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The deformation spaces of geodesic triangulations of flat tori

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We prove that the deformation space of geodesic triangulations of a flat torus is homotopically equivalent to a torus. This solves an open problem proposed by Connelly et al. in 1983 in the case of flat tori. A key tool of the proof is a generalization of Tutte's embedding theorem for flat tori. While this paper was under preparation, Erickson and Lin proved a similar result, which works for all convex drawings.

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1 Introduction

This paper is a continuation of the previous work [Luo et al. 2023], where we proved that the deformation space of geodesic triangulations of a surface with negative curvature is contractible. The purpose of this paper is to identify the homotopy type of the deformation space of geodesic triangulations of a flat torus. This solves an open question proposed in [Connelly et al. 1983]. The main result of this paper is:

Theorem 1.1 *The deformation space of geodesic triangulations of a flat torus is homotopically equivalent to a torus.*

It is conjectured in [Connelly et al. 1983] that the space of geodesic triangulations of a closed orientable surface S with constant curvature deformation retracts to the group of orientation-preserving isometries of S homotopic to the identity. This paper affirms this conjecture in the case of flat tori. The case of hyperbolic surfaces was proved in [Luo et al. 2023]. In a very recent work, Erickson and Lin [2021] proved independently a generalized version of our Theorem 1.1 for general graph drawings on a flat torus.

The study of the homotopy types of spaces of geodesic triangulations stemmed from [Cairns 1944]. A brief history of this problem can be found in [Luo et al. 2023]. These spaces are closely related to diffeomorphism groups of surfaces. Bloch, Connelly and Henderson [Bloch et al. 1984] proved that the space of geodesic triangulations of a convex polygon is contractible. The space of geodesic triangulations of a planar polygon is equivalent to the space of simplexwise linear homeomorphisms. Hence, the Bloch–Connelly–Henderson theorem can be viewed as a discrete analogue of Smale's theorem, which states that the diffeomorphism group of the closed 2–disk fixing the boundary pointwise is contractible. Earle and Eells [1969] proved that the group of orientation-preserving diffeomorphisms of a torus isotopic to the identity is homotopically equivalent to a torus. Theorem 1.1 can be regarded as a discrete version of this theorem.

Similar to the previous work [Luo et al. 2023], our key idea to prove Theorem 1.1 originates from Tutte's embedding theorem.

1.1 Set up and the main theorem

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 = [0,1]^2/\sim$ be the flat torus constructed by gluing the opposite sides of the unit square in $\mathbb{R}^2$.

A topological triangulation of $\mathbb{T}^2$ can be identified as a homeomorphism ψ from $|\mathcal{T}|$ to $\mathbb{T}^2$, where $|\mathcal{T}|$ is the carrier of a 2–dimensional simplicial complex $\mathcal{T} = (V, E, F)$ with the vertex set V, and the edge set E, and the face set F. For convenience, we label the vertices as $v_1, \dots, v_n$, where $n = |V|$ is the number of the vertices. Denote by $|E|$ the number of edges, by $\vec{e}_{ij}$ the directed edge from the vertex i to its neighbor j, by $\vec{E} = \{\vec{e}_{ij} \mid ij \in E\}$ the set of directed edges, and by $N(i)$ the indices of neighboring vertices of $v_i \in V$.

A *geodesic triangulation* with the combinatorial type $(\mathcal{T}, \psi)$ is an embedding φ from the one-skeleton $\mathcal{T}^{(1)}$ to $\mathbb{T}^2$ satisfying that

(a) the restriction φ_{ij} of φ on each edge e_{ij}, identified with a unit interval $[0,1]$, is a geodesic of constant speed, and

(b) φ is homotopic to the restriction of ψ on $\mathcal{T}^{(1)}$.

Let $X = X(\mathbb{T}^2, \mathcal{T}, \psi)$ denote the set of all such geodesic triangulations, which is called a *deformation space of geodesic triangulations of* $\mathbb{T}^2$. This space can be defined for other flat tori in a similar fashion. Perturbing each vertex locally, we can construct a family of geodesic triangulations from an initial geodesic triangulation. Therefore, the space X is naturally a $2n$–dimensional manifold.

For any geodesic triangulation $\varphi \in X$, we can always translate φ on $\mathbb{T}^2$ to make the image $\varphi(v_1)$ of the first vertex v_1 be at the (quotient of the) origin $(0,0)$. By this normalization, we can decompose X as $X = X_0 \times \mathbb{T}^2$, where

$$X_0 = X_0(\mathbb{T}^2, \mathcal{T}, \psi) = \{\varphi \in X \mid \varphi(v_1) = (0,0)\}.$$

Since there are affine transformations between any two flat tori, and an affine transformation always preserves the geodesic triangulations, Theorem 1.1 reduces to the following.

Theorem 1.2 *Given a topological triangulation $(\mathcal{T}, \psi)$ of $\mathbb{T}^2$, the space $X_0 = X_0(\mathbb{T}^2, \mathcal{T}, \psi)$ is contractible.*

1.2 Key tool: generalized Tutte's embedding theorem

Let φ be a map from $\mathcal{T}^{(1)}$ to $\mathbb{T}^2$. Assume φ maps every edge in E to a geodesic arc parametrized by $[0,1]$ with constant speed on $\mathbb{T}^2$. A positive assignment $w \in \mathbb{R}_+^{\vec{E}}$ on the set of directed edges is called a *weight* of $\mathcal{T}$. We say φ is *w–balanced* at v_i if

$$\sum_{j \in N(i)} w_{ij} \dot{\varphi}_{ij} = 0,$$

where $\dot{\varphi}_{ij} = \dot{\varphi}_{ij}(0) \in T_{\varphi(v_i)}\mathbb{T}^2 \cong \mathbb{R}^2$. Then $\dot{\varphi}_{ij}$ indicates the direction of the edge $\varphi(e_{ij})$ and $\|\dot{\varphi}_{ij}\|$ equals the length of $\varphi(e_{ij})$. A map φ is called *w–balanced* if it is w–balanced at each vertex in V. We have the following version of Tutte's embedding theorem, which is a special case of Gortler, Gotsman and Thurston's embedding result in [Gortler et al. 2006] and Theorem 1.6 in [Luo et al. 2023].

Theorem 1.3 *Assume $(\mathcal{T}, \psi)$ is a topological triangulation of $\mathbb{T}^2$, and φ is a map from $\mathcal{T}^{(1)}$ to $\mathbb{T}^2$ such that φ is homotopic to $\psi|_{\mathcal{T}^{(1)}}$ and the restriction φ_{ij} of φ on each edge e_{ij} is a geodesic parametrized by constant speed. If φ is w–balanced for some weight $w \in \mathbb{R}_+^{\vec{E}}$, then φ is an embedding, or equivalently φ is a geodesic triangulation.*

To be self-contained, we will give a simple proof for Theorem 1.3, which is adapted from the argument in [Gortler et al. 2006].

The classical Tutte's embedding theorem [1963] states that a straight-line embedding of a simple 3–vertex-connected planar graph can be constructed by fixing an outer face as a convex polygon and solving interior vertices on the condition that each vertex is in the convex hull of its neighbors. Various new proofs of Tutte's embedding theorem have been proposed by Floater [2003b], Gortler, Gotsman and Thurston [Gortler et al. 2006], et al.

Tutte's embedding theorem has been generalized by Colin de Verdière [1991], Delgado and Friedrichs [2005], and Hass and Scott [2015] to surfaces with nonpositive Gaussian curvatures. They showed that the minimizer of a discrete Dirichlet energy is a geodesic triangulation. Here the fact that φ is a minimizer of a discrete Dirichlet energy means that φ is w–balanced for some symmetric weight w in $\mathbb{R}_+^{\vec{E}}$ with $w_{ij} = w_{ji}$. Their result also implies that $X = X(\mathbb{T}^2, \mathcal{T}, \psi)$ is not an empty set for any topological triangulation $(\mathcal{T}, \psi)$. Recently, Luo, Wu and Zhu [Luo et al. 2023] proved a new version of Tutte's embedding theorem for nonsymmetric weights and triangulations of orientable closed surfaces with nonpositive Gaussian curvature.

Gortler, Gotsman and Thurston [Gortler et al. 2006] generalized Tutte's embedding theorem to flat tori. In contrast to the case of convex polygons and surfaces of negative curvatures, it is not always possible to construct a geodesic triangulation of $\mathbb{T}^2$ such that it is w–balanced with respect to a given nonsymmetric weight w. See [Chambers et al. 2021, Section 1.1] for a detailed discussion.

1.3 Outline of the proof

Fix a lifting $(x_i, y_i) \in \mathbb{R}^2$ of $\psi(v_i) \in \mathbb{T}^2$ for each $i = 1, \dots, n$. Then for any $\vec{e}_{ij} \in \vec{E}$, there exists a unique lifting $\tilde{\psi}_{ij} : [0,1] \to \mathbb{R}^2$ of $\psi_{ij} = \psi|_{e_{ij}} : [0,1] \to \mathbb{T}^2$ such that $\tilde{\psi}_{ij}(0) = (x_i, y_i)$. Then

$$\tilde{\psi}_{ij}(1) = (x_j, y_j) + (b_{ij}^x, b_{ij}^y)$$

for some lattice point $(b_{ij}^x, b_{ij}^y) \in \mathbb{Z}^2$. Notice that for any $j \in N(i)$, the liftings $\tilde{\psi}_{ij}$ have the same base point (x_i, y_i), and b_{ij}^x and b_{ij}^y are determined by the liftings $\tilde{\psi}_i$, (x_i, y_i), and (x_j, y_j).

A geodesic triangulation φ in X_0 can be represented by $(x_i, y_i) \in \mathbb{R}^2$ for $i = 1, \ldots, n$ with $(x_1, y_1) = (0, 0)$. Under this representation, $\varphi_{ij} : [0, 1] \to \mathbb{T}^2$ is the quotient of the linear map

$$\varphi_{ij}(t) = t(x_j + b_{ij}^x, y_j + b_{ij}^y) + (1-t)(x_i, y_i),$$

and the equations for w–balanced conditions at all the vertices can be written as

$$\sum_{j \in N(i)} w_{ij}(x_j - x_i + b_{ij}^x) = 0 \quad \text{and} \quad \sum_{j \in N(i)} w_{ij}(y_j - y_i + b_{ij}^y) = 0.$$

In a closed matrix form, we can write

$$A(w)\boldsymbol{x} = \boldsymbol{b}(w), \tag{1}$$

where the weight matrix $A(w)$ is

$$A(w) = \begin{pmatrix} -\sum_{j=1}^n w_{1j} & w_{12} & w_{13} & \cdots & w_{1n} \\ w_{21} & -\sum_{j=1}^n w_{2j} & w_{23} & \cdots & w_{2n} \\ w_{31} & w_{32} & -\sum_{j=1}^n w_{3j} & \cdots & w_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & w_{n3} & \cdots & -\sum_{j=1}^n w_{nj} \end{pmatrix},$$

and

$$\boldsymbol{x} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{pmatrix}, \quad \boldsymbol{b}(w) = \begin{pmatrix} -\sum_{j=1}^n w_{1j} b_{1j}^x & -\sum_{j=1}^n w_{1j} b_{1j}^y \\ -\sum_{j=1}^n w_{2j} b_{2j}^x & -\sum_{j=1}^n w_{2j} b_{2j}^y \\ \vdots & \vdots \\ -\sum_{j=1}^n w_{nj} b_{nj}^x & -\sum_{j=1}^n w_{nj} b_{nj}^y \end{pmatrix}.$$

Here we write $w_{ij} = 0$ if $e_{ij} \notin E$.

A weight w in $\mathbb{R}_+^{\vec{E}}$ is called *admissible* if (1) is solvable. Let W be the space of admissible weights. For any $w \in W$, a solution $\boldsymbol{x}$ to (1) uniquely determines the coordinates of the vertices and a w–balanced map φ that is homotopic to $\psi|_{\mathcal{T}^{(1)}}$. By Theorem 1.3, such a φ is an embedding, and $\varphi \in X$. Noticing that $A(w)$ is weakly diagonally dominant and the graph $\mathcal{T}^{(1)}$ is connected, the solution to (1) is unique up to a 2–dimensional translation, and is unique if we require $(x_1, y_1) = (0, 0)$. Define the *Tutte map* as

$$\Psi : W \to X_0$$

sending an admissible weight w to the unique w–balanced geodesic triangulation in X_0. The Tutte map is continuous by the continuous dependence of the solutions to the coefficients in a linear system.

The Tutte map is also surjective, since there exists a smooth map σ from X_0 to W by the "mean value coordinates"

$$w_{ij} = \frac{\tan(\alpha_i^j/2) + \tan(\beta_i^j/2)}{l_{ij}}$$

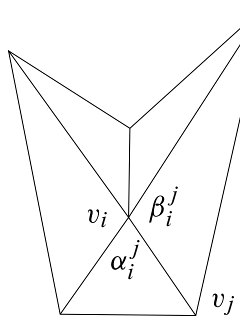

Figure 1: The mean value coordinate.

introduced in [Floater 2003a]. Here α_i^j, β_i^j are two inner angles adjacent to e_{ij} at the vertex v_i, and l_{ij} is the edge length of φ_{ij}. See Figure 1 for an illustration. Floater [2003a] showed that any geodesic triangulation φ is $\sigma(\varphi)$–balanced, ie $\Psi \circ \sigma = \mathrm{Id}_{X_0}$.

Having the knowledge of the Tutte map and the mean value coordinates, Theorem 1.2 reduces to the following proposition.

Proposition 1.4 *There exists a continuous map* $\Phi\colon \mathbb{R}_+^{\vec{E}} \to W$ *such that*

$$\Phi|_W = \mathrm{Id}_W.$$

Proof of Theorem 1.2 assuming Proposition 1.4 By Proposition 1.4, W is contractible since there exists a retraction Φ from the contractible space $\mathbb{R}_+^{\vec{E}}$ to W. So $\sigma \circ \Psi$ is homotopic to the identity map on W. On the other hand $\Psi \circ \sigma = \mathrm{Id}_{X_0}$, and thus X_0 is homotopic to W and contractible. □

1.4 Organization of this paper

In Section 2, we will prove Proposition 1.4 by constructing a flow. In Section 3, we prove Theorem 1.3 following the idea in [Gortler et al. 2006].

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2 Proof of Proposition 1.4

Set an energy function on the weight space $\mathbb{R}_+^{\vec{E}}$ as

$$\mathcal{E}(w) = \min_{\boldsymbol{x} \in \mathbb{R}^{n\times 2}} \|A(w)\boldsymbol{x} - \boldsymbol{b}(w)\|^2 = \min_{\boldsymbol{x} \in \mathbb{R}^{n\times 2} | x_1 = y_1 = 0} \|A(w)\boldsymbol{x} - \boldsymbol{b}(w)\|^2,$$

where the norm is the Frobenius norm of a matrix. The second minimization problem above is a least square problem with $2(n-1)$ real variables and a nondegenerate coefficient matrix. By the standard formula in linear least squares (LLS) or quadratic programming (QP), the minimizer, denoted by $\boldsymbol{x}(w)$, is a smooth function of w, and thus $\mathcal{E}(w)$ is also a smooth function of w. Note that $\mathcal{E}(w)=0$ if and only if w is admissible, and intuitively $\mathcal{E}(w)$ measures the deviation of w from being admissible. The key idea of the proof is to construct a flow on $\mathbb{R}_+^{\vec{E}}\backslash W$ to minimize $\mathcal{E}(w)$, as in the following lemma.

Lemma 2.1 *There exists a smooth function $\Theta\colon \mathbb{R}_+^{\vec{E}}\backslash W \to \mathbb{R}^{\vec{E}}$ and a positive continuous function $C(w)$ on $\mathbb{R}_+^{\vec{E}}$ such that, for any initial value $w^0 \in \mathbb{R}_+^{\vec{E}}$, the flow $w(t)$ defined by*

$$\begin{cases} \dot{w}(t) = \Theta(w(t)), \\ w(0) = w^0 \end{cases} \tag{2}$$

satisfies that, for any t in the maximum existing interval $[0,T)$,

(a) $0 \le \dot{w}_{ij}(t) \le w_{ij}(t)$, *and*

(b) $\dfrac{d\mathcal{E}(w(t))}{dt} \le -C(w^0)\sqrt{\mathcal{E}(w(t))}$.

Proposition 1.4 is proved in Section 2.1, assuming Lemma 2.1. Then we construct a flow in Section 2.2, and in Section 2.3 show that this flow is satisfactory for Lemma 2.1.

2.1 Proof of Proposition 1.4 assuming Lemma 2.1

Assume $\Theta(w)$ and $C=C(w^0)$ are as in Lemma 2.1. Given $w^0 \in \mathbb{R}_+^{\vec{E}}\backslash W$, assume $w(t)$ is the flow defined by equation (2), and $[0,T)$ is the maximum existing interval.

We claim that $T=T(w^0)<\infty$ and $w(t)$ converges to some $\bar{w}$ as $t\to T$. Since

$$\frac{d\mathcal{E}}{dt} \le -C\sqrt{\mathcal{E}},$$

we have

$$\frac{d(\sqrt{\mathcal{E}})}{dt}(t) \le -\frac{C}{2}$$

and

$$\sqrt{\mathcal{E}(w(t))} \le \sqrt{\mathcal{E}(w(0))} - \frac{Ct}{2},$$

which implies

$$T \le \frac{2\sqrt{\mathcal{E}(w^0)}}{C(w^0)} < \infty.$$

Since

$$0 \le \dot{w}_{ij}(t) \le w_{ij}(t),$$

we have

$$w_{ij}(t) \le w_{ij}^0 e^t \le w_{ij}^0 e^T. \tag{3}$$

Then by the monotone convergence theorem, $w(t)$ converges to some $\bar{w}$. By the maximality of T, $\bar{w}$ has to be in W. Let $\Phi\colon \mathbb{R}_+^{\vec{E}} \to W$ be such that $\Phi(w^0) = \bar{w}$ if $w^0 \notin W$, and $\Phi(w) = w$ if $w \in W$.

Now we prove that Φ is continuous, ie for any $w \in \mathbb{R}_+^{\vec{E}}$ and $\epsilon > 0$ there exists $\delta > 0$ such that $|\Phi(w') - \Phi(w)|_\infty \le \epsilon$ for any w' with $|w' - w|_\infty < \delta$. We consider the two cases $w \in W$ and $w \notin W$.

2.1.1 $\boldsymbol{w \in W}$ Since $C(w)$ is continuous, there exist $C_1 > 0$ and $\delta_1 > 0$ such that $C(w') \ge C_1$ for any w' with $|w - w'|_\infty \le \delta_1$. Since $\mathcal{E}$ is continuous, there exists $\delta_2 \in (0, \delta_1)$ such that

$$\mathcal{E}(w') \le \left[\frac{C_1}{2}\log\left(1 + \frac{\epsilon}{2|w|_\infty + \epsilon}\right)\right]^2$$

if $|w' - w|_\infty < \delta_2$. Then we will show that $\delta = \min\{\delta_2, \epsilon/2\}$ is satisfactory. Assume w' satisfies that $|w' - w|_\infty < \delta$. If $w' \in W$, then $|\Phi(w') - \Phi(w)|_\infty = |w' - w|_\infty < \delta \le \epsilon$. If $w' \notin W$,

$$T(w') \le \frac{2\sqrt{\mathcal{E}(w')}}{C(w')} \le \frac{C_1 \log(1 + \epsilon/(2|w|_\infty + \epsilon))}{C_1} = \log\left(1 + \frac{\epsilon}{2|w|_\infty + \epsilon}\right) \le \log\left(1 + \frac{\epsilon}{2|w'|_\infty}\right).$$

So by inequality (3),

$$|\Phi(w')_{ij} - \Phi(w)_{ij}| \le |\Phi(w')_{ij} - w'_{ij}| + |w'_{ij} - w_{ij}| < w'_{ij}(e^{T(w')} - 1) + \frac{\epsilon}{2} \le \epsilon.$$

2.1.2 $\boldsymbol{w \notin W}$ Assume $\bar{w} = \Phi(w) \in W$. Then by the result of the previous case, there exists $\delta_1 > 0$ such that $|\Phi(w') - \Phi(w)|_\infty < \epsilon$ for any w' with $|w' - \bar{w}|_\infty < \delta_1$. Assume $w(t)$ is the flow determined by (2) with the initial value $w^0 = w$. Then there exists some t_0, such that $|w(t_0) - \bar{w}|_\infty < \delta_1/2$. By the continuous dependence of the solutions of ODEs on the initial values, there exists $\delta > 0$ such that if $|w' - w|_\infty < \delta$, then $|w'(t_0) - w(t_0)| < \delta_1/2$, where $w'(t)$ is the flow determined by (2) with the initial value $w^0 = w'$. So if $|w - w'|_\infty < \delta$, we have $|w'(t_0) - \bar{w}| < \delta_1$ and

$$|\Phi(w') - \Phi(w)|_\infty = |\Phi(w'(t_0)) - \Phi(w)|_\infty < \epsilon.$$

2.2 Construction of the flow

Denote by $\boldsymbol{x}(w)$ the minimizer of the second minimization problem in the definition of the energy function $\mathcal{E}(w)$. Define the residual $\boldsymbol{r}(w)$ as

$$\boldsymbol{r}(w) = A(w)\boldsymbol{x}(w) - \boldsymbol{b}(w),$$

where

$$\boldsymbol{r}(w) = \begin{pmatrix} r_1^x(w) & r_1^y(w) \\ r_2^x(w) & r_2^y(w) \\ \vdots & \vdots \\ r_n^x(w) & r_n^y(w) \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}^x(w) & \boldsymbol{r}^y(w) \end{pmatrix} = \begin{pmatrix} \boldsymbol{r}_1(w) \\ \boldsymbol{r}_2(w) \\ \vdots \\ \boldsymbol{r}_n(w) \end{pmatrix} \in \mathbb{R}^{n \times 2}.$$

The vector $\boldsymbol{r}_i$ is the residual at the vertex $v_i \in V$, and $\boldsymbol{r}^x$, $\boldsymbol{r}^y$ are the projections of the total residual in the directions of the x–axis and the y–axis respectively. Then by the minimality of $\boldsymbol{x}(w)$,

$$A^T(w)\boldsymbol{r}^x(w) = 0 \quad \text{and} \quad A^T(w)\boldsymbol{r}^y(w) = 0.$$

Equivalently,

$$\boldsymbol{r}^x(w) \perp A(w)(\mathbb{R}^n) \quad \text{and} \quad \boldsymbol{r}^y(w) \perp A(w)(\mathbb{R}^n).$$

Since $\operatorname{rank}(A(w)) = n-1$, we have $\boldsymbol{r}^x \parallel \boldsymbol{r}^y$. So $\operatorname{rank}(\boldsymbol{r}) \le 1$, and $\boldsymbol{r}_i \parallel \boldsymbol{r}_j$ for any $1 \le i, j \le n$. Here $u \parallel v$ means that vectors u and v are parallel, ie linearly dependent.

Lemma 2.2 *Assume $\boldsymbol{r} \neq 0$ and the following properties hold for $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n$:*

(a) *The vectors have the same direction, namely,*

$$\langle \boldsymbol{r}_i, \boldsymbol{r}_j \rangle > 0 \quad \text{for all } 1 \le i, j \le n.$$

(b) *If*

$$C = \max_{\vec{e}_{ij} \in \vec{E}} \frac{w_{ij}}{w_{ji}},$$

then

$$\frac{\|\boldsymbol{r}_i\|_2}{\|\boldsymbol{r}_j\|_2} \le C^{n-1} \quad \text{for all } 1 \le i, j \le n.$$

Proof Without loss of generality, after a rotation we can assume that all the vectors $\boldsymbol{r}_i$ are parallel to the x–axis, namely $\boldsymbol{r}^y = 0$.

To prove part (a), assume that $\langle \boldsymbol{r}_i, \boldsymbol{r}_j \rangle = r_i^x \cdot r_j^x \le 0$ for some $1 \le i, j \le n$. Then one can find a nonzero vector $\boldsymbol{p} = (p_1, \dots, p_n)^T \in \mathbb{R}^n_{\ge 0}$ so that $\boldsymbol{p} \perp \boldsymbol{r}^x$. Then $\boldsymbol{p} \in A(w)(\mathbb{R}^n)$ and there exists $\boldsymbol{q} = (q_1, \dots, q_n)^T \in \mathbb{R}^n$ with $\boldsymbol{p} = A(w)\boldsymbol{q}$. Then if $q_i = \max_j q_j$ for some i,

$$0 \le p_i = \sum_{j=1}^n w_{ij}(q_j - q_i) \le 0,$$

and thus $q_j = q_i$ if $j \in N(i)$. By this maximum principle and the connectedness of the graph, $q_j = q_i$ for any $j \in V$, and $\boldsymbol{p} = A(w)\boldsymbol{q} = 0$. This contradicts that $\boldsymbol{p}$ is nonzero.

If part (b) is not true, ordering the set V based on the values of r_i^x monotonically, one can find a nonempty proper subset $V_0 \subsetneq V$ such that

$$\frac{\min_{i \in V_0}\{r_i^x\}}{\max_{i \in V - V_0}\{r_i^x\}} > C.$$

Choose a vector $\boldsymbol{p} \in \mathbb{R}^n$ such that $p_i = 1$ if $i \in V_0$, and $p_i = 0$ otherwise. Then the contradiction follows from

$$\begin{aligned} 0 = \langle \boldsymbol{r}^x, A(w)\boldsymbol{p} \rangle &= \sum_{i \in V} r_i^x \sum_{j \in N(i)} w_{ij}(p_j - p_i) = \sum_{\vec{e}_{ij} \in \vec{E}} w_{ij} r_i^x p_j - \sum_{i \in V} r_i^x p_i \sum_{j \in N(i)} w_{ij} \\ &= \sum_{\substack{\vec{e}_{ij} \in \vec{E} \\ j \in V_0}} w_{ij} r_i^x - \sum_{i \in V_0} r_i^x \sum_{j \in N(i)} w_{ij} = \sum_{\substack{i \in V_0, j \in V - V_0 \\ j \in N(i)}} (r_j^x w_{ji} - r_i^x w_{ij}) < 0. \end{aligned}$$ □

Assume $\boldsymbol{n} = \boldsymbol{n}(w) \in \mathbb{R}^2$ is the unit vector that is parallel to $\boldsymbol{r}_1$ and $\langle \boldsymbol{n}, \boldsymbol{r}_1 \rangle > 0$. Define for each directed edge

$$u_{ij} = u_{ij}(w) = \boldsymbol{n} \cdot (\boldsymbol{x}_j - \boldsymbol{x}_i + (b_{ij}^x, b_{ij}^y)) \quad \text{for all } \vec{e}_{ij} \in \vec{E},$$

where $\boldsymbol{x}_i = (x_i, y_i)$ is given by the minimizer $\boldsymbol{x}(w)$. Note that

$$\|\boldsymbol{r}_i\|_2 = \boldsymbol{n} \cdot \boldsymbol{r}_i = \sum_{j \in N(i)} w_{ij} u_{ij}.$$

Lemma 2.3 *There exists a constant $\beta = \beta(T, \psi) > 0$ such that for any $w \in \mathbb{R}_+^{\vec{E}}$, there exists $\vec{e}_{ij} \in \vec{E}$ such that $u_{ij}(w) \leq -\beta$.*

Proof Since $u_{ij} = -u_{ji}$ for any $ij \in E$, it suffices to find $\vec{e}_{ij}$ such that $|u_{ij}(w)| \geq \beta$. Assume $\boldsymbol{n} = (n_1, n_2) \in \mathbb{R}^2$. Then $|n_1| \geq 1/\sqrt{2}$ or $|n_2| \geq 1\sqrt{2}$.

If $|n_1| \geq 1/\sqrt{2}$, let $\gamma_1 = \mathbb{T} \times \{0\}$ be a horizontal simple loop in $\mathbb{T}^2$. Then $\psi^{-1}(\gamma_1)$ is a simple loop in the carrier of $\mathcal{T}$, and it is not difficult to show that there exists a sequence of vertices $v(1), \dots, v(k) = v(0)$ such that $v(i) \sim v(i+1)$ for any $i = 0, \dots, k-1$, and the union $\bigcup_{i=0}^{k-1} e_{v(i)v(i+1)}$ is a piecewise linear loop in $|\mathcal{T}|$, which is homotopic to $\psi^{-1}(\gamma_1)$. By choosing an appropriate orientation, we have

$$\sum_{i=0}^{k-1} (\boldsymbol{x}_{v(i)+1} - \boldsymbol{x}_{v(i)} + (b^x_{v(i+1)v(i)}, b^y_{v(i+1)v(i)})) = (1, 0).$$

So

$$\sum_{i=0}^{k-1} u_{v(i)v(i+1)} = \boldsymbol{n} \cdot (1, 0) = n_1,$$

and there exists some i such that $|u_{v(i)v(i+1)}| \geq |n_1|/k \geq 1/(\sqrt{2}k)$. Notice that here k is a constant depending only on $\mathcal{T}$ and ψ.

Similarly, if $|n_2| \geq 1/\sqrt{2}$, there exists some $\vec{e}_{ij} \in \vec{E}$ such that $|u_{ij}| \geq 1/(\sqrt{2}k')$ for some constant $k' = k'(\mathcal{T}, \psi)$. ◻

We define the smooth flow Θ on the domain $\mathbb{R}_+^{\vec{E}} \backslash W$ on each edge as

$$\begin{cases} \dot{w}_{ij} = w_{ij} \cdot g\Big(\dfrac{1}{\alpha}(w_{ij} + w_{ji}) u_{ij}\Big) \cdot h(w_{ij} - w_{ji}), \\ w_{ij}(0) = w_{ij}^0, \end{cases} \tag{4}$$

where g and h are smooth nonincreasing functions such that

(a) $g \equiv 1$ on $(-\infty, -1)$ and $g \equiv 0$ on $[0, +\infty)$, and

(b) $h \equiv 1$ on $(-\infty, 1)$ and $h \equiv 0$ on $[2, +\infty)$, and

(c) $\alpha = \alpha(w) = \beta \cdot \Big(2|E| + \sum_{\vec{e}_{ij} \in \vec{E}} w_{ij}^{-1}\Big)^{-1}$.

Roughly speaking, the function g tends be positive if $u_{ij} < 0$, meaning that $w_{ij} u_{ij}$ will decrease so as to reduce the residual $\|\boldsymbol{r}_i\|_2$. The function h controls the difference between w_{ij} and w_{ji}, and note that $\alpha(w)$ is smooth and very small. Specifically we have

$$\alpha(w) \leq \frac{\beta}{2|E|} \quad \text{and} \quad \alpha(w) \leq \beta L(w), \tag{5}$$

where $L(w) = \min_{\vec{e}_{ij} \in \vec{E}} w_{ij}$ is a continuous function on $w \in \mathbb{R}_{+}^{\vec{E}}$. Define

$$M(w) = \max\Big\{2, \max_{\vec{e}_{ij} \in \vec{E}} |w_{ij} - w_{ji}|\Big\},$$

which is another continous function on $\mathbb{R}_{+}^{\vec{E}}$.

Lemma 2.4 *Assume the flow $w(t)$ satisfies (4). Then we have the following*:

(a) $0 \le \dot{w}_{ij} \le w_{ij}$.

(b) *$u_{ij} \ge 0$ implies $\dot{w}_{ij} = 0$. Then $\dot{w}_{ij} u_{ij} \le 0$ for all directed edges.*

(c) *$w_{ij} - w_{ji} \ge 2$ implies $\dot{w}_{ij} = 0$.*

(d) *$L(w(t))$ is nondecreasing and $M(w(t))$ is nonincreasing.*

(e) *For any edge ij,*

$$\frac{w_{ij}(t)}{w_{ji}(t)} \le 1 + \frac{M}{L}.$$

(f) *The residual vectors $\boldsymbol{r}_i(t)$ satisfy*

$$\frac{\max_{i \in V} \|\boldsymbol{r}_i(t)\|_2}{\min_{i \in V} \|\boldsymbol{r}_i(t)\|_2} \le (1 + \frac{M}{L})^{n-1} \quad \text{for all } 1 \le i, j \le n.$$

(f) *The residual vectors $\boldsymbol{r}_i(t)$ satisfy*

$$\frac{\sqrt{\mathcal{E}(w(t))}}{\sqrt{n}(1 + M/L)^{n-1}} \le \|\boldsymbol{r}_i(t)\|_2 \quad \text{for all } 1 \le i \le n. \tag{6}$$

Proof Parts (a)–(d) are straightforward from (4) and the defining properties of smooth functions g and h. Part (e) follows from

$$\frac{w_{ij}(t)}{w_{ji}(t)} = 1 + \frac{w_{ij}(t) - w_{ji}(t)}{w_{ji}(t)} \le 1 + \frac{M}{L}.$$

Part (f) follows from part (e) and Lemma 2.2. For part (g), by definition

$$\mathcal{E}(t) = \sum_{j=1}^{n} \|\boldsymbol{r}_j(t)\|_2^2.$$

Part (f) implies that

$$\mathcal{E}(t) \le n\Big(1 + \frac{M}{L}\Big)^{2n-2} \|\boldsymbol{r}_i(t)\|_2^2 \quad \text{and} \quad \frac{\sqrt{\mathcal{E}(t)}}{\sqrt{n}(1 + M/L)^{n-1}} \le \|\boldsymbol{r}_i(t)\|_2 \quad \text{for all } 1 \le i \le n. \qquad \square$$

2.3 Proof of Lemma 2.1

Proof Let

$$C(w) = \frac{\beta L/M}{2\sqrt{n}(1 + M/L)^{n-1}},$$

where $L = L(w)$ and $M = M(w)$ are continous functions on $\mathbb{R}_+^{\vec{E}}$, on which $C(w)$ is also continuous. We claim that such a function $C(w)$ and the flow Θ defined as (4) are satisfactory. Assume $w^0 \in \mathbb{R}_+^{\vec{E}} \backslash W$ and $w(t)$ is a flow defined by (4). By part (a) of Lemma 2.4 we only need to prove part (b) of Lemma 2.1. By part (d) of Lemma 2.4, it is easy to see that $C(w(t))$ is nondecreasing on t. So we only need to prove that

$$\frac{d\mathcal{E}(w(t))}{dt} \le -C(w(t))\sqrt{\mathcal{E}(w(t))}.$$

Given $w \in R_+^{\vec{E}}$ and $\boldsymbol{x} \in \mathbb{R}^{n\times 2}$, define

$$\widetilde{\mathcal{E}}(w,\boldsymbol{x}) = \|A(w)\boldsymbol{x} - \boldsymbol{b}(w)\|^2,$$

and then

$$\begin{aligned}\frac{d\mathcal{E}(w(\cdot))}{dt}(t) &= \lim_{\epsilon\to 0}\frac{\mathcal{E}(w(t+\epsilon)) - \mathcal{E}(w(t))}{\epsilon}\\ &\le \lim_{\epsilon\to 0}\frac{\widetilde{\mathcal{E}}(w(t+\epsilon),\boldsymbol{x}(w(t))) - \widetilde{\mathcal{E}}(w(t),\boldsymbol{x}(w(t)))}{\epsilon} = \frac{\partial\widetilde{\mathcal{E}}}{\partial w}(w(t),\boldsymbol{x}(w(t)))\cdot\dot{w}.\end{aligned}$$

So it suffices to show

$$\frac{\partial\widetilde{\mathcal{E}}}{\partial w}(w(t),\boldsymbol{x}(w(t)))\cdot\dot{w} \le -C\sqrt{\mathcal{E}(w(t))}.$$

Notice that

$$\begin{aligned}\frac{\partial\widetilde{\mathcal{E}}}{\partial w_{ij}}(w(t),\boldsymbol{x}(w(t))) &= \left(\frac{\partial}{\partial w_{ij}}\sum_{i=1}^{n}\Big\|\sum_{j\in N(i)} w_{ij}(\boldsymbol{x}_j - \boldsymbol{x}_i + (b_{ij}^x, b_{ij}^y))\Big\|_2^2\right)\Big|_{(w,\boldsymbol{x}(w))}\\ &= 2\boldsymbol{r}_i\cdot(\boldsymbol{x}_j - \boldsymbol{x}_i + (b_{ij}^x, b_{ij}^y))\\ &= 2\|\boldsymbol{r}_i\|_2\boldsymbol{n}\cdot(\boldsymbol{x}_j - \boldsymbol{x}_i + (b_{ij}^x, b_{ij}^y)) = 2\|\boldsymbol{r}_i\|_2\cdot u_{ij}\end{aligned}$$

and then,

$$\frac{\partial\widetilde{\mathcal{E}}}{\partial w}(w(t),\boldsymbol{x}(w(t)))\cdot\dot{w} = 2\sum_{\vec{e}_{ij}\in\vec{E}}\|\boldsymbol{r}_i\|_2 u_{ij}\dot{w}_{ij} \le \frac{2\sqrt{\mathcal{E}(w(t))}}{\sqrt{n}(1+M/L)^{n-1}}\sum_{\vec{e}_{ij}\in\vec{E}} u_{ij}\dot{w}_{ij}. \tag{7}$$

Here we use the fact that $\dot{w}_{ij}u_{ij} \le 0$ for all directed edges (part (b) of Lemma 2.4), and inequality (6). It remains to show that

$$\sum_{\vec{e}_{ij}\in\vec{E}} u_{ij}\dot{w}_{ij} \le -\frac{\beta L}{2M}.$$

By Lemma 2.3 there exists a directed edge $\vec{e}_{i'j'}$ with $u_{i'j'} \le -\beta$. Then we will consider the following two cases:

Case 1 $(w_{i'j'} - w_{j'i'} \le 1)$ By the definition of the function h,

$$h(w_{i'j'} - w_{j'i'}) = 1.$$

We also have

$$g\Big(\frac{1}{\alpha}(w_{i'j'} + w_{j'i'})u_{i'j'}\Big) = 1,$$

since

$$(w_{i'j'}+w_{j'i'})u_{i'j'} \le -2\beta L \le -\alpha.$$

By (4), $\dot{w}_{i'j'} = w_{i'j'}$. Notice that $\dot{w}_{ij}u_{ij} \le 0$ for any $\vec{e}_{ij} \in \vec{E}$ by part (b) of Lemma 2.4, so

$$\sum_{\vec{e}_{ij}\in\vec{E}} u_{ij}\dot{w}_{ij} \le u_{i'j'}\dot{w}_{i'j'} = u_{i'j'}w_{i'j'} \le -\beta L \le -\frac{\beta L}{2M}.$$

Case 2 $(w_{i'j'} - w_{j'i'} \ge 1)$ Define

$$\vec{E}_0 = \{\vec{e}_{ij} \in \vec{E} \mid u_{ij} < 0,\ (w_{ij}-w_{ji})u_{ij} \ge \alpha\}.$$

If $\vec{e}_{ij} \in \vec{E}_0$, then obviously $w_{ij} - w_{ji} < 0$ and

$$h(w_{ij}-w_{ji}) = 1.$$

Also,

$$g\Big(\frac{1}{\alpha}(w_{ij}+w_{ji})u_{ij}\Big) = 1$$

since

$$(w_{ij}+w_{ji})u_{ij} \le (w_{ji}-w_{ij})u_{ij} \le -\alpha.$$

By (4), $\dot{w}_{ij} = w_{ij}$ and

(8) $$\sum_{\vec{e}_{ij}\in\vec{E}} \dot{w}_{ij}u_{ij} \le \sum_{\vec{e}_{ij}\in\vec{E}_0} \dot{w}_{ij}u_{ij} = \sum_{\vec{e}_{ij}\in\vec{E}_0} w_{ij}u_{ij} \le -\frac{L}{M}\sum_{e_{ij}\in E_0}(w_{ij}-w_{ji})u_{ij}.$$

The last inequality uses the fact that $w_{ij} \ge -L(w_{ij}-w_{ji})/M$, which is equivalent to $w_{ji}/w_{ij} \le 1+M/L$.

By the fact that $u_{i'j'} \le -\beta$, and the assumption $w_{i'j'} - w_{j'i'} \ge 1$,

$$(w_{i'j'}-w_{j'i'})u_{i'j'} \le -\beta < 0 < \alpha,$$

and thus $\vec{e}_{i'j'} \notin \vec{E}_0$. Notice that

$$\sum_{\vec{e}_{ij}\in\vec{E}} w_{ij}u_{ij} = \sum_{i=1}^{n}\sum_{j\in N(i)} w_{ij}u_{ij} = \sum_{i=1}^{n} \|\boldsymbol{r}_i\|_2 \ge 0,$$

and

$$\begin{aligned}\sum_{\vec{e}_{ij}\in\vec{E}} w_{ij}u_{ij} &= \sum_{\vec{e}_{ij}\in\vec{E}:u_{ij}<0} (w_{ij}-w_{ji})u_{ij}\\ &= \sum_{\vec{e}_{ij}\in\vec{E}_0}(w_{ij}-w_{ji})u_{ij} + \sum_{e_{ij}\in\vec{E}-\vec{E}_0-\{e_{i'j'}\}:u_{ij}<0}(w_{ij}-w_{ji})u_{ij} + (w_{i'j'}-w_{j'i'})u_{i'j'}\\ &\le \sum_{\vec{e}_{ij}\in\vec{E}_0}(w_{ij}-w_{ji})u_{ij} + |E|\alpha - \beta.\end{aligned}$$

Then

$$\sum_{\vec{e}_{ij}\in\vec{E}_0}(w_{ij}-w_{ji})u_{ij}\geq\beta-|E|\alpha\geq\frac{\beta}{2},$$

and

$$\sum_{\vec{e}_{ij}\in\vec{E}}u_{ij}\dot{w}_{ij}\leq-\frac{\beta L}{2M}.$$

□

3 Proof of Theorem 1.3

We will first introduce the concept of discrete one-forms, and the index theorem proposed in [Gortler et al. 2006].

3.1 Discrete one-forms and the index theorem

A *discrete one-form* is a real-valued function η on the set of directed edges such that it is antisymmetric on each undirected edge. Specifically, let $\eta_{ij}=\eta(\vec{e}_{ij})$ be the value of η on the directed edge from v_i to v_j; then we have $\eta_{ij}=-\eta_{ji}$.

For a discrete one-form, an edge is degenerate (resp nonvanishing) if the one-form is zero (resp nonzero) on it. A vertex is degenerate (resp nonvanishing) if all of edges connected to it are degenerate (resp nonvanishing). A face is degenerate (resp nonvanishing) if all of its three edges are degenerate (resp nonvanishing). A one-form is degenerate (resp nonvanishing) if all the edges are degenerate (resp nonvanishing). Each edge is either degenerate or nonvanishing. However, vertices or faces can be degenerate, nondegenerate but vanishing on some edges, or nonvanishing.

Assume η is a discrete one-form. Denote by $\mathrm{sc}(\eta,v)$ the number of *sign changes* of the nonzero values of η on the directed edges starting from v, counted in counterclockwise order. For a vertex $v\in V$, define the *index* of v as $\mathrm{Ind}(\eta,v)=(2-\mathrm{sc}(\eta,v))/2$. Similarly, for a nondegenerate face $t\in F$, the index of t is $\mathrm{Ind}(\eta,t)=(2-\mathrm{sc}(\eta,t))/2$, where $\mathrm{sc}(\eta,t)$ is the number of sign changes of the nonzero values of η on the three edges of t, counted in counterclockwise order.

The following theorem is a special case of the index theorem from [Gortler et al. 2006], which is a discrete version of the Poincaré–Hopf theorem for discrete one-forms.

Theorem 3.1 *Let η be a nonvanishing discrete one-form on a triangulation of a torus. Then*

$$\sum_{v_i\in V}\mathrm{Ind}(\eta,v_i)+\sum_{t_{ijk}\in F}\mathrm{Ind}(\eta,t_{ijk})=0.$$

Assume φ satisfies the assumption in Theorem 1.3; then for any unit vector $\boldsymbol{n}\in\mathbb{R}^2$ we can naturally construct a discrete one-form η by letting $\eta_{ij}=\dot{\varphi}_{ij}\cdot\boldsymbol{n}$. If $\varphi\in X$, a generic unit vector determines a nonvanishing discrete one-form η. Further, if $\varphi\in X$ and such a constructed η is nonvanishing, it is

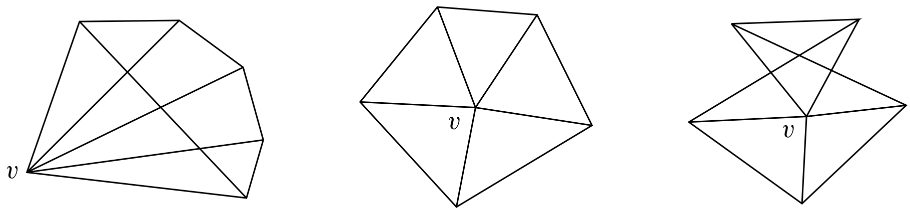

Figure 2: Typical vertex with positive (left), zero (middle), and negative (right) index.

not difficult to show that all the indices of the vertices and faces are zero. Figure 2 illustrates how the neighborhood of v looks if it has positive, or zero, or negative index for the case $\boldsymbol{n} = (1, 0)$.

Based on this construction, we have:

Lemma 3.2 *Given a triangulation $(\mathcal{T}, \psi)$ of $\mathbb{T}^2$, denote by $t_{ijk} \in F$ the triangle with three vertices v_i, v_j, and v_k. There exists a nonvanishing discrete one-form η such that $\eta_{ij} > 0$ and $\eta_{jk} > 0$. Moreover, all the indices of the vertices and faces of η are zero.*

Proof By the result of [Colin de Verdière 1991] and [Hass and Scott 2015], the space $X(\mathcal{T}, \psi)$ is not empty for any $(\mathcal{T}, \psi)$. Let φ be a geodesic triangulation in X. Then it is not difficult to find a unit vector $\boldsymbol{n}$ such that $\dot{\varphi}_{ij} \cdot \boldsymbol{n} > 0$ and $\dot{\varphi}_{jk} \cdot \boldsymbol{n} > 0$. Define the discrete one-form η as $\eta_{ij} = \dot{\varphi}_{ij} \cdot \boldsymbol{n}$. We can perturb the unit vector $\boldsymbol{n}$ a little bit to make η nonvanishing, and then such an η is satisfactory. □

3.2 The proof of Theorem 1.3

Assume $\varphi\colon \mathcal{T}^{(1)} \to \mathbb{T}^2$ satisfies the assumption of Theorem 1.3; then there exists a unique extension $\bar{\varphi}\colon |\mathcal{T}| \to \mathbb{T}^2$ such that the restriction of $\bar{\varphi}$ to every face is linear. Such a $\bar{\varphi}$ is homotopic to ψ, and φ is a geodesic triangulation in X if and only if $\bar{\varphi}$ is a homeomorphism.

For any triangle $t_{ijk} \in X$, we say that $\bar{\varphi}(t_{ijk})$ is *degenerate* if $\bar{\varphi}(t_{ijk})$ is contained in some geodesic λ. If $\bar{\varphi}(t_{ijk})$ is not degenerate, we can naturally define its inner angle θ^i_{jk} at $\varphi(v_i)$. We claim that

(a) $\bar{\varphi}(t_{ijk})$ is not degenerate for any $t_{ijk} \in F$, and

(b) $\bar{\varphi}$ is locally a homeomorphism.

Then $\bar{\varphi}$ is a proper local homeomorphism, and thus is a covering map. Since $\bar{\varphi}$ is homotopic to the homeomorphism ψ, we know $\bar{\varphi}$ is indeed a degree-1 covering map, ie a homeomorphism.

3.2.1 Proof of claim (a) Assume there is some triangle $t \in F$ such that $\bar{\varphi}(t)$ is degenerate and hence contained in a geodesic λ. Here λ is assumed to be a closed geodesic, or a densely immersed complete geodesic. Let $\mathcal{C}$ be the union of all triangles t such that $\bar{\varphi}(t) \subset \lambda$. Then $\mathcal{C}$ is not the whole complex $\mathcal{T}$; otherwise $\bar{\varphi}$ is not homotopic to the homeomorphism ψ. So, we can find a vertex $v_0 \in \partial\mathcal{C}$. Denote by star$(v_0)$ the star-neighborhood of v_0 in $\mathcal{T}$. Then $\bar{\varphi}(\text{star}(v_0))$ is not in λ, but $\bar{\varphi}(t_0) \subset \lambda$ for some triangle t_0 in star(v_0).

Let $\boldsymbol{n}$ be a unit vector that is orthogonal to the geodesic λ, and define η as $\eta_{ij} = \dot{\varphi}_{ij} \cdot \boldsymbol{n}$. Then the vertex v_0 is nondegenerate with respect to η, but the face t_0 is degenerate. Let ξ be a discrete one-form in Lemma 3.2 with the triangle t_0 and $v_j = v_0$. Scale ξ to make it very small so that $\eta + \xi$ has the same signs with η on the nondegenerate edges of η.

Notice that $\mathrm{sc}(\eta, v_0) \neq 0$, or equivalently $\mathrm{sc}(\eta, v_0) \geq 2$; otherwise all the edges connecting v_0 lie on a half-space, which contradicts the assumption that φ is balanced. Since t_0 is degenerate in η, taking the opposite $-\xi$ instead of ξ if necessary, we can assume that $\mathrm{sc}(\eta, v_0) < \mathrm{sc}(\eta + \xi, v_0)$, and thus $\mathrm{Ind}(\eta + \xi, v_0) < 0$.

Noticing that $\eta + \xi$ is nonvanishing, we will derive a contradiction with the index theorem (Theorem 3.1), by showing that the index of $\eta + \xi$ is nonpositive for any vertex and face.

If a face t is degenerate in η, then $\mathrm{Ind}(\eta + \xi, t) = \mathrm{Ind}(\xi, t) = 0$. If a face t is nondegenerate in η, then $\mathrm{Ind}(\eta + \xi, t) = \mathrm{Ind}(\eta) = 0$. In fact, the index of any nondegenerate face is zero.

If a vertex v is degenerate in η, then $\mathrm{Ind}(\eta + \xi, v) = \mathrm{Ind}(\xi, v) = 0$. If a vertex v is nonvanishing, then $\mathrm{Ind}(\eta + \xi, v) = \mathrm{Ind}(\eta, v)$. Since φ is balanced, $\mathrm{Ind}(\eta, v) \leq 0$.

If a vertex v is nondegenerate but vanishing at some edges in η, then

$$\mathrm{Ind}(\eta + \xi, v) \leq \mathrm{Ind}(\eta, v) \leq 0,$$

since adding ξ can only introduce more sign changes.

3.2.2 Proof of claim (b) Since φ is w–balanced, it is not difficult to show that for any vertex i,

$$\sum_{jk:t_{ijk} \in F} \theta^i_{jk} \geq 2\pi,$$

and the equality holds if and only if all the edges around v_i do not "fold" under the map $\bar{\varphi}$. By the Gauss–Bonnet theorem,

$$\sum_{i=1}^{n} \Bigl(2\pi - \sum_{jk:t_{ijk} \in F} \theta^i_{jk} \Bigr) = 0.$$

So

$$\sum_{jk:t_{ijk} \in F} \theta^i_{jk} = 2\pi$$

for any vertex v_i, and all the edges in E do not fold. Thus, $\bar{\varphi}$ is a local homeomorphism.

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