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# Horizontal decompositions, I

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We show that every smooth closed orientable 4–manifold $X$ admits a special kind of handlebody decomposition that we call *horizontal*. We classify the closed 4–manifolds with the simplest horizontal decompositions and we describe all such decompositions of $\mathbb{CP}^2$, showing that they give rise to infinitely many of the known embeddings of rational homology balls in the complex projective plane.

57K40; 57R40

## 1 Introduction

Let $p > q \geq 0$ be coprime integers and $B_{p,q}$ the rational homology ball smoothing of the quotient singularity $(1/p^2)(pq-1,1)$ used by Fintushel and Stern in the rational blow-down construction [7]. We were led to the results here by the realization that the smooth embeddings $B_{p,q} \hookrightarrow \mathbb{CP}^2$ we constructed in [17] were obtained using certain special handlebody decompositions of $\mathbb{CP}^2$, and that every smooth closed orientable 4–manifold admits such decompositions. Our purpose here is to define horizontal decompositions, prove the general existence result just mentioned, study the simplest cases and illustrate their potential applications by showing how to recover infinitely many of the known embeddings of the $B_{p,q}$ into $\mathbb{CP}^2$ in a "systematic" way.

Let $X\colon \partial_- X \to \partial_+ X$ be a smooth oriented 4–dimensional cobordism, ie a smooth oriented compact 4–manifold with oriented boundary $\partial X = \partial_+ X \cup -\partial_- X$. We always assume that $\partial_- X$ and $\partial_+ X$ are nonempty and connected. By eg Gompf and Stipsicz [9, Proposition 4.2.13], $X$ admits a handlebody decomposition relative to $\partial_- X$ without 0– or 4–handles. Suppose such a decomposition has $u$ 1–handles and $h$ 3–handles. The 1–handles give a cobordism $\partial_- X \to \partial_-^u X := \partial_- X \#^u S^1 \times S^2$, and the 3–handles a cobordism $\partial_+^h X := \partial_+ X \#^h S^1 \times S^2 \to \partial_+ X$. The cobordism $\partial_-^u X \to \partial_+^h X$ given by the 2–handles determines and is determined by a framed link $L \subset \partial_-^u X$. We denote it by $X_L$.

**Definition 1.1** Let $Y$ be a closed oriented 3–manifold and $L = \bigcup_{i=1}^k L_i \subset Y$ a $k$–component framed link. We say that $L$ is a *horizontal link of type* $g$ if $g > 0$, there is a decomposition $Y = H_g \cup \Sigma_g \times [0,1] \cup H_g'$, where $H_g$ and $H_g'$ are 3–dimensional genus-$g$ handlebodies, and there are real numbers $0 < t_1 < \cdots < t_k < 1$ such that, for each $i = 1, \dots, k$,

- $L_i$ is a homotopically nontrivial simple closed curve on $\mathfrak{S}_i := \Sigma_g \times \{t_i\}$,

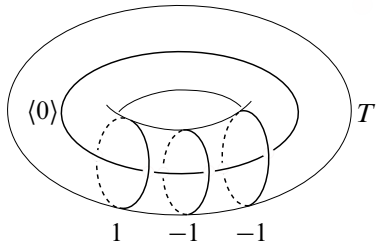


Figure 1: A framed link giving $(1,1,2,0)$–decompositions of $S^2\times S^2$.

- $|\mathrm{fr}(L_i)-\mathrm{fr}_{\mathfrak{S}_i}(L_i)|=1$, where $\mathrm{fr}(L_i)$ is the given framing of $L_i$ and $\mathrm{fr}_{\mathfrak{S}_i}(L_i)$ the framing induced on $L_i$ by $\mathfrak{S}_i$.

We call the difference $\mathrm{fr}(L_i)-\mathrm{fr}_{\mathfrak{S}_i}(L_i)$ the *relative framing* of $L_i$.

**Definition 1.2** When $L\subset\partial^u_- X$ is a horizontal link of type $g$ with $\ell=|L|$ components and $X_L\colon\partial^u_- X\to\partial^h_+ X$, we call a handlebody decomposition of an oriented cobordism $X\colon\partial_- X\to\partial_+ X$ as above a *horizontal decomposition* of $X$ of type $(g,u,\ell,h)$, or simply a $(g,u,\ell,h)$*–decomposition* of $X$. We say that a smooth closed 4–manifold $\widehat{X}$ has a horizontal decomposition of type $(g,u,\ell,h)$ if $X:=\widehat{X}\setminus\{B^4\cup B^4\}$ has one as a cobordism $S^3\to S^3$.

Note that $g\geq b_1(\partial^u_- X)=u+b_1(\partial_- X)\geq u$. Moreover, it is not difficult to check that if a link $L\subset\partial^u_- X$ is horizontal with respect to a genus-$g$ Heegaard splitting $H_g\cup_{\Sigma_g}H'_g=\partial^u_- X$, then $\partial^h_+ X$ has a genus-$g$ Heegaard splitting obtained by cutting along $\Sigma_g$ and regluing via a diffeomorphism; see Lemma 3.1. In particular, the Heegaard genus of $\partial^h_+ X$ is at most $g$. As above, this number cannot be less than $h$. This shows that *for every horizontal decomposition of type* $(g,u,\ell,h)$, *the inequality* $g\geq\max(u,h)$ *holds*.

**Example 1.3** The following simple example illustrates the above definitions. View the $\langle 0\rangle$–labeled unknot in Figure 1 as a surgery description of $S^1\times S^2$. The torus $T$ in Figure 1 is a Heegaard surface, the framed link $\mathscr{L}$ given by the three framed knots is framed isotopic to several type-1 horizontal links $L$, and in each case $X_L\colon S^1\times S^2\to S^3$. By Cerf [2] and Laudenbach and Poénaru [14], there is a canonical way to obtain a closed 4–manifold $\widehat{X}_L$ gluing $S^1\times D^3$ to $X_L$ along $S^1\times S^2$ and a 4–ball along $S^3$. An easy application of Kirby calculus shows that $\widehat{X}_L=S^2\times S^2$.

The following establishes a basic existence result for horizontal decompositions:

**Theorem 1.4** *Let $X\colon\partial_- X\to\partial_+ X$ be a smooth oriented connected 4–dimensional cobordism such that $\partial_- X$ and $\partial_+ X$ are nonempty and connected. Then $X$ admits a horizontal decomposition of type $(u+g_H(\partial_- X),u,\ell,h)$, where $g_H(\partial_- X)$ is the Heegaard genus of $\partial_- X$, and such that each component of the associated horizontal link is nonseparating in the Heegaard surface.*

**Remark 1.5** The referee pointed out that the existence of horizontal decompositions also follows from the fact that each framed link is horizontal with respect to some Heegaard splitting. At the end of Section 2 we sketch a proof of this fact following the referee's suggestion.

We prove Theorem 1.4 by showing that any handlebody decomposition of $X$ can be made horizontal by adding $(1,2)$–canceling pairs and sliding 2–handles; see Section 2. Our proof bears some resemblance to John Harer's argument [12] showing that any 4–dimensional 2–handlebody admits an achiral Lefschetz fibration with bounded fibers over $D^2$ (see also Etnyre and Fuller [5, Theorem 7]). Theorem 1.4 may also be compared to the main result in [5], where Etnyre and Fuller combine Harer's result with Giroux's [8] to prove that the complement of a smoothly embedded circle in a closed 4–manifold admits an achiral fibration over $S^2$. Nevertheless, the two results appear to us genuinely different, and in any case we do not rely on Giroux's correspondence.

**Remark 1.6** In view of Definition 1.2 and Theorem 1.4, it makes sense to define the *genus* of a smooth oriented cobordism or closed 4–manifold $X$ as the smallest $g \geq 1$ such that $X$ admits a $(g,u,\ell,h)$–decomposition. It is easy to check that a cobordism (or closed 4–manifold) $X$ with genus $g(X)=1$ and first Betti number $b_1(X)=0$ must have finite cyclic fundamental group. This implies that there are many smooth closed 4–manifolds $X$ with $b_1(X)=0$, $g(X)>1$ and $\chi(X)=2$, which is the smallest possible value of the Euler characteristic under these assumptions. As kindly pointed out by the referee, simple examples are given by spun aspherical homology 3–spheres; see Gordon [10] and Suciu [21].

In order to get the first classification results for cobordisms with horizontal decompositions, we introduce some simplifying assumptions. A natural choice is to look at cobordisms $X\colon S^3 \to S^3$. The following result deals with the simplest cases of $(1,u,\ell,0)$–decompositions. In a forthcoming paper [18], we deal with the simplest $(1,u,\ell,1)$–decompositions.

**Theorem 1.7** *Let $X\colon S^3 \to S^3$ be an oriented cobordism with a $(1,u,\ell,0)$–decomposition. Then, after possibly reversing the orientation of $X$, we have*

$$X \cong \begin{cases} S^3 \times [0,1] & \text{if } \chi(X)=0, \\ \mathbb{CP}^2 \setminus (B^4 \sqcup B^4) & \text{if } \chi(X)=1. \end{cases}$$

A key fact used in the proof of Theorem 1.7 is that to a horizontal framed link $L \subset \Sigma_g \times [0,1]$ one can naturally associate an element of the mapping class group $\mathrm{Map}(\Sigma_g)$ factorized as a product of Dehn twists. In fact, suppose $L = \bigcup_{i=1}^k L_i$, with

$$L_i \subset \mathfrak{S}_i := \Sigma_g \times \{t_i\} \quad \text{for } 0 < t_1 < \cdots < t_k < 1.$$

Since $\pi$ identifies each $\mathfrak{S}_i$ with $\Sigma_g$, one can associate to each component $L_i$ the Dehn twist

$$\tau_i := \tau_{\pi(L_i)} \in \mathrm{Map}(\Sigma_g),$$

where $\pi\colon \Sigma_g \times [0,1] \to \Sigma_g$ is the projection onto the first factor. We define the *factorization* of $L$ to be the $k$–tuple

$$F_L = (\tau_k^{\delta_k}, \tau_{k-1}^{\delta_{k-1}}, \dots, \tau_1^{\delta_1}),$$

where the exponent $\delta_i := \mathrm{fr}_{\mathfrak{S}_i}(L_i) - \mathrm{fr}(L_i) \in \{\pm 1\}$ is equal to *minus* the relative framing of $L_i$ for each $i$. The product $m_L = \tau_k^{\delta_k} \cdots \tau_1^{\delta_1} \in \mathrm{Map}(\Sigma_g)$ will be called the *monodromy* of $L$. Sometimes, when the

horizontal link $L$ is clear from context, we will abuse notation and talk about the factorization or the monodromy of the decomposition of the associated cobordisms $X_L$ or $X$.

**Example 1.8** The concept of factorization can be exemplified via the link $\mathscr{L}$ of Example 1.3. As we already observed, $\mathscr{L}$ is framed isotopic to several type-1 horizontal links $L = \bigcup_{i=1}^{3} L_i \subset S^1 \times S^2$. For example, we can arrange that the 1–framed unknot becomes the component $L_1 \subset \Sigma_g \times \{t_1\}$ at the "lowest level". In that case, the horizontal link $L$ has factorization $(\tau_\mu, \tau_\mu, \tau_\mu^{-1})$, where $\mu \subset T$ is the meridian of the 0–framed unknot. If we replace $\mathscr{L}$ with any sublink $\mathscr{L}' \subset \mathscr{L}$ consisting of two components of $\mathscr{L}$, depending on the choice of $\mathscr{L}'$ and the isotopies, the corresponding horizontal link $L'$ has factorization $(\tau_\mu^{-1}, \tau_\mu)$, $(\tau_\mu, \tau_\mu^{-1})$ or $(\tau_\mu, \tau_\mu)$.

In the proof of Theorem 1.7 we determine the factorizations of all $(1, u, \ell, 0)$–decompositions of $\mathbb{CP}^2$ minus two disjoint balls, viewed as an oriented cobordism $S^3 \to S^3$. We do that by relating such factorizations with the solutions of certain degree-2 Diophantine equations. We exploit a certain structure of the solutions of those equations to show that any horizontal decomposition of $X$ can be simplified in a "systematic" way. For this approach to the proof of Theorem 1.7 we took inspiration from Denis Auroux's paper [1].

As we mentioned above, we discovered horizontal decompositions while trying to construct smooth embeddings of the rational balls $B_{p,q}$ into $\mathbb{CP}^2$. Hence, it should not be surprising that the proof of Theorem 1.7 yields such embeddings, leading to Theorem 1.9. In order to give some motivation, we provide the following brief survey of results about embeddings of rational homology balls in $\mathbb{CP}^2$.

It is well known that $\partial B_{p,q}$ is the lens space $L(p^2, pq-1)$. Moreover, $B_{p,p-q}$ is orientation-preserving diffeomorphic to $B_{p,q}$ by Evans and Smith [6, Remark 2.8]. Using results by Khodorovskiy [13], it is not hard to show [6, Section 2.1] that if the positive integers $p_1$, $p_2$ and $p_3$ form a *Markov triple*, that is $p_1^2 + p_2^2 + p_3^2 = 3p_1p_2p_3$, then there is a symplectic embedding of the disjoint union $\bigcup_{i=1}^{3} B_{p_i,q_i}$ into $\mathbb{CP}^2$, where $q_i = \pm 3p_j/p_k \bmod p_i$ with $\{i,j,k\} = \{1,2,3\}$. Conversely, Evans and Smith [6, Theorem 1.2] generalized results of Hacking and Prokhorov [11] in complex algebraic geometry by showing that if $B_{p_i,q_i} \subset \mathbb{CP}^2$ for $i = 1, \dots, N$ is a collection of pairwise disjoint symplectic embeddings, then $N \le 3$ and the $p_i$ and the $q_i$ satisfy some constraints which coincide with the ones above when $N = 3$. Moreover, they showed that if $B_{p,q} \subset \mathbb{CP}^2$ is a symplectic embedding, then $\mathbb{CP}^2 \setminus B_{p,q}$ contains a disjoint union $B_{p',q'} \sqcup B_{p'',q''}$, therefore $p$ must belong to a Markov triple and divide $q^2+9$. Let $\{F_m\}_{m\in\mathbb{Z}}$ be the classical Fibonacci sequence defined by $F_{-1} = 1$, $F_0 = 0$ and $F_{m+1} = F_m + F_{m-1}$. Owens [20, Theorem 1] recently proved the existence of smooth embeddings $B_{F_{2m+1},F_{2m-1}} \subset \mathbb{CP}^2$ for each $m \ge 1$. He also observed that $F_{2m+1}$ divides $F_{2m-1}^2 + 9$ only if $m = 1$, so it follows from [6] that $B_{F_{2m+1},F_{2m-1}}$ does not embed *symplectically* in $\mathbb{CP}^2$ for $m > 1$. Moreover, Owens used work of Donaldson [4] to show that it is not possible to embed in $\mathbb{CP}^2$ a disjoint union of two or more of the rational homology balls $B_{F_{2m+1},F_{2m-1}}$ with $m \ge 1$. In [17] we extend Owens' family of smooth embeddings to a 2–parameter family $\{B(k,m)\} \subset \{B_{p,q}\}$ such that $B(k,m)$ cannot

be symplectically embedded in $\mathbb{CP}^2$. Moreover, in [19] we prove the nonexistence of almost complex embeddings $B(k,m) \subset \mathbb{CP}^2$ without relying on [6].

The proof of Theorem 1.7 shows that when $\chi(X) = 1$, most of the horizontal decompositions of $X$ contain one 1–handle and two 2–handles. As explained in Section 5, this implies that many horizontal decompositions of $\mathbb{CP}^2$ yield a smooth embedding of a disjoint union of $B_{p,q}$'s into $\mathbb{CP}^2$. In Section 5 we use this fact to prove Theorem 1.9. Note that $F_{2m+1} > 0$ for each $m \in \mathbb{Z}$, and see Theorem 5.2 for a more precise version of the statement.

**Theorem 1.9** *The $(1,1,2,0)$–decompositions of $\mathbb{CP}^2$ give rise to smooth orientation-preserving embeddings into $\mathbb{CP}^2$ of the disjoint unions*

$$B_{m+1} \sqcup B_m \quad \text{and} \quad B'_{m+1} \sqcup B'_m$$

*for each $m \geq 0$, where $B_m = B_{F_{2m-1},F_{2m-5}}$ and $B'_m = (-1)^m B_{F_{m+1},F_m}$.*

**Remark 1.10** The rational balls $B_m$ form a subfamily of the $B_{p,q}$ that embed symplectically in $\mathbb{CP}^2$. More precisely, for any $m \geq 0$ the triple $(1, F_{2m-1}, F_{2m+1})$ is a Markov triple producing a symplectic embedding of $B^4 \cup B_m \cup B_{m+1}$. On the other hand, the balls $B'_{2k} = B_{F_{2k+1},F_{2k}} \cong B_{F_{2k+1},F_{2k+1}-F_{2k}} = B_{F_{2k+1},F_{2k-1}}$ are exactly the rational balls shown by Owens to embed smoothly but not symplectically.

In [18] we use horizontal decompositions to prove the existence of many more smooth embeddings of the rational balls $B_{p,q}$ into $\mathbb{CP}^2$.

A few questions are naturally raised by our results. First, it is natural to wonder about the relationship between the various horizontal decompositions of a given cobordism. More precisely, one can ask whether there exists a uniqueness result for horizontal decompositions, perhaps up to the Hurwitz moves of Section 4.3 combined with some kind of "horizontal stabilizations". Second, we could ask what the possible generalizations of Theorem 1.7 are. For instance, we are planning to establish the analogue of Theorem 1.7 for the simplest cobordisms $S^3 \to S^3$ with a $(1,u,\ell,1)$–decomposition [18]. Finally, we expect that the techniques of the present paper can be applied to find new smooth embeddings of rational balls and/or 3–manifolds in $\mathbb{CP}^2$ and/or other smooth 4–manifolds with small Euler characteristic.

The paper is organized as follows. In Section 2 we prove Theorem 1.4, in Sections 3 and 4 we prove Theorem 1.7 and in Section 5 we prove Theorem 5.2, which implies Theorem 1.9.

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## 2 Proof of Theorem 1.4

We refer the reader to [9, Chapter 4] for basic facts about handlebody decompositions. The plan of the proof is as follows. We start from any handle decomposition of a smooth 4–dimensional cobordism

$X\colon \partial_- X \to \partial_+ X$ and we modify it so that the framed link $L$ consisting of the attaching curves of the 2–handles satisfies the conditions of Definition 1.2.

We start by recalling a few facts about the operation of addition of a $(1,2)$–canceling pair. Let $\mathscr{H}$ be a handle decomposition of $X$ without 0– or 4–handles, and with $u$ 1–handles and $h$ 3–handles. Then attaching the 1–handles gives a cobordism $\partial_- X \to \partial_-^u X$. A 4–dimensional $(1,2)$–canceling pair consists of a 1–handle $\mathfrak{h}$ and a 2–handle $\mathfrak{h}'$ such that the attaching sphere of $\mathfrak{h}'$ intersects the belt sphere of $\mathfrak{h}$ transversely in a single point. The introduction of a $(1,2)$–canceling pair does not alter $X\colon \partial_- X \to \partial_+ X$, but turns $\mathscr{H}$ into a new handlebody decomposition $\mathscr{H}'$ of $X$ in such a way that the 1–handles of $\mathscr{H}'$ give a subcobordism $X' \subset X$ from $\partial_- X$ to the 3–manifold $Y$ obtained from $\partial_-^u X$ by removing the interior of two disjoint 3–balls $B_1$ and $B_2$ and gluing to each other the corresponding boundary 2–spheres via an orientation-reversing diffeomorphism. In other words, $Y$ is diffeomorphic to $\partial_-^{u+1} X$ and $X'\colon \partial_- X \to Y$. The canceling 2–handle is attached along any arbitrarily framed curve in $\partial_-^u X \setminus (B_1 \cup B_2)$ that connects two identified points of $\partial B_1$ and $\partial B_2$. Such a curve determines a component of the new framed link in $\partial_-^{u+1} X$.

Now let $\Sigma \subset \partial_-^u X$ be a Heegaard surface of genus $g_H(\partial_-^u X) = u + g_H(\partial_- X)$. Since we can always add a $(1,2)$–canceling pair, from now on we assume $g > 0$ without loss of generality. The complement of $\Sigma$ in $\partial_-^u X$ is a disjoint union $H_g \cup H_g'$ of two genus-$g$ handlebodies. Moreover, the handlebodies $H_g$ and $H_g'$ are regular neighborhoods of graphs $G \subset H_g$ and $G' \subset H_g'$, and $\partial_-^u X \setminus \{G \cup G'\}$ is diffeomorphic to the product of $\Sigma$ with an open interval. Up to framed isotopy, we may assume $L$ disjoint from $G \cup G'$, and thus contained in a closed regular neighborhood $N$ of $\Sigma$. After fixing a diffeomorphism between $N$ and $\Sigma \times [0,1]$, and consequently a projection $\pi\colon N \to \Sigma$, we may also assume that the restriction $\pi|_L\colon L \to \Sigma$ is an immersion, and that each self-intersection of $\pi(L) \subset \Sigma$ is a transverse double point. Hence, we can represent $L$ using its diagram $D_L \subset \Sigma$, consisting of $\pi(L)$ together with the over–under information at each point of self-intersection. Recall that, for any framed simple closed curve $\gamma \subset N$ contained in a horizontal copy of $\Sigma$, ie a surface of the form $\Sigma \times \{t\} \subset \Sigma \times [0,1]$, the relative framing of $\gamma$ is the integer $\mathrm{fr}(\gamma) - \mathrm{fr}_{\Sigma\times\{t\}}(\gamma)$. Observe that a framed link $L = \bigcup_{i=1}^k L_i \subset N$ is horizontal if each connected component $L_i \subset L \subset N$ sits on a horizontal copy of $\Sigma$ and

(1) the relative framing of $L_i$ is $\pm 1$,

(2) the diagram $D_{L_i}$ of $L_i$ has no crossings and is therefore a simple closed curve in $\Sigma$,

(3) $D_{L_i}$ is nonseparating in $\Sigma$.

Moreover, for each $i < j$,

(4) at each crossing between $D_{L_i}$ and $D_{L_j}$ the overpassing arc belongs to $D_{L_j}$.

We say that a crossing of $D_L$ is *bad* if it involves a diagram $D_{L_i}$ which violates either (2) or (4). We are going to change the handle decomposition $\mathscr{H}$ into a horizontal decomposition of $X$ via a finite sequence of steps. At each step we shall either eliminate a bad crossing or adjust the framing of some $L_i$, making sure that the genus of $\Sigma$ is $u + g_H(\partial_- X)$ at each step, and at the last step (1)–(4) are satisfied. In the

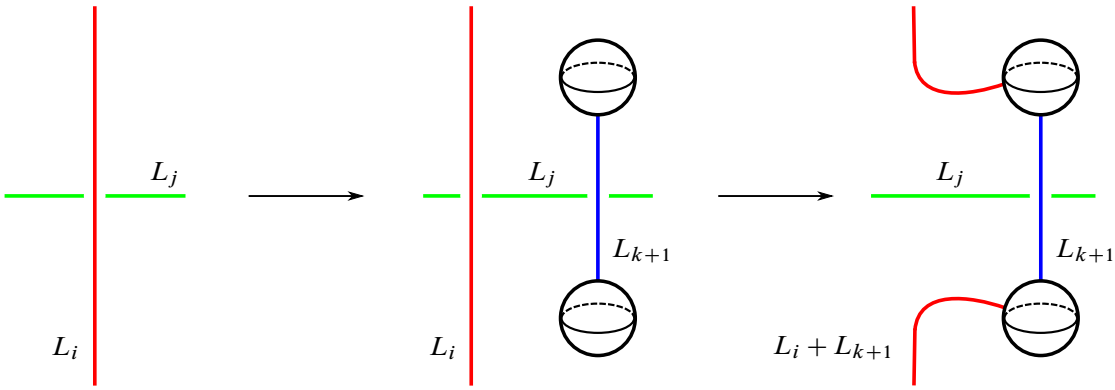


Figure 2: Elimination of a bad crossing.

process of changing $\mathscr{H}$ we will add 4–dimensional $(1,2)$–canceling pairs using the following *stabilization* procedure for each pair:

- Choose two disks $\Delta_1$ and $\Delta_2$ in $\Sigma$ disjoint from each other and from the projections of $L_1,\dots,L_n$, as well as an orientation-reversing diffeomorphism $\varphi$ between their boundaries.
- Let $B_i \subset \partial^u_- X$, for $i = 1,2$, be disjoint 3–balls such that $B_i \cap N$ corresponds to $\Delta_i \times [0,1]$ under the identification of $N$ with $\Sigma \times [0,1]$.
- Attach a 1–handle $D^3 \times [0,1]$ to $\partial^u_- X$ along $B_1$ and $B_2$, giving a cobordism from $\partial^u_- X$ to $\partial^{u+1}_- X$, with the latter viewed as the quotient of $\partial^u_- X \setminus (B_1 \cup B_2)$ obtained by gluing $\partial B_1$ to $\partial B_2$ via a diffeomorphism which looks like $\varphi \times \mathrm{id}_{[0,1]}$ when restricted to $N$.
- Attach the 2–handle along a curve contained in a horizontal copy of $\Sigma$ inside $N$.

Note that the image $N'$ of $N \setminus (B_1 \cup B_2)$ inside $\partial^{u+1}_- X$ is diffeomorphic to $\Sigma' \times [0,1]$, where $\Sigma'$ is the quotient of $\Sigma \setminus (\Delta_1 \cup \Delta_2)$ obtained by gluing $\partial\Delta_1$ to $\partial\Delta_2$ via the diffeomorphism $\varphi$. Also, we have a projection $\pi' \colon N' \to \Sigma'$ induced by the restriction of $\pi$ to $N \setminus (B_1 \cup B_2)$, and $\Sigma' \subset \partial^{u+1}_- X$ is a Heegaard surface of genus $g(\Sigma) + 1 = u + 1 + g_H(\partial_- X)$.

We are now going to start the modification of $\mathscr{H}$. Suppose that $i < j$ and let $c$ be a bad crossing between $D_{L_i}$ and $D_{L_j}$ inside $\Sigma$. Then the overpassing arc at $c$ belongs to $L_i$, as in Figure 2, left. By a stabilization as above, we introduce a canceling pair near $c$, so that the attaching circle of the new 2–handle, which we call $L_{k+1}$, has relative framing 0 and passes over $L_j$ once, as in Figure 2, center. Sliding $L_i$ over $L_{k+1}$ we eliminate the bad crossing between $L_i$ and $L_j$, replacing it with a good crossing between $L_j$ and $L_{k+1}$, as in Figure 2, right. The same argument works for self intersections, ie when $i = j$. We can repeat this procedure until we get a diagram with no bad crossings, so that each component of $L$ can be assumed to sit on a horizontal copy of $\Sigma$.

Now we proceed to adjust the framings. In order to do that, we use the Kirby calculus operation of twisting a 1–handle by 360 degrees, for which we refer to [9, Section 5.4]. Figure 3 illustrates the fact that a 360–degree twist of one of the two attaching spheres of a 1–handle changes the relative framings

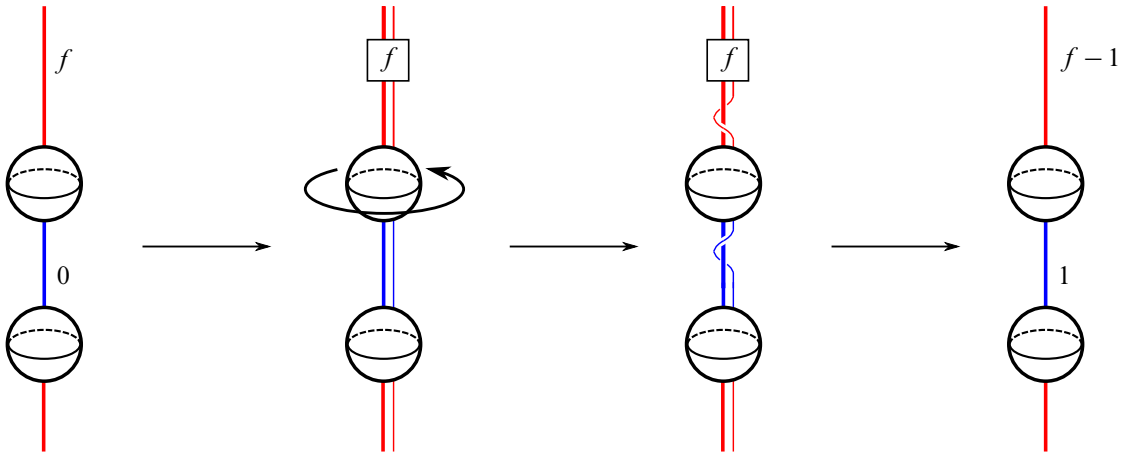


Figure 3: Twisting a 1–handle. The relative framing of a knot can be represented by either an integer or a longitude: in the first and last step we are just switching from one notation to the other, while in the middle we are twisting a sphere as indicated by the arrow. Of course, the twist can also be performed in the opposite direction. In that case, the new framings would be $f+1$ and $-1$.

of the 2–handles going over it. If we do this for all the 1–handles of the canceling pairs introduced to eliminate bad crossings, we can change the relative framings of the canceling 2–handles from 0 to $\pm 1$. At that point, if 2–handles with relative framing not equal to $\pm 1$ are still present, we can first perform more stabilizations and handle slides as in Figure 2 — just imagine removing $L_j$ from the pictures. Then apply twists with suitable signs as in Figure 3. Every time, we introduce a new $\pm 1$–framed 2–handle and change by $\mp 1$ the relative framing of an already existing 2–handle. We can clearly keep going like this until all the relative framings are $\pm 1$. Therefore so far we have ensured that $D_L$ satisfies (1), (2) and (4) above. If a component of the diagram happens to be homologically trivial, we can make it homologically nontrivial with a single stabilization. In fact, consider the first two pictures from the left in Figure 2 without the component $L_i$ and choosing the framing on $L_{k+1}$ so that its relative framing is $\pm 1$. If $L_j$ is homologically trivial in the first picture, it is certainly not so in the second picture, and its relative framing has not changed. This shows that, after possibly performing a few more stabilizations, $(1)-(4)$ can all be satisfied. Moreover, we made sure that the genus of $\Sigma$ is $u + g_H(\partial_- X)$ at each step. This concludes the proof of Theorem 1.4.

As anticipated in Remark 1.5, we now sketch an alternative proof of the existence of handlebody decompositions suggested by the referee. The idea is based on a construction by B Clark [3] showing that any link has finite tunnel number. Let $L \subset \partial^u_- X$ be a framed link. As before, up to isotopy we may assume that $L$ sits in a neighborhood $N \cong \Sigma \times [0,1]$ of a Heegaard surface $\Sigma \subset \partial^u_- X$ and that we have fixed a projection $\pi : N \to \Sigma$ providing us with a diagram $D_L \subset \Sigma$. We may also assume that $D_L$ is in general position with respect to the set $\{\alpha_1, \dots, \alpha_g\} \subset \Sigma$ of boundaries of a maximal set of disjoint compressing disks for the handlebody bounded by $\Sigma \times \{0\}$. For each crossing of $D_L$ and $D_L \cap \left(\bigcup_{i=1}^g \alpha_i\right)$, we add a "vertical segment" to $L \cup \left(\bigcup_{i=1}^g \alpha_i\right)$, obtaining an embedded trivalent graph $\Gamma$ with the property that the

complement in $\partial^u_- X$ of a regular neighborhood $N(\Gamma)$ is a handlebody. A pushed-off copy $L'$ of $L$ sits on the Heegaard surface $\partial N(\Gamma)$ in such a way that each connected component of $L'$ is homologically nontrivial, and its relative framing can be adjusted by an isotopy which adds to it an appropriate number of "twists" around $\Gamma$. This shows that the whole framed link $L$ sits horizontally on a Heegaard surface for $\partial^u_- X$ having genus equal to $g$ plus the number of crossings of $D_L$ and $D_L \cap \big(\bigcup_{i=1}^g \alpha_i\big)$. It follows from this argument that every smooth cobordism as in Theorem 1.4 admits a horizontal decomposition of type $(g, u, \ell, h)$ for some $g \geq u$. Note that, although this alternative proof gives a stronger conclusion on $L$, one loses control of the difference $g - u$.

## 3 Proof of Theorem 1.7: $\chi(X) = 0$

We state and prove Lemma 3.1 below in slightly greater generality than needed, for possible future use. The construction used in the lemma is essentially the same one used by Lickorish in his proof of the Lickorish–Wallace theorem [16].

**Lemma 3.1** *Let $H_g \cup (\Sigma_g \times [0,1]) \cup H'_g$ be a Heegaard decomposition of a closed 3–manifold $N$. Let $L \subset \Sigma_g \times [0,1]$ be a horizontal framed link with monodromy $m_L \in \mathrm{Map}(\Sigma_g)$ and associated cobordism $X_L \colon \partial_- X_L \to \partial_+ X_L$. Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_g)$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_g)$ be $g$–tuples of simple closed curves in $\Sigma_g$ bounding compressing discs in $H_g$ and $H'_g$, respectively, so that $(\boldsymbol{\mu}, \boldsymbol{\lambda})$ is a Heegaard diagram for $N$. Then $(m_L(\boldsymbol{\mu}), \boldsymbol{\lambda})$ is a Heegaard diagram for $\partial_+ X_L$, where $m_L(\boldsymbol{\mu})$ denotes the $g$–tuple $(m_L(\mu_1), \dots, m_L(\mu_g))$.*

**Proof** Given any orientation-preserving diffeomorphism $f \colon \Sigma_g \to \Sigma_g$, we can glue $\Sigma_g \times [0,1]$ to $H_g$ via $f$:

$$H_g \cup_f (\Sigma_g \times [0,1]) := H_g \cup (\Sigma_g \times [0,1]) / (x \in \partial H_g = \Sigma_g \sim f(x) \in \Sigma_g \times \{0\}).$$

The result is diffeomorphic to $H_g$ via the map

$$\psi = \mathrm{id}_{H_g} \cup (f^{-1} \times \mathrm{id}_{[0,1]}) \colon H_g \cup_f (\Sigma_g \times [0,1]) \to H_g \cup_{\mathrm{id}_{\Sigma_g}} (\Sigma_g \times [0,1]) \cong H_g.$$

Let $\Delta_i \subset H_g$ be a compressing disk with $\partial \Delta_i = \lambda_i$ for $i = 1, \dots, g$. Then

$$\partial_+ X_L = H_g \cup_{m_L} (\Sigma_g \times [0,1]) \cup_{\mathrm{id}_{\Sigma_g}} H'_g,$$

and $\Delta'_i = \Delta_i \cup m_L(\mu_i) \times [0,1]$ is a compressing disk in $H_g \cup_{m_L} (\Sigma_g \times [0,1]) \cong H_g$ with $\partial \Delta'_i = m_L(\mu_i)$. Therefore $(m_L(\boldsymbol{\mu}), \lambda)$ is a Heegaard diagram for $\partial_+ X_L$. □

The following proposition implies the first part of Theorem 1.7:

**Proposition 3.2** *Let $X \colon S^3 \to \partial_+ X$ be a smooth oriented cobordism with a horizontal decomposition of type $(1, u, \ell, 0)$, having Euler characteristic $\chi(X) = 0$ and $b_1(\partial_+ X) = 0$. Then $u \in \{0, 1\}$. If $u = 0$ then $X \cong S^3 \times [0,1]$; if $u = 1$ and the monodromy is $\tau_\gamma^\delta$, $X$ is orientation-preserving diffeomorphic to*

*$\pm B_{p,q} \setminus B^4$ for some $p > q \geq 0$, with the plus sign occurring if and only if $(p,q) = (1,0)$ or $\delta = 1$. In particular, $\partial_+ X = S^3$ if and only if $X$ is diffeomorphic to $S^3 \times [0,1]$.*

**Proof** Since $g = 1 \geq u$ and $0 = \chi(X) = \ell - u$, we have $\ell = u \in \{0,1\}$. If $u = \ell = 0$ clearly $X \cong S^3 \times [0,1]$. If $u = \ell = 1$, the decomposition of $X$ consists of a 1–handle and a 2–handle attached along a simple closed curve $\gamma \subset \partial_+(S^1 \times D^3 \setminus B^4) = S^1 \times S^2$ sitting on the standard genus-1 Heegaard torus $T$. In this case, the monodromy is simply $\tau_\gamma^\delta$ with $\delta \in \{\pm 1\}$ and it coincides with its factorization. After choosing a suitable orientation, we may assume that $\gamma$ supports the homology class $p\mu + q\lambda \in H_1(T;\mathbb{Z})$ with $p \geq 0$ and $p$, $q$ coprime, where $(\mu, \lambda)$ is a pair of oriented simple closed curves such that $\mu \cdot \lambda = 1$. Moreover, we can choose $\lambda$ as the boundary of a compressing disc in the solid torus $H_1$ and we are free to replace $\mu$ with $\mu + m\lambda$ for any $m \in \mathbb{Z}$. Thus, we may assume either $pq \neq 0$ or $(p,q) \in \{(1,0),(0,1)\}$. Clearly, if $(p,q) = (1,0)$ then $\gamma = \mu$ and $X \cong B_{1,0} \setminus B^4 \cong S^3 \times [0,1]$, while if $(p,q) = (0,1)$ then $\gamma = \lambda$ and by Lemma 3.1 $\partial_+ X = S^1 \times S^2$, and hence $b_1(\partial_+ X) \neq 0$. Therefore, we may assume $pq \neq 0$. If we view $S^1 \times S^2$ as 0–surgery on an unknot in $S^3$, we can think of $\gamma$ as sitting inside $S^3$. Then the Seifert framing induced by $T$ on $\gamma$ is equal to $\mathrm{fr}_T(\gamma) = pq$. To see this, it suffices to compute the linking number of $\gamma$ with its push-off in the direction of the negative normal to $T$. Since $\gamma = p\mu + q\lambda$, we have $\mathrm{fr}_T(\gamma) = lk(\gamma, p\mu^- + q\lambda^-) = lk(\gamma, q\lambda^-) = pq$. Up to replacing $q$ with its remainder modulo $p$ and $\mu$ with $\mu$ plus a multiple of $\lambda$, we may assume $p > q > 0$. If $X_L$ has factorization $(\tau_\gamma^\delta)$, the framing of $\gamma$ prescribed by the 2–handle is $\mathrm{fr}(\gamma) = pq - \delta$. By [15, Section 3.2], $X$ is orientation-preserving diffeomorphic to $B_{p,q} \setminus B^4$ if $\delta = 1$, $q > 0$ and $\gamma = p\mu - q\lambda$. Therefore $\gamma = p\mu + q\lambda = p(\mu+\lambda) - (p-q)\lambda$, so that $X$ is orientation-preserving diffeomorphic to $B_{p,p-q} \setminus B^4$. Note that taking the mirror image of the resulting Kirby calculus picture changes the sign of $\mathrm{fr}(\gamma)$ and $\mathrm{fr}_T(\gamma)$, and therefore the sign of $\delta$ as well. Thus, if $\delta = -1$ after changing orientation, the same argument shows that $-X$ is orientation-preserving diffeomorphic to $B_{p,q} \setminus B^4$, and hence $X \cong -B_{p,q} \setminus B^4$. To conclude the proof it suffices to observe that $B_{p,q}$ is orientation-preserving diffeomorphic to $-B_{p,q}$ if and only if $p = 1$. Indeed, the corresponding oriented boundaries $\partial B_{p,q} = L(p^2, pq-1)$ and $\partial(-B_{p,q}) = L(p^2, p^2-pq+1)$ are orientation-preserving diffeomorphic if and only if $p^2 - pq + 1 \equiv (pq-1)^{\pm 1} \mod p^2$, which never holds if $p > 1$. The last portion of the statement follows from the fact that $\partial B_{p,p-q} = \partial B_{p,q}$ is $S^3$ if and only if $B_{p,q} = B_{1,0} = B^4$. □

## 4 Proof of Theorem 1.7: $\chi(X) = 1$

The proof is organized as follows. In Section 4.1 we start the analysis of a smooth oriented cobordism $X : S^3 \to S^3$ having Euler characteristic $\chi(X) = 1$. We show that each decomposition determines a triple of integers which is a solution of a Diophantine equation satisfying certain extra constraints. In Section 4.2 we completely determine the sets of all such triples. In Section 4.3 we show that when applying Hurwitz moves to the factorization of a horizontal framed link, the associated horizontal decomposition changes by a sequence of handle slides while staying horizontal. Finally, in Section 4.4 we put everything together to finish the proof.

### 4.1 First step

Let $X\colon S^3 \to S^3$ be a smooth oriented cobordism with Euler characteristic $\chi(X) = 1$ having a horizontal decomposition of type $(1, u, \ell, 0)$. Since the number $u$ of 1–handles satisfies $u \le g = 1$, there is at most one 1–handle and no 3–handles.

If the decomposition contains no 1–handles, then $\chi(X) = 1$ implies that there is a single 2–handle attached along a simple closed curve $\gamma \subset S^3$ sitting on a genus-1 Heegaard torus $T$. Suppose that, with a suitable orientation, $\gamma$ supports the homology class $p\mu + q\lambda \in H_1(T;\mathbb{Z})$ with $p \ge 0$ and $p, q$ coprime, where $(\mu, \lambda)$ is a pair of oriented simple closed curves such that $\mu \cdot \lambda = 1$. We can choose $\lambda$ as the boundary of a compressing disc in the solid torus $H_1$, and we are free to replace $\mu$ with $\mu + m\lambda$ for any $m \in \mathbb{Z}$. Therefore, we may assume either $p > q > 0$ or $(p, q) \in \{(1, 0), (0, 1)\}$. Since the 2–handle is attached to $S^3$ with framing $\pm 1$ with respect to the framing $pq$ induced by $T$ and $H_1(S^3;\mathbb{Z}) = 0$, we must have $|pq \pm 1| = 1$. The only possibilities are $(2, 1)$, $(1, 0)$ and $(0, 1)$, and in every case we clearly have $X \cong \pm\mathbb{CP}^2 \setminus \{B^4 \cup B^4\}$. This concludes the proof when $\chi(X) = 1$ if there are no 1–handles.

From now on, we assume that the decomposition contains one 1–handle. In the notation of Section 1, the 2–handles define a cobordism $X_L\colon S^1 \times S^2 \to S^3$, where $L \subset S^1 \times S^2$ is a link which is horizontal with respect to the standard genus-1 Heegaard decomposition. The assumption $\chi(X) = 1$ implies that $L$ has two components $L_1$ and $L_2$ sitting on parallel copies of the standard Heegaard torus $T \subset S^1 \times S^2$. Let $N \cong T \times [0, 1]$ be a regular neighborhood of $T$ containing $L$, so that $S^1 \times S^2 = H_1 \cup N \cup H_1'$, and let $\pi : N \to T$ be the projection map. Let $\gamma_1 = \pi(L_1)$ and $\gamma_2 = \pi(L_2)$. By definition of horizontal decomposition, $\gamma_1$ and $\gamma_2$ are nonseparating simple closed curves. Setting $\tau_i := \tau_{\gamma_i}$, the factorization of $L$ is given by

$$F_L = (\tau_2^{\delta_2}, \tau_1^{\delta_1}), \tag{4-1}$$

where $\delta_i \in \{\pm 1\}$ is equal to minus the relative framing of $L_i$ and the monodromy of $L$ is $m_L = \tau_2^{\delta_2}\tau_1^{\delta_1}$.

Let $\lambda \subset T = \partial H_1$ be a simple closed curve which bounds a disc in $H_1$. We are going to show that, assuming the curves are oriented, the integers $\gamma_1 \cdot \lambda$, $\gamma_2 \cdot \lambda$ and $\gamma_2 \cdot \gamma_1$ satisfy a certain Diophantine equation. In Section 4.2 we determine the solutions of the equation and in Section 4.4 we use that knowledge, together with the results of Section 4.3, to finish the proof of Theorem 1.7.

We want to apply Lemma 3.1 to our cobordism $X_L\colon S^1 \times S^2 \to S^3$. As before, let $\lambda \subset T = \partial H_1$ be the arbitrarily oriented boundary of a compressing disc in $H_1$. Then $(\lambda, \lambda)$ is a Heegaard diagram for $S^1 \times S^2$, and it follows from Lemma 3.1 that $(m_L(\lambda), \lambda)$ is a Heegaard diagram for $\partial_+ X_L = S^3$, which immediately implies

$$|\lambda \cdot m_L(\lambda)| = 1. \tag{4-2}$$

From now on we assume $\gamma_1$ and $\gamma_2$ are oriented, and we abuse notation by also using $\gamma_1$ and $\gamma_2$ to denote the corresponding homology classes in $H_1(T;\mathbb{Z})$. Define

$$x := \gamma_1 \cdot \lambda, \quad y := \gamma_2 \cdot \lambda \quad \text{and} \quad n := \gamma_2 \cdot \gamma_1.$$

Note that $\tau_1$ and $\tau_2$ do not depend on the orientations of $\gamma_1$ and $\gamma_2$. Since

$$m_L(\lambda) = \tau_2^{\delta_2}(\tau_1^{\delta_1}(\lambda)) = \tau_2^{\delta_2}(\lambda + \delta_1 x\gamma_1) = \lambda + \delta_1 x\gamma_1 + \delta_2 y\gamma_2 + \delta_1\delta_2 nx\gamma_2,$$

taking the intersection product of both sides with $\delta_1\delta_2\lambda$ yields

$$\delta_2 x^2 + \delta_1 y^2 + nxy = \varepsilon, \tag{4-3}$$

where $\varepsilon = \delta_1\delta_2 m_L(\lambda)\cdot\lambda$, and by (4-2) $|\varepsilon| = 1$. Note that $\varepsilon$ is independent of the orientations of $\lambda$, $\gamma_1$ and $\gamma_2$.

**Definition 4.1** Define $S_{n,\varepsilon}^{\delta_2,\delta_1}$ to be the set of pairs $(x, y) \in \mathbb{Z}^2$ such that $(x, y)$ is a solution of (4-3) and satisfies the extra condition

$$n = ay - bx \quad \text{for some } a, b \in \mathbb{Z} \text{ with } \gcd(x, a) = \gcd(y, b) = 1. \tag{4-4}$$

Observe that a pair $(x, y)$ associated to a horizontal decomposition must satisfy (4-4), because if $(\mu, \lambda)$ is a symplectic basis of $H_1(T; \mathbb{Z})$ with $\mu\cdot\lambda = 1$ and we write $\gamma_1 = x\mu + a\lambda$, $\gamma_2 = y\mu + b\lambda$, we must have $\gcd(x, a) = \gcd(y, b) = 1$ and $n = \gamma_2\cdot\gamma_1 = ay - bx$.

**Remark 4.2** It easily follows from (4-3) that $x$ and $y$ are coprime. Moreover, (4-4) is equivalent to $\gcd(x, n) = \gcd(y, n) = 1$. Indeed:

- If (4-4) holds, then $\gcd(x, n) = \gcd(x, ay - bx) = \gcd(x, ay) = 1$, and similarly $\gcd(y, n) = 1$.
- Conversely, suppose $\gcd(x, n) = \gcd(y, n) = 1$. By Bézout's theorem there are $a, b \in \mathbb{Z}$ with $n = ay - bx$. Then $\gcd(x, a) = 1$, otherwise a prime factor of both $x$ and $a$ would also divide $n$, contradicting $\gcd(x, n) = 1$. Similarly, $\gcd(y, b) = 1$.

We analyze the sets $S_{n,\varepsilon}^{\delta_2,\delta_1}$ in Section 4.2 to prove Theorem 1.7 and in Section 5 to prove Theorem 5.2.

## 4.2 Second step

In this section we determine the set $S_{n,\varepsilon}^{\delta_2,\delta_1}$ of Definition 4.1. Note that if $x$ and $y$ solve (4-3), then they are necessarily coprime. Fix a curve $\mu$ with $\mu\cdot\lambda = 1$. Then

$$\gamma_1 = p_1\mu + q_1\lambda \quad \text{and} \quad \gamma_2 = p_2\mu + q_2\lambda$$

for some $p_i$ and $q_i$ with $(p_i, q_i) = 1$ for $i = 1, 2$. Moreover,

$$x = \gamma_1\cdot\lambda = p_1, \quad y = \gamma_2\cdot\lambda = p_2 \quad \text{and} \quad n = \gamma_2\cdot\gamma_1 = p_2q_1 - p_1q_2.$$

**Remark 4.3** If $x = 0$ then (4-3) implies $|y| = |p_2| = 1$, and since $\gamma_1 = \pm\lambda$ we have $|q_1| = 1$. Therefore $|n| = |\gamma_2\cdot\gamma_1| = |p_2q_1| = 1$. Similarly, $y = 0$ implies $|n| = 1$.

Given $(x, y) \in S_{n,\varepsilon}^{\delta_2,\delta_1}$, define

$$\hat{x} := -x - n\delta_2 y \quad \text{and} \quad \hat{y} := -y - n\delta_1 x. \tag{4-5}$$

We call the pairs $(\hat{x}, y)$ and $(x, \hat{y})$ *mutations* of $(x, y)$. Moreover, let $m_{n,\varepsilon}^{\delta_2,\delta_1} \in \mathbb{N}$ be the minimum of the function $S_{n,\varepsilon}^{\delta_2,\delta_1} \to \mathbb{N}$ given by $(x, y) \mapsto |xy|$. We define the *bottom subset* $b(S_{n,\varepsilon}^{\delta_2,\delta_1}) \subset S_{n,\varepsilon}^{\delta_2,\delta_1}$ as the set of pairs $(x, y)$ where the minimum $m_{n,\varepsilon}^{\delta_2,\delta_1}$ is attained:

$$b(S_{n,\varepsilon}^{\delta_2,\delta_1}) := \{(x, y) \in S_{n,\varepsilon}^{\delta_2,\delta_1} \mid |xy| = m_{n,\varepsilon}^{\delta_2,\delta_1}\} \subset S_{n,\varepsilon}^{\delta_2,\delta_1}.$$

**Lemma 4.4** *If $(x, y) \in S_{n,\varepsilon}^{\delta_2,\delta_1}$, both mutations of $(x, y)$ belong to $S_{n,\varepsilon}^{\delta_2,\delta_1}$. Moreover,*

(1) *there is a finite sequence of mutations which sends $(x, y)$ into $b(S_{n,\varepsilon}^{\delta_2,\delta_1})$,*

(2) *if $S_{n,\varepsilon}^{\delta_2,\delta_1} \neq \varnothing$ then $|n| \in \{1, 3\}$ and $m_{n,\varepsilon}^{\delta_2,\delta_1} = \frac{1}{2}(|n| - 1)$.*

**Proof** The proof that $(\hat{x}, y) \in S_{n,\varepsilon}^{\delta_2,\delta_1}$ is symmetric to the proof that $(x, \hat{y}) \in S_{n,\varepsilon}^{\delta_2,\delta_1}$, so we provide the argument only for $(\hat{x}, y)$. One can easily check that if $(x, y) \in S_{n,\varepsilon}^{\delta_2,\delta_1}$ then $(\hat{x}, y)$ is a solution of (4-3). Moreover, since $(x, y) \in S_{n,\varepsilon}^{\delta_2,\delta_1}$, by Remark 4.2 we have $\gcd(x, n) = \gcd(y, n) = 1$, which implies $\gcd(\hat{x}, n) = \gcd(-x - n\delta_2 y, n) = \gcd(x, n) = 1$, so that $(\hat{x}, y) \in S_{n,\varepsilon}^{\delta_2,\delta_1}$ as well. This proves the first part of the statement.

Now observe that if $(x, y) \in S_{n,\varepsilon}^{\delta_2,\delta_1}$ and $|x| = |y|$, then

$$x^2(\delta_2 + \delta_1 \pm n) = \varepsilon,$$

which implies $|x| = |y| = 1$ and $|n| \in \{1, 3\}$. Hence, if $(x, y) \in S_{n,\varepsilon}^{\delta_2,\delta_1}$ and $|xy| > 1$, we must have $|x| \neq |y|$. To prove (1) we are going to use infinite descent to reduce to the case $|xy| \leq 1$. We claim that if $|x| > |y|$ then $|\hat{x}| < |x|$, while if $|x| < |y|$ then $|\hat{y}| < |y|$. Since the arguments in the two cases are similar, we only go through the case $|x| > |y|$. It suffices to observe that, if $|\hat{x}| \geq |x|$, then using (4-3) we get

$$y^2 + 1 \geq |\delta_1\delta_2 y^2 - \delta_2\varepsilon| = |x\hat{x}| \geq x^2 \geq (|y| + 1)^2 = y^2 + 2|y| + 1,$$

which is impossible because $y \neq 0$. Therefore $|\hat{x}| < |x|$ and the claim is established. Inducting on $|xy|$, we can apply a sequence of mutations until we obtain a solution $(x_0, y_0)$ with $|x_0y_0| \leq 1$. If $|x_0y_0| = 0$, clearly $m_{n,\varepsilon}^{\delta_2,\delta_1} = 0$ and $(x_0, y_0) \in b(S_{n,\varepsilon}^{\delta_2,\delta_1})$, and by Remark 4.3 $(x_0, y_0) \in \pm\{(1, 0), (0, 1)\}$ and $|n| = 1$, so that $m_{n,\varepsilon}^{\delta_2,\delta_1} = \frac{1}{2}(|n| - 1)$. If $|x_0y_0| = 1$, then we saw above that $|n| \in \{1, 3\}$. Moreover, if $|n| = 3$ then $m_{n,\varepsilon}^{\delta_2,\delta_1} \neq 0$ (again by Remark 4.3) and $|x_0y_0| = 1$ implies $m_{n,\varepsilon}^{\delta_2,\delta_1} = 1 = \frac{1}{2}(|n| - 1)$ and $(x_0, y_0) \in b(S_{n,\varepsilon}^{\delta_2,\delta_1})$. If $|n| = 1$, then

$$\hat{x}_0 = 0 \iff n\delta_2 y_0 = -x_0 \iff \delta_2 = -nx_0y_0,$$

and similarly, $\hat{y}_0 = 0$ if and only if $\delta_1 = -nx_0y_0$; however, we cannot have $\delta_1 = \delta_2 = nx_0y_0$ because it is inconsistent with (4-3), so either $\hat{x}_0 = 0$ or $\hat{y}_0 = 0$, which implies that $m_{n,\varepsilon}^{\delta_2,\delta_1} = 0 = \frac{1}{2}(|n| - 1)$ and that a further mutation sends $(x, y)$ into $b(S_{n,\varepsilon}^{\delta_2,\delta_1})$. □

Observe that the map $(x, y) \mapsto (x, -y)$ defines a bijection between $S_{n,\varepsilon}^{\delta_2,\delta_1}$ and $S_{-n,\varepsilon}^{\delta_2,\delta_1}$, while $S_{n,\varepsilon}^{\delta_2,\delta_1} = S_{n,-\varepsilon}^{-\delta_2,-\delta_1}$. Therefore it will suffice to determine $S_{n,-1}^{\delta_2,\delta_1}$ for $n \in \{1, 3\}$.

**Lemma 4.5** *The following hold*:

(1) *If* $S^{\delta_2,\delta_1}_{3,-1} \neq \varnothing$ *then* $\delta_1 = \delta_2 = 1$ *and* $b(S^{1,1}_{3,-1}) = \pm\{(1,-1)\}$.

(2) *If* $S^{\delta_2,\delta_1}_{1,-1} \neq \varnothing$ *then* $\delta_1 + \delta_2 \in \{0,-2\}$.

(3) $b(S^{-1,1}_{1,-1}) = \pm\{(1,0)\}$.

(4) $b(S^{1,-1}_{1,-1}) = \pm\{(0,1)\}$.

(5) $b(S^{-1,-1}_{1,-1}) = \pm\{(1,0),(0,1)\}$.

**Proof** By Lemma 4.4, if $S^{\delta_2,\delta_1}_{n,-1} \neq \varnothing$ then $b(S^{\delta_2,\delta_1}_{n,-1}) \neq \varnothing$ and $m^{\delta_2,\delta_1}_{n,-1} = \frac{1}{2}(|n|-1)$. If $n = 3$ then $m^{\delta_2,\delta_1}_{3,-1} = 1$. Therefore there is $(x,y) \in S^{\delta_2,\delta_1}_{3,-1}$ with $|xy| = 1$. Then (4-3) reads

$$\delta_1 + \delta_2 = -(1 + 3xy),$$

which forces $xy = -1$, $\delta_1 = \delta_2 = 1$ and $b(S^{1,1}_{3,-1}) = \pm\{(1,-1)\}$. This proves (1). If $n = 1$ then $m^{\delta_2,\delta_1}_{1,-1} = 0$, and there is $(x,y) \in S^{\delta_2,\delta_1}_{1,-1}$ with $|xy| = 0$. Then (4-3) implies $(x,y) \in \{(\pm 1,0),(0,\pm 1)\}$ and either $\delta_1 = -1$ or $\delta_2 = -1$. Therefore (2) holds. If $\delta_1 = \delta_2 = -1$, (4-3) implies $x^2 + y^2 = 1$ and (5) holds. If $\delta_1\delta_2 = -1$, either $(x,y) = \pm(0,1)$ and $(\delta_1,\delta_2) = (1,-1)$, or $(x,y) = \pm(1,0)$ and $(\delta_1,\delta_2) = (-1,1)$. Hence, (3) and (4) hold. □

## 4.3 Hurwitz moves and handle slides

In the last step of the proof of Theorem 1.7, Section 4.4, we will repeatedly use Proposition 4.6, which shows that if one applies a Hurwitz move to the factorization of a horizontal framed link $L$, the associated horizontal decomposition changes by a sequence of handle slides while staying horizontal. This result is essentially an adaptation and an extension of [22, Lemma 5.1]. In the following, we use notation from Section 1.

**Proposition 4.6** *Let* $L = \bigcup_{i=1}^k L_i \subset \Sigma_g \times [0,1]$ *be a horizontal link with* $L_i \subset \mathfrak{S}_i := \Sigma_g \times \{t_i\}$ *for* $i = 1,\dots,k$. *Let* $F_L = (\tau_k^{\delta_k},\dots,\tau_1^{\delta_1})$ *be the factorization of* $L$. *Define the horizontal link* $L' = \bigcup_{i=1}^k L'_i$ *by setting* $L'_j = L_j$ *as framed knots for* $j \neq i, i+1$ *and*

$$\begin{aligned} L'_i &:= L_{i+1} \times \{t_i\}, & \operatorname{fr}(L'_i) &:= \operatorname{fr}_{\mathfrak{S}_i}(L'_i) - \delta_{i+1}, \\ L'_{i+1} &:= \tau_{i+1}^{\delta_{i+1}}(\pi(L_i)) \times \{t_{i+1}\}, & \operatorname{fr}(L'_{i+1}) &:= \operatorname{fr}_{\mathfrak{S}_{i+1}}(L'_{i+1}) - \delta_i. \end{aligned}$$

*Then the factorization* $F_{L'}$ *is obtained from* $F_L$ *by the Hurwitz move*

$$F_L = (\dots,\tau_{i+1}^{\delta_{i+1}},\tau_i^{\delta_i},\dots) \to F_{L'} = (\dots,\tau_{i+1}^{\delta_{i+1}}\tau_i^{\delta_i}\tau_{i+1}^{-\delta_{i+1}},\tau_{i+1}^{\delta_{i+1}},\dots).$$

*If we instead set* $L'_j = L_j$ *for* $j \neq i, i+1$ *and*

$$\begin{aligned} L'_i &:= \tau_i^{\delta_i}(\pi(L_{i+1})) \times \{t_i\}, & \operatorname{fr}(L'_i) &:= \operatorname{fr}_{\mathfrak{S}_i}(L'_i) - \delta_{i+1}, \\ L'_{i+1} &:= L_i \times \{t_{i+1}\}, & \operatorname{fr}(L'_{i+1}) &:= \operatorname{fr}_{\mathfrak{S}_{i+1}}(L'_{i+1}) - \delta_i, \end{aligned}$$

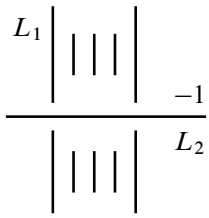


Figure 4: The local configuration of $\alpha$ and $\beta$.

*the factorization $F_{L'}$ is obtained from $F_L$ by the Hurwitz move*

$$F_L = (\dots, \tau_{i+1}^{\delta_{i+1}}, \tau_i^{\delta_i}, \dots) \to F_{L'} = (\dots, \tau_i^{\delta_i}, \tau_i^{-\delta_i} \tau_{i+1}^{\delta_{i+1}} \tau_i^{\delta_i}, \dots).$$

*In both cases, let $X$ and $X'$ be cobordisms having horizontal decompositions with associated horizontal links $L$ and $L'$, respectively. Then the handlebody decomposition of $X_{L'}$ is obtained from the one induced on $X_L$ by a sequence of handle slides. In particular, $X$ and $X'$ are orientation-preserving diffeomorphic.*

**Proof** Given simple closed curves $a, b \subset \Sigma_g$, it is a well-known and easily checked fact that $\tau_{\tau_a(b)} = \tau_a \tau_b \tau_a^{-1}$. From this and the definition of $L'$, it follows immediately that the factorization of $L'$ is obtained from that of $L$ as described. Therefore we only need to check that the horizontal decomposition of $X_{L'}$ is obtained from that of $X_L$ by a sequence of handle slides. It clearly suffices to show this for a link with two components, so we may assume $L = L_1 \cup L_2$, with $L_i \subset \mathfrak{S}_i := \Sigma_g \times \{t_i\}$ for $i = 1, 2$ and $t_1 < t_2$. Suppose first that $\mathrm{fr}(L_2) = \mathrm{fr}_{\mathfrak{S}_2}(L_2) - 1$. After an isotopy, the diagram of the projection of $L$ to $\Sigma_g$ will appear inside a disk $D \subset \Sigma_g$ as in Figure 4, and will have no crossings outside of $D$. Because of our assumption on the framings, indicated by the label $-1$ in the picture, the factorization of $L$ is given by $(\tau_{L_2}, \tau_{L_1}^{\pm 1})$. Assuming there are $c$ crossings in Figure 4, the left-hand side of Figure 5 shows

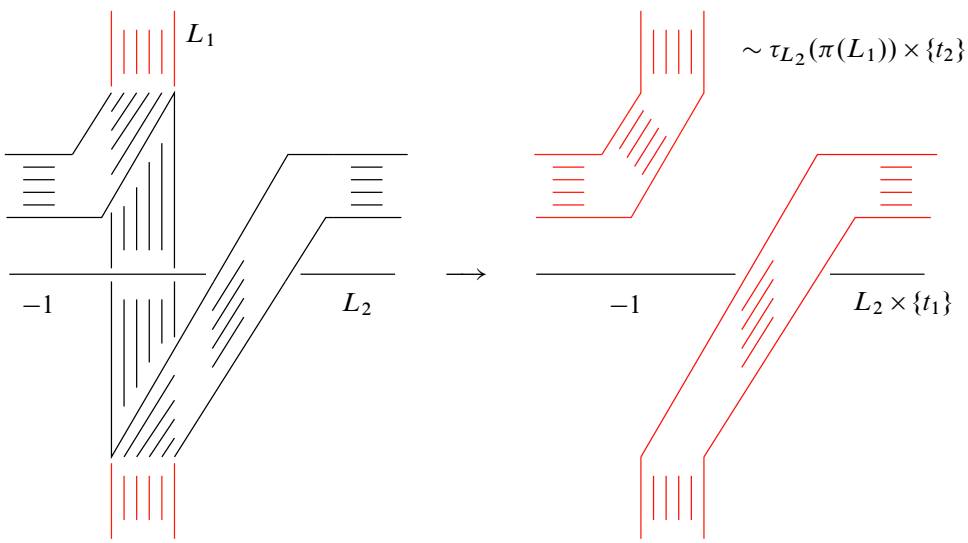


Figure 5: The result of $n$ handle slides.

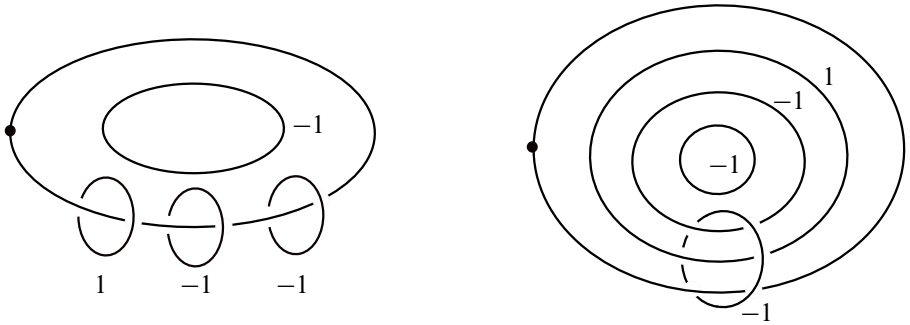


Figure 6: The diffeomorphic 4–manifolds $X$ and $X'$.

$c$ pushed-off copies of $L_2$ drawn in black. The link $L_1$ is shown in red, underneath $L_2$. Sliding the 2–handle attached along $L_1$ over the 2–handle attached along $L_2$ $c$ times, we obtain a curve of the form $L_1 \setminus \{\bigcup_{i=1}^{c} I_1^{(i)}\} \cup L_2 \setminus \{\bigcup_{i=1}^{c} I_2^{(i)}\}$, where $I_2^{(1)}, \dots, I_2^{(c)} \subset L_2$ are the $c$ vertical black segments visible on the left-hand side of Figure 5 and $I_1^{(1)}, \dots, I_1^{(c)} \subset L_1$ the vertical red segments on $L_1$ directly underneath the $I_2^{(i)}$ (not visible in the picture). The right-hand side of Figure 5 shows the resulting framed link, which is clearly framed isotopic to the horizontal framed link $L' = L_2 \times \{t_1\} \cup \tau_{L_2}(\pi(L_1)) \times \{t_2\}$, with $\mathrm{fr}(L_2 \times \{t_1\}) = \mathrm{fr}_{\mathfrak{S}_1}(L_2 \times \{t_1\}) - 1$ and

$$\mathrm{fr}(L_1 \times \{t_2\}) = \mathrm{fr}_{\mathfrak{S}_2}(L_1 \times \{t_2\}) - \mathrm{fr}_{\mathfrak{S}_1}(L_1) + \mathrm{fr}(L_1).$$

The factorization of $L'$ is $(\tau_{L_2}\tau_{L_1}^{\pm 1}\tau_{L_2}^{-1}, \tau_{L_2})$. This concludes the argument for $\mathrm{fr}(L_2) = \mathrm{fr}_{\mathfrak{S}_2}(L_2) - 1$. The case $\mathrm{fr}(L_2) = \mathrm{fr}_{\mathfrak{S}_2}(L_2) + 1$ can be treated similarly or reduced to the previous case by mirroring. □

As an illustration of Proposition 4.6, we show how to use Hurwitz moves to prove the existence of an orientation-preserving diffeomorphism between the 4–manifolds $X$ and $X'$ described by Figure 6. The same diffeomorphism can of course also be easily established by Kirby calculus. Both parts of Figure 6 can be viewed as describing 4–component horizontal framed links inside $S^2 \times S^1$, giving $(1,1,3,0)$–decompositions of $X \setminus (B^4 \sqcup B^4)$ and $X' \setminus (B^4 \sqcup B^4)$ viewed as cobordisms $S^3 \to S^3$. Let $\mu$ and $\lambda$ be the meridian and the Seifert longitude, respectively, of the dotted unknot $U$. If we view $\mu$ and $\lambda$ as simple closed curves on the boundary of a standard neighborhood of $U$, then we see that the link $L$ on the left-hand side is horizontal with respect to the Heegaard decomposition described in Example 1.3, with factorization $F_L = (\tau_\lambda, \tau_\mu, \tau_\mu^{-1}, \tau_\mu)$. On the other hand, the link $L'$ on the right-hand side has factorization $F_{L'} = (\tau_\lambda, \tau_\mu, \tau_\lambda, \tau_\lambda^{-1})$. We shall use the known fact that, given a diffeomorphism $f : \Sigma_g \to \Sigma_g$ and a simple closed curve $a \subset \Sigma_g$, we have $\tau_{f(a)} = f \tau_a f^{-1}$. Denoting Hurwitz equivalence by $\overset{H}{\sim}$, we have

$$F_L \overset{H}{\sim} (\tau_\mu, \tau_{\tau_\mu^{-1}(\lambda)}, \tau_\mu^{-1}, \tau_\mu) \overset{H}{\sim} (\tau_\mu, \tau_\mu^{-1}, \tau_\lambda, \tau_\mu) \overset{H}{\sim} (\tau_\mu, \tau_\lambda, \tau_{\tau_\lambda^{-1}(\mu)}^{-1}, \tau_\mu) \overset{H}{\sim} (\tau_\mu, \tau_\lambda, \tau_\mu, \tau_{\tau_\mu^{-1}\tau_\lambda^{-1}(\mu)}^{-1})$$

while

$$F_{L'} \overset{H}{\sim} (\tau_\mu, \tau_{\tau_\mu^{-1}(\lambda)}, \tau_\lambda, \tau_\lambda^{-1}) \overset{H}{\sim} (\tau_\mu, \tau_\lambda, \tau_{\tau_\lambda^{-1}\tau_\mu^{-1}(\lambda)}, \tau_\lambda^{-1}) = (\tau_\mu, \tau_\lambda, \tau_\mu, \tau_{\tau_\mu^{-1}\tau_\lambda^{-1}(\mu)}^{-1}),$$

because $\tau_\lambda^{-1}\tau_\mu^{-1}(\lambda)$ is isotopic to $\mu$ and $\tau_\mu^{-1}\tau_\lambda^{-1}(\mu)$ to $-\lambda$. This shows $F_L \overset{H}{\sim} F_{L'}$, and therefore $X$ and $X'$ are diffeomorphic by Proposition 4.6.

## 4.4 Last step

In this section we conclude the proof of Theorem 1.7. We assume familiarity with the notation introduced in the previous sections. Let $X\colon S^3 \to S^3$ be the cobordism of Section 4.1, and $L$ the horizontal 2–component link associated to a $(1,1,2,0)$–decomposition of $X$. We assume that $L$ has factorization $F_L = (\tau_2^{\delta_2}, \tau_1^{\delta_1})$ for some $\delta_1, \delta_2 \in \{\pm 1\}$. The oriented curves $\lambda, \gamma_1, \gamma_2 \subset T$ determine $(x, y) \in S_{n,\varepsilon}^{\delta_2,\delta_1}$ for some $\varepsilon \in \{\pm 1\}$.

By Proposition 4.6, we know that, applying Hurwitz moves to $F_L$, we obtain factorizations of links associated to $(1,1,2,0)$–decompositions of $X$. Therefore our plan will be to show that via Hurwitz moves we can change $F_L$, and consequently $(x, y)$, until we get a pair $(x', y')$ with $|x'y'|$ minimal. If we succeed, to determine the diffeomorphism type of $X$ it suffices to do it when the pair $(x, y)$ is minimal. We start with a couple of observations showing that we may assume $n \in \{1, 3\}$ and $\varepsilon = -1$.

First, since reversing the orientation of a component of $L$ changes the sign of $n$ without altering $X_L$, from now on we may (and will) assume without loss of generality $n > 0$. In view of Lemma 4.4, the number $n$ will be either 3 or 1. Next, if we represent $S^1 \times S^2$ as a dotted unknot $U \subset S^3$, we may assume that $T$ is the boundary of a regular neighborhood of $U$. We may also assume that the curves $\mu$ and $\lambda$ of the symplectic basis used to define $x$ and $y$ are a meridian and a Seifert longitude of $U$, respectively. Taking the union with $L$ gives a Kirby diagram for $\widehat{X} := B^4 \cup_{\partial_- X} X$. Note that taking the mirror image of such a diagram amounts to replacing the curves $\gamma_i = p_i\mu + q_i\lambda$ with $p_i\mu - q_i\lambda$ and changing the signs of their relative framings. The result is a diagram of a $(1,1,2,0)$–decomposition of $-X$ with associated pair $(-x, -y)$. If we also reverse the orientation of $\gamma_2$, we obtain a bijection between $S_{n,\varepsilon}^{\delta_2,\delta_1}$ and $S_{n,-\varepsilon}^{-\delta_2,-\delta_1}$ given by $(x, y) \leftrightarrow (x, -y)$. This means that, up to reversing the orientation of $X$, we may assume $\varepsilon = -1$.

Let $U, V\colon S_{n,\varepsilon}^{\delta_2,\delta_1} \to S_{n,\varepsilon}^{\delta_2,\delta_1}$ be given by $U(x, y) = (\hat{x}, y)$ and $V(x, y) = (x, \hat{y})$, where

$$\begin{pmatrix}\hat{x}\\ y\end{pmatrix} = \begin{pmatrix}-1 & -n\delta_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}x\\ \hat{y}\end{pmatrix} = \begin{pmatrix}1 & 0\\ -n\delta_1 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

We warn the reader that, to avoid heavy notation, we are writing $U$ instead of $U_{n,\varepsilon}^{\delta_2,\delta_1}$, and similarly for $V$. Define $S\colon S_{n,\varepsilon}^{\delta_2,\delta_1} \to S_{n,\varepsilon}^{\delta_1,\delta_2}$ by $S(x, y) = (y, x)$. Then $U$, $V$ and $S$ act on $S_{3,-1}^{1,1}$ and $S_{1,-1}^{-1,1} \cup S_{1,-1}^{1,-1}$ and satisfy the relations $US = SV$ and $VS = SU$.

We now check what happens to the quantities $x$, $y$ and $n$ when we apply a Hurwitz move to $F_L$. The two possible Hurwitz moves are

$$(\tau_2^{\delta_2}, \tau_1^{\delta_1}) \to (\tau_1^{\delta_1}, \tau_1^{-\delta_1}\tau_2^{\delta_2}\tau_1^{\delta_1} = \tau_{-\gamma_2 - n\delta_1\gamma_1}^{\delta_2}) \quad \text{and} \quad (\tau_2^{\delta_2}, \tau_1^{\delta_1}) \to (\tau_2^{\delta_2}\tau_1^{\delta_1}\tau_2^{-\delta_2} = \tau_{-\gamma_1 - n\delta_2\gamma_2}^{\delta_1}, \tau_2^{\delta_2}).$$

So we get, respectively,

$$(x, y, n) \to (-y - n\delta_1 x, x, n) = (\hat{y}, x, n) \quad \text{and} \quad (x, y, n) \to (y, -x - n\delta_2 y, n) = (y, \hat{x}, n). \tag{4-6}$$

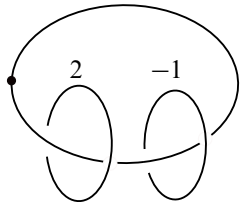


Figure 7: Kirby diagram in the case $n = 3$.

Thus, a Hurwitz move transforms $F_L$ into $F_{L'}$, where the triple $(x', y', n')$ associated to $L'$ satisfies $n' = n$ and, using notation from Section 4.2, either $(x', y') = SU(x, y)$ or $(x', y') = SV(x, y)$. In particular, if $(x, y) \in S^{\delta_2,\delta_1}_{n,-1}$ then $(x', y') \in S^{\delta_1,\delta_2}_{n,-1}$.

**Case $n = 3$** Since $\varepsilon = -1$, in view of Lemma 4.5(1) we may assume that the pair $(x, y)$ associated to $L$ is in $S^{1,1}_{3,-1}$, which is preserved by both $SU$ and $SV$. By Lemma 4.5, a sequence of Hurwitz moves transforms $F_L$ into $F_{L'}$, where $L'$ is a horizontal framed link whose associated pair is $(1, -1)$ and by Proposition 4.6 we have $X_L \cong X_{L'}$. Thus, it suffices to show that if $n = 3$ and the pair is $(1, -1) \in S^{1,1}_{3,-1}$, then $X$ is diffeomorphic to $\mathbb{CP}^2 \setminus (B^4 \cup B^4)$. From $x = \gamma_1 \cdot \lambda = 1$ we deduce that $\gamma_1 = \mu + c\lambda$ for some $c \in \mathbb{Z}$. We can further simplify the handlebody decomposition as follows. Any self-diffeomorphism $T \to T$ of the form $\tau^m_\lambda$ for $m \in \mathbb{Z}$ extends to a diffeomorphism $\bar\tau\colon H_1 \to H_1$ and therefore to $\bar\tau \times \mathrm{id}\colon H_1 \times [0, 1] \to H_1 \times [0, 1]$. We can extend $\bar\tau \times \mathrm{id}$ over the two 2–handles $h_1$ and $h_2$ attached along $\gamma_1$ and $\gamma_2$ to an orientation-preserving diffeomorphism

$$\tilde\tau\colon (H_1 \times [0, 1]) \cup h_1 \cup h_2 \xrightarrow{\cong} (H_1 \times [0, 1]) \cup \tilde\tau(h_1) \cup \tilde\tau(h_2),$$

where the 2–handles $\tilde\tau(h_i)$ are attached along $L' = \tilde\tau(L)$. The factorization of $F_{L'}$ is obtained by conjugating the elements of $F_L$ by $\tau^m_\lambda$ and the triple associated to $L'$ is still $(x, y, n)$. This shows that we may assume $\gamma_1 = \mu$ without loss of generality. Now $y = \gamma_2 \cdot \lambda = -1$ and $n = \gamma_2 \cdot \gamma_1 = 3$ force $\gamma_2 = -\mu - 3\lambda$. The associated Kirby diagram is isotopic to the one given in Figure 7. It can be transformed into the standard diagram of $\mathbb{CP}^2$ by sliding the 2–framed handle over the $-1$–framed handle and then canceling the 1–handle with the $-1$–framed 2–handle.

**Case $n = 1$** Since $\varepsilon = -1$, in view of Lemma 4.5 we may assume that the pair $(x, y)$ associated to $L$ is either in $S^{1,-1}_{1,-1} \cup S^{-1,1}_{1,-1}$ or $S^{-1,-1}_{1,-1}$. In the latter case, a sequence of Hurwitz moves transforms $F_L$ into $F_{L'}$, where $L'$ is a horizontal framed link with associated pair $(x, y) \in \pm\{(1, 0), (0, 1)\}$, and by Proposition 4.6 we have $X_L \cong X_{L'}$. If $(x, y)$ is, up to sign, equal to $(1, 0)$, then since $\gamma_2$ is nonseparating it must be parallel to $\lambda$. Moreover, $x = \gamma_1 \cdot \lambda = 1$ and $n = 1 = \gamma_2 \cdot \gamma_1$ imply $(\gamma_1, \gamma_2) = \pm(\mu, \lambda)$. Since $\delta_1 = \delta_2 = -1$, a Kirby calculus picture is given by a canceling $(1, 2)$–pair with $+1$–framed meridian 2–handle together with an unlinked $+1$–framed unknot. Therefore, in this case, $X \cong \mathbb{CP}^2 \setminus (B^4 \cup B^4)$. If $(x, y) = (0, 1)$ we deduce $(\gamma_1, \gamma_2) = \pm(\lambda, \mu)$ and reach the same conclusion as before. Now suppose $(x, y) \in S^{1,-1}_{1,-1} \cup S^{-1,1}_{1,-1}$. By Lemma 4.5, a sequence of Hurwitz moves transforms $F_L$ into $F_{L'}$, where $L'$ is a horizontal framed link with associated pair equal to either $(\pm 1, 0) \in S^{-1,1}_{1,-1}$ or $(0, \pm 1) \in S^{1,-1}_{1,-1}$. In

the first case, as before, $(\gamma_1,\gamma_2)=\pm(\mu,-\lambda)$; in the second case $(\gamma_1,\gamma_2)=\pm(\lambda,\mu)$. In both cases the associated Kirby diagram is easily shown to represent $\mathbb{CP}^2$. This concludes the proof of Theorem 1.7.

**Remark 4.7** The above proof shows that when the pair $(x,y)$ associated to the $(1,1,2,0)$–decomposition of $X$ and the oriented curves $\lambda$, $\gamma_1$ and $\gamma_2$ belong to $S^{\delta_2,\delta_1}_{n,\varepsilon}$ with $n\in\{1,3\}$ and $\varepsilon=-1$, $X$ is *orientation-preserving* diffeomorphic to $\mathbb{CP}^2\setminus(B^4\cup B^4)$. Since the quantity $\varepsilon=\delta_1\delta_2 m_L(\lambda)\cdot\lambda$ is invariant under orientation changes of $\lambda$, $\gamma_1$ and $\gamma_2$ and Hurwitz moves, while it changes sign if the orientation of $X$ is reversed, we conclude that $\varepsilon=-\sigma(X)$. This fact can be also established directly by applying Wall's nonadditivity formula [23].

## 5 Proof of Theorem 1.9

In this section we determine the smooth embeddings $B_{p,q}\subset\mathbb{CP}^2$ implicit in our proof of Theorem 1.7. Recall that, given a $(g,u,\ell,h)$–decomposition of an oriented cobordism $X\colon\partial_-X\to\partial_+X$, we denote by $X_L$ the cobordism determined by attaching the 2–handles to the horizontal link $L\subset\partial^u_-X$. More generally, if $L'\subset L$ is a sublink of $L$, we denote by $X_{L'}$ the cobordism determined by the corresponding 2–handles. The following lemma is key to our arguments:

**Lemma 5.1** *Let $\gamma_1\cup\gamma_2\subset S^1\times S^2$ be a 2–component link, horizontal with respect to the standard genus-1 Heegaard decomposition and associated to a $(1,1,2,0)$–decomposition of $X=\mathbb{CP}^2\setminus(B^4\cup B^4)$. Then, setting $\widehat{X}_{\gamma_i}:=S^1\times D^3\cup_{\partial_-X_{\gamma_i}}X_{\gamma_i}$ for $i=1,2$, the disjoint union $\widehat{X}_{\gamma_1}\sqcup\widehat{X}_{\gamma_i}$ admits a smooth orientation-preserving embedding in $\mathbb{CP}^2$.*

**Proof** Each 2–handle $h_i$ for $i=1,2$ is attached along $\gamma_i\times\{t_i\}\subset\Sigma_1\times\{t_i\}\subset S^1\times S^2$, with $0<t_1<t_2<1$. We may clearly assume that the attaching loci of the handles are disjoint. Hence, in these cases $\mathbb{CP}^2$ contains $S^1\times D^3\cup X_{\gamma_1\cup\gamma_2}$. Moreover, $S^1\times D^3\approx H_1\times[0,1]$ and $\partial(H_1\times\{t_i\})=\Sigma_1\times\{t_i\}$, so $\mathbb{CP}^2$ contains $H_1\times[t_i-\varepsilon,t_i+\varepsilon]\cup X_{\gamma_i}$ for $i=1,2$ and some small $\varepsilon>0$. Therefore the disjoint union $(S^1\times D^3\cup X_{\gamma_1})\cup(S^1\times D^3\cup X_{\gamma_2})$ admits a smooth orientation-preserving embedding in $\mathbb{CP}^2$. □

In view of Lemma 5.1, to prove Theorem 1.9 we need to identify the 4–manifolds $\widehat{X}_{\gamma_i}$ for $i=1,2$. Note that $X_{\gamma_i}\colon S^3\to\partial_+X_{\gamma_i}$ is a cobordism having $\chi(X_{\gamma_i})=0$ and a $(1,1,1,0)$–decomposition. Therefore, if $b_1(\partial_+X_{\gamma_i})=0$ and the factorization is $(\tau_2^{\delta_2},\tau_1^{\delta_1})$, it follows from Proposition 3.2 that $\widehat{X}_{\gamma_i}$ is diffeomorphic to either $B^4$ or $\delta_i B_{p,q}$ for some $p>q>0$. The following result clearly implies Theorem 1.9:

**Theorem 5.2** *If a horizontal decomposition of $\mathbb{CP}^2\setminus(B^4\cup B^4)$ has type $(1,1,2,0)$, factorization $(\tau_2^{\delta_2},\tau_1^{\delta_1})$ and $b_1(\partial\widehat{X}_{\gamma_1})=b_1(\partial\widehat{X}_{\gamma_2})=0$, then the set $\{\widehat{X}_{\gamma_1},\widehat{X}_{\gamma_2}\}$ is one of*

(1) *$\{B_{m+1},B_m\}$ for some $m\geq 0$, where $B_m=B_{F_{2m-1},F_{2m-5}}$ if $\delta_1=\delta_2=1$,*

(2) *$\{B'_{m+1},B'_m\}$ for some $m\geq 0$, where $B'_m=(-1)^m B_{F_{m+1},F_m}$ if $\delta_1+\delta_2=0$, or*

(3) *$\{B^4\}$ if $\delta_1=\delta_2=-1$.*

*Moreover, in (1) we have $|\gamma_1\cdot\gamma_2|=3$ and in (2) and (3) we have $|\gamma_1\cdot\gamma_2|=1$.*

For the proof of Theorem 5.2 we need the following lemma. Note that the involutions $S$ given by $S(x,y)=(y,x)$ and minus the identity $-I$ generate $G$–actions on $S^{1,1}_{3,-1}$ and $S^{-1,1}_{1,-1}\cup S^{1,-1}_{1,-1}$, where $G=\langle S,-I\rangle\cong\mathbb{Z}/2Z\times\mathbb{Z}/2\mathbb{Z}$. We denote by $H=\{I,-I\}\cong\mathbb{Z}/2\mathbb{Z}$ the subgroup of $G$ generated by $-I$.

**Lemma 5.3** *Let $\{F_m\}_{m\in\mathbb{Z}}\subset\mathbb{Z}$ be the Fibonacci sequence, given by $F_{-1}=1$, $F_0=0$ and $F_{m+1}=F_m+F_{m-1}$. Then*

$$S^{1,1}_{3,-1}=G\cdot\mathscr{S},\quad S^{-1,1}_{1,-1}=H\cdot(\mathscr{T}_1\cup\mathscr{T}_2),\quad S^{1,-1}_{1,-1}=S\cdot S^{-1,1}_{1,-1}\quad\text{and}\quad S^{-1,-1}_{1,-1}=\pm\{(1,0),(0,1),(1,1)\},$$

*where*

$$\begin{aligned}\mathscr{S}&=\{(-F_{2k-1},F_{2k+1})\in\mathbb{Z}^2\mid k\geq 0\},\\ \mathscr{T}_1&=\{(F_{2k+1},F_{2k})\mid k\geq 0\},\\ \mathscr{T}_2&=\{(F_{2k+1},-F_{2k+2})\mid k\geq 0\}.\end{aligned}$$

**Proof** **First case $n=3$, $\delta_1=\delta_2=1$** By Lemmas 4.4 and 4.5, each $(x,y)\in S^{1,1}_{3,-1}$ is sent to either $(1,-1)$ or $(-1,1)$ by a sequence of mutations. Then $(x,y)$ is obtained from either $(1,-1)$ or $(-1,1)$ by applying a map of the form $\cdots UVU$ or $\cdots VUV$. Since $S\colon S^{1,1}_{3,-1}\to S^{1,1}_{3,-1}$ is an involution which intertwines the actions of $U$ and $V$ and $(1,-1)=S(-1,1)$, if $(x,y)\in S^{1,1}_{3,-1}$ then $(x,y)$ belongs to the $G$–orbit of either $(VU)^h(-1,1)$ or $U(VU)^h(-1,1)$ for some $h\geq 0$. Since $VU=(VS)^2=(-VS)^2$ and $U=S^2V=S(VS)$, we can rephrase the condition on $(x,y)$ by saying that $(x,y)$ belongs to the $G$–orbit of $(-VS)^k(-1,1)$ for some $k\geq 0$. The map $-VS$ acts as multiplication by $\left(\begin{smallmatrix}0&-1\\1&3\end{smallmatrix}\right)$; therefore $(x,y)$ belongs to the $G$–orbit of $\{(x_k,y_k)\in\mathbb{Z}^2\mid k\geq 0\}$, where

$$\begin{pmatrix}x_k\\y_k\end{pmatrix}=\begin{pmatrix}0&-1\\1&3\end{pmatrix}^k\begin{pmatrix}-1\\1\end{pmatrix}.$$

The relation $(x_{k+1},y_{k+1})=(-YS)(x_k,y_k)$ is equivalent to $x_{k+1}=-y_k$ and $y_{k+1}=x_k+3y_k$, which imply $x_{k+2}=3x_{k+1}-x_k$. Since $(x_0,y_0)=(-1,1)$ and $(x_1,y_1)=(-1,2)$, we have $x_k=-F_{2k-1}$ and $y_k=F_{2k+1}$ for each $k\geq 0$; therefore $(x,y)$ belongs to $G\cdot\mathscr{S}$. This shows $S^{1,1}_{3,-1}\subset G\cdot\mathscr{S}$. Conversely, since $V$, $S$ and $-I$ preserve $S^{1,1}_{3,-1}$ and $S^{1,1}_{3,-1}$ contains $(-1,1)$, we have $G\cdot\mathscr{S}\subset S^{1,1}_{3,-1}$.

**Second case $n=1$, $\delta_1+\delta_2=0$** By Lemmas 4.4 and 4.5, a sequence of mutations sends each $(x,y)\in S^{-1,1}_{1,-1}$ to $(1,0)$ or $(-1,0)$ and each $(x,y)\in S^{1,-1}_{1,-1}$ to $(0,1)$ or $(0,-1)$. Since $S\colon S^{-1,1}_{1,-1}\to S^{1,-1}_{1,-1}$ defines a bijection intertwining $U$ and $V$ and commuting with $-I$, it will suffice to determine the elements of $S^{-1,1}_{1,-1}$. Since $U(1,0)=(-1,0)$, if $(x,y)\in S^{-1,1}_{1,-1}$ then $(x,y)$ belongs to the $H$–orbit of either $(UV)^h(1,0)$ or $V(UV)^h(1,0)$ for some $h\geq 0$. Moreover,

$$UV=-\begin{pmatrix}1&1\\1&0\end{pmatrix}^2$$

and so

$$(UV)^h=(-1)^h\begin{pmatrix}1&1\\1&0\end{pmatrix}^{2h}.$$

Hence $(UV)^h(1,0) = (-1)^h(x_{2h}, y_{2h})$, where the sequence $\{(x_k, y_k)\}$ is defined by $(x_0, y_0) = (1,0)$ and

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} \quad \text{for } k \geq 0.$$

But this is equivalent to $y_k = x_{k-1}$ and $x_{k+1} = x_k + x_{k-1}$; therefore $x_k = F_{k+1}$, where $F_k$ is the $k^{\text{th}}$ Fibonacci number. We conclude that $(UV)^h(1,0) = (-1)^h(F_{2h+1}, F_{2h})$. Since $V(F_{2h+1}, F_{2h}) = (F_{2h+1}, -F_{2h+1} - F_{2h}) = (F_{2h+1}, -F_{2h+2}) \in S^{-1,1}_{1,-1}$, the proof of this case is finished.

**Third case $n = 1$, $\delta_1 = \delta_2 = -1$** By Lemmas 4.4 and 4.5, a sequence of mutations sends each $(x, y) \in S^{-1,-1}_{1,-1}$ to $(\pm 1, 0)$ or $(0, \pm 1)$. It is easy to check that $\pm(1,1)$ are the only other pairs one can get by acting with $U$ and $V$ on $\pm(1,0)$ and $\pm(0,1)$. □

**Proof of Theorem 5.2** In view of Proposition 3.2, identifying the possible $X_{\gamma_1}$ and $X_{\gamma_2}$ amounts to identifying (up to sign) the possible homology classes of the curves $\gamma_1$ and $\gamma_2$. Using the notation of Section 4, let us fix a symplectic basis $(\mu, \lambda)$ of $H_1(T;\mathbb{Z})$ with $\mu \cdot \lambda = 1$, so that $\gamma_i = p_i\mu + q_i\lambda$ for some $p_i$ and $q_i$ with $(p_i, q_i) = 1$ for $i = 1, 2$. Note that by Proposition 3.2 the assumption $b_1(\partial \widehat{X}_{\gamma_i}) = b_1(\partial_+ X_{\gamma_i}) = 0$ implies $p_1 p_2 \neq 0$.

We have already observed that the map $(x, y) \mapsto (x, -y)$ defines a bijection between $S^{\delta_2,\delta_1}_{n,\varepsilon}$ and $S^{\delta_2,\delta_1}_{-n,\varepsilon}$. Therefore, to list the possible $(|p_1|, |p_2|)$ it suffices to assume $n \in \{1, 3\}$. Moreover, by Remark 4.7 $\varepsilon = -\sigma(X) = -1$. Then, by Lemma 5.3, each pair $(|p_1|, |p_2|)$ is one of

- $(F_{2k+1}, F_{2k-1})$ or $(F_{2k-1}, F_{2k+1})$ for some $k \geq 0$ when $\delta_1 = \delta_2 = 1$ and $n = 3$,
- $(F_{2k+1}, F_{2k})$ or $(F_{2k+1}, F_{2k+2})$ for some $k \geq 0$ when $(\delta_1, \delta_2) = (1, -1)$ and $n = 1$, or
- $(F_{2k}, F_{2k+1})$ or $(F_{2k+2}, F_{2k+1})$ for some $k \geq 0$ when $(\delta_1, \delta_2) = (-1, 1)$ and $n = 1$.

Recall that the Fibonacci numbers satisfy Vajda's identity:

$$F_r F_{m+j} - F_m F_{r+j} = (-1)^{r+1} F_{m-r} F_j.$$

Choosing $j = 4$, $m = 2k - 3$ and $r = 2k - 5$, we obtain the relation

$$(\pm F_{2k-5}) F_{2k+1} - (\pm F_{2k-3}) F_{2k-1} = \pm 3,$$

which implies

- $q_1 \equiv \pm F_{2k-3} \bmod p_1$ and $q_2 \equiv \pm F_{2k-5} \bmod p_2$ when $(|p_1|, |p_2|) = (F_{2k+1}, F_{2k-1})$,
- $q_1 \equiv \pm F_{2k-5} \bmod p_1$ and $q_2 \equiv \pm F_{2k-3} \bmod p_2$ when $(|p_1|, |p_2|) = (F_{2k-1}, F_{2k+1})$.

Since $B_{p,q} \cong B_{p,p-q}$, when $\delta_1 = \delta_2 = 1$ and $|n| = 3$ by Proposition 3.2 the 4–manifolds $\widehat{X}_{\gamma_1}$ and $\widehat{X}_{\gamma_2}$ are

$$B_{F_{2k+1}, F_{2k-3}} = B_{k+1} \quad \text{and} \quad B_{F_{2k-1}, F_{2k-5}} = B_k \quad \text{for } k \geq 0.$$

Choosing $j = 1$, $m = 2k$ and $r = 2k - 1$ in Vajda's identity we obtain the relation

$$(\pm F_{2k-1}) F_{2k+1} - (\pm F_{2k}) F_{2k} = \pm 1,$$

which provides us with

- $q_1 \equiv \pm F_{2k} \mod p_1$ and $q_2 \equiv \pm F_{2k-1} \mod p_2$ when $(|p_1|, |p_2|) = (F_{2k+1}, F_{2k})$,
- $q_1 \equiv \pm F_{2k-1} \mod p_1$ and $q_2 \equiv \pm F_{2k} \mod p_2$ when $(|p_1|, |p_2|) = (F_{2k}, F_{2k+1})$.

In view of Proposition 3.2, the corresponding rational balls are

$$B_{F_{2k+1},F_{2k}} = B'_{2k} \quad \text{and} \quad -B_{F_{2k},F_{2k-1}} = B'_{2k-1} \quad \text{for } k > 0.$$

Choosing $j = 1$, $m = 2k+1$ and $r = 2k$ in Vajda's identity we obtain the relation

$$(\pm F_{2k})F_{2k+2} - (\pm F_{2k+1})F_{2k+1} = \mp 1,$$

which gives

- $q_1 \equiv \pm F_{2k} \mod p_1$ and $q_2 \equiv \pm F_{2k+1} \mod p_2$ when $(|p_1|, |p_2|) = (F_{2k+1}, F_{2k+2})$,
- $q_1 \equiv \pm F_{2k+1} \mod p_1$ and $q_2 \equiv \pm F_{2k} \mod p_2$ when $(|p_1|, |p_2|) = (F_{2k+2}, F_{2k+1})$.

The rational balls are

$$B_{F_{2k+1},F_{2k}} = B'_{2k} \quad \text{and} \quad -B_{F_{2k+2},F_{2k+1}} = B'_{2k+1} \quad \text{for } k \geq 0.$$

Finally, if $(x, y) \in S^{-1,-1}_{1,-1}$, then by Lemma 5.3 $(p_1, p_2) = \pm(1, 1)$ and the statement follows. □

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