AGT

Algebraic & Geometric Topology

Volume 24 (2024)

The homology of a Temperley–Lieb algebra on an odd number of strands

ROBIN J SROKA

msp

msp

Algebraic & Geometric Topology **24**:6 (2024) 3527–3541
DOI: 10.2140/agt.2024.24.3527
Published: 7 October 2024

The homology of a Temperley–Lieb algebra on an odd number of strands

ROBIN J SROKA

We show that the homology of any Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on an odd number of strands vanishes in positive degrees. This improves a result obtained by Boyd and Hepworth. In addition, we present alternative arguments for two vanishing results of Boyd and Hepworth: the stable homology of Temperley–Lieb algebras is trivial, and if the parameter $a \in R$ is a unit, then the homology of any Temperley–Lieb algebra is concentrated in degree zero.

16E40, 20J05; 20F55

1 Introduction

Let $a \in R$ be an element in a commutative unital ring. Intuitively, the Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on n strands is the R–algebra whose underlying R–module has a basis given by isotopy classes of planar diagrams. The multiplication of two planar diagrams is given by gluing them together. Any circles appearing in the resulting diagram correspond to multiplication by $a \in R$. This is illustrated in Figure 1, and a precise definition is given in the next section (see Definition 1).

The Temperley–Lieb algebra $\mathcal{TL}_n(a)$ has a natural augmentation $\epsilon_n : \mathcal{TL}_n(a) \to R$ that maps all nonidentity planar diagrams to zero. The augmentation gives R the structure of a $\mathcal{TL}_n(a)$–module, which we call the trivial module and denote by $\mathbb{1}$. In this work, we examine the homology of Temperley–Lieb algebras with coefficients in this module:

$$H_\star(\mathcal{TL}_n(a), \mathbb{1}) = \operatorname{Tor}_\star^{\mathcal{TL}_n(a)}(\mathbb{1}, \mathbb{1}).$$

Recently, these homology groups have been studied by Boyd and Hepworth [2] and Randal-Williams [18]. Our new main finding is the following theorem, which removes all conditions in the vanishing results for the homology of Temperley–Lieb algebras on an odd number of strands obtained by Boyd and Hepworth:

Theorem A *Let R be a commutative unital ring, $a \in R$ and $n \in \mathbb{N}$ be odd. Consider the Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on n strands. Then*

$$H_0(\mathcal{TL}_n(a), \mathbb{1}) = R \quad \text{and} \quad H_\star(\mathcal{TL}_n(a), \mathbb{1}) = 0 \quad \text{for } \star > 0.$$

© 2024 The Author, under license to MSP (Mathematical Sciences Publishers). Distributed under the Creative Commons Attribution License 4.0 (CC BY). Open Access made possible by subscribing institutions via Subscribe to Open.

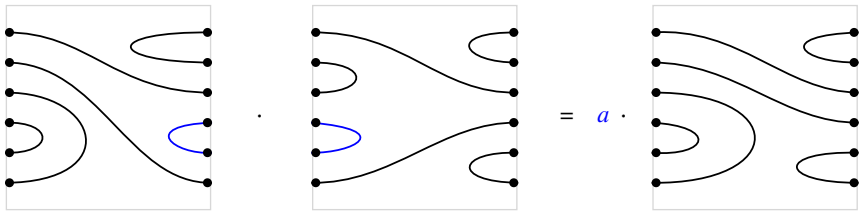

Figure 1: Multiplication of two planar diagrams in $\mathcal{TL}_6(a)$.

Previously known results Assuming that the parameter $a \in R$ used to define $\mathcal{TL}_n(a)$ is of the form $a = v + v^{-1}$ for a unit $v \in R$, Boyd and Hepworth proved that the homology of a Temperley–Lieb algebra on an odd number n of strands is trivial in homological degrees $1 \leq \star \leq n-1$ [2, Theorem B]. Theorem D of [2] furthermore establishes that the homology groups $H_\star(\mathcal{TL}_n(a), \mathbb{1})$ are trivial in all positive degrees $\star > 0$ if $n = 2k+1$ is odd, the parameter $a = v + v^{-1}$ is zero, and R is a field whose characteristic does not divide $\binom{k}{r}$ for any $1 \leq r \leq k$. In [18], Randal-Williams builds on the work of Boyd and Hepworth and shows that the assumptions on the parameter $a \in R$ can be removed using base change techniques [18, Theorem B']. However, an invertibility condition on the binomials $\binom{k}{r}$ remains [18, item (ii) before Corollary 3.2].

Our proof of Theorem A removes all of these conditions and gives an alternative argument for the vanishing results due to Boyd and Hepworth, as well as the strengthening obtained by Randal-Williams.

In addition to Theorem A, our methods allow us to prove a vanishing line for the homology of Temperley–Lieb algebras on an even number of strands. This vanishing line is weaker than the one obtained by Boyd and Hepworth [2, Theorem B], who prove a slope-1 vanishing line; the one we prove is of slope $\frac{1}{2}$.

Theorem B *Let R be a commutative unital ring, $a \in R$ and $n \in \mathbb{N}$ be even. Consider the Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on n strands. Then $H_0(\mathcal{TL}_n(a), \mathbb{1}) = R$,*

$$H_\star(\mathcal{TL}_n(a), \mathbb{1}) = 0 \quad \text{for } 0 < \star < \frac{n}{2}$$

and

$$H_{\star+n/2}(\mathcal{TL}_n(a), \mathbb{1}) \cong H_\star(\mathcal{TL}_n(a), \mathrm{Cup}(M)) \quad \text{for } \star \geq 0,$$

where $\mathrm{Cup}(M)$ is a certain $\mathcal{TL}_n(a)$–module (see Definitions 7 and 11).

The coefficient $\mathcal{TL}_n(a)$–module $\mathrm{Cup}(M)$ in Theorem B is, in general, not isomorphic to the Fineberg module in the description of the higher homology groups obtained by Boyd and Hepworth [2, Theorem 5.1].

Remark In contrast with Temperley–Lieb algebras on an odd number of strands, the homology of a Temperley–Lieb algebras on an even number n of strands can be nontrivial in positive degrees. This was established in [2, Theorem C], which states that $H_{n-1}(\mathcal{TL}_n(a), \mathbb{1})$ is nonzero if n is even and $a = v + v^{-1} \in R$ is not a unit.

Together, Theorems A and B give an alternative proof of the fact that the family of Temperley–Lieb algebras $\{\mathcal{TL}_n(a)\}_{n\in\mathbb{N}}$ satisfies homological stability and that the stable homology is trivial:

$$\operatorname*{colim}_{n\to\infty} H_\star(\mathcal{TL}_n(a),\mathbb{1}) = \begin{cases} R & \text{if } \star = 0,\\ 0 & \text{if } \star > 0.\end{cases}$$

This also follows from the work of Boyd and Hepworth and is discussed in [2]. Homological stability questions for groups play a fundamental role in algebraic topology and algebraic K–theory, eg [4; 6; 7; 8; 9; 10; 11; 14; 21; 22]. This article can be seen as a contribution to the set of ideas formulated by Hepworth [12], Boyd and Hepworth [2] and Boyd, Hepworth and Patzt [3], which aim to extend these techniques to families of abstract algebras.

Building on Theorems A and B, we can formulate an alternative argument for [2, Theorem A], which shows that the homology of any Temperley–Lieb algebra is trivial in positive degrees if the parameter $a \in R$ is a unit.

Theorem C ([2], Boyd and Hepworth) *Let R be a commutative unital ring, $a \in R$ and $n \in \mathbb{N}$. Consider the Temperley–Lieb algebra $\mathcal{TL}_n(a)$ on n strands. If $a \in R$ is a unit, then*

$$H_0(\mathcal{TL}_n(a),\mathbb{1}) = R \qquad \text{and} \qquad H_\star(\mathcal{TL}_n(a),\mathbb{1}) = 0 \quad \text{for } \star > 0.$$

Comparison with work of Boyd and Hepworth [2] and Boyd [1] The general strategy of proof that we employ is similar to the one used by Boyd and Hepworth, ie we construct a certain highly connected $\mathcal{TL}_n(a)$–chain complex and study spectral sequences attached to it. The chain complex that we use is different from the complex of planar injective words studied by Boyd and Hepworth [2]. Indeed, we will use the "cellular Davis complex" $\mathcal{TLD}$ of the Temperley–Lieb algebra $\mathcal{TL}_n(a)$. The structure of the complex $\mathcal{TLD}$ is "sensitive" to the question of whether the number of strings of a Temperley–Lieb algebra is even or odd. This is what enables us to prove Theorem A. The chain complex $\mathcal{TLD}$ has a simple diagrammatic description and can be seen as an algebraic analogue of the classical Davis complex of the symmetric group equipped with the "Coxeter cell" CW–structure; see Davis [5, Chapters 7 and 8]. The classical Davis complex of a finite symmetric group S_n is a contractible CW–complex with S_n–action. We prove the following Temperley–Lieb analogue:

Theorem D *The "cellular Davis complex" $\mathcal{TLD}$ of a Temperley–Lieb algebra* (see Definition 8) *is a contractible $\mathcal{TL}_n(a)$–chain complex with $H_0(\mathcal{TLD}) = \mathbb{1}$.*

From this perspective, the content of this paper is inspired by and in direct analogy with the approach that Boyd employed in [1] to derive formulas for the low-dimensional homology of Coxeter groups. The first two chapters of the author's thesis [19] make this analogy precise and explain how the chain complex $\mathcal{TLD}$ was discovered via this analogy.

Outline In Section 2, we define the Temperley–Lieb algebra $\mathcal{TL}_n(a)$ and its trivial module $\mathbb{1}$. At the beginning of Section 3, we introduce a class of left submodules $\mathrm{Cup}(F)$ of $\mathcal{TL}_n(a)$ which will play an important role. In Section 3.1, we use these modules to define the "cellular Davis complex" $\mathcal{TLD}$ of a Temperley–Lieb algebra and establish that $\mathcal{TLD}$ is contractible (Theorem D). In Section 3.2, we study the homology of the Temperley–Lieb algebra $\mathcal{TL}_n(a)$ with coefficients in $\mathrm{Cup}(F)$. These homology groups will appear on the E_1–page of the spectral sequences that we use in Section 3.3 to formulate the proofs of Theorems A, B and C.

Acknowledgements This article is based on the second chapter of my PhD thesis [19] written at the University of Copenhagen under the direction of Nathalie Wahl. All results presented here are contained in [19]. I am grateful to Nathalie Wahl, Richard Hepworth and Rachael Boyd for their constructive feedback and many enlightening discussions while completing this work. I would like to thank Richard Hepworth for fruitful conversations about related ideas [1; 12; 19, Chapter 3] that eventually inspired this project. For clarifying conversations and helpful comments, I would like to thank Jonas Stelzig. I am grateful to the referee for their suggestions and careful reading of this work. This research was primarily supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement 772960), as well as by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92) and the Copenhagen Centre for Geometry and Topology (DNRF151). It was partially supported by NSERC Discovery Grant A4000 in connection with a Postdoctoral Fellowship at McMaster University, and by the Swedish Research Council under grant 2016-06596 while the author was in residence at Institut Mittag-Leffler in Djursholm Sweden during the semester *Higher algebraic structures in algebra, topology and geometry*. I gratefully acknowledge this support.

2 Temperley–Lieb algebras

This section contains necessary background knowledge on Temperley–Lieb algebras that we mainly learned from Kassel and Turaev [15] and the exposition of Boyd and Hepworth [2]. In 1971, Temperley and Lieb introduced these algebras in their article [20].

Definition 1 [2, Definition 2.1; 15, Definition 5.24] Let R be a commutative unital ring, $a \in R$ and $n \in \mathbb{N}$. The *Temperley–Lieb algebra* $\mathcal{TL}_n(a)$ *with parameter* $a \in R$ is the R–algebra with generators $U_1, \dots, U_{n-1}$ and the relations

(i) $U_i U_j = U_j U_i$ if $|i-j| \geq 2$,

(ii) $U_i U_j U_i = U_i$ if $j = i \pm 1$,

(iii) $U_i^2 = a U_i$ for all i.

The unit 1 corresponds to the empty product of the generators U_i. Note that

$$\mathcal{TL}_0(a) = \mathcal{TL}_1(a) = R.$$

Jones [13] used Temperley–Lieb algebras to define the polynomial invariant for knots, which we now call the Jones polynomial. The following diagrammatic interpretation is due to Kauffman [16; 17]. We follow [15, Section 5.7.4] in our exposition.

Definition 2 A *planar diagram of* $n \geq 1$ *arcs* $D = \{\gamma_1, \dots, \gamma_n\}$ in $[0,1] \times \mathbb{R}$ is a disjoint union of n smoothly embedded arcs $\gamma_i : [0,1] \to [0,1] \times \mathbb{R}$ such that

(i) the images of any two arcs γ_i and γ_j are pairwise disjoint,

(ii) the points $\gamma_i(0)$ and $\gamma_i(1)$ are a subset of the points

$$\{(0,1), \dots, (0,n), (1,1), \dots, (1,n)\},$$

(iii) the tangent vectors at $\gamma_i(0)$ and $\gamma_i(1)$ are parallel to the x–axis $\mathbb{R} \times 0$.

Let $a \in R$ and $P_n(a)$ be the free R–module spanned by the set of isotopy classes $[D]$ of planar diagrams D. We will now explain how the module $P_n(a)$ can be equipped with the structure of an associative R–algebra. Given two isotopy classes of planar diagrams $[D]$ and $[D']$, we obtain a diagram in $[0,1] \times \mathbb{R}$ by pasting D into $\left[0, \frac{1}{2}\right] \times \mathbb{R}$ and D' into $\left[\frac{1}{2}, 1\right] \times \mathbb{R}$. We can choose representatives $D \in [D]$ and $D' \in [D']$ such that the resulting diagram consists of a planar diagram $D \circ D'$ and $k(D, D') \geq 0$ circles. The product of $[D]$ and $[D']$ is defined as

$$[D] \cdot [D'] = a^{k(D,D')}[D \circ D'].$$

The reader is invited to revisit Figure 1, which illustrates this definition.

Theorem 3 [2, Section 2; 15, Theorem 5.34] *The Temperley–Lieb algebra $\mathcal{TL}_n(a)$ is isomorphic to the R–algebra of planar diagrams $P_n(a)$. The isomorphism is given by mapping a generator $U_i \in \mathcal{TL}_n(a)$ to the isotopy class of the planar diagram in Figure 2.*

Definition 4 An element $A \in \mathcal{TL}_n(a)$ is called a *planar diagram* in $\mathcal{TL}_n(a)$ if the isomorphism in Theorem 3 identifies A with the isotopy class $[D] \in P_n(a)$ of some planar diagram D of n arcs.

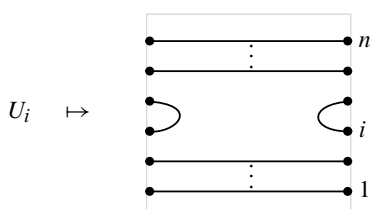

Figure 2: Translating between algebraic and diagrammatic definition of $\mathcal{TL}_n(a)$.

Let $\widehat{\mathcal{TL}}_n$ be the two-sided ideal of $\mathcal{TL}_n = \mathcal{TL}_n(a)$ generated by the set

$$\{U_i : 1 \leq i \leq n-1\}.$$

In the diagrammatic interpretation (see Theorem 3), $\widehat{\mathcal{TL}}_n$ is spanned by all planar diagrams except the identity diagram.

Definition 5 The *trivial module* $\mathbb{1}$ of the Temperley–Lieb algebra $\mathcal{TL}_n = \mathcal{TL}_n(a)$ is defined via the exact sequence

$$0 \to \widehat{\mathcal{TL}}_n \hookrightarrow \mathcal{TL}_n \twoheadrightarrow \mathbb{1} \to 0.$$

Equivalently, the trivial module $\mathbb{1}$ is the 1–dimensional $\mathcal{TL}_n$–module coming from the augmentation $\epsilon : \mathcal{TL}_n \to R$ that sends every generator U_i to zero.

3 Vanishing theorems for the homology of Temperley–Lieb algebras

This section builds on ideas contained in [1; 2]. We introduce the cellular Davis complex for Temperley–Lieb algebras $\mathcal{TL}\mathfrak{D}$ and prove that it is contractible. We then use it to derive the vanishing results for the homology of Temperley–Lieb algebras, which we stated as Theorems A, B and C in Section 1.

Throughout this section, let $\mathcal{TL} = \mathcal{TL}_n(a)$ denote the Temperley–Lieb algebra on n strands with parameter $a \in R$ and let $U_1, \ldots, U_{n-1}$ be the standard generators. We sometimes identify $\{U_1, \ldots, U_{n-1}\}$ with the set $\langle n-1 \rangle = \{1, \ldots, n-1\}$.

Definition 6 (i) A (possibly empty) subset $F \subseteq \langle n-1 \rangle$ is called *innermost* if for any

$$i \neq j \in F \quad \text{we have that} \quad |i-j| \geq 2.$$

(ii) Let $A \in \mathcal{TL}$ be a planar diagram in the sense of Definition 4, with corresponding isotopy class $[A] \in P_n(a)$. $A \in \mathcal{TL}$ is represented by certain monomials in the generating set $\{U_1, \ldots, U_{n-1}\}$. We write

$$F(A) = \{U_{i_k} : A = U_{i_1} \cdot \cdots \cdot U_{i_k}\}$$

for the set of possible last letters in a monomial representing A. Using the identification in Theorem 3, the set $F(A)$ has the following diagrammatic interpretation:

$$F(A) = \{U_i : [A] \text{ has an arc connecting } (1,i) \text{ and } (1,i+1)\}.$$

We call an arc connecting $(1,i)$ and $(1,i+1)$ an *innermost right cup* at position i and $F(A)$ *the set of innermost right cups* of the planar diagram $A \in \mathcal{TL}$; see Figure 3 for an example. Note that $F(A)$ is an innermost set.

Observe that by Definition 1(i) all generators in an innermost set commute with each other. The following modules are therefore well defined:

Definition 7 Let $F \subseteq \langle n-1\rangle$ be an innermost set. Then we write $\mathrm{Cup}(F)$ for the left submodule of $\mathcal{TL}$ generated by $\prod_{i\in F} U_i$. Using Theorem 3, this is the $\mathcal{TL}$–submodule of $\mathcal{TL}_n(a) \cong P_n(a)$ spanned by all isotopy classes of planar diagrams $[A] \in P_n(a)$ that have an innermost right cup at position i for any $i \in F$, ie that satisfy $F \subseteq F(A)$.

3.1 The cellular Davis complex of a Temperley–Lieb algebra

Using the definitions above, we will now introduce and study the $\mathcal{TL}$–chain complex that enables us to formulate our proofs for Theorems A, B and C.

Definition 8 The *cellular Davis complex* $(\mathcal{TLD}, \delta)$ of $\mathcal{TL}$ is the chain complex whose chain module in degree α is given by

$$\mathcal{TLD}_\alpha = \bigoplus_{\substack{F\subseteq\langle n-1\rangle \text{ innermost}\\ |F|=\alpha}} \mathrm{Cup}(F)$$

and whose differential

$$\delta_\alpha \colon \mathcal{TLD}_\alpha \to \mathcal{TLD}_{\alpha-1}$$

factorizes summandwise as

$$\mathrm{Cup}(F) \to \bigoplus_{s\in F} \mathrm{Cup}(F_s) \hookrightarrow \mathcal{TLD}_{\alpha-1},$$

where $F_s = F - \{s\}$ for $s \in F$. The first arrow in this factorization is the map

$$A \mapsto \sum_{s\in F} (-1)^{\gamma_F(s)} \iota_F^{F_s}(A),$$

where $\iota_F^{F_s} \colon \mathrm{Cup}(F) \hookrightarrow \mathrm{Cup}(F_s)$ is the inclusion and $\gamma_F(s) = |\{s' \in F : s' < s\}|$.

In other words, the chain module $\mathcal{TLD}_\alpha$ of the cellular Davis complex $\mathcal{TLD}$ has an R–module basis consisting of planar diagrams A with $\alpha = |F|$ marked innermost right cups $F \subseteq F(A)$ (see Definition 7) and the value of the differential $\delta \colon \mathcal{TLD}_\alpha \to \mathcal{TLD}_{\alpha-1}$ on such a marked planar diagram (A, F) is an

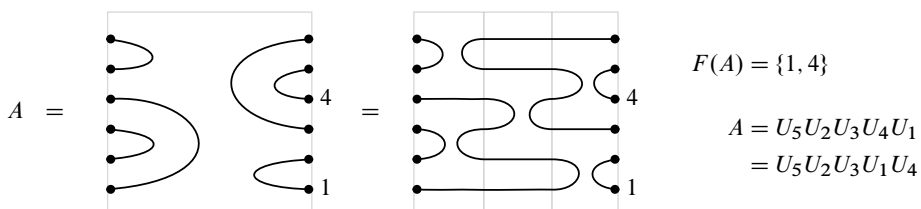

Figure 3: The set of innermost right cups of an element in $\mathcal{TL}_6$.

alternating sum of the marked planar diagrams (A, F_s), ie copies of the same planar diagram A with one mark $s \in F$ erased. See Figure 4 for an example.

We start by verifying that this really defines a chain complex. This argument is standard.

Lemma 9 *The cellular Davis complex $(\mathcal{TLD}, \delta)$ is a chain complex of left $\mathcal{TL}$–modules.*

Proof Let $A \in \mathrm{Cup}(F) \subset \mathcal{TLD}_\alpha$ be a planar diagram. We need to argue that

$$\delta^2(A) = 0 \in \mathcal{TLD}_{\alpha-2}.$$

Observe that

$$\begin{aligned}\delta^2(A) &= \delta\Big(\sum_{s\in F}(-1)^{\gamma_F(s)}\iota_F^{F_s}(A)\Big) = \sum_{s\in F}(-1)^{\gamma_F(s)}\Big(\sum_{t\in F_s}(-1)^{\gamma_{F_s}(t)}\iota_F^{F_{s,t}}(A)\Big)\\ &= \sum_{\substack{(s,t)\in F\times F\\ s\neq t}}(-1)^{\gamma_F(s)+\gamma_{F_s}(t)}\iota_F^{F_{s,t}}(A),\end{aligned}$$

where $\iota_F^{F_{s,t}} : \mathrm{Cup}(F) \hookrightarrow \mathrm{Cup}(F_{s,t})$ denotes the inclusion. Clearly, $F_{s,t} = F_{t,s}$. It therefore suffices to show that if $s < t$, then

$$(-1)^{\gamma_F(s)+\gamma_{F_s}(t)} + (-1)^{\gamma_F(t)+\gamma_{F_t}(s)} = 0.$$

This holds because $s < t$ implies that

$$\gamma_F(t) = \gamma_{F_s}(t) + 1 \quad \text{and} \quad \gamma_F(s) = \gamma_{F_t}(s).$$

The $\mathcal{TL}$–equivariance of the differential δ follows from the $\mathcal{TL}$–equivariance of the inclusion maps. □

Theorem 10 *The cellular Davis complex $(\mathcal{TLD}, \delta)$ is contractible with*

$$H_0(\mathcal{TLD}, \delta) = \mathbb{1}.$$

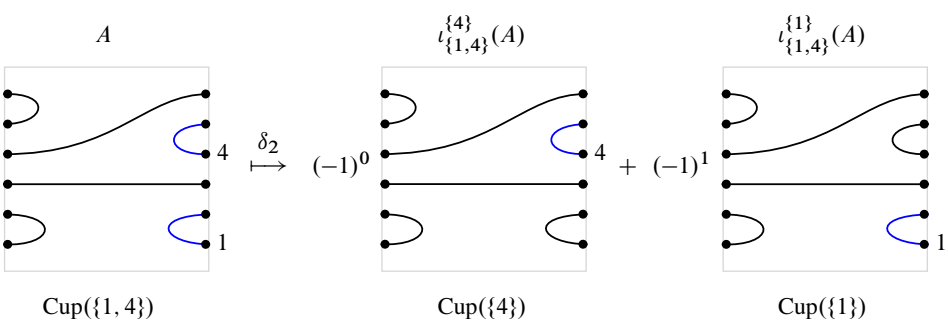

Figure 4: The differential of $(\mathcal{TL}_6\mathfrak{D}, \delta)$ evaluated on an element in $\mathrm{Cup}(\{1, 4\})$.

Proof Let $A \in \mathcal{TL}$ be a planar diagram and $F \subseteq \langle n-1 \rangle$ be an innermost set. Recall from Definition 7 that

$$A \in \mathrm{Cup}(F) \iff F \subseteq F(A).$$

Let $R\{(A,F)\} \subseteq \mathrm{Cup}(F)$ denote the R–linear summand spanned by the isotopy class of A in the diagrammatic picture using the convention that $R\{(A,F)\} = 0$ if $F \not\subseteq F(A)$. We think of (A,F) as a planar diagram with $|F|$–marked innermost right cups. Then the R–module $\mathrm{Cup}(F)$ admits the decomposition

$$\mathrm{Cup}(F) = \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} R\{(A,F)\}.$$

Hence, the R–module $\mathcal{TLD}_\alpha$ can be written as

$$\begin{aligned}\mathcal{TLD}_\alpha &= \bigoplus_{\substack{F \subseteq \langle n-1 \rangle \text{ innermost} \\ |F|=\alpha}} \mathrm{Cup}(F) = \bigoplus_{\substack{F \subseteq \langle n-1 \rangle \text{ innermost} \\ |F|=\alpha}} \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} R\{(A,F)\} \\ &= \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} \bigoplus_{\substack{F \subseteq \langle n-1 \rangle \text{ innermost} \\ |F|=\alpha}} R\{(A,F)\} = \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} \mathcal{TLD}(A)_\alpha,\end{aligned}$$

where we define

$$\mathcal{TLD}(A)_\alpha = \bigoplus_{\substack{F \subseteq F(A) \\ |F|=\alpha}} R\{(A,F)\}.$$

Observe that for any planar diagram $A \in \mathcal{TL}$, the sequence of submodules $\{\mathcal{TLD}(A)_\alpha\}_{\alpha \in \mathbb{N}}$ forms an R–module subcomplex $\mathcal{TLD}(A)$ of the cellular Davis complex $\mathcal{TLD}$ because the partial differentials in $\mathcal{TLD}$ are given by inclusion maps $\mathrm{Cup}(F) \hookrightarrow \mathrm{Cup}(F_s)$ with $F_s \subset F$ (see Definition 8). It follows that, as a chain complex in R–modules, the cellular Davis complex is a coproduct:[1]

$$\mathcal{TLD} = \bigoplus_{\substack{A \in \mathcal{TL} \\ \text{planar diagram}}} \mathcal{TLD}(A).$$

We will calculate the homology of each subcomplex $\mathcal{TLD}(A)$. If $A = \mathrm{id}$, then $F(A) = \varnothing$ and $\mathcal{TLD}(A) = R\{(\mathrm{id}, \varnothing)\}[0]$ is concentrated in degree 0. If $A \neq \mathrm{id}$, then $\mathcal{TLD}(A)$ is exactly the augmented chain complex $\widetilde{C}(\Delta)[+1]$ of the simplex Δ on the vertex set $F(A)$, where the vertices are ordered using $F(A) \subset \langle n-1 \rangle$ and the chain complex is shifted up by one degree. Therefore the homology of the complex $\mathcal{TLD}(A)$ is zero in every degree if $A \neq \mathrm{id}$. The identification $H_0(\mathcal{TLD}, \delta) = \mathbb{1}$ as a $\mathcal{TL}$–module holds because $\mathrm{im}(\delta_1) = \widehat{\mathcal{TL}}$, where the right side is as in Definition 5, and $\mathcal{TLD}_0 = \mathrm{Cup}(\varnothing) = \mathcal{TL}$. □

[1] A similar splitting was used by Boyd, Hepworth and Patzt in their work on the homology of Brauer algebras [3, Definition 5.5].

3.2 Homology with coefficients in Cup(F)

Let $F \subseteq \langle n-1 \rangle$ be an innermost set. The homology groups $H_\star(\mathcal{TL}, \mathrm{Cup}(F))$ of the Temperley–Lieb algebra $\mathcal{TL}$ with coefficients in $\mathrm{Cup}(F)$ will occur on the E_1–page of the spectral sequences, which we will use to derive Theorem A, B and C. For this reason, we will now study the modules $\mathrm{Cup}(F)$ and the homology groups $H_\star(\mathcal{TL}, \mathrm{Cup}(F))$. We begin by collecting several important observations about innermost sets.

The following notion explains why the cellular Davis complex is "sensitive" to the question of whether the underlying Temperley–Lieb algebra is defined on an even or an odd number of strands:

Definition 11 If n is even, we call $M = \{1, 3, \dots, n-1\} \subset \langle n-1 \rangle$ the *unique maximal innermost set*.

Note that if n is even, then $M = \{1, 3, \dots, n-1\} \subset \langle n-1 \rangle$ really is the unique innermost set of maximal cardinality. If the number of strands $n \geq 3$ is odd, then in contrast to the even case there exist multiple different innermost sets of maximal cardinality (see Figure 5).

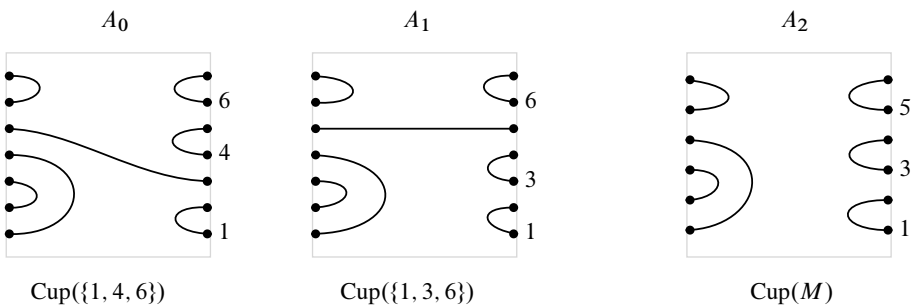

Figure 5: Innermost sets of maximal cardinality for $\langle 6 \rangle$ (left) and $\langle 5 \rangle$ (right).

In particular the next observation always applies, if the Temperley–Lieb algebra is defined on an odd number $n \geq 3$ of strands:

Observation 12 *If $F \subset \langle n-1 \rangle$ is nonempty and innermost, but not the unique maximal innermost set M, then there exists an index $1 \leq i \leq n$ such that either $i-1 \notin F$ and $s = i+1 \in F$, or $i \notin F$ and $s = i-2 \in F$.*

Proof Since F is nonempty, there exists a smallest $z \in F$. If $z \neq 1$, we set $1 \leq i := z-1 \leq n$. Then $i-1 = z-2 \notin F$ by the minimality of z and $s = i+1 = z \in F$ by definition. Hence, the first case applies. Assume that $z = 1 \in F$. Because $F \neq M$, the set $\{1, \dots, n\} \setminus \{j, j+1 : j \in F\}$ is nonempty and contains a smallest index i. The assumption $1 \in F$ and the minimality of i imply that $i \geq 3$ is an odd number such that $s = i-2 \in F$. Therefore the second case applies. □

In the setting of Observation 12 we can think of the index $1 \le i \le n$ as the vertex $(1, i)$ on the right side of *any* planar diagram A in $\mathrm{Cup}(F)$. Then, this vertex has the property that it is next to the *marked* innermost right cup at position $s \in F$ but not itself the start or endpoint of a *marked* innermost right cup.

The following proposition is closely related to [2, Section 3] and replaces the "inductive resolutions", which Boyd and Hepworth introduced, in our setting:

Proposition 13 *Let $F \subset \langle n-1 \rangle$ be nonempty and innermost but not unique maximal innermost, and choose $i \in \{1, \dots, n\}$ and $s \in \{i-2, i+1\}$ as in Observation 12. Let $F_s = F - \{s\}$. Then*

$$\mathrm{Cup}(F) \hookrightarrow \mathrm{Cup}(F_s)$$

is a retract of left $\mathcal{TL}$–modules.

Proof Assume that $i \notin F$ and $s = i-2 \in F$. Consider the map of left modules (see Figure 6)

$$\mathrm{Cup}(F_s) \to \mathrm{Cup}(F) \quad \text{given by } \; c \mapsto c \cdot (U_{i-1}U_{i-2}).$$

We need to check that this map is well defined. If $c \in \mathrm{Cup}(F_s)$, then $c = A \cdot \prod_{j \in F_s} U_j$ for some $A \in \mathcal{TL}$. Therefore

$$c \cdot (U_{i-1}U_{i-2}) = \Big(A \cdot \prod_{j \in F_s} U_j\Big)(U_{i-1}U_{i-2}) = (A \cdot U_{i-1})\Big(\prod_{j \in F} U_j\Big),$$

where we used that U_{i-1} commutes with all $\{U_z\}_{z \in F_s}$ and that $s = i-2 \in F$. The commutativity of U_{i-1} with $\{U_z\}_{z \in F_s}$ follows from Observation 12 because

$$|(i-1) - z| \ge \min\{|(i-1) - (i-4)|, |(i-1) - (i+1)|\} \ge 2 \quad \text{for any } \; z \in F_s.$$

We therefore proved that $c \cdot (U_{i-1}U_{i-2}) \in \mathrm{Cup}(F)$, and hence that the map is well defined. To see that it defines a retraction, we observe that if $c \in \mathrm{Cup}(F)$, then $c = c' \cdot U_{i-2}$. Therefore

$$c \cdot (U_{i-1}U_{i-2}) = c' \cdot (U_{i-2}U_{i-1}U_{i-2}) = c' \cdot U_{i-2} = c.$$

The argument for the case where $i-1 \notin F$ and $s = i+1 \in F$ is similar. We consider the map of left modules

$$\mathrm{Cup}(F_s) \to \mathrm{Cup}(F) \quad \text{given by } \; c \mapsto c \cdot (U_i U_{i+1}).$$

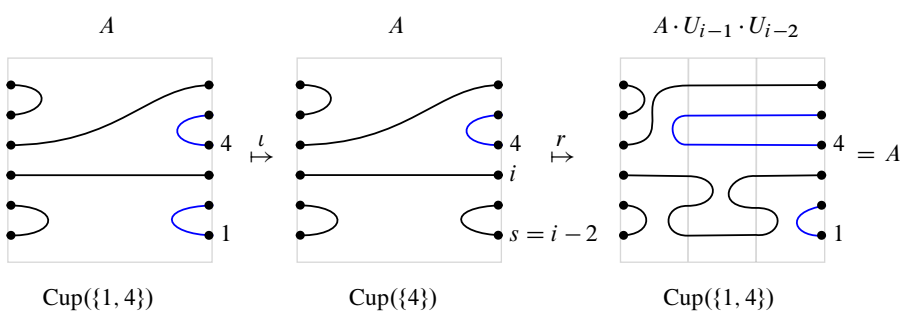

Figure 6: Illustration of the retraction constructed in Proposition 13.

By Observation 12, $|i-z| \geq \min\{|i-(i-2)|, |i-(i+3)|\} \geq 2$ for $z \in F_s$. This again implies that the map is well defined. To see that it defines a retraction, we use that if $c \in \mathrm{Cup}(F)$ then $c = c' \cdot U_{i+1}$. Therefore

$$c \cdot (U_i U_{i+1}) = c' \cdot (U_{i+1} U_i U_{i+1}) = c' \cdot U_{i+1} = c. \qquad \square$$

Remark 14 The elements that we use to define the retractions in Proposition 13 are exactly the same elements that Boyd and Hepworth use in [2, Section 3] to define "inductive resolutions".

The role of the following corollary in our arguments is similar to the role of Theorem F in [2].

Corollary 15 *Let $F \subseteq \langle n-1 \rangle$ be innermost, but not unique maximal innermost. Then*

$$\mathrm{Cup}(F) \hookrightarrow \mathcal{TL}$$

is a retract of left $\mathcal{TL}$–modules and $H_\star(\mathcal{TL}, \mathrm{Cup}(F)) = 0$ for $\star > 0$.

Proof If $F = \varnothing$, the retraction statement is trivial because $\mathrm{Cup}(\varnothing) = \mathcal{TL}$. If $F \neq \varnothing$, it follows from Proposition 13 by induction. The fact that $\mathrm{Cup}(F)$ is a retract of the free $\mathcal{TL}$–module $\mathcal{TL}$ implies that $\mathrm{Cup}(F)$ is a projective $\mathcal{TL}$–module. Therefore $H_\star(\mathcal{TL}, \mathrm{Cup}(F)) = 0$ for $\star > 0$. $\square$

In the next lemma, we compute the degree-zero homology of the Temperley–Lieb algebras with coefficients in $\mathrm{Cup}(F)$:

Lemma 16 *Assume that $F \subseteq \langle n-1 \rangle$ is an innermost set. Then*

$$H_0(\mathcal{TL}, \mathrm{Cup}(F)) \cong \mathbb{1} \otimes_{\mathcal{TL}} \mathrm{Cup}(F) \cong \begin{cases} R & \text{if } F = \varnothing, \\ R/a & \text{if } n = 2 \text{ and } F = \{1\}, \\ 0 & \text{if } n > 2 \text{ and } F \neq \varnothing. \end{cases}$$

Proof The short exact sequence defining the trivial module $\mathbb{1}$ (Definition 5) and the right exactness of $- \otimes_{\mathcal{TL}} \mathrm{Cup}(F)$ yield an exact sequence

$$\widehat{\mathcal{TL}} \otimes_{\mathcal{TL}} \mathrm{Cup}(F) \to \mathcal{TL} \otimes_{\mathcal{TL}} \mathrm{Cup}(F) \to \mathbb{1} \otimes_{\mathcal{TL}} \mathrm{Cup}(F) \to 0.$$

It follows that

$$\mathbb{1} \otimes_{\mathcal{TL}} \mathrm{Cup}(F) \cong \frac{\mathrm{Cup}(F)}{\widehat{\mathcal{TL}} \cdot \mathrm{Cup}(F)}.$$

We use this to compute $\mathbb{1} \otimes_{\mathcal{TL}} \mathrm{Cup}(F)$ in each case. If $F = \varnothing$, then $\mathrm{Cup}(F) = \mathcal{TL}$ and we find that

$$\mathbb{1} \otimes_{\mathcal{TL}} \mathrm{Cup}(F) = \mathbb{1} \otimes_{\mathcal{TL}} \mathcal{TL} \cong R.$$

If $n = 2$ and $F = \{1\}$, then $\mathrm{Cup}(F) = \widehat{\mathcal{TL}} = R\{U_1\}$ and $\widehat{\mathcal{TL}} \cdot \widehat{\mathcal{TL}} = aR\{U_1\}$. Hence,

$$\mathbb{1} \otimes_{\mathcal{TL}} \mathrm{Cup}(F) \cong R/a.$$

For the last case, we assume that $n > 2$ and $F \neq \varnothing$. We will prove that

$$\mathrm{Cup}(F) \subseteq \widehat{\mathcal{TL}} \cdot \mathrm{Cup}(F).$$

Consider an element $A \in \mathrm{Cup}(F)$. The definition of $\mathrm{Cup}(F)$ implies that $A = A' \cdot \left(\prod_{j \in F} U_j\right)$ for some $A' \in \mathcal{TL}$. We fix a generator U_i, whose index i has the property that $i \in F \neq \varnothing$. Because $\mathcal{TL}$ has $n-1 > 1$ generators, $i-1$ or $i+1$ also indexes a generator U_{i-1} or U_{i+1}. By Definition 1(ii), it follows that

$$A' \cdot (U_i U_{i-1}) \cdot \prod_{j \in F} U_j = A' \cdot \prod_{j \in F} U_j \quad \text{or} \quad A' \cdot (U_i U_{i+1}) \cdot \prod_{j \in F} U_j = A' \cdot \prod_{j \in F} U_j.$$

Note in either case, $B := A' \cdot (U_i U_{i-1})$ or $B := A' \cdot (U_i U_{i+1})$, that $B \in \widehat{\mathcal{TL}}$. Therefore

$$A = B \cdot \left(\prod_{j \in F} U_j \right) \in \widehat{\mathcal{TL}} \cdot \mathrm{Cup}(F).$$

It follows that $\mathrm{Cup}(F) \subseteq \widehat{\mathcal{TL}} \cdot \mathrm{Cup}(F)$, and hence that $\mathbb{1} \otimes_{\mathcal{TL}} \mathrm{Cup}(F) \cong 0$. □

Remark 17 The homology with trivial coefficients of the Temperley–Lieb algebra $\mathcal{TL}$ on one generator has been completely computed by Boyd and Hepworth in [2, Proposition 7.1]. For the case $n = 2$ and $F = \{1\}$ in Lemma 16, it holds that $H_0(\mathcal{TL}, \mathrm{Cup}(F)) \cong H_1(\mathcal{TL}, \mathbb{1})$. In particular, this case can also be deduced from [2, Proposition 7.1].

3.3 Proof of Theorems A, B and C

We will now prove the three main theorems. The following result, stated as Theorem A in Section 1, generalizes [2, Theorem D] and the "odd" part of [2, Theorem B].

Theorem 18 *If n is odd, then $H_0(\mathcal{TL}, \mathbb{1}) = R$ and $H_\star(\mathcal{TL}, \mathbb{1}) = 0$ for $\star > 0$.*

Proof Let $P_\star$ be a free resolution of the trivial $\mathcal{TL}$–module $\mathbb{1}$, and consider the double complex $P_\star \otimes_{\mathcal{TL}} \mathcal{TLD}$, where $\mathcal{TLD}$ is the cellular Davis complex for Temperley–Lieb algebras (see Definition 8). The horizontal and vertical filtration of $P_\star \otimes_{\mathcal{TL}} \mathcal{TLD}$ give rise to two spectral sequences. The vE^1–page of the vertical spectral sequence is given by

$$vE^1_{\alpha,\beta} = H_\beta(P_\alpha \otimes_{\mathcal{TL}} \mathcal{TLD}) \cong P_\alpha \otimes_{\mathcal{TL}} H_\beta(\mathcal{TLD}).$$

It follows from Theorem 10 that the vE^1–page is concentrated in degree $\beta = 0$, where it is given by the complex $vE^1_{\star,0} \cong P_\star \otimes_{\mathcal{TL}} \mathbb{1}$. Then by definition,

$$vE^2_{\alpha,0} = H_\alpha(\mathcal{TL}, \mathbb{1}).$$

The collapsing of the vertical spectral sequence on the vE^2–page implies that the horizontal spectral sequence converges to $H_{\alpha+\beta}(\mathcal{TL},\mathbb{1})$. The hE^1–page of the horizontal spectral sequence is given by

$$hE^1_{\alpha,\beta} = H_\beta(P_\star \otimes_{\mathcal{TL}} \mathcal{TL}\mathfrak{D}_\alpha) \cong \bigoplus_{\substack{F\subseteq\langle n-1\rangle \text{ innermost}\\ |F|=\alpha}} H_\beta(P_\star \otimes_{\mathcal{TL}} \mathrm{Cup}(F))$$
$$\cong \bigoplus_{\substack{F\subseteq\langle n-1\rangle \text{ innermost}\\ |F|=\alpha}} H_\beta(\mathcal{TL},\mathrm{Cup}(F)).$$

Because n is odd, Corollary 15 applies to any innermost subset $F\subset\langle n-1\rangle$. Together with Lemma 16, this implies that

$$hE^1_{\alpha,\beta} = \begin{cases} R & \text{if } (\alpha,\beta)=(0,0),\\ 0 & \text{if } (\alpha,\beta)\neq(0,0).\end{cases}$$

The theorem follows. □

The next vanishing result, stated as Theorem B in Section 1, is similar to the "even" part of [2, Theorem B]. The vanishing line which Boyd and Hepworth obtain is stronger than ours — slope 1 versus slope $\frac{1}{2}$ — and in fact optimal [2, Theorem C]. The description of the high-dimensional homology in terms of homology with coefficients is new in the sense that the coefficient system is different from the one in [2].

Theorem 19 *If n is even, then $H_0(\mathcal{TL},\mathbb{1})=R$, $H_\star(\mathcal{TL},\mathbb{1})=0$ for $0<\star<\frac{n}{2}$ and*

$$H_{\star+n/2}(\mathcal{TL},\mathbb{1}) \cong H_\star(\mathcal{TL},\mathrm{Cup}(M))$$

for $\star\geq 0$ and where $M=\{1,3,\dots,n-1\}\subseteq\langle n-1\rangle$ is the unique maximal innermost set.

Proof By the same argument as in the proof of Theorem 18, we obtain a spectral sequence converging to $H_{\alpha+\beta}(\mathcal{TL},\mathbb{1})$ with hE^1–page

$$hE^1_{\alpha,\beta} \cong \bigoplus_{\substack{F\subseteq\langle n-1\rangle \text{ innermost}\\ |F|=\alpha}} H_\beta(\mathcal{TL},\mathrm{Cup}(F)).$$

Because n is even, there exists a unique innermost set $M=\{1,3,\dots,n-1\}\subset\langle n-1\rangle$ of maximal cardinality $|M|=\frac{n}{2}$. Corollary 15 applies to all innermost subsets $F\subset\langle n-1\rangle$ except M. Together with Lemma 16, this implies that

$$hE^1_{\alpha,\beta} = \begin{cases} R & \text{if } (\alpha,\beta)=(0,0),\\ H_\beta(\mathcal{TL},\mathrm{Cup}(M)) & \text{if } (\alpha,\beta)=\left(\frac{n}{2},\beta\right),\\ 0 & \text{else.}\end{cases}$$

If $n=2$, the only possibly nontrivial differential on the hE^1–page

$$d_1\colon H_0(\mathcal{TL},\mathrm{Cup}(M)) = \mathbb{1}\otimes_{\mathcal{TL}}\mathrm{Cup}(M) \to \mathbb{1}\otimes_{\mathcal{TL}}\mathcal{TL} = H_0(\mathcal{TL},\mathcal{TL}), \quad I\otimes A\mapsto I\otimes A = I\cdot A\otimes 1 = 0$$

is the zero map. We conclude that the spectral sequence collapses on the hE^1–page. □

Remark 20 In the setting of Theorem 19, Lemma 16 implies that $H_{n/2}(\mathcal{TL}, \mathbb{1}) = R/a$ if $n = 2$, and that $H_{n/2}(\mathcal{TL}, \mathbb{1}) = 0$ if $n > 2$. This is consistent with [2, Theorem B and C] mentioned above.

We finish by presenting an alternative proof of [2, Theorem A], ie Theorem C stated in Section 1.

Theorem 21 [2, Theorem A] *If the parameter $a \in R$ is a unit, then*

$$H_0(\mathcal{TL}, \mathbb{1}) = R \quad \text{and} \quad H_\star(\mathcal{TL}, \mathbb{1}) = 0 \quad \text{for } \star > 0.$$

Proof By Theorems 18 and 19, it suffices to prove that, for $n \geq 2$ even and $M \subset \langle n-1 \rangle$ the unique maximal innermost set, $H_\star(\mathcal{TL}, \mathrm{Cup}(M)) = 0$ for $\star \geq 0$. To see this, we consider the innermost set $M_1 = M - \{1\}$. There is a map of left $\mathcal{TL}$–modules

$$\mathrm{Cup}(M_1) \to \mathrm{Cup}(M) \quad \text{given by } c \mapsto c \cdot U_1.$$

Observe that the map obtained by precomposition with the inclusion

$$\mathrm{Cup}(M) \hookrightarrow \mathrm{Cup}(M_1) \to \mathrm{Cup}(M)$$

is multiplication by a. If a is a unit, the inclusion induced map

$$H_\star(\mathcal{TL}, \mathrm{Cup}(M)) \to H_\star(\mathcal{TL}, \mathrm{Cup}(M_1))$$

must therefore be an injection. By Corollary 15, the target of this map is zero if $\star > 0$. For $\star = 0$, we invoke Lemma 16. □

References

[1] **R Boyd**, *The low-dimensional homology of finite-rank Coxeter groups*, Algebr. Geom. Topol. 20 (2020) 2609–2655 MR Zbl

[2] **R Boyd**, **R Hepworth**, *The homology of the Temperley–Lieb algebras*, Geom. Topol. 28 (2024) 1437–1499 MR Zbl

[3] **R Boyd**, **R Hepworth**, **P Patzt**, *The homology of the Brauer algebras*, Selecta Math. 27 (2021) art. id. 85 MR Zbl

[4] **R Charney**, *A generalization of a theorem of Vogtmann*, J. Pure Appl. Algebra 44 (1987) 107–125 MR Zbl

[5] **M W Davis**, *The geometry and topology of Coxeter groups*, Lond. Math. Soc. Monogr. Ser. 32, Princeton Univ. Press (2008) MR Zbl

[6] **W G Dwyer**, *Twisted homological stability for general linear groups*, Ann. of Math. 111 (1980) 239–251 MR Zbl

[7] **S Galatius**, **A Kupers**, **O Randal-Williams**, *E_2–cells and mapping class groups*, Publ. Math. Inst. Hautes Études Sci. 130 (2019) 1–61 MR Zbl

[8] **J L Harer**, *Stability of the homology of the mapping class groups of orientable surfaces*, Ann. of Math. 121 (1985) 215–249 MR Zbl

[9] **A Hatcher**, **K Vogtmann**, *Homology stability for outer automorphism groups of free groups*, Algebr. Geom. Topol. 4 (2004) 1253–1272 MR Zbl

[10] **A Hatcher**, **K Vogtmann**, **N Wahl**, *Erratum to: Homology stability for outer automorphism groups of free groups*, Algebr. Geom. Topol. 6 (2006) 573–579 MR Zbl

[11] **R Hepworth**, *Homological stability for families of Coxeter groups*, Algebr. Geom. Topol. 16 (2016) 2779–2811 MR Zbl

[12] **R Hepworth**, *Homological stability for Iwahori–Hecke algebras*, J. Topol. 15 (2022) 2174–2215 MR Zbl

[13] **V F R Jones**, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. 12 (1985) 103–111 MR Zbl

[14] **W van der Kallen**, *Homology stability for linear groups*, Invent. Math. 60 (1980) 269–295 MR Zbl

[15] **C Kassel**, **V Turaev**, *Braid groups*, Graduate Texts in Math. 247, Springer (2008) MR Zbl

[16] **L H Kauffman**, *State models and the Jones polynomial*, Topology 26 (1987) 395–407 MR Zbl

[17] **L H Kauffman**, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. 318 (1990) 417–471 MR Zbl

[18] **O Randal-Williams**, *A remark on the homology of Temperley–Lieb algebras* (2021) Available at `https://www.dpmms.cam.ac.uk/~or257/notes/Descent.pdf`

[19] **R J Sroka**, *Patterns in the homology of algebras*: *vanishing*, *stability*, *and higher structures*, PhD thesis, University of Copenhagen (2021) Available at `https://noter.math.ku.dk/phd21rs.pdf`

[20] **H N V Temperley**, **E H Lieb**, *Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem*, Proc. A 322 (1971) 251–280 MR Zbl

[21] **K Vogtmann**, *Homology stability for* $O_{n,n}$, Comm. Algebra 7 (1979) 9–38 MR Zbl

[22] **N Wahl**, *Homological stability for the mapping class groups of non-orientable surfaces*, Invent. Math. 171 (2008) 389–424 MR Zbl

Department of Mathematics and Statistics, McMaster University
Hamilton, ON, Canada

Current address: *Mathematisches Institut, Universität Münster*
Münster, Germany

`robinjsroka@uni-muenster.de`

Received: 31 July 2022 Revised: 29 August 2022

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Kathryn Hess
kathryn.hess@epfl.ch
École Polytechnique Fédérale de Lausanne

BOARD OF EDITORS

Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu	Robert Lipshitz	University of Oregon lipshitz@uoregon.edu
Tara E Brendle	University of Glasgow tara.brendle@glasgow.ac.uk	Norihiko Minami	Yamato University minami.norihiko@yamato-u.ac.jp
Indira Chatterji	CNRS & Univ. Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr	Andrés Navas	Universidad de Santiago de Chile andres.navas@usach.cl
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Robert Oliver	Université Paris 13 bobol@math.univ-paris13.fr
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Jessica S Purcell	Monash University jessica.purcell@monash.edu
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Birgit Richter	Universität Hamburg birgit.richter@uni-hamburg.de
David Futer	Temple University dfuter@temple.edu	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk	Vesna Stojanoska	Univ. of Illinois at Urbana-Champaign vesna@illinois.edu
Ian Hambleton	McMaster University ian@math.mcmaster.ca	Zoltán Szabó	Princeton University szabo@math.princeton.edu
Matthew Hedden	Michigan State University mhedden@math.msu.edu	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Chris Wendl	Humboldt-Universität zu Berlin wendl@math.hu-berlin.de
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu	Daniel T Wise	McGill University, Canada daniel.wise@mcgill.ca
Thomas Koberda	University of Virginia thomas.koberda@virginia.edu	Lior Yanovski	Hebrew University of Jerusalem lior.yanovski@gmail.com
Markus Land	LMU München markus.land@math.lmu.de		

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US $705/year for the electronic version, and $1040/year (+$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

https://msp.org/

© 2024 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 6 (pages 2971–3570) 2024