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*Algebraic & Geometric  
Topology*

Volume 24 (2024)

**L-spaces, taut foliations and the Whitehead link**

DIEGO SANTORO



# L–spaces, taut foliations and the Whitehead link

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We prove that if  $M$  is a rational homology sphere that is a Dehn surgery on the Whitehead link, then  $M$  is not an L–space if and only if  $M$  supports a coorientable taut foliation. The left orderability of some of these manifolds is also proved, by determining which of the constructed taut foliations have vanishing Euler class.

We also present some more general results about the structure of the L–space surgery slopes for links with two unknotted components and linking number zero, and about the existence of taut foliations on the fillings of a  $k$ –holed torus bundle over the circle with some prescribed monodromy. Our results, combined with some results of Roberts, Shareshian and Stein (2003), also imply that all the rational homology spheres that arise as integer surgeries on the Whitehead link satisfy the L–space conjecture.

[57K10](#), [57K18](#), [57K32](#), [57R30](#)

## 1 Introduction

In this paper we study the rational homology spheres obtained as surgery on the Whitehead link, motivated by the following conjecture.

**Conjecture 1** (L–space conjecture) For an irreducible oriented rational homology 3–sphere  $M$ , the following are equivalent:

- (1)  $M$  supports a cooriented taut foliation;
- (2)  $M$  is not an L–space, ie its Heegaard Floer homology is not minimal;
- (3)  $M$  is left orderable, ie  $\pi_1(M)$  is left orderable.

The equivalence between (1) and (2) was conjectured by Juhász [2015], while the equivalence between (2) and (3) was conjectured by Boyer, Gordon and Watson [Boyer et al. 2013].

Although the properties involved in the conjecture are very different in flavour and nature, it follows by the works of Ozsváth and Szabó [2004], Bowden [2016] and Kazez and Roberts [2017] that (1) implies (2). Moreover it is now known that the conjecture holds for all the graph manifolds [Boyer and Clay 2017; Rasmussen 2017; Hanselman et al. 2020].

It is therefore interesting to study the conjecture in the case of hyperbolic manifolds. In this direction, in [Zung 2020] the conjecture is proved for some manifolds obtained by considering mapping tori of pseudo-Anosov diffeomorphisms of closed surfaces and then by surgering on some collections of closed orbits.

In addition, in [Dunfield 2020] the conjecture is tested on more than 300 000 hyperbolic rational homology spheres and proved for more than 60% of these manifolds.

One way of producing rational homology spheres is via Dehn surgery on knots or links in  $S^3$ . When dealing with surgeries on knots, the different aspects of this conjecture have been studied separately in several papers. For example it has been proved that if a knot  $K$  admits a positive surgery that is an  $L$ -space then  $K$  is fibered [Ghiggini 2008; Ni 2007], strongly quasipositive [Hedden 2010] and the  $r$ -framed surgery along  $K$  is an  $L$ -space if and only if  $r \geq 2g(K) - 1$ , where  $g(K)$  denotes the genus of the knot  $K$  [Kronheimer et al. 2007].

Taut foliations on surgeries on knots are constructed, for example, in [Delman and Roberts 2020; 2021; Krishna 2020; Li and Roberts 2014; Roberts 2001b] and it is possible to prove the left orderability of some of these manifolds by determining which of these foliations have vanishing Euler class, as done in [Hu 2019]. Another approach to study the left orderability of surgeries on knots is via representation theoretic methods, as presented in [Culler and Dunfield 2018].

On the other hand, not much is known when it comes to the study of surgeries on links. Some results regarding integer  $L$ -space surgeries on links in  $S^3$  are presented in [Gorsky and Hom 2017; Gorsky and Némethi 2016; 2018; Gorsky et al. 2020; Liu 2017], while in [Rasmussen 2020] rational  $L$ -space surgeries on satellites by algebraic links are studied. Concerning foliations, Kalelkar and Roberts [2015] construct coorientable taut foliations on some fillings of 3-manifolds that fiber over the circle. In particular, their methods can also be applied to surgeries on fibered links.

As far as we know, in this paper we provide the first example of the equivalence between conditions (1) and (2) of the conjecture for manifolds obtained via Dehn surgery on a hyperbolic link with at least two components.

We focus our attention on the Whitehead link, which is depicted in Figure 1.

We will denote the Whitehead link by WL and the  $(p_1/q_1, p_2/q_2)$ -surgery on the Whitehead link by  $S^3_{p_1/q_1, p_2/q_2}(\text{WL})$ . Notice that since the Whitehead link has linking number zero, the homology of this manifold is isomorphic to  $\mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2}$ . In particular, the manifold  $S^3_{p_1/q_1, p_2/q_2}(\text{WL})$  is a rational homology sphere if and only if  $p_1 \neq 0$  and  $p_2 \neq 0$ .

Recall that the Whitehead link exterior supports a complete hyperbolic structure and therefore, by virtue of Thurston's hyperbolic Dehn surgery theorem [1979], "most" of its fillings are hyperbolic.

The main result of this paper is the following:

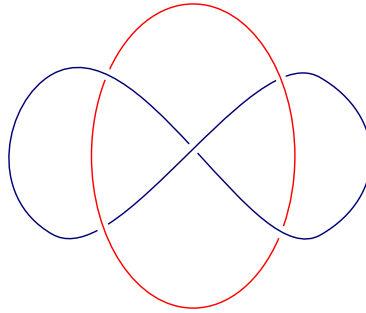


Figure 1: The Whitehead link.

**Theorem 1.1** *Let  $p_1, q_1$  and  $p_2, q_2$  be two pairs of nonvanishing coprime integers. Let  $S^3_{p_1/q_1, p_2/q_2}$  (WL) be the  $(p_1/q_1, p_2/q_2)$ -surgery on the Whitehead link. Then*

- $S^3_{p_1/q_1, p_2/q_2}$  (WL) is an *L*-space if and only if  $p_1/q_1 \geq 1$  and  $p_2/q_2 \geq 1$ ;
- $S^3_{p_1/q_1, p_2/q_2}$  (WL) supports a cooriented taut foliation if and only if  $p_1/q_1 < 1$  or  $p_2/q_2 < 1$ .

*In particular, for all these manifolds we have equivalence between (1) and (2) of Conjecture 1.*

Recall that since the Whitehead link has linking number zero, if  $p_1 = 0$  or  $p_2 = 0$  then  $S^3_{p_1/q_1, p_2/q_2}$  (WL) is not a rational homology sphere. This implies that when some of the parameters  $p_1, p_2, q_1$  and  $q_2$  vanish, the only rational homology spheres that can be obtained are  $S^3$  and lens spaces and it is known that they satisfy the *L*-space conjecture.

Also notice that the statement of Theorem 1.1 is invariant under the switch  $p_1/q_1 \leftrightarrow p_2/q_2$ . This is a consequence of the fact that the Whitehead link is symmetric, ie that there exists an isotopy exchanging its two components.

In the proof of the theorem we study the conditions of being an *L*-space and of supporting a taut foliation separately.

The key idea in the proof of the first part of the theorem is to use the results of J Rasmussen and SD Rasmussen [2017] and Gorsky, Liu and Moore [Gorsky et al. 2020] to prove the following more general fact:

**Theorem 1.2** *Suppose that  $\mathcal{L}$  is a nontrivial link in  $S^3$  with two unknotted components and linking number zero. Suppose that there exist rationals  $p_1/q_1 > 0$  and  $p_2/q_2 > 0$  such that  $S^3_{p_1/q_1, p_2/q_2}$  ( $\mathcal{L}$ ) is an *L*-space. Then there exist nonnegative integers  $b_1$  and  $b_2$  such that*

$$L(\mathcal{L}) = ([2b_1 + 1, \infty] \times [2b_2 + 1, \infty]) \cup (\{\infty\} \times \mathbb{Q}^*) \cup (\mathbb{Q}^* \times \{\infty\}).$$

In the previous statement the symbol  $L(\mathcal{L})$  denotes the set of the *L*-space filling slopes of the exterior of  $\mathcal{L}$ . This is the set of slopes such that filling the exterior of  $\mathcal{L}$  with such slopes yields *L*-spaces.

**Remark 1.3** We will not make use of this fact, but the integers  $b_1$  and  $b_2$  can be explicitly computed with the  $H$  function associated to  $\mathcal{L}$ ; see [Gorsky et al. 2020].

The foliations, on the other hand, are obtained by constructing branched surfaces without sink discs, using the results of Li [2002; 2003] and inspired by the works of Kalelkar and Roberts [Kalelkar and Roberts 2015; Roberts 2001a; 2001b]. Also in this case, part of the proof of [Theorem 1.1](#) will follow from a more general result that allows us to construct taut foliations on some fillings of manifolds that fiber over the circle with fiber a  $k$ -holed torus and some particular type of monodromy. This is the content of [Theorem 3.18](#), which seems to be interesting in itself.

We are also able to determine which of the taut foliations of [Theorem 1.1](#) have zero Euler class, by adapting the ideas of Hu [2019] to our case. This implies that the manifolds supporting such taut foliations have left-orderable first fundamental group. More precisely:

**Theorem 1.4** *Let  $S_{p_1/q_1, p_2/q_2}^3(\text{WL})$  be the  $(p_1/q_1, p_2/q_2)$ -surgery on the Whitehead link, with  $q_1, q_2 \neq 0$  and  $p_1, p_2 > 0$ . Then the foliations constructed in the proof of [Theorem 1.1](#) have vanishing Euler class if and only if  $|q_i| \equiv 1 \pmod{p_i}$  for  $i = 1, 2$ .*

*In particular, for all these manifolds the  $L$ -space conjecture holds.*

Moreover, [Theorem 1.1](#), together with some results proved in [Roberts et al. 2003], implies the following:

**Theorem 1.5** *All the rational homology spheres obtained by integer surgery on WL satisfy the  $L$ -space conjecture.*

We refer to the last section of this paper for a more detailed statement of [Theorem 1.5](#), which also combines results from [Zung 2020].

**Structure of the paper** The paper is organised as follows. In [Section 2](#) we recall the main result of [Rasmussen and Rasmussen 2017] and we prove [Theorem 1.2](#) and the first part of [Theorem 1.1](#). In [Section 3](#) we focus our attention on taut foliations. In [Section 3.1](#) we recall some basic notions on branched surfaces and the main result of [Li 2003]. In [Section 3.2](#) we prove [Theorem 3.18](#) and start the proof of the second part of [Theorem 1.1](#), which is concluded in [Section 3.3](#). In the last section we prove [Theorem 1.4](#) and collect from [Dunfield 2020; Roberts et al. 2003; Zung 2020] some other results about orderability, and nonorderability, of some surgeries on the Whitehead link.

**Acknowledgements** I warmly thank my advisors Bruno Martelli and Paolo Lisca for having presented this problem to me, for their support and for their useful comments on this paper. I also thank Alice Merz and Ludovico Battista for the several helpful discussions. I also thank the referee for the many valuable comments and suggestions.

## 2 L-spaces

In this section we prove the first part of [Theorem 1.1](#). We start by recalling some definitions and the main result of [\[Rasmussen and Rasmussen 2017\]](#). Let  $Y$  be a rational homology solid torus, ie  $Y$  is a compact oriented 3-manifold with toroidal boundary such that  $H_*(Y; \mathbb{Q}) \cong H_*(\mathbb{D}^2 \times S^1; \mathbb{Q})$ .

We are interested in the study of the Dehn fillings on  $Y$ . We define the *set of slopes in  $Y$*  as

$$\text{Sl}(Y) = \{\alpha \in H_1(\partial Y; \mathbb{Z}) \mid \alpha \text{ is primitive}\} / \pm 1.$$

It is a well known fact that each element  $[\alpha] \in \text{Sl}(Y)$  determines a Dehn filling on  $Y$ , which we will denote by  $Y(\alpha)$ .

Notice that since  $Y$  is a rational homology solid torus, there is a distinguished slope in  $\text{Sl}(Y)$  that we call the *homological longitude* of  $Y$  and define in the following way. We denote by  $i : H_1(\partial Y; \mathbb{Z}) \rightarrow H_1(Y; \mathbb{Z})$  the map induced by the inclusion  $\partial Y \subset Y$  and we consider a primitive element  $l \in H_1(\partial Y; \mathbb{Z})$  such that  $i(l)$  is torsion in  $H_1(Y; \mathbb{Z})$ . The element  $l$  is unique up to sign, and its equivalence class  $[l] \in \text{Sl}(Y)$  is the homological longitude of  $Y$ . This definition, which may seem to be counterintuitive, is given so that when  $Y$  is the complement of a knot in  $S^3$ , the homological longitude of  $Y$  coincides with the slope defined by the longitude the knot.

We want to study the fillings on  $Y$  that are L-spaces. For this reason we define the set of the *L-space filling slopes* by

$$L(Y) = \{[\alpha] \in \text{Sl}(Y) \mid Y(\alpha) \text{ is an L-space}\}$$

and we say that  $Y$  is *Floer simple* if  $Y$  admits multiple L-space filling slopes, ie if  $|L(Y)| > 1$ .

It turns out that if  $Y$  is Floer simple then the set  $L(Y)$  has a simple structure, and this can be computed by knowing the *Turaev torsion* of  $Y$ . We only recall some properties of the Turaev torsion, and we refer the reader to [\[Turaev 2002\]](#) for the precise definitions.

Fix an identification  $H_1(Y; \mathbb{Z}) = \mathbb{Z} \oplus T$ , where  $T$  is the torsion subgroup, and denote by  $\phi : H_1(Y; \mathbb{Z}) \rightarrow \mathbb{Z}$  the projection induced by this identification. Then the Turaev torsion of  $Y$  can be normalised to be written as a formal sum

$$\tau(Y) = \sum_{\substack{h \in H_1(Y; \mathbb{Z}) \\ \phi(h) \geq 0}} a_h h,$$

where  $a_h$  is an integer for each  $h$ ,  $a_0 \neq 0$  and  $a_h = 1$  for all but finitely many  $h$ . For example (see [\[Turaev 2002, Section II.5\]](#)), if  $H_1(Y; \mathbb{Z}) = \mathbb{Z}$ , the Turaev torsion of  $Y$  can be written as

$$\tau(Y) = \frac{\Delta(Y)}{1-t} \in \mathbb{Z}[[t]],$$

where  $(1-t)^{-1}$  is expanded as an infinite sum in positive powers of  $t$  and  $\Delta(Y)$  is the Alexander polynomial of  $Y$  normalised so that  $\Delta(Y) \in \mathbb{Z}[t]$ ,  $\Delta(Y)(0) \neq 0$  and  $\Delta(Y)(1) = 1$ . In fact, in this case the coefficients of  $\tau(Y)$  are eventually constant and equal to the sum of all the coefficients of  $\Delta(Y)$ , and this value is exactly  $\Delta(Y)(1) = 1$ .

We define  $S[\tau(Y)] = \{h \in H_1(Y; \mathbb{Z}) \mid a_h \neq 0\}$  to be the *support* of  $\tau(Y)$ .

We also define the subset of  $H_1(Y; \mathbb{Z})$ ,

$$D_{>0}^\tau(Y) = \{x - y \mid x \notin S[\tau(Y)], y \in S[\tau(Y)] \text{ and } \phi(x) > \phi(y)\} \cap i(H_1(\partial Y; \mathbb{Z})),$$

where  $i : H_1(\partial Y; \mathbb{Z}) \rightarrow H_1(Y; \mathbb{Z})$  is induced from the inclusion.

**Lemma 2.1** *The set  $D_{>0}^\tau$  is always finite.*

**Proof** Recall that we have fixed an identification  $H_1(Y, \mathbb{Z}) = \mathbb{Z} \oplus T$ , where  $T$  is the torsion subgroup, and we have denoted by  $\phi : H_1(Y, \mathbb{Z}) \rightarrow \mathbb{Z}$  the projection induced by this identification. Also recall that the Turaev torsion of  $Y$  is normalised so to be written as

$$\tau(Y) = \sum_{\substack{h \in H_1(Y; \mathbb{Z}) \\ \phi(h) \geq 0}} a_h h,$$

where  $a_h$  is an integer for each  $h$ ,  $a_0 \neq 0$  and  $a_h = 1$  for all but finitely many  $h$ . This implies in particular that if  $h \in S[\tau(Y)]$  then  $\phi(h) \geq 0$ . Moreover since  $a_h = 1$  for all but finitely many  $h$  with  $\phi(h) \geq 0$ , we also deduce that there exists a positive constant  $c \in \mathbb{Z}$  such that if  $h' \notin S[\tau(Y)]$  then  $\phi(h') \leq c$ .

We now prove that  $D_{>0}^\tau$  is finite. To do this, we define for each  $x \notin S[\tau(Y)]$  the set

$$\mathcal{G}_x^\tau = \{y \in S[\tau(Y)] \mid \phi(x) > \phi(y)\}.$$

We show that  $\mathcal{G}_x^\tau$  is always finite and that it is nonempty only for finitely many  $x \notin S[\tau(Y)]$ . It follows from the definition of  $D_{>0}^\tau(Y)$  that this implies that  $D_{>0}^\tau(Y)$  is finite. We fix  $x \notin S[\tau(Y)]$  and we have two cases:

- If  $\phi(x) \leq 0$  then, since all the  $y \in S[\tau(Y)]$  satisfy  $\phi(y) \geq 0$ , we have that  $\mathcal{G}_x^\tau$  is empty.
- If  $\phi(x) > 0$  then we use again the fact that all the  $y \in S[\tau(Y)]$  satisfy  $\phi(y) \geq 0$  to deduce that

$$\mathcal{G}_x^\tau \subset \{0, 1, \dots, \phi(x) - 1\} \oplus T \subset \mathbb{Z} \oplus T = H_1(Y, \mathbb{Z}).$$

Since the torsion subgroup  $T$  is finite,  $S_x^\tau$  is finite.

To conclude the proof we show that the latter case occurs only for finitely many  $x \notin S[\tau(Y)]$ . In fact, since there exists a positive constant  $c \in \mathbb{Z}$  such that if  $x \notin S[\tau(Y)]$  then  $\phi(x) \leq c$ , we have that the set  $\{x \notin S[\tau(Y)] \mid \phi(x) > 0\}$  is contained in  $\{0, 1, \dots, c\} \oplus T$ , and this is a finite set. □

We are now ready to state the main result of [\[Rasmussen and Rasmussen 2017\]](#).

**Theorem 2.2** [\[Rasmussen and Rasmussen 2017\]](#) *If  $Y$  is Floer simple, then either*

- $D_{>0}^\tau(Y)$  is empty and  $L(Y) = \text{Sl}(Y) \setminus [l]$ , or
- $D_{>0}^\tau(Y)$  is nonempty and  $L(Y)$  is a closed interval whose endpoints are consecutive elements in  $i^{-1}(D_{>0}^\tau(Y))$ .

We explain more precisely the second part of the statement of this theorem. Once we fix a basis  $(\mu, \lambda)$  for  $H_1(\partial Y; \mathbb{Z})$  we can associate to each element  $a\mu + b\lambda \in H_1(\partial Y; \mathbb{Z})$  the element  $a/b \in \overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\} \subset S^1$ . This association defines a map onto  $\overline{\mathbb{Q}}$  that yields an identification between  $\text{Sl}(Y)$  and  $\overline{\mathbb{Q}}$ .

If the set  $D_{>0}^\tau$  is not empty, then we can apply this map to the set  $i^{-1}(D_{>0}^\tau) \subset H_1(\partial Y; \mathbb{Z})$  and [Theorem 2.2](#) states that if  $Y$  is Floer simple then  $L(Y)$  is a closed interval in  $\text{Sl}(Y) = \overline{\mathbb{Q}}$  whose endpoints are consecutive elements in the image of  $i^{-1}(D_{>0}^\tau)$  in  $\overline{\mathbb{Q}}$ .

In the case of our interest we consider a link  $\mathcal{L} = K_1 \sqcup K_2 \subset S^3$  such that

- $\mathcal{L}$  has two unknotted components, and
- $\mathcal{L}$  has linking number zero.

By analogy with the definitions given for rational homology solid tori, we denote by  $\text{Sl}(\mathcal{L})$  the set of slopes of the exterior of  $\mathcal{L}$  and by  $L(\mathcal{L})$  the set of *L*-space filling slopes of the exterior of  $\mathcal{L}$ .

We fix an orientation of the components of  $\mathcal{L}$  and in this way we obtain canonical meridian-longitude bases  $(\mu_i, \lambda_i)_{i=1,2}$  of the first homology groups of the boundary tori of its exterior. The choice of these bases also determines an identification  $\text{Sl}(\mathcal{L}) = \overline{\mathbb{Q}} \times \overline{\mathbb{Q}}$ .

We denote by  $S^3_{p_1/q_1, \bullet}(\mathcal{L})$  the manifold obtained by drilling the second component of  $\mathcal{L}$  and by performing  $(p_1/q_1)$ -surgery on the first. Analogously, we denote by  $S^3_{\bullet, p_2/q_2}(\mathcal{L})$  the manifold obtained by drilling the first component of  $\mathcal{L}$  and by performing  $(p_2/q_2)$ -surgery on the second. When it will not be important which component we are drilling and on which component we are surgering, we will simply use the symbol  $S^3_{p/q}(\mathcal{L})$ .

Notice that since  $\mathcal{L}$  has linking number zero, we have an isomorphism

$$H_1(S^3_{p_1/q_1, \bullet}(\mathcal{L}); \mathbb{Z}) \cong \mathbb{Z}_{p_1} \oplus \mathbb{Z},$$

where the image of the meridian  $\mu_1$  in  $H_1(S^3_{p_1/q_1, \bullet}(\mathcal{L}); \mathbb{Z})$  is mapped to  $(1, 0)$  and the image of the meridian  $\mu_2$  in  $H_1(S^3_{p_1/q_1, \bullet}(\mathcal{L}); \mathbb{Z})$  is mapped to  $(0, 1)$ . An analogous result holds for  $S^3_{\bullet, p_2/q_2}(\mathcal{L})$ .

**Lemma 2.3** *Fix  $p \neq 0$  and  $q$  coprime integers. Suppose that  $S^3_{p/q}(\mathcal{L})$  is Floer simple. Then either*

- $L(S^3_{p/q}(\mathcal{L})) = \overline{\mathbb{Q}} \setminus \{0\}$ , or
- there is a natural number  $k > 0$  such that either  $L(S^3_{p/q}(\mathcal{L})) = [k, \infty)$  or  $L(S^3_{p/q}(\mathcal{L})) = [\infty, -k]$ .

**Proof** We suppose that  $S^3_{p/q}(\mathcal{L}) = S^3_{\bullet, p/q}(\mathcal{L})$ ; the case  $S^3_{p/q}(\mathcal{L}) = S^3_{p/q, \bullet}(\mathcal{L})$  being analogous. We denote  $S^3_{p/q}(\mathcal{L})$  by  $M$ .

The lemma follows from [Theorem 2.2](#) together with a simple inspection of the possible forms of the set  $D_{>0}^\tau(M)$ . In fact, if  $D_{>0}^\tau(M)$  is empty, then  $L(M) = \overline{\mathbb{Q}} \setminus \{0\}$ . Otherwise, recall that  $D_{>0}^\tau(M)$  is the subset of  $H_1(M; \mathbb{Z})$  defined as

$$D_{>0}^\tau(M) = \{x - y \mid x \notin S[\tau(M)], y \in S[\tau(M)] \text{ and } \phi(x) > \phi(y)\} \cap i(H_1(\partial M; \mathbb{Z})).$$



In our case the projection  $\phi$  associated to the identification

$$H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_p$$

is simply the map  $\phi(x_1, x_2) = x_1$  and therefore the condition  $\phi(x) > \phi(y)$  in the definition of  $D_{>0}^\tau(M)$  implies that

$$D_{>0}^\tau(M) \subset (\mathbb{Z}_{>0} \times \mathbb{Z}_p) \cap i(H_1(\partial M; \mathbb{Z})).$$

Moreover, since  $i(H_1(\partial M; \mathbb{Z})) = \mathbb{Z} \times \{0\}$  we have that  $D_{>0}^\tau(M)$  is a subset of  $\mathbb{Z}_{>0} \times \{0\}$ , and we denote by  $S = \{n_1, \dots, n_h\} \subset \mathbb{Z}_{>0}$  the first coordinates of its elements, listed in ascending order. Recall from [Lemma 2.1](#) that  $D_{>0}^\tau(M)$  is always a finite set.

We have that

$$i^{-1}(D_{>0}^\tau(M)) = \{(n_i, m) \in \mathbb{Z} \times \mathbb{Z} \mid n_i \in S \text{ and } m \in \mathbb{Z}\}$$

and we know by [Theorem 2.2](#) that  $L(M)$  is a closed interval in  $\overline{\mathbb{Q}}$  whose endpoints are consecutive elements in the set  $\{n_i/m \mid n_i \in S \text{ and } m \in \mathbb{Z}\}$ . Since the components of  $\mathcal{L}$  are unknotted, we know that  $S_{\infty, p/q}^3(\mathcal{L})$  is an L-space (it is indeed a lens space) and therefore that  $\infty$  belongs to  $L(M)$ . Hence we can conclude that either  $L(M) = [n_h, \infty]$  or  $L(M) = [\infty, -n_h]$ . □

As we already anticipated, the first part of [Theorem 1.1](#) will be a corollary of the following more general result, which we will prove soon:

**Theorem 1.2** *Suppose that  $\mathcal{L}$  is a nontrivial link with two unknotted components and linking number zero. Suppose that there exist rationals  $p_1/q_1 > 0$  and  $p_2/q_2 > 0$  such that  $S_{p_1/q_1, p_2/q_2}^3(\mathcal{L})$  is an L-space. Then there exist nonnegative integers  $b_1$  and  $b_2$  such that*

$$L(\mathcal{L}) = ([2b_1 + 1, \infty] \times [2b_2 + 1, \infty]) \cup (\{\infty\} \times \mathbb{Q}^*) \cup (\mathbb{Q}^* \times \{\infty\}).$$

As a corollary of [Theorem 1.2](#) we have:

**Corollary 2.4** (first part of [Theorem 1.1](#)) *The 3-manifold  $S_{p_1/q_1, p_2/q_2}^3(\text{WL})$  is an L-space if and only if  $p_1/q_1 \geq 1$  and  $p_2/q_2 \geq 1$ .*

**Proof** The  $(1, 1)$ -surgery on the Whitehead link is the Poincaré homology sphere, which has finite fundamental group and is therefore an L-space. In other words  $(1, 1)$  belongs to  $L(\text{WL})$ . The Whitehead link also has unknotted components and linking number zero and we can therefore apply [Theorem 1.2](#), which immediately implies the thesis. □

To prove [Theorem 1.2](#) we recall the definition of L-space link, as given by Gorsky and Némethi [2016]. We give the definition for a 2-component link, but it is generalisable to links with more components.

**Definition 2.5** [Gorsky and Némethi 2016] A link  $\mathcal{L} \subset S^3$  is an *L-space link* if all sufficiently large integer surgeries are L-spaces, ie if there exist integers  $p_1$  and  $p_2$  such that  $S_{d_1, d_2}^3(\mathcal{L})$  is an L-space for all integers  $d_1 > p_1$  and  $d_2 > p_2$ .

For knots, the existence of a positive rational L-space surgery implies the existence of arbitrarily large L-space surgeries, but this fails in the case of links, as [Liu 2017, Example 2.4] shows.

Nevertheless, the following lemma shows that such a generalisation holds if  $\mathcal{L}$  has unknotted components and linking number zero, or more generally if it is an  $n$ -component *Brunnian* link. Recall that a link  $L$  with three or more components is Brunnian if all of its sublinks are unlinks. A link with two components is Brunnian when its components are unknotted and have linking number zero. We will use the symbols  $\lfloor x \rfloor$  and  $\lceil x \rceil$ , where  $x$  is a rational number, to denote the integers

$$\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}, \quad \lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}.$$

**Lemma 2.6** Let  $\mathcal{L}$  be an  $n$ -component Brunnian link and suppose that there exist positive rational numbers  $r_1, r_2, \dots, r_n$  such that  $S_{r_1, \dots, r_n}^3(\mathcal{L})$  is an L-space. Then  $S_{s_1, \dots, s_n}^3(\mathcal{L})$  is an L-space for all  $(s_1, \dots, s_n)$  satisfying

$$\begin{aligned} s_i &\geq \lfloor r_i \rfloor && \text{if } r_i \geq 1, \\ s_i > \lfloor r_i \rfloor &= 0 && \text{if } 0 < r_i < 1. \end{aligned}$$

In particular,  $\mathcal{L}$  is an L-space link.

**Proof** We suppose that  $\mathcal{L}$  has two components, the proof being analogous in the general case. We have the following cases.

If  $r_1 \geq 1$  and  $r_2 \geq 1$ , then we start by considering the set  $L(S_{r_1, \bullet}^3(\mathcal{L}))$ . We know by hypothesis that this set contains  $r_2$  and since  $r_2$  is positive it follows from Lemma 2.3 that  $L(S_{r_1, \bullet}^3(\mathcal{L}))$  must be either  $\overline{\mathbb{Q}} \setminus \{0\}$  or  $[k, \infty]$  for some positive natural number  $k$ . In both of these cases, since  $r_2 \geq 1$  we can deduce that

$$[\lfloor r_2 \rfloor, \infty) \subset L(S_{r_1, \bullet}^3(\mathcal{L})).$$

We now consider the set  $L(S_{\bullet, \lfloor r_2 \rfloor}^3(\mathcal{L}))$ . We have just proved that it contains  $r_1$  and with the same argument as before we can deduce that

$$[\lceil r_1 \rceil, \infty) \subset L(S_{\bullet, \lfloor r_2 \rfloor}^3(\mathcal{L})).$$

By applying the same reasoning it follows that for every  $s_1 \geq \lceil r_1 \rceil$  we have that

$$[\lfloor r_2 \rfloor, \infty) \subset L(S_{s_1, \bullet}^3(\mathcal{L}))$$

and this is exactly what we wanted. A pictorial sketch of the proof is shown in Figure 2.

If  $r_1 \geq 1$  and  $r_2 < 1$ , then since  $0 < r_2 < 1$  we have  $L(S_{r_1, \bullet}^3(\mathcal{L})) = \overline{\mathbb{Q}} \setminus \{0\}$  and in particular

$$(0, \infty) = (\lfloor r_2 \rfloor, \infty) \subset L(S_{r_1, \bullet}^3(\mathcal{L})).$$

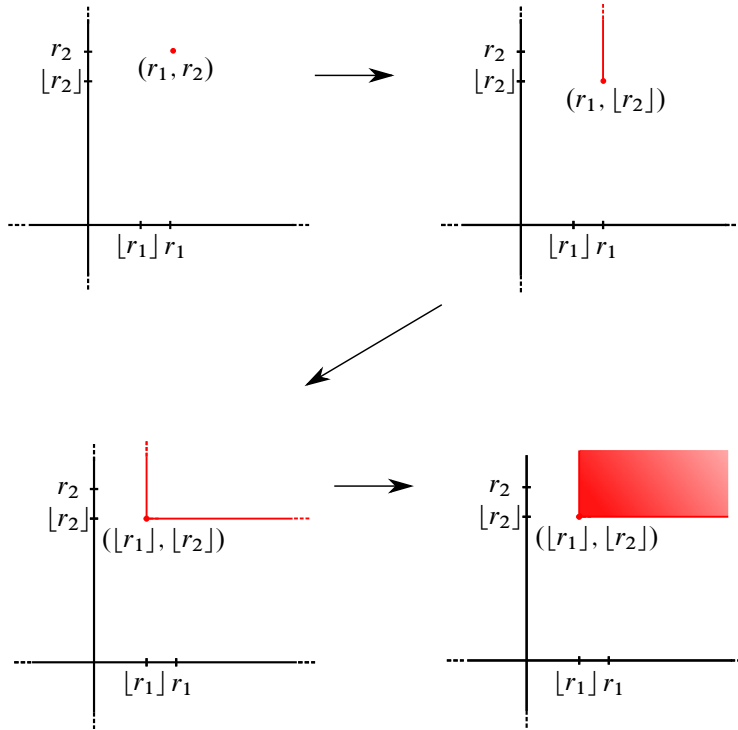


Figure 2: A pictorial sketch of the proof of Lemma 2.6.

This implies that for any  $s_2 > 0$  we have that  $S_{r_1, s_2}^3(\mathcal{L})$  is an L-space and therefore by applying again Lemma 2.3, since  $r_1 \geq 1$ ,

$$[[r_1], \infty) \subset L(S_{\bullet, s_2}^3(\mathcal{L}))$$

and this is exactly what we wanted.

The case of  $r_1 < 1$  and  $r_2 \geq 1$  is completely analogous to the previous one.

If  $r_1 < 1$  and  $r_2 < 1$ , then we again have

$$(0, \infty) = ([r_2], \infty) \subset L(S_{r_1, \bullet}^3(\mathcal{L})) = \overline{\mathbb{Q}} \setminus \{0\}.$$

As a consequence, for any  $s_2 > 0$  we have that  $S_{r_1, s_2}^3(\mathcal{L})$  is an L-space and therefore by applying again Lemma 2.3, since  $0 < r_1 < 1$ ,

$$(0, \infty) = ([r_1], \infty) \subset L(S_{\bullet, s_2}^3(\mathcal{L})) = \overline{\mathbb{Q}} \setminus \{0\}. \quad \square$$

**Remark 2.7** It follows from Theorem 1.2 that if  $\mathcal{L}$  has two components, then in the previous lemma the case  $0 < r_1 < 1$  or  $0 < r_2 < 1$  cannot occur.

Before proving Theorem 1.2 we recall the following theorem from [Gorsky et al. 2020].

**Theorem 2.8** [Gorsky et al. 2020] Assume that  $\mathcal{L}$  is a nontrivial *L*-space link with unknotted components and linking number zero. Then there exist nonnegative integers  $b_1$  and  $b_2$  such that for  $p_1, p_2 \in \mathbb{Z}$  we have that  $S_{p_1, p_2}^3(\mathcal{L})$  is an *L*-space if and only if  $p_1 > 2b_1$  and  $p_2 > 2b_2$ .

**Proof of Theorem 1.2** We know from Lemma 2.6 that  $\mathcal{L}$  is an *L*-space link. Therefore we can apply Theorem 2.8 and deduce that there exist nonnegative integers  $b_1$  and  $b_2$  such that

$$L(\mathcal{L}) \cap \mathbb{Z}^2 = [2b_1 + 1, \infty) \times [2b_2 + 1, \infty) \cap \mathbb{Z}^2.$$

Exactly as in the proof of Lemma 2.6, we can use Lemma 2.3 to deduce that

$$L(\mathcal{L}) \supset ([2b_1 + 1, \infty) \times [2b_2 + 1, \infty)) \cup (\{\infty\} \times \mathbb{Q}^*) \cup (\mathbb{Q}^* \times \{\infty\}),$$

and therefore we only have to prove that this inclusion is an equality.

Suppose on the contrary that there exists an *L*-space surgery slope  $(r_1, r_2)$ , with  $r_1$  and  $r_2$  rationals, such that

$$(r_1, r_2) \notin ([2b_1 + 1, \infty) \times [2b_2 + 1, \infty)) \cup (\{\infty\} \times \mathbb{Q}^*) \cup (\mathbb{Q}^* \times \{\infty\}).$$

We suppose that  $r_1 < 2b_1 + 1$ . The case  $r_2 < 2b_2 + 1$  can be solved in the same way. We analyse the possible cases.

If  $1 \leq r_1 < 2b_1 + 1$  then, by virtue of Lemma 2.3, we have that  $[\lceil r_1 \rceil, \infty)$  is contained in  $L(S_{\bullet, r_2}^3)$ . This implies that  $S_{\lceil r_1 \rceil, \bullet}^3$  is Floer simple and therefore, by applying again Lemma 2.3, we deduce that it admits integral *L*-space filling slopes. In this way we produce a point in

$$(L(\mathcal{L}) \cap \mathbb{Z}^2) \setminus (([2b_1 + 1, \infty) \times [2b_2 + 1, \infty)) \cap \mathbb{Z}^2),$$

contradicting Theorem 2.8.

If  $r_1 \in (-1, 1)$  then, as a consequence of Lemma 2.3, we have that  $L(S_{\bullet, r_2}^3) = \overline{\mathbb{Q}} \setminus \{0\}$ . Therefore if we fix any negative integer  $-m < 0$  we have that  $r_2 \in L(S_{-m, \bullet}^3)$  and by applying again Lemma 2.3 we deduce that there exist integral *L*-space filling slopes on  $S_{-k, \bullet}^3$ , contradicting Theorem 2.8.

If  $r_1 \leq -1$  then, by applying the same argument used in the first case,  $L(S_{\bullet, r_2}^3)$  contains  $[\infty, \lceil r_1 \rceil]$ . Therefore  $S_{\lceil r_1 \rceil, \bullet}^3$  admits integral *L*-space filling slopes, contradicting Theorem 2.8.  $\square$

### 3 Coorientable taut foliations

In this section we study the existence of taut foliations on the Dehn fillings on the Whitehead link exterior. The main theorem in this section is the following:

**Theorem 3.1** Let  $p_1, q_1$  and  $p_2, q_2$  be two pairs of nonvanishing coprime integers. Let  $S_{p_1/q_1, p_2/q_2}^3(\text{WL})$  be the  $(p_1/q_1, p_2/q_2)$ -surgery on the Whitehead link. Then  $S_{p_1/q_1, p_2/q_2}^3(\text{WL})$  supports a cooriented taut foliation if and only if  $p_1/q_1 < 1$  or  $p_2/q_2 < 1$ .

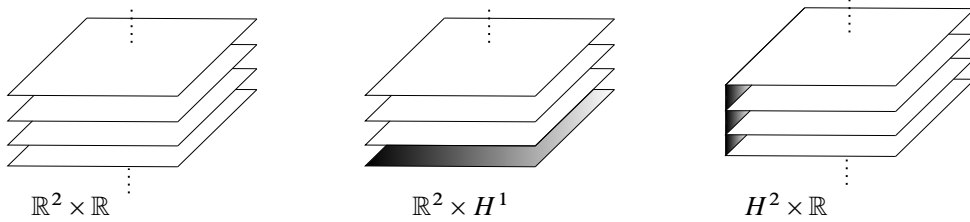


Figure 3: Local models for a foliation.

In this paper the term *foliation* will refer to codimension-1 foliations of class  $C^{\infty,0}$ , as defined for example in [Candel and Conlon 2000; Kazez and Roberts 2017]. We recall the definition here. We denote by  $H^k$  the  $k$ -dimensional Euclidean closed half space

$$H^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k \mid x_k \geq 0\}.$$

**Definition 3.2** A  $C^{\infty,0}$  codimension-1 foliation  $\mathcal{F}$  of a smooth 3-manifold  $M$  with (possibly empty) boundary is a decomposition of  $M$  into the union of disjoint smoothly injectively immersed surfaces, called the *leaves* of  $\mathcal{F}$ , together with a collection of charts  $(U_i, \phi_i)_{i \in \mathcal{I}}$  covering  $M$  such that

- $\phi_i : U_i \rightarrow \mathcal{X}$  is a homeomorphism, where  $\mathcal{X}$  is either  $\mathbb{R}^2 \times \mathbb{R}$ ,  $\mathbb{R}^2 \times H^1$  or  $H^2 \times \mathbb{R}$ , with the property that the image of each component of a leaf intersected with  $U_i$  is a slice  $\mathbb{R}^2 \times \{\text{point}\}$  or  $H^2 \times \{\text{point}\}$ ;
- all partial derivatives of any order in the variables  $x$  and  $y$  on the domain of each transition function  $\phi_j \phi_i^{-1}$  are continuous (here we have fixed coordinates  $(x, y, z)$  on  $\mathcal{X}$ ).

The three local models for a foliation are depicted in Figure 3, where  $\partial\mathcal{X}$  is shaded.

**Remark 3.3** The tangent planes to the leaves of a foliation  $\mathcal{F}$  of a 3-manifold  $M$  define a continuous plane subbundle of  $TM$ , which we denote by  $T\mathcal{F}$ .

**Definition 3.4** A foliation  $\mathcal{F}$  of a 3-manifold  $M$  is *orientable* if the plane bundle  $T\mathcal{F}$  is orientable and is *coorientable* if the line bundle  $TM/T\mathcal{F}$  is orientable.

**Definition 3.5** A foliation  $\mathcal{F}$  of a 3-manifold  $M$  is *taut* if every leaf of  $\mathcal{F}$  intersects a closed transversal, ie a smooth simple closed curve in  $M$  that is transverse to  $\mathcal{F}$ .

There are several definitions of tautness and in general they are not equivalent. For details we refer to [Colin et al. 2019], where the relations among these different notions are also discussed.

We recall that in order to prove Theorem 3.1 it is enough to prove the “if” part, since L-spaces do not support taut foliations [Bowden 2016; Kazez and Roberts 2017; Ozsváth and Szabó 2004], and we have already proved in the previous section that if  $p_i/q_i \geq 1$  for  $i = 1, 2$ , then  $S^3_{p_1/q_1, p_2/q_2}$  (WL) is an L-space.

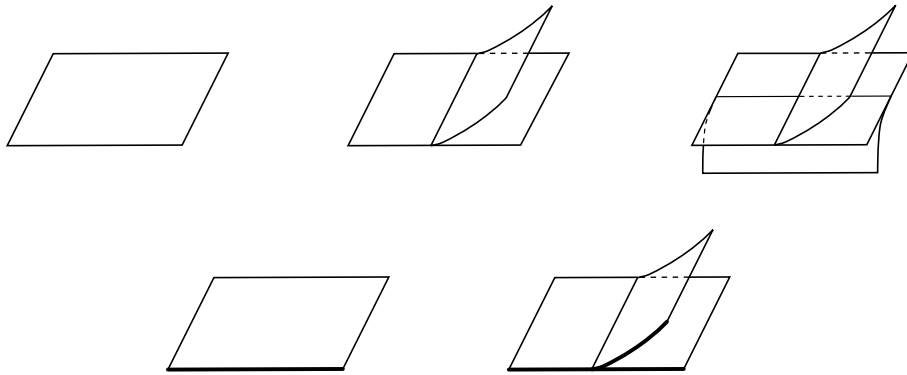


Figure 4: Local models for a branched surface.

This section is organised as follows. In [Section 3.1](#) we recall the required background regarding branched surfaces and we state the main theorem of [\[Li 2003\]](#). In [Section 3.2](#) we prove [Theorem 3.18](#), regarding the existence of taut foliations on Dehn fillings on manifolds that fiber over the circle with fiber a  $k$ -holed torus and with some prescribed monodromy. This theorem will be useful to prove that many of the manifolds of [Theorem 3.1](#) support taut foliations. In [Section 3.3](#) we conclude the proof of [Theorem 3.1](#).

### 3.1 Background

In this and the next sections we will assume familiarity with the basic notions of the theory of train tracks; see [\[Penner and Harer 1992\]](#) for reference. We only point out that in the cases of our interest, train tracks can also have bigons as complementary regions.

We now recall some basic facts about branched surfaces. We refer to [\[Floyd and Oertel 1984; Oertel 1984\]](#) for more details.

**Definition 3.6** A branched surface with boundary in a 3–manifold  $M$  is a closed subset  $B \subset M$  that is locally diffeomorphic to one of the models in  $\mathbb{R}^3$  of [Figure 4](#), top, or to one of the models in the closed half space of [Figure 4](#), bottom, where  $\partial B := B \cap \partial M$  is represented with a bold line.

Branched surfaces generalise the concept of train tracks from surfaces to 3–manifolds and when the boundary of  $B$  is nonempty it defines a train track  $\partial B$  in  $\partial M$ .

If  $B$  is a branched surface it is possible to identify two subsets of  $B$ : the *branch locus* and the set of *triple points* of  $B$ . The branch locus is defined as the set of points where  $B$  is not locally homeomorphic to a surface. It is self-transverse and intersects itself in double points only. The set of triple points of  $B$  can be defined as the points where the branch locus is not locally homeomorphic to an arc. For example, the rightmost model of [Figure 4](#), top, contains a triple point.

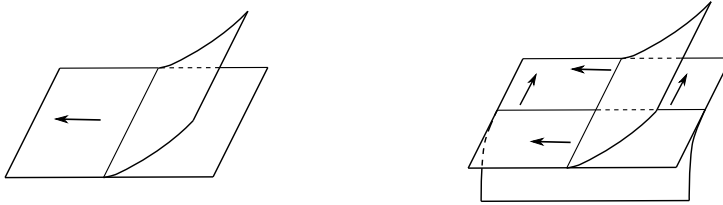


Figure 5: Some examples of cusp directions.

The complement of the branch locus in  $B$  is a union of connected surfaces. The abstract closures of these surfaces under any path metric on  $M$  are called the *branch sectors* of  $B$ . Analogously the complement of the set of the triple points inside the branch locus is a union of 1–dimensional connected manifolds. Moreover to each of these manifolds we can associate an arrow in  $B$  pointing in the direction of the smoothing, as we did in Figure 5. We call these arrows *branch directions*, or also *cusp directions*.

If  $B$  is a branched surface in  $M$ , we denote by  $N_B$  a fibered regular neighbourhood of  $B$  constructed as suggested in Figure 6.

The boundary of  $N_B$  decomposes naturally into the union of three compact subsurfaces,  $\partial_h N_B$ ,  $\partial_v N_B$  and  $N_B \cap \partial M$ . We call  $\partial_h N_B$  the *horizontal boundary* of  $N_B$  and  $\partial_v N_B$  the *vertical boundary* of  $N_B$ . The horizontal boundary is transverse to the interval fibers of  $N_B$  while the vertical boundary intersects, if at all, the fibers of  $N_B$  in one or two proper closed subintervals contained in their interior. If we collapse each interval fiber of  $N_B$  to a point, we obtain a branched surface in  $M$  that is isotopic to  $B$ , and the image of  $\partial_v N_B$  coincides with the branch locus of such a branched surface.

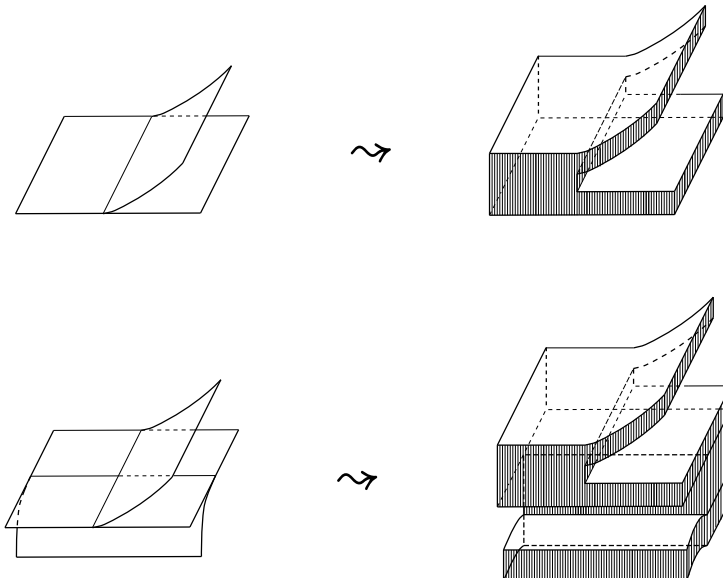


Figure 6: Regular neighbourhood of a branched surface.

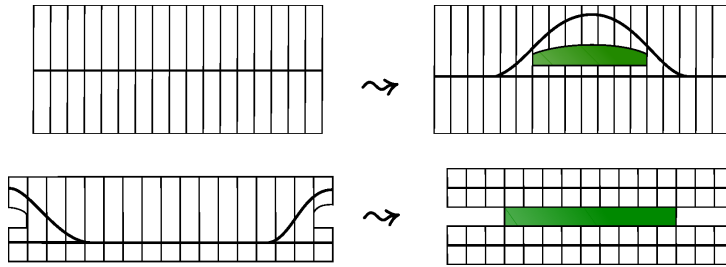


Figure 7: Some examples of splittings. The coloured region is the interval bundle  $J$ .

We also recall the definition of *splitting*.<sup>1</sup>

**Definition 3.7** Given two branched surfaces  $B_1$  and  $B_2$  in  $M$  we say that  $B_2$  is obtained by *splitting*  $B_1$  if  $N_{B_1}$  can be obtained as  $N_{B_2} \cup J$ , where  $J$  is a  $[0, 1]$ -bundle such that  $\partial_h J \subset \partial_h N_{B_2}$ ,  $\partial_v J \cap \partial N_{B_2} \subset \partial_v N_{B_2}$  and  $\partial J$  meets  $\partial N_{B_2}$  such that the fibers agree.

Figure 7 shows two examples of splittings, depicted for the case of 1-dimensional branched manifolds, ie train tracks.

Branched surfaces provide a useful tool to construct *laminations* on 3-manifolds.

**Definition 3.8** [Gabai and Oertel 1989] Let  $B$  be a branched surface in a 3-manifold  $M$ . A *lamination carried by  $B$*  is a closed subset  $\Lambda$  of some regular neighbourhood  $N_B$  of  $B$  such that  $\Lambda$  is a disjoint union of smoothly injectively immersed surfaces, called leaves, that intersect the fibers of  $N_B$  transversely. We say that  $\Lambda$  is *fully carried by  $B$*  if  $\Lambda$  is carried by  $B$  and intersects every fiber of  $N_B$ .

**Remark 3.9** Analogously to Definition 3.8, if  $S$  is a closed oriented surface and  $\tau$  is a train track in  $S$ , we can define what is a lamination (fully) carried by  $\tau$ . In this case we say that an oriented simple closed curve  $\gamma$  is *realised* by  $\tau$  if  $\tau$  fully carries a union of finitely many disjoint curves that are parallel to  $\gamma$  inside  $S$ .

Li [2002] introduces the notion of *sink disc*.

**Definition 3.10** Let  $B$  be a branched surface in  $M$  and let  $S$  be a branch sector in  $B$ . We say that  $S$  is a *sink disc* if  $S$  is a disc,  $S \cap \partial M = \emptyset$  and the branch direction of any smooth curve or arc in its boundary points into  $S$ . We say that  $S$  is a *half sink disc* if  $S$  is a disc,  $S \cap \partial M \neq \emptyset$  and the branch direction of any smooth arc in  $\partial S \setminus \partial M$  points into  $S$ .

In Figure 8 some examples of sink discs and half sink discs are depicted. The bold lines represent the intersection of the branched surface with  $\partial M$ . Notice that if  $S$  is a half sink disc the intersection  $\partial S \cap \partial M$  can also be disconnected.

<sup>1</sup>This operation is referred to as *restriction* in [Oertel 1984].



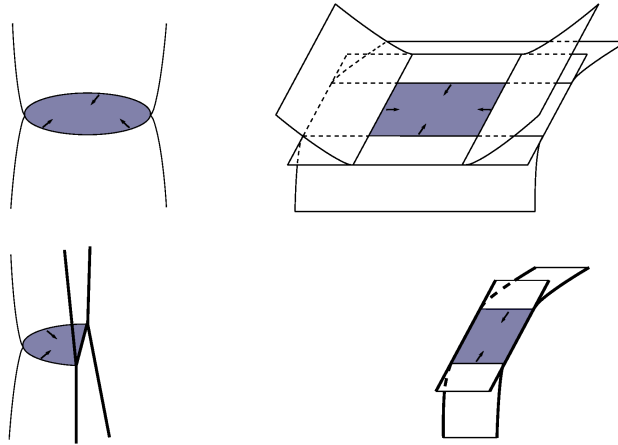


Figure 8: Examples of sink discs (top) and half sink discs (bottom).

If  $B$  contains a sink disc or a half sink disc there is a very simple way to eliminate it, namely it is enough to blow an air bubble in its interior, as in Figure 9.

We want to avoid this situation. We say that a connected component of  $M \setminus \text{int}(N_B)$  is a  $\mathbb{D}^2 \times [0, 1]$  region if it is homeomorphic to a ball and its boundary can be subdivided into an annular region, corresponding to a component of  $\partial_v N_B$ , and two  $\mathbb{D}^2$  regions corresponding to components of  $\partial_h N_B$ . We say that a  $\mathbb{D}^2 \times [0, 1]$  region is *trivial* if the map collapsing the fibers of  $N_B$  is injective on  $\text{int}(\mathbb{D}^2) \times \{0, 1\}$ . In this case the image of  $\mathbb{D}^2 \times \{0, 1\}$  via the collapsing map is called a *trivial bubble* in  $B$ . Trivial bubbles and trivial  $\mathbb{D}^2 \times [0, 1]$  regions are created when we eliminate sink discs as in Figure 9.

When  $M$  and  $B$  have boundary these definitions generalise straightforwardly to the relative case; see [Li 2003].

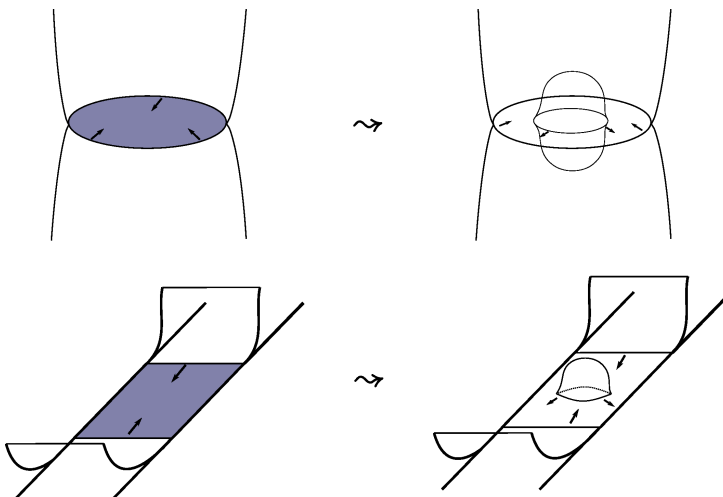


Figure 9: How to eliminate a sink disc or a half sink disc by blowing an air bubble.

Li [2002] introduces the definition of laminar branched surface and shows that laminar branched surfaces fully carry essential laminations.<sup>2</sup> In [Li 2003] he generalises this definition to branched surfaces with boundary as follows:

**Definition 3.11** [Li 2003] Let  $B$  be a branched surface in a 3-manifold  $M$ . We say that  $B$  is *laminar* if  $B$  has no trivial bubbles and

- (1)  $\partial_h N_B$  is incompressible and  $\partial$ -incompressible in  $M \setminus \text{int}(N_B)$ , and no component of  $\partial_h N_B$  is a sphere or a properly embedded disc in  $M$ ;
- (2) there is no monogon in  $M \setminus \text{int}(N_B)$ , ie no disc  $D \subset M \setminus \text{int}(N_B)$  such that  $\partial D = D \cap N_B = \alpha \cup \beta$ , where  $\alpha$  is in an interval fiber of  $\partial_v N_B$  and  $\beta$  is an arc in  $\partial_h N_B$ ;
- (3)  $M \setminus \text{int}(N_B)$  is irreducible and  $\partial M \setminus \text{int}(N_B)$  is incompressible in  $M \setminus \text{int}(N_B)$ ;
- (4)  $B$  contains no Reeb branched surfaces (see [Gabai and Oertel 1989] for the definition);
- (5)  $B$  has no sink discs or half sink discs.

Since  $\partial_h N_B$  is not properly embedded in  $M \setminus \text{int}(N_B)$  we explain more precisely the request of  $\partial$ -incompressibility in (1): we require that if  $D$  is a disc in  $M \setminus \text{int}(N_B)$  with  $\text{int}(D) \subset M \setminus N_B$  and  $\partial D = \alpha \cup \beta$  where  $\alpha$  is an arc in  $\partial_h N_B$  and  $\beta$  is an arc in  $\partial M$ , then there is a disc  $D' \subset \partial_h N_B$  with  $\partial D' = \alpha \cup \beta'$  where  $\beta' = \partial D' \cap \partial M$ .

The following theorem of [Li 2003] will be used profusely in this section.

**Theorem 3.12** [Li 2003] Let  $M$  be an irreducible and orientable 3-manifold whose boundary is the union of  $k$  incompressible tori  $T_1, \dots, T_k$ . Suppose that  $B$  is a laminar branched surface in  $M$  such that  $\partial M \setminus \partial B$  is a union of bigons. Then for any multislope  $(s_1, \dots, s_k) \in \overline{\mathbb{Q}}^k$  that is realised by the train track  $\partial B$ , if  $B$  does not carry a torus that bounds a solid torus in  $M(s_1, \dots, s_k)$ , there exists an essential lamination  $\Lambda$  in  $M$  fully carried by  $B$  that intersects  $\partial M$  in parallel simple curves of multislope  $(s_1, \dots, s_k)$ . Moreover this lamination extends to an essential lamination of the filled manifold  $M(s_1, \dots, s_k)$ .

**Remark 3.13** The statement of Theorem 3.12 is slightly more detailed than the version of [Li 2003]. The details we have added come from the proof of Theorem 3.12. In fact the idea of the proof is to split the branched surface  $B$  in a neighbourhood of  $\partial M$  so that it intersects  $T_i$  in parallel simple closed curves of slopes  $s_i$ , for  $i = 1, \dots, k$ . In this way, when gluing the solid tori, we can glue meridional discs of these tori to  $B$  to obtain a branched surface  $B(s_1, \dots, s_k)$  in  $M(s_1, \dots, s_k)$  that is laminar and that therefore, by virtue of [Li 2002, Theorem 1], fully carries an essential lamination. In particular, this essential lamination is obtained by gluing the meridional discs of the solid tori to an essential lamination in  $M$  that intersects  $T_i$  in parallel simple closed curves of slopes  $s_i$ , for  $i = 1, \dots, k$ .

<sup>2</sup>For the definition of essential lamination see [Gabai and Oertel 1989], but we will not need their properties for our purposes.

**Remark 3.14** In [Li 2003] the statement of the theorem is given for  $M$  with connected boundary, but as already noticed in [Kalelkar and Roberts 2015] if  $M$  has multiple boundary components we can split  $B$  in a neighbourhood of each boundary tori  $T_i$  and the same proof of [Li 2003] works.

### 3.2 Constructing taut foliations

The aim of this subsection is to prove [Theorem 3.18](#), that concerns the existence of taut foliations on Dehn fillings on manifolds that fiber over the circle with fiber a  $k$ -holed torus and with some prescribed monodromy. To do this we will recall a very simple, yet useful, way to build branched surfaces. First of all we fix some notation and recall the definition of fibered link.

Given an oriented surface  $S$  with (possibly empty) boundary and  $h: S \rightarrow S$  an orientation-preserving homeomorphism of  $S$  fixing  $\partial S$  pointwise we denote by  $M_h$  the mapping torus of  $h$ ,

$$M_h = \frac{S \times [0, 1]}{(h(x), 0) \sim (x, 1)}.$$

We orient  $S \times [0, 1]$  as a product and we orient  $M_h$  with the orientation induced by  $S \times [0, 1]$ . We also identify  $S$  with its image in  $M_h$  via the map

$$S \rightarrow S \times \{0\} \subset M_h, \quad x \mapsto (x, 0).$$

The homeomorphism  $h$  is called the *monodromy* of  $M_h$ .

**Definition 3.15** Let  $L$  be an oriented link in  $S^3$ . We say that  $L$  is *fibered* if there exists a Seifert surface  $S$  for  $L$ , an orientation-preserving homeomorphism  $h$  of  $S$  fixing  $\partial S$  pointwise and an orientation-preserving homeomorphism

$$\chi: S^3 \setminus \text{int}(N_L) \rightarrow M_h,$$

where  $N_L$  denotes a tubular neighbourhood of  $L$  in  $S^3$ , such that

- $\chi|_S$  is the inclusion  $S \subset M_h$ ;
- $\chi(m_i) = \{x_i\} \times [0, 1]$ , where  $m_i$  is a meridian for the  $i^{\text{th}}$  component of  $L$  and  $x_i \in \partial S$  is a point.

We want to apply [Theorem 3.12](#) to construct laminations on  $S^3_{p_1/q_1, p_2/q_2}$  (WL) when  $p_i/q_i < 1$  for at least one  $i \in \{1, 2\}$  and then promote them to taut foliations. To do this we will define some branched surfaces in the exterior of the Whitehead link and then study their boundary train tracks.

The construction of these branched surfaces relies on the fact that the Whitehead link is a fibered link. In fact, as [Figure 10](#) shows, the Whitehead link can be obtained as the boundary of a surface  $F$  that is a torus with two open discs removed. This torus is obtained by a sequence of three Hopf plumbings and this implies by standard results [[Gabai 1986](#); [Stallings 1978](#)] that  $F$  is a fiber surface for WL with monodromy  $h$  given by  $h = \tau_0 \tau_1 \tau_2^{-1}$ , where  $\tau_i$  denotes the positive Dehn twist along the curve  $\gamma_i$  and where the factorisation of  $h$  should be read from right to left.

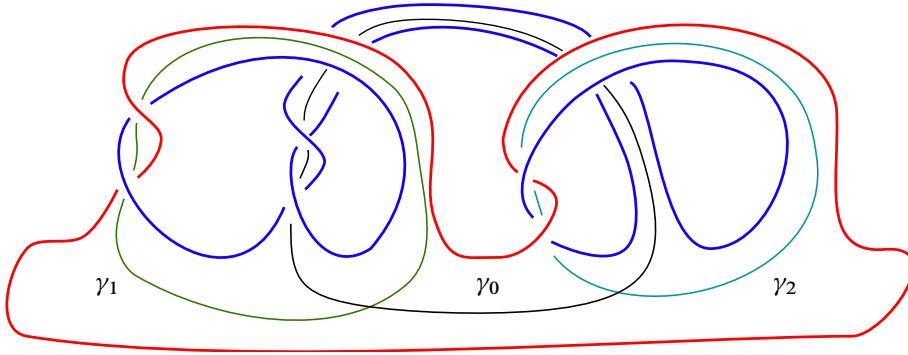


Figure 10: The Whitehead link is the union of the red curve and the blue curve. The curves  $\gamma_0$ ,  $\gamma_1$  and  $\gamma_2$  lie on the Seifert surface that should be evident from the picture.

Before focusing on the specific example of the Whitehead link exterior, we work in a more general setting.

Let  $S$  be an oriented surface with boundary and let  $h$  be an orientation-preserving homeomorphism of  $S$  fixing  $\partial S$  pointwise. We consider some pairwise disjoint properly embedded arcs  $\alpha_1, \dots, \alpha_k$  in  $S$  and the discs  $\bar{D}_i = \alpha_i \times [0, 1] \subset S \times [0, 1]$ . Each of these discs has a “bottom” boundary,  $\alpha_i \times \{0\}$ , and a “top” boundary,  $\alpha_i \times \{1\}$ . When we consider the images of these discs in  $M_h$  under the projection map

$$S \times [0, 1] \rightarrow M_h,$$

the bottom and top boundaries become respectively  $\bigcup_i \alpha_i \subset S$  and  $\bigcup_i h(\alpha_i) \subset S$ .

We can isotope simultaneously the discs  $\bar{D}_i$  in a neighbourhood of  $S \times \{1\} \subset S \times [0, 1]$  so that when projected to  $M_h$  their top boundaries define a family of arcs  $\{\widetilde{h(\alpha_i)}\}_{i=1, \dots, k}$  in  $S$  such that for each  $i, j \in \{1, \dots, k\}$  the intersection between  $\alpha_i$  and  $\widetilde{h(\alpha_j)}$  is transverse and minimal. Notice that each arc  $\widetilde{h(\alpha_i)}$  is isotopic as a properly embedded arc to  $h(\alpha_i)$ . We also denote by  $D_i$  the projected perturbed disc contained in  $M_h$ .

If we assign (co)orientations to these discs, since  $S$  is (co)oriented, we can smooth  $S \cup D_1 \cup \dots \cup D_k$  to a branched surface  $B$  by imposing that the smoothing preserves the coorientation of  $S$  and of the discs. In particular, each disc has two possible coorientations and therefore it can be smoothed in two different ways. This operation is demonstrated in Figure 11, where  $S$  is a torus with an open disc removed.

We prove the following lemma, which is only implicit in [Kalelkar and Roberts 2015].

**Lemma 3.16** *Let  $S$  be a connected and oriented surface with boundary and let  $h$  be an orientation-preserving homeomorphism of  $S$  fixing  $\partial S$  pointwise. Let  $\{\alpha_i\}_{i=1, \dots, k} \subset S$  be pairwise disjoint properly embedded arcs in  $S$  and suppose that  $S \setminus \bigcup_{i=1}^k \alpha_i$  has no disc components. Denote by  $D_i$  the discs in  $M_h$  associated to the arcs  $\alpha_i$  in the way described above and fix a coorientation for these discs. Let  $B = S \cup D_1 \cup \dots \cup D_k$  denote the branched surface in  $M_h$  obtained by smoothing according to these coorientations. Then  $B$  has no trivial bubbles and satisfies conditions (1), (2), (3) and (4) of Definition 3.11.*

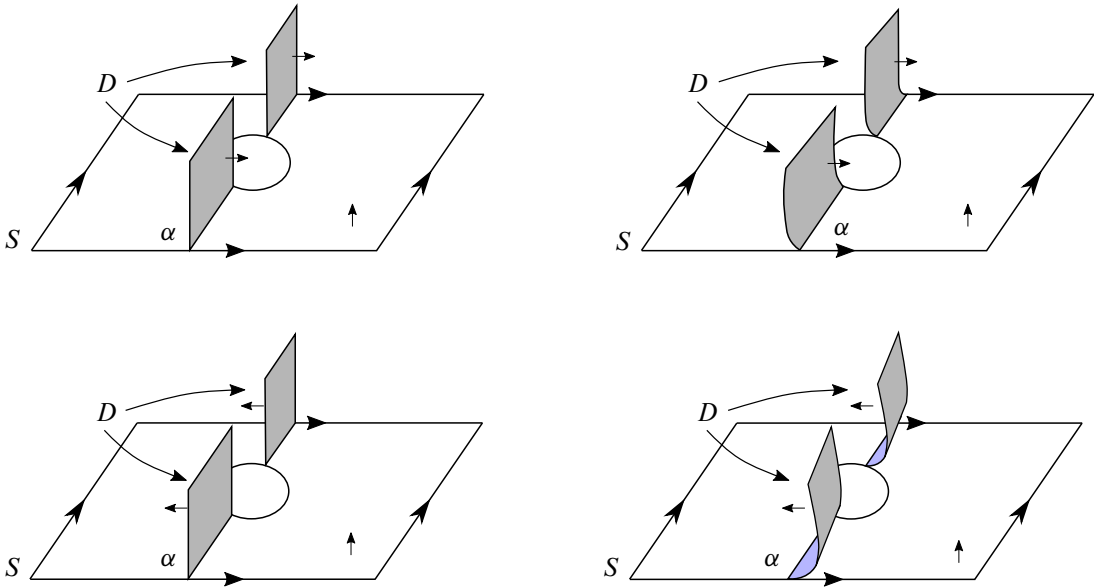


Figure 11: How to smooth  $S \cup D$  according to the coorientations.

**Proof** We denote the mapping torus  $M_h$  by  $M$ . We fix for each arc  $\alpha_i$  a tubular neighbourhood  $N_{\alpha_i}$  in  $S$  and we denote by  $S'$  the surface  $S \setminus \bigcup_{i=1}^k \text{int}(N_{\alpha_i})$ . The first observation is that by construction,

$$M \setminus \text{int}(N_B) \cong S' \times [0, 1]$$

with a homeomorphism that identifies

$$\partial_h N_B = S' \times \{0, 1\} \quad \text{and} \quad \partial_v N_B = \partial' S' \times [0, 1],$$

where  $\partial' S'$  denotes the closure of  $\partial S' \setminus \partial M$ .

Basically, the proof follows from the fact that  $M \setminus \text{int}(N_B)$  is homeomorphic to  $S' \times [0, 1]$  and that  $S'$  has no discs components.

First of all, we notice that since by hypothesis  $S \setminus \bigcup_{i=1}^k \alpha_i$  has no discs components, there are no  $\mathbb{D}^2 \times [0, 1]$  regions in  $M \setminus \text{int}(N_B)$  and in particular no trivial bubbles. We now verify that conditions (1)–(4) of [Definition 3.11](#) hold.

(1) The horizontal boundary  $\partial_h N_B$  is incompressible in  $M \setminus \text{int}(N_B)$ : This follows from the fact that the inclusions of  $S' \times \{0\}$  and  $S' \times \{1\}$  in  $M \setminus \text{int}(N_B)$  are homotopy equivalences. In particular, if a simple closed curve in  $\partial_h N_B$  bounds a disc in  $M \setminus \text{int}(N_B)$  then it must be nullhomotopic in  $\partial_h N_B$  and nullhomotopic simple closed curves in surfaces always bound embedded discs.

The horizontal boundary  $\partial_h N_B$  is  $\partial$ -incompressible in  $M \setminus \text{int}(N_B)$ : Suppose that there is a disc  $\Delta \subset M \setminus \text{int}(N_B)$  such that  $\text{int}(\Delta) \subset M \setminus N_B$  and  $\partial\Delta = a \cup b$ , where  $a$  is an arc in  $\partial_h N_B$  and  $b = \partial\Delta \cap \partial M$ . We have to find a disc  $\Delta' \subset \partial_h N_B$  with  $\partial\Delta' = a \cup b'$  where  $b' = \partial\Delta' \cap \partial M$ .

Without loss of generality we can suppose that  $a \subset S' \times \{0\}$ . The arc  $b$  is an arc in  $\partial M \setminus \text{int}(N_B)$  with both endpoints in  $S' \times \{0\}$  and since the connected components of  $\partial M \setminus \text{int}(N_B)$  are either discs or annuli, there exists a homotopy in  $\partial M \setminus \text{int}(N_B)$ , relative to the boundary, from the arc  $b$  to an arc  $b' \subset (S' \times \{0\}) \cap \partial M$ . In particular since the simple closed curve  $a \cup b$  is nullhomotopic in  $M \setminus \text{int}(N_B)$ , the curve  $a \cup b'$  is nullhomotopic as well.

To conclude it is enough to observe that since the inclusion of  $S' \times \{0\}$  in  $M \setminus \text{int}(N_B)$  is a homotopy equivalence, the simple closed curve  $a \cup b'$  bounds a disc  $\Delta'$  in  $S' \times \{0\}$ .

No component of the horizontal boundary is a sphere or a properly embedded disc: this follows by our hypotheses.

(2) There is no monogon in  $M \setminus \text{int}(N_B)$ : this is a consequence of the fact that the branched surface  $B$  admits a coorientation.

(3)  $M \setminus \text{int}(N_B)$  is irreducible: this is a consequence of the fact that each component of  $M \setminus \text{int}(N_B)$  is the product of a surface with boundary with  $[0, 1]$ .

$\partial M \setminus \text{int}(N_B)$  is incompressible in  $M \setminus \text{int}(N_B)$ : Consider any boundary component  $T$  of  $M$ . By construction  $T \setminus \text{int}(N_B)$  is a union of discs or an annulus (in case there are no endpoints of the arcs  $\alpha_i$  on  $T$ ). In the former case,  $T \setminus \text{int}(N_B)$  is obviously incompressible in  $M \setminus N_B$ , while in the latter it is compressible if and only if it is the boundary of  $S' \times [0, 1]$  and  $S' \times [0, 1]$  is diffeomorphic to  $\mathbb{D}^2 \times [0, 1]$ , but this would contradict our hypotheses.

(4)  $B$  contains no Reeb branched surfaces: the presence of a Reeb branched surface would imply that some of the complementary regions of  $\text{int}(N_B)$  are  $\mathbb{D}^2 \times [0, 1]$  regions [Gabai and Oertel 1989] and we have already observed that there are no such regions. □

All the branched surfaces we will use are obtained with the previous construction. One problem that one has to face is that there could be sink discs in such branched surfaces. Kalelkar and Roberts [2015] present a useful procedure to build splittings of branched surfaces constructed in this way that are without sink discs, and part of the results that we are going to obtain can also be proved with the methods there presented (see Remark 3.24). However, to be able to construct taut foliations on all the manifolds of the statement of Theorem 3.1 we will need to find a way to build our branched surfaces that is slightly different from the one presented in [Kalelkar and Roberts 2015].

We fix some notation. We suppose that  $S$  is a torus with  $k$  open discs removed. We consider the curves  $\gamma_0, \gamma_1, \dots, \gamma_k$  and we label the boundary components of  $S$  with numbers in  $\{1, \dots, k\}$  as in Figure 12. We also orient  $S$  so that the orientation induced on the boundary components is the one of the figure.

We denote by  $\tau_i$  the positive Dehn twist along the curve  $\gamma_i$ . Notice that since  $\gamma_i \cap \gamma_j = \emptyset$  for  $i \neq j$  and  $i, j \in \{1, \dots, k\}$  we have that  $\tau_i \tau_j = \tau_j \tau_i$  for  $i, j = 1, \dots, k$ .

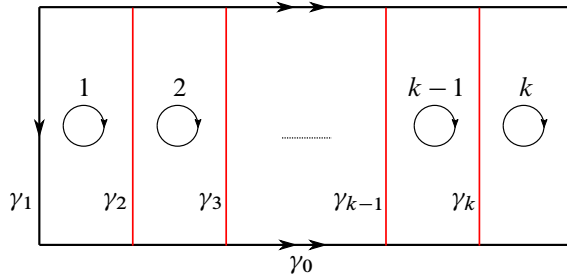


Figure 12: The oriented torus  $S$  with the labelled boundary components.

We focus on homeomorphisms of  $S$  of the type

$$h = \tau_0^{a_0} \tau_1^{a_1} \cdots \tau_k^{a_k}, \quad a_0 \in \mathbb{Z}, a_i \in \mathbb{Z} \setminus \{0\} \text{ for } i \neq 0,$$

where the factorisation of  $h$  should be read from right to left.

We fix the following convention:

**Convention** The indices  $1, \dots, k$  have to be considered ordered cyclically; so we set  $a_{k+1} = a_1$  and think of  $a_1$  as consecutive to  $a_k$ .

Let  $\partial_i S$  denote the boundary component of  $S$  labelled with  $i$ . Given such a homeomorphism  $h$  we assign to  $\partial_i S$  the label

- $p_+$  if  $a_i$  and  $a_{i+1}$  are both positive;
- $p_-$  if  $a_i$  and  $a_{i+1}$  are both negative;
- $n$  if  $a_i$  and  $a_{i+1}$  have different signs.

Figure 13 shows an example. Notice that in this example, following our convention, to assign a label to  $\partial_3 S$  we have to check the signs of  $a_3$  and  $a_1$ , since  $a_1$  is consecutive to  $a_3$ . Also notice that when  $k = 1$ , ie  $S$  has only one boundary component, we have that  $\partial_1 S$  has label  $p_+$  when  $a_1$  is positive and label  $p_-$  when  $a_1$  is negative.

Finally we assign to each boundary component  $\partial_i S$  two intervals  $I_i$  and  $J_i$  in  $\overline{\mathbb{Q}}$  in the following way:

- If  $\partial_i S$  has label  $p_+$  we set  $I_i = J_i = (\infty, 1)$ .
- If  $\partial_i S$  has label  $p_-$  we set  $I_i = J_i = (-1, \infty)$ .
- Let  $i_1 < \dots < i_{2c}$  the indices of the boundary components labelled with  $n$ . We set  $I_{i_a} = (\infty, 0)$  when  $a \in \{1, \dots, 2c\}$  is odd and we set  $I_{i_a} = (0, \infty)$  when  $a \in \{1, \dots, 2c\}$  is even. Therefore,  $I_{i_1} = (\infty, 0)$ ,  $I_{i_2} = (0, \infty)$ ,  $I_{i_3} = (\infty, 0)$  and so on. On the contrary, we set  $J_{i_a} = (0, \infty)$  when  $a \in \{1, \dots, 2c\}$  is odd and  $J_{i_a} = (\infty, 0)$  when  $a \in \{1, \dots, 2c\}$  is even.

**Example 3.17** In the example of Figure 13,

$$I_1 = J_1 = (\infty, 1), \quad I_2 = (\infty, 0), \quad J_2 = (0, \infty), \quad I_3 = (0, \infty), \quad J_3 = (\infty, 0).$$

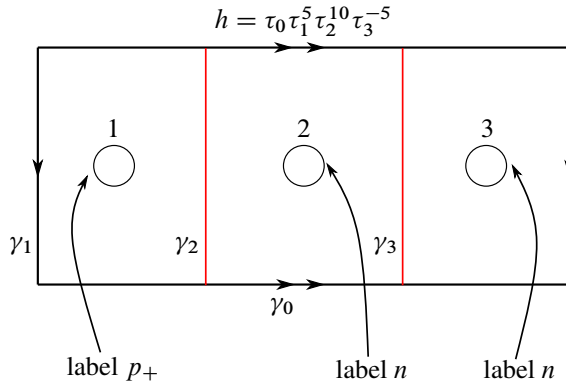


Figure 13: Example with  $h = \tau_0 \tau_1^5 \tau_2^{10} \tau_3^{-5}$ .

We are now ready to state the theorem. In the statement of the theorem, for each boundary torus  $T_i$  of the manifold  $M_h$  we have fixed as longitude the oriented curve  $\partial_i S$  and as meridian the image in  $M_h$  of the curve  $\{x_i\} \times [0, 1]$ , oriented as  $[0, 1]$ , where  $x_i \in \partial_i S$ .

**Theorem 3.18** *Let  $S$  be a  $k$ -holed torus as in Figure 12 and let  $h$  be a homeomorphism of  $S$  of the form*

$$h = \tau_0^{a_0} \tau_1^{a_1} \dots \tau_k^{a_k},$$

where  $a_0 \in \mathbb{Z}$  and  $a_i \in \mathbb{Z} \setminus \{0\}$  for  $i = 1, \dots, k$ .

- (1) *If  $a_0 > 0$  (resp.  $a_0 < 0$ ) then  $M_h(s_1, \dots, s_k)$  supports a coorientable taut foliation for each multislope  $(s_1, \dots, s_k) \in (\infty, 1)^k$  (resp.  $(-1, \infty)^k$ ).*
- (2) *For any multislope  $(s_1, \dots, s_k) \in (I_1 \times \dots \times I_k) \cup (J_1 \times \dots \times J_k)$ , the filled manifold  $M_h(s_1, \dots, s_k)$  supports a coorientable taut foliation, where the intervals  $I_i$  and  $J_i$  are the ones described above.*

**Remark 3.19** *If  $h'$  is conjugated in  $\text{MCG}(S, \partial S)$  to a homeomorphism  $h$  that satisfies the hypotheses of Theorem 3.18, then the conclusion of the theorem holds also for  $M_{h'}$ .*

**Remark 3.20** *The first part of the theorem does not cover the case  $a_0 = 0$ . We did not investigate further this case, but for our purpose the statement of Theorem 3.18 will be sufficient.*

**3.2.1 Proof of the first part of Theorem 3.18** *To prove Theorem 3.18 we will build branched surfaces in  $M_h$  satisfying the hypotheses of Theorem 3.12 by following the construction presented before Lemma 3.16.*

We start by proving the first part of the theorem. We define a branched surface as follows. We consider the parallel arcs  $\alpha_1, \dots, \alpha_k$  depicted in Figure 14.

We consider the discs  $\alpha_i \times [0, 1]$ , perturb them in a neighbourhood of  $S \times \{1\}$  as explained in the discussion before Lemma 3.16, and project them to the mapping torus  $M_h$ . We consider the (co)oriented branched



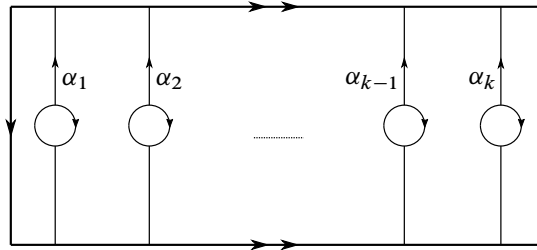


Figure 14: The parallel arcs  $\alpha_1, \dots, \alpha_k$ .

surface  $B$  in  $M_h$  obtained by adding these discs  $D_i$  to the surface  $S$ . The discs  $D_i$  are oriented such that the orientation on their boundary induces the given orientation on the arcs  $\alpha_i$ . For an example, see Figure 15, where also some cusp directions are shown. A good way to deduce the cusp directions along the arcs  $\alpha_i$  and  $h(\alpha_i)$  is the following: they point to the right along the arcs  $\alpha_i$  and they point to the left along the arcs  $h(\alpha_i)$ , where the latter are oriented as the image of the arcs  $\alpha_i$ .

**Lemma 3.21** *The branched surface  $B$  is laminar and satisfies the hypotheses of Theorem 3.12.*

**Proof** Thanks to Lemma 3.16, to prove that  $B$  is laminar it is enough to prove that  $B$  contains no sink discs or half sink discs. We prove that:

- $B$  contains no sink discs or half sink discs: there are  $k$  sectors of  $B$  that are half discs and that coincide with the discs  $D_i$ ; these sectors are never sink by construction (see Figure 11).

The other sectors coincide with the abstract closures of the connected components of  $S \setminus (\bigcup_i \alpha_i \cup \widetilde{h(\alpha_i)})$ . Since  $a_0$  is nonzero, these sectors are discs and half discs.<sup>3</sup> We can organise these sectors in the following way. We refer to Figure 15 to visualise the situation. If we cut  $S$  along the arcs  $\alpha_i$  we obtain  $k$  oriented annuli  $A_1, \dots, A_k$  such that  $\partial A_i \supset -\alpha_i \cup \alpha_{i+1}$  for  $i \in \{1, \dots, k\}$ , where  $-\alpha_i$  denotes the arc  $\alpha_i$  with the opposite orientation. Also notice that the cusp directions along  $\alpha_i$  point inside  $A_i$  and the cusp directions along  $\alpha_{i+1}$  point outside  $A_i$ .

It follows by the definition of the arcs  $\alpha_i$  that  $h(\alpha_i) = \tau_0^{a_0}(\alpha_i)$ . Each of these annuli intersects  $h(\alpha_j)$  in  $|a_0|$  subarcs, for each  $j = 1, \dots, k$ . Therefore, when we cut along the  $h(\alpha_j)$  we subdivide each of this annuli in  $k|a_0|$  discs and these discs coincide with the sectors of  $B$  in  $S$ . By construction each of these discs is contained in an annulus, say  $A_i$ , and intersects both  $\alpha_i$  and  $\alpha_{i+1}$  and therefore there is a cusp direction pointing outside it.

- The only connected compact surface properly embedded in  $M_h$  carried by  $B$  is  $S$ : if  $\Sigma$  is a compact surface properly embedded in  $M_h$  carried by  $B$ , then  $\Sigma$  induces an integral weight system on  $B$ ; that is to say,  $\Sigma$  defines a way to assign to each branch sector of  $B$  a nonnegative integer such that along each connected component of the branch locus minus the set of triple points of  $B$  the weights sum according to the cusp directions, as represented in Figure 16.

<sup>3</sup>When the product  $k|a_0|$  satisfies  $k|a_0| \leq 3$  there are only half-disc sectors.

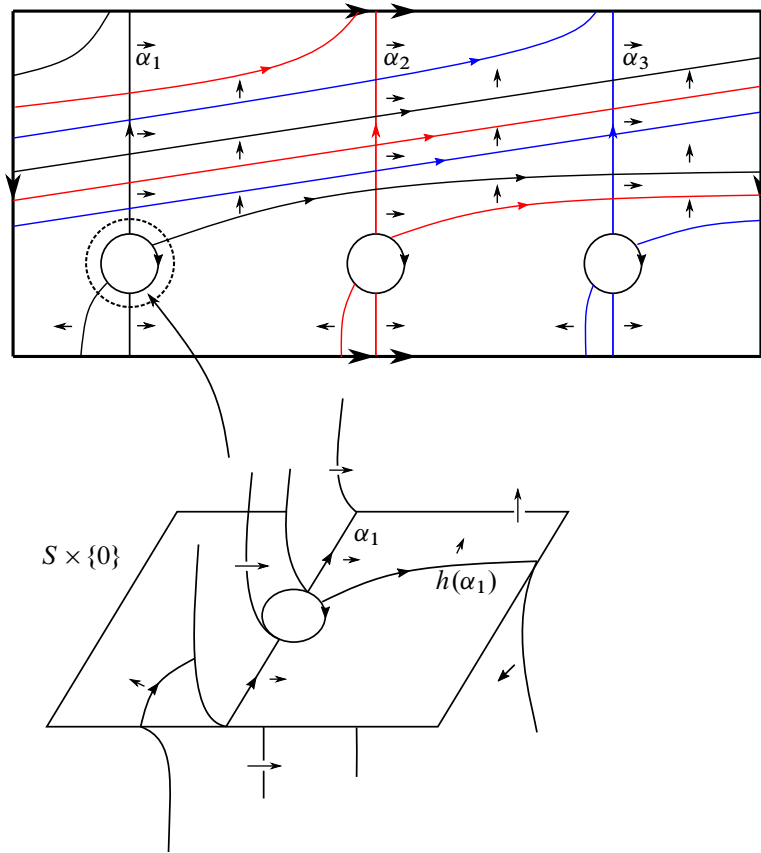


Figure 15: In this example,  $a_0 = 2$  and  $k = 3$ . We also show the details of our branched surface in a neighbourhood of  $\partial_1 S$ .

The way  $\Sigma$  induces a weight system is the following: we fix a point in the interior of each sector and we assign to each sector the number of intersections between  $\Sigma$  and the fiber of  $N_B$  over this fixed point. We denote by  $\omega_i$  the weights of the half-disc sectors  $D_i$ .

We have already observed that the other sectors of  $B$  are organised such that each of the annuli  $A_1, \dots, A_k$  contains  $k|a_0|$  discs and half-disc sectors. When  $a_0 > 0$  (resp.  $a_0 < 0$ ), for each  $i = 1, \dots, k$ , we order in each annulus these sectors by following the direction of  $\alpha_i$  (resp.  $-\alpha_i$ ) and denote by  $\Delta_{i,j}$  the discs

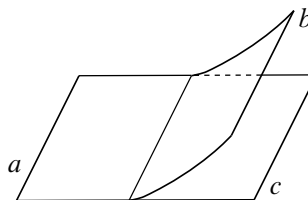


Figure 16: The weights  $a$ ,  $b$  and  $c$  must satisfy the equality  $a = b + c$ .

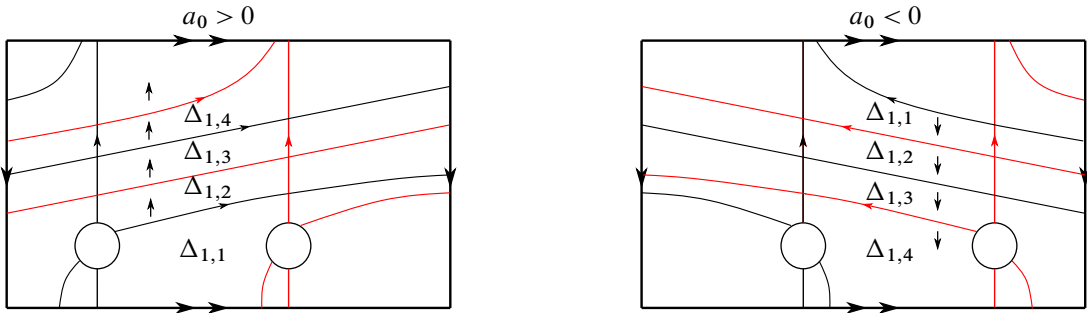


Figure 17: In this example it is shown how to order the discs inside the annulus  $A_1$  in the case  $a_0 > 0$  and in the case  $a_0 < 0$ .

contained in the annulus  $A_i$ , with  $j = 1, \dots, ka_0$ . We denote the weight of the disc  $\Delta_{i,j}$  by  $\omega_{i,j}$ . See Figure 17 for an example.

Let us fix an annulus  $A_{i_0}$ . Each pair of consecutive discs  $\Delta_{i_0,j}, \Delta_{i_0,j+1}$  is separated by a subarc of  $h(\alpha_l)$ , for some  $l = 1, \dots, k$ . Since the arcs  $\alpha_i$  (and therefore also the  $h(\alpha_i)$ ) are parallel it follows by the orientation of the discs  $D_i$  that

$$\omega_{i_0,j} + \omega_l = \omega_{i_0,j+1}.$$

This implies the chain of inequality

$$\omega_{i_0,1} \leq \omega_{i_0,2} \leq \dots \leq \omega_{i_0,k|a_0|} \leq \omega_{i_0,1}.$$

Therefore the weights  $\omega_{i_0,j}$  are all equal and since all the arcs  $h(\alpha_i)$  intersect the annulus  $A_{i_0}$  we have that  $\omega_l = 0$  for each  $l = 1, \dots, k$ . Therefore the sectors of  $B$  contained in  $S$  all have the same weight and the discs  $D_i$  have weight zero. This means that  $\Sigma$  is a finite number of parallel copies of  $S$ .

By construction,  $\partial M_h \setminus \partial B$  is a union of bigons (see Figure 18). Moreover  $B$  does not carry any closed surface, and therefore it does not carry tori bounding a solid torus in  $M_h(s_1, \dots, s_k)$ , for any multislope  $(s_1, \dots, s_k)$ . Since  $M_h$  is irreducible and its boundary is union of incompressible tori, the hypotheses of Theorem 3.12 are fulfilled. □

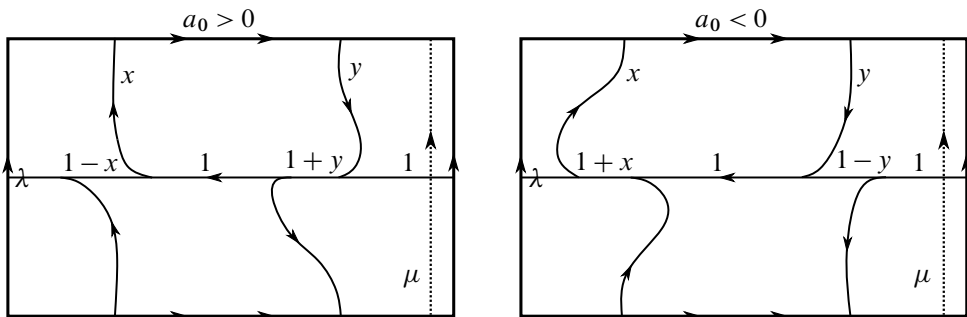


Figure 18: The two possible boundary train tracks with weight systems.

**Proposition 3.22** *If  $a_0 > 0$ , for any multislope  $(s_1, \dots, s_k) \in (\infty, 1)^k$  the branched surface  $B$  fully carries an essential lamination intersecting the boundary of  $M_h$  in parallel curves of multislope  $(s_1, \dots, s_k)$ . If  $a_0 < 0$  the same happens for any multislope  $(s_1, \dots, s_k) \in (-1, \infty)^k$ .*

**Proof** We study the multislopes realised by the boundary train tracks of  $B$ . In order to do this we assign rational weight systems to our boundary train tracks. Since our train tracks are oriented, we can associate to such a weight system the rational number  $w_{\mu_i}/w_{\lambda_i}$ , where  $w_{\mu_i}$  and  $w_{\lambda_i}$  are the *weighted* intersections of the train tracks with our fixed meridians  $\mu_i$  and longitudes  $\lambda_i$ , as we would do with oriented simple closed curves. This quotient can be interpreted as a *slope* in the  $i^{\text{th}}$  boundary component of  $M_h$ . In fact it is can be shown that each slope  $p/q$  obtained in this way is realised by the train track. Since we want to study slopes *fully* carried by these train tracks, we have to require that each weight is strictly positive: if the weight of an arc is zero, the associated slope will not intersect the fibers over that arc. For details, see [Penner and Harer 1992].

The boundary train tracks of  $B$  are all the same for each boundary tori, and only depend on the sign of  $a_0$ . The two possible types of boundary train tracks are depicted in Figure 18.

We also endowed the two train tracks with weight systems. The slopes of these weight systems are always  $x - y$ , but since we have to impose that each sector of the train tracks has positive weight, we have that

- if  $a_0 > 0$ ,  $x$  can vary in  $(0, 1)$  and  $y$  can vary in  $(0, +\infty)$ ;
- if  $a_0 < 0$ ,  $x$  can vary in  $(0, +\infty)$  and  $y$  can vary in  $(0, 1)$ .

By letting  $x$  and  $y$  vary, when  $a_0 > 0$  the boundary train tracks realise all multislopes in  $(\infty, 1)^k$  and when  $a_0 < 0$  the boundary train tracks realise all slopes in  $(-1, \infty)^k$ . Thanks to Lemma 3.21 we can apply Theorem 3.12 to obtain the desired essential laminations. □

**Lemma 3.23** *All the laminations constructed in the previous proposition extend to taut foliations of the filled manifolds.*

**Proof** Let  $\Lambda$  be one of the laminations constructed in Proposition 3.22 and suppose that  $\Lambda$  intersect  $\partial M_h$  in parallel curves of multislope  $(s_1, \dots, s_k)$ . We can also suppose that  $\partial_h N_B \subset \Lambda$ . First of all we notice that if the multislope is different from  $(0, \dots, 0)$  then  $\Lambda$  does not have any compact leaves. In fact any leaf of  $\Lambda$  is carried by  $B$  and we showed in the proof of Lemma 3.21 that the only connected compact surface carried by  $B$  is the  $k$ -holed torus  $S$ . Therefore if  $\Lambda$  has a compact leaf then it should intersect  $\partial M_h$  in parallel curves of multislope  $(0, \dots, 0)$ .

We now consider the abstract closures (in a path metric on  $M_h$ ) of the complementary regions of  $\Lambda$ . These closures are  $[0, 1]$ -bundles; in fact they are unions, along  $\partial_v N_B$ , of

- components of  $M_h \setminus \text{int}(N_B)$ , that are products of the type  $F \times [0, 1]$ , where  $F$  is a surface, with

$$\partial_h N_B \cap (F \times [0, 1]) = F \times \{0, 1\} \quad \text{and} \quad \partial_v N_B \cap (F \times [0, 1]) = \partial' F \times [0, 1],$$

where  $\partial' F$  is the closure of  $\partial F \setminus \partial M_h$ ;

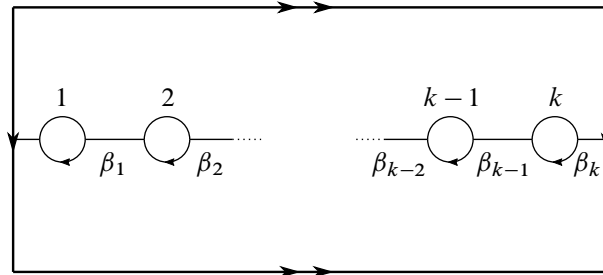


Figure 19: The arcs  $\beta_i$ .

- abstract closures of the components of  $N_B \setminus \Lambda$  — since  $\Lambda$  intersects transversely the fibers of  $N_B$  these closures are also products with the same properties of the components of  $M_h \setminus \text{int}(N_B)$ .

Each component of the vertical boundary of  $N_B$  is an annulus  $S^1 \times [0, 1]$  or a disc  $[0, 1] \times [0, 1]$ , where each interval  $\{*\} \times [0, 1]$  is contained in a fiber of  $N_B$ . Both the product structures of the components of  $M_h \setminus \text{int}(N_B)$  and of the abstract closures of the components of  $N_B \setminus \Lambda$  define a foliation of the vertical boundary transverse to the interval fibers. Any of two such foliations are isotopic and therefore also the abstract closures of the complementary regions of  $\Lambda$  are products.

In particular, since the horizontal boundary of the closures of these complementary regions are leaves of  $\Lambda$ , we can foliate these bundles with parallel leaves to obtain a foliation of the whole  $M_h$ . This foliation has no compact leaves and intersects the boundary of  $M$  in parallel curves of multislope  $(s_1, \dots, s_k)$ . Therefore the leaves of this foliation can be capped with the meridional discs of the solid tori to obtain a foliation of the filled manifold  $M_h(s_1, \dots, s_k)$  that has no compact leaves as well, and that is therefore taut [Calegari 2007, Example 4.23]. □

**Remark 3.24** In the terminology of [Kalelkar and Roberts 2015], if  $|a_0| = 1$  then the pair of parallel  $k$ -tuples

$$(\widetilde{h(\alpha)}, \alpha)$$

is good and oriented, where  $\alpha = (\alpha_1, \dots, \alpha_k)$  and  $\widetilde{h(\alpha)} = (\widetilde{h(\alpha_1)}, \dots, \widetilde{h(\alpha_k)})$ . In this case the branched surface constructed in the previous discussion coincides with the branched surface associated to the sequence  $(\widetilde{h(\alpha)}, \alpha)$  by Kalelkar and Roberts [2015].

**3.2.2 Proof of the second part of Theorem 3.18** We now focus our attention on the second part of Theorem 3.18 and we define a new branched surface. We fix a new set of arcs  $\alpha_1, \dots, \alpha_k$  in the following way: we consider arcs  $\beta_1, \dots, \beta_k$  as in Figure 19 and choose  $\alpha_i$  so that  $\widetilde{h(\alpha_i)} = \beta_i$ . One example is depicted in Figure 20.

We now give orientations to the arcs  $\alpha_i$  in order to build our branched surfaces. It will be simpler to state how to assign orientations to the  $\beta_i$  and we will orient each  $\alpha_i$  as isotopic to  $h^{-1}(\beta_i)$ , for  $i = 1, \dots, k$ .

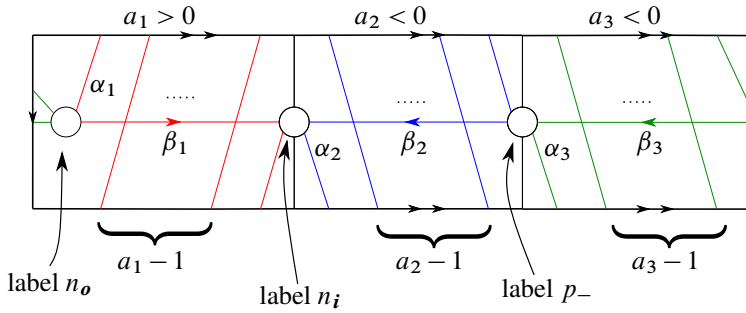


Figure 20: An example of a coherent orientation. In this case the arcs  $\beta_2$  and  $\beta_3$  have the same direction, while the arcs  $\beta_1$  and  $\beta_2$  and the arcs  $\beta_3$  and  $\beta_1$  have opposite directions.

**Definition 3.25** We say that an orientation of the arcs  $\beta_i$  is *coherent* if the following hold:

- If a boundary component  $\partial_i S$  has label  $p_+$  or  $p_-$ , the arcs  $\beta_{i-1}$  and  $\beta_i$  intersecting  $\partial_i S$  are oriented such that the first starts at  $\partial_i S$  and the second ends at  $\partial_i S$ , or vice versa. In this case we say that  $\beta_{i-1}$  and  $\beta_i$  have the same direction.
- If a boundary component  $\partial_i S$  has label  $n$ , the arcs  $\beta_{i-1}$  and  $\beta_i$  intersecting  $\partial_i S$  are oriented such that both start or both end at  $\partial_i S$ . In this case we say that  $\beta_{i-1}$  and  $\beta_i$  have opposite directions. In case the arcs both start at  $\partial_i S$  we say that the component is of type  $n_o$  (the subscript  $o$  stands for “out”) and if both end at  $\partial_i S$  we say that is of type  $n_i$  ( $i$  standing for “in”).

See Figure 20 for an example.<sup>4</sup> Notice that there is always an even number of boundary components of  $S$  with label  $n$ . Moreover the boundary components with label  $n$  are alternately of type  $n_o$  and  $n_i$ . We will soon use coherent orientations to build branched surfaces. First of all we prove the following lemma.

**Lemma 3.26** *There always exist exactly two different coherent orientations of the arcs  $\beta_i$ .*

**Proof** We fix an orientation of the arc  $\beta_1$ . We prove that there exists a unique coherent orientation of the arcs  $\beta_i$  agreeing with the fixed orientation on  $\beta_1$  and this implies the thesis. We orient the arcs  $\beta_i$  inductively. Suppose that we have oriented  $\beta_1, \dots, \beta_j$ . Then

- if  $\partial_{j+1} S$  has label  $p_+$  or  $p_-$ , we orient  $\beta_{j+1}$  such that it has the same direction of  $\beta_j$ ;
- if  $\partial_{j+1} S$  has label  $n$ , we orient  $\beta_{j+1}$  such that its direction is opposite to the one of  $\beta_j$ .

In other words, once we have fixed an orientation on  $\beta_1$  the coherence condition completely determines the orientations of  $\beta_2, \dots, \beta_k$ . The only thing to be checked in order to prove that this orientation is actually coherent is the behaviour of  $\beta_k$  and  $\beta_1$  at  $\partial_1 S$ . Since there is always an even number of boundary components of  $S$  with label  $n$  it follows that

- if  $\partial_1 S$  has label  $p_+$  or  $p_-$  then the direction changes an even number of times between  $\beta_1$  and  $\beta_k$  and therefore  $\beta_1$  and  $\beta_k$  have the same direction;

<sup>4</sup>Recall that the factorisation of the monodromy  $h$  is to be read from right to left; this should help to figure out why  $h(\alpha_i) = \beta_i$ .

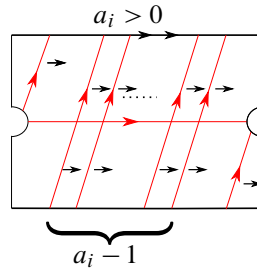


Figure 21: The annulus  $A_i$ .

- if  $\partial_1 S$  has label  $n$  then the direction changes an odd number of times between  $\beta_1$  and  $\beta_k$  and therefore  $\beta_1$  and  $\beta_k$  have opposite directions.

Therefore the orientation defined in this way is coherent. □

We fix a coherent orientation and as usual we consider the branched surface  $B$  that is the union of  $S$  and the images in  $M_h$  of the discs  $\alpha_i \times [0, 1] \subset S \times [0, 1]$ . We denote the image of  $\alpha_i \times [0, 1]$  by  $D_i$  and orient the discs  $D_i$  such that the orientation on their boundary induces the given orientation on the  $\alpha_i$ . Exactly as before we have:

**Lemma 3.27** *The branched surface  $B$  is laminar and satisfies the hypotheses of Theorem 3.12.*

**Proof** The proof is analogous to that of Lemma 3.21. We only need to prove that  $B$  contains no sink discs or half sink discs, and that the only connected surface properly embedded in  $M_h$  carried by  $B$  is  $S$ .

- $B$  contains no sink discs or half sink discs: There are  $k$  sectors of  $B$  that coincide with the discs  $D_i$  and they always have cusp directions pointing outside. We focus our attention on the sectors contained in  $S$ . We consider the  $k$  annuli  $A_i$  obtained by cutting  $S$  along the arcs of Figure 14. Each of these annuli contains in its interior some disc and half-disc sectors and intersects two other half-disc sectors. The former are never sink because each of these sectors has in its boundary two parallel subarcs of some arc of the  $\alpha_i$ , as for example Figure 21 shows.

We now claim the following:

**Claim** *Since we have fixed a coherent orientation of the arcs  $\beta_i$ , the cusp directions along the arcs  $\alpha_i$  all point in the same direction.*

The claim implies that the sectors belonging to two consecutive annuli are never sink because each of these sectors has in its boundary two subarcs of two consecutive arcs of the  $\alpha_i$ . For an example, see Figure 22.

**Proof** We first notice that when  $a_i > 0$  (resp.  $a_i < 0$ ) the cusp direction along the arc  $\alpha_i$  has the same (resp. opposite) direction of  $\beta_i$  (recall that the cusp direction always points to the right along the oriented arcs  $\alpha_i$ ). Therefore to prove the claim it is sufficient to prove that  $\beta_i$  and  $\beta_j$  have the same direction if and only if  $a_i a_j > 0$ , and to prove this it is enough to prove that  $a_1 a_i > 0$  if and only if  $\beta_1$  and  $\beta_i$

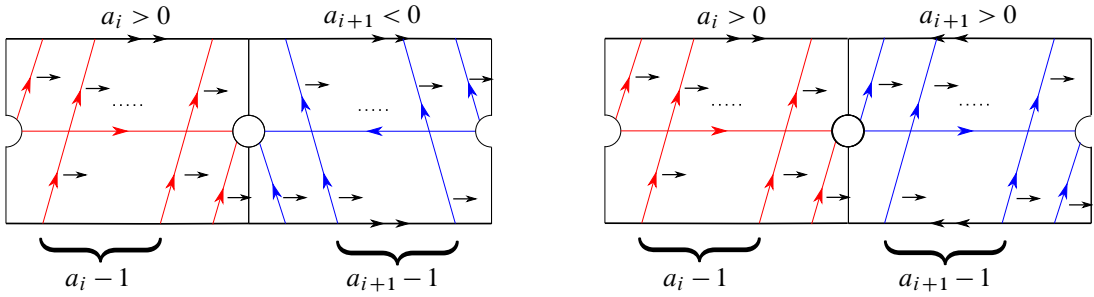


Figure 22: The figure shows how the choice of a coherent orientation implies that the cusp directions along the arcs  $\alpha_i$  all have the same direction.

have the same direction. We prove this by induction on  $i$ . If  $i = 2$  this follows from the definition of coherent orientation. We suppose now that the thesis is true for  $i$  and we prove it for  $i + 1$ . Suppose that  $a_1 a_{i+1} > 0$ ; then if  $a_1 a_i > 0$  we know by inductive hypothesis that  $\beta_1$  and  $\beta_i$  have the same direction. Moreover we deduce that  $a_i a_{i+1} > 0$  and by the definition of coherent orientation that  $\beta_i$  and  $\beta_{i+1}$  have the same direction and therefore also  $\beta_1$  and  $\beta_{i+1}$  have the same direction. The other cases can be analysed similarly.  $\square$

- The only connected compact surface properly embedded in  $M_h$  carried by  $B$  is  $S$ : Suppose that  $\Sigma$  is a compact surface carried by  $B$ .  $\Sigma$  induces an integral weight system on  $B$ . We denote by  $\omega_i$  the weight of the discs  $D_i$ . The number of sectors contained in  $S$  is equal to  $N = \sum_{i=1}^k |a_i|$ . Since we have fixed a coherent orientation of the arcs  $\beta_i$ , the cusp directions along the arcs  $\alpha_i$  all point in the same direction. We order the sectors in  $S$  according to this direction as depicted in Figure 23; we denote them by  $\Delta_l$  and we denote their weights by  $\delta_l$ , where  $1 \leq l \leq N$ .

As in the proof of the first part of the theorem, we have that

$$\delta_1 \leq \delta_2 \leq \dots \leq \delta_N \leq \delta_1.$$

Therefore these weights are all equal and this implies that the discs  $D_i$  all have weight zero; that is to say,  $\Sigma$  is a finite number of parallel copies of  $S$ .  $\square$

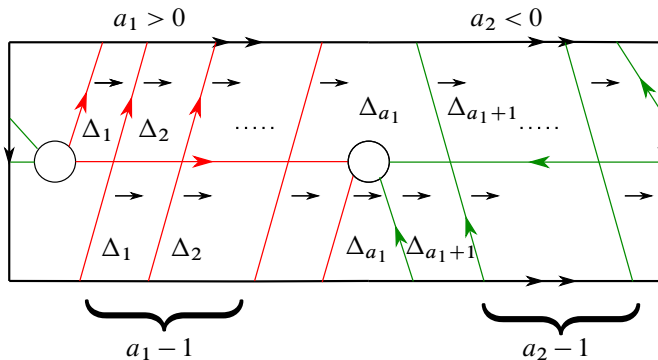


Figure 23: An example that shows how to label the sectors  $\Delta_l$ .



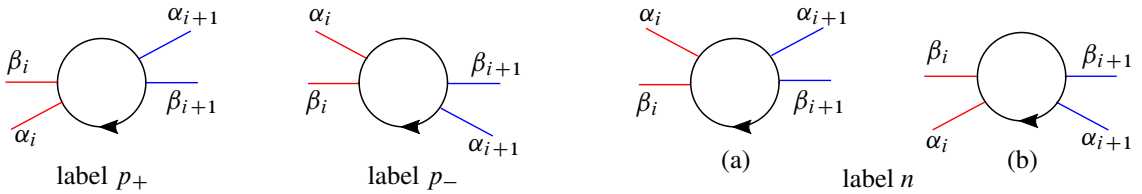


Figure 24: The four possible configuration of arcs in a neighbourhood of  $\partial_{i+1}S$ .

**Proposition 3.28** *Let  $I_1, \dots, I_k$  and  $J_1, \dots, J_k$  be the intervals defined in the discussion before the statement of Theorem 3.18 and let  $B$  denote the branched surface associated to a coherent orientation of the arcs  $\beta_i$ . Then for one choice of coherent orientation of the arcs  $\beta_i$ ,  $B$  fully carries essential laminations intersecting the boundary of  $M_h$  in parallel curves of multislope  $(s_1, \dots, s_k)$ , for  $(s_1, \dots, s_k) \in I_1 \times \dots \times I_k$ . Choosing the other coherent orientation yields  $B$  that fully carries essential laminations intersecting the boundary of  $M_h$  in parallel curves of multislope  $(s_1, \dots, s_k)$ , for  $(s_1, \dots, s_k) \in J_1 \times \dots \times J_k$ . Moreover, these laminations extend to taut foliations of the filled manifold  $M_h(s_1, \dots, s_k)$ .*

**Proof** We focus our attention on the boundary train tracks of  $B$ . For a fixed boundary component of  $S$  we have the four possible configurations shown in Figure 24 and for each of these configurations we have two possible ways to fix a coherent orientation.

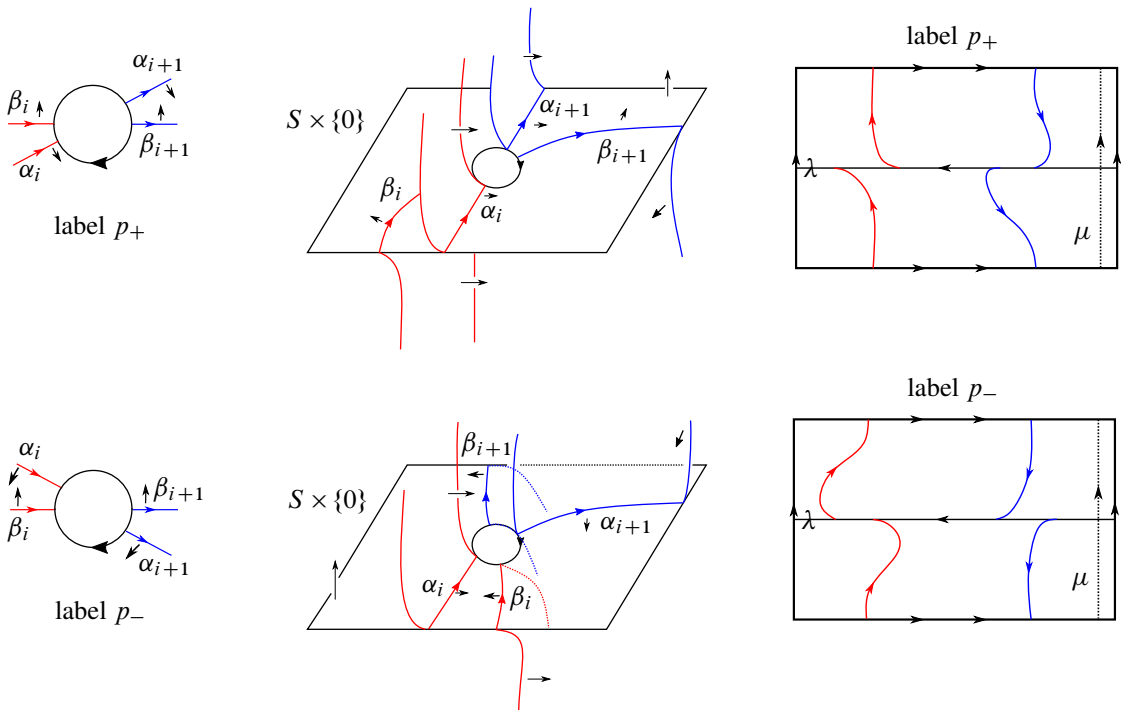


Figure 25: In this figure we describe the branched surface in a neighbourhood of the  $(i + 1)^{st}$  boundary component of  $M_h$  and its boundary train track in the case of label  $p_+$  and  $p_-$ .

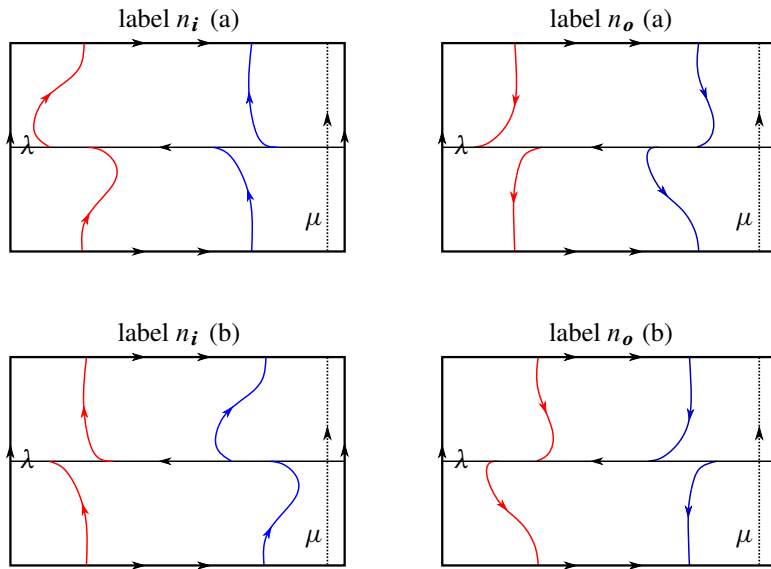


Figure 26: The possible boundary train tracks associated to a boundary component with label  $n$ . Notice that the train tracks depend only on the orientation of the arcs, and not on the label (a) or (b) of the configuration.

If the boundary component has label  $p_+$  or  $p_-$ , the type of the boundary train track does not depend on the choice of the coherent orientation, and is described in Figure 25, where for concreteness we have assigned an orientation to the arcs, and where in the middle picture we have also described the branched surface in a neighbourhood of the boundary component. If we consider the other coherent orientation, we obtain the same train tracks. By assigning weights to these train tracks as we have already done in the proof of Proposition 3.22 we have that if the label is  $p_+$ , the train track realises all the slopes in the interval  $(\infty, 1)$ , while if the label is  $p_-$ , the slopes realised are those in the interval  $(-1, \infty)$ .

On the other hand if the boundary component has label  $n$  the choice of the orientation yields two different train tracks. We represent the possible train tracks in Figure 26. Notice that the train tracks depend only on the orientation of the arcs, and not on the label (a) or (b) of the configuration.

The train tracks on the left of Figure 26 realise all the slopes in the interval  $(0, \infty)$ , while those on the right realise the slopes in the interval  $(\infty, 0)$ .

By fixing one or the other of the two possible coherent orientations, the boundary train tracks of  $B$  realise all the multislopes in  $I_1 \times \dots \times I_k$  and  $J_1 \times \dots \times J_k$ . By virtue of Lemma 3.27, we can apply Theorem 3.12 to obtain the desired essential laminations, and Lemma 3.23 implies that these laminations extends to taut foliations of the filled manifolds. □

**Proof of Theorem 3.18** The first part of the theorem is the content of Proposition 3.22 and Lemma 3.23. The second part is the content of Proposition 3.28. □

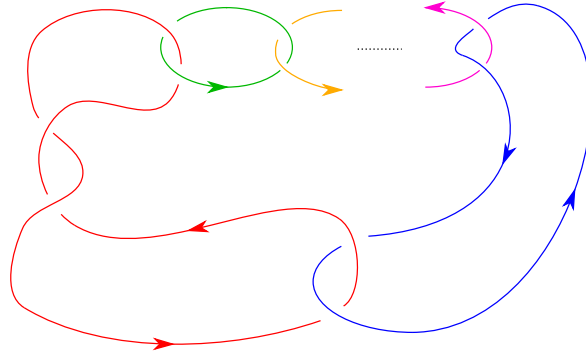


Figure 27: The link  $\mathcal{L}_n$ .

**Example 3.29** For each natural number  $n$  we consider the  $n$ -component oriented link  $\mathcal{L}_n$  in Figure 27.

We can represent the link  $\mathcal{L}_n$  in a different way, as in Figure 28. With this representation, it is evident that  $\mathcal{L}_n$  can be realised as a plumbing of Hopf bands. Therefore,  $\mathcal{L}_n$  is a fibered link, with fiber surface a torus with  $n$  open discs removed, and the monodromy associated to this fiber is

$$h = \tau_0^{-1} \tau_1 \cdots \tau_n,$$

where  $\tau_i$  is the positive Dehn twist along the curve  $\gamma_i$  [Gabai 1986; Stallings 1978].

We now prove that  $\mathcal{L}_n$  is a hyperbolic link. We recall the following theorem of Penner [1988]:

**Theorem 3.30** [Penner 1988] Suppose that  $\mathcal{C}$  and  $\mathcal{D}$  are each disjointly embedded collections of essential simple closed curves (with no parallel components) in an oriented surface  $F$  such that  $\mathcal{C}$  hits  $\mathcal{D}$  efficiently and  $\mathcal{C} \cup \mathcal{D}$  fills  $F$ . Let  $R(\mathcal{C}^+, \mathcal{D}^-)$  be the free semigroup generated by the Dehn twists

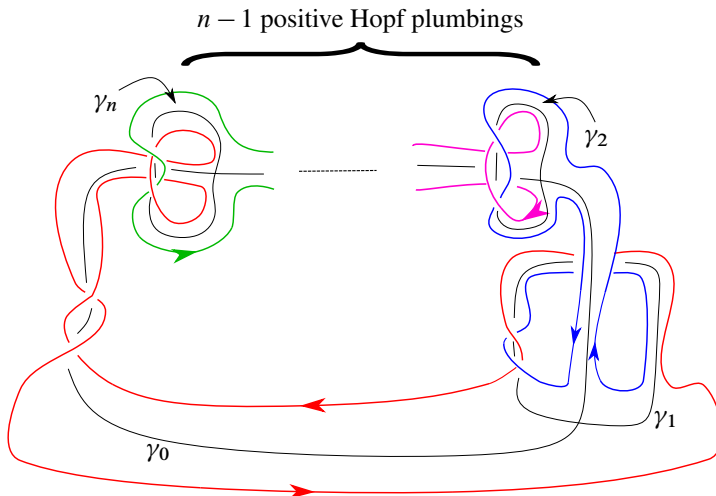


Figure 28: A description of  $\mathcal{L}_n$  as a plumbing of Hopf bands.

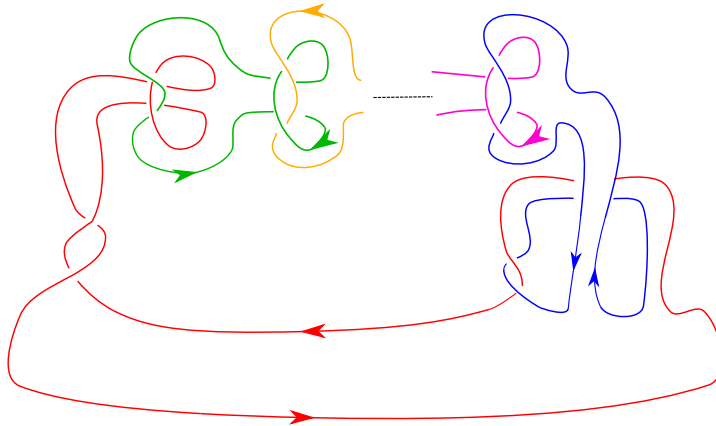


Figure 29: Another picture of the link  $\mathcal{L}_n$ . Without the cores of the Hopf bands it may be easier to see the isotopy to the link depicted in Figure 27.

$\{\tau_c^{+1} \mid c \in \mathcal{C}\} \cup \{\tau_d^{-1} \mid d \in \mathcal{D}\}$ . Each component map of the isotopy class of  $w \in R(\mathcal{C}^+, \mathcal{R}^-)$  is either the identity or pseudo-Anosov, and the isotopy class of  $w$  is itself pseudo-Anosov if each  $\tau_c^{+1}$  and  $\tau_d^{-1}$  occurs at least once in  $w$ .

In the statement of the previous theorem, “ $\mathcal{C} \cup \mathcal{D}$  fills  $F$ ” means that each component of the complement of  $\mathcal{C} \cup \mathcal{D}$  is a disc, a boundary-parallel annulus, or a puncture-parallel punctured disc. Moreover “ $\mathcal{C}$  hits  $\mathcal{D}$  efficiently” if there is no bigon in  $F$  with boundary made of one arc of a curve  $c \in \mathcal{C}$  and one arc of a curve  $d \in \mathcal{D}$ .

In our case we set  $\mathcal{C} = \{\gamma_1, \dots, \gamma_n\}$  and  $\mathcal{D} = \{\gamma_0\}$  and we can apply this theorem to deduce that the monodromy associated to  $\mathcal{L}_n$  is a pseudo-Anosov map; applying [Thurston 1998] we deduce these links are hyperbolic.

Moreover Theorem 3.18 applies to these links and we can deduce that for any multislope

$$(s_1, \dots, s_n) \in (\infty, 1)^n \cup (-1, \infty)^n,$$

the filling of the exterior of  $\mathcal{L}_n$  with multislope  $(s_1, \dots, s_n)$  supports a coorientable taut foliation. Recall that these slopes are referred to the meridian-longitude bases given by the mapping torus; since the components of the link  $\mathcal{L}_n$  do not have pairwise linking number zero, the longitudes of these bases do not coincide with the canonical longitudes of the link.

### 3.3 The Whitehead link case

We now return to the Whitehead link exterior. Recall from the discussion preceding Figure 10 that the Whitehead link is a fibered link with fiber surface a 2–holed torus and monodromy  $h = \tau_0 \tau_1 \tau_2^{-1}$ , where the curves  $\gamma_i$  are represented in Figure 30. In what follows we will identify the exterior of the Whitehead link with the mapping torus  $M_h$ .

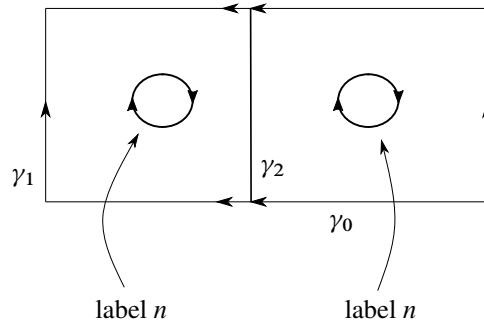


Figure 30: An abstract description of the fiber surface for the Whitehead link. The monodromy is given by  $h = \tau_0 \tau_1 \tau_2^{-1}$ . We have also indicated the labels of the boundary components of the torus.

As a consequence of [Theorem 3.18](#) and of the fact that the components of the Whitehead link have linking number zero, we have:

**Corollary 3.31** *Let  $(p_1/q_1, p_2/q_2)$  be a multislope in*

$$(\infty, 1)^2 \cup (0, \infty) \times (\infty, 0) \cup (\infty, 0) \times (0, \infty).$$

*Then  $S^3_{p_1/q_1, p_2/q_2}$  (WL) supports a coorientable taut foliation.*

We have been able to prove that for slopes in the region depicted in [Figure 31](#), the corresponding filling on the Whitehead exterior supports a coorientable taut foliation.

We now cover the remaining regions of [Figure 31](#). We define a branched surface by considering the pair of arcs  $\alpha$  and  $\beta$  shown in [Figure 32](#), left. We apply the monodromy  $h$  and we obtain what is depicted in [Figure 32](#).

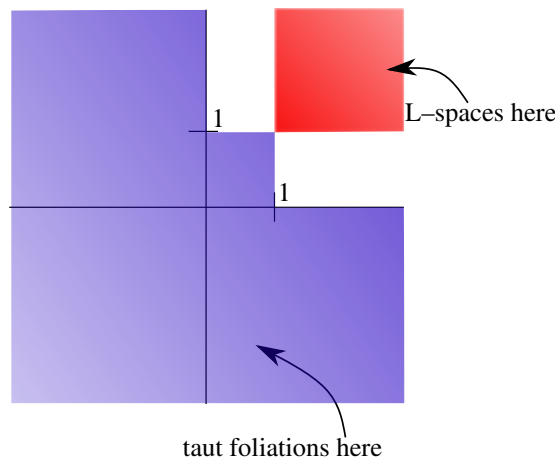


Figure 31: The figure describes what we have been able to prove up to now. The blue points are the slopes whose corresponding filling supports a coorientable taut foliation; the red points are those whose corresponding filling is an L-space.

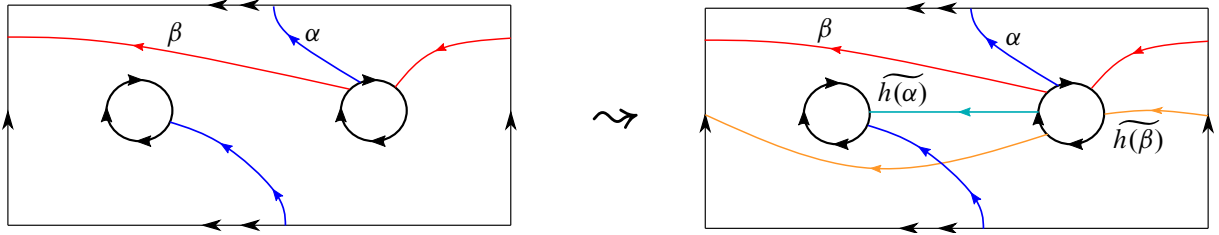


Figure 32: The arcs  $\alpha$  and  $\beta$  and their (perturbed) images via  $h$ .

As usual, we consider the branched surface  $B$  associated to the arcs  $\alpha$  and  $\beta$ .

**Lemma 3.32** *The branched surface  $B$  is laminar and satisfies the hypotheses of Theorem 3.12.*

**Proof** Since  $S \setminus (\alpha \cup \beta)$  has no disc components, by virtue of Lemma 3.16 we only need to prove that  $B$  contains no sink disc or half sink discs and this is showed in Figure 33.

To prove that  $B$  satisfies the hypotheses of Theorem 3.12 we have to show that  $\partial M_h \setminus \partial B$  is a union of bigons and that  $B$  does not carry a torus. By construction  $\partial M_h \setminus \partial B$  is a union of bigons (see Figure 34) and since each sector of  $B$  intersect  $\partial M_h$  it follows that any surface carried by  $B$  must intersect  $\partial M_h$ . Therefore  $B$  does not carry any closed surfaces and in particular it does not carry tori.  $\square$

**Corollary 3.33** *Let  $(p_1/q_1, p_2/q_2)$  be slopes in*

$$(0, \infty) \times (-1, 1) \cup (-1, 1) \times (0, \infty).$$

*Then the filled manifold  $S^3_{p_1/q_1, p_2/q_2}$  (WL) supports a coorientable taut foliation.*

**Proof** By virtue of Theorem 3.12, for any multislope  $(s_1, s_2)$  realised by the boundary train tracks of  $B$  there exists an essential lamination  $\Lambda$  fully carried by  $B$  intersecting the boundary of the exterior of the Whitehead link in parallel curves of multislope  $(s_1, s_2)$ . The boundary train tracks of  $B$  are shown in Figure 34.

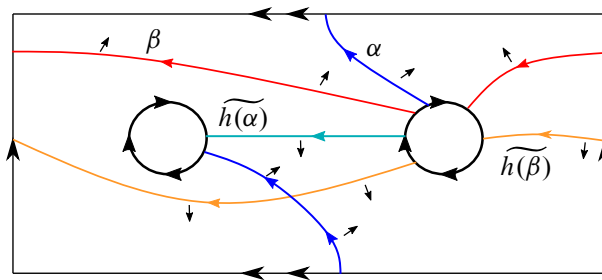


Figure 33: The sectors of  $B$  are five half discs. The figure also describes the cusp directions of  $B$  and it is easy to check that none of these half discs is sink.

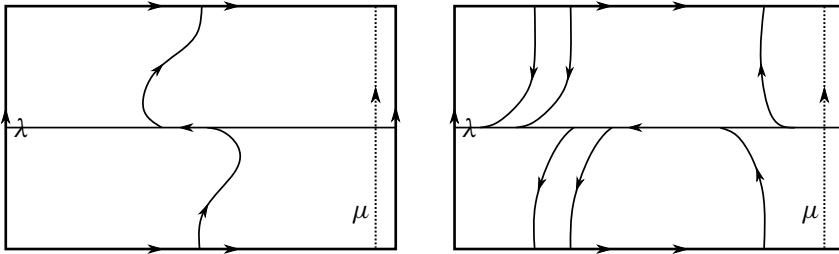


Figure 34: The boundary train tracks of  $B$ .

By assigning weights to these train tracks as in Figure 35 it follows that these train tracks realise all the slopes in  $(0, \infty) \times (-1, 1)$ .

Let  $\Lambda$  be an essential lamination intersecting the boundary of  $M_h$  in parallel curves of one of these multislopes  $(s_1, s_2)$ . Since each leaf of  $\Lambda$  is carried by  $B$  and each sector of  $B$  intersects  $\partial M_h$  it follows that all the leaves of  $\Lambda$  intersect  $\partial M_h$ . It follows by the proof of Lemma 3.23 that we can construct a foliation of  $M_h$  such that each leaf of this foliation is parallel to some leaf of  $\Lambda$ . Therefore all the leaves of this foliation intersect  $\partial M$  and as a consequence when we cap these leaves with the meridional discs of the solid tori we obtain a foliation of  $M_h(s_1, s_2)$  with the property that the cores of the tori are transversals intersecting all the leaves.

We have proved that for any  $(p_1/q_1, p_2/q_2) \in (0, \infty) \times (-1, 1)$  the manifold  $S^3_{p_1/q_1, p_2/q_2}$  (WL) supports a coorientable taut foliation. Since the Whitehead link is symmetric we deduce that the same result holds also for any  $(p_1/q_1, p_2/q_2) \in (-1, 1) \times (0, \infty)$ . □

**Proof of Theorem 3.1** It is a consequence of Corollaries 3.31 and 3.33. □

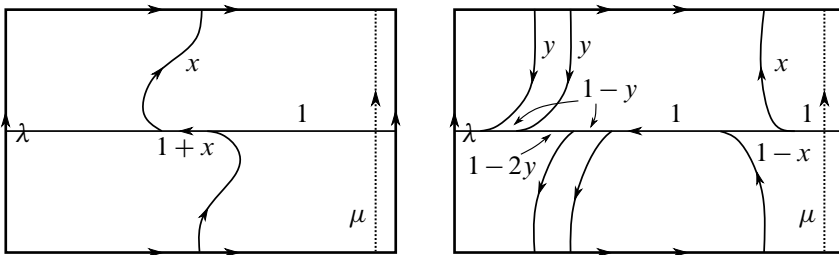


Figure 35: The boundary train tracks of  $B$  with weight systems.

### 4 Orderability

In this last section we will prove Theorem 1.4 and discuss some results about the orderability, and nonorderability, of some surgeries on the Whitehead link. We recall the following definition:

**Definition 4.1** Let  $G$  be a group.  $G$  is *left orderable* if there exists a total order  $<$  on  $G$  that is invariant under left multiplication by elements in  $G$ , ie such that for any  $g, g' \in G$  we have that  $g < g'$  if and only if  $hg < hg'$  for all  $h \in G$ .

If  $G$  is the fundamental group of a closed, orientable, irreducible 3-manifold, left orderability translates in the following dynamical property.

**Theorem 4.2** [Boyer et al. 2005] *Let  $N$  be a closed, irreducible, orientable 3-manifold. Then  $\pi_1(N)$  is left orderable if and only if there exists a nontrivial homomorphism  $\varphi: \pi_1(N) \rightarrow \text{Homeo}^+(\mathbb{R})$ .*

This result yields a theoretical way to connect taut foliations to left orderability in the following way. Suppose that  $\mathcal{F}$  is a cooriented taut foliation on a rational homology 3-sphere  $N$ . We can associate to  $\mathcal{F}$  its tangent bundle  $T\mathcal{F}$ , that is a plane bundle over  $N$ . Being a plane bundle, we can associate to  $T\mathcal{F}$  its Euler class  $e(T\mathcal{F}) \in H^2(N; \mathbb{Z})$ . Moreover, by a construction of Thurston — see [Calegari and Dunfield 2003] — it is possible to associate to  $\mathcal{F}$  a nontrivial homomorphism

$$\varphi: \pi_1(N) \rightarrow \text{Homeo}^+(S^1).$$

Since there is an injective homomorphism from the universal cover  $\widetilde{\text{Homeo}^+(S^1)}$  of  $\text{Homeo}^+(S^1)$  into  $\text{Homeo}^+(\mathbb{R})$ , one would like to lift  $\varphi$  to a homomorphism

$$\tilde{\varphi}: \pi_1(N) \rightarrow \widetilde{\text{Homeo}^+(S^1)}.$$

The obstruction to find such a lift is again a cohomology class in  $H^2(N; \mathbb{Z})$ , and it turns out that this class vanishes if and only if  $e(T\mathcal{F}) = 0$ . For more details we refer to [Boyer and Hu 2019].

The upshot of the previous discussion is the following.

**Theorem 4.3** [Boyer and Hu 2019] *Let  $N$  be a rational homology sphere and let  $\mathcal{F}$  be a coorientable taut foliation on  $N$ . If the Euler class of  $\mathcal{F}$  vanishes then  $\pi_1(N)$  is left orderable.*

We now consider the taut foliations obtained in the previous section and determine which of them have vanishing Euler class. To do this we will adapt part of the content of [Hu 2019] to our context.

We fix some notation. We denote by  $M$  the exterior of WL and we denote by  $S$  the 2-holed torus of Figure 10 that is a Seifert surface for WL. We fix a multislope  $(p_1/q_1, p_2/q_2)$ , with  $p_1/q_1 < 1$  or  $p_2/q_2 < 1$ , and we denote by  $\mathcal{F}$  the foliation in  $M$  intersecting  $\partial M$  in parallel curves of multislope  $(p_1/q_1, p_2/q_2)$ , as constructed in the proof of Theorem 3.1. This foliation extends to a foliation  $\hat{\mathcal{F}}$  of the filled manifold  $S^3_{p_1/q_1, p_2/q_2}(\text{WL})$  such that in the glued solid tori  $N_1$  and  $N_2$  the foliation  $\hat{\mathcal{F}}$  restricts to the standard foliations  $\mathcal{D}_1$  and  $\mathcal{D}_2$ , which are the foliations by meridional discs. We can suppose without loss of generality that  $p_1, p_2 > 0$ . We orient the meridional disc  $D_i$  of  $N_i$  so that the gluing map identifies  $\partial D_i$  with the oriented curve  $p_i \mu_i + q_i \lambda_i$  in  $\partial M$ .



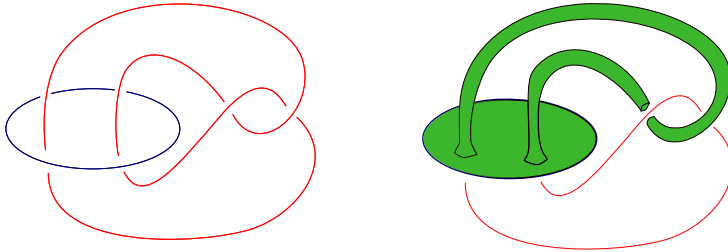


Figure 36: The 1–holed torus depicted in this figure is one of the two generators of  $H_2(M, \partial M; \mathbb{Z})$ .

The second homology group  $H_2(M, \partial M; \mathbb{Z})$  is isomorphic to  $\mathbb{Z}^2$  and in particular we can fix as generators two properly embedded surfaces  $S_1$  and  $S_2$  that are dual to the meridians of the two components of the Whitehead link. Since the Whitehead link has linking number zero, these surfaces can be taken to be Seifert surfaces for the components of the link. In particular, these can be chosen to be tori with one disc removed, so that  $\partial S_i = \lambda_i$ . One of these tori is showed in Figure 36 and the other can be obtained by an isotopy of  $S^3$  exchanging the two components of WL.

We fix a nowhere vanishing section  $\sigma$  of  $(T\mathcal{F})|_{\partial M}$  that is everywhere pointing outside of  $M$ . Hence the restrictions of  $\sigma$  to the boundary components of  $M$  also define nowhere vanishing sections  $\sigma_i$  of  $(T\mathcal{D}_i)|_{\partial N_i}$  everywhere pointing inside  $N_i$ , for  $i = 1, 2$ . These sections yield us relative Euler classes in  $H^2(M, \partial M; \mathbb{Z})$ ,  $H^2(N_1, \partial N_1; \mathbb{Z})$  and  $H^2(N_2, \partial N_2; \mathbb{Z})$ , which we denote respectively by  $e_\sigma(T\mathcal{F})$ ,  $e_{\sigma_1}(T\mathcal{D}_1)$  and  $e_{\sigma_2}(T\mathcal{D}_2)$ . See [Hu 2019] for details.

Finally, we set  $a_i = \langle e_\sigma(T\mathcal{F}), [S_i] \rangle$  and  $b_i = \langle e_{\sigma_i}(T\mathcal{D}_i), [D_i] \rangle$ , where  $D_i$  is a meridional disc in  $N_i$ .

**Remark 4.4** Since  $\mathcal{D}_i$  is the standard foliation of the solid torus by meridional discs, we have that  $b_i$  coincides with  $\pm \langle e_{\sigma_i}(TD_i), [D_i] \rangle = \pm \chi(D_i) = \pm 1$ , where  $TD_i$  denotes the tangent bundle of  $D_i$  and where the sign depends on the orientation of the foliation  $\mathcal{D}_i$ .

We are interested in knowing when  $e(T\widehat{\mathcal{F}})$  vanishes. The following proposition tells us exactly when this happens. Recall that without loss of generality we are supposing  $p_1, p_2 > 0$ , whereas the signs of  $q_1$  and  $q_2$  are arbitrary.

**Proposition 4.5** We have that  $e(T\widehat{\mathcal{F}}) = 0$  if and only if  $e(T\mathcal{F}) = 0$  and  $a_i q_i \equiv b_i \pmod{p_i}$ .

**Proof** The statement of this proposition is the generalisation to our case of the statements of [Hu 2019, Lemma 3.1 and Theorem 1.4] and the proof that is presented there adapts almost unaltered. We give a brief sketch of the proof and refer to [Hu 2019] for the details. In what follows the cohomology and homology groups are all implicitly assumed with integer coefficients and we will denote by  $\overline{M}$  the  $(p_1/q_1, p_2/q_2)$ –surgery on  $M$ . Since  $\overline{M}$  is a rational homology sphere, as a consequence of the long exact sequence of the pair  $(\overline{M}, \partial M)$ , we have

$$(1) \quad 0 \rightarrow H^1(\partial M) \xrightarrow{\delta} H^2(\overline{M}, \partial M) \xrightarrow{\iota} H^2(\overline{M}) \rightarrow 0.$$

Moreover as a consequence of the Mayer–Vietoris sequence there is an isomorphism

$$(2) \quad H^2(M, \partial M) \oplus H^2(N_1, \partial N_1) \oplus H^2(N_2, \partial N_2) \cong H^2(\bar{M}, \partial M)$$

defined by mapping the relative classes  $(c_M, c_{N_1}, c_{N_2})$  to the sum  $\bar{c}_M + \bar{c}_{N_1} + \bar{c}_{N_2}$ , where each of these cohomology classes is obtained by extending to  $\bar{M}$  the corresponding relative class by the zero map.

By using the identification given by the isomorphism in (2) we obtain a short exact sequence

$$(3) \quad 0 \rightarrow H^1(\partial M) \xrightarrow{\psi} H^2(M, \partial M) \oplus H^2(N_1, \partial N_1) \oplus H^2(N_2, \partial N_2) \xrightarrow{\varphi} H^2(\bar{M}) \rightarrow 0,$$

where

$$\psi(\beta) = (\delta_M \beta, (\delta_{N_1} \circ f_1^*)(\beta), (\delta_{N_2} \circ f_2^*)(\beta))$$

with  $f_1: \partial N_1 \hookrightarrow \partial M$  and  $f_2: \partial N_2 \hookrightarrow \partial M$  denoting the gluing maps of the solid tori and with

$$\delta_M: H^1(\partial M) \rightarrow H^2(M, \partial M), \quad \delta_{N_1}: H^1(\partial N_1) \rightarrow H^2(N_1, \partial N_1), \quad \delta_{N_2}: H^1(\partial N_2) \rightarrow H^2(N_2, \partial N_2)$$

denoting the maps appearing in the long exact sequences of the pairs  $(M, \partial M)$ ,  $(N_1, \partial N_1)$  and  $(N_2, \partial N_2)$ .

We suppose now that  $e(T\hat{\mathcal{F}}) = 0$ . By naturality of the Euler class,  $e(T\mathcal{F})$  is the image of  $e(T\hat{\mathcal{F}}) = 0$  under the map induced by the inclusion  $M \hookrightarrow \bar{M}$  and therefore  $e(T\mathcal{F}) = 0$ .

Moreover, it also holds that

$$\varphi(e_\sigma(T\mathcal{F}), e_{\sigma_1}(T\mathcal{D}_1), e_{\sigma_2}(T\mathcal{D}_2)) = e(T\hat{\mathcal{F}}) = 0$$

and therefore there exists  $\beta \in H^1(\partial M)$  such that  $\psi(\beta) = (e_\sigma(T\mathcal{F}), e_{\sigma_1}(T\mathcal{D}_1), e_{\sigma_2}(T\mathcal{D}_2))$ ; in other words  $\beta$  satisfies

$$\delta_M \beta = e_\sigma(T\mathcal{F}), \quad (\delta_{N_1} \circ f_1^*)(\beta) = e_{\sigma_1}(T\mathcal{D}_1), \quad (\delta_{N_2} \circ f_2^*)(\beta) = e_{\sigma_2}(T\mathcal{D}_2).$$

We verify that  $a_i q_i \equiv b_i \pmod{p_i}$ :

$$\begin{aligned} b_i &= \langle e_{\sigma_i}(T\mathcal{D}_i), [D_i] \rangle = \langle (\delta_{N_i} \circ f_i^*)(\beta), [D_i] \rangle = \langle \beta, [f_i(\partial D_i)] \rangle \\ &= \langle \beta, p_i \mu_i + q_i \lambda_i \rangle = p_i \langle \beta, \mu_i \rangle + q_i \langle \beta, \lambda_i \rangle = p_i \langle \beta, \mu_i \rangle + q_i a_i, \end{aligned}$$

where in the last equality we have used that

$$\langle \beta, \lambda_i \rangle = \langle \beta, [\partial S_i] \rangle = \langle \delta_M \beta, [S_i] \rangle = \langle e_\sigma(T\mathcal{F}), [S_i] \rangle = a_i.$$

We now prove that if  $e(T\mathcal{F}) = 0$  and  $a_i q_i \equiv b_i \pmod{p_i}$  for  $i = 1, 2$ , then  $e(T\hat{\mathcal{F}}) = 0$ .

We consider again the short exact sequence in (1). The nowhere vanishing section  $\sigma$  defines an element  $e_\sigma(T\hat{\mathcal{F}}) \in H^2(\bar{M}, \partial M)$  that satisfies  $\iota(e_\sigma(T\hat{\mathcal{F}})) = e(T\hat{\mathcal{F}})$  and therefore if we prove that  $e_\sigma(T\hat{\mathcal{F}})$  belongs to the image of  $\delta: H^1(\partial M) \rightarrow H^2(\bar{M}, \partial M)$  we obtain the thesis. Moreover, under the isomorphism (2) the element  $(e_\sigma(T\mathcal{F}), e_{\sigma_1}(T\mathcal{D}_1), e_{\sigma_2}(T\mathcal{D}_2))$  corresponds to  $e_\sigma(T\hat{\mathcal{F}})$  and therefore it is enough to prove that  $(e_\sigma(T\mathcal{F}), e_{\sigma_1}(T\mathcal{D}_1), e_{\sigma_2}(T\mathcal{D}_2))$  belongs to the image of  $\psi$  in the short exact sequence (3).

If we consider the long exact sequence of the pair  $(M, \partial M)$ , we have

$$H^1(M; \mathbb{Z}) \xrightarrow{\iota'_M} H^1(\partial M) \xrightarrow{\delta_M} H^2(M, \partial M) \xrightarrow{\iota''_M} H^2(M),$$

and since  $\iota''_M(e_\sigma(T\mathcal{F})) = e(T\mathcal{F}) = 0$  we deduce that there exists  $\beta_0 \in H^1(\partial M)$  such that

$$\delta_M(\beta_0) = e_\sigma(T\mathcal{F}) \in H^2(M, \partial M).$$

We now want to modify  $\beta_0$  in order to find  $\beta \in H^1(\partial M)$  that satisfies

$$\psi(\beta) = (e_\sigma(T\mathcal{F}), e_{\sigma_1}(T\mathcal{D}_1), e_{\sigma_2}(T\mathcal{D}_2)),$$

that is to say such that

$$\delta_M \beta = e_\sigma(T\mathcal{F}), \quad (\delta_{N_1} \circ f_1^*)(\beta) = e_{\sigma_1}(T\mathcal{D}_1), \quad (\delta_{N_2} \circ f_2^*)(\beta) = e_{\sigma_2}(T\mathcal{D}_2).$$

We denote by  $\mu_i^* \in H^1(\partial M)$  the dual of  $\mu_i \in H_1(\partial M)$  and we define

$$\beta = \beta_0 + n_1 \mu_1^* + n_2 \mu_2^* \quad \text{where } n_i = -\langle \beta_0, \mu_i \rangle - \frac{a_i q_i - b_i}{p_i}.$$

Since  $a_i q_i \equiv b_i \pmod{p_i}$  for  $i = 1, 2$  it follows that  $n_i$  is an integer. Moreover, since  $\beta - \beta_0 \in \iota'_M(H^1(M))$  we have that  $\delta_M(\beta_0) = \delta_M(\beta) = e_\sigma(T\mathcal{F}) \in H^2(M, \partial M)$ . We have to prove that  $(\delta_{N_i} \circ f_i^*)(\beta) = e_{\sigma_i}(T\mathcal{D}_i)$  for  $i = 1, 2$ . Since

$$H^2(N_i, \partial N_i) \cong \text{Hom}(H_2(N_i, \partial N_i), \mathbb{Z})$$

it is enough to prove that  $\langle (\delta_{N_i} \circ f_i^*)(\beta), [D_i] \rangle = \langle e_{\sigma_i}(T\mathcal{D}_i), [D_i] \rangle$  and this is a consequence of the computation (the case  $i = 2$  is analogous)

$$\begin{aligned} \langle (\delta_{N_1} \circ f_1^*)(\beta), [D_1] \rangle &= \langle \beta, f_1(\partial D_1) \rangle \\ &= \langle \beta_0, p_1 \mu_1 + q_1 \lambda_1 \rangle + n_1 \langle \mu_1^*, p_1 \mu_1 + q_1 \lambda_1 \rangle + n_2 \langle \mu_2^*, p_1 \mu_1 + q_1 \lambda_1 \rangle \\ &= p_1 \langle \beta_0, \mu_1 \rangle + a_1 q_1 + p_1 \left( -\langle \beta_0, \mu_1 \rangle - \frac{a_1 q_1 - b_1}{p_1} \right) \\ &= b_1 \\ &= \langle e_{\sigma_1}(T\mathcal{D}_1), [D_1] \rangle, \end{aligned}$$

where in the last line we have used again that

$$\langle \beta_0, \lambda_1 \rangle = \langle \beta_0, [\partial S_1] \rangle = \langle \delta_M \beta_0, [S_1] \rangle = \langle e_\sigma(T\mathcal{F}), [S_1] \rangle = a_1$$

and that  $\langle \mu_2^*, \mu_1 \rangle = \langle \mu_2^*, \lambda_1 \rangle = 0$ . □

**Theorem 1.4** *Let  $S^3_{p_1/q_1, p_2/q_2}$  (WL) be the  $(p_1/q_1, p_2/q_2)$ -surgery on the Whitehead link, with  $q_1, q_2 \neq 0$  and  $p_1, p_2 > 0$ . Then the taut foliations constructed in the proof of [Theorem 1.1](#) have vanishing Euler class if and only if  $|q_i| \equiv 1 \pmod{p_i}$  for  $i = 1, 2$ .*

*In particular, for all these manifolds the L-space conjecture holds.*

**Proof** First of all we prove that  $e(T\mathcal{F}) = 0$ . In fact, let  $T$  denote one of the boundary components of  $M$ ; the inclusion of  $T$  in  $M$  induces an isomorphism  $\iota: H^2(M; \mathbb{Z}) \rightarrow H^2(T; \mathbb{Z})$ . Therefore,

$$e(T\mathcal{F}) = 0 \iff \iota(e(T\mathcal{F})) = 0.$$

By naturality of the Euler class,

$$\iota(e(T\mathcal{F})) = e(T(\mathcal{F}|_T))$$

and since  $\mathcal{F}|_T$  admits a nowhere vanishing section, the last quantity is zero.

We now want to compute the numbers  $a_i$ . As a consequence of the proof of Theorem 1.7 in [Hu 2019],

- $b_i = 1$  if  $q_i < 0$  for  $i = 1, 2$ ;
- $b_i = -1$  if  $q_i > 0$  for  $i = 1, 2$ ;
- $\langle e_\sigma(T\mathcal{F}), [S] \rangle = \chi(S) = -2$ .

Since by construction  $S$  intersects positively in one point the meridians of the components of the Whitehead link, we have the equality  $[S] = [S_1 + S_2]$  in  $H_2(M, \partial M; \mathbb{Z})$  and hence

$$a_1 + a_2 = \langle e_\sigma(T\mathcal{F}), [S_1] \rangle + \langle e_\sigma(T\mathcal{F}), [S_2] \rangle = \chi(S) = -2.$$

As a consequence of [Thurston 1986, Corollary 1, page 118] for any  $[F] \in H_2(M, \partial M; \mathbb{Z})$  we have the inequality

$$|\langle e_\sigma(T\mathcal{F}), [F] \rangle| \leq |\chi(F)|$$

and since  $S_1$  and  $S_2$  are 1–holed tori, this implies that  $a_i = \langle e_\sigma(T\mathcal{F}), [S_i] \rangle = -1$  for  $i = 1, 2$ . Therefore, by virtue of Proposition 4.5 we have  $e(T\hat{\mathcal{F}}) = 0$  if and only if for each  $i = 1, 2$  either

- $q_i$  is positive and  $q_i \equiv 1 \pmod{p_i}$ , or
- $q_i$  is negative and  $q_i \equiv -1 \pmod{p_i}$ .

In other words  $e(T\hat{\mathcal{F}}) = 0$  if and only if

$$|q_i| \equiv 1 \pmod{p_i} \quad \text{for } i = 1, 2,$$

which is exactly what we wanted. □

We point out the following straightforward consequence of Theorem 1.4.

**Corollary 4.6** *Let  $d_1$  and  $d_2$  be two integers such that  $d_1 < 0$  or  $d_2 < 0$ . Then the manifold  $S_{d_1, d_2}^3(\text{WL})$  satisfies the *L*-space conjecture. □*

We conclude by collecting from the literature some results regarding the orderability (or nonorderability) of some surgeries on the Whitehead link, obtaining a generalisation of Corollary 4.6.

**Theorem 1.5** *Let  $m \neq 0$  be an integer.*

- *If  $m \leq -1$  then the manifolds  $S_{m,p/q}^3(\text{WL})$  and  $S_{p/q,m}^3(\text{WL})$  have left-orderable fundamental groups for all rationals  $p/q$ .*
- *If  $m \geq 1$  then the manifolds  $S_{m,p/q}^3(\text{WL})$  and  $S_{p/q,m}^3(\text{WL})$  have non-left-orderable fundamental groups for all rationals  $p/q \geq 1$ .*

*In particular, all the rational homology spheres obtained by integer surgery on WL satisfy the L–space conjecture.*

**Proof** The manifold  $S_{m,\bullet}^3(\text{WL})$  fibers over the circle if and only if  $m$  is an integer [Hodgson et al. 1992]. Moreover, in this case the fiber is a punctured torus.

When  $m \leq -1$  the monodromy of  $S_{m,\bullet}^3(\text{WL})$  can be extended to an Anosov diffeomorphism  $\phi$  of the torus that preserves the orientations of its stable and unstable foliations [Hodgson et al. 1992]. The manifold  $S_{m,p/q}^3(\text{WL})$  can be obtained by surgery along a closed orbit of  $\phi$  in the mapping torus  $M_\phi$  and as a consequence of [Zung 2020, Theorem 1] we have that all the nontrivial fillings of  $S_{m,\bullet}^3(\text{WL})$  have left-orderable fundamental group. Since WL is symmetric, the same result holds for  $S_{\bullet,m}^3(\text{WL})$ .

When  $m \geq 3$  and  $p/q \geq 1$ , the fundamental group of the manifold  $S_{m,p/q}^3(\text{WL})$  was studied by Roberts, Shareshian and Stein [Roberts et al. 2003, Proposition 3.1], where they prove that it is not orderable. The technical details of their result are contained in the proofs of [Roberts et al. 2003, Lemma 3.5 and Corollary 3.6] and these also work in the case  $m = 2$  (notice that in their notation, this is the case “ $m = 0$ ”). When  $m = 1$  the manifold  $S_{1,\bullet}^3(\text{WL})$  is the exterior of the right-handed trefoil knot and its surgeries are well known [Moser 1971]. In fact all surgeries but one yield Seifert fibered manifolds and we have already proved that these are L–spaces; since the L–space conjecture holds for Seifert fibered manifolds we deduce that these surgeries are all nonorderable. The remaining surgery is a connected sum of two lens spaces and since its fundamental group has torsion it is not orderable as well.  $\square$

**Remark 4.7** Even if the content of Corollary 4.6 is generalised by Theorem 1.5, the statement of Theorem 1.4 is not; in fact there are also noninteger rationals  $p_1/q_1$  and  $p_2/q_2$  that satisfy the hypotheses of Theorem 1.4.

Dunfield [2020] considers of more than 300 000 hyperbolic rational homology spheres, testing the conjecture on this collection. These manifolds are obtained by filling 1–cusped hyperbolic 3–manifolds that can be triangulated with at most 9 ideal tetrahedra; see [Burton 2014]. We checked whether some of these manifolds studied by Dunfield arise as Dehn surgery on the Whitehead link and obtained the following.

**Proposition 4.8** *Among the 307 301 rational homology spheres studied in [Dunfield 2020], at least 625 are obtained as Dehn surgery on the Whitehead link. In [Dunfield 2020] it is proved that 300 of these manifolds are orderable and 250 are nonorderable.*

*It follows from Theorem 1.5 that 16 of the remaining 75 manifolds are orderable and 10 are nonorderable.*

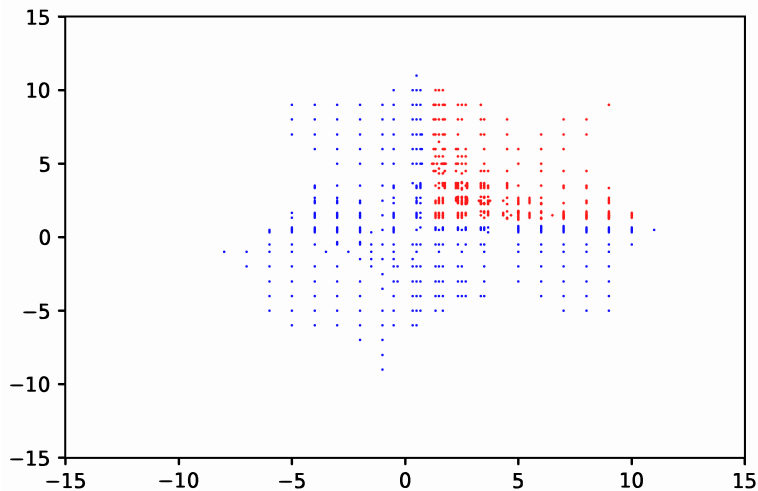


Figure 37: In this figure the red dots represent the coefficients whose corresponding surgery is nonorderable and the blue dots represent the coefficients whose corresponding surgery is orderable.

The code of the program can be found at <https://sites.google.com/view/santoro/research>. The surgery coefficients yielding these manifolds are plotted in Figure 37.

The examples of Theorem 1.5 and Proposition 4.8 are consistent with the conjecture, and together with Theorem 1.1 confirm that the L-space conjecture holds also for all these manifolds.

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*Scuola Normale Superiore*  
*Pisa, Italy*

[diego.santoro95@gmail.com](mailto:diego.santoro95@gmail.com)

Received: 1 February 2022      Revised: 19 September 2022

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Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

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AGT peer review and production are managed by EditFlow<sup>®</sup> from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
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