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Hyperbolic homology 3–spheres from drum polyhedra

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We construct explicit families of hyperbolic homology spheres, by surgery on links with a large number of components or by surgery on knots. In both cases the original cusped manifolds are obtained from basic ideal polyhedra, which allows us to get further geometric properties, such as geometric convergence to $\mathbb{H}^3$ and arbitrarily large Heegaard genus. In the same construction we also find a family of hyperbolic knots converging geometrically to $\mathbb{H}^3$.

57K10, 57K32

1 Introduction

An integer homology 3–sphere is a closed 3–manifold M having the same integer homology groups as the 3–dimensional sphere S^3, that is, $H_\bullet(M;\mathbb{Z}) \simeq H_\bullet(S^3;\mathbb{Z})$. Although it may be considered homology with another coefficients, mainly rational, throughout this paper we shall deal with the definition over the integers.

These manifolds were under consideration previously, in the setting of the Poincaré conjecture. Homology spheres manifested themselves as a much more extensive type of manifold.

In the setting of hyperbolic geometry we may apply the general method of performing $1/n$–Dehn filling to the complement of a knot in S^3 to get a homology sphere. If the knot is hyperbolic, then for sufficiently large n the resulting homology sphere becomes hyperbolic by Thurston's Dehn surgery theorem.

There are, to our knowledge, three main constructions of hyperbolic homology spheres. One of them is due to Baker, Boileau and Wang [Baker et al. 2001], which carry out the construction of an infinite tower of finite-index coverings of hyperbolic homology spheres. This construction makes use of the 120–sheeted covering of the sphere S^3 over the Poincaré homology sphere Σ.

Also noteworthy is the construction by Brock and Dunfield [2015]. In this case they deal with a sequence of homology spheres focusing on the injectivity radius behavior rather than on topological aspects. Namely, their sequence converges to $\mathbb{H}^3$ in the *Benjamini–Schramm* sense. Their construction involves modern Kleinian group theory and is based on the residual finiteness of 3–dimensional hyperbolic fundamental groups.

From the Brock–Dunfield construction, Kent and Purcell [2018] were able to obtain a family of knots that also Benjamini–Schramm converge to $\mathbb{H}^3$. Performing $1/n$–Dehn filling they obtain a sequence of homology spheres with the same convergence.

The aim of this paper is to make an elementary and explicit construction of families of hyperbolic homology spheres — Theorems 5.2 and 5.3. From the geometric properties analyzed in the construction, we show in Theorem 7.6 sequences of hyperbolic homology spheres converging geometrically to $\mathbb{H}^3$, and in Corollary 8.3 sequences of homology spheres with large Heegaard genus.

We provide two different families of such hyperbolic homology spheres, which are obtained from Dehn filling two different types of links: the first type consists of links with many but "short" components, in the sense that each component is linked with few other components, and the other type of links are just knots. The first family of hyperbolic links have the property that the cusp shapes are uniformly bounded in the Teichmüller space of the torus, so the manifolds obtained after Dehn filling might be candidates for hyperbolic homology spheres with injectivity radius uniformly bounded below.

From this construction it is manifest that there are hyperbolic manifolds obtained by Dehn filling links and hyperbolic manifolds obtained by Dehn filling knots which have close volume — Corollary 7.7. This motivates the question of whether the former manifolds can also be obtained from surgery on knots in S^3.

Purcell and Souto [2010] proved the existence of a family of hyperbolic knots converging geometrically to $\mathbb{H}^3$ and in their paper they ask for explicit examples. Spacious knots constructed by Kent and Purcell [2018] provide such an example. As an aside to our construction we also obtain a family of hyperbolic knots converging geometrically to $\mathbb{H}^3$ — Theorem 7.8.

Using the relation proved by Bachman, Cooper and White [Bachman et al. 2004] between the Heegaard genus of a negatively curved manifold and the radius of balls contained in this manifold, we find, as another consequence, homology spheres with large Heegaard genus — Corollary 8.3.

The idea of our constructions is as follows. To get the geometric convergence to $\mathbb{H}^3$ we consider hyperbolic links constructed from big hyperbolic polyhedra. In this way, our starting point is to obtain big polyhedra by stacking a large number of ideal drum polyhedra (Thurston's notes [1979]) with basis a polygon with many edges. So in this first step we obtain hyperbolic links containing big balls, and so converging to $\mathbb{H}^3$ — Corollary 7.5. The homology spheres are obtained from the previous links by hyperbolic Dehn surgery, using the 2π theorem and the Perelman geometrization theorem.

To obtain the two types of links involved in the constructions, we make use of some operations referred in hyperbolic geometry literature, as Bridgeman surgery and Adams' moves.

We remark that the links in our constructions are examples of *fully augmented links* studied by, among others, Adams, Lackenby, Agol, D Thurston, Futer and Purcell; see for instance [Purcell 2011]. In particular, their hyperbolicity is well known. In our construction we start with the hyperbolic polyhedra to keep control of the symmetry and more explicit description of the geometry of the resulting links and manifolds.

1.1 Organization of the paper

We recall some preliminaries in Section 2. Section 3 is devoted to constructing the hyperbolic polyhedra and links we will work with: the polyhedra $Q(p,q)$ obtained by performing Bridgeman surgery on a stack of drum polyhedra; and the links $\mathcal{M}(p,q)$, $\mathcal{H}(p,q)$ and $\mathcal{K}(p,q)$ obtained by gluing faces of two copies of $Q(p,q)$, sometimes after Adams' moves. The link $\mathcal{H}(p,q)$ has many short components while the link $\mathcal{K}(p,q)$ has a very "long" component.

In Section 4.1 we show which surgery coefficients we need to add to $\mathcal{H}(p,q)$ in order to obtain homology spheres — Theorem 4.1. In Section 5 we use the 2π theorem to determine which of the homology spheres obtained from $\mathcal{H}(p,q)$ and $\mathcal{K}(p,q)$ are negatively curved — Theorems 5.2 and 5.3. We notice that all cusps shapes in $\mathcal{H}(p,q)$ are uniformly bounded in Section 6.

In Section 7 we extract geometric properties of our manifolds; mainly we find big half-balls contained in the polyhedra $Q(p,q)$ and, consequently, we find big balls in the hyperbolic links $\mathcal{H}(p,q)$ and $\mathcal{K}(p,q)$ — Corollary 7.5. From this, we obtain geometric convergence of homology spheres and of hyperbolic knots to $\mathbb{H}^3$ in Theorems 7.6 and 7.8. In Section 8 we show that the balls contained in the previous links remain after applying the 2π theorem, and, as an application, we obtain hyperbolic homology spheres with large genus — Corollary 8.3.

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2 Preliminaries

2.1 Operations on hyperbolic links

There are some operations in the literature that we will use to obtain further hyperbolic manifolds from previous ones.

Bridgeman surgery [1994; 1996; 1998]: If P is an ideal polyhedron with all dihedral angles equal to $\pi/2$ and F is a nontriangular face, one can draw a line joining two points in the interior of two nonconsecutive edges of F and collapse it to a point. The result is a new combinatorial polyhedron which has a realization as an ideal hyperbolic polyhedron P' with all dihedral angles equal to $\pi/2$.

Adams' moves [1985, Corollary 5.1]: Whenever we have a link with an unknotted component which bounds a disc intersecting two strands of the link, one can cut along this disc, do a half-twist and glue it again. Adams proved that a link obtained in this way from a hyperbolic link is also hyperbolic with the same volume.

2.2 Dehn surgery (Dehn filling)

We recall the standard terminology. A *framed link* is a link $\mathscr{L} = \{K_1, \dots, K_n\}$ in S^3 with rational numbers p_i/q_i attached to each component K_i. A framed link represents the manifold obtained by doing Dehn surgery in each component with surgery instructions the given coefficients. The linking matrix of a framed oriented link is a matrix $B = (b_{ij})$ where the diagonal entries are the surgery coefficients and the off-diagonal entries are the linking numbers, ie $b_{ii} = p_i/q_i$ and $b_{ij} = \mathrm{lk}(K_i, K_j)$. A result by Hoste [1986] asserts that the framed link $\mathscr{L}$ represents a homology sphere if and only if $\det B = \pm 1/(q_1 \cdots q_n)$. This result will be used in Section 4.

2.3 Hyperbolic Dehn surgery

There are two results that we would like to mention in this point. First, the Gromov–Thurston 2π theorem provides sufficient conditions in order to guarantee that a closed manifold obtained by Dehn filling a hyperbolic link is negatively curved. Some years later, Agol [2000] and Lackenby [2000] obtained a similar result, the 6 theorem, to assert that the resulting filled manifold is *hyperbolike*. The term hyperbolike refers to a closed, orientable 3–manifold that is irreducible with infinite word-hyperbolic fundamental group. After the geometrization theorem, being negatively curved and being hyperbolike are each equivalent to being hyperbolic manifold. We notice here that the 6 theorem was improved in terms of a real parameter by Lackenby and Meyerhoff [2013] in the context of Mom Technology.

In our paper we will use the 2π theorem because the negatively curved structure obtained there is needed in order to apply the Bachman, Cooper and White result in Section 8.

2π Theorem [Bleiler and Hodgson 1996] *Let M be a complete hyperbolic 3–manifold of finite volume and $P_1, \dots, P_\nu$ be disjoint horoball neighborhoods of the cusps of M. Suppose r_i is a slope on ∂P_i represented by a geodesic α_i with length in the Euclidean metric satisfying $\mathrm{length}(\alpha_i) > 2\pi$ for each $i = 1, \dots, \nu$. Then $M(r_1, \dots, r_\nu)$ has a metric of negative curvature, where $M(r_1, \dots, r_\nu)$ is the manifold obtained by Dehn filling M with the instructions that the slopes r_i will bound discs.*

2.4 Heegaard genus

We will use the following result by Bachman, Cooper and White to obtain hyperbolic homology spheres with large genus.

Theorem 2.1 [Bachman et al. 2004] *If M is a closed, orientable, connected Riemannian 3–manifold with all sectional curvatures less than or equal to -1, then its Heegaard genus g satisfies $g \geq \frac{1}{2}\cosh(r)$, where r is the radius of any isometrically embedded ball in M.*

2.5 Geometric convergence

A sequence of pointed hyperbolic 3–manifolds (M_i, p_i) *converges geometrically* or equivalently in the pointed Gromov–Hausdorff topology to a pointed manifold (M, p) if for every ϵ and every compact set $K \subset M$ with $p \in K$ there exists i_0 such that for all $i \geq i_0$, there is a $(1+\epsilon)$-bi-Lipschitz embedding

$$f_i : (K, p) \to (M_i, p_i).$$

In particular, a sequence of hyperbolic 3–manifolds (M_i, p_i) containing isometrically embedded balls $B(p_i, r_i)$ with $r_i \to \infty$ converges geometrically to $(\mathbb{H}^3, p)$ for any $p \in \mathbb{H}^3$.

3 Hyperbolic polyhedra and hyperbolic links

The main constructions of this section are the hyperbolic polyhedra $Q(p,q)$ and the hyperbolic links $\mathcal{M}(p,q)$ for any $p \geq 3$ and $q \geq 1$. The construction of $Q(5,3)$ and $\mathcal{M}(5,3)$ is illustrated in Figures 2 and 3.

3.1 Drum polyhedra and polyhedra $Q(p,q)$

Drum polyhedra and links constructed from them appear in Thurston's notes [1979, Chapter 6]. We will consider a mild generalization of these examples, in the sense that we do not consider just drum polyhedra but those obtained by stacking drums one over the other.

The *n–drum* also known as *antiprism* is a polyhedron with two n–gonal faces called *bases* and $2n$ *lateral* triangular faces. Each lateral face is a triangle with an edge common to one of the bases and the opposite vertex common to the other basis; see Figure 1. By explicit construction one can show that there is an ideal drum polyhedra $D(n)$ with all dihedral angles equal to $\pi/2$, with cyclic symmetry of order n and with the two bases regular ideal n–gons.

Next we can take m copies of $D(n)$ and glue them together along their bases. The result is again an ideal polyhedron $D(n,m)$ with all dihedral angles equal to $\pi/2$. Notice that most lateral faces are ideal squares.

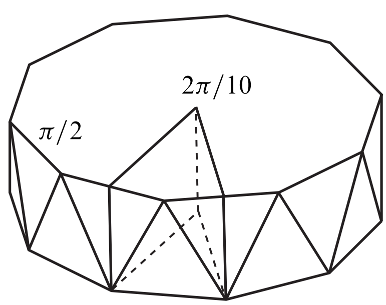

Figure 1: The schematic picture of the drum polyhedron $D(10)$.

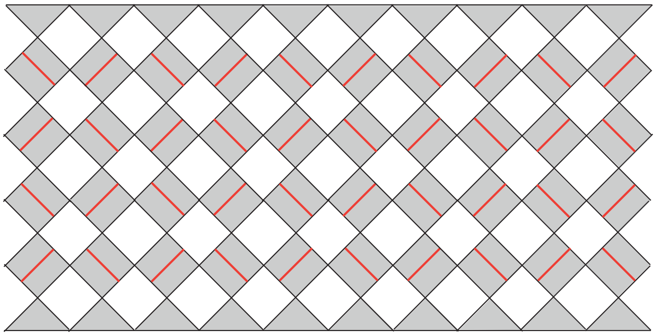

Figure 2: This picture shows a diagram of $D(10,10)$. Also shown is the pattern followed by the segments in quadrangular faces in which Bridgeman's surgery is to be performed to obtain the polyhedron $Q(5,3)$.

To gain flexibility, one can use Bridgeman surgery to split some of the squares into 2 triangles. From now on, we will consider $n = 2p$ and $m = 2(2q-1)$ for some $p \geq 3$ and $q \geq 1$. Figure 2 shows the precise lines on the quadrangular faces of $D(2p, 4q-2)$ which we will collapse to points. Each quadrangular face will give rise to a pair of new triangular faces once the segment is collapsed. We obtain the polyhedron $Q(p,q)$, which is again ideal and with all dihedral angles equal to $\pi/2$. See Figure 15 at the end of the paper for a more "realistic" view.

Unlike $D(n,m)$, which is composed by isometric slices combinatorially equivalent to drum polyhedra, the polyhedron $Q(p,q)$ can be divided into slices but they are no longer isometric. In Section 7 we will work with these slices.

Notice that $Q(p,q)$ has dihedral symmetry of order p about the common perpendicular axis to the two basis (ie generated by symmetry planes containing this axis) and it also has a horizontal symmetry plane interchanging the two basis. To convince oneself of the existence of these symmetries in the hyperbolic polyhedron, one can prove the existence in hyperbolic space of the fundamental domain for these symmetries with the appropriate dihedral angles so that $Q(p,q)$ may be obtained by gluing a finite number of copies of it.

Remark The existence and explicit construction of the hyperbolic polyhedra $Q(p,q)$ can also be shown by means of circle packings; see the appendix. We will use this technique in Section 7 to prove some properties of these polyhedra.

3.2 Links $\mathcal{M}(p,q)$

We now proceed to identify the faces of two copies of $Q(p,q)$ to obtain a 3–dimensional manifold, following the construction carried out by Thurston [1979, Example 6.8.8 of Chapter 6]. For this purpose, we take two equal copies of $Q(p,q)$ and we identify their faces in two stages. In the first one we make the identifications in each copy of $Q(p,q)$ separately, gluing all the triangular faces as shown in the

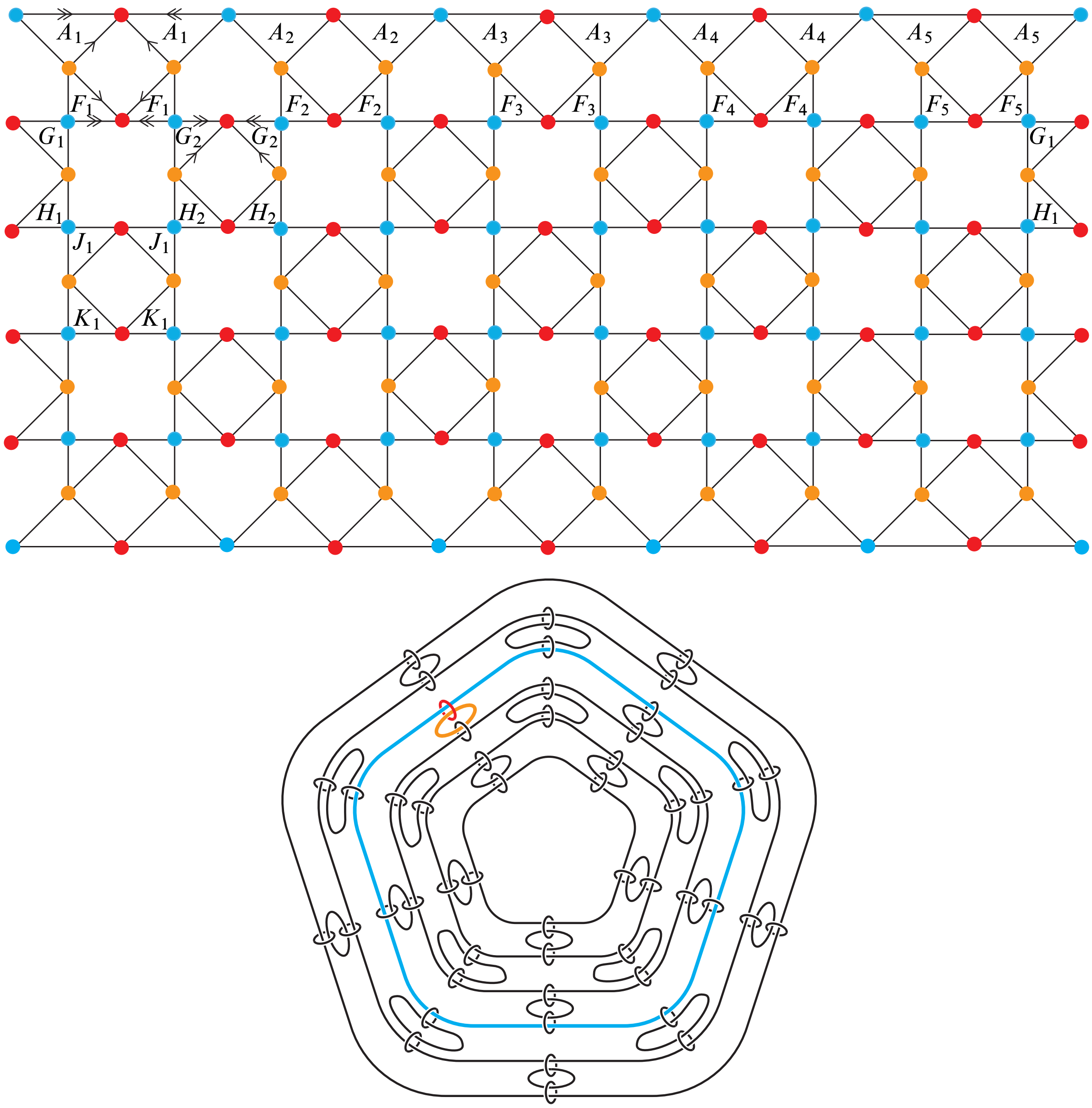

Figure 3: The top picture shows the way to identify faces of the polyhedron $Q(5,3)$. Gluing two identical copies produces the link $\mathbb{M}(5,3)$ on the bottom. Blue, orange and red vertices correspond, respectively to large horizontal, small horizontal and vertical components.

pattern at the top of Figure 3. In the second step we identify each nontriangular face of the first copy with its corresponding one in the second copy.

After the first step, what is obtained is topologically a ball with some arcs removed from its interior and some groves removed from its boundary. The second step corresponds to gluing these two balls to

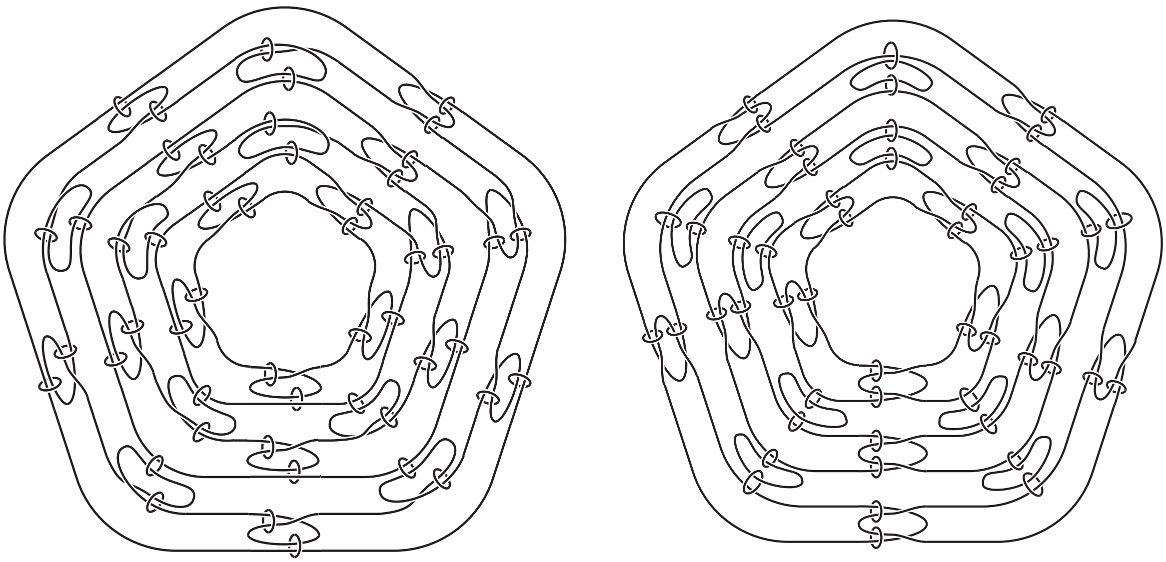

Figure 4: Left: the link $\mathcal{K}(5,3)$. If we perform $1/n$ Dehn filling on the vertical components, we obtain the knot $\mathcal{K}(5,3,n)$. Right: the link $\mathcal{H}(5,3)$.

obtain the complement of the link $\mathcal{M}(p,q)$ shown at the bottom part of Figure 3. The ideal vertices of the original polyhedra give rise to the components of the link.

The link $\mathcal{M}(p,q)$ has some horizontal components (ie contained in the projection plane) and other vertical components, all of them unknotted. For a more precise description of this link, see Section 4.1.

3.3 Adams' moves

A way of modifying the link $\mathcal{M}(p,q)$ is by performing Adams' moves in the vertical components of $\mathcal{M}(p,q)$. We will do it in two different ways, in order to obtain two new links with opposite properties:

- In the first, we will obtain the link $\mathcal{K}(p,q)$ of Figure 4 where all the horizontal components of $\mathcal{M}(p,q)$ have been connected to give a very long component which is linked to the vertical components.
- In the second, we will obtain the link $\mathcal{H}(p,q)$ of Figure 5, with many components but all of them short in the sense that they are linked with few components (at most 8, independent of p and q).

The effect of the Adams' moves in the gluing of the polyhedra is only that the gluing pattern is different. In particular, some triangular faces of one copy of $Q(p,q)$ are identified with triangular faces of the other copy of $Q(p,q)$.

Corollary 3.1 *The link exteriors $\mathcal{M}(p,q)$, $\mathcal{K}(p,q)$ and $\mathcal{H}(p,q)$ are hyperbolic, ie they have a complete hyperbolic structure.*

Proof We will prove that $\mathcal{M}(p,q)$ has a complete hyperbolic structure. From this it will follow that the same is true for $\mathcal{K}(p,q)$ and $\mathcal{H}(p,q)$, as Adam's moves preserve both hyperbolicity and completeness.

To prove hyperbolicity of $\mathcal{M}(p,q)$ we first notice that all the face identifications are done through hyperbolic isometries. Then, as a result of this face pairing, each resulting edge in $\mathcal{M}(p,q)$ coming from identifying polyhedral edges add up to 2π, since the edges of $Q(p,q)$ are identified in groups of four and all dihedral angles are $\pi/2$.

To prove completeness, we may apply [Thurston 1997, Proposition 3.4.15(d)], which characterizes the completeness of a hyperbolic manifold M in terms of the existence of a family of compact subsets S_t of M for $t \in \mathbb{R}^+$, such that $\bigcup S_t = M$ and S_{t+a} contains a neighborhood of radius a about S_t.

In our case we shall consider the middle point M of the common perpendicular of the two bases of $Q(p,q)$, which is a symmetry center of this polyhedron (recall the symmetries mentioned in Section 3.1). Let B_t be the intersection of $Q(p,q)$ with the ball of center M and radius t. The set S_t is the union of the copies of B_t in the two copies of $Q(p,q)$. The two isometric copies of B_t become identified by isometries which are the restriction of the identifications of the faces of the two copies of $Q(p,q)$. Hence, the family of subsets S_t satisfies the required condition. □

4 Topological homology spheres

To finish our constructions, we will perform Dehn filling in order to obtain closed manifolds, especially homology spheres.

- Starting from $\mathcal{K}(p,q)$, we first perform $1/n$–Dehn filling on the vertical components in order to obtain a knot, denoted by $\mathcal{K}(p,q,n)$. Then we just need to do $1/m$–Dehn surgery on this knot to get a homology sphere, which we denote by $\mathcal{K}(p,q,n)(m)$
- To obtain homology spheres from $\mathcal{H}(p,q)$ we need to choose the surgery coefficient with some more care — Theorem 4.1.

4.1 Homology spheres from $\mathcal{H}(p,q)$

In order to obtain homology spheres from $\mathcal{H}(p,q)$ we will carefully choose the surgery coefficients in Theorem 4.1 so that the linking matrix satisfies the condition stated in Section 2.2.

We will label the components of $\mathcal{H}(p,q)$. In all the links appearing in the paper, the vertical components fall into two types: those where Adams' moves have been applied when passing from the link $\mathcal{M}(p,q)$ to another one, and those where no Adams' move has been applied. We will define *type A* to be the first ones and *type B* to be the second ones.

Notice that this link has rotational symmetry of order p and q "levels". The components of this link fall into four types (we will only label components which are not equivalent under the rotational symmetry; see Figure 5):

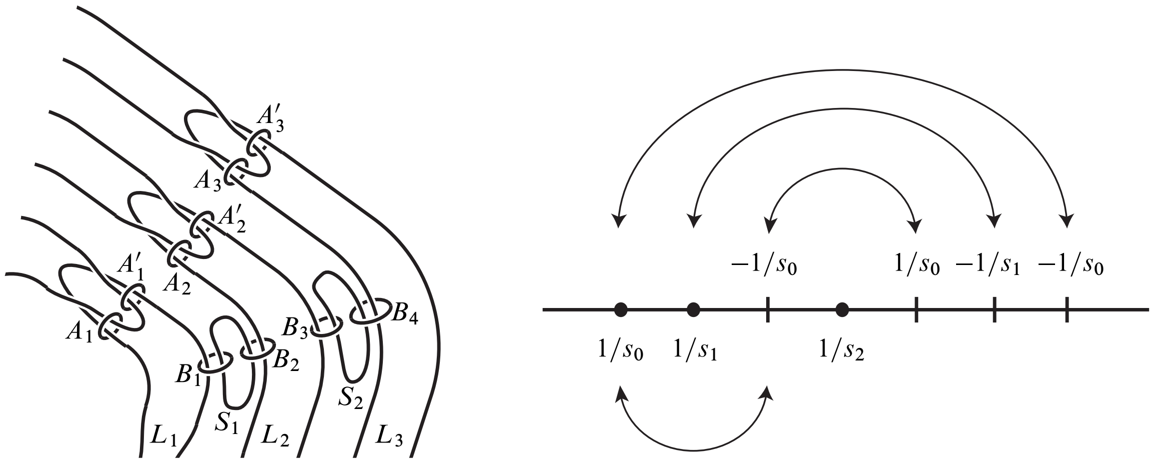

Figure 5: Labeling the components of the link $\mathcal{H}(p,3)$. On the right, the rule to assign coefficients to the small horizontal components.

- pq large horizontal components distributed in q levels, denoted by L^i_j for $j = 1, \dots, q$;
- $p(q-1)$ small horizontal components, denoted by S^i_j for $i = 1, \dots, p$ and $j = 1, \dots, q-1$;
- $2pq$ vertical components of type A, denoted by A^i_j and A'^i_j for $j = 1, \dots, q$;
- $2p(q-1)$ vertical components of type B, denoted by B_j for $j = 1, \dots, 2(q-1)$.

In the above, the superindex i always runs from 1 to p and will be suppressed when no confusion arises.

Each component L of the link has attached a surgery coefficient $c(L)$. Equivalent components of the link under the rotational symmetry are given the same coefficient. In this way we obtain a framed link representing a manifold that we denote by $\mathcal{H}(p,q)(X)$, where X is the collection of surgery coefficients.

Theorem 4.1 *For $p \geq 1$ and $q = 2^k$ with $k \geq 0$, consider the link $\mathcal{H}(p, 2^k)$ with the following surgery coefficients*:

(a) *If $k = 0$, then there are no S_i or B_i components. We take $c(L_1) = 1/\ell$ and $c(A_1) = -c(A'_1) = 1/a_i$, for any integers ℓ and a.*

(b) *If $k > 0$, so $q > 1$, then*

  (i) $c(A_j) = -c(A'_j) = 1/a_j$ *for* $j = 1, \dots, q$;

  (ii) $c(B_j) = 1/b$ *if j is even, and* $c(B_j) = -1/b$ *if j is odd*;

  (iii) $c(L_j) = 1/\ell$ *if* $j \neq 1, q$ *and j is even,* $c(L_j) = -1/\ell$ *if* $j \neq 1, q$ *and j is odd, and* $c(L_1) = -c(L_q) = -1/\ell - b$

  (iv) $c(S_{2^i}) = 1/s_i$ *for* $i = 0, \dots, k-1$, *and if* $2^i < j < 2^{i+1}$ *then* $c(S_j) = -c(S_{(2^{i+1}-j)})$,

where a_j, b, ℓ and s_i are arbitrary integers. See Figure 5.

Then the manifold $\mathcal{H}(p,q)(X)$ is a homology sphere.

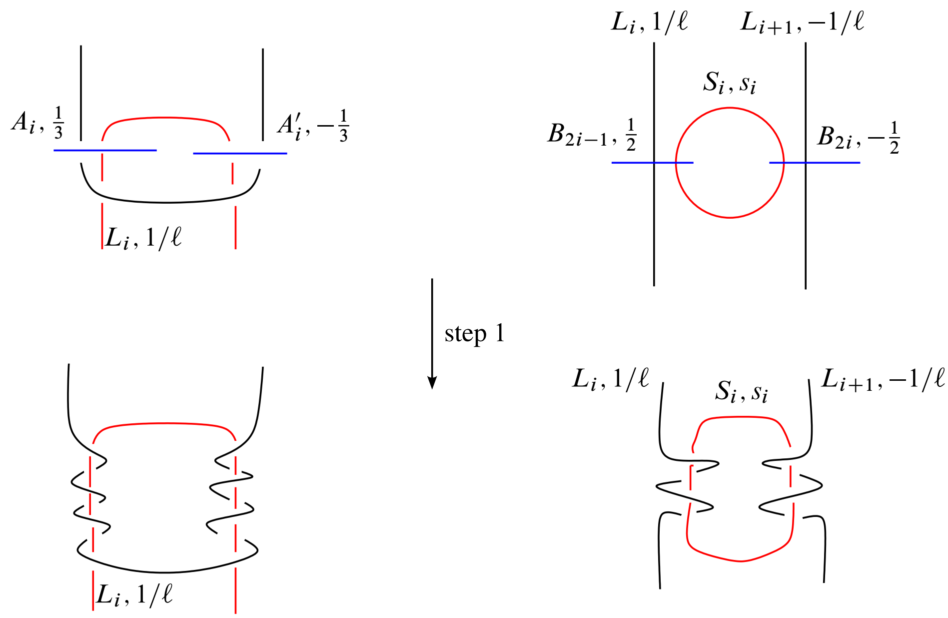

Figure 6: Type A and B components of $\mathcal{H}(p,q)$: an example of surgery coefficients is assigned and step 1 of the proof has been applied.

Notation When the integers a_j, b, ℓ and s_i in the above theorem are all equal to n, the manifold $\mathcal{H}(p,q)(X)$ will be denoted by $\mathcal{H}(p,q)(n)$.

Proof We do the proof in two steps: first we pass to another framed link $\mathcal{H}'(p,q)(X')$ representing the same manifold as $\mathcal{H}(p,q)(X)$, and then we compute the linking matrix of this new framed link and its determinant to show that the manifold it represents is a homology sphere.

For the first step, we twist about each vertical component V in order to change their surgery coefficient to ∞ (precisely, if the coefficient of a vertical component is $1/a$, we perform $-a$ twists, where negative twists means left handed twists). After doing this and removing the components with coefficient equal to ∞, we obtain the framed link $\mathcal{H}'(p,q)(X')$. In this new link, a large horizontal component L_j is linked to two other large horizontal components in the same level (except when $p=1,2$, when it is linked just to itself or to one other component), but with linking number equal to 0, since $c(A_j) = -c(A_j)$; see Figure 6. Also, the large component L_i gets linked to S_{i-1} and S_i.

To find the new surgery coefficients, recall the formula $x' = x + t(\mathrm{lk}(L,V))^2$ which relates the old and new surgery coefficients x and x' of a component L different from the component V we are twisting about.

Applying this formula to a large horizontal component L_j for $j \neq 1, q$, we see that there are cancellations coming from the fact that $c(A'_j) = -c(A_j)$ and $c(B_{j+1}) = -c(B_j)$. Thus we have that $c(L_j)' = c(L_j)$

for all $j \neq 1, q$. The components L_1 and L_q are only linked to one type B vertical component (if $k > 0$), and thus we obtain

$$c(L_1)' = c(L_1) + b = -\frac{1}{\ell}, \quad c(L_q)' = c(L_q) - b = \frac{1}{\ell}.$$

If $k = 0$, there are no B–vertical components at all, so $c(L_1)' = c(L_1)$. Similarly, the new coefficients of the small horizontal components are equal to the old ones.

In conclusion, the frame link $\mathcal{H}'(p,q)(X')$ only has horizontal large and small components, two large components have linking number 0, and the surgery coefficients are equal to the old ones except when $k > 0$ for the large components in the first and last level.

Next we compute the linking matrix of $\mathcal{H}'(p,q)(X')$.

(a) In the case $k = 0$, by the above, the linking matrix is diagonal with entries integer inverses; so $\mathcal{H}'(p,1)$ is a homology sphere by Hoste's theorem (see Section 2.2).

(b) Let us see the general case $q = 2^k$ for $k > 0$. We order its components in p groups as

$$L_1^i, S_1^i, L_2^i, S_2^i, \dots, S_{q-1}^i, L_q^i, \quad i = 1, \dots, p,$$

and the linking matrix of $\mathcal{H}'(p,q)(X')$ can be thought of as decomposed into square block submatrices of size $2q - 1$. Since $\mathrm{lk}(K_j^i, K_{j'}^{i'}) = 0$ when $i \neq i'$ (where K is either L or S), only the blocks in the diagonal are nonzero. Moreover, by the rotational symmetry, all these diagonal blocks are equal. We denote this block B_k, and notice that it has order $2q - 1 = 2^{k+1} - 1$. To obtain the off diagonal entries of B_k we orientate $\mathcal{H}'(p,q)$ with the orientation induced by a fixed orientation of the plane (namely, orientate each planar component of $\mathcal{H}(p,q)$ with the orientation induced by a orientation of the plane containing them; this orientation induces one in $\mathcal{H}'(p,q)$). Then we have

$$B_k = \begin{pmatrix} -1/\ell & b & & & & & \\ b & 1/s_0 & -b & & & & \\ & -b & 1/\ell & b & & & \\ & & & \ddots & & & \\ & & & & -1/\ell & b & \\ & & & & b & c(S_{q-1}) & -b \\ & & & & & -b & 1/\ell \end{pmatrix}.$$

Since all the diagonal entries of B_k are of the form $\pm 1/a$ with a some integer, to finish the proof we need to show that the determinant of B_k is equal, in absolute value, to the product of its diagonal entries. We prove it by induction on k (recall that $q = 2^k$).

If $k = 0$ then $B_0 = (-1/\ell)$, so $\det B_0 = \pm 1/\ell$. Let us compute the case $k = 1$ by Laplace expansion through the second row,

$$\begin{vmatrix} -1/\ell & b & 0 \\ b & 1/s_0 & -b \\ 0 & -b & 1/\ell \end{vmatrix} = -bb\frac{1}{\ell} + \frac{1}{s_0}\Big(-\frac{1}{\ell}\Big)\frac{1}{\ell} - (-b)(-b)\Big(-\frac{1}{\ell}\Big) = -\frac{1}{s_0 \ell^2}.$$

Assume the result is true for all $k \leq h$ and let us see for $k = h+1$. Notice that the submatrix of B_{h+1} consisting of the first $2^{h+1}-1$ rows and columns is the matrix B_h. Also notice that

$$\operatorname{order}(B_{h+1}) = 2^{h+2}-1 = 2(2^{h+1}-1)+1 = 2 \operatorname{order}(B_h)+1.$$

Now, the key observation is that the surgery coefficients in (iii) and (iv) have been chosen so that the matrix B_{h+1} has the structure

$$B_{h+1} = \left(\begin{array}{c|c|c} B_h & \begin{matrix} \\ -b \end{matrix} & \\ \hline -b & 1/s_{h+1} & b \\ \hline & \begin{matrix} b \\ \\ \end{matrix} & SB_h \end{array}\right)$$

where we have used the notation SA for the *symmetric matrix* of a square matrix A, ie for $A = (a_{ij})$, its symmetric matrix is defined to be $SA = (-a_{d-i,d-j})$. Notice that $\det SA = \det A$ if d is even, while $\det SA = -\det A$ if d is odd.

Finally we denote by $A[i,j]$ the i,j minor of A, and call $H = 2^{h+1}-1$, the order of B_h. Expanding the determinant of B_{h+1} by its $(H+1)^{\text{st}}$ row, we have

$$\det B_{h+1} = -(-b)^2 B_h[H,H] \det(SB_h) + \frac{1}{s_{h+1}} \det(B_h)\det(SB_h) - b^2 \det(B_h) SB_h[1,1].$$

Since H is odd, $\det SB_h = -\det B_h$. On the other hand, notice that the submatrix of SB_h obtained by deleting the first row and column is the symmetric of the submatrix of B_h obtained by deleting its last row and column. It follows that $\det SB_h[1,1] = \det B_h[H,H]$. Therefore, the first and last summands of the above expression for $\det B_{h+1}$ cancel out and we get

$$\det B_{h+1} = \frac{1}{s_{h+1}} \det(B_h)\det(SB_h).$$

Then, by the induction hypothesis, we obtain the desired result. □

5 Hyperbolicity of the homology spheres

The boundary of a horoball neighborhood of any vertex of $Q(p,q)$ is a rectangle with two parallel edges contained in triangular faces and the other two parallel edges contained in nontriangular faces. Following [Futer and Purcell 2007] we define *shadow* and *white* edges to be the edges of the rectangle contained, respectively, in triangular and nontriangular faces. Therefore, the torus boundary of a horoball neighborhood of a cusp of $\mathcal{M}(p,q)$, $\mathcal{K}(p,q)$ or $\mathcal{H}(p,q)$ is made by gluing these rectangles, white edges with white edges and shadow ones with shadow ones.

To apply the 2π theorem it is convenient to find maximal horoball neighborhoods. In the context of fully augmented links, Futer and Purcell [2007] describe a way to obtain a maximal horoball neighborhood, which we describe in Lemma 5.1, particularized to our links.

Notice that each edge of an ideal hyperbolic triangle has a *middle point* defined as the intersection point of this edge with the orthogonal line from the opposite vertex. Because any edge of $Q(p,q)$ belongs to a unique triangular face, any edge of $Q(p,q)$ has a well defined middle point.

Lemma 5.1 [Futer and Purcell 2007] (i) *There is a maximal horoball neighborhood of $\mathcal{M}(p,q)$ which intersects any edge of $Q(p,q)$ in its middle point.*

(ii) *At the boundary of this maximal horoball neighborhood, $\ell(\boldsymbol{s}) = 1$ and $\ell(\boldsymbol{w}) \geq 1$, where $\boldsymbol{s}$ is the shadow edge of the rectangle and $\boldsymbol{w}$ is the white edge.*

Next we analyze the combinatorics of each torus at the boundary of the cusps neighborhood of $\mathcal{H}(p,q)$ or $\mathcal{K}(p,q)$ in terms of the rectangles it is made of. Any such torus T corresponds to a component K of the link. In each torus we will identify its *meridian*, ie the curve which is also a meridian of K (and so the one which produces trivial surgery).

Suppose first that K is a horizontal component and suppose it is divided into t strands when cutting along the discs bounded by the vertical components. Then there are t vertices in each copy of $Q(p,q)$ identified in the corresponding cusp so that the torus T is made by gluing $2t$ rectangles. Taking the two rectangles corresponding to the same vertex in the two copies of $Q(p,q)$ and identifying them by their white edges produces a cylinder whose meridian coincides with the meridian of the component K. The torus T is then the result of gluing these cylinders, and has meridian μ of length 2 and shortest longitude λ with length greater than t. Notice that λ may not be the preferred longitude, that is, the curve λ' such that the surgery coefficient m/n means that the slope to be killed is $n\lambda' + m\mu$, but $\ell(\lambda') \geq \ell(\lambda)$.

When K is a vertical component, its corresponding cusp is composed only of one vertex in each copy of $Q(p,q)$, so T is made by gluing two rectangles. If K is of type B (recall, no Adams' move on K to produce $\mathcal{L}$ from $\mathcal{M}(p,q)$), we glue first the shadow edges of the rectangles and this produces two cylinders whose meridians are the meridian of K, with length $\ell(\mu) \geq 1$. If the vertical component is of type A, the corresponding torus is obtained from the previous one by cutting it along its longitude and performing a half-twist.

In Figure 7 we can see the three types of cusp tori. We have represented the shaded edges of the rectangles with a vector $\boldsymbol{s}$ and the white edges with a vector $\boldsymbol{w}$. The vector $\boldsymbol{s}$ has length 1, while $\boldsymbol{w}$ may represent different (but proportional) vectors with length greater or equal to 1.

We can now estimate the lengths of curves on the boundary of the horoball neighborhood of the cusps and apply the 2π theorem. We obtain the following results.

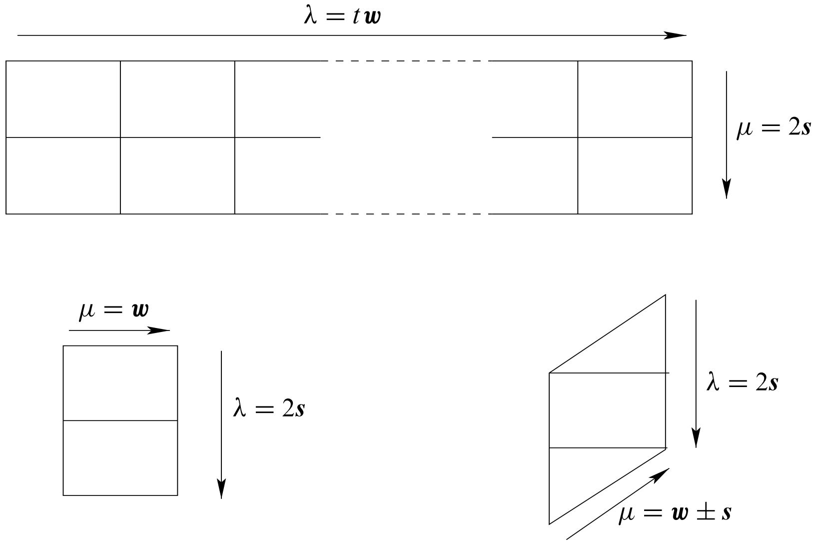

Figure 7: Cusp shapes of the components of $\mathscr{L}$. Top: the horizontal component. Bottom left: the vertical components of type B. Bottom right: the vertical component of type A.

Theorem 5.2 *Let $p \geq 3$ and $k \geq 0$. Let $X = (a_i, b, \ell, s_i)$ for $i = 1, \dots, 2^k$ be integers with $|\ell| \geq 1$, $|b| \geq 4$, $|a_i| \geq 4$ and $|s_i| \geq 3$. Then the homology spheres $\mathscr{H}(p, 2^k)(X)$ constructed in Theorem 4.1 are hyperbolic. In particular, $\mathscr{H}(p, 2^k)(n)$ is negatively curved, and hence hyperbolic, for all $p \geq 3$, $k \geq 0$ and $|n| \geq 4$.*

Proof Consider a maximal horoball neighborhood V of the cusps of $\mathscr{H}(p, q)$ as described in Lemma 5.1. Its boundary ∂V is a union of tori made by gluing triangles as described above and in Figure 7.

Now we compute the length of the slope $\beta = m\mu + n\lambda$ in each of the above torus boundaries T using the estimates in Lemma 5.1. Since most of the surgery coefficients are inverses of integers, we will do the computations for $m = 1$ and check at the end that this is enough. If $n > 0$, notice that β lifts to the curve $\mu + \lambda'$ in the n–sheeted cover $\widetilde{T}$ of T determined by the subgroup generated by μ and $n\lambda$. Therefore, the length of β is the length of a diagonal of the fundamental domain of $\widetilde{T}$. If $n < 0$, the length of β would be that of the other diagonal; thus, we will always consider $n > 0$ and argument with the two diagonals.

For the cases of horizontal components and vertical components of type B, the torus is rectangular, so both diagonals are equal and their length is $\sqrt{\ell(\mu)^2 + (n\ell(\lambda))^2}$. Thus, calling $s = \ell(\boldsymbol{s})$, $w = \ell(\boldsymbol{w})$ and recalling $s = 1$ and $w \geq 1$,

$$\ell(\beta) = \sqrt{4s^2 + n^2t^2w^2} \geq \sqrt{4 + n^2t^2}$$

in the case of horizontal components, and

$$\ell(\beta) = \sqrt{w^2 + 4n^2s^2} \geq \sqrt{1 + 4n^2}$$

for vertical components of type B.

For type A vertical components, the tori T are not rectangular; consequently, the two diagonals of $\widetilde{T}$ are different, and notice that the shortest one has the same length as the diagonal of the rectangular torus with $\mu = \boldsymbol{w}$ and $\lambda = (2n-1)\boldsymbol{s}$. Hence,

$$\ell(\beta) = \sqrt{w^2 + (2n-1)^2 s^2} \geq \sqrt{1 + (2n-1)^2}.$$

Now we just find n such that the lower bound we have obtained for $\ell(\beta)$ is greater than 2π. For horizontal components we obtain $n > 2\sqrt{\pi^2 - 1}/t$, so $n \geq 3$ for $t = 2$ (small horizontal components) and $n \geq 1$ for $t = 6$ (large horizontal components L_j with $j \neq 1, q$). Hence we take $s_i \geq 3$ and $\ell \geq 1$. For type B vertical components we obtain $n > \frac{1}{2}\sqrt{4\pi^2 - 1} \approx 3.1$. And for type A vertical components we obtain $n \geq \frac{1}{2}\sqrt{4\pi^2 - 1} + 1 \approx 3.6$. Hence we take $b, a_i \geq 4$.

To finish the proof it is only left to check that with the above bounds, the length of the slopes to be killed in the horizontal components L_j with $j = 1, q$ is greater than 2π. Since the surgery coefficient is $-1/\ell - b$,

$$\ell(\beta) = \sqrt{(-1 - b\ell)^2 4s^2 + \ell^2 5^5 w^2} \geq \sqrt{(-1 - b\ell)^2 4 + \ell^2 5^5} \geq \sqrt{125} > 2\pi. \qquad \square$$

Theorem 5.3 (a) *The knot exterior $\mathcal{K}(p, q, n)$ is hyperbolic for $|n| \geq 4$.*

(b) *The homology spheres $\mathcal{K}(p, q, n)(m)$ are hyperbolic for $|n| \geq 4$ and $|m| \geq 1$.*

Proof (a) The link $\mathcal{K}(p, q, n)$ is the result of doing $1/n$–Dehn filling at all the vertical components of $\mathcal{K}(p, q)$. Some of these components are of type A and the others of type B. So looking at the computations on the proof of Theorem 5.2, we obtain that it is enough to take $n \geq 4$.

(b) The manifold $\mathcal{K}(p, q, n)(m)$, obtained by $1/m$–Dehn filling the knot $\mathcal{K}(p, q, n)$, is also the manifold obtained by doing $1/n$–Dehn filling at all the vertical components of $\mathcal{K}(p, q)$ and r–Dehn filling at the horizontal component, where $1/m$ is the new surgery coefficient modified from r when performing first the surgery instructions on the vertical components. Precisely, after noticing that the linking number of any vertical component with the horizontal component of $\mathcal{K}(p, q)$ is equal to 2, we have $1/m = r + 4nv$, where v is the number of vertical components (ie $v = 2p(2q - 1)$). So $r = 1/m - 4nv = (1 - 4nmv)/m$.

On the other hand, by the analysis of the cusps done before Theorem 5.2, the torus corresponding to the horizontal component H is rectangular with $\ell(\mu) = 2$ and $\ell(\lambda) \geq 2v$. In this torus, the length of the slope β corresponding to the surgery coefficient $r = a/b$ is equal to the length of the diagonal of a rectangle with sides $a\mu$ and $b\lambda$. Hence,

$$\ell(\beta) \geq 2\sqrt{(1 - 4nmv)^2 + (vm)^2} \geq 2\sqrt{(3nmv)^2 + (vm)^2} \geq 2\sqrt{(3nmv)^2} = 6|nmv|.$$

Since, as in part (a), we must take $n \geq 4$, we have that $\ell(\beta) \geq 2\pi$ for any $|m| \geq 1$. $\square$

6 Uniformly bounded cusp shapes

In this section we show the following result.

Proposition 6.1 *The hyperbolic links $\mathcal{H}(p,q)$ have cusp shapes uniformly bounded for all p and q.*

Proof This is a consequence of two properties. The first one is that any component of $\mathcal{H}(p,q)$ is linked to at most 6 vertical components, which makes the boundary tori composed of at most 12 rectangles, as shown in Section 5. The second one is that the similarity structure of each rectangle is bounded above and below. One estimate is the one given in Lemma 5.1; the other is Lemma 6.2 below. □

Lemma 6.2 *The vertex figure (boundary of horoball cusp) of any vertex of the polyhedron $Q(p,q)$ is a Euclidean rectangle similar to one with side lengths 1 and a satisfying $1 \le a \le 7$.*

Proof A vertex V of $Q(p,q)$ is incident to two triangular faces A and A', and two white faces B and B' with $b, b' \ge 3$ vertices. Thinking in the Poincaré halfspace model, and after normalization, we can assume that one of the triangular faces, A, has vertices 0, 1 and ∞, with $V = \infty$. Thus, the vertex figure of V is a rectangle R with vertices 0, 1, ri and $1 + ri$, the faces B and B' are contained in the planes $x = 0$ and $x = 1$ respectively, the face A' is contained in the plane $y = r$ and the whole polyhedron is contained in the region bounded by these four planes; see Figure 8.

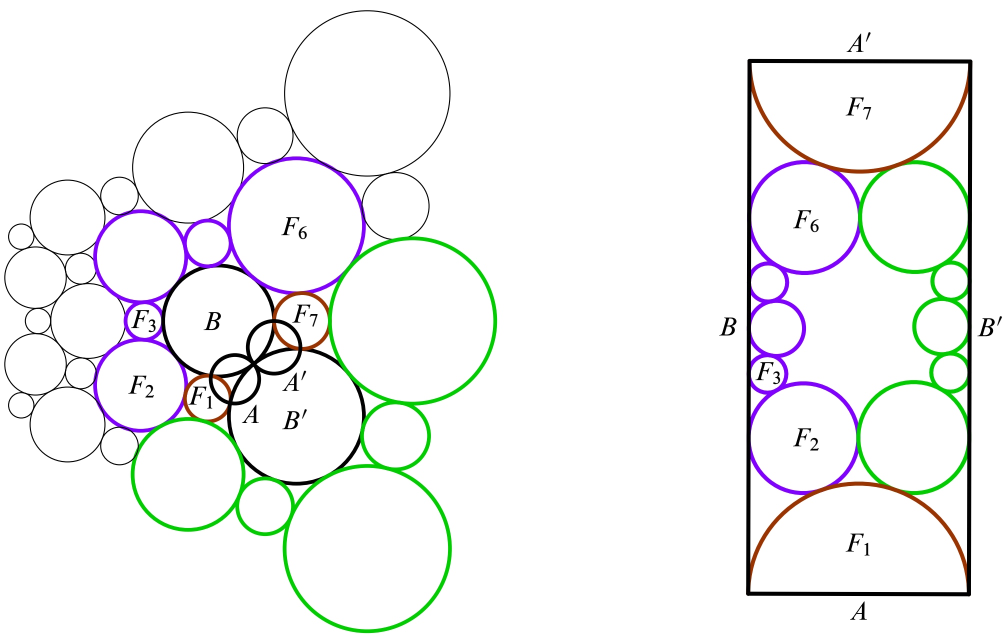

Figure 8: To the left, the four planes B, B', A and A' that meet the ideal vertex V at infinity shown in the circle packing. To the right, the polyhedron seen from the infinity: the planes B, B', A and A' cut a horosphere in a rectangle.

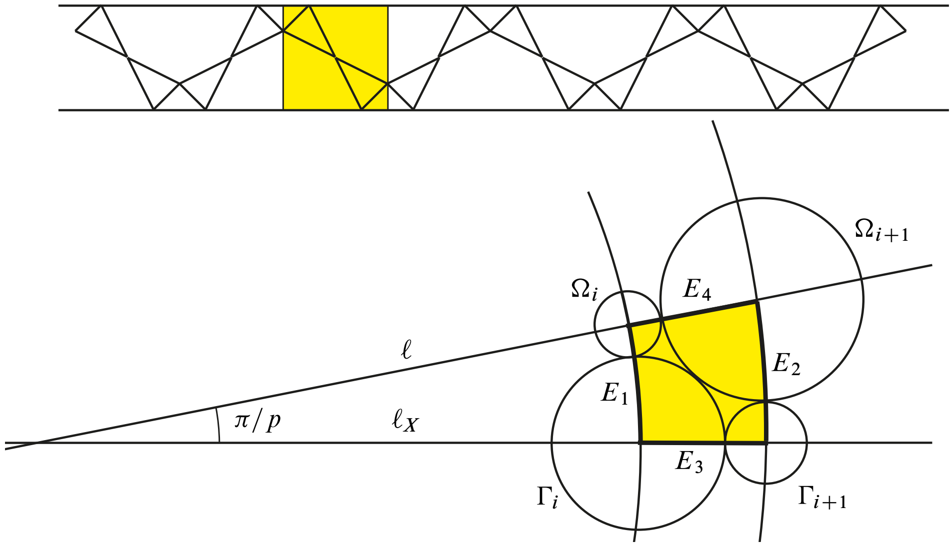

Figure 9: Central slice of $Q(p,q)$.

Let $V = V_0, V_1, \dots, V_k, V_{k+1} = V$ be the vertices of B cyclically ordered and let $F_1, F_2, \dots, F_k$ be the white faces different from B containing these vertices. Abusing notation, we also denote by F_i the circle containing F_i. Then the circle F_1 is tangent to the (unique) circle tangent to $x = 0$ and $x = 1$ and orthogonal to $y = 0$, and F_k is the circle tangent to $x = 0$ and $x = 1$ and orthogonal to $y = a$. Next, the circles $F_2, \dots, F_{k-1}$ are tangent to $x = 0$, disjoint from $x = 1$ and mutually tangent to its previous and to its next ones. Since the diameter of a circle contained in R is at most 1, we have that $a-1 \le k-1$, so $a \le k$. Taking into account that each face of $Q(p,q)$ has at most 8 vertices, we have the result. □

Remark Neumann and Zagier [1985] give an asymptotic estimation of the length of the core of the filled solid torus which is inversely proportional to the normalized length of the slope killed. So if the normalized length goes to infinity, then the length of the core of the filled solid torus tends to zero. Since in our case the normalized length of all the slopes to be killed are bounded above, by Proposition 6.1, we may hope that the homology spheres $\mathscr{H}(p,q)(n)$ have uniformly bounded below injectivity radius.

7 Geometric properties of $Q(p,q)$ and consequences

In this section we will show that the polyhedra $Q(p,q)$ contain big balls. The technical tool consists of cutting $Q(p,q)$ into slices and controlling their height.

The rotational symmetry of the polyhedron $Q(p,q)$ produces sections by planes orthogonal to the rotation axis and going through some of the vertices of the polyhedron. In particular, the planes containing the vertices corresponding to the small horizontal components cut $Q(p,q)$ into $2q$ slices: the top and the bottom ones, which are isometric by the horizontal reflection plane, and are combinatorially equivalent to a

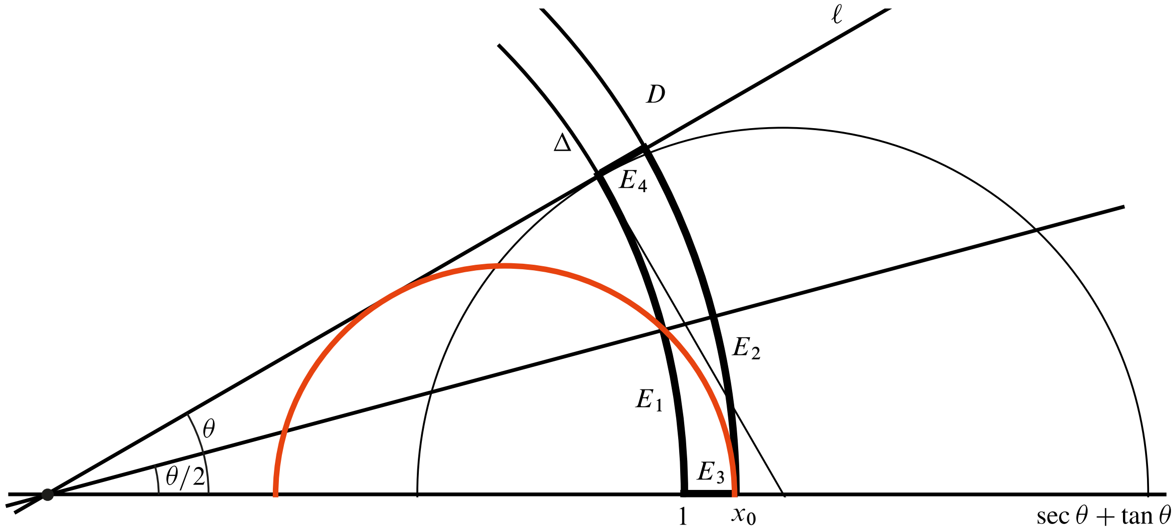

Figure 10: Proof of Lemma 7.1: finding the lower bound.

p–gon drum polyhedron $D(p)$; the remaining slices, which we call *central* are combinatorially equivalent to the polyhedron $S(p)$ shown in Figure 9. One should compare this figure with Figures 14 and 15, which represent the whole polyhedron $Q(p,q)$. Each slice is ideal, with dihedral p–gonal symmetry, and the dihedral angles between lateral faces are equal to $\pi/2$; furthermore, in the top and bottom slices the dihedral angles between one of the basis and the lateral faces are equal to $\pi/2$.

Lemma 7.1 (height of central slices) *For $p \geq 3$ let $S(p)$ be a central slice of $Q(p,q)$. Then its height h is bounded above and below by constants $0 < r_0 < r_1$ independent of q. Moreover, $r_1 \to 0$ as $p \to \infty$.*

Proof The polyhedron $S(p)$ is shown in Figure 9. We normalize so that the fundamental region, painted in yellow, is bounded by the unit circle Δ, a circle D of radius r, the X–axis ℓ_X and the line ℓ of slope $\theta = \pi/p$. These four curves determine a quadrilateral whose edges are denoted by E_1, E_2, E_3 and E_4. The white faces are given by the circles Ω_i, Γ_i, Ω_{i+1} and Γ_{i+1}, obtained as explained in the appendix. We remark that the circles Γ do not intersect ℓ, while the circles Ω do not intersect ℓ_X (see the appendix).

The height of $S(p)$ is the hyperbolic distance between the geodesics determined by Δ and D. Thus, it is enough to show that there are constants $1 < x_0 < x_1$ such that the radius r of D satisfies $x_0 < r < x_1$.

To find the lower bound, first notice that the circles centered at ℓ_X, tangent to ℓ and that intersect the edge E_1 are those whose rightmost intersection point $(t,0)$ with ℓ_X satisfy $1 < t < \sec\theta + \tan\theta$. Let $e^{i\alpha(t)}$ be the intersection point of these circles with E_1. As t decreases in the above interval, $\alpha(t)$ decreases from θ to 0. Consider x_0 such that $\alpha(x_0) = \theta/2$; see Figure 10. We claim that x_0 is a lower bound for the radius r of D. Indeed, if $r < x_0$, then any circle Γ intersecting the edge E_3 intersects the edge E_1 in a point $e^{i\alpha}$ with $\alpha < \theta/2$, while any circle Ω intersecting E_4 intersects E_1 in a point $e^{i\alpha'}$ with $\alpha' > \theta/2$. Thus Ω and Γ are not tangent at a point of E_1, which is a contradiction.

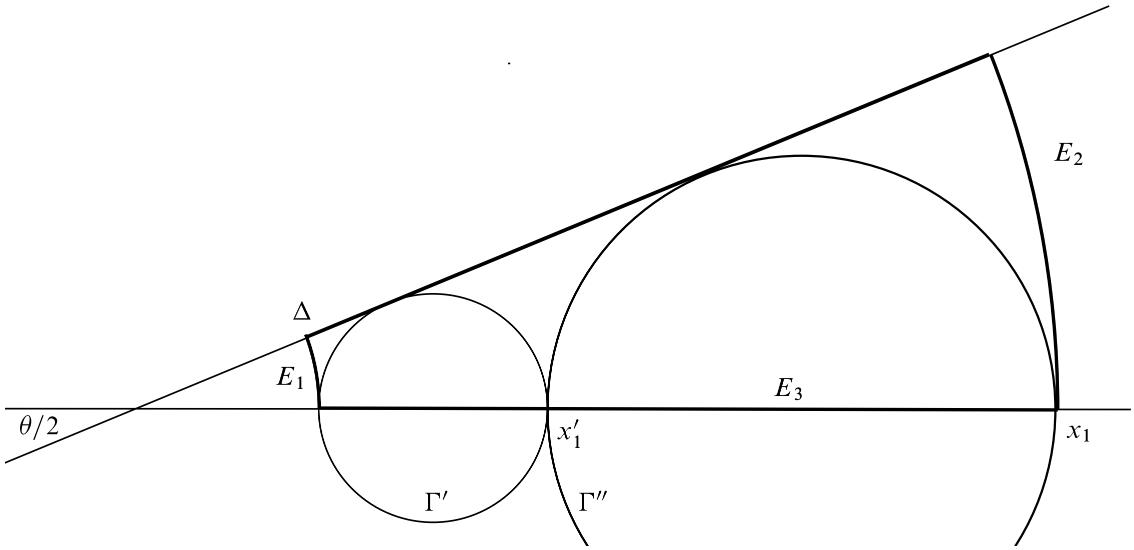

Figure 11: Proof of Lemma 7.1: finding the upper bound.

For the upper bound, consider the circle Γ' centered at ℓ_X and tangent to ℓ and Δ. Let $(x_1',0)$ be point in $\ell_X \cap \Gamma'$ furthest from the origin and let Γ'' be the circle centered at ℓ_X and tangent to ℓ and Γ' (see Figure 11). Let $(x_1,0)$ be the point in $\ell_X \cap \Gamma''$ furthest from the origin. We claim that x_1 is an upper bound for the radius r of D. Indeed, suppose $r > x_1$. Any circle Γ orthogonal to ℓ_X, disjoint from ℓ and intersecting the edge E_1 intersects E_3 at a point $(x,0)$ with $x < x_1'$, so is contained in the disc D' centered at the origin and of radius x_1'. On the other hand, any circle Ω orthogonal to ℓ, disjoint from ℓ_X and intersecting the edge E_2 is outside D'. Thus, Γ and Ω are disjoint, which is a contradiction.

Finally it is clear that as $p \to \infty$, $x_1 \to 1$ and so $r_1 \to 0$. □

Corollary 7.2 *For each p, the height of $Q(p,q)$ tends to infinity as $q \to \infty$.*

Proof The polyhedron $Q(p,q)$ contains $2q-2$ slices combinatorially equivalent to $S(p)$, and the common perpendicular to the two bases of $Q(p,q)$ contains the common perpendiculars to the two basis of any of these slices. Therefore, the height of $Q(p,q)$ is greater than $(2q-2)r_0$, by Lemma 7.1. □

Lemma 7.3 (height of drum slices) *For each p there is a constant $r_2(p) > 0$ such that the height of the top slice of any $Q(p,q)$ is bounded above by $r_2(p)$. Moreover, $r_2(p) \to 0$ as $p \to \infty$.*

Proof Recall that these are the top and bottom slices of the polyhedron $Q(p,q)$ and that one of the bases is orthogonal to the lateral faces. We normalize so that this basis is the unit circle Δ. By the symmetry, we can assume that two consecutive vertices in this basis are the points $P_1 = 1$ and $P_2 = e^{i\theta}$. Then the lateral face containing these vertices is given by the circle C orthogonal to Δ at P_1 and P_2. The other basis is given by a circle D centered at the origin and which intersects C, so it is furthest from Δ when

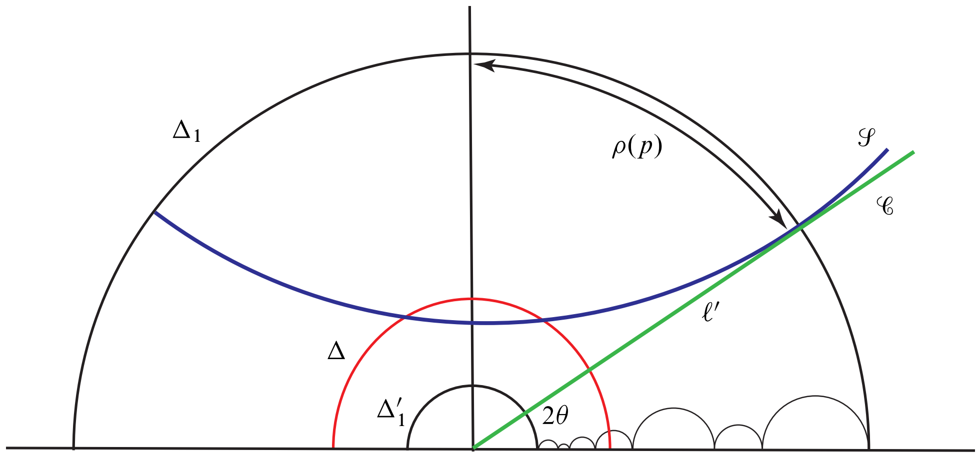

Figure 12

D is tangent to C. In other words, the height of this drum polyhedron is bounded above by the distance r_2 between Δ and the circle D' centered at the origin and tangent to C. By the construction it is clear that $r_2 \to 0$ as $p \to \infty$. □

Proposition 7.4 *For each $r > 0$ there are p and q such that the polyhedron $Q(p,q)$ contains a half-ball of radius r orthogonal to one of its bases.*

Proof Consider the circle packing determining $Q(p,q)$ as explained in the appendix, where Δ, Δ_1 and Δ_1' are the circles centered at the origin with respective radii 1, R and $1/R$, and let ℓ' be the line through the origin with slope $2\theta = 2\pi/p$.

In the upper half-space, let $\mathcal{C}$ be the euclidean cone generated by ℓ' and let $\mathcal{S}$ be the sphere tangent to $\mathcal{C}$ and orthogonal to the half-sphere over Δ_1 (see Figure 12). It is clear that the cone $\mathcal{C}$ does not intersect any of the hyperbolic planes bounded by the circles of the circle packing with the exception of Δ_1 and Δ_1' (Figure 12 shows a vertical section of it), so the same is true for $\mathcal{S}$.

When viewed in $\mathbb{H}^3$, the cone $\mathcal{C}$ is the surface equidistant to the z–axis at distance $\rho(p)$ equal to the hyperbolic distance between the z–axis and ℓ'. Clearly, as p goes to infinity, $\rho(p)$ also goes to infinity, independently of q. Notice that $\rho(p)$ is also the hyperbolic radius of $\mathcal{S}$.

Let $C = (0,0,c)$ be the lowest intersection point of $\mathcal{S}$ with the z–axis. Small computations give that

$$c = R\frac{1-\cos 2\theta}{\sin 2\theta}$$

and that

$$c \geq \frac{1}{R} \iff R^2 \geq \frac{\sin 2\theta}{1-\cos 2\theta}.$$

Thus, if the above inequality holds, $\mathcal{S}$ does not intersect Δ_1' and hence half of the ball bounded by $\mathcal{S}$ is contained in the polyhedron $Q(p,q)$.

To conclude, given r, we first take p big enough that $\rho(p) \geq r$. Once p is fixed, by Corollary 7.2, there is a q such that the height of $Q(p,q)$ is greater than

$$\sqrt{\frac{\sin 2\theta}{1-\cos 2\theta}}.$$

For these values of p and q, $Q(p,q)$ satisfies what is required. □

Corollary 7.5 *For each $r > 0$ there are p and q such that $\mathcal{M}(p,q)$ (resp. $\mathcal{H}(p,q)$ or $\mathcal{K}(p,q)$) contains an isometrically embedded ball of radius r. As a consequence, there is a sequence of hyperbolic link exteriors $\mathcal{M}(p,q)$ (resp. $\mathcal{H}(p,q)$ or $\mathcal{K}(p,q)$) converging geometrically to $\mathbb{H}^3$.*

Proof Since the half-ball contained in $Q(p,q)$ found in Proposition 7.4 is orthogonal to one of the bases, it is enough to glue two copies of $Q(p,q)$. Depending on the pattern used for gluing the triangular faces, the quotient manifold may be $\mathcal{M}(p,q)$, $\mathcal{H}(p,q)$ or $\mathcal{K}(p,q)$, among others. □

Theorem 7.6 (a) *There is a sequence of hyperbolic homology spheres $\mathcal{H}(p,q)(n)$ converging geometrically to $\mathbb{H}^3$.*

(b) *There is a sequence of hyperbolic homology spheres $\mathcal{K}(p,q,n)(m)$ converging geometrically to $\mathbb{H}^3$.*

Proof (a) By Corollary 7.5, there is a sequence of hyperbolic links $\mathcal{H}(p,q)$ converging geometrically to $\mathbb{H}^3$. By the hyperbolic Dehn surgery theorem, for each p and q, there is a sequence of closed hyperbolic manifolds $\mathcal{H}(p,q)(n)$ converging geometrically to $\mathcal{H}(p,q)$ as $n \to \infty$. By Theorem 4.1 we can chose those n such that $\mathcal{H}(p,q)(n)$ is a homology sphere. Thus, we obtain the result by a diagonal argument.

(b) The argument is similar, since $\mathcal{K}(p,q,n)(m)$ can be obtained by Dehn filling $\mathcal{K}(p,q)$. □

Corollary 7.7 *For each ϵ there is an N such that $|\mathrm{vol}(\mathcal{H}(p,q)(n)) - \mathrm{vol}(\mathcal{K}(p,q,n)(m))| < \epsilon$ for each $n, m > N$.*

Proof The exterior links $\mathcal{H}(p,q)$ and $\mathcal{K}(p,q)$ have the same volume V since they are the union of two copies of the same polyhedron $Q(p,q)$. Taking n and m sufficiently large, the volume of the Dehn filled manifolds $\mathcal{H}(p,q)(n)$ and $\mathcal{K}(p,q,n)(m)$ converge to the volume V of the links and so the result follows. □

The above corollary motivates the following question.

Question Can the manifold $\mathcal{H}(p,q)(n)$ be obtained as Dehn surgery on a knot in S^3?

Using the same argument of Theorem 7.6 we also obtain an explicit sequence of knots converging geometrically to $\mathbb{H}^3$, answering a question of Purcell and Souto [2010].

Theorem 7.8 *There is a sequence of hyperbolic knots $\mathcal{K}(p,q,n)$ which converge geometrically to $\mathbb{H}^3$.*

Proof By Corollary 7.5 there is a sequence of links $\mathcal{H}(p,q)$ converging geometrically to $\mathbb{H}^3$. For any p and q there is a sequence of knots $\mathcal{K}(p,q,n)$ converging to $\mathcal{K}(p,q)$. Thus, the result follows from a diagonal argument. □

8 Homology spheres with large genus

In the previous section we have proved that the exterior of the links $\mathcal{M}(p,q)$ contain big balls. Now we will prove that this property remains true after removing a maximal horoball neighborhood of the cusps — Lemma 8.1. Then using the 2π theorem [Bleiler and Hodgson 1996] we will obtain closed negatively curved manifolds containing big balls.

Lemma 8.1 *The half-balls contained in $Q(p,q)$ found in Proposition 7.4 are also contained in these polyhedra minus the maximal horoball neighborhoods described in Lemma 5.1.*

Proof We will prove that the maximal horoball neighborhood $\mathcal{N}$ described in Lemma 5.1 does not intersect the cone $\mathcal{C}$ defined in the proof of Proposition 7.4. Because of the geometric description of $\mathcal{N}$ (Lemma 5.1(i)), notice that this neighborhood is invariant under the symmetries of $Q(p,q)$.

We introduce the following terminology. If H is a horoball with center P, we define the *horizontal projection* of H as its vertical projection $p_h(H)$ over the plane $z=0$. We define the *vertical section* as the section with the vertical plane going through P and the origin $O=(0,0,0)$. We define the *amplitude* of a circle C with respect to a exterior point Q as the angle between the two tangents from Q to C. For a horoball, we define the *horizontal* and *vertical* amplitude as the amplitude of the horizontal projection or the vertical section with respect to the origin O. Basic trigonometry implies that the vertical amplitude of a horosphere is smaller than its horizontal amplitude.

Using this terminology, it is enough to show that the vertical amplitude of any horosphere of the maximal horoball neighborhood $\mathcal{N}$ is smaller than $2\pi/p$.

Let H be a horoball of $\mathcal{N}$ whose center is not in the x–axis ℓ_X nor in ℓ. By contradiction, suppose its horizontal projection intersects ℓ. By reflection on ℓ we obtain another horoball H' of $\mathcal{N}$, different to H, such that $p_h(H)\cap p_h(H')\neq\varnothing$. This implies that $H\cap H'\neq\varnothing$, since both horospheres have the same euclidean radius. Thus, $H\cap H'\cap Q(p,q)\neq\varnothing$, which is a contradiction. The argument is the same if $p_h(H)$ intersects ℓ_X or other symmetry line of the circle packing defining $Q(p,q)$. Thus, $p_h(H)$ is contained in a sector of angle π/p, which implies that the horizontal amplitude, and hence the vertical amplitude, of H is smaller than or equal to π/p.

Consider now H a horoball of $\mathcal{N}$ with center at ℓ_X and suppose that its horizontal projection intersects ℓ. Reflecting in ℓ we obtain another horoball H' of $\mathcal{N}$ such that $p_h(H)\cap p_h(H')\cap Q(p,q)\neq 0$, which is a contradiction. Then, $p_h(H)$ is contained in a sector of angle $2\pi/p$ and hence its horizontal and vertical amplitude are smaller than or equal to $2\pi/p$. □

Proposition 8.2 (a) *Let $|n| \geq 5$. For any $r > 0$, there exists a homology sphere $\mathcal{H}(p,q)(n)$ with sectional curvature less than or equal to -1 containing an isometrically embedded ball of radius greater than r.*

(b) *For any $r > 0$ there exists a homology sphere $\mathcal{H}(p,q,n)(m)$ with sectional curvature less than or equal to -1 containing an isometrically embedded ball of radius greater than r.*

Proof We prove (a); (b) is equal. By Lemma 8.1 there are p and q such that the hyperbolic link $\mathcal{H}(p,q)$ minus a maximal system $C_1,\dots,C_k$ of horoballs contains a ball B of radius greater than Ar for any arbitrarily chosen $A > 0$. The Dehn filling consists now on gluing negatively curved tori V_i to the boundary components of the horoball neighborhood removed. To guarantee that the sectional curvature is bounded above by -1 we need a little modification, as follows.

By the proof of Theorem 5.2, we can check that for $n \geq 5$ the minimal length, $\ell_{\min}$, of the slopes β_i in ∂C_i is greater than 9. Then, using [Futer et al. 2008, Theorem 2.1], we obtain that the Dehn filled manifold $\mathcal{H}(p,q)(n)$ admits a Riemannian metric for each $\zeta \in (0,1)$, which agrees with the hyperbolic metric in $\mathcal{H}(p,q)$ outside of C_i and with sectional curvatures bounded above by

$$\zeta\Bigl(\Bigl(\frac{2\pi}{\ell_{\min}}\Bigr)^2 - 1\Bigr) < \zeta\Bigl(\Bigl(\frac{2\pi}{9}\Bigr)^2 - 1\Bigr) = -\zeta\cdot 0.512 < 0.$$

Now we rescale the metric on $\mathcal{H}(p,q)(n)$ by multiplying by the factor $\lambda = -\zeta\bigl(\bigl(\frac{2\pi}{9}\bigr)^2 - 1\bigr)$, so that the lengths get multiplied by $\sqrt{\lambda}$ and the sectional curvatures get multiplied by $1/\lambda$. Thus, with this new metric, the homology sphere $\mathcal{H}(p,q)(n)$ has sectional curvature less than or equal to -1.

Finally, choosing A at the beginning of the proof to be equal to $1/\sqrt{\lambda}$, we obtain that the ball B embedded in $\mathcal{H}(p,q) \setminus \bigcup_i C_i$ has, in this new metric, radius greater than r. □

Corollary 8.3 *For any $n \geq 5$ and for any $g > 0$ there exists a hyperbolic homology sphere $\mathcal{H}(p,q)(n)$ with genus greater than or equal to g. For any $g > 0$ there exists a hyperbolic homology sphere $\mathcal{H}(p,q,n)(m)$ with genus greater than or equal to g*

Proof By Proposition 8.2, there exists a homology sphere $\mathcal{H}(p,q)(n)$ with sectional curvature less than or equal to -1 which contains an embedded ball of radius greater than $\operatorname{arccosh} 2g$. Then by Theorem 2.1, the genus of $\mathcal{H}(p,q)(n)$ is greater or equal to g. By the geometrization theorem, the manifold $\mathcal{H}(p,q)(n)$ is hyperbolic. □

Appendix Circle packing construction of $Q(p,q)$

It is a property of fully augmented links that their corresponding polyhedra can be constructed by means of circle packings [Purcell 2011]. Precisely, think of $Q(p,q)$ as placed in the Poincaré upper half-space. Remove the triangular faces and consider the planes containing the remaining faces. The boundaries of these planes enclose disjoint discs in $\mathbb{C}$ which are tangent if and only if the two faces share a vertex of $Q(p,q)$.

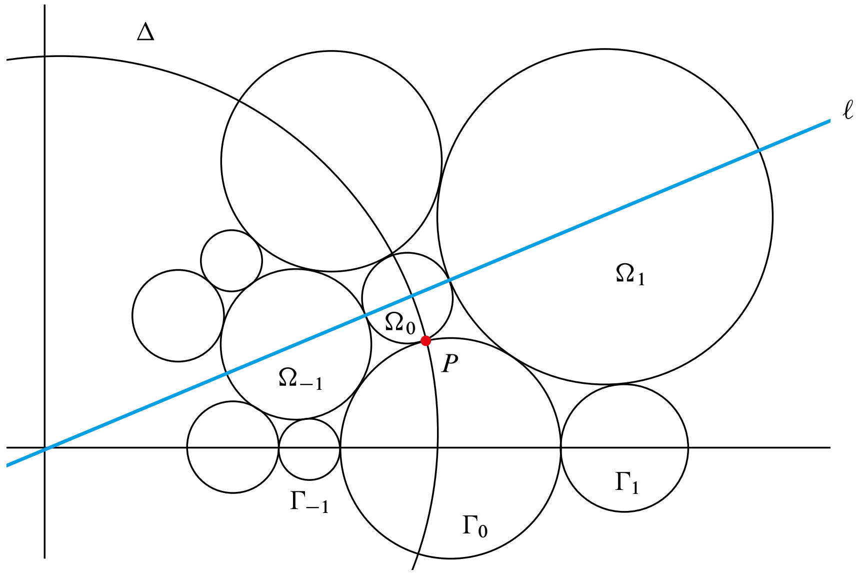

Figure 13: Iterative construction of the circle packing. The tangency point P in Δ of the starting circles Γ_0 and Ω_0 acts as a parameter of deformation of the configuration shown in the picture.

In this appendix we explain how to construct the circle packing corresponding to $Q(p,q)$. We normalize so that the reflection plane interchanging the bases is the (half-sphere over the) unit circle Δ. Then, the bases will be circles Δ_1 and Δ_1' centered at the origin with radius R and $1/R$ for some $R>1$ which will be determined in the construction. On the other hand, since $Q(p,q)$ has dihedral symmetry or order p, the circle packing constructed will have this symmetry. Then the x–axis and the line ℓ with slope $\theta=\pi/p$ are symmetry planes and a fundamental domain of the circle packing is given by the region enclosed by these two lines, and the circles Δ and Δ_1.

The construction of the fundamental domain is iterative; see Figure 13. The iterative step is as follows:

We have two mutually tangent circles Ω_i and Γ_i having their centers respectively at ℓ and the x–axis. Accordingly with the combinatorics of $Q(p,q)$, the next circle is tangent to Ω_i and Γ_i, and centered either at ℓ or the x–axis. Without loss of generality, suppose it is ℓ. Let P_i be the rightmost intersection point of Ω_i and ℓ and let Ω_{i+1} be the unique circle tangent to Ω_i at P_i and tangent to Γ_i. Notice that in order that the discs bounded by Ω_{i+1} and Ω_i be disjoint, some inequality involving the centers and radii of Ω_i and Γ_i, denoted by $c(\Omega_i)$, $r(\Omega_i)$, etc, must be satisfied. Precisely,

$$|c(\Omega_i)|+r(\Omega_i)<(|c(\Gamma_i)|+r(\Gamma_i))\cos\frac{\pi}{p}.$$

In the first step we take Ω_0 and Γ_0 orthogonal to Δ. Their tangency point $P=e^{it}$ is taken as a parameter of the construction, so that for each t we have a family $C(p,q)(t)$ of circles with the required tangency conditions. Nevertheless, it is not necessarily a circle packing because it is not guaranteed that the discs enclosed by the circles constructed are disjoint.

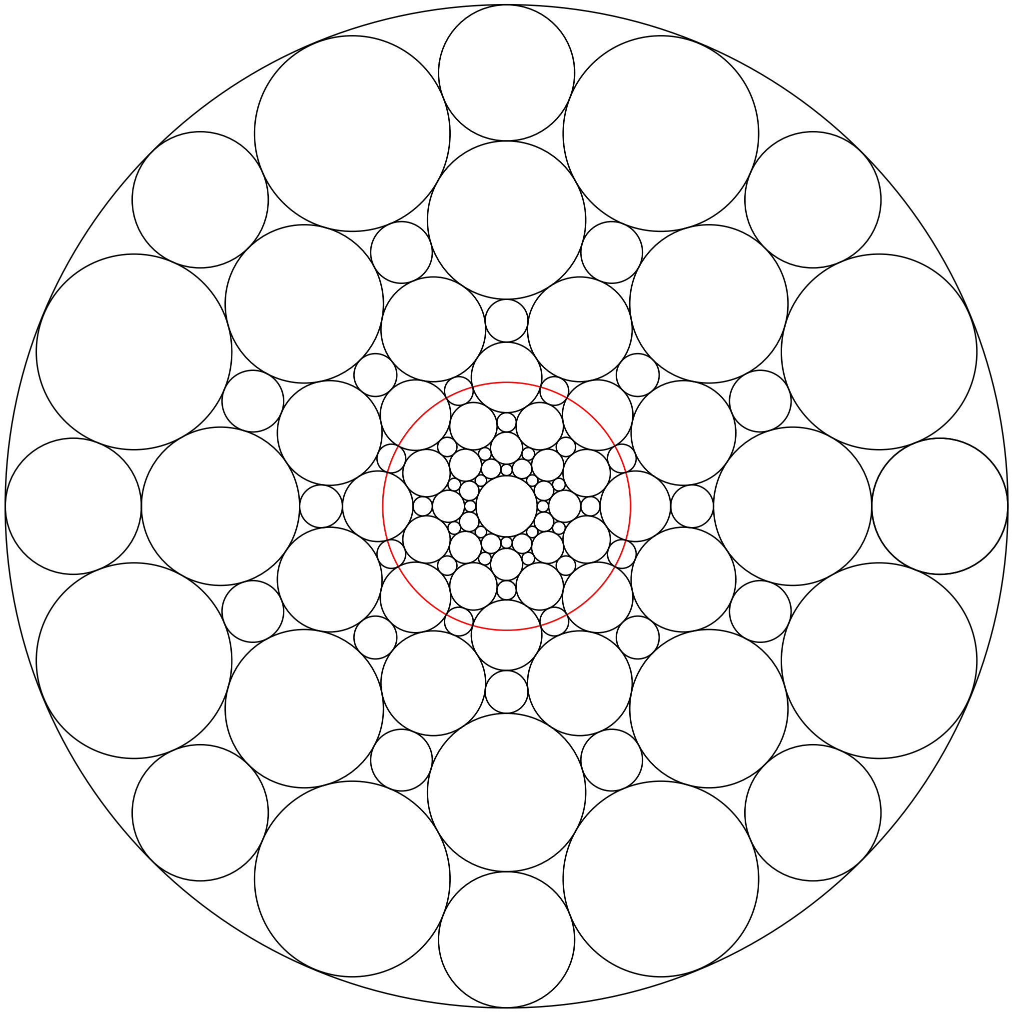

Figure 14: Circle packing corresponding to $Q(8,4)$. Circles inside the annulus represent white faces of the polyhedron. The circle in red is the boundary of a symmetry plane of the circle packing and thus of the polyhedron by reflection.

The polyhedron $Q(p,q)$ will be achieved when the parameter t satisfies

$$|c(\Omega_{q-1})| + r(\Omega_{q-1}) = \big(|c(\Gamma_{q-1})| + r(\Gamma_{q-1})\big).$$

Notice that the above value is the radius R of Δ_1.

The existence of the right value of t is guaranteed by the existence of $Q(p,q)$ (see Section 3.1). Alternatively, it can be shown by a careful analysis of the one parameter family $C(p,q)(t)$.

Remark A consequence of the invariance of the circle packing $Q(p,q)$ under the symmetries at the lines ℓ and x–axis, is that all the circles centered at ℓ are disjoint to the x–axis and vice versa. This is useful in Section 7.

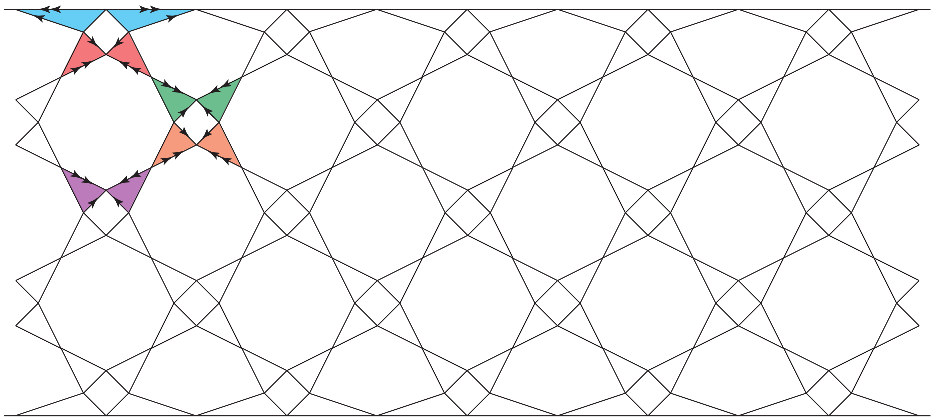

Figure 15: This is a more realistic view of the polyhedron $Q(5,3)$. Colored faces are to be identified following the pattern of Figure 3 to obtain the link $\mathcal{M}(5,3)$.

We end by showing the existence of the infinite polyhedra $Q(p,\infty)$.

Lemma A.1 *There is a value $t = t_\infty$ of the parameter described above for which the corresponding circle packing $C(p,\infty)$ is an infinite circle packing of the Euclidean plane minus the origin.*

Proof Notice that the slice $S(p)$ of $Q(p,q)$ defined in Section 7 still has some combinatorial symmetry, corresponding to involutions with axes going through vertices in the lateral faces. To construct $Q(p,\infty)$ it is enough to construct the fundamental region $R(p)$ of this new slice. The parameter t_∞ can be found by noticing that the tangent point of the circles Ω_1 and Γ_1 is in the bisector line of the sector $\widehat{\ell\ell_X}$. $\square$

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