

AG
T

*Algebraic & Geometric
Topology*

Volume 24 (2024)

Geography of bilinearized Legendrian contact homology

FRÉDÉRIC BOURGEOIS

DAMIEN GALANT

Geography of bilinearized Legendrian contact homology

FRÉDÉRIC BOURGEOIS

DAMIEN GALANT

We study the geography of bilinearized Legendrian contact homology for closed connected Legendrian submanifolds with vanishing Maslov class in 1–jet spaces. We show that this invariant detects whether the two augmentations used to define it are DGA homotopic or not. We describe a collection of graded vector spaces containing all possible values for bilinearized Legendrian contact homology and then show that all these vector spaces can be realized.

53D42, 57R17

1 Introduction

Let Λ be a closed Legendrian submanifold of the 1–jet space $J^1(M)$ of a manifold M . Given a generic complex structure for the canonical contact structure on $J^1(M)$, one can associate to Λ its Chekanov–Eliashberg differential graded algebra $(\mathcal{A}(\Lambda), \partial)$; see Chekanov [3] and Ekholm, Etnyre and Sullivan [7; 9]. The homology of $(\mathcal{A}(\Lambda), \partial)$, called Legendrian contact homology, is an invariant of the Legendrian isotopy class of Λ , but it is often hard to compute. It is therefore useful to consider augmentations of $(\mathcal{A}(\Lambda), \partial)$, because such an augmentation ε can be used to define a linearized complex $(C(\Lambda), \partial^\varepsilon)$. The homology is denoted by $\text{LCH}^\varepsilon(\Lambda)$ and called the linearized Legendrian contact homology of Λ with respect to ε . The collection of these homologies for all augmentations of $(\mathcal{A}(\Lambda), \partial)$ is also an invariant of the Legendrian isotopy class of Λ . The geography (i.e. the determination of all possible values) of a similar homological invariant defined using generating families was described by the first author with Sabloff and Traynor [2]. Using the work of Dimitroglou Rizell [4] on the effect of embedded surgeries on Legendrian contact homology, this geography can be shown to hold for linearized Legendrian contact homology as well. On the other hand, the first author and Chantraine [1] showed that it is possible to use two augmentations ε_1 and ε_2 of the Chekanov–Eliashberg DGA to define a bilinearized differential $\partial^{\varepsilon_1, \varepsilon_2}$ on $C(\Lambda)$. The corresponding homology is called bilinearized Legendrian contact homology and is denoted by $\text{LCH}^{\varepsilon_1, \varepsilon_2}(\Lambda)$. Our object is to describe the geography of bilinearized Legendrian contact homology. In other words, our goal is to describe a collection of Legendrian submanifolds equipped with two augmentations such that their bilinearized Legendrian contact homologies realize all possible values for this invariant.

When $\varepsilon_1 = \varepsilon_2$, bilinearized Legendrian contact homology coincides with linearized Legendrian contact homology. More generally, if the two augmentations are DGA homotopic, $\text{LCH}^{\varepsilon_1, \varepsilon_2}(\Lambda)$ is isomorphic to

$\text{LCH}^{\varepsilon_1}(\Lambda)$. Our first result describes a crucial difference in the behavior of bilinearized Legendrian contact homology depending whether the two augmentations are DGA homotopic or not. More precisely, this different behavior is detected by a map $\tau_n : \text{LCH}_n^{\varepsilon_1, \varepsilon_2}(\Lambda) \rightarrow H_n(\Lambda)$ appearing in the duality exact sequence for Legendrian contact homology (see Ekholm, Etnyre and Sabloff [6]) and described in Sections 2 and 3.

Theorem 1.1 *Let Λ be a closed connected Legendrian submanifold of $J^1(M)$ with $\dim M = n$. Let ε_1 and ε_2 be two augmentations of the Chekanov–Eliashberg DGA of Λ with coefficients in \mathbb{Z}_2 . Then ε_1 and ε_2 are DGA homotopic if and only if the map $\tau_n : \text{LCH}_n^{\varepsilon_1, \varepsilon_2}(\Lambda) \rightarrow H_n(\Lambda)$ is surjective.*

In other words, the fundamental class of Λ induces a class in linearized Legendrian contact homology, while the class of the point in Λ induces a class in bilinearized Legendrian contact homology with respect to non-DGA homotopic augmentations.

Corollary 1.2 *Bilinearized Legendrian contact homology is a complete invariant for DGA homotopy classes of augmentations of the Chekanov–Eliashberg DGA.*

The strength of this result will be illustrated in Section 3 by revisiting an important example of a Legendrian knot featuring only a partial study of its augmentations; see Melvin and Shrestha [14]. We complete the study of this Legendrian knot with a full description of its DGA homotopy classes of augmentations.

Our second result describes the geography of the Laurent polynomials that can be obtained as a Poincaré polynomial for bilinearized Legendrian contact homology. We will introduce in Definition 4.1 the explicit notion of a BLCH–admissible Laurent polynomial, and prove that only these polynomials can be obtained as the Poincaré polynomial of bilinearized Legendrian contact homology.

Theorem 1.3 *For any BLCH–admissible Laurent polynomial P , there exists a closed connected Legendrian submanifold Λ of $J^1(M)$ and there exist two non-DGA homotopic augmentations ε_1 and ε_2 of the Chekanov–Eliashberg DGA of Λ , with the property that the Poincaré polynomial of $\text{LCH}^{\varepsilon_1, \varepsilon_2}(\Lambda)$ with coefficients in \mathbb{Z}_2 is equal to P .*

We also will establish a similar result, namely Theorem 4.17, in the specific case of Legendrian spheres.

The collection of Poincaré polynomials that is realized by bilinearized Legendrian contact homology is considerably wider than the corresponding collection for the geography of linearized Legendrian contact homology [2, Theorem 1.1]. For this reason, the examples of Legendrian submanifolds that are constructed here in order to realize the geography of bilinearized Legendrian contact homology differ substantially from those considered in [2] and exhibit new interesting phenomena. In particular, while connected sums of Legendrian submanifolds played an important role in [2], such constructions cannot be used here because these tend to produce pairs of unwanted generators in bilinearized Legendrian contact homology. Moreover, we introduce a completely new construction in order to create pairs of generators in arbitrary degrees, instead of degrees summing to $n - 1$ as in linearized Legendrian contact homology. We also introduce another completely new construction in order to obtain bilinearized Legendrian contact

homologies of different ranks, depending on the ordering of the two non-DGA homotopic augmentations. Note that the examples we construct are convenient to work with, as they only have cusp singularities.

This paper is organized as follows. In [Section 2](#) we review the definition of bilinearized Legendrian contact homology and state its main properties. In [Section 3](#) we study fundamental classes in bilinearized Legendrian contact homology, prove [Theorem 1.1](#) and [Corollary 1.2](#) and study the effect of connected sums on bilinearized Legendrian contact homology. In [Section 4](#) we study the geography of bilinearized Legendrian contact homology and prove [Theorem 1.3](#) and its counterpart [Theorem 4.17](#) for Legendrian spheres.

Acknowledgments We are indebted to Josh Sabloff for providing us computer code that computes linearized Legendrian contact homology of Legendrian knots in \mathbb{R}^3 , using techniques of Henry and Rutherford [[11](#); [12](#)]. Although our exposition is independent from these sources, the generalization of this computer code by Galant to the calculation of bilinearized Legendrian contact homology played an essential role at the beginning of this work, before its generalization to higher dimensions. We thank Georgios Dimitroglou Rizell for a productive discussion of DGA homotopies of augmentations. An important refinement in the constructions from [Section 4](#) emerged after an interesting conversation with Sylvain Courte. Special thanks go to Filip Strakoš for spotting a mistake in the proof of [Proposition 4.2](#) impacting other parts of an earlier version of the paper. We also thank Cyril Falcon for his remarks on the original manuscript. Bourgeois was partially supported by the Institut Universitaire de France and by the ANR projects Quantact (16-CE40-0017), Microlocal (15-CE40-0007) and COSY (21-CE40-0002). Galant is an FRS-FNRS research fellow.

2 Bilinearized Legendrian contact homology

The 1-jet space $J^1(M) = T^*M \times \mathbb{R}$ of a smooth n -dimensional manifold M is equipped with a canonical contact structure $\xi = \ker(dz - \lambda)$, where λ is the Liouville 1-form on T^*M and z is the coordinate along \mathbb{R} . Let Λ be a closed Legendrian submanifold of this contact manifold, i.e. a closed embedded submanifold of dimension n such that $T_p\Lambda \subset \xi_p$ for any $p \in \Lambda$.

We first describe the definition of a differential graded algebra associated to Λ , following its construction by Ekholm, Etnyre and Sullivan [[7](#)]. The Reeb vector field associated to the contact form $\alpha = dz - \lambda$ for ξ is simply $R_\alpha = \partial/\partial z$. A Reeb chord of Λ is a finite nontrivial piece of integral curve for R_α with endpoints on Λ . After performing a Legendrian isotopy, we can assume that all Reeb chords of Λ are nondegenerate, i.e. the canonical projections to the tangent space of T^*M of the tangent spaces to Λ at the endpoints of each chord intersect transversally. Let us assume that the Maslov class $\mu(\Lambda)$ of Λ vanishes; see [[7](#), section 2.2].

We denote by $\mathcal{A}(\Lambda)$ the unital noncommutative algebra freely generated over \mathbb{Z}_2 by the Reeb chords of Λ . Each Reeb chord c is graded by its Conley–Zehnder $\nu(c) \in \mathbb{Z}$; when Λ is connected, this does not depend on any additional choice since $\mu(\Lambda) = 0$. The grading of c is defined as $|c| = \nu(c) - 1$. Hence, in this case, the algebra $\mathcal{A}(\Lambda)$ is naturally graded.

Let J be a complex structure on ξ which is compatible with its conformal symplectic structure. This complex structure naturally extends to an almost complex structure, which we still denote by J , on the symplectization $(\mathbb{R} \times J^1(M), d(e^t \alpha))$ by $J\partial/\partial t = R_\alpha$. We consider the moduli space $\tilde{\mathcal{M}}(a; b_1, \dots, b_k)$ of J -holomorphic disks in $\mathbb{R} \times J^1(M)$ with boundary on $\mathbb{R} \times \Lambda$ and with $k + 1$ punctures on the boundary that are asymptotic at the first puncture to the Reeb chord a at $t = +\infty$ and at the other punctures to the Reeb chords b_1, \dots, b_k at $t = -\infty$. For a generic choice of J , this moduli space is a smooth manifold of dimension $|a| - \sum_{i=1}^k |b_i|$; see [7, Proposition 2.2]. This moduli space carries a natural \mathbb{R} -action corresponding to the translation of J -holomorphic disks along the t -coordinate. When $\{b_1, \dots, b_k\} \neq \{a\}$, let us denote by $\mathcal{M}(a; b_1, \dots, b_k)$ the quotient of this moduli space by this free action.

We define a differential ∂ on $\mathcal{A}(\Lambda)$ by

$$\partial a = \sum_{\substack{b_1, \dots, b_k \\ \dim \mathcal{M}(a; b_1, \dots, b_k) = 0}} \#_2 \mathcal{M}(a; b_1, \dots, b_k) b_1 \cdots b_k,$$

where $\#_2 \mathcal{M}(a; b_1, \dots, b_k)$ is the number of elements in the corresponding moduli space, modulo 2. This differential has degree -1 and satisfies $\partial \circ \partial = 0$.

The resulting differential graded algebra $(\mathcal{A}(\Lambda), \partial)$ is called the Chekanov–Eliashberg DGA, and its homology is called Legendrian contact homology and denoted by $\text{LCH}(\Lambda)$. This graded algebra over \mathbb{Z}_2 depends only on the Legendrian isotopy class of Λ .

Let us now turn to the definition of a linearized version of Legendrian contact homology. An augmentation of $(\mathcal{A}(\Lambda), \partial)$ is a unital DGA map $\varepsilon: (\mathcal{A}(\Lambda), \partial) \rightarrow (\mathbb{Z}_2, 0)$. In other words, it is a choice of $\varepsilon(c) \in \mathbb{Z}_2$ for all Reeb chords c of Λ , it satisfies $\varepsilon(1) = 1$, it extends to $\mathcal{A}(\Lambda)$ multiplicatively and additively, and it satisfies $\varepsilon \circ \partial = 0$.

Such an augmentation can be used to define a linearization of $(\mathcal{A}(\Lambda), \partial)$. Let $C(\Lambda)$ be the vector space over \mathbb{Z}_2 freely generated by all Reeb chords of Λ . We also define the linearized differential ∂^ε on $C(\Lambda)$ by

$$\partial^\varepsilon a = \sum_{\substack{b_1, \dots, b_k \\ \dim \mathcal{M}(a; b_1, \dots, b_k) = 0}} \#_2 \mathcal{M}(a; b_1, \dots, b_k) \sum_{i=1}^k \varepsilon(b_1) \cdots \varepsilon(b_{i-1}) b_i \varepsilon(b_{i+1}) \cdots \varepsilon(b_k).$$

This differential has degree -1 and satisfies $\partial^\varepsilon \circ \partial^\varepsilon = 0$. The homology of the resulting linearized complex $(C(\Lambda), \partial^\varepsilon)$ is called linearized Legendrian contact homology (with respect to ε) and denoted by $\text{LCH}^\varepsilon(\Lambda)$. The collection of these graded modules over \mathbb{Z}_2 for all augmentations of Λ depends only on the Legendrian isotopy class of Λ .

Linearized Legendrian contact homology fits into a duality long exact sequence [6] together with its cohomological version $\text{LCH}_\varepsilon(\Lambda)$ and with the singular homology $H(\Lambda)$ of the underlying n -dimensional manifold Λ :

$$\cdots \rightarrow \text{LCH}_\varepsilon^{n-k-1}(\Lambda) \rightarrow \text{LCH}_\varepsilon^k(\Lambda) \xrightarrow{\tau_k} H_k(\Lambda) \rightarrow \text{LCH}_\varepsilon^{n-k}(\Lambda) \rightarrow \cdots.$$

Moreover, the map τ_n in the above exact sequence does not vanish. These properties induce constraints on the graded modules over \mathbb{Z}_2 that can be realized as the linearized Legendrian contact homology of some Legendrian submanifold, with respect to some augmentation. These constraints can be formulated in terms of the Poincaré polynomial of $\text{LCH}^\varepsilon(\Lambda)$, which is the Laurent polynomial defined by

$$P_{\Lambda,\varepsilon}(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} \text{LCH}_k^\varepsilon(\Lambda) t^k.$$

When Λ is connected, the duality exact sequence and the nonvanishing of τ_n imply that the above Poincaré polynomial has the form

$$(2-1) \quad P_{\Lambda,\varepsilon}(t) = q(t) + p(t) + t^{n-1} p(t^{-1}),$$

where q is a monic polynomial of degree n with integral coefficients (corresponding to the image of the maps τ_k) and p is a Laurent polynomial with integral coefficients (corresponding to the kernel of the maps τ_k). We shall say that a Laurent polynomial of this form is LCH–admissible.

The first author together with Sabloff and Traynor [2] studied generating family homology $\text{GH}(f)$, an invariant for isotopy classes of Legendrian submanifolds $\Lambda \subset (J^1(M), \xi)$ admitting a generating family f . This invariant is also a graded module over \mathbb{Z}_2 and satisfies the same duality exact sequence as above. In this study, the effect of Legendrian ambient surgeries on this invariant was determined and these operations were used to produce many interesting examples of Legendrian submanifolds admitting generating families. More precisely, for any LCH–admissible Laurent polynomial P , a connected Legendrian submanifold Λ_P admitting a generating family f_P realizing P as the Poincaré polynomial of $\text{GH}(f_P)$ was constructed using these operations. On the other hand, Dimitroglou Rizell [4] showed in particular that Legendrian ambient surgeries have the same effect as above on linearized Legendrian contact homology (for more details in the case of the connected sum, see the proof of Proposition 3.5). This result can be used step by step in the constructions of [2] to show that, for any LCH–admissible Laurent polynomial P , there exists an augmentation ε_P for Λ_P such that $\text{LCH}^{\varepsilon_P}(\Lambda_P) \cong \text{GH}(f_P)$. Therefore, the geography question for linearized Legendrian contact homology is completely determined by the above LCH–admissible Laurent polynomials.

Finally, we turn to a generalization of linearized LCH introduced by the first author together with Chantraine [1]. Using two augmentations ε_1 and ε_2 of $(\mathcal{A}(\Lambda), \partial)$, we can define another differential $\partial^{\varepsilon_1, \varepsilon_2}$ on $C(\Lambda)$, called the bilinearized differential:

$$\partial^{\varepsilon_1, \varepsilon_2} a = \sum_{\substack{b_1, \dots, b_k \\ \dim \mathcal{M}(a; b_1, \dots, b_k) = 0}} \#_2 \mathcal{M}(a; b_1, \dots, b_k) \sum_{i=1}^k \varepsilon_1(b_1) \cdots \varepsilon_1(b_{i-1}) b_i \varepsilon_2(b_{i+1}) \cdots \varepsilon_2(b_k).$$

As above, this differential has degree -1 and satisfies $\partial^{\varepsilon_1, \varepsilon_2} \circ \partial^{\varepsilon_1, \varepsilon_2} = 0$. The homology of the resulting bilinearized complex $(C(\Lambda), \partial^{\varepsilon_1, \varepsilon_2})$ is called bilinearized Legendrian contact homology (with respect to ε_1 and ε_2) and denoted by $\text{LCH}^{\varepsilon_1, \varepsilon_2}(\Lambda)$. The collection of these graded modules over \mathbb{Z}_2 for all pairs of augmentations of Λ depends only on the Legendrian isotopy class of Λ .

Bilinearized Legendrian contact homology also satisfies a duality exact sequence [1], but one has to take care of the ordering of the augmentations:

$$(2-2) \quad \cdots \rightarrow \text{LCH}_{\varepsilon_2, \varepsilon_1}^{n-k-1}(\Lambda) \rightarrow \text{LCH}_k^{\varepsilon_1, \varepsilon_2}(\Lambda) \xrightarrow{\tau_k} H_k(\Lambda) \xrightarrow{\sigma_{n-k}} \text{LCH}_{\varepsilon_2, \varepsilon_1}^{n-k}(\Lambda) \rightarrow \cdots.$$

Moreover, unlike in the linearized case, there exist [1, Section 5] connected Legendrian submanifolds Λ with augmentations ε_1 and ε_2 such that the map τ_n vanishes. Our goal here is to understand when the map τ_n vanishes, and to study the geography of the Poincaré polynomials

$$P_{\Lambda, \varepsilon_1, \varepsilon_2}(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} \text{LCH}_k^{\varepsilon_1, \varepsilon_2}(\Lambda) t^k$$

for bilinearized Legendrian contact homology.

3 Fundamental classes in bilinearized Legendrian contact homology

There are several notions of equivalence for augmentations of DGAs that were introduced in the literature and used in the context of the Chekanov–Eliashberg DGA. As the results of this section will show, it turns out that the equivalence relation among augmentations that best controls the behavior of bilinearized LCH is the notion of DGA homotopic augmentations [16, Definition 5.13]. Let ε_1 and ε_2 be two augmentations of the DGA (\mathcal{A}, ∂) over \mathbb{Z}_2 . Recall that a linear map $K: \mathcal{A} \rightarrow \mathbb{Z}_2$ is said to be an $(\varepsilon_1, \varepsilon_2)$ –derivation if $K(ab) = \varepsilon_1(a)K(b) + K(a)\varepsilon_2(b)$ for any $a, b \in \mathcal{A}$. We say that ε_1 is DGA homotopic to ε_2 , and we write $\varepsilon_1 \sim \varepsilon_2$, if there exists an $(\varepsilon_1, \varepsilon_2)$ –derivation $K: \mathcal{A} \rightarrow \mathbb{Z}_2$ of degree +1 such that $\varepsilon_1 - \varepsilon_2 = K \circ \partial$. It is a standard fact that DGA homotopy is an equivalence relation [10, Lemma 26.3].

Note that the defining condition for a DGA homotopy admits a beautiful and convenient reformulation in terms of the bilinearized complex.

Lemma 3.1 *Two augmentations ε_1 and ε_2 are DGA homotopic if and only if there exists a linear map $\bar{K}: C(\Lambda) \rightarrow \mathbb{Z}_2$ of degree +1 such that $\varepsilon_1 - \varepsilon_2 = \bar{K} \circ \partial^{\varepsilon_1, \varepsilon_2}$ on $C(\Lambda)$.*

Proof Suppose first that ε_1 is DGA homotopic to ε_2 . This implies in particular that $\varepsilon_1(c) - \varepsilon_2(c) = K \circ \partial c$ for any $c \in C(\Lambda)$. Since K is an $(\varepsilon_1, \varepsilon_2)$ –derivation, it directly follows from the definition of the bilinearized differential that $K \circ \partial c = K \circ \partial^{\varepsilon_1, \varepsilon_2} c$. It then suffices to take \bar{K} to be the restriction of K to $C(\Lambda)$.

Suppose now that there exists a linear map $\bar{K}: C(\Lambda) \rightarrow \mathbb{Z}_2$ of degree +1 such that $\varepsilon_1 - \varepsilon_2 = \bar{K} \circ \partial^{\varepsilon_1, \varepsilon_2}$ on $C(\Lambda)$. The map \bar{K} determines a unique $(\varepsilon_1, \varepsilon_2)$ –derivation $K: \mathcal{A} \rightarrow \mathbb{Z}_2$ via the relation

$$K(a_1 \cdots a_n) = \sum_{i=1}^n \varepsilon_1(a_1 \cdots a_{i-1}) \bar{K}(a_i) \varepsilon_2(a_{i+1} \cdots a_n)$$

for all $a_1, \dots, a_n \in \mathcal{A}$. As above, these maps satisfy $K \circ \partial c = \bar{K} \circ \partial^{\varepsilon_1, \varepsilon_2} c$, so that $\varepsilon_1 - \varepsilon_2 = \bar{K} \circ \partial$ on $C(\Lambda)$. Now observe that $\varepsilon_1(ab) - \varepsilon_2(ab) = \varepsilon_1(a)(\varepsilon_1(b) - \varepsilon_2(b)) + (\varepsilon_1(a) - \varepsilon_2(a))\varepsilon_2(b)$, and on the other hand $K \circ \partial(ab) = \varepsilon_1(\partial a)K(b) + \varepsilon_1(a)K(\partial b) + K(\partial a)\varepsilon_2(b) + K(a)\varepsilon_2(\partial b) = \varepsilon_1(a)K(\partial b) + K(\partial a)\varepsilon_2(b)$.

Hence if a and b satisfy the DGA homotopy relation, then ab satisfies it as well. Since this relation holds on $C(\Lambda)$, it follows that it is also satisfied on \mathcal{A} . \square

Note that, in the above proof, the extension of the linear map \bar{K} to a unique $(\varepsilon_1, \varepsilon_2)$ -derivation on \mathcal{A} as well as the extension of the homotopy relation from $C(\Lambda)$ to \mathcal{A} were first established in a more general setup by Kálmán in [13, Lemma 2.18].

With this suitable notion of equivalence for augmentations, we can now turn to the study of the fundamental class in bilinearized LCH, via the maps τ_0 and τ_n from the duality long exact sequence. The following proposition generalizes [6, Theorem 5.5].

Proposition 3.2 *Let ε_1 and ε_2 be augmentations of the Chekanov–Eliashberg DGA (\mathcal{A}, ∂) of a closed connected n -dimensional Legendrian submanifold Λ in $(J^1(M), \xi)$. The map $\tau_0: \text{LCH}_0^{\varepsilon_1, \varepsilon_2}(\Lambda) \rightarrow H_0(\Lambda)$ from the duality long exact sequence vanishes if and only if ε_1 and ε_2 are DGA homotopic.*

Proof Let f be a Morse function on Λ with a unique minimum at point m , and g be a Riemannian metric on Λ . Since the stable manifold of m is open and dense in Λ , for a generic choice of the Morse–Smale pair (f, g) , the endpoints of all Reeb chords of Λ are in this stable manifold. The vector space $H_0(\Lambda)$ is generated by m and we identify it with \mathbb{Z}_2 . By the results of [6], the map τ_0 counts rigid J -holomorphic disks with boundary on Λ , with a positive puncture on the boundary and with a marked point on the boundary mapping to the stable manifold of m . This disk can have extra negative punctures on the boundary; these are augmented by ε_1 if they sit between the positive puncture and the marked point, and by ε_2 if they sit between the marked point and the positive puncture. Since mapping to m is an open condition on Λ , such rigid configurations can only occur when the image of the disk boundary is discrete in Λ . In other words, the holomorphic disk maps to the symplectization of a Reeb chord c of Λ . Since there is a unique positive puncture, this map is not a covering, and there is a unique negative puncture at c . There is a unique such J -holomorphic disk for any chord c of Λ . The marked point maps to the starting point or to the ending point of the chord c in Λ . If the marked point maps to the starting point of c , the negative puncture sits between the positive puncture and the marked point on the boundary of the disk, which therefore contributes $\varepsilon_2(c)$ to $\tau_0(c)$ at chain level. If the marked point maps to the ending point of c , the negative puncture sits between the marked point and the positive puncture on the boundary of the disk, which therefore contributes $\varepsilon_1(c)$ to $\tau_0(c)$. We conclude that the map τ_0 is given at chain level by $\varepsilon_1 - \varepsilon_2$.

If ε_1 and ε_2 are DGA homotopic, then by Lemma 3.1 the map τ_0 is nullhomotopic and therefore vanishes in homology. On the other hand, if ε_1 and ε_2 are not DGA homotopic, Lemma 3.1 implies that the map $\varepsilon_1 - \varepsilon_2: C_0(\Lambda) \rightarrow \mathbb{Z}_2$ does not factor through the bilinearized differential $\partial^{\varepsilon_1, \varepsilon_2}$. In other words, there exists $a \in C_0(\Lambda)$ such that $\partial^{\varepsilon_1, \varepsilon_2} a = 0$ but $\varepsilon_1(a) - \varepsilon_2(a) \neq 0$. But then the homology class $[a] \in \text{LCH}_0^{\varepsilon_1, \varepsilon_2}(\Lambda)$ satisfies $\tau_0([a]) \neq 0$, so that τ_0 does not vanish in homology. \square

We are now in position to prove our first main result.

Proof of Theorem 1.1 In the duality long exact sequence (2-2) for bilinearized LCH, the maps τ_k and σ_k are adjoint in the sense of [6, Proposition 3.9] as in the linearized case. The proof of this fact is essentially identical in the bilinearized case: the holomorphic disks counted by τ_k are still in bijective correspondence with those counted by σ_k . In the bilinearized case, it is also necessary to use the fact that the extra negative punctures on corresponding disks are augmented with the same augmentations in order to reach the conclusion.

In particular, τ_n vanishes if and only if σ_n vanishes. Since $H_0(\Lambda) \cong \mathbb{Z}_2$, the exactness of the duality sequence (2-2) implies that σ_n vanishes if and only if τ_0 does not vanish. By Proposition 3.2, this means that τ_n vanishes if and only if the augmentations ε_1 and ε_2 are not DGA homotopic. \square

This difference in the behavior of bilinearized LCH can be used to determine DGA homotopy classes of augmentations. More precisely, the next proposition shows that bilinearized LCH provides an explicit criterion to decide whether two augmentations are DGA homotopic or not.

Proposition 3.3 *Let ε_1 and ε_2 be augmentations of the Chekanov–Eliashberg DGA (\mathcal{A}, ∂) of a closed connected n -dimensional Legendrian submanifold Λ in $(J^1(M), \xi)$. Then*

$$\dim_{\mathbb{Z}_2} \text{LCH}_n^{\varepsilon_2, \varepsilon_1}(\Lambda) - \dim_{\mathbb{Z}_2} \text{LCH}_{-1}^{\varepsilon_1, \varepsilon_2}(\Lambda) = \begin{cases} 0 & \text{if } \varepsilon_1 \sim \varepsilon_2, \\ 1 & \text{if } \varepsilon_1 \not\sim \varepsilon_2. \end{cases}$$

Proof By the duality exact sequence (2-2), we have

$$H_0(\Lambda) \cong \mathbb{Z}_2 \xrightarrow{\sigma_n} \text{LCH}_{\varepsilon_2, \varepsilon_1}^n(\Lambda) \rightarrow \text{LCH}_{-1}^{\varepsilon_1, \varepsilon_2}(\Lambda) \rightarrow H_{-1}(\Lambda) = 0.$$

In other words, $\text{LCH}_{\varepsilon_2, \varepsilon_1}^n(\Lambda) / \text{im } \sigma_n \cong \text{LCH}_{-1}^{\varepsilon_1, \varepsilon_2}(\Lambda)$. Taking into account that

$$\dim_{\mathbb{Z}_2} \text{LCH}_{\varepsilon_2, \varepsilon_1}^n(\Lambda) = \dim_{\mathbb{Z}_2} \text{LCH}_n^{\varepsilon_2, \varepsilon_1}(\Lambda),$$

we obtain the desired result since, as in the proof of Theorem 1.1, the rank of σ_n is 1 when $\varepsilon_1 \sim \varepsilon_2$ and vanishes when $\varepsilon_1 \not\sim \varepsilon_2$. \square

Corollary 1.2 follows immediately from the above proposition.

Example 3.4 Let us consider the Legendrian knot K_2 studied by Melvin and Shrestha in [14, Section 3], which is topologically the mirror image of the knot 8_{21} , and illustrated in Figure 1.

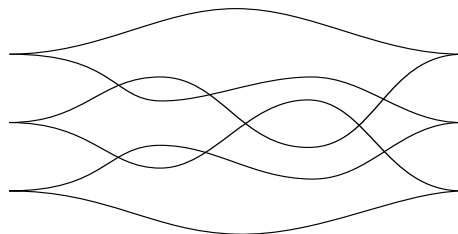


Figure 1: Front projection of the Legendrian knot K_2 .

It is shown in [14, Section 3] that the Chekanov–Eliashberg DGA of this Legendrian knot K_2 has exactly 16 augmentations, which split into a set A of 4 augmentations and a set B of 12 augmentations, such that $P_{K_2,\varepsilon}(t) = 2t + 4 + t^{-1}$ if $\varepsilon \in A$ and $P_{K_2,\varepsilon}(t) = t + 2$ if $\varepsilon \in B$. This implies that augmentations in A are not DGA homotopic to augmentations in B . However, the number of DGA homotopy classes of augmentations for K_2 was not determined in [14], as linearized LCH does not suffice to obtain this information.

Using Proposition 3.3, these DGA homotopy classes can be determined systematically. It turns out that the augmentations in A are pairwise not DGA homotopic, because the Poincaré polynomial of any such pair of augmentations is $t + 3 + t^{-1}$. On the other hand, the set B splits into six DGA homotopy classes $\mathcal{C}_1, \dots, \mathcal{C}_6$ of augmentations. The BLCH Poincaré polynomials are given by $t + 2$ for two DGA homotopic augmentations in B , by 1 for two non-DGA homotopic augmentations both in $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ or in $\mathcal{C}_4 \cup \mathcal{C}_5 \cup \mathcal{C}_6$, and by $t + 2$ and $2 + t^{-1}$ otherwise.

These calculations are straightforward but tedious. Suitable Python code gives the above answer instantly.

We conclude our study of the fundamental classes in bilinearized LCH with a useful description of their behavior when performing a connected sum. To this end, it is convenient to introduce some additional notation about the map τ_n in the duality exact sequence (2-2). Its target space $H_n(\Lambda)$ is spanned by the fundamental classes $[\Lambda_i]$ of the connected components Λ_i of the Legendrian submanifold Λ . We can therefore decompose τ_n as $\sum_i \tau_{n,i}[\Lambda_i]$, where the maps $\tau_{n,i}$ take their values in \mathbb{Z}_2 .

Proposition 3.5 *Let Λ be a Legendrian link in $J^1(M)$ equipped with two augmentations ε_1 and ε_2 . Let $\bar{\Lambda}$ be the Legendrian submanifold obtained by performing a connected sum between two connected components Λ_1 and Λ_2 of Λ , and let $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$ be the augmentations induced by ε_1 and ε_2 .*

If the map $\tau_{n,1} - \tau_{n,2}$ constructed from the map τ_n in the duality exact sequence (2-2) vanishes, then $P_{\bar{\Lambda},\bar{\varepsilon}_1,\bar{\varepsilon}_2}(t) = P_{\Lambda,\varepsilon_1,\varepsilon_2}(t) + t^{n-1}$. Otherwise, $P_{\bar{\Lambda},\bar{\varepsilon}_1,\bar{\varepsilon}_2}(t) = P_{\Lambda,\varepsilon_1,\varepsilon_2}(t) - t^n$.

Proof As explained in [1, Section 3.2.5], the map τ_n in the duality exact sequence (2-2) for Λ counts holomorphic disks in the symplectization of $J^1(M)$ with boundary on the symplectization of Λ , having a positive puncture asymptotic to a chord c of Λ and a marked point on the boundary mapped to a fixed generic point p_j of a connected component Λ_j of Λ . This disk can also carry negative punctures on the boundary; let us say that those located between the positive puncture and the chord (with respect to the natural orientation of the boundary) are asymptotic to chords c_1^-, \dots, c_r^- , while those between the marked point and the positive puncture are asymptotic to $c_{r+1}^-, \dots, c_{r+s}^-$. Let us denote by $\mathcal{M}(c; c_1^-, \dots, c_r^-, p_j, c_{r+1}^-, \dots, c_{r+s}^-)$ the moduli space of such holomorphic disks, modulo translation in the \mathbb{R} direction of the symplectization. The map τ_n is then given by

$$\tau_n(c) = \sum_j \#_2 \mathcal{M}(c; c_1^-, \dots, c_r^-, p_j, c_{r+1}^-, \dots, c_{r+s}^-) \varepsilon_1(c_1^-) \cdots \varepsilon_1(c_r^-) \varepsilon_2(c_{r+1}^-) \cdots \varepsilon_2(c_{r+s}^-) [\Lambda_j].$$

On the other hand, the effect of a connected sum on bilinearized LCH can be deduced from the results of Dimitroglou Rizell on the full Chekanov–Eliashberg DGA [4, Theorem 1.6]. There is an isomorphism

of DGAs $\Psi: (\mathcal{A}(\bar{\Lambda}), \partial_{\bar{\Lambda}}) \rightarrow (\mathcal{A}(\Lambda; S), \partial_S)$ between the Chekanov–Eliashberg DGA of $\bar{\Lambda}$ and the DGA $(\mathcal{A}(\Lambda; S), \partial_S)$ generated by the Reeb chords of Λ as well as a formal generator s of degree $n - 1$, equipped with a differential ∂_S satisfying in particular $\partial_S s = 0$. In this notation, S stands for the pair of points $\{p_1 \in \Lambda_1, p_2 \in \Lambda_2\}$ in a neighborhood of which the connected sum is performed. Any augmentation ε of the Chekanov–Eliashberg DGA of Λ can be extended to an augmentation of $(\mathcal{A}(\Lambda; S), \partial_S)$ by setting $\varepsilon(s) = 0$. Moreover, the pullback $\Psi^* \varepsilon$ of this extension to the Chekanov–Eliashberg DGA of $\bar{\Lambda}$ coincides with the augmentation induced on $\bar{\Lambda}$ from the original augmentation ε for Λ via the surgery Lagrangian cobordism between $\bar{\Lambda}$ and Λ . In particular, we have $\bar{\varepsilon}_1 = \Psi^* \varepsilon_1$ and $\bar{\varepsilon}_2 = \Psi^* \varepsilon_2$. Applying the bilinearization procedure to the map Ψ , we obtain a chain complex isomorphism $\Psi^{\varepsilon_1, \varepsilon_2}$ between the bilinearized chain complex for $\bar{\Lambda}$ and the chain complex $(C(\Lambda, S), \partial_S^{\varepsilon_1, \varepsilon_2})$ generated by Reeb chords of Λ and the formal generator s . Since $\partial_S^{\varepsilon_1, \varepsilon_2} s = 0$, the line spanned by s forms a subcomplex of $(C(\Lambda, S), \partial_S^{\varepsilon_1, \varepsilon_2})$. Moreover, the quotient complex is exactly the bilinearized chain complex for Λ . We therefore obtain a long exact sequence in homology

$$\dots \rightarrow \text{LCH}_k^{\bar{\varepsilon}_1, \bar{\varepsilon}_2}(\bar{\Lambda}) \rightarrow \text{LCH}_k^{\varepsilon_1, \varepsilon_2}(\Lambda) \xrightarrow{\rho_k} \mathbb{Z}_2[s]_{k-1} \rightarrow \text{LCH}_{k-1}^{\bar{\varepsilon}_1, \bar{\varepsilon}_2}(\bar{\Lambda}) \rightarrow \dots$$

that corresponds to the long exact sequence obtained in [2, Theorem 2.1] for generating family homology. This exact sequence implies that bilinearized LCH remains unchanged by a connected sum, except possibly in degrees $n - 1$ and n . The map ρ_n is the part of the bilinearized differential $\partial_S^{\varepsilon_1, \varepsilon_2}$ from the bilinearized complex for Λ to the line spanned by s . According to the definition [4, Section 1.1.3] of ∂_S and the above description of τ_n , this map is given by $\rho_n = (\tau_{n,1} - \tau_{n,2})s$.

If $\rho_n = 0$, the generator s injects into $\text{LCH}_{n-1}^{\bar{\varepsilon}_1, \bar{\varepsilon}_2}(\bar{\Lambda})$, resulting in an exact term t^{n-1} in the Poincaré polynomial. If $\rho_n \neq 0$, the map $\text{LCH}_n^{\bar{\varepsilon}_1, \bar{\varepsilon}_2}(\bar{\Lambda}) \rightarrow \text{LCH}_n^{\varepsilon_1, \varepsilon_2}(\Lambda)$ has a 1-dimensional cokernel, resulting in the loss of a term t^n in the Poincaré polynomial. □

4 Geography of bilinearized Legendrian contact homology

In this section we study the possible values for the Poincaré polynomial $P_{\Lambda, \varepsilon_1, \varepsilon_2}$ of the bilinearized LCH for a closed connected Legendrian submanifold Λ in $J^1(M)$ with $\dim M = n$, equipped with two augmentations ε_1 and ε_2 of its Chekanov–Eliashberg DGA.

When $\varepsilon_1 = \varepsilon_2$, this geography question was completely answered in [2] for generating family homology. As explained in Section 2, this result extends to linearized LCH via the work of Dimitroglou Rizell [4]. Moreover, bilinearized LCH is invariant under changes of augmentations within their DGA homotopy classes [16, Section 5.3]. Therefore, the geography of bilinearized LCH is already known when $\varepsilon_1 \sim \varepsilon_2$.

4.1 Basic properties of BLCH Poincaré polynomials

We now turn to the case $\varepsilon_1 \sim \varepsilon_2$, and describe the possible Poincaré polynomials for bilinearized LCH.

Definition 4.1 A BLCH–admissible polynomial is the data of a Laurent polynomial P with nonnegative integral coefficients together with a splitting $P = q + p$ involving two Laurent polynomials with nonnegative integral coefficients p and q such that

- (i) q is a polynomial of degree at most $n - 1$ with $q(0) = 1$, and
- (ii) $p(-1)$ is even if $n = 1$ and $p(-1) \leq \frac{1}{2}(1 - q(-1))$ if $n = 2$.

We first show that the Poincaré polynomial of bilinearized LCH always has this form.

Proposition 4.2 Let ε_1 and ε_2 be augmentations of the Chekanov–Eliashberg DGA (\mathcal{A}, ∂) of a closed connected n –dimensional Legendrian submanifold Λ with vanishing Maslov class in $(J^1(M), \xi)$. If ε_1 and ε_2 are not DGA homotopic, then the Poincaré polynomial $P_{\Lambda, \varepsilon_1, \varepsilon_2}$ corresponding to $\text{LCH}^{\varepsilon_1, \varepsilon_2}(\Lambda)$ is BLCH–admissible.

Proof Considering the map τ_k from the duality exact sequence (2-2), we have $\dim_{\mathbb{Z}_2} \text{LCH}_k^{\varepsilon_1, \varepsilon_2}(\Lambda) = \dim_{\mathbb{Z}_2} \ker \tau_k + \dim_{\mathbb{Z}_2} \text{im } \tau_k$. Let p and q be the Poincaré polynomials constructed using the terms in the right-hand side of this relation: $p(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} \ker \tau_k t^k$ and $q(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} \text{im } \tau_k t^k$. This provides the desired splitting $P_{\Lambda, \varepsilon_1, \varepsilon_2} = q + p$.

Let us prove (i). Since $\text{im } \tau_k \subset H_k(\Lambda)$, q is a polynomial of degree at most n . By Proposition 3.2, since $\varepsilon_1 \sim \varepsilon_2$, $\text{im } \tau_0 \neq 0$. But $H_0(\Lambda) = \mathbb{Z}_2$ as Λ is connected, so that $q(0) = 1$. On the other hand, by Theorem 1.1, since $\varepsilon_1 \sim \varepsilon_2$ we have that $\tau_n = 0$. Therefore the term of degree n in q vanishes and q is a polynomial of degree at most $n - 1$.

Let us now prove (ii). Assume first that n is odd. Since the generators of the chain complex $C(\Lambda)$ do not depend on the augmentations, the Euler characteristic $P_{\Lambda, \varepsilon_1, \varepsilon_2}(-1)$ does not depend on the augmentations either. Equation (2-1) then implies that $P_{\Lambda, \varepsilon_1, \varepsilon_2}(-1)$ has the same parity as $\frac{1}{2} \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} H_k(\Lambda)$, since $(-1)^{n-1} = 1$ when n is odd. If $n = 1$, then condition (i) sets $q(t) = 1$ so that $q(-1) = 1$ while $\frac{1}{2} \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} H_k(\Lambda) = 1$. By subtraction, we deduce that $p(-1)$ must be even. Note that if $n \geq 3$, this does not impose any condition on $p(-1)$ since $q(-1)$ can take arbitrary integer values.

Assume now that n is even. By [8, Proposition 3.3], the Thurston–Bennequin invariant of Λ is given by $\text{tb}(\Lambda) = (-1)^{(n-1)(n-2)/2} P_{\Lambda, \varepsilon_1, \varepsilon_2}(-1)$. On the other hand, $\text{tb}(\Lambda) = (-1)^{n/2+1} \frac{1}{2} \mathcal{X}(\Lambda)$ when n is even by [8, Proposition 3.2]. Hence $P_{\Lambda, \varepsilon_1, \varepsilon_2}(-1) = \frac{1}{2} \mathcal{X}(\Lambda)$. When $n = 2$, we have that $\frac{1}{2} \mathcal{X}(\Lambda) = \frac{1}{2}(1 - \dim_{\mathbb{Z}_2} H_1(\Lambda) + 1) \leq \frac{1}{2}(1 + q(-1))$. By subtraction, we get that $p(-1) \leq \frac{1}{2}(1 - q(-1))$. Note that if $n \geq 4$, this does not impose any condition on $p(-1)$ since $\frac{1}{2} \mathcal{X}(\Lambda)$ can take arbitrary integer values. \square

Remark 4.3 If we restrict ourselves to Legendrian spheres Λ , the Laurent polynomials $P = q + p$ that can arise as the Poincaré polynomial of bilinearized LCH can also be characterized. More precisely, revisiting the proof of Proposition 4.2 shows that in this case q and p satisfy the more restrictive conditions

- (i') $q(t) = 1$, and
- (ii') $p(-1)$ is even if n is odd and $p(-1) = 0$ if n is even.

The duality exact sequence imposes fewer restrictions on $\text{LCH}^{\varepsilon_1, \varepsilon_2}(\Lambda)$ than in the case of linearized LCH because it mainly relates this invariant to $\text{LCH}^{\varepsilon_2, \varepsilon_1}(\Lambda)$ with exchanged augmentations. This fact, however, means that one of these invariants determines the other one. In order to formulate this more precisely, let us consider the duality exact sequence obtained from (2-2) after reversing the ordering of the augmentations:

$$(4-1) \quad \cdots \rightarrow \text{LCH}_{\varepsilon_1, \varepsilon_2}^{n-k-1}(\Lambda) \rightarrow \text{LCH}_k^{\varepsilon_2, \varepsilon_1}(\Lambda) \xrightarrow{\tilde{\tau}_k} H_k(\Lambda) \xrightarrow{\tilde{\sigma}_{n-k}} \text{LCH}_{\varepsilon_1, \varepsilon_2}^{n-k}(\Lambda) \rightarrow \cdots.$$

In the next proposition, we denote by $P_\Lambda(t)$ the Poincaré polynomial for the singular homology of Λ with coefficients in \mathbb{Z}_2 .

Proposition 4.4 *Let ε_1 and ε_2 be non-DGA homotopic augmentations of the Chekanov–Eliashberg DGA (\mathcal{A}, ∂) of a closed connected n -dimensional Legendrian submanifold Λ with vanishing Maslov class in $(J^1(M), \xi)$. If $P_{\Lambda, \varepsilon_1, \varepsilon_2}$ decomposes as $q + p$ as in Definition 4.1, then $P_{\Lambda, \varepsilon_2, \varepsilon_1}$ decomposes as $\tilde{q} + \tilde{p}$ with $\tilde{q}(t) = P_\Lambda(t) - t^n q(t^{-1})$ and $\tilde{p}(t) = t^{n-1} p(t^{-1})$.*

Proof Let us decompose $P_{\Lambda, \varepsilon_2, \varepsilon_1}(t) = \tilde{q}(t) + \tilde{p}(t)$ as in Definition 4.1. The polynomial p was defined as $p(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} \ker \tau_k t^k$ in the proof of Proposition 4.2. But $\ker \tau_k$ is the image of the map $\text{LCH}_{\varepsilon_2, \varepsilon_1}^{n-k-1}(\Lambda) \rightarrow \text{LCH}_k^{\varepsilon_1, \varepsilon_2}(\Lambda)$, which is isomorphic to a supplementary subspace of $\text{im } \sigma_{n-k-1}$ in $\text{LCH}_{\varepsilon_2, \varepsilon_1}^{n-k-1}(\Lambda)$. Since σ_{n-k-1} is the adjoint in the sense of [6, Proposition 3.9] of the map $\tilde{\tau}_{n-k-1} : \text{LCH}_{n-k-1}^{\varepsilon_2, \varepsilon_1}(\Lambda) \rightarrow H_{n-k-1}(\Lambda)$, the spaces $\ker \tau_k$ and $\ker \tilde{\tau}_{n-k-1}$ are isomorphic. Therefore, the polynomial \tilde{p} is given by $\tilde{p}(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} \ker \tau_k t^{n-k-1} = t^{n-1} p(t^{-1})$.

On the other hand, we have $\tilde{q}(t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{Z}_2} \text{im } \tilde{\tau}_k t^k$ as in the proof of Proposition 4.2. But $\text{im } \tilde{\tau}_k = \ker \tilde{\sigma}_{n-k}$ and since τ_{n-k} is the adjoint in the sense of [6, Proposition 3.9] of the map τ_{n-k} , we have that $\ker \tilde{\sigma}_{n-k}$ is isomorphic to a supplementary subspace of $\text{im } \tau_{n-k}$ in $H_{n-k}(\Lambda)$. Hence

$$\tilde{q}(t) = \sum_{k \in \mathbb{Z}} (\dim_{\mathbb{Z}_2} H_{n-k}(\Lambda) - \dim_{\mathbb{Z}_2} \text{im } \tau_{n-k}) t^k = P_\Lambda(t) - t^n q(t^{-1}),$$

as announced. □

Note that, since the data of $P_{\Lambda, \varepsilon_1, \varepsilon_2}$ and $P_{\Lambda, \varepsilon_2, \varepsilon_1}$ determine P_Λ , the question of finding Λ , ε_1 and ε_2 with prescribed polynomials $P_{\Lambda, \varepsilon_1, \varepsilon_2}$ and $P_{\Lambda, \varepsilon_2, \varepsilon_1}$ is more complicated than our geography question. We will not address this more complicated question.

4.2 Motivating example

We now describe a fundamental example in view of the construction of Legendrian submanifolds and augmentations realizing BLCH–admissible polynomials.

Example 4.5 With $n = 1$, consider the right-handed trefoil knot Λ with maximal Thurston–Bennequin invariant, depicted in its front projection in Figure 2. The same Legendrian knot was studied in Section 5.1 of [1]. We consider it this time in the front projection, after applying Ng’s resolution procedure [15].

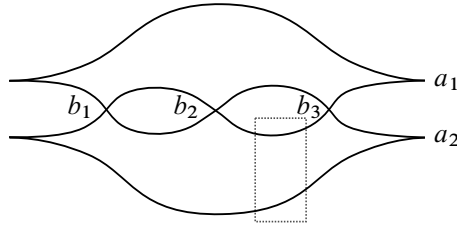


Figure 2: Front projection of the maximal tb right-handed trefoil.

The Chekanov–Eliashberg DGA has five generators: a_1 and a_2 correspond to right cusps and have grading 1, while b_1, b_2 and b_3 correspond to crossings and have grading 0. The differential is given by

$$\partial a_1 = 1 + b_1 + b_3 + b_1 b_2 b_3 \quad \text{and} \quad \partial a_2 = 1 + b_1 + b_3 + b_3 b_2 b_1.$$

This DGA admits 5 augmentations $\varepsilon_1, \dots, \varepsilon_5$ given by

	b_1	b_2	b_3
ε_1	1	1	1
ε_2	1	0	0
ε_3	1	1	0
ε_4	0	0	1
ε_5	0	1	1

A straightforward calculation shows that $P_{\Lambda, \varepsilon_i, \varepsilon_j}(t) = 1$ for all $i \neq j$. In view of [Definition 4.1](#) and [Proposition 4.2](#), this is the simplest Poincaré polynomial that can be obtained using bilinearized LCH.

In order to produce other terms in this Poincaré polynomial, let us replace the portion of Λ contained in the dotted rectangle in [Figure 2](#) by the fragment represented in [Figure 3](#). This produces a Legendrian link Λ' .

The additional generator a_3 corresponds to a right cusp and has grading 1. The four mixed chords between the unknot and the trefoil have a grading that depends on a shift $k \in \mathbb{Z}$ between the Maslov potentials of the trefoil and of the unknot. These gradings are given by

$$|c_1| = k - 1, \quad |c_2| = k, \quad |d_1| = 1 - k \quad \text{and} \quad |d_2| = -k.$$

The augmentations $\varepsilon_1, \dots, \varepsilon_5$ can be extended to this enlarged DGA by sending all new generators to 0. The bilinearized differential of the original generators is therefore unchanged. The differential of the new generators is, on the other hand, given by

$$\partial c_1 = 0, \quad \partial c_2 = (1 + b_2 b_1) c_1, \quad \partial d_1 = d_2 (1 + b_2 b_1), \quad \partial d_2 = 0 \quad \text{and} \quad \partial a_3 = d_1 c_1 + d_2 c_2.$$

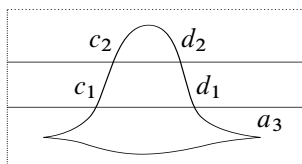


Figure 3: Replacement for the dotted rectangle in [Figure 2](#).

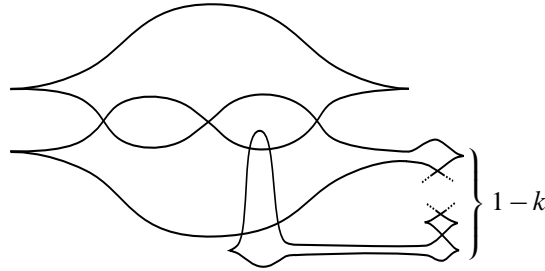


Figure 4: Front projection of the Legendrian knot Λ'' .

If we choose $\varepsilon_L = \varepsilon_1$ or ε_3 and $\varepsilon_R = \varepsilon_2, \varepsilon_4$ or ε_5 , then the bilinearized differential is

$$\partial^{\varepsilon_L, \varepsilon_R} c_1 = 0, \quad \partial^{\varepsilon_L, \varepsilon_R} c_2 = 0, \quad \partial^{\varepsilon_L, \varepsilon_R} d_1 = d_2, \quad \partial^{\varepsilon_L, \varepsilon_R} d_2 = 0 \quad \text{and} \quad \partial^{\varepsilon_L, \varepsilon_R} a_3 = 0.$$

The Poincaré polynomial of the resulting homology is therefore $P_{\Lambda', \varepsilon_L, \varepsilon_R}(t) = t^k + t^{k-1} + t + 1$. We now perform a connected sum between the right cusps corresponding to a_2 and a_3 in order to obtain the connected Legendrian submanifold Λ'' represented by Figure 4. A Legendrian isotopy involving a number of first Reidemeister moves is performed before the connected sum in order to ensure that the Maslov potentials agree on the cusps to be merged. This connected sum induces a Lagrangian cobordism L from Λ'' to Λ' , and we can use this cobordism to pull back the augmentations ε_L and ε_R to the Chekanov–Eliashberg DGA of Λ'' .

By Proposition 3.5, since $[a_3] \in \text{LCH}_1^{\varepsilon_L, \varepsilon_R}(\Lambda')$ corresponds to the fundamental class of the Legendrian unknot depicted in Figure 4, we obtain the Poincaré polynomial $P_{\Lambda'', \varepsilon_L, \varepsilon_R}(t) = t^k + t^{k-1} + 1$. This corresponds to $q(t) = 1$ and $p(t) = t^k + t^{k-1}$ in Definition 4.1.

4.3 A family of Legendrian spheres with a basic BLCH Poincaré polynomial

In order to generalize Example 4.5 to higher dimensions, let us consider the standard Legendrian Hopf link, or in other words the 2-copy of the standard Legendrian unknot $\Lambda^{(2)} \subset J^1(\mathbb{R}^n)$. This will lead to a generalization of the trefoil knot from Figure 2, since it can be obtained from the standard Legendrian Hopf link in \mathbb{R}^3 via a connected sum. Let us denote by l the length of the unique Reeb chord of the standard Legendrian unknot and by ε the positive shift (much smaller than l) in the Reeb direction between the two components Λ_1 and Λ_2 of $\Lambda^{(2)}$. We assume that the top component is perturbed by a Morse function of amplitude δ much smaller than ε with exactly one maximum M and one minimum m . In particular, among the continuum of Reeb chords of length ε between the two components, only two chords corresponding to these extrema persist after perturbation. We also assume that thanks to this perturbation, all Reeb chords of $\Lambda^{(2)}$ lie above distinct points of \mathbb{R}^n . In order to define the grading of mixed Reeb chords in this link, we choose the Maslov potential of the upper component Λ_2 to be given by the Maslov potential of the lower component Λ_1 plus k .

Proposition 4.6 *The Chekanov–Eliashberg DGA of $\Lambda^{(2)} \subset J^1(\mathbb{R}^n)$ has the following six generators:*

	grading	length
c_{11}	n	l
c_{22}	n	l
c_{12}	$n + k$	$l + \varepsilon$
c_{21}	$n - k$	$l - \varepsilon$
m_{12}	$k - 1$	$\varepsilon - \delta$
M_{12}	$n + k - 1$	$\varepsilon + \delta$

Its differential is given by

$$\partial c_{12} = M_{12} + m_{12}c_{11} + c_{22}m_{12}, \quad \partial c_{11} = c_{21}m_{12} \quad \text{and} \quad \partial c_{22} = m_{12}c_{21},$$

and $\partial M_{12} = \partial m_{12} = \partial c_{21} = 0$.

Proof The front projection of each component in $\Lambda^{(2)}$ consists of two sheets, having parallel tangent hyperplanes above a single point of \mathbb{R}^n before the perturbation by the Morse function. The number of Reeb chords above that point is the number of pairs of sheets, which is $\frac{1}{2}4(4 - 1) = 6$. The chords between the two highest or the two lowest sheets belong to a continuum of chords of length ε between the two components, which is replaced by two chords M_{12} for the maximum M and m_{12} for the minimum m after the perturbation by the Morse function. Their lengths are therefore $\varepsilon \pm \delta$. Their gradings are given by the Morse index of the corresponding critical point plus the difference of Maslov potentials minus one, so that we obtain $n + k - 1$ and $k - 1$.

The four other chords will be denoted by c_{ij} , where i numbers the component of origin for the chord and j numbers the component of the endpoint of the chord. Each of these chords corresponds to a maximum of the local difference function between the heights of the sheets it joins. We therefore obtain the announced gradings and lengths.

The link $\Lambda^{(2)}$ and its Reeb chords determine a quiver represented in Figure 5, in which each component of the link corresponds to a vertex and each Reeb chord corresponds to an oriented edge. When computing the differential of a generator, the terms to be considered correspond to paths formed by a sequence of edges in this quiver with the same origin and endpoint as the generator, with total grading one less than the grading of the generator and with total length strictly smaller than the length of the generator.

For ∂c_{12} , the only possible terms are M_{12} , $m_{12}c_{11}$ and $c_{22}m_{12}$. Such terms cannot contain c_{21} because two other chords from Λ_1 to Λ_2 would be needed as well. The resulting total length would be smaller than

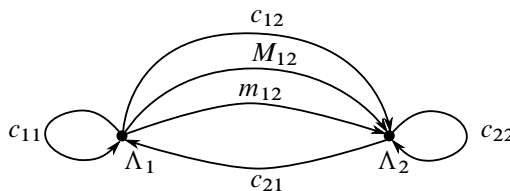


Figure 5: Quiver corresponding to the standard Hopf link.

the length of c_{12} only in the case of $m_{12}c_{21}m_{12}$, but this term is of grading 2 lower than c_{12} . The generators c_{11} and c_{22} can appear at most once due to their length, and due to total length constraint, only m_{12} can appear (only once) as a factor, leading to the possibilities $m_{12}c_{11}$ and $c_{22}m_{12}$. Finally, if M_{12} appears, then no other chord can appear as a factor by the previous discussion, leading to the possibility M_{12} .

Let us show that each possible term in ∂c_{12} is realized by exactly one Morse flow tree [5], which in turn corresponds to a unique holomorphic curve. To obtain M_{12} , we start at the chord c_{12} and follow the negative gradient of the local height difference function in the unique direction leading to the chord M_{12} . At this chord, we have a 2-valent puncture of the Morse flow tree and we continue by following the negative gradient of the local height difference function corresponding to one of the components Λ_1 or Λ_2 (depending on which hemisphere the maximum M is located on). This gradient trajectory will generically not hit any other Reeb chord and will finally hit the cusp equator of that component, which is the end of the Morse flow tree. To obtain $m_{12}c_{11}$, we start at the chord c_{12} and follow the negative gradient of the local height difference function in the unique direction leading to the chord c_{11} . At this chord, we have a 2-valent puncture of the Morse flow tree and we continue by following the negative gradient of the local height difference function corresponding to the highest two sheets, which is the Morse function used to perturb the Hopf link. Generically, this gradient trajectory will reach the minimum m so that we obtain a 1-valent puncture of the Morse flow tree at m_{12} . The term $c_{22}m_{12}$ is obtained similarly.

For ∂c_{11} , the only possible term is $c_{21}m_{12}$. Indeed, when $n > 1$, the chord c_{21} is the only one available to start an admissible path from Λ_1 to itself, because the empty path is not admissible. When $n = 1$, the empty path is admissible but there are two holomorphic disks having c_{11} as a positive puncture and no negative puncture, which cancel each other. Due to its length, the only chord we can still use is m_{12} , and after this no other chord can be added. Let us show that this possible term for ∂c_{11} is realized by exactly one Morse flow tree. We start at the chord c_{11} and follow the negative gradient of the local height difference function in the unique direction leading to the chord c_{21} . At this chord, we have a 2-valent puncture of the Morse flow tree and we continue by following the negative gradient of the local height difference function corresponding to the lowest two sheets, which is the Morse function used to perturb the Hopf link. Generically, this gradient trajectory will reach the minimum m so that we obtain a 1-valent puncture of the Morse flow tree at m_{12} . The calculation of ∂c_{22} is analogous.

For ∂c_{12} , there are no possible terms because no other chord can lead from Λ_1 to Λ_2 . For ∂M_{12} , the only chord which is short enough to appear is m_{12} but its grading $k - 1$ is strictly smaller when $n > 1$ than the necessary grading $n + k - 2$. When $n = 1$, there are two gradient trajectories from the maximum to the minimum of a Morse function on the circle, which cancel each other. Finally, $\partial m_{12} = 0$ because it is the shortest chord and it joins different components. \square

Corollary 4.7 *If $k = 1$, the Chekanov–Eliashberg DGA of $\Lambda^{(2)} \subset J^1(\mathbb{R}^n)$ has two augmentations ε_L and ε_R such that $\varepsilon_L(m_{12}) = 0$ and $\varepsilon_R(m_{12}) = 1$, and that vanish on the other Reeb chords. When $n > 1$, there are no other augmentations.*

Proof When $n > 1$, m_{12} is the only generator of degree 0, so that the maps ε_L and ε_R are the only two degree-preserving algebra morphisms $\mathcal{A} \rightarrow \mathbb{Z}_2$. In order to show that these are augmentations, we need to check that $1, m_{12} \notin \text{im } \partial$. This follows from the fact that there is no term 1 and that m_{12} always appears as a factor of another generator in the expression of ∂ in Proposition 4.6. \square

The above augmentations ε_L and ε_R can be used in order to obtain a bilinearized differential associated to the differential from Proposition 4.6. We obtain $\partial^{\varepsilon_L, \varepsilon_R} c_{12} = M_{12} + c_{22}$ and $\partial^{\varepsilon_L, \varepsilon_R} c_{11} = c_{21}$, while the differential of the other four generators vanishes. The corresponding homology is therefore generated by $[M_{12}] = -[c_{22}]$ in degree n and by $[m_{12}]$ in degree 0. Hence, the Poincaré polynomial $P_{\Lambda^{(2)}, \varepsilon_L, \varepsilon_R}(t)$ is given by $1 + t^n$.

After this preliminary calculation, let us consider a combination of several such links in view of obtaining more general Poincaré polynomials than those in Example 4.5. To this end, consider the $2N$ -copy of the standard Legendrian unknot $\Lambda^{(2N)} \subset J^1(\mathbb{R}^n)$ for $N \geq 1$. This link contains the components $\Lambda_1, \dots, \Lambda_{2N}$ numbered from bottom to top. If l denotes the length of the unique Reeb chord of Λ_i and ε denotes the positive shift between any two consecutive components, we require that $2N\varepsilon$ is much smaller than l . We perturb the component Λ_i for $i = 2, \dots, 2N$ by a Morse function f_i with two critical points and of amplitude δ much smaller than ε such that all differences $f_i - f_j$ with $i \neq j$ are Morse functions with two critical points. In order to define the gradings of mixed Reeb chords in this link, we choose the Maslov potential of the component Λ_i to be given by the Maslov potential of the lowest component Λ_1 plus $i - 1$. A direct application of Proposition 4.6 to each pair of components Λ_i and Λ_j gives the chords of $\Lambda^{(2N)}$:

	grading	length
$c_{i,i}$	n	l
$c_{i,j}$	$n + j - i$	$l + \varepsilon(j - i)$
$c_{j,i}$	$n - j + i$	$l - \varepsilon(j - i)$
$m_{i,j}$	$j - i - 1$	$\varepsilon(j - i) - \delta$
$M_{i,j}$	$n + j - i - 1$	$\varepsilon(j - i) + \delta$

Here the indices i and j take all possible values between 1 and $2N$ such that $i < j$.

Proposition 4.8 *The algebra morphisms ε_L and ε_R defined by $\varepsilon_L(m_{i,i+1}) = 1$ when i is even, $\varepsilon_R(m_{i,i+1}) = 1$ when i is odd and that vanish on all other chords are augmentations of the Chekanov–Eliashberg DGA of $\Lambda^{(2N)}$.*

Proof Let us show that $m_{i,i+1} \notin \text{im } \partial$ for all $i = 1, \dots, 2N - 1$. If $m_{i,i+1}$ was a term in ∂a for some a in the Chekanov–Eliashberg of $\Lambda^{(2N)}$, then a would have to be a linear combination of chords from Λ_i to Λ_{i+1} . Indeed, ∂c does not contain the term 1 for any chord c of $\Lambda^{(2N)}$, say from Λ_i to Λ_j , because it would give rise to a term 1 in Proposition 4.6 for the Legendrian Hopf link composed of Λ_i and Λ_j . Therefore ∂ does not decrease the number of factors in terms it acts on. Since a must be a single chord from Λ_i to Λ_{i+1} , if there were a term $m_{i,i+1}$ in ∂a , then there would already be such a term in Proposition 4.6 for the Legendrian Hopf link composed of Λ_i and Λ_{i+1} . Hence $m_{i,i+1} \notin \text{im } \partial$, as announced.

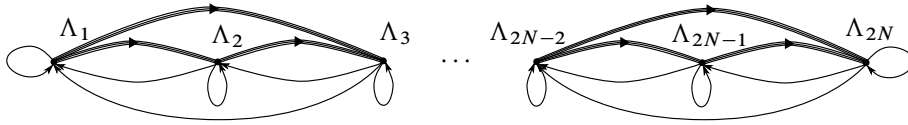


Figure 6: Quiver corresponding to the $2N$ -copy of the standard Legendrian unknot.

This implies that ε_L and ε_R are augmentations, because any element of $\text{im } \partial$ is composed of monomials having at least one factor which is not of the form $m_{i,i+1}$, and in particular not augmented, so that ε_L and ε_R vanish on $\text{im } \partial$. \square

Proposition 4.9 *The bilinearized differential $\partial^{\varepsilon_L, \varepsilon_R}$ of $\Lambda^{(2N)}$ is given by*

$$\begin{aligned} \partial^{\varepsilon_L, \varepsilon_R} c_{i,i} &= \bar{i} c_{i,i-1} + \bar{i} c_{i+1,i}, & \partial^{\varepsilon_L, \varepsilon_R} c_{i,j} &= M_{i,j} + \bar{j} c_{i,j-1} + \bar{i} c_{i+1,j}, \\ \partial^{\varepsilon_L, \varepsilon_R} c_{j,i} &= \bar{i} c_{j,i-1} + \bar{j} c_{j+1,i}, & \partial^{\varepsilon_L, \varepsilon_R} m_{i,j} &= \bar{j} m_{i,j-1} + \bar{i} m_{i+1,j}, \\ \partial^{\varepsilon_L, \varepsilon_R} M_{i,j} &= \bar{j} M_{i,j-1} + \bar{i} M_{i+1,j}, \end{aligned}$$

with $i < j$ and where \bar{i} and \bar{j} are the modulo-2 reductions of i and j . In the above formulas, any generator with one of its indices equal to 0 or $2N + 1$ or of the form $m_{i,i}$ or $M_{i,i}$ should be replaced by zero.

Proof The link $\Lambda^{(2N)}$ and its Reeb chords determine a quiver represented in Figure 6, and as in the proof of Proposition 4.6, the terms in the differential of a chord from Λ_i to Λ_j must form a path from vertex i to vertex j .

Let us compute $\partial^{\varepsilon_L, \varepsilon_R} c_{i,i}$. The only possible terms in $\partial c_{i,i}$ that could lead to a nonzero contribution to $\partial^{\varepsilon_L, \varepsilon_R} c_{i,i}$ are $c_{i+1,i} m_{i,i+1}$ and $m_{i-1,i} c_{i,i-1}$. Indeed, there are no other chords of Λ_i , so a change of component is needed. Since only chords of the form $m_{i,i+1}$ are augmented by ε_L and ε_R , there must be exactly one chord from Λ_j to Λ_k with $j > k$. Moreover, since neither ε_L nor ε_R augment consecutive chords in the quiver determined by $\Lambda^{(2N)}$, we must have $|j - k| = 1$ and $j = i$ or $k = i$. Considering the Legendrian Hopf link composed of Λ_i and Λ_{i+1} , Proposition 4.6 gives the term $c_{i+1,i} m_{i,i+1}$, while considering the Legendrian Hopf link composed of Λ_{i-1} and Λ_i , it gives the term $m_{i-1,i} c_{i,i-1}$. With the first term, since $m_{i,i+1}$ has to be augmented by ε_R , we obtain the contribution $c_{i+1,i}$ when i is odd. With the second term, since $m_{i-1,i}$ has to be augmented by ε_L , we obtain the contribution $c_{i,i-1}$ when $i - 1$ is even. In other words, we obtain $\partial^{\varepsilon_L, \varepsilon_R} c_{i,i} = \bar{i} c_{i,i-1} + \bar{i} c_{i+1,i}$, as announced.

Let us compute $\partial^{\varepsilon_L, \varepsilon_R} c_{i,j}$ with $i < j$. All terms in $\partial c_{i,j}$ involving a single chord from Λ_i to Λ_j correspond to terms with a single factor in the expression for ∂c_{12} in Proposition 4.6. We therefore obtain the term $M_{i,j}$. The other terms must involve augmented chords; since ε_L and ε_R do not have consecutive augmented chords, these other terms could come from $m_{j-1,j} c_{i,j-1}$, $c_{i+1,j} m_{i,i+1}$, $m_{j-1,j} c_{i+1,j-1} m_{i,i+1}$ or analogous terms with $c_{k,l}$ replaced with $m_{k,l}$ or $M_{k,l}$. The last two possibilities lead to elements with a too small grading, so that the unaugmented chord is of the type $c_{k,l}$. The possibilities $m_{j-1,j} c_{i,j-1}$ and $c_{i+1,j} m_{i,i+1}$ are each realized by a single holomorphic disk, corresponding to the contribution $m_{12} c_{11} + c_{22} m_{12}$ in the expression for ∂c_{12} in Proposition 4.6. The remaining possibility

$m_{j-1,j}c_{i+1,j-1}m_{i,i+1}$ has a too small grading. Summing up, the possibility $m_{j-1,j}c_{i,j-1}$ leads to the term $c_{i,j-1}$ when j is odd and the possibility $c_{i+1,j}m_{i,i+1}$ leads to the term $c_{i+1,j}$ when i is odd, so that we obtain $\partial^{\varepsilon_L, \varepsilon_R} c_{j,i} = M_{i,j} + \bar{j}c_{i,j-1} + \bar{i}c_{i+1,j}$, as announced.

The computation of $\partial^{\varepsilon_L, \varepsilon_R} c_{j,i}$ with $i < j$ is similar. Since there are no other chords from Λ_i to Λ_j , the only contributions involve augmented chords and come from $m_{i-1,i}c_{j+1,i-1}m_{j,j+1}$, $m_{i-1,i}c_{j,i-1}$ or $c_{j+1,i}m_{j,j+1}$. The first possibility has a too small grading, while the last two possibilities are each realized by a single holomorphic disk, corresponding to the contributions $c_{21}m_{12}$ and $m_{12}c_{21}$ in the expressions for ∂c_{11} and ∂c_{22} in Proposition 4.6. The possibility $m_{i-1,i}c_{j,i-1}$ leads to the term $c_{j,i-1}$ when i is odd and the possibility $c_{j+1,i}m_{j,j+1}$ leads to the term $c_{j+1,i}$ when j is odd, so that we obtain $\partial^{\varepsilon_L, \varepsilon_R} c_{j,i} = \bar{i}c_{j,i-1} + \bar{j}c_{j+1,i}$, as announced.

The computation of $\partial^{\varepsilon_L, \varepsilon_R} m_{i,j}$ and $\partial^{\varepsilon_L, \varepsilon_R} M_{i,j}$ with $i < j - 1$ involves only chords of the type $m_{k,l}$ and $M_{k,l}$ since all other chords are much longer. Let us start with $\partial^{\varepsilon_L, \varepsilon_R} m_{i,j}$. Arguing as above, since $m_{i,j}$ is the shortest chord from Λ_i to Λ_j , the only contributions involve augmented chords and come from $m_{i-1,i}m_{j,i-1}$, $m_{j+1,i}m_{j,j+1}$ or $m_{i-1,i}m_{j+1,i-1}m_{j,j+1}$. The last possibility has a too small grading, and the first two possibilities are each realized by a unique Morse flow tree [5], which in turn corresponds to a unique holomorphic curve. Both Morse flow trees start with a constant gradient trajectory at $m_{i,j}$, which is the minimum of the difference function $f_j - f_i$. The only possibility to leave $m_{i,j}$ is to have a 3-valent vertex, corresponding to the splitting of the gradient trajectory into two gradient trajectories, for $f_j - f_k$ and for $f_k - f_i$, for some k strictly between i and j . These trajectories converge to the corresponding minima $m_{k,j}$ and to $m_{i,k}$, so we obtain the desired trees for $k = i + 1$ and $k = j - 1$. Summing up, we obtain as above $\partial^{\varepsilon_L, \varepsilon_R} m_{j,i} = \bar{i}m_{j,i-1} + \bar{j}m_{j+1,i}$, as announced. The computation of $\partial^{\varepsilon_L, \varepsilon_R} M_{i,j}$ is completely analogous, except for the description of the Morse flow trees. Both Morse flow trees start with a gradient trajectory from $M_{i,j}$ to a priori any point of the sphere. In order to reach $M_{i+1,j}$ or $M_{i,j-1}$ it is necessary for the gradient trajectory to end exactly at the maximum of the corresponding height difference function. There, we have a 2-valent puncture of the Morse flow tree and we continue with a gradient trajectory converging to the minimum $m_{i,i+1}$ or $m_{j-1,j}$. Again, $\partial^{\varepsilon_L, \varepsilon_R} M_{j,i} = \bar{i}M_{j,i-1} + \bar{j}M_{j+1,i}$, as announced. \square

Proposition 4.10 *The Poincaré polynomial of $\Lambda^{(2N)}$ with respect to the augmentations ε_L and ε_R is given by $P_{\Lambda^{(2N)}, \varepsilon_L, \varepsilon_R}(t) = N(1 + t^N)$.*

Proof We need to compute the homology of the complex described in Proposition 4.9.

Let us first consider the subcomplex spanned by the chords $m_{i,j}$ with $i < j$. For any $k, l = 1, \dots, N$ with $k < l - 1$, the generators $m_{2k-1,2l-1}$, $m_{2k,2l-1}$, $m_{2k-1,2l-2}$ and $m_{2k,2l-2}$ form an acyclic subcomplex. When $k = l - 1$, we just have a subcomplex with the three generators $m_{2l-3,2l-1}$, $m_{2l-2,2l-1}$ and $m_{2l-3,2l-2}$, which has homology spanned by $[m_{2l-2,2l-1}] = [m_{2l-3,2l-2}]$ in degree 0. We therefore obtain $N - 1$ such generators. For any $k = 1, \dots, N - 1$, the generators $m_{2k-1,2N}$ and $m_{2k,2N}$ form an acyclic subcomplex. Finally, the generator $m_{2N-1,2N}$ survives in homology and has degree 0. The total contribution of the chords $m_{i,j}$ to the polynomial $P_{\Lambda^{(2N)}, \varepsilon_L, \varepsilon_R}$ is therefore the term N .

Consider now the subcomplex spanned by the chords $M_{i,j}$ with $i < j$ and $c_{i,j}$ for all $i, j = 1, \dots, 2N$. For any $k, l = 1, \dots, N$ with $k < l - 1$, the generators $c_{2k-1,2l-1}, c_{2k,2l-1}, c_{2k-1,2l-2}, c_{2k,2l-2}, M_{2k-1,2l-1}, M_{2k,2l-1}, M_{2k-1,2l-2}$ and $M_{2k,2l-2}$ form an acyclic subcomplex. When $k = l - 1$, we just have a subcomplex with the seven generators $c_{2k-1,2l-1}, c_{2k,2l-1}, c_{2k-1,2l-2}, c_{2k,2l-2}, M_{2k-1,2l-1}, M_{2k,2l-1}$ and $M_{2k-1,2l-2}$, which has homology spanned by $c_{2l-2,2l-2}$ in degree n . We therefore obtain $N - 1$ such generators. For any $k = 1, \dots, N - 1$, the generators $c_{2k-1,2N}, c_{2k,2N}, M_{2k-1,2N}$ and $M_{2k,2N}$ form an acyclic subcomplex. But the subcomplex spanned by the three generators $c_{2N-1,2N}, c_{2N,2N}, M_{2N-1,2N}$ has homology generated by $[c_{2N,2N}] = [M_{2N-1,2N}]$ in degree n . For any $k, l = 1, \dots, N$ with $k \leq l$ and $k > 1$, the generators $c_{2l-1,2k-1}, c_{2l,2k-1}, c_{2l-1,2k-2}$ and $c_{2l,2k-2}$ form an acyclic subcomplex. When $k = 1$, we just have an acyclic subcomplex with the 2 generators $c_{2l-1,1}$ and $c_{2l,1}$. The total contribution of the chords $M_{i,j}$ with $i < j$ and $c_{i,j}$ to the polynomial $P_{\Lambda^{(2N)}, \varepsilon_L, \varepsilon_R}$ is therefore the term Nt^n .

The sum of the above two contributions therefore gives $P_{\Lambda^{(2N)}, \varepsilon_L, \varepsilon_R}(t) = N(1 + t^n)$, as announced. \square

The next step is to perform some type of connected sum on the Legendrian link $\Lambda^{(2N)}$ in order to obtain a Legendrian sphere $\tilde{\Lambda}^{(2N)} \subset J^1(\mathbb{R}^n)$. More precisely, for each $i = 1, \dots, N - 1$, we consider the Legendrian link formed by $\Lambda_{2i-1}, \Lambda_{2i}, \Lambda_{2i+1}$ and Λ_{2i+2} as the 2-copy of the Legendrian link formed by Λ_{2i-1} and Λ_{2i+1} , and we perform the 2-copy of the connected sum of Λ_{2i-1} and Λ_{2i+1} as follows:

We deform Λ_{2i-1} by a Legendrian isotopy corresponding to the spinning of two iterated first Reidemeister moves on one half of the standard Legendrian unknot in $J^1(\mathbb{R})$. Since this front in $J^0(\mathbb{R})$ has a vertical symmetry axis, we can spin it around this axis to produce a Legendrian surface in $J^1(\mathbb{R}^2)$ as in [2, Section 3.2]. The resulting front has vertical symmetry planes, and hence is spinnable around such a plane; iterating the spinning construction, we obtain the desired 2-component Legendrian link in $J^1(\mathbb{R}^n)$ with cusp edges from (the deformation of) Λ_{2i-1} and Λ_{2i+1} facing each other and having the same Maslov potentials. This is illustrated by Figure 7.

In this figure, we consider the rectangular area limited by a dashed line: its image in $J^0(\mathbb{R}^+) \subset J^0(\mathbb{R}^n)$, i.e. with all spinning angles set to zero, is a rectangular area intersecting $\Lambda_{2i-1}, \Lambda_{2i}, \Lambda_{2i+1}$ and Λ_{2i+2} in the 2-copy of two cusps facing each other. We then replace a neighborhood of this rectangular area with the 2-copy of a connecting tube, as shown in Figure 8. This operation is equivalent to the 2-copy of the connected sum operation described in [2, Section 4].

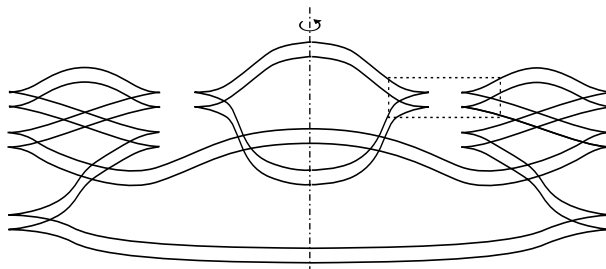


Figure 7: Isotopy of $\Lambda_{2i-1}, \Lambda_{2i}, \Lambda_{2i+1}$ and Λ_{2i+2} .

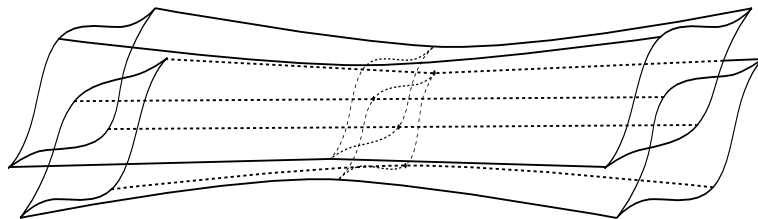


Figure 8: Double tube.

Finally, after performing $N - 1$ times these 2-copies of connected sums, we are left with a Legendrian link composed of two connected components: Λ_{odd} , resulting from the connected sum of Λ_{2i-1} for $i = 1, \dots, N$, and Λ_{even} , resulting from the connected sum of Λ_{2i} for $i = 1, \dots, N$. We then perform an (ordinary) connected sum between these components in order to obtain the Legendrian sphere $\tilde{\Lambda}^{(2N)}$.

Proposition 4.11 *The augmentations ε_L and ε_R of $\Lambda^{(2N)}$ induce augmentations $\tilde{\varepsilon}_L$ and $\tilde{\varepsilon}_R$ of $\tilde{\Lambda}^{(2N)}$.*

Proof It suffices to show that an augmentation induces another augmentation after a single 2-copy of a connected sum. To this end, we describe this operation differently, in order to gain a better control on the Reeb chords during this process. Before performing the 2-copy connected sum connecting Λ_{2i-1} and Λ_{2i} to Λ_{2i+1} and Λ_{2i+2} , respectively, we deform these components by a Legendrian isotopy in order to create a pair of canceling critical points $m'_{2i-1,2i}$ of index 0 and $s_{2i-1,2i}$ of index 1 for the Morse function $f_{2i} - f_{2i-1}$, and a similar pair $m'_{2i+1,2i+2}$ and $s_{2i+1,2i+2}$ for $f_{2i+2} - f_{2i+1}$ near the attaching locus of the connecting double tube. More precisely, the chords $m'_{2i-1,2i}$ and $m'_{2i+1,2i+2}$ are contained in the small balls that are removed during the connected sums, while the chords $s_{2i-1,2i}$ and $s_{2i+1,2i+2}$ are just outside these balls. The connecting double tube is the thickening of an $(n-1)$ -dimensional standard Legendrian Hopf link, and we shape each tube so that its thickness in the z -direction is minimal in the middle. We extend the Morse functions $f_{2i} - f_{2i-1}$ and $f_{2i+2} - f_{2i+1}$ by a Morse function on the connecting tube decreasing towards its middle and having exactly two critical points (of index 0 and $n - 1$) in its middle slice. All Reeb chords for the connecting double tube are contained in this middle slice and correspond to the generators described in Proposition 4.6 with $k = 1$ and n replaced with $n - 1$:

	grading	length
$c_{2i-1,2i-1}^h$	$n - 1$	$l' < l$
$c_{2i,2i}^h$	$n - 1$	l'
$c_{2i-1,2i}^h$	n	$l' + \varepsilon$
$c_{2i,2i-1}^h$	$n - 2$	$l' - \varepsilon$
$m_{2i-1,2i}^h$	0	$\varepsilon - \delta$
$M_{2i-1,2i}^h$	$n - 1$	$\varepsilon + \delta$

The last two generators correspond to the critical points of the Morse function on the connecting tube mentioned above. The unital subalgebra \mathcal{A}^h generated by these six generators is a subcomplex of the Chekanov–Eliashberg DGA, because Morse flow trees are pushed towards the middle of the double

connecting tube due to its shape. By Corollary 4.7, this subcomplex has two augmentations such that only $m'_{2i-1,2i}$ is possibly augmented. On the other hand, we have $\partial s_{2i-1,2i} = m_{2i-1,2i} + m'_{2i-1,2i}$ with no other terms because the length of $s_{2i-1,2i}$ is very short. Hence, for any augmentation ε , we must have $\varepsilon(m'_{2i-1,2i}) = \varepsilon(m_{2i-1,2i})$ and this forces the choice of the augmentation for \mathcal{A}^h . More precisely, the map $\tilde{\varepsilon}$ induced by ε must satisfy $\tilde{\varepsilon}(m'_{2i-1,2i}) = \varepsilon(m_{2i-1,2i})$. Similarly, arguing with $s_{2i+1,2i+2}$, we also have $\tilde{\varepsilon}(m'_{2i+1,2i+2}) = \varepsilon(m_{2i+1,2i+2})$. Note that these relations are compatible since each of ε_L and ε_R have the same value on $m_{2i-1,2i}$ and $m_{2i+1,2i+2}$.

Let us check that the resulting maps $\tilde{\varepsilon}_L, \tilde{\varepsilon}_R: \mathcal{A}(\tilde{\Lambda}^{(2N)}) \rightarrow \mathbb{Z}_2$ satisfy $\tilde{\varepsilon}_L \circ \partial = 0 = \tilde{\varepsilon}_R \circ \partial$. We already saw that these relations are satisfied on \mathcal{A}^h as well as on $s_{2i-1,2i}$ and $s_{2i+1,2i+2}$. On any other chord c , the relation was satisfied before the 2-copy of connected sum. We claim that the augmented terms in ∂c are modified by the 2-copy of connected sum in the following way: all occurrences of $m'_{2i-1,2i}$ and $m'_{2i+1,2i+2}$ are replaced with $m^h_{2i-1,2i}$. In particular, the maps $\tilde{\varepsilon}_L$ and $\tilde{\varepsilon}_R$ keep the same value on these terms and the augmentation relation continues to hold after the 2-copy of connected sum.

To verify the claim, note that the region in which the 2-copy of connected sum is taking place is a trap for Morse flow trees: any portion of such a tree entering this region cannot leave it, because all relevant gradient vector fields are pointing inwards. We only have to consider augmented terms, since these are the only ones that could harm the augmentation relation. We first consider an augmented term that contains neither $m'_{2i-1,2i}$ nor $m'_{2i+1,2i+2}$. If the corresponding Morse flow tree enters the region in which the 2-copy of connected sum is taking place, it must end at a cusp edge. Moreover, it cannot contain any trivalent vertex, otherwise it would not be rigid. Hence, it is a single gradient trajectory ending at a cusp edge. After the 2-copy of connected sum, it becomes another gradient trajectory, also ending at a cusp edge. Hence the corresponding term is not affected by the 2-copy of connected sum. Consider now an augmented term containing $m'_{2i-1,2i}$ or $m'_{2i+1,2i+2}$. A rigid Morse flow tree cannot have a 2-valent negative puncture at such a chord, since it is a minimum of the Morse function $f_{2i} - f_{2i-1}$ or $f_{2i+2} - f_{2i+1}$ [5, Lemma 3.7], so that these chords are 1-valent negative punctures. The only other way a fragment of Morse flow tree contained in the region in which the 2-copy of connected sum is taking place can end is at a cusp edge. As above, it cannot contain any trivalent vertex, otherwise it would not be rigid. Hence, it is a single gradient trajectory ending at a minimum $m'_{2i-1,2i}$ or $m'_{2i+1,2i+2}$. After the 2-copy of connected sum, it becomes another gradient trajectory, also ending at a minimum $m^h_{2i-1,2i}$. Conversely, consider an augmented term containing $m^h_{2i-1,2i}$ after the 2-copy of connected sum. In particular, the corresponding Morse flow tree can only end at the chord $m^h_{2i-1,2i}$ (at a 1-valent negative puncture, as above) or at a cusp edge. For the same reason as above, such a rigid tree cannot contain a trivalent vertex in the 2-copy of the connecting tube. Hence, it is just a single gradient trajectory ending at $m^h_{2i-1,2i}$. If we remove the 2-copy of the connecting tube and replace it with the regions containing the minima $m'_{2i-1,2i}$ and $m'_{2i+1,2i+2}$, this gradient trajectory is replaced with a single gradient trajectory ending at one of these minima. In other words, such an augmented term involving $m^h_{2i-1,2i}$ always comes from the substitution of $m'_{2i-1,2i}$ and $m'_{2i+1,2i+2}$ with $m^h_{2i-1,2i}$, proving the claim. \square

We are now in position to show that these 2–copies of connected sums destroy almost all terms in the Poincaré polynomial for bilinearized LCH.

Proposition 4.12 *The Poincaré polynomial $P_{\tilde{\Lambda}^{(2N)}, \tilde{\varepsilon}_L, \tilde{\varepsilon}_R}$ is equal to 1.*

Proof Let us show by induction that, after applying i successive 2–copies of connected sums on $\Lambda^{(2N)}$, its Poincaré polynomial is given by $(N - k)(1 + t^n)$. Proposition 4.10 corresponds to the case $i = 0$. As shorthand, we denote by C_* the BLCH chain complex after $i - 1$ successive 2–copies of connected sums, and by \tilde{C}_* the BLCH chain complex after i successive 2–copies of connected sums. Using the description of the i^{th} 2–copy of connected sum in the proof of Proposition 4.11, we see that this operation has two effects on the complex C_* . First, the generators $m'_{2i-1,2i}$ and $m'_{2i+1,2i+2}$ are removed. Second, we add generators of the BLCH complex C_*^h of the $(n-1)$ –dimensional standard Legendrian Hopf link with distinct augmentations. Recall that C_*^h forms a subcomplex of \tilde{C}_* (see the proof of Proposition 4.11).

Since the 2–copy of connected sum is performed away from rigid holomorphic disks connecting generators of \tilde{C}_*/C_*^h , the differential on this quotient complex is directly induced from that of C_* . In particular, we have $\partial s_{2i-1,2i} = m_{2i-1,2i}$ and $\partial s_{2i+1,2i+2} = m_{2i+1,2i+2}$ in \tilde{C}_*/C_*^h . Hence, its homology coincides with the homology of C_* , except in degree 0, where it has two fewer generators. Hence, its Poincaré polynomial is $(N - i - 1) + (N - i + 1)t^n$. On the other hand, the homology of C_*^h is given by Proposition 4.10 with $N = 1$ and n replaced with $n - 1$. Hence its Poincaré polynomial is $1 + t^{n-1}$.

In order to deduce the homology of \tilde{C}_* , we consider the long exact sequence

$$\dots \rightarrow H_{k+1}(\tilde{C}_*/C_*^h) \rightarrow H_k(C_*^h) \rightarrow H_k(\tilde{C}_*) \rightarrow H_k(\tilde{C}_*/C_*^h) \rightarrow H_{k-1}(C_*^h) \rightarrow \dots$$

When $k = 0$, we see that the generator $[m'_{2i-1,2i}]$ of $H_0(C_*^h)$ injects into $H_0(\tilde{C}_*)$, as it can only be hit by $s_{2i-1,2i}$ and $s_{2i+1,2i+2}$, but these do not survive in the homology of the quotient complex. Hence the rank of $H_0(\tilde{C}_*)$ is $N - i$.

When $k = n$, we see that the generator $[c_{2i+2,2i+2}]$ in $H_n(\tilde{C}_*/C_*^h)$, which was not affected by the $i - 1$ first 2–copies of connected sums, hits the generator $[c_{2i,2i}^h]$ of $H_{n-1}(C_*^h)$, because there exists a single Morse flow tree connecting them. Indeed, in Figure 7 the chord $c_{2i+2,2i+2}$ is in the middle of the uppermost connected component, and the Morse flow tree starts from there to the right in the plane of the figure (corresponding to all spinning angles set to zero), then enters the dotted rectangle (hence the upper tube in Figure 8), until it reaches the chord $c_{2i,2i}^h$ sitting in the middle of that tube. Hence, the rank of $H_n(\tilde{C}_*)$ is $N - i$. The Poincaré polynomial for the homology of \tilde{C}_* is therefore $(N - i)(1 + t^n)$, as announced.

After these $N - 1$ operations, we are therefore left with the Poincaré polynomial $1 + t^n$. The last step in the construction of $\tilde{\Lambda}^{(2N)}$ is an ordinary connected sum between the remaining two connected components Λ_{even} (the connected sum of Λ_{2i} for $i = 1, \dots, N$) and Λ_{odd} (the connected sum of Λ_{2i-1} for $i = 1, \dots, N$). Let us denote the corresponding 2–component Legendrian link by Λ' .

As in the proof of Proposition 3.2, the map $\tilde{\tau}_0$ from the duality exact sequence (2-2) with $\varepsilon_1 = \tilde{\varepsilon}_R$ and $\varepsilon_2 = \tilde{\varepsilon}_L$ is given at chain level by $\tilde{\varepsilon}_R - \tilde{\varepsilon}_L$, except that we must refine according to the connected component

Λ_{even} or Λ_{odd} which is hit. Note that all chords augmented by $\tilde{\varepsilon}_L$ start on Λ_{odd} and all chords augmented by $\tilde{\varepsilon}_R$ end on Λ_{odd} . This means that $\tilde{\tau}_0$ necessarily takes its values in $H_0(\Lambda_{\text{odd}})$. By Proposition 4.4, since $P_{\Lambda', \tilde{\varepsilon}_L, \tilde{\varepsilon}_R}(t) = 1 + t^n$ and $H_*(\Lambda')$ has rank 4, we must have $p = 0$, and hence $P_{\Lambda', \tilde{\varepsilon}_R, \tilde{\varepsilon}_L}(t) = 1 + t^n$ as well. Therefore, the image of the map $\tilde{\tau}_0: \text{LCH}_0^{\tilde{\varepsilon}_R, \tilde{\varepsilon}_L}(\Lambda') \rightarrow H_0(\Lambda')$ is equal to $H_0(\Lambda_{\text{odd}})$.

We deduce that $\ker \tilde{\sigma}_n = H_0(\Lambda_{\text{odd}})$ in the duality exact sequence (2-2) with $\varepsilon_1 = \tilde{\varepsilon}_R$ and $\varepsilon_2 = \tilde{\varepsilon}_L$. Consider now the map τ_n in the duality exact sequence (2-2) with $\varepsilon_1 = \tilde{\varepsilon}_L$ and $\varepsilon_2 = \tilde{\varepsilon}_R$. Since $\tilde{\sigma}_n$ and τ_n are adjoint in the sense of [6, Proposition 3.9], $\text{im } \tau_n$ is the annihilator of $H_0(\Lambda_{\text{odd}})$ for the intersection pairing, which is $H_n(\Lambda_{\text{even}})$. In particular, the map $\tau_{n,1} - \tau_{n,2} = \tau_{n,\text{odd}} - \tau_{n,\text{even}}$ from Proposition 3.5 does not vanish, so that this last connected sum modifies the Poincaré polynomial by $-t^n$. We are therefore left with $P_{\tilde{\Lambda}^{(2N)}, \tilde{\varepsilon}_L, \tilde{\varepsilon}_R}(t) = 1$, as announced. □

4.4 Geography of BLCH for Legendrian spheres

The next step in our construction is to add to $\tilde{\Lambda}^{(2N)}$ a standard Legendrian unknot Λ_0 which forms with the bottom k components $\Lambda_1, \dots, \Lambda_k$ a Legendrian link isotopic to the $(k+1)$ -copy of the standard Legendrian unknot, but which is unlinked with the $2N - k$ top components $\Lambda_{k+1}, \dots, \Lambda_{2N}$. We fix the Maslov potential of the component Λ_0 to be given by the Maslov potential of Λ_1 plus $m - 1$, for some integer m . We can deform this link by a Legendrian isotopy in order to widen the components $\Lambda_1, \dots, \Lambda_k \subset J^1(\mathbb{R}^n)$ so that their projection to \mathbb{R}^n becomes much larger than the projection of the components $\Lambda_{k+1}, \dots, \Lambda_{2N}$. We further narrow the component Λ_0 so that its projection to \mathbb{R}^n does not intersect the projection of the components $\Lambda_{k+1}, \dots, \Lambda_{2N}$. We denote the resulting Legendrian link by $\tilde{\Lambda}_{(k,m)}^{(2N)}$.

The addition of Λ_0 to $\tilde{\Lambda}^{(2N)}$ is illustrated by Figure 9 in the case $k = 4$, where the picture zooms in on the bottom strata of the k components $\Lambda_1, \dots, \Lambda_k$, which are represented as portions of horizontal planes. This Legendrian link $\tilde{\Lambda}_{(k,m)}^{(2N)}$ has several additional Reeb chords compared to $\tilde{\Lambda}^{(2N)}$. These are easily identified within the $(k+1)$ -copy of the standard Legendrian unknot formed by $\Lambda_0, \Lambda_1, \dots, \Lambda_k$ and are given by

	grading	length
$c_{0,0}$	n	l
$c_{0,j}$	$n + j - m$	$l + \varepsilon j$
$c_{j,0}$	$n - j + m$	$l - \varepsilon j$
$m_{0,j}$	$j - m - 1$	$\varepsilon j - \delta$
$M_{0,j}$	$n + j - m - 1$	$\varepsilon j + \delta$

where the index j takes all possible values between 1 and k .

We extend the augmentations $\tilde{\varepsilon}_L$ and $\tilde{\varepsilon}_R$ by zero on these additional chords in order to define augmentations, still denoted by $\tilde{\varepsilon}_L$ and $\tilde{\varepsilon}_R$, on the Chekanov–Eliashberg DGA of $\tilde{\Lambda}_{(k,m)}^{(2N)}$. Since the mixed chords involving Λ_0 are not augmented, it follows that the vector space generated by the above chords is a direct summand of the bilinearized complex with respect to the differential $\partial^{\tilde{\varepsilon}_L, \tilde{\varepsilon}_R}$.

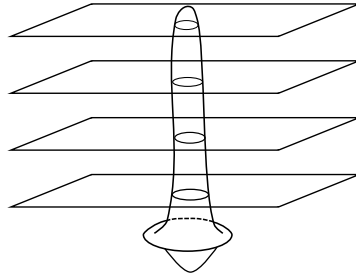


Figure 9: Additional component Λ_0 with $k = 4$.

Proposition 4.13 The bilinearized differential $\partial^{\tilde{\varepsilon}_L, \tilde{\varepsilon}_R}$ of $\tilde{\Lambda}_{(k,m)}^{(2N)}$ on the subcomplex generated by the chords involving the component Λ_0 is given by

$$\begin{aligned} \partial^{\tilde{\varepsilon}_L, \tilde{\varepsilon}_R} c_{0,0} &= 0, & \partial^{\tilde{\varepsilon}_L, \tilde{\varepsilon}_R} c_{0,j} &= M_{0,j} + \bar{j} c_{0,j-1}, & \partial^{\tilde{\varepsilon}_L, \tilde{\varepsilon}_R} c_{j,0} &= \bar{j} c_{j+1,0}, \\ \partial^{\tilde{\varepsilon}_L, \tilde{\varepsilon}_R} m_{0,j} &= \bar{j} m_{0,j-1}, & \partial^{\tilde{\varepsilon}_L, \tilde{\varepsilon}_R} M_{0,j} &= \bar{j} M_{0,j-1}, \end{aligned}$$

for $j = 1, \dots, k$, where \bar{j} is the modulo-2 reduction of j and where in the right-hand sides $c_{k+1,0}$, $c_{0,0}$, $m_{0,0}$ and $M_{0,0}$ should be replaced by zero.

Proof This result follows from the same computations as in Proposition 4.9, in which we replace $2N$ with k , i with 0 and where all terms obtained by changing the index i are omitted since the mixed Reeb chords involving Λ_0 are not augmented. □

Proposition 4.14 Consider the Legendrian link $\tilde{\Lambda}_{(k,m)}^{(2N)} \subset J^1(\mathbb{R}^n)$. Its Poincaré polynomial with respect to the augmentations $\tilde{\varepsilon}_L$ and $\tilde{\varepsilon}_R$ is given by

$$P_{\tilde{\Lambda}_{(k,m)}^{(2N)}, \tilde{\varepsilon}_L, \tilde{\varepsilon}_R}(t) = 1 + t^n + t^{-m} + t^a,$$

where

$$(4-2) \quad a = \begin{cases} k - m - 1 & \text{if } k \text{ is even,} \\ n - k + m & \text{if } k \text{ is odd.} \end{cases}$$

Proof Let us compute the homology of the subcomplex generated by all Reeb chords involving the component Λ_0 . First note that $c_{0,0}$ is always a generator in homology, leading to the term t^n in the Poincaré polynomial. Moreover, the complex generated by the chords $c_{0,1}, \dots, c_{0,k}$ and $M_{0,1}, \dots, M_{0,k}$ is acyclic. If k is even, the complex generated by the chords $c_{1,0}, \dots, c_{k,0}$ is acyclic. On the other hand, the complex generated by the chords $m_{0,1}, \dots, m_{0,k}$ has its homology generated by $m_{0,1}$ and $m_{0,k}$. These lead to the terms t^{-m} and t^{k-m-1} in the Poincaré polynomial.

If k is odd, the complex generated by the chords $c_{1,0}, \dots, c_{k,0}$ has its homology generated by $c_{k,0}$. This leads to the term t^{n-k+m} in the Poincaré polynomial. On the other hand, the complex generated by the chords $m_{0,1}, \dots, m_{0,k}$ has its homology generated by $m_{0,1}$. This leads to the term t^{-m} in the Poincaré polynomial.

Adding these contributions to the Poincaré polynomial of $\tilde{\Lambda}^{(2N)}$ from Proposition 4.12, we obtain the announced result. □

Remark 4.15 As a variant of the above construction, if we choose Λ_0 to be unlinked with Λ_1 in addition to $\Lambda_{k+1}, \dots, \Lambda_{2N}$, then we obtain instead the Poincaré polynomial $1 + t^n + t^{n+m-2} + t^a$ with the same a as in Proposition 4.14. This is because the subcomplex generated by all Reeb chords involving the component Λ_0 considered in the above proof does not contain the generators $c_{1,0}$ and $m_{0,1}$ anymore. Therefore, when k is even its homology is generated by $c_{2,0}$ and $m_{0,k}$, and when k is odd it is generated by $c_{2,0}$ and $c_{k,0}$. Hence, in the Poincaré polynomial the exponent $-m = |m_{0,1}|$ is replaced with $n + m - 2 = |c_{2,0}|$.

The next step in our construction is to perform a connected sum between the component Λ_0 and the original knot $\tilde{\Lambda}^{(2N)}$. This can be done after a Legendrian isotopy of Λ_0 similar to the one depicted in Figure 7, so that a piece of cusp in the deformed Λ_0 faces a piece of cusp from the component Λ_1 . In this case, it will be necessary to use a different number of first Reidemeister moves as in Figure 4 before spinning the resulting front, so that the Maslov potentials near the facing cusps agree. We denote by $\bar{\Lambda}_{(k,m)}^{(2N)}$ the resulting Legendrian knot in $J^1(\mathbb{R}^n)$. We denote by $\bar{\varepsilon}_L$ and $\bar{\varepsilon}_R$ the augmentations induced from $\tilde{\varepsilon}_L$ and $\tilde{\varepsilon}_R$ via the exact Lagrangian cobordism between $\bar{\Lambda}_{(k,m)}^{(2N)}$ and $\tilde{\Lambda}_{(k,m)}^{(2N)}$.

Proposition 4.16 Consider the Legendrian knot $\bar{\Lambda}_{(k,m)}^{(2N)} \subset J^1(\mathbb{R}^n)$. We have

$$P_{\bar{\Lambda}_{(k,m)}^{(2N)}, \bar{\varepsilon}_L, \bar{\varepsilon}_R}(t) = 1 + t^{-m} + t^a,$$

where a is given by (4-2).

Proof By Proposition 4.14, the generator $[c_{0,0}] \in \text{LCH}_n^{\tilde{\varepsilon}_L, \tilde{\varepsilon}_R}(\tilde{\Lambda}_{(k,m)}^{(2N)})$ corresponds to the fundamental class $[\Lambda_0]$ of the component Λ_0 of the Legendrian link $\tilde{\Lambda}_{(k,m)}^{(2N)}$. By Proposition 3.5, the effect of the connected sum with this component is to remove the term t^n from the Poincaré polynomial, so that we obtain the announced result. □

Note that, instead of adding a single component Λ_0 to the Legendrian knot $\tilde{\Lambda}^{(2N)}$, we can add a collection of components $\Lambda_{0,1}, \dots, \Lambda_{0,r} \subset J^1(\mathbb{R}^n)$ with similar properties. More precisely, for all $i = 1, \dots, r$, $\Lambda_{0,i}$ forms with the bottom k_i components $\Lambda_1, \dots, \Lambda_{k_i}$ a Legendrian link isotopic to the $(k_i + 1)$ -copy of the standard Legendrian unknot, but the projection of $\Lambda_{0,i}$ to \mathbb{R}^n is disjoint from the projection of the other components $\Lambda_{k_i+1}, \dots, \Lambda_{2N}$. The Maslov potential of $\Lambda_{0,i}$ is fixed as the Maslov potential of Λ_1 plus $m_i - 1$, for some integer m_i . With $\bar{k} = (k_1, \dots, k_r)$ and $\bar{m} = (m_1, \dots, m_r)$, we denote the resulting Legendrian link by $\tilde{\Lambda}_{(\bar{k}, \bar{m})}^{(2N)}$.

The addition of $\Lambda_{0,1}, \dots, \Lambda_{0,r}$ to $\tilde{\Lambda}^{(2N)}$ is illustrated by Figure 10 in the case $r = 3$ and $\{k_1, k_2, k_3\} = \{1, 3, 4\}$, where the picture zooms in on the bottom strata of the k components $\Lambda_1, \dots, \Lambda_k$, which are represented as portions of horizontal planes.

Each additional component $\Lambda_{0,i}$ gives rise to an additional subcomplex in the bilinearized complex as in Proposition 4.13, and hence to additional terms in the Poincaré polynomial of the form $t^n + t^{-m_i} + t^{a_i}$ with a_i given by (4-2). After the connected sum of these components with $\tilde{\Lambda}^{(N)}$, we obtain a Legendrian

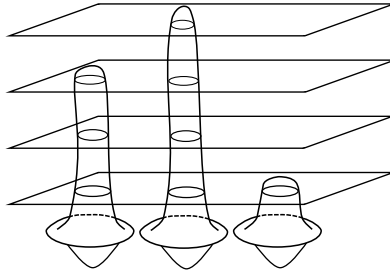


Figure 10: Additional components $\Lambda_{0,i}$ with $r = 3$ and $\{k_1, k_2, k_3\} = \{1, 3, 4\}$.

knot $\bar{\Lambda}_{(k,m)}^{(2N)}$ and, arguing as in Proposition 4.16, its Poincaré polynomial is given by

$$(4-3) \quad P_{\bar{\Lambda}_{(\bar{k},\bar{m})}^{(2N)}, \bar{\varepsilon}_L, \bar{\varepsilon}_R}(t) = 1 + \sum_{i=1}^r (t^{-m_i} + t^{a_i}).$$

At this point of our constructions we have realized the geography of BLCH for Legendrian spheres Λ .

Theorem 4.17 *Let $P = q + p$ be the sum of Laurent polynomials with nonnegative integral coefficients satisfying conditions (i') and (ii') from Remark 4.3. Then there exists a Legendrian sphere Λ in $J^1(\mathbb{R}^n)$ and two non-DGA homotopic augmentations ε_1 and ε_2 of the Chekanov–Eliashberg DGA of Λ , with the property that the Poincaré polynomial of $\text{LCH}^{\varepsilon_1, \varepsilon_2}(\Lambda)$ with coefficients in \mathbb{Z}_2 is equal to P .*

Proof Let us show that the Poincaré polynomials obtained in (4-3) realize all polynomials $P = q + p$ satisfying conditions (i') and (ii').

Indeed, let $q(t) = 1$ and p be a Laurent polynomial satisfying (ii'). If n is even, $p(-1) = 0$, so the polynomial p can be expressed as a sum of polynomials of the form $\sum_{i=1}^r (t^{u_i} + t^{v_i})$, where $u_i < v_i$ have different parities. If n is odd, $p(-1)$ is even, so the polynomial p can be expressed as the sum of polynomials of the form $\sum_{i=1}^r (t^{u_i} + t^{v_i})$, with no parity conditions on u_i and v_i .

In order to realize the polynomial $t^{u_i} + t^{v_i}$ when u_i and v_i have different parities, we can choose $m_i = -u_i$ and $k_i = v_i - u_i + 1$, which is even. When u_i and v_i have the same parity, which can happen only if n is odd, we proceed as follows. If $u_i + v_i \leq n - 1$, we can choose $m_i = -u_i$ and $k_i = n - u_i - v_i$, which is odd. If $u_i + v_i \geq n - 1$, we use the variant of the construction with Λ_0 described in Remark 4.15 with $m_i = u_i + 2 - n$ and $k_i = u_i + v_i + 3 - n$, which is even.

Let us define $\bar{k} = (k_1, \dots, k_r)$ and $\bar{m} = (m_1, \dots, m_r)$, and let N be the smallest even integer such that $k_i \leq 2N$ for all $i = 1, \dots, r$. Then, in view of (4-3), the Legendrian sphere $\bar{\Lambda}_{(\bar{k},\bar{m})}^{(2N)}$ satisfies

$$P_{\bar{\Lambda}_{(\bar{k},\bar{m})}^{(2N)}, \bar{\varepsilon}_L, \bar{\varepsilon}_R}(t) = 1 + p(t) = q(t) + p(t),$$

as desired. □

4.5 Geography of BLCH for general Legendrian submanifolds

In order to obtain Poincaré polynomials with all possible polynomials q satisfying condition (i) from Definition 4.1, we use the following construction from [2, Corollary 6.7]:

Proposition 4.18 For any monic polynomial \bar{q} of degree n satisfying $\bar{q}(0) = 0$, there exists a connected Legendrian submanifold $\Lambda_{\bar{q}} \subset J^1(\mathbb{R}^n)$ equipped with an augmentation ε such that $P_{\Lambda_{\bar{q}}, \varepsilon} = \bar{q}$.

If q is a polynomial satisfying condition (i) from Definition 4.1, then the polynomial \bar{q} given by $\bar{q}(t) = q(t) + t^n - 1$ satisfies the assumptions of Proposition 4.18.

Let $\Lambda_{\bar{q}, (\bar{k}, \bar{m})}^{(2N)}$ be the disjoint union of the Legendrian knots $\bar{\Lambda}_{\bar{q}, (\bar{k}, \bar{m})}^{(2N)}$ and $\Lambda_{\bar{q}}$ such that the projections of these components to \mathbb{R}^n are disjoint. We denote by $\hat{\varepsilon}_L$ and $\hat{\varepsilon}_R$ the augmentations for $\Lambda_{\bar{q}, (\bar{k}, \bar{m})}^{(2N)}$ induced by the augmentation ε for $\Lambda_{\bar{q}}$ and the augmentations $\bar{\varepsilon}_L$ and $\bar{\varepsilon}_R$ for $\bar{\Lambda}_{\bar{q}, (\bar{k}, \bar{m})}^{(2N)}$. The Poincaré polynomial of $\Lambda_{\bar{q}, (\bar{k}, \bar{m})}^{(2N)}$ is given by the sum of the Poincaré polynomials of its components:

$$P_{\Lambda_{\bar{q}, (\bar{k}, \bar{m})}^{(2N)}, \bar{\varepsilon}_L, \bar{\varepsilon}_R}(t) = t^n + q(t) + \sum_{i=1}^r (t^{-m_i} + t^{a_i}).$$

We then perform a connected sum on the Legendrian link $\Lambda_{\bar{q}, (\bar{k}, \bar{m})}^{(2N)}$ in order to obtain a Legendrian knot $\tilde{\Lambda}_{\bar{q}, (\bar{k}, \bar{m})}^{(2N)}$, equipped with two augmentations still denoted by $\hat{\varepsilon}_L$ and $\hat{\varepsilon}_R$. Since the augmentations $\bar{\varepsilon}_L$ and $\bar{\varepsilon}_R$ coincide (with ε) on the component $\Lambda_{\bar{q}}$, by Proposition 3.2 the fundamental class $[\Lambda_{\bar{q}}]$ of this component is in the image of the map τ_n in the duality exact sequence (2-2). By Proposition 3.5, the effect of the connected sum with $\Lambda_{\bar{q}}$ is to remove a term t^n from the Poincaré polynomial. We therefore obtain

$$P_{\tilde{\Lambda}_{\bar{q}, (\bar{k}, \bar{m})}^{(2N)}, \bar{\varepsilon}_L, \bar{\varepsilon}_R}(t) = q(t) + \sum_{i=1}^r (t^{-m_i} + t^{a_i}).$$

Although these Poincaré polynomials realize all polynomials q satisfying condition (i) from Definition 4.1, we are still missing some Laurent polynomials p , since these can be arbitrary when $n > 2$. In order to realize these more general Laurent polynomials p , we describe a generalization of the embedded surgery construction on which Proposition 4.18 and its proof in [2, Corollary 6.7] are based.

From now on, assume that $n \geq 2$. Consider a point on the cusp locus of the component Λ_1 of the $2N$ -copy of the standard Legendrian unknot $\Lambda^{(2N)} \subset J^1(\mathbb{R}^n)$. By a Legendrian isotopy, it is always possible to arrange so that, in a neighborhood of this point, the front of $\Lambda^{(2N)}$ in $J^0(\mathbb{R}^n)$ with local coordinates (x_1, \dots, x_n, z) is locally described as follows: the fragment of Λ_1 in this neighborhood is composed of a bottom stratum $z = 0$ and of a top stratum satisfying $z^2 = x_n^3$, both for $x_n \geq 0$. Moreover, the fragments of the bottom strata of the components Λ_i in this neighborhood satisfy $z = (i - 1)\varepsilon$ for $i = 2, \dots, 2N$, and no other parts of the front of $\Lambda^{(2N)}$ lie in this neighborhood. Note that it is possible to arrange so that this local model still holds for the more sophisticated Legendrian $\Lambda_{\bar{q}, (\bar{k}, \bar{m})}^{(2N)}$ after our above constructions.

For a given $m' \in \{0, \dots, n - 2\}$, we consider an embedded sphere $S^{m'}$ of dimension m' in the cusp locus $\{x_n = z = 0\}$ of Λ_1 . In view of our assumptions on the front of $\Lambda^{(2N)}$, this sphere bounds an embedded disk of dimension $m' + 1$ with its interior disjoint from the front of $\Lambda^{(2N)}$. For a given $k' \in \{2, \dots, 2N\}$, we define a function f on the cusp locus of Λ_1 , equal to $((k' + \frac{2}{3})\varepsilon)^{2/3}$ along $S^{m'}$, given by $((k' + \frac{2}{3})\varepsilon)^{2/3} / r_0^{1/2} \sqrt{r_0 - r}$ at distance $r \in (0, r_0]$ from $S^{m'}$ and extended by 0 everywhere else. We

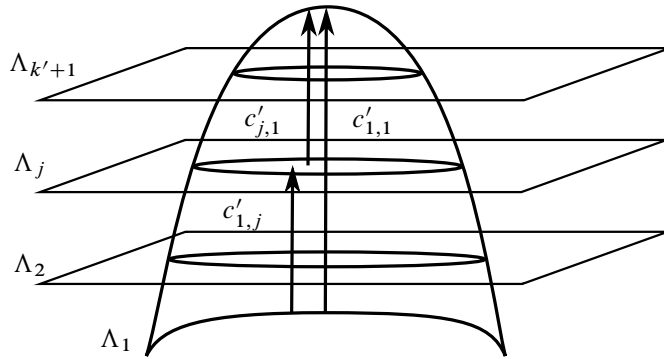


Figure 11: Center of a generalized handle.

remove from the front of Λ_1 the region satisfying $x_n < f(x_1, \dots, x_{n-1})$; the resulting front has boundary diffeomorphic to the cartesian product of $S^{m'}$ with a standard Legendrian sphere of dimension $n - m' - 1$, with a flat bottom stratum. We now perform an m' -surgery on $\Lambda^{(2N)}$ by attaching a standard Legendrian handle diffeomorphic to $D^{m'+1} \times S^{n-m'-1}$ to the above front along its boundary. By construction, along the boundary of this handle, the standard Legendrian sphere of dimension $n - m' - 1$ has height $(k' + \frac{2}{3})\varepsilon$. We shape the handle so that this height decreases monotonically from the boundary of $D^{m'+1}$ to its center, where it takes the minimal value $(k' + \frac{1}{3})\varepsilon$. This is a standard Legendrian surgery on Λ_1 , but it is of a more general nature if we consider the whole $\Lambda^{(2N)}$, since the front of the attached handle intersects the front of the components $\Lambda_2, \dots, \Lambda_{k'+1}$ (but not of the components $\Lambda_{k'+2}, \dots, \Lambda_{2N}$). When this operation is performed on the Legendrian submanifold $\Lambda_{\bar{q},(\bar{k},\bar{m})}^{(2N)}$, we denote the resulting Legendrian submanifold by $\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}^{(2N)}$.

In order to minimize the number of Reeb chords created by this operation, we shape the standard Legendrian sphere of dimension $n - m' - 1$ as shown in Figure 11, with both top and bottom strata being the graphs of concave functions. Assuming for simplicity that the minima of the perturbing Morse functions $f_i - f_j$ for $i \neq j$ are located in the bottom strata and that the corresponding maxima are located in the top strata, the bottom strata of the Λ_i are slightly moving away from each other in the z -direction as x_n decreases to 0. Hence, the bottom stratum of the standard Legendrian sphere of dimension $n - m' - 1$ is slightly moving down from the boundary of $D^{m'+1}$ to its center. In particular, all new Reeb chords are located very close to the center of the handle: $c'_{1,1}$ with endpoints on the handle, $c'_{1,j}$ from the handle to Λ_j and $c'_{j,1}$ from Λ_j to the handle, for $j = 2, \dots, k' + 1$, as shown in Figure 11. On the other hand, we can perturb the resulting Legendrian submanifold so that there are no Reeb chords between the attached handle and the components $\Lambda_{k'+2}, \dots, \Lambda_{2N}$. Summarizing, the gradings and lengths of the new Reeb chords are given by

	grading	length
$c'_{1,1}$	$n - m' - 1$	$(k' + \frac{1}{3})\varepsilon$
$c'_{j,1}$	$n - m' - j$	$(k' - j + \frac{4}{3})\varepsilon$
$c'_{1,j}$	$m' + j - 1$	$(j - 1)\varepsilon$

Proposition 4.19 *The augmentations $\hat{\varepsilon}_L$ and $\hat{\varepsilon}_R$ can be extended by zero on the new chords to augmentations of $\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}$. The vector space spanned by the new chords $c'_{1,1}$, $c'_{1,j}$ and $c'_{j,1}$ is a subcomplex with respect to the bilinearized differentials $\partial^{\hat{\varepsilon}_L, \hat{\varepsilon}_R}$ and $\partial^{\hat{\varepsilon}_R, \hat{\varepsilon}_L}$. These differentials are given by*

$$\partial^{\hat{\varepsilon}_L, \hat{\varepsilon}_R} c'_{1,j+1} = \overline{j+1} c'_{1,j}, \quad \partial^{\hat{\varepsilon}_L, \hat{\varepsilon}_R} c'_{j,1} = \bar{j} c'_{j+1,1} \quad \text{and} \quad \partial^{\hat{\varepsilon}_L, \hat{\varepsilon}_R} c'_{k'+1,1} = 0,$$

and respectively by

$$\partial^{\hat{\varepsilon}_R, \hat{\varepsilon}_L} c'_{1,j+1} = \begin{cases} \bar{j} c'_{1,j} & \text{if } j \neq 1, \\ 0 & \text{if } j = 1, \end{cases} \quad \partial^{\hat{\varepsilon}_R, \hat{\varepsilon}_L} c'_{j,1} = \overline{j+1} c'_{j+1,1} \quad \text{and} \quad \partial^{\hat{\varepsilon}_R, \hat{\varepsilon}_L} c'_{k'+1,1} = 0$$

for $j = 1, \dots, k'$, where \bar{j} is the modulo-2 reduction of j .

Proof We first show that $\hat{\varepsilon}_L \circ \partial c = \hat{\varepsilon}_R \circ \partial c = 0$ for any Reeb chord c of $\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}$. If c is a Reeb chord of $\Lambda_{\bar{q},(\bar{k},\bar{m})}$, then ∂c consists of terms from the differential for $\Lambda_{\bar{q},(\bar{k},\bar{m})}$, and hence in the kernel of $\hat{\varepsilon}_L$ and $\hat{\varepsilon}_R$, and of terms involving at least one new chord of $\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}$. Since $\hat{\varepsilon}_L$ and $\hat{\varepsilon}_R$ vanish on these new chords, we obtain the desired relations.

If c is a new chord of $\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}$, we claim that any term in ∂c contains an unaugmented chord as a factor, and hence is in the kernel of $\hat{\varepsilon}_L$ and $\hat{\varepsilon}_R$. Indeed, the only augmented chords go from Λ_j to Λ_{j+1} , with a parity condition on j depending on the augmentation. Moreover, Morse flow trees cannot entirely go across a connecting tube (since they are attracted to its center), so chords are the only way to jump from Λ_i to Λ_j with $i \neq j$. Since the new chords have at least one endpoint on Λ_1 , if a Morse flow tree has all negative ends at augmented chords, it must start at $c'_{1,1}$ or at $c'_{1,2}$. But $|c'_{1,1}| = n - m' - 1$ equals 1 if and only if $m' = n - 2$, and in that case a Morse flow tree with endpoints remaining on Λ_1 must remain in the center of the handle, which is a 1-dimensional standard Legendrian knot, so that there are 2 such Morse flow trees with no negative end, canceling each other. On the other hand, $|c'_{1,2}| = m' + 1$ equals 1 if and only if $m' = 0$, and in that case a Morse flow tree with endpoints remaining on Λ_1 and Λ_2 must connect the critical point $c'_{1,2}$ of $f_2 - f_1$ of index 1 to the critical point $m_{1,2}$ of $f_2 - f_1$ of index 0. There are two such Morse flow trees, corresponding to the two sides of the 1-dimensional unstable manifold of $c'_{1,2}$, and these cancel each other.

Let us now compute the bilinearized differentials. If a rigid Morse flow tree starting at $c'_{1,j}$ with $j = 1, \dots, k' + 1$, has only one negative end, it will leave the handle radially and then flow to the minimum $m_{1,j}$ of $f_j - f_1$. Such a configuration is rigid if and only if $|m_{1,j}| = j - 2 = |c'_{1,j}| - 1 = m' + j - 2$, but when $m' = 0$ there are two such Morse flow trees as above, canceling each other. If it has more negative ends and contributes to the bilinearized differential of $c'_{1,j}$, it can only have a negative end at $m_{j-1,j}$, and the other one must then be at $c'_{1,j-1}$. There is a unique such Morse flow tree, flowing from $c'_{1,j}$ to the position of $c'_{1,j-1}$ in the $D^{m'+1}$ -factor of the handle, then splitting at the bottom stratum of Λ_{j-1} , so that one part flows in the $S^{n-m'-1}$ -factor of the handle to $c'_{1,j-1}$ and the other part flows to the minimum $m_{j-1,j}$ of $f_j - f_{j-1}$. This term $m_{j-1,j} c'_{1,j-1}$ gives rise to the term $c'_{1,j-1}$ in $\partial^{\hat{\varepsilon}_L, \hat{\varepsilon}_R} c'_{1,j}$ if and only if $\hat{\varepsilon}_L(m_{j-1,j}) = 1$, i.e. when j is odd and > 1 . It gives rise to the term $c'_{1,j-1}$ in $\partial^{\hat{\varepsilon}_R, \hat{\varepsilon}_L} c'_{1,j}$ if and only if $\hat{\varepsilon}_R(m_{j-1,j}) = 1$, i.e. when j is even.

Let us now consider a rigid Morse flow tree starting at $c'_{j,1}$ with $j = 2, \dots, k' + 1$. Such a tree cannot have only one negative end, and if it contributes to the bilinearized differential of $c'_{j,1}$, it must have two negative ends, one at $m_{j,j+1}$ and the other one at $c'_{j+1,1}$. There is a unique such Morse flow tree, flowing from $c'_{j,1}$ to the position of $c'_{j+1,1}$ in the $S^{n-m'-1}$ -factor of the handle, then splitting at the bottom stratum of Λ_{j+1} , so that one part flows in the $D^{m'+1}$ -factor of the handle to $c'_{j+1,1}$ and the other part flows to the minimum $m_{j,j+1}$ of $f_{j+1} - f_j$. This term $c'_{j+1,1}m_{j,j+1}$ gives rise to the term $c'_{j+1,1}$ in $\partial^{\hat{e}_L, \hat{e}_R} c'_{j,1}$ if and only if $\hat{e}_R(m_{j,j+1}) = 1$, i.e. when j is odd and $< k' + 1$. It gives rise to the term $c'_{j+1,1}$ in $\partial^{\hat{e}_R, \hat{e}_L} c'_{j,1}$ if and only if $\hat{e}_L(m_{j,j+1}) = 1$, i.e. when j is even and $< k' + 1$. \square

As an immediate consequence of Proposition 4.19, the homology with respect to $\partial^{\hat{e}_L, \hat{e}_R}$ of the subcomplex generated by the new Reeb chords is generated by $[c'_{k'+1,1}]$ in degree $n - m' - k' - 1$ if k' is even, and by $[c'_{1,k'+1}]$ in degree $m' + k'$ if k' is odd. Similarly, the homology with respect to $\partial^{\hat{e}_R, \hat{e}_L}$ of this subcomplex is generated by $[c'_{1,1}]$ in degree $n - m' - 1$, $[c'_{1,2}]$ in degree $m' + 1$, and by $[c'_{1,k'+1}]$ in degree $m' + k'$ if k' is even, and by $[c'_{k'+1,1}]$ in degree $n - m' - k' - 1$ if k' is odd.

Proposition 4.20 *The BLCH Poincaré polynomials of $\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}^{(2N)}$ are given by*

$$P_{\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}, \hat{e}_L, \hat{e}_R}^{(2N)}(t) = P_{\Lambda_{\bar{q},(\bar{k},\bar{m}), \hat{e}_L, \hat{e}_R}^{(2N)}}(t) + t^b$$

and by

$$P_{\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}, \hat{e}_R, \hat{e}_L}^{(2N)}(t) = P_{\Lambda_{\bar{q},(\bar{k},\bar{m}), \hat{e}_R, \hat{e}_L}^{(2N)}}(t) + t^{n-m'-1} + t^{m'+1} + t^{n-1-b},$$

where $b = n - m' - k' - 1$ if k' is even and $b = m' + k'$ if k' is odd.

Proof Observe that the image of $[c'_{1,1}]$ by the map

$$\tilde{\tau}_{n-m'-1} : \text{LCH}_{n-m'-1}^{\hat{e}_R, \hat{e}_L}(\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}^{(2N)}) \rightarrow H_{n-m'-1}(\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}^{(2N)})$$

from the duality exact sequence (4-1) is the homology class of the cocore sphere of the attached handle. Indeed, all Morse flow trees starting at $c'_{1,1}$ and with no negative end must remain in the cocore sphere of the handle, since it is narrowest there. The resulting Morse flow trees start at $c'_{1,1}$ in any direction and finish at the cusp of the cocore sphere. The boundary of the corresponding holomorphic disks foliate the cocore sphere minus the endpoints of $c'_{1,1}$ so that the image of the cycle $c'_{1,1}$ in the bilinearized complex is the cycle corresponding to the cocore sphere in the singular complex of $\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}^{(2N)}$. Since the corresponding homology class does not vanish in $H_{n-m'-1}(\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}^{(2N)})$, it follows that $[c'_{1,1}]$ does not vanish in bilinearized homology either.

Similarly, observe that the image of $[c'_{1,2}]$ by the map

$$\tilde{\tau}_{m'+1} : \text{LCH}_{m'+1}^{\hat{e}_R, \hat{e}_L}(\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}^{(2N)}) \rightarrow H_{m'+1}(\Lambda_{\bar{q},(\bar{k},\bar{m}), (k',m')}^{(2N)})$$

from the duality exact sequence (4-1) is the Poincaré dual of the homology class of the cocore sphere of the attached handle. Indeed, all Morse flow trees starting at $c'_{1,2}$ and with no negative end must follow radii of the disk factor $D^{m'+1}$ for the handle. Once such a Morse flow tree exits the handle, it will flow

down the chord $m_{1,2}$ corresponding to the minimum of the perturbing Morse function $f_2 - f_1$. The chord $m_{1,2}$ is augmented for $\hat{\varepsilon}_R$ so that the image by $\tilde{\tau}_{m'+1}$ is obtained by considering the part of the boundary of the corresponding holomorphic disks lying in Λ_1 . This is a sphere of dimension $m' + 1$, intersecting the cocore sphere at the endpoint of $c'_{1,2}$ in Λ_1 . Since the corresponding homology class does not vanish in $H_{m'+1}(\Lambda_{\tilde{q},(\tilde{k},\tilde{m}),(\tilde{k}',\tilde{m}')}^{(2N)})$, it follows that $[c'_{1,2}]$ does not vanish in bilinearized homology either.

In view of the long exact sequence relating the bilinearized homology of our subcomplex with the bilinearized homologies of our Legendrian submanifold before and after the generalized handle attachment, the effect of $[c'_{k'+1,1}]$ or $[c'_{1,k'+1}]$ could either be to add a term in the BLCH Poincaré polynomial in the degree of this generator, or to remove a term in the degree of this generator, plus one.

In terms of Proposition 4.4, we have just shown that the polynomial \tilde{q} gains the terms $t^{n-m'-1} + t^{m'+1}$ as an effect of this generalized handle attachment. Since the dimension of the singular homology of the Legendrian submanifold increased by 2, it follows that the modifications due to $[c'_{k'+1,1}]$ and $[c'_{1,k'+1}]$ are affecting the polynomials p and \tilde{p} . Since the relation $\tilde{p}(t) = t^{n-1} p(t^{-1})$ must hold at all times, it follows that the changes to both BLCH Poincaré polynomials must occur in degrees that add up to $n - 1$. But since the sum of the gradings of $[c'_{k'+1,1}]$ and of $[c'_{1,k'+1}]$ is $n - 1$, it follows that the effect of these generators is necessarily to add a term in their corresponding BLCH Poincaré polynomial.

Since the four generators

$$[c'_{1,1}], \quad [c'_{1,2}], \quad [c'_{k'+1,1}] \quad \text{and} \quad [c'_{1,k'+1}]$$

each give rise to an additional term in one of the BLCH Poincaré polynomials of $\Lambda_{\tilde{q},(\tilde{k},\tilde{m}),(\tilde{k}',\tilde{m}')}^{(2N)}$, the announced relations follow. □

We can repeat the above generalized handle attachment as many times as we want, with different values of k' and m' . Repeating it s times with parameters k'_i and m'_i , let us define $\tilde{k}' = (k'_1, \dots, k'_s)$ and $\tilde{m}' = (m'_1, \dots, m'_s)$, and after choosing N so that $k'_i + 1 \leq 2N$ for all $i = 1, \dots, s$. Applying these operations on $\Lambda_{\tilde{q},(\tilde{k},\tilde{m})}^{(2N)}$, we denote the resulting Legendrian submanifold by $\Lambda_{\tilde{q},(\tilde{k},\tilde{m}),(\tilde{k}',\tilde{m}')}^{(2N)}$.

Corollary 4.21 *The BLCH Poincaré polynomial of $\Lambda_{\tilde{q},(\tilde{k},\tilde{m}),(\tilde{k}',\tilde{m}')}^{(2N)}$ is given by*

$$P_{\Lambda_{\tilde{q},(\tilde{k},\tilde{m}),(\tilde{k}',\tilde{m}')}^{(2N)},\hat{\varepsilon}_L,\hat{\varepsilon}_R}(t) = q(t) + \sum_{i=1}^r (t^{-m_i} + t^{a_i}) + \sum_{i=1}^s t^{b_i},$$

where

$$a_i = \begin{cases} k_i - m_i - 1 & \text{if } k_i \text{ is even,} \\ n - k_i + m_i & \text{if } k_i \text{ is odd,} \end{cases} \quad \text{and} \quad b_i = \begin{cases} n - k'_i - m'_i - 1 & \text{if } k'_i \text{ is even,} \\ k'_i + m'_i & \text{if } k'_i \text{ is odd.} \end{cases}$$

Proof of Theorem 1.3 Note that if $n = 1$, any connected Legendrian submanifold Λ is a circle. Since we already showed that the BLCH geography for spheres is realized by the submanifolds $\bar{\Lambda}_{(\tilde{k},\tilde{m})}^{(2N)}$ with Poincaré polynomial given by (4-3) with $q(t) = 1$ and $p(-1)$ even, we can assume that $n \geq 2$.

Assume first that $n > 2$. Let $q + p$ be a BLCH–admissible polynomial in the sense of [Definition 4.1](#). Writing $p(t) = \sum_{i=1}^s t^{w_i}$, for any term $i = 1, \dots, s$ we can find $k'_i \geq 1$ and $0 \leq m'_i \leq n - 2$ such that $b_i = w_i$ as in [Corollary 4.21](#): if $w_i > 0$ is odd we can choose $m'_i = 0$ and $k'_i = w_i$, if $w_i > 0$ is even we can choose $m'_i = 1$ and $k'_i = w_i - 1$, if $w_i \leq 0$ has the same parity as n we can choose $m'_i = 1$ and $k'_i = n - 2 - w_i$, and if $w_i \leq 0$ has the same parity as $n - 1$ we can choose $m'_i = 0$ and $k'_i = n - 1 - w_i$. Then the Legendrian submanifold $\Lambda_{\bar{q}, (\bar{k}', \bar{m}')}^{(2N)}$ has the desired BLCH Poincaré polynomial $q + p$.

Finally, in the case $n = 2$, we cannot use the above choices of parameters since we must have $m'_i = 0$ for all $i = 1, \dots, s$. Let $q + p$ be a BLCH–admissible polynomial in the sense of [Definition 4.1](#). Let us decompose p as $p_0 + p_1$ where p_0 and p_1 are Laurent polynomials with nonnegative integral coefficients, $p_0(-1) = 0$ and $p_1(1)$ is minimal with respect to these properties. We have already showed that there exists a Legendrian sphere $\bar{\Lambda}_{(\bar{k}, \bar{m})}^{(2N)}$ with BLCH Poincaré polynomial given by $1 + p_0$ in view of [\(4-3\)](#). Since $p_1(1)$ is minimal, it follows that all terms in p_1 have degrees of the same parity.

If this parity is odd, all terms in p_1 are of the form t^{w_i} with w_i odd. If $w_i \geq 1$, we choose $k'_i = w_i$ odd, and if $w_i \leq -1$, we choose $k'_i = 1 - w_i$ even, as in [Corollary 4.21](#). Therefore, using as many generalized handle attachments as needed, we can realize the BLCH Poincaré polynomial $1 + p_0 + p_1$, regardless of the value of $p_1(-1) \leq 0$. Then, by a connected sum with the Legendrian submanifold $\Lambda_{\bar{q}}$ from [Proposition 4.18](#), we realize the BLCH Poincaré polynomial $q + p$ as desired.

If the terms in p_1 have degrees of even parity, we use generalized handle attachments on Λ_2 instead of Λ_1 : the effect of this modified operation will be as described by [Proposition 4.20](#), with the ordering of the augmentations reversed. In other words, each such generalized handle attachment will add $2t + t^{1-b_i}$ to the BLCH Poincaré polynomial of our Legendrian submanifold, with $1 - b_i = 1 - k'_i$ even, as in [Corollary 4.21](#). If $q(t) = 1 + at$ then we can perform up to $\lfloor \frac{1}{2}a \rfloor$ such attachments. Therefore, for any polynomial p_1 such that $p_1(-1) = p_1(1) \leq \frac{1}{2}a = \frac{1}{2}(1 - q(-1))$, we can realize the BLCH Poincaré polynomial $1 + 2p_1(1)t + p_0 + p_1$. Setting $q_0(t) = q(t) - 2p_1(1)t$, we then perform a connected sum with the Legendrian submanifold $\Lambda_{\bar{q}_0}$ from [Proposition 4.18](#) in order to realize the BLCH Poincaré polynomial $q + p$, as desired. □

References

- [1] **F Bourgeois, B Chantraine**, *Bilinearized Legendrian contact homology and the augmentation category*, J. Symplectic Geom. 12 (2014) 553–583 [MR](#) [Zbl](#)
- [2] **F Bourgeois, J M Sabloff, L Traynor**, *Lagrangian cobordisms via generating families: construction and geography*, Algebr. Geom. Topol. 15 (2015) 2439–2477 [MR](#) [Zbl](#)
- [3] **Y Chekanov**, *Differential algebra of Legendrian links*, Invent. Math. 150 (2002) 441–483 [MR](#) [Zbl](#)
- [4] **G Dimitroglou Rizell**, *Legendrian ambient surgery and Legendrian contact homology*, J. Symplectic Geom. 14 (2016) 811–901 [MR](#) [Zbl](#)

- [5] **T Ekholm**, *Morse flow trees and Legendrian contact homology in 1–jet spaces*, Geom. Topol. 11 (2007) 1083–1224 [MR](#) [Zbl](#)
- [6] **T Ekholm**, **J B Etnyre**, **J M Sabloff**, *A duality exact sequence for Legendrian contact homology*, Duke Math. J. 150 (2009) 1–75 [MR](#) [Zbl](#)
- [7] **T Ekholm**, **J Etnyre**, **M Sullivan**, *The contact homology of Legendrian submanifolds in \mathbb{R}^{2n+1}* , J. Differential Geom. 71 (2005) 177–305 [MR](#) [Zbl](#)
- [8] **T Ekholm**, **J Etnyre**, **M Sullivan**, *Non-isotopic Legendrian submanifolds in \mathbb{R}^{2n+1}* , J. Differential Geom. 71 (2005) 85–128 [MR](#) [Zbl](#)
- [9] **T Ekholm**, **J Etnyre**, **M Sullivan**, *Legendrian contact homology in $P \times \mathbb{R}$* , Trans. Amer. Math. Soc. 359 (2007) 3301–3335 [MR](#) [Zbl](#)
- [10] **Y Félix**, **S Halperin**, **J-C Thomas**, *Rational homotopy theory*, Graduate Texts in Math. 205, Springer (2001) [MR](#) [Zbl](#)
- [11] **M B Henry**, **D Rutherford**, *A combinatorial DGA for Legendrian knots from generating families*, Commun. Contemp. Math. 15 (2013) art. id. 1250059 [MR](#) [Zbl](#)
- [12] **M B Henry**, **D Rutherford**, *Equivalence classes of augmentations and Morse complex sequences of Legendrian knots*, Algebr. Geom. Topol. 15 (2015) 3323–3353 [MR](#) [Zbl](#)
- [13] **T Kálmán**, *Contact homology and one parameter families of Legendrian knots*, Geom. Topol. 9 (2005) 2013–2078 [MR](#) [Zbl](#)
- [14] **P Melvin**, **S Shrestha**, *The nonuniqueness of Chekanov polynomials of Legendrian knots*, Geom. Topol. 9 (2005) 1221–1252 [MR](#) [Zbl](#)
- [15] **L L Ng**, *Computable Legendrian invariants*, Topology 42 (2003) 55–82 [MR](#) [Zbl](#)
- [16] **L Ng**, **D Rutherford**, **V Shende**, **S Sivek**, **E Zaslow**, *Augmentations are sheaves*, Geom. Topol. 24 (2020) 2149–2286 [MR](#) [Zbl](#)

FB: *Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d’Orsay*
91405 Orsay, France

DG: *INSA Hauts-de-France, CERAMATHS – Laboratoire de Matériaux Céramiques et de Mathématiques,*
Université Polytechnique Hauts-de-France
F-59313 Valenciennes, France

DG: *FR.S.–FNRS, Département de Mathématiques, Université de Mons*
Place du Parc, 20, B-7000 Mons, Belgium

frederic.bourgeois@universite-paris-saclay.fr, damien.galant@umons.ac.be

<https://www.imo.universite-paris-saclay.fr/~frederic.bourgeois/>,
<https://damien-gal.github.io/>

Received: 7 August 2020 Revised: 25 August 2022

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Kathryn Hess
kathryn.hess@epfl.ch
École Polytechnique Fédérale de Lausanne

BOARD OF EDITORS

Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu	Robert Lipshitz	University of Oregon lipshitz@uoregon.edu
Tara E Brendle	University of Glasgow tara.brendle@glasgow.ac.uk	Norihiko Minami	Yamato University minami.norihiko@yamato-u.ac.jp
Indira Chatterji	CNRS & Univ. Côte d'Azur (Nice) indira.chatterji@math.cnrs.fr	Andrés Navas	Universidad de Santiago de Chile andres.navas@usach.cl
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Robert Oliver	Université Paris 13 bobol@math.univ-paris13.fr
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Jessica S Purcell	Monash University jessica.purcell@monash.edu
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Birgit Richter	Universität Hamburg birgit.richter@uni-hamburg.de
David Futer	Temple University dfuter@temple.edu	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
John Greenlees	University of Warwick john.greenlees@warwick.ac.uk	Vesna Stojanoska	Univ. of Illinois at Urbana-Champaign vesna@illinois.edu
Ian Hambleton	McMaster University ian@math.mcmaster.ca	Zoltán Szabó	Princeton University szabo@math.princeton.edu
Matthew Hedden	Michigan State University mhedden@math.msu.edu	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Chris Wendl	Humboldt-Universität zu Berlin wendl@math.hu-berlin.de
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu	Daniel T Wise	McGill University, Canada daniel.wise@mcgill.ca
Thomas Koberda	University of Virginia thomas.koberda@virginia.edu	Lior Yanovski	Hebrew University of Jerusalem lior.yanovski@gmail.com
Markus Land	LMU München markus.land@math.lmu.de		

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2024 is US \$705/year for the electronic version, and \$1040/year (+\$70, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [Current Mathematical Publications](#) and the [Science Citation Index](#).

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 9 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Oakland, CA 94615-9651, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**

nonprofit scientific publishing

<https://msp.org/>

© 2024 Mathematical Sciences Publishers

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 24 Issue 7 (pages 3571–4137) 2024

Geography of bilinearized Legendrian contact homology	3571
FRÉDÉRIC BOURGEOIS and DAMIEN GALANT	
The deformation spaces of geodesic triangulations of flat tori	3605
YANWEN LUO, TIANQI WU and XIAOPING ZHU	
Finite presentations of the mapping class groups of once-stabilized Heegaard splittings	3621
DAIKI IGUCHI	
On the structure of the top homology group of the Johnson kernel	3641
IGOR A SPIRIDONOV	
The Heisenberg double of involutory Hopf algebras and invariants of closed 3-manifolds	3669
SERBAN MATEI MIHALACHE, SAKIE SUZUKI and YUJI TERASHIMA	
A closed ball compactification of a maximal component via cores of trees	3693
GIUSEPPE MARTONE, CHARLES OUYANG and ANDREA TAMBURELLI	
An algorithmic discrete gradient field and the cohomology algebra of configuration spaces of two points on complete graphs	3719
EMILIO J GONZÁLEZ and JESÚS GONZÁLEZ	
Spectral diameter of Liouville domains	3759
PIERRE-ALEXANDRE MAILHOT	
Classifying rational G -spectra for profinite G	3801
DAVID BARNES and DANNY SUGRUE	
An explicit comparison between 2-complicial sets and Θ_2 -spaces	3827
JULIA E BERGNER, VIKTORIYA OZORNOVA and MARTINA ROVELLI	
On products of beta and gamma elements in the homotopy of the first Smith–Toda spectrum	3875
KATSUMI SHIMOMURA and MAO-NO-SUKE SHIMOMURA	
Phase transition for the existence of van Kampen 2-complexes in random groups	3897
TSUNG-HSUAN TSAI	
A qualitative description of the horoboundary of the Teichmüller metric	3919
AITOR AZEMAR	
Vector fields on noncompact manifolds	3985
TSUYOSHI KATO, DAISUKE KISHIMOTO and MITSUNOBU TSUTAYA	
Smallest nonabelian quotients of surface braid groups	3997
CINDY TAN	
Lattices, injective metrics and the $K(\pi, 1)$ conjecture	4007
THOMAS HAETTEL	
The real-oriented cohomology of infinite stunted projective spaces	4061
WILLIAM BALDERRAMA	
Fourier transforms and integer homology cobordism	4085
MIKE MILLER EISMEIER	
Profinite isomorphisms and fixed-point properties	4103
MARTIN R BRIDSON	
Slice genus bound in $DT S^2$ from s -invariant	4115
QIUYU REN	
Relatively geometric actions of Kähler groups on $\text{CAT}(0)$ cube complexes	4127
COREY BREGMAN, DANIEL GROVES and KEJIA ZHU	