlarge h and s. In Figure 8 the surgeries are realized by adding C_l'' and $\beta_i\,(i\ge 1)$, ie enlarging the parameters L and I for the graph. For boundaries, we need two more surgeries:

- **Boundary surgery I** When $\partial|S^3/\phi|\neq\varnothing$, applying a connected sum on $|\Sigma/\phi|$ with a properly embedded disk in $|(S^3-Z)/\phi|$, we can increase b by 1 without changing t. In the example, we add C_k', ie enlarge K, to accomplish it.

case (in Theorem 2.4)	initial example (in Example 3.2)
(1) with s even and $t=0$	(4) with $h=1, b=1, s=0, t=0$
(1) with $s=2t>0$	(3) with $h=0, b=1, s=2, t=1$
(1) with s even and $t>s/2$	(2)(ii) with $h=0, b=2, s=0, t=1$
(1) with s odd	(1)(ii) with $h=0, b=1, s=1, t=1$
(2) with s even	(2)(i) with $n/2$ odd and $h=2, s=0$
(2) with s odd	(1)(i) with $n/2$ odd and $h=1, s=1$
(3) with s even	(2)(i) with $n/2$ even and $h=2, s=0$
(3) with s odd	(1)(i) with $n/2$ even and $h=1, s=1$

Table 2: Extendable periodic maps in type $(-,-)$.

- **Boundary surgery II** When $\partial|S^3/\phi| \neq \varnothing$, applying a connected sum on $|\Sigma/\phi|$ with a properly embedded annulus A in $|(S^3-Z)/\phi|$ such that ∂A surrounds $|Z/\phi|$, we can increase b and t by 2 and 1, respectively. In the example, we add C_j, ie enlarge J, to accomplish it.

The same trick works for the remaining cases in type $(-,-)$. Table 2 lists the constructions of all extendable maps, where the fourth case has been discussed in detail as above. Note that if Theorem 2.4(1) holds with $s=2t=0$, h must be positive for otherwise ψ cannot be surjective, and if Theorem 2.4(3) holds, h and s must have the same parity, according to the equation

$$2\left(\sum_{i=1}^{h}\psi(\delta_i)\right)+\sum_{k=1}^{s}\psi(\xi_k)\equiv 0 \pmod n$$

and $2\psi(\delta_i)\equiv\psi(\xi_k)\equiv 2 \pmod 4$.

4.3 Type $(+,-)$

Proposition 4.8 *Suppose f satisfies the conditions in Theorem 2.5. Then the conjugacy class of $\langle f\rangle$ is uniquely determined by n, h and s.*

This proposition is an immediate consequence of Theorem 2.2(1). So we just turn to the realizations of all extendable maps in type $(+,-)$. If s is even, take Example 3.3(1) with $h=0$ and $s=2$; and if s is odd, take Example 3.3(2) with $h=0$ and $s=3$. Similarly, we can apply genus surgery and singular surgery to increase h and s in either case and complete the construction.

4.4 Type $(-,+)$

Example 4.9 We first construct a family of extendable maps such that the quotient surfaces are Klein bottles.

In Example 3.6(2), we have an automorphism ϕ acting on (Σ, S^3) as

$$\phi(w_1, w_2) = (w_1 e^{2\pi i/l}, w_2 e^{2\pi i(l/2+1)/l})$$

with $l \equiv 0 \pmod 4$. We are now to do branch surgery on it. As the branch axis S^1 is on Σ, generally Σ does not lift to a surface. Though, we can disturb Σ ϕ–equivariantly to get over it. For example, take the solid tori V_1 and V_2 as in Example 3.6, and let V_0 be a thinner solid torus in V_1:

$$V_0 = \{(w_1, w_2) \in S^3 \subset \mathbb{C}^2 : |w_2| \le \tfrac{1}{2}\}.$$

We disturb Σ in V_0 so that it looks like the same as in V_2. That is to say, we define

$$\tau\colon V_2 \to V_0, \quad (w_1, w_2) \mapsto \left(\frac{\sqrt{1-\frac{1}{2}|w_1|^2}}{|w_2|} w_2, \frac{\sqrt{2}}{2} w_1\right),$$

and then replace Σ with the surface

$$\hat{\Sigma} = (\Sigma - \Sigma \cap V_0) \cup \tau(\Sigma \cap V_2).$$

Suppose p and q are coprime positive integers. Without loss of generality, assume p is odd. Choose the smallest $m_0 \in \mathbb{N}$ such that

$$\begin{cases} m_0 \equiv \frac{1}{2}l + 1 \pmod l, \\ \gcd(m_0, p) = 1. \end{cases}$$

By Lemma 4.3(1), such an m_0 exists. Let $\phi'\colon S^3 \to S^3$ be the map defined by

$$\phi'(w_1, w_2) = (w_1 e^{2\pi i/ql}, w_2 e^{2\pi i m_0/pl}).$$

Then $(\phi')^l$ induces a branched covering map

$$\omega\colon S^3 \to |S^3/(\phi')^l| \cong S^3, \quad (w_1, w_2) \mapsto (|w_1| e^{i \cdot q \cdot \mathrm{Arg}(w_1)}, |w_2| e^{i \cdot p \cdot \mathrm{Arg}(w_2)}).$$

For the ϕ–invariant surface $\hat{\Sigma} \subset |S^3/(\phi')^l| \cong S^3$, write $\Sigma' = \omega^{-1}(\hat{\Sigma})$. Obviously, $|\Sigma'/\phi'| \cong |\hat{\Sigma}/\phi| \cong |\Sigma/\phi|$, so the restriction of ϕ' on Σ' is an extendable map with $n = pql$, $h = 2$, $s = 2$ and $t = 1$. In addition, we can also use singular surgery to make $t > 1$ or $s - t > 1$.

Denote the extendable surface map $\phi'|_{\Sigma'}$ by f'. Now we compute its isotropy invariant as well as $h_1(f')$ and $h_2(f')$. Denote the epimorphism for f' by ψ', and let $i_*\colon \pi_1(\Sigma'/f') \to \pi_1(S^3/\phi')$ be the homomorphism induced by the inclusion. Then there is a commutative diagram with the two rows exact:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\Sigma') & \longrightarrow & \pi_1(\Sigma'/f') & \xrightarrow{\psi'} & \mathbb{Z}_{pql} & \longrightarrow & 1 \\ & & & & \Big\downarrow i_* & & = \Big\downarrow f' \leftrightarrow 1 \leftrightarrow \phi' & & \\ 1 & \longrightarrow & \pi_1(S^3) & \longrightarrow & \pi_1(S^3/\phi') & \longrightarrow & \mathbb{Z}_{pql} & \longrightarrow & 1 \end{array}$$

With an abuse of notation, denote the epimorphism for the ϕ'–action on S^3 by ψ' too,

$$\psi'\colon \pi_1(S^3/\phi') \to \langle \phi' \rangle = \mathbb{Z}_{pql}, \quad \phi' \leftrightarrow 1.$$

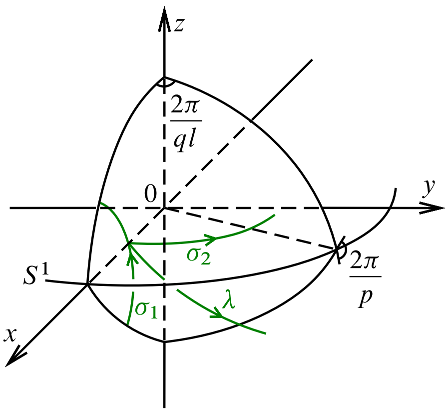

Figure 9: A fundamental domain for a cyclic action on S^3.

The ϕ'–action on $S^3 \subset \mathbb{C}^2$ has a fundamental domain F, which is the convex part bounded by the disks $\mathrm{Arg}(w_1) = 0$, $\mathrm{Arg}(w_1) = 2\pi/ql$, $\mathrm{Arg}(w_2) = -\pi/p$ and $\mathrm{Arg}(w_2) = \pi/p$; see Figure 9. $|S^3/\phi'|$ is homeomorphic to a lens space $L(l, m_0) \cong L(l, l/2 + 1)$, which can be obtained from F by gluing its boundary.

There are some typical elements in $\pi_1(S^3/\phi')$:

- σ_1, represented by the image of the oriented arc
$$\left\{\left(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}e^{i\theta}\right) : -\frac{\pi}{p} \le \theta \le \frac{\pi}{p}\right\},$$
which has order p, and $\psi'(\sigma_1)$ corresponds to the automorphism $(\phi')^{\psi'(\sigma_1)}$ of S^3 defined by
$$(w_1, w_2) \mapsto (w_1, w_2 e^{2\pi i/p}).$$
So $\psi'(\sigma_1) = \alpha_1 ql$, where α_1 is a solution of the congruence equation
$$\alpha_1 q m_0 \equiv 1 \pmod{p}.$$
- σ_2, represented by the image of the oriented arc
$$\left\{\left(\tfrac{\sqrt{2}}{2}e^{i\theta}, \tfrac{\sqrt{2}}{2}\right) : 0 \le \theta \le \frac{2\pi}{q}\right\},$$
which has order q, and $\psi'(\sigma_2)$ corresponds to the map defined by
$$(w_1, w_2) \mapsto (w_1 e^{2\pi i/q}, w_2).$$
So $\psi'(\sigma_2) = \alpha_2 pl$, where α_2 is a solution of the congruence equation
$$\alpha_2 p \equiv 1 \pmod{q}.$$

- λ, represented by the image of the oriented curve
$$\left\{\left(\tfrac{\sqrt{2}}{2}e^{i\theta/q}, \tfrac{\sqrt{2}}{2}e^{im_0\theta/p}\right) : 0 \le \theta \le \frac{2\pi}{l}\right\},$$
which has order n, and $\psi'(\lambda)$ corresponds to the map defined by
$$(w_1, w_2) \mapsto (w_1 e^{2\pi i/ql}, w_2 e^{2\pi i m_0/pl}).$$
So $\psi'(\lambda) = 1$. Note that in $|S^3/\phi'| \cong L(l, l/2+1)$, λ is homotopic to a core curve.

By choosing a canonical generator system
$$\mathcal{G} = \{\delta_1, \delta_2, \xi_1, \xi_2, \ldots, \xi_s\}$$
of $\pi_1(\Sigma'/f')$, we may assume $i_*(\xi_i) = (-1)^i \sigma_1$ for $1 \le i \le t$ and $i_*(\xi_j) = (-1)^{j-t}\sigma_2$ for $t < j \le s$. Then we have
$$\begin{aligned} \psi'(\xi_i) &= (-1)^i \alpha q l \qquad \text{for } i = 1, 2, \ldots, t, \\ \psi'(\xi_j) &= (-1)^{j-t}\alpha p l \quad \text{for } j = t+1, t+2, \ldots, s, \end{aligned}$$
where α is a solution of the congruence equations
$$\begin{cases} \alpha q m_0 \equiv 1 & \pmod p, \\ \alpha p \equiv 1 & \pmod q. \end{cases}$$
So the isotropy invariant of f' is
$$(\underbrace{\pm\alpha q l, \ldots, \pm\alpha q l}_{t}, \underbrace{\pm\alpha p l, \ldots, \pm\alpha p l}_{s-t}).$$
According to Lemma 6.6, in $|S^3/\phi'| \cong L(l, l/2+1) \cong L(l, l/2-1)$, $i_*(\delta_1)$ and $i_*(\delta_2)$ are homotopic to $\pm\lambda$ and $\pm(l/2-1)\lambda$, respectively. That means
$$\psi'(\delta_1) \equiv \pm 1 \quad \text{and} \quad \psi'(\delta_2) \equiv \pm(l/2-1) \pmod l.$$
So we have
$$h_2(f') = \{\pm 1, \pm(l/2-1)\} \subset \mathbb{Z}_{l/2}.$$
If $2 \notin \{p, q\}$, then $h_1(f')$ is defined. To compute it, as $\psi'(\xi_1), \ldots, \psi'(\xi_s)$ are fixed, we only need to determine the value of $\psi'(\delta_1\delta_2)$. It satisfies
$$2(\psi'(\delta_1\delta_2)) - \alpha q l - \alpha p l = 0 \in \mathbb{Z}_{pql}.$$
The equation has two solutions in $\mathbb{Z}_{pql}$: $\alpha(p+q)l/2$ and $\alpha(p+q)l/2 + pql/2$. We can exclude one of them with one more branch surgery as follows.

Let $\widetilde{\phi}\colon S^3 \to S^3$ be the map defined by
$$\widetilde{\phi}(w_1, w_2) = (w_1 e^{2\pi i/2ql}, w_2 e^{2\pi i m_0/pl}).$$

Then $(\widetilde{\phi})^{pql}$ is a π–rotation around Z and induces a 2–fold branched covering map

$$\widetilde{\omega}\colon S^3 \to |S^3/(\widetilde{\phi})^{pql}| \cong S^3.$$

Let $\widetilde{\Sigma}$ be the surface $\widetilde{\omega}^{-1}(\Sigma')$. Denote the epimorphism for $\widetilde{\phi}$ by $\widetilde{\psi}$.

If $q \neq 1$, $\widetilde{\Sigma}/\widetilde{\phi}$ has the same singular points with Σ'/ϕ', though the index-q singular points now have index $2q$. So $\mathscr{G}$ provides a canonical generator system for $\pi_1(\widetilde{\Sigma}/\widetilde{\phi})$. With an abuse of notation, similarly we can assume

$$\widetilde{\psi}(\xi_i) = (-1)^i \widetilde{\alpha} \cdot 2ql \quad \text{for } i = 1, 2, \dots, t,$$
$$\widetilde{\psi}(\xi_j) = (-1)^{j-t} \widetilde{\alpha} pl \quad \text{for } j = t+1, t+2, \dots, s,$$

where $\widetilde{\alpha}$ is a solution of the congruence equations

$$\begin{cases} \widetilde{\alpha} \cdot 2qm_0 \equiv 1 \pmod{p}, \\ \qquad \widetilde{\alpha} p \equiv 1 \pmod{2q}. \end{cases}$$

Thus

$$\widetilde{\psi}(\delta_1\delta_2) = \tfrac{1}{2}\widetilde{\alpha}(p+2q)l \ \text{ or } \ \tfrac{1}{2}\widetilde{\alpha}(p+2q)l + pql \in \mathbb{Z}_{2pql}.$$

By Lemma 4.2,

$$\psi'(\delta_1\delta_2) \equiv \widetilde{\psi}(\delta_1\delta_2) \equiv \tfrac{1}{2}\widetilde{\alpha}(p+2q)l \pmod{pql}.$$

Since $\widetilde{\alpha}$ is unique modulo $2pq$, $\psi'(\delta_1\delta_2) \in \mathbb{Z}_n$ is determined.

If $q = 1$, then $s = t$. The regular points on $(\Sigma' \cap S^1)/\phi'$ now become index-two singular points in $\widetilde{\Sigma}/\widetilde{\phi}$. The number of them is odd, according to Lemma 6.2. So there are $\widetilde{\xi}_{t+1}, \dots, \widetilde{\xi}_{\widetilde{s}}$, with $\widetilde{s} > s = t$ and $\widetilde{s} - t$ odd, such that

$$\mathscr{G} \cup \{\widetilde{\xi}_{t+1}, \dots, \widetilde{\xi}_{\widetilde{s}}\}$$

is a canonical generator system for $\pi_1(\widetilde{\Sigma}/\widetilde{\phi})$. Similarly we assume

$$\widetilde{\psi}(\xi_i) = (-1)^i \widetilde{\alpha} \cdot 2ql = (-1)^i \widetilde{\alpha} \cdot 2l \in \mathbb{Z}_{2pql} \quad \text{for } i = 1, 2, \dots, t,$$
$$\widetilde{\psi}(\widetilde{\xi}_j) = (-1)^{j-t} \widetilde{\alpha} pl = pl \in \mathbb{Z}_{2pql} \quad \text{for } j = t+1, t+2, \dots, \widetilde{s}.$$

Thus we still have

$$\widetilde{\psi}(\delta_1\delta_2) = \tfrac{1}{2}\widetilde{\alpha}(p+2q)l \ \text{ or } \ \tfrac{1}{2}\widetilde{\alpha}(p+2q)l + pql \in \mathbb{Z}_{2pql},$$
$$\psi'(\delta_1\delta_2) \equiv \widetilde{\psi}(\delta_1\delta_2) \equiv \tfrac{1}{2}\widetilde{\alpha}(p+2q)l \pmod{pql}.$$

So in either case, $\psi'(\delta_1\delta_2) \equiv \widetilde{\alpha}(p+2q)l/2 \pmod{pql}$, and $h_1(f')$ can be computed by definition.

Now take the smallest positive integer k that satisfies

$$\begin{cases} k \equiv qm_0 \pmod{p}, \\ k \equiv p \pmod{2q}, \\ \gcd(k, pql) = 1. \end{cases}$$

With Lemma 4.3(1) we see such a k exists. We can simply verify $k\alpha \equiv 1 \pmod p$, $k\alpha \equiv 1 \pmod q$ and $k\widetilde{\alpha} \equiv (p+1)/2 \pmod p$, $k\widetilde{\alpha} \equiv 1 \pmod{2q}$. Take k_0 such that $kk_0 \equiv 1 \pmod{pql}$. Let $f_0 = (f')^{k_0}$ and denote the corresponding epimorphism by ψ_0. By Lemma 4.1,

$$\begin{aligned} \psi_0(\xi_i) &= k\psi'(\xi_i) = (-1)^i k\alpha ql = (-1)^i ql \in \mathbb{Z}_{pql} && \text{for } i = 1, 2, \dots, t, \\ \psi_0(\xi_j) &= k\psi'(\xi_j) = (-1)^{j-t} k\alpha pl = (-1)^{j-t} pl \in \mathbb{Z}_{pql} && \text{for } j = t+1, t+2, \dots, s. \end{aligned}$$

So f_0 has isotropy invariant

$$(\underbrace{\pm ql, \dots, \pm ql}_{t}, \underbrace{\pm pl, \dots, \pm pl}_{s-t}),$$

and

$$h_2(f_0) = \left\{\pm k, \pm\left(\tfrac{1}{2}l - k\right)\right\} \subset \mathbb{Z}_{l/2}.$$

If $h_1(f_0)$ is defined, ie $2 \notin \{p, q\}$, then we can compute it according to the definition. For example, when $p > 2, q > 2$, as $ql, pl \in \{0, 2, \dots, pql/2 - 2\}$, we have

$$\begin{aligned} \chi_k &= \begin{cases} 0 & \text{if } \psi_0(\xi_k) \in \left\{2, 4, \dots, \frac{1}{2}pql - 2\right\} \subset \mathbb{Z}_{pql}, \\ 1 & \text{if } \psi_0(\xi_k) \in \left\{\frac{1}{2}pql + 2, \frac{1}{2}pql + 4, \dots, pql - 2\right\} \subset \mathbb{Z}_{pql}, \end{cases} \\ &= \begin{cases} 1 & \text{if } 1 \le k \le t \text{ and } k \text{ is odd}, \\ 0 & \text{if } 1 \le k \le t \text{ and } k \text{ is even}, \\ 1 & \text{if } t < k \le s \text{ and } k - t \text{ is odd}, \\ 0 & \text{if } t < k \le s \text{ and } k - t \text{ is even}, \end{cases} \end{aligned}$$

$$\begin{aligned} h_1(f_0) = \psi_0(\delta_1\delta_2) + \sum_{k=1}^{s} \chi_k \psi_0(\xi_k) &= k\widetilde{\alpha} \cdot \tfrac{1}{2}(p+2q)l + \tfrac{1}{2}(t+1)\cdot(-ql) + \tfrac{1}{2}(s-t+1)\cdot(-pl) \\ &= \tfrac{1}{2}pl + \tfrac{1}{2}(p+1)ql - \tfrac{1}{2}(t+1)ql - \tfrac{1}{2}(s-t+1)pl = \tfrac{1}{2}pql - \tfrac{1}{2}tql - \tfrac{1}{2}(s-t)pl \in \mathbb{Z}_{pql}. \end{aligned}$$

With similar checks for other cases, we finally conclude that

$$h_1(f_0) = \tfrac{1}{2}pql - \max\{t, 1\} \cdot \tfrac{1}{2}ql - \max\{s-t, 1\} \cdot \tfrac{1}{2}pl \in \mathbb{Z}_{pql}.$$

Proposition 4.10 (1) *Suppose f satisfies Theorem 2.6(1). Then the conjugacy class of $\langle f \rangle$ is uniquely determined by n, h and s.*

(2) *Suppose f satisfies Theorem 2.6(2). Then the conjugacy class of $\langle f \rangle$ is uniquely determined by n, h, s and l.*

(3) *Suppose f satisfies Theorem 2.6(3). Then the conjugacy class of $\langle f \rangle$ is uniquely determined by n, h, s, t, p and q.*

Proof Also, g can be figured out from the Riemann–Hurwitz formula, and we may assume $\alpha = 1 \in \mathbb{Z}_n$. Then (1) follows from Theorem 2.2. For (2), h_1 does not exist, and in the case $h = 2$ we must have $h_2(f) = \{\pm 1, \pm 1\} \subset \mathbb{Z}_2$ for $\gcd(2\alpha, 2l\alpha, n) = 2$. Therefore (2) is also verified, and we turn to (3).

The following cases are straightforward:

- $n/2$ is odd. Then $l/2$ is odd and $h \neq 2$. So f has no invariants h_1, h_2.
- $n/2$ is even, $h \neq 2$ and $2 \in \{p, q\}$. Without loss of generality, suppose $q = 2$. Then by the assumptions in the theorem we have $t \neq s$, and there exists $\pm pl = n/2$ in the isotropy invariant. So f has no invariants h_1, h_2.
- $h = 2$.

There is only one remaining case, where $n/2$ is even, $h \neq 2$ and $2 \notin \{p, q\}$. We only need to focus on the normalization of $h_1(f)$ for that case.

Fix a canonical generator system

$$\mathcal{G} = \{\delta_1, \dots, \delta_h, \xi_1, \dots, \xi_s\}$$

such that $\psi(\xi_i) = \alpha q l$ for $1 \le i \le t$ and $\psi(\xi_j) = \alpha p l$ for $t+1 \le j \le s$. The relation

$$\prod_{i=1}^{h} \delta_i^2 \prod_{k=1}^{s} \xi_k = 1 \in \pi_1(\Sigma_g/f)$$

implies that $x = \sum_{i=1}^{h} \psi(\delta_i)$ is a solution for the equation

$$2x + t\alpha q l + (s-t)\alpha p l \equiv 0 \pmod{n}.$$

There are two solutions,

$$x_1 = -\tfrac{1}{2}(t\alpha q l + (s-t)\alpha p l) \quad \text{and} \quad x_2 = x_1 + \tfrac{1}{2}n.$$

So

$$\sum_{i=1}^{h} \psi(\delta_i) = x_1 \quad \text{or} \quad \sum_{i=1}^{h} \psi(\delta_i) = x_1 + \tfrac{1}{2}n.$$

If pq is odd, $n/2 \equiv l/2 \pmod{l}$. By definition,

$$h_1(f) = \sum_{i=1}^{h} \psi(\delta_i) + t\chi\alpha q l + (s-t)\chi'\alpha p l \in \mathbb{Z}_n,$$

where

$$\chi = \begin{cases} 0 & \text{if } \alpha q l \in \left\{0, 2, 4, \dots, \tfrac{1}{2}n - 2\right\} \subset \mathbb{Z}_n, \\ 1 & \text{if } \alpha q l \in \left\{\tfrac{1}{2}n + 2, \tfrac{1}{2}n + 4, \dots, n-2\right\} \subset \mathbb{Z}_n, \end{cases}$$

$$\chi' = \begin{cases} 0 & \text{if } \alpha p l \in \left\{0, 2, 4, \dots, \tfrac{1}{2}n - 2\right\} \subset \mathbb{Z}_n, \\ 1 & \text{if } \alpha p l \in \left\{\tfrac{1}{2}n + 2, \tfrac{1}{2}n + 4, \dots, n-2\right\} \subset \mathbb{Z}_n. \end{cases}$$

Note that $n/2 \equiv pql/2 \equiv l/2 \pmod{l}$. With the condition $h_1(f) \equiv l/2 \pmod{l}$, we see that one of x_1 or $x_1 + n/2$ is impossible for $\sum_{i=1}^{h} \psi(\delta_i)$. Then $h_1(f)$, and therefore $\langle f \rangle$, are determined.

case (in Theorem 2.6)	initial example
(1)	Example 3.4 with $h=0$, $s=1$
(2) with h odd	Example 3.6(1) with $n=2l$, $h=1$, $s=0$
(2) with h even	Example 3.7(1) with $n=2l$, $h=2$, $s=0$
(3) with $h=1$	Example 3.5 with $h=1$, $s=t=0$, $p=q=1$
(3) with h even	Example 3.6(2) with $h=2$, $s=t=0$, $p=q=1$
(3) with h odd and $h\geq 3$	Example 3.7(2) with $h=3$, $s=t=0$, $p=q=1$

Table 3: Extendable periodic maps in type $(-,+)$.

If pq is even, it suffices to show that there exists a positive integer m such that

$$\langle f^m\rangle=\langle f\rangle \quad \text{and} \quad h_1(f^m)=h_1(f)+\tfrac{1}{2}n,$$

and f^m and f have the same isotropy invariant. In fact, if it holds, then we can always normalize $h_1(f)$ to x_1 and thus $\langle f\rangle$ is determined up to conjugacy. By Lemma 4.3(1), we choose a positive integer d such that $\gcd((pq+1)+(2pq)d,n)=1$. Choose $m\in\mathbb{Z}$ such that $m\cdot((pq+1)+(2pq)d)\equiv 1 \pmod n$. Let ψ' be the corresponding epimorphism for f^m. By Lemma 4.1, for each $\gamma\in\pi_1(\Sigma/f)$, we have $\psi'(\gamma)=((pq+1)+(2pq)d)\psi(\gamma)$. Without loss of generality, assume p is odd and q is even, hence $s-t$ is odd. If

$$\sum_{i=1}^{h}\psi(\delta_i)=x_1+\tfrac{1}{2}n=(pq-t\alpha q-(s-t)\alpha p)\cdot\tfrac{1}{2}l,$$

we have

$$\begin{aligned}\sum_{i=1}^{h}\psi'(\delta_i)&=((pq+1)+(2pq)d)\sum_{i=1}^{h}\psi(\delta_i)=(pq+1)\sum_{i=1}^{h}\psi(\delta_i)+0\\&=\tfrac{1}{2}pq(pql)-t\alpha q\cdot\tfrac{1}{2}pql-(s-t)\alpha(pql)\cdot\tfrac{1}{2}p+\sum_{i=1}^{h}\psi(\delta_i)\\&=0+0+\tfrac{1}{2}n+\sum_{i=1}^{h}\psi(\delta_i)\in\mathbb{Z}_n.\end{aligned}$$

If $\sum_{i=1}^{h}\psi(\delta_i)=x_1$, a similar check also shows

$$\sum_{i=1}^{h}\psi'(\delta_i)=\sum_{i=1}^{h}\psi(\delta_i)+\tfrac{1}{2}n.$$

Moreover, for $i=1,2,\dots,s$, as $\psi(\xi_i)=\alpha ql$ or αpl, we have

$$\psi'(\xi_i)=((pq+1)+(2pq)d)\psi(\xi_i)=\psi(\xi_i)\in\mathbb{Z}_n.$$

So f^m and f have the same isotropy invariant and $h_1(f^m)=h_1(f)+\tfrac{1}{2}n$. □

Similarly, all extendable maps in type $(-,+)$ can be constructed from Examples 3.4–3.7 with the surgeries; see Table 3.

5 Necessary conditions for extendability

A periodic automorphism of S^2 is conjugate to a rational rotation, maybe composed with a reflection, hence must be extendable in each type with respect to the standard embedding $S^2 \hookrightarrow S^3$. So in this section, we assume the genus g is no less than 1.

Suppose a periodic map $f \in \mathrm{Aut}(\Sigma_g)$ of order n is extendable over S^3 with respect to an embedding $e\colon \Sigma_g \hookrightarrow S^3$ and an automorphism $\phi \in \mathrm{Aut}(S^3)$. For convenience, we identify Σ_g with its image $e(\Sigma_g)$. As we work in the smooth category, ϕ can be assumed to be a torsion in the orthogonal group $O(4)$. By the assumption $g \geq 1$ we see that $\phi^m = \mathrm{id}$ if and only if $f^m = \mathrm{id}$, so ϕ is also of order n.

Extendable maps in type $(+,+)$ are classified in [7]. So we only verify the necessity of the listed conditions in Theorems 2.4, 2.5 and 2.6.

We denote the fixed-point set of an automorphism by $\mathrm{fix}(\cdot)$, and for a group action $G \curvearrowright X$, we use $\mathrm{Fix}(G, X)$ to denote the set

$$\{x \in X : \gamma(x) = x \text{ for some } \gamma \in G \backslash \{1_G\}\}.$$

For a periodic map $f \in \mathrm{Aut}(\Sigma_g)$, if $\mathrm{Fix}(\langle f \rangle, \Sigma_g)$ has dimension 1, then f must be orientation-reversing; if $\mathrm{fix}(f)$ has dimension 1, then f must be an orientation-reversing involution; and if there exists an isolated point in $\mathrm{fix}(f)$, then f must be orientation-preserving.

5.1 Type (−, −)

Proposition 5.1 *Suppose $f \in \mathrm{Aut}(\Sigma_g)$ is orientation-reversing. Then f is extendable over S^3 in type $(-,-)$ if and only if it is extendable over $\mathbb{R}^3$.*

Proof Suppose f extends to some orientation-reversing automorphism ϕ of S^3 with respect to some embedding $\Sigma_g \hookrightarrow S^3$. Up to similarity, $\phi \in O(4)$ has the standard form

$$\begin{pmatrix} \cos(2p\pi/n) & -\sin(2p\pi/n) & 0 & 0 \\ \sin(2p\pi/n) & \cos(2p\pi/n) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{where } \gcd(n,p) = 1 \text{ or } 2.$$

Identify S^3 with $\mathbb{R}^3 \cup \{\infty\}$; then ϕ corresponds to the matrix

$$\begin{pmatrix} \cos(2p\pi/n) & -\sin(2p\pi/n) & 0 \\ \sin(2p\pi/n) & \cos(2p\pi/n) & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which fixes two points, 0 and ∞. If $\infty \notin \Sigma_g$, f is naturally extendable over $\mathbb{R}^3$. Otherwise $\infty \in \mathrm{fix}(f)$, so f has a one-dimensional fixed-point set, for an orientation-reversing map has no isolated fixed point. It implies f is an involution and $\phi = \mathrm{diag}(1,1,-1)$. So $|\Sigma_g/f|$ is homeomorphic to the closure of a fundamental domain, which is an orientable surface with nonempty boundary. By [4, Proposition 1], f is extendable over $\mathbb{R}^3$ as well. □

The extendability over $\mathbb{R}^3$ of an orientation-reversing map has been discussed by Costa [4]; Theorem 2.4 is essentially the same as his conclusions. We omit the details in the calculation here.

5.2 Type $(+,-)$

The automorphism ϕ has the same matrix as in the proof of Proposition 5.1 with n even. For orientation reasons, ϕ exchanges the two components of $S^3-\Sigma_g$, so the fixed points 0 and ∞ must be on Σ_g. Note that $\mathrm{Fix}(\langle f\rangle,\Sigma_g)$ is discrete and $|\Sigma_g/f|$ is a closed orientable surface. If $\gcd(n,p)=2$, then $n/2$ must be odd as ϕ is of order n. Then $\mathrm{fix}(\phi^{n/2})$ is the sphere $\{\infty\}\cup xy$–plane. This implies $\mathrm{fix}(f^{n/2})=\Sigma_g\cap\mathrm{fix}(\phi^{n/2})$ has dimension 1 and $f^{n/2}$ is orientation-reversing, a contradiction. So $\gcd(n,p)=1$. Without loss of generality, we assume $p=1$, for otherwise we can consider f^k instead of f, where $k\in\mathbb{Z}$ satisfies $kp\equiv 1 \pmod n$. Then ϕ is the composition of a $(2\pi/n)$–rotation around the z–axis and reflection across the xy–plane.

If $n=2$, $\mathrm{Fix}(\langle\phi\rangle,S^3)=\{0,\infty\}$. Hence Σ_g/f has only two singular points, and their corresponding elements $\xi_1,\xi_2\in\pi_1(\Sigma_g/f)$ are sent to the generator of $\mathbb{Z}_2$ by the epimorphism $\psi:\pi_1(\Sigma_g/f)\to\mathbb{Z}_2$.

If $n>2$, Σ_g intersects the circle $\{\infty\}\cup z$–axis at $2s-2$ points: 0, ∞ and $(0,0,\pm z_i)$ for $1\le i\le s-2$, where $0<z_1<z_2<\cdots<z_{s-2}$. Their images in Σ_g/f are the singular points. Let 0, ∞ and $(0,0,z_i)$ correspond to ξ_1, ξ_2 and $\xi_{i+2}\in\pi_1(\Sigma_g/f)$, respectively. Then $\psi(\xi_1)=1$ and $\psi(\xi_i)=(-1)^i\times 2$ for $3\le i\le s$, while $\psi(\xi_2)$ is determined by

$$0=\psi\biggl(\prod_{i=1}^{h}[\alpha_i,\beta_i]\prod_{k=1}^{s}\xi_k\biggr)=\sum_{k=1}^{s}\psi(\xi_k),$$

so equals -1 if s is even, and otherwise equals 1.

5.3 Type $(-,+)$

As $\phi\in O(4)$ is orientation-preserving, up to similarity it has the standard form

$$\begin{pmatrix}\cos(2m_1\pi/n) & -\sin(2m_1\pi/n) & 0 & 0\\ \sin(2m_1\pi/n) & \cos(2m_1\pi/n) & 0 & 0\\ 0 & 0 & \cos(2m_2\pi/n) & -\sin(2m_2\pi/n)\\ 0 & 0 & \sin(2m_2\pi/n) & \cos(2m_2\pi/n)\end{pmatrix},$$

where $\gcd(n,m_1,m_2)=1$.

Let $p=\gcd(n,m_1)$ and $q=\gcd(n,m_2)$. Then p,q are coprime as $\gcd(n,m_1,m_2)=1$. Without loss of generality, we set p to be odd. Moreover, we assume $m_1=p$, for otherwise we can consider ϕ^k and f^k instead of ϕ and f, where k satisfies $\gcd(k,n)=1$ and $km_1\equiv p \pmod n$, by Lemma 4.3(2). Assume that $n=pql$ and $m_2=mq$. Using the model

$$S^3=\{(w_1,w_2)\in\mathbb{C}^2:|w_1|^2+|w_2|^2=1\},$$

ϕ is the map

$$(w_1, w_2) \mapsto (w_1 e^{2\pi i/ql}, w_2 e^{2\pi i m/pl}).$$

The ϕ–action on S^3 has a fundamental domain F as in Figure 9, and $|S^3/\phi|$ is homeomorphic to a lens space $L(l, m)$.

If $|\Sigma_g/f|$ is orientable with nonempty boundary, then the boundary comes from the one-dimensional part of $\mathrm{fix}(f^{n/2}) \subseteq \mathrm{fix}(\phi^{n/2})$. As

$$\phi^{n/2}(w_1, w_2) = (w_1 e^{2\pi i p/2}, w_2 e^{2\pi i m q/2}) = (w_1 e^{2\pi i/2}, w_2 e^{2\pi i m q/2}),$$

$\mathrm{fix}(\phi^{n/2})$ can only be the axis Z. So $Z \subset \Sigma_g$ and $\partial|\Sigma_g/f|$ is connected. With the fundamental domain F we see $q = 2$, for otherwise Σ_g cannot be a closed surface. Moreover, by Proposition 6.1(1) we have $l = 1$. Take $\epsilon_1 \in \pi_1(\Sigma_g/f)$ to be the element represented by an oriented curve which is parallel to the boundary Z/ϕ. Then ϵ_1 is sent to 2α by ψ, where α is a generator of $\mathbb{Z}_n$. If $n > 2$, the singular points come from $\Sigma_g \cap S^1$, so $\psi(\xi_i) = \pm 2\alpha$. The signs of $\psi(\xi_1), \dots, \psi(\xi_s)$ must be alternating, as $|\Sigma_g/f|$ is orientable. From the relation

$$\psi(\epsilon_1) + \psi(\xi_1) + \cdots + \psi(\xi_s) = 0 \in \mathbb{Z}_n,$$

we see s is odd and $\psi(\xi_1), \dots, \psi(\xi_s)$ are $-2\alpha, 2\alpha, -2\alpha, 2\alpha, \dots, -2\alpha$.

If $|\Sigma_g/f|$ is nonorientable with nonempty boundary, similarly we have $|Z/\phi| = \partial|\Sigma_g/f|$, $q = 2$ and $\psi(\epsilon_1) = \pm 2\alpha$, where $\langle\alpha\rangle = \mathbb{Z}_n$. Moreover, if $p > 1$, s is odd and $\pm\psi(\xi_i) = \pm 2l\alpha$ for each $i = 1, 2, \dots, s$.

Finally we suppose $|\Sigma_g/f|$ is a closed (thus nonorientable) surface of genus h, embedded in the lens space $L(l, m)$. According to [2] (see Lemmas 6.3 and 6.4), l is even, and $l/2$ and h have the same parity. Moreover, if $h = 1$, l must be 2. The circles S^1/ϕ and Z/ϕ in S^3/ϕ have indices p and q, respectively. Suppose that in $\Sigma_g \cap F$ there are t points on S^1/ϕ and $s - t$ points on Z/ϕ. They are the singular points of indices p and q, respectively (though they degenerate when $p = 1$ or $q = 1$). So the isotropy invariant is

$$(\underbrace{\pm\beta, \dots, \pm\beta}_{t}, \underbrace{\pm\gamma, \dots, \pm\gamma}_{s-t}),$$

where $\beta, \gamma \in \mathbb{Z}_n$ have orders p, q respectively. Moreover, t and $s - t$ should be odd, by Lemma 6.2.

When $n/2$ is even and $2 \notin \{p, q\}$, $h_1(f)$ is defined. Let

$$\phi_l \colon (w_1, w_2) \mapsto (w_1 e^{2\pi i/l}, w_2 e^{2\pi i m/l})$$

be the automorphism of $|S^3/\phi^l| \cong S^3$ induced by ϕ, and f_l be induced on $|\Sigma_g/f^l|$, with the corresponding epimorphism ψ_l. As $\mathrm{Fix}(\langle f_l\rangle, |\Sigma_g/f^l|) \subseteq \mathrm{Fix}(\langle\phi_l\rangle, S^3)$ and $\mathrm{Fix}(\langle\phi_l\rangle, S^3)$ is empty, the map

$$(|\Sigma_g/f^l|, S^3) \xrightarrow{(f_l, \phi_l)} (|\Sigma_g/f|, L(l, m))$$

is a covering, which induces a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(|\Sigma_g/f^l|) & \longrightarrow & \pi_1(|\Sigma_g/f|) & \xrightarrow{\psi_l} & \mathbb{Z}_l & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle = \,|\, f_l \leftrightarrow 1 \leftrightarrow \phi_l} & & \\ 1 & \longrightarrow & \pi_1(S^3) & \longrightarrow & \pi_1(L(l,m)) & \xrightarrow{\cong} & \mathbb{Z}_l & \longrightarrow & 1 \end{array}$$

By Proposition 6.5, $i_*\big(\prod_{i=1}^h \delta_i\big)$ is nontrivial and represents the order-2 element in $H_1(L(l,m);\mathbb{Z}) \cong \pi_1(L(l,m))$. Hence, by Lemma 4.2,

$$\psi\left(\prod_{i=1}^h \delta_i\right) \equiv \psi_l\left(\prod_{i=1}^h \delta_i\right) \equiv \tfrac{1}{2}l \pmod{l}.$$

So by definition we have $h_1(f) \equiv \frac{1}{2}l \pmod{l}$.

When $h=2$, by Lemma 6.4 we have $l \equiv 0 \pmod 4$ and $m \equiv \frac{1}{2}l \pm 1 \pmod{l}$. Suppose $m \equiv \frac{1}{2}l+1 \pmod{l}$. Take the smallest $m_0 \in \mathbb{N}$ such that

$$\begin{cases} m_0 \equiv \frac{1}{2}l+1 \pmod{l}, \\ \gcd(m_0, p) = 1. \end{cases}$$

Choose $\kappa \in \mathbb{N}$ that satisfies the congruence equations

$$\begin{cases} \kappa \equiv 1 & \pmod{ql}, \\ \kappa m \equiv m_0 & \pmod{pl}. \end{cases}$$

With Lemma 4.3(3) and a few basic calculations, such a κ exists and is coprime to n. Take κ' with $\kappa\kappa' \equiv 1 \pmod{n}$. By replacing ϕ and f by $\phi^{\kappa'}$ and $f^{\kappa'}$, we may assume $m = m_0$. Note that in Example 4.9 the embedding of surface has nothing to do with the computation of the invariants. Therefore we can follow it directly and see that f must have the same isotropy invariant, h_1 and h_2 as the f' in the example. So $\langle f \rangle$ is conjugate to $\langle f_0 \rangle$. If $m \equiv \frac{1}{2}l-1 \pmod{l}$, as the quotient space $|S^3/\phi| \cong L(l, l/2-1) \cong L(l, l/2+1)$, it must be conjugate to the last case, so we finish the proof.

6 Surfaces in lens spaces

In this section, we present some facts about embedded surfaces in lens spaces.

A lens space $L(l,m)$, with $l \geq 1$ and $\gcd(l,m) = 1$, can be constructed by gluing two solid tori $V_1 = D^2 \times S^1$ and $V_2 = S^1 \times D^2$ with a homeomorphism $\omega\colon \partial V_2 \to \partial V_1$ whose restriction to the meridian $\{1\} \times \partial D^2$ of V_2 is

$$\{1\} \times S^1 \to S^1 \times S^1 = \partial V_1, \quad (1, e^{2\pi i t}) \mapsto (e^{2\pi i t m}, e^{2\pi i t l}).$$

Cutting V_1 along the disk $D^2 \times \{1\}$, we obtain a cylinder $C = D^2 \times I$. Its boundary ∂C consists of two disks $D^2 \times \{0\}$ and $D^2 \times \{1\}$ and an annulus $\partial D^2 \times I = S^1 \times I$, denoted by D_0, D_1 and A, respectively.

Remark To some degree, $V_1, V_2 \subset L(l,m)$ are unique. In fact, for a given lens space, its Heegaard splitting of a fixed genus is unique up to isotopy [1].

A simple closed curve in a lens space is a *core* if its complement is homeomorphic to a solid torus.

Proposition 6.1 *Let γ be a core curve of a lens space $L(l,m)$ with $\gcd(l,m)=1$.*

(1) *If $l>1$, γ cannot bound an embedded orientable surface in $L(l,m)$.*

(2) *If l is even, γ can neither bound an embedded nonorientable surface in $L(l,m)$.*

(3) *If l is odd, γ bounds an embedded nonorientable surface in $L(l,m)$. Moreover, if $m=2$, γ bounds a Möbius band; and if $m=4$, γ bounds a one-holed Klein bottle.*

Proof (1) If γ bounds an embedded orientable surface then it must be trivial in $H_1(L(l,m);\mathbb{Z}) \cong \pi_1(L(l,m))$, a contradiction.

(2) If γ bounds an embedded nonorientable surface then it must be trivial in $H_1(L(l,m);\mathbb{Z}_2) \cong \mathbb{Z}_2$, a contradiction.

(3) It suffices to consider the quotient space pair $(|\Sigma/\phi|, |S^3/\phi|)$ in Examples 3.6(1) and 3.7(1). □

Lemma 6.2 [2, page 97, lines 24–28] *Suppose a closed surface Σ is embedded in a lens space, and Σ intersects a core curve transversely. Then Σ is nonorientable if and only if the number of their intersection points is odd.*

Lemma 6.3 [2, page 88, lines 11–24] *Suppose a lens space $L(l,m)$, with $\gcd(l,m)=1$, admits an embedded nonorientable closed surface of genus h. Then l is even and $l/2$ has the same parity as h.*

Lemma 6.4 [2, Theorem 6.1] (1) *A lens space $L(l,m)$ admits an embedded projective plane if and only if $L(l,m) \cong L(2,1)$.*

(2) *$L(l,m)$ admits an embedded Klein bottle if and only if $L(l,m) \cong L(4r, 2r \pm 1)$ for some positive integer r.*

(3) *A nonorientable closed surface of genus 3 can be embedded into $L(4r+2, 2r-1)$ for each positive integer r.*

According to [2], the embeddings can be constructed as follows. For $L(2,1)$, see Figure 10. For $L(4r, 2r-1)$, choose the arcs on ∂C as in Figure 11. Their union consists of $2r-1$ closed curves, which bound disjoint disks in C. Gluing these disks and $\{1\} \times D^2 \subset V_2$ back, we get an embedded closed surface in $L(4r, 2r-1)$. From the gluing we see its Euler number is $(2r-1)-(4r)/2+1=0$. As the

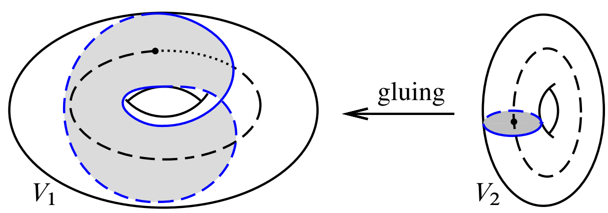

Figure 10: A projective plane in $L(2,1)$.

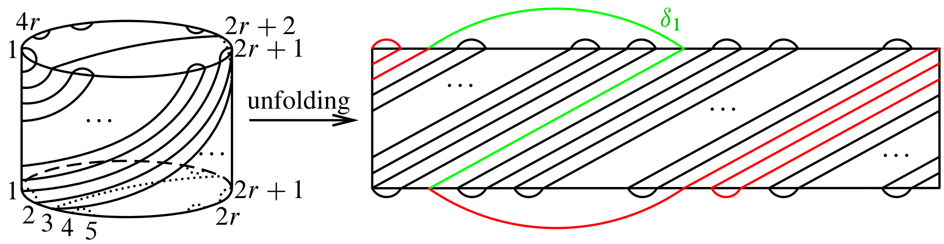

Figure 11: A Klein bottle in $L(4r, 2r-1)$.

surface intersects the core circle $S^1 \times \{0\} \subset V_2$ at only one point, it must be nonorientable, thus a Klein bottle. Similarly, with the closed curves on ∂C shown in Figure 12, we obtain a nonorientable closed surface of genus 3, embedded in $L(4r+2, 2r-1)$.

Remark Every green curve in Figures 11 and 12 generates the fundamental group of the corresponding lens space. Thus the embeddings above induce surjective homomorphisms on π_1.

Proposition 6.5 *Suppose K is a nonorientable closed surface, and $i: K \to L(l, m)$ is an embedding. Let $\delta \in H_1(K;\mathbb{Z})$ be the order-2 element. Then $i_*(\delta)$ is nontrivial in $H_1(L(l, m);\mathbb{Z})$.*

Proof We will choose a union of oriented curves on K so that it represents δ. Then we will verify it is nontrivial in the homology of $L(l, m)$.

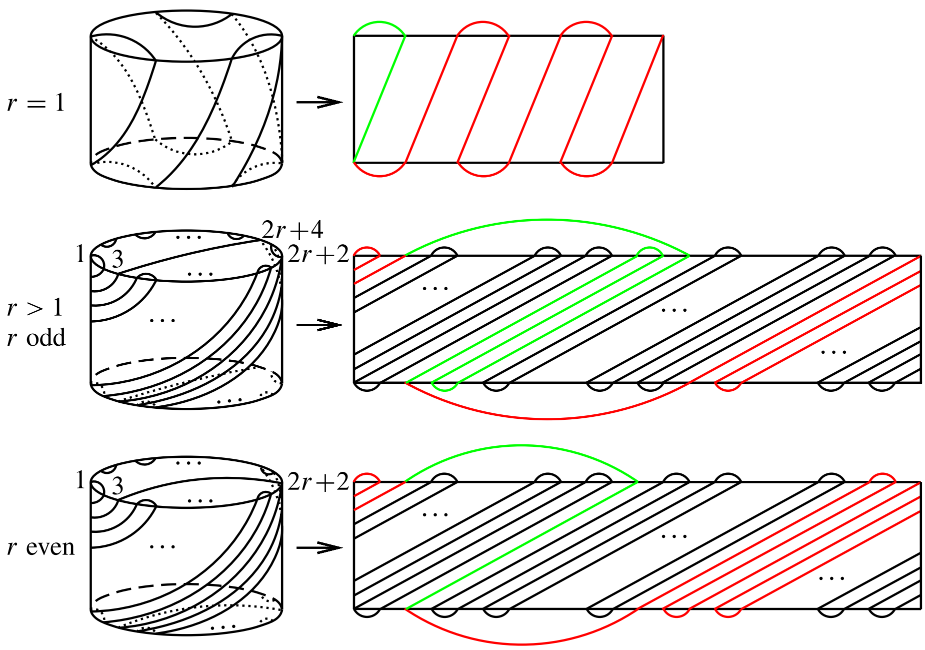

Figure 12: A genus 3 nonorientable surface in $L(4r+2, 2r-1)$.

We follow the arguments in [2, Section 7]. Suppose K intersects the core $S^1 \times \{0\} \subset V_2$ transversely; thus $K \cap V_2$ can be assumed to be a union of d meridian disks. We may reduce to the case $d = 1$ by isotopy as [2] does. Now assume $K \cap V_2 = \{1\} \times D^2$ and $K \cap \partial C$ consists of l parallel arcs on A, $l/2$ arcs on D_0, $l/2$ arcs on D_1, and maybe some closed curves in the interior of D_0 and D_1. Thus $K \cap C$ consists of some compact surfaces properly embedded in C. They must be orientable, otherwise a double of C would give an S^3 admitting embedded nonorientable closed surfaces. Now K can be constructed by gluing back these surfaces and the disk $K \cap V_2$. We just glue part of them back as follows, to obtain a compact orientable surface K_0 such that K_0 can also be constructed by cutting K along a union of curves which represents δ in homology.

Fix an orientation on $K \cap V_2$. Each arc component α of $K \cap D_1$ connects two parallel components β_1, β_2 of $K \cap A$. Let K_α be the component of $K \cap C$ such that $\partial K_\alpha \supset \alpha \cup \beta_1 \cup \beta_2$. Any given orientation on K_α induces opposite orientations on the parallel arcs β_1, β_2. Therefore, we can glue part of $\partial(K \cap V_2)$ back to one of β_1, β_2 so that the orientation of $K \cap V_2$ and that of K_α coincide; thus $(K \cap V_2) \cup K_\alpha$ is oriented. In this way, half of the l parallel arcs $K \cap A$ can be glued onto $\partial(K \cap V_2)$ so that the obtained surface, denoted by K_0, is still oriented.

The oriented boundary of K_0 consists of the following arcs and curves:

(1) $l/2$ parallel arcs on A, with accordant orientations.

(2) $l/2$ oriented arcs on $\partial(K \cap D_2)$, of equal length.

(3) $K_0 \cap D_0$ with orientation.

(4) $K_0 \cap D_1$ with orientation.

The Klein bottle K can be obtained by gluing ∂K_0: the oriented arcs in (1) are identified with those in (2), and (3) is identified with (4). Therefore, the union of (1) and (3) represents the order-2 element δ in $H_1(K; \mathbb{Z})$.

Let c be the oriented core $\{0\} \times S^1$ in V_1, then $H_1(L(l,m); \mathbb{Z}) \cong \mathbb{Z}_l$ is generated by $[c]$. As only the arcs in (1) contribute to $H_1(V_1; \mathbb{Z})$, the union of (1) and (3) represents $(l/2)[c]$ in $H_1(V_1; \mathbb{Z})$, as well as in $H_1(L(l,m); \mathbb{Z})$. Thus $i_*(\delta) = (l/2)[c] \neq 0$ in $H_1(L(l,m); \mathbb{Z})$. □

The following lemma can also be deduced by the analysis in [2, Section 7] and Figure 11. Note that the green curve in Figure 11 represents δ_1.

Lemma 6.6 *The embedding of a Klein bottle into $L(4r, 2r-1)$, where $r \geq 1$, is unique up to isotopy. Denote it by i and identify $H_1(L(4r, 2r-1); \mathbb{Z})$ with $\mathbb{Z}_{4r}$ by mapping an oriented core to 1. Let $\langle \delta_1, \delta_2 \mid \delta_1^2 \delta_2^2 = 1 \rangle$ be the fundamental group of the Klein bottle. Then*

$$\{i_*([\delta_1]), i_*([\delta_2])\} = \{1, 2r-1\} \text{ or } \{-1, 2r+1\} \subset \mathbb{Z}_{4r}$$

in homology.

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School of Mathematical Sciences & Shanghai Key Laboratory of PMMP, East China Normal University
Shanghai, China

School of Mathematical Sciences, Peking University
Beijing, China

Current address: *School of Mathematics and Statistics, Central South University*
Changsha, China

chao_wang_1987@126.com, wwb@pku.edu.cn

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