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Finite presentations of the mapping class groups of once-stabilized Heegaard splittings

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Let $g \geq 2$ and assume that we are given a genus g Heegaard splitting of a closed orientable 3–manifold with distance greater than $2g+2$. We prove that the mapping class group of the once-stabilization of such a Heegaard splitting is finitely presented.

57K30, 57M60

1 Introduction

Let (M, Σ) be a Heegaard splitting of a compact orientable 3–manifold M. The *mapping class group* $\mathrm{MCG}(M, \Sigma)$ of the Heegaard splitting (M, Σ) is defined to be the group $\pi_0(\mathrm{Diff}(M, \Sigma))$ of path-connected components of the group $\mathrm{Diff}(M, \Sigma)$, where we denote by $\mathrm{Diff}(M, \Sigma)$ the group of diffeomorphisms of M that preserve Σ setwise. There is a natural homomorphism from $\mathrm{MCG}(M, \Sigma)$ to the mapping class group $\mathrm{MCG}(M)$ of M. Following Johnson [2011], we call the kernel of this natural homomorphism the *isotopy subgroup* of $\mathrm{MCG}(M, \Sigma)$, and denote it by $\mathrm{Isot}(M, \Sigma)$.

In this paper, we are interested in the isotopy subgroup of the mapping class group of a once-stabilized Heegaard splitting. Let (M, Σ') be a genus $g(\Sigma') \geq 2$ Heegaard splitting of a closed orientable 3–manifold M. We say that a Heegaard splitting (M, Σ) is a (once-)*stabilization* of (M, Σ') if it is obtained from (M, Σ') by adding a 1–handle whose core is parallel into Σ'. Corresponding to two handlebodies $V^-_{\Sigma'}$ and $V^+_{\Sigma'}$ in M with $\partial V^-_{\Sigma'} = \partial V^+_{\Sigma'} = \Sigma'$, there are two obvious subgroups of $\mathrm{Isot}(M, \Sigma)$: one is $\mathrm{Isot}(V^-_{\Sigma'}, \Sigma^-)$ and the other is $\mathrm{Isot}(V^+_{\Sigma'}, \Sigma^+)$, where Σ^- (resp. Σ^+) is the Heegaard surface obtained by pushing Σ into $V^-_{\Sigma'}$ (resp. $V^+_{\Sigma'}$) slightly. It is natural to ask when these subgroups generate $\mathrm{Isot}(M, \Sigma)$. Johnson [2011] proved that if the distance $d(\Sigma')$ of the Heegaard splitting (M, Σ') is greater than $2g(\Sigma')+2$, then the two subgroups defined above generate $\mathrm{Isot}(M, \Sigma)$. As a consequence of this fact, together with a result of Scharlemann [2013] that says $\mathrm{Isot}(V^{\pm}_{\Sigma'}, \Sigma^{\pm})$ are finitely generated, it follows that $\mathrm{Isot}(M, \Sigma)$ and $\mathrm{MCG}(M, \Sigma)$ are finitely generated. In that paper, Johnson conjectured that $\mathrm{Isot}(M, \Sigma)$ is an amalgamation of the two groups $\mathrm{Isot}(V^-_{\Sigma'}, \Sigma^-)$ and $\mathrm{Isot}(V^+_{\Sigma'}, \Sigma^+)$. This is the main result of the paper:

Theorem 1.1 *Suppose that (M, Σ') is Heegaard splitting of a closed orientable 3–manifold M with $d(\Sigma') > 2g(\Sigma')+2$, and that (M, Σ) is a once-stabilization of (M, Σ'). Suppose that $(V^-_{\Sigma'}, \Sigma^-)$ (resp.*

$(V^+_{\Sigma'}, \Sigma^+)$) *is the Heegaard splitting of* $V^-_{\Sigma'}$ (*resp.* $V^+_{\Sigma'}$) *obtained by pushing* Σ *into* $V^-_{\Sigma'}$ (*resp.* $V^+_{\Sigma'}$) *slightly, where* $V^-_{\Sigma'}$ *and* $V^+_{\Sigma'}$ *are handlebodies in* M *bounded by* Σ'*. Then* $\mathrm{Isot}(M, \Sigma)$ *is isomorphic to an amalgamation of the two groups* $\mathrm{Isot}(V^-_{\Sigma'}, \Sigma^-)$ *and* $\mathrm{Isot}(V^+_{\Sigma'}, \Sigma^+)$.

One might expect that the above theorem has something to do with van Kampen's theorem. This idea can be justified as follows. Following Johnson and McCullough [2013], we define the space $\mathcal{H}(M, \Sigma)$ to be $\mathrm{Diff}(M)/\mathrm{Diff}(M, \Sigma)$ and call it the *space of Heegaard splittings* equivalent to (M, Σ). Let $\mathcal{H}$ denote the path-connected component of $\mathcal{H}(M, \Sigma)$ containing the left coset $\mathrm{id}_M \cdot \mathrm{Diff}(M, \Sigma)$. It is known that if a 3–manifold admits a Heegaard splitting with the distance greater than two, then such a 3–manifold must be hyperbolic. By a result in [Johnson and McCullough 2013] (see Theorem 2.1 below for more details) together with this fact, it follows that $\mathrm{Isot}(M, \Sigma)$ is isomorphic to $\pi_1(\mathcal{H})$.

Now fix a *spine* $K = K^- \cup K^+$ of the Heegaard splitting (M, Σ'), that is, K^- and K^+ are finite graphs embedded in M such that the complement $M \setminus K$ is diffeomorphic to $\Sigma' \times (-1, 1)$ and Σ' is a slice of this product structure. Denote by $\mathcal{H}^-$ (resp. $\mathcal{H}^+$) the subspace of $\mathcal{H}$ consisting of those elements represented by a Heegaard surface T such that T is a genus $g(\Sigma') + 1$ Heegaard surface of the genus $g(\Sigma')$ handlebody $M \setminus \mathrm{Int}(N(K^+))$ (resp. $M \setminus \mathrm{Int}(N(K^-))$), where $N(K^+)$ (resp. $N(K^-)$) is a small neighborhood of K^+ (resp. K^-). By the similar reason as above (see Theorem 2.2 below), we can identify $\mathrm{Isot}(V^-_{\Sigma'}, \Sigma^-)$ and $\mathrm{Isot}(V^+_{\Sigma'}, \Sigma^+)$ with the fundamental groups $\pi_1(\mathcal{H}^-)$ and $\pi_1(\mathcal{H}^+)$ respectively. Set $\mathcal{H}^\cup := \mathcal{H}^- \cup \mathcal{H}^+$. Theorem 1.1 is a corollary of the following.

Theorem 1.2 *The inclusion* $\mathcal{H}^\cup \to \mathcal{H}$ *is a homotopy equivalence.*

It is well known that a genus $g+1$ Heegaard splitting of a genus g handlebody is unique up to isotopy. Similarly, a genus $g+1$ Heegaard splitting of the space $F_g \times [-1, 1]$ is unique up to isotopy, where we denote by F_g a closed genus g surface. In other words, $\mathcal{H}^+$, $\mathcal{H}^-$ and $\mathcal{H}^- \cap \mathcal{H}^+$ are all connected, and hence van Kampen's theorem applies to the triple $(\mathcal{H}^-, \mathcal{H}^+, \mathcal{H}^- \cap \mathcal{H}^+)$.

The proof of Theorem 1.2 is based on the concept of *graphics*, which was first introduced by Cerf [1968] and then successfully applied to the study of Heegaard splittings by Rubinstein and Scharlemann [1996]. More precisely, we prove Theorem 1.2 by generalizing the method developed by Johnson [2010; 2011]. We also use an argument due to Hatcher [1976] crucially, which is a parametrized version of the innermost disk argument.

In Section 5, we confirm that the isotopy subgroup of a genus $g+1$ Heegaard splitting of a genus g handlebody is finitely presented:

Theorem 1.3 *Let* V *be a handlebody of genus* $g(V) \geq 2$*, and let* (V, Σ) *be a genus* $g(V)+1$ *Heegaard splitting of* V*. Then* $\mathrm{Isot}(V, \Sigma)$ *is finitely presented.*

It follows from Theorem 1.3 that $\pi_1(\mathcal{H}^-)$ and $\pi_1(\mathcal{H}^+)$ are finitely presented. As a consequence, we have:

Corollary 1.4 *Let (M,Σ') be a Heegaard splitting of a closed orientable 3–manifold M with*

$$d(\Sigma') > 2g(\Sigma') + 2.$$

Let (M,Σ) be a once-stabilization of (M,Σ'). Then $\mathrm{Isot}(M,\Sigma)$ *and* $\mathrm{MCG}(M,\Sigma)$ *are finitely presented.*

We remark that a problem related to this work was treated by Koda and Sakuma [2023]. In that paper, the concept of the "homotopy motion group" was introduced, and they considered the question that asks when the homotopy motion group $\Pi(M,\Sigma)$ of a Heegaard surface in a 3–manifold M can be written as an amalgamation of the two homotopy motion groups $\Pi(U_\Sigma^-,\Sigma)$ and $\Pi(U_\Sigma^+,\Sigma)$ corresponding to the two handlebodies U_Σ^- and U_Σ^+ with $\partial U_\Sigma^- = \partial U_\Sigma^+ = \Sigma$.

The paper is organized as follows. In Section 2, we recall from [Johnson and McCullough 2013] some facts about the space of Heegaard splittings. We also recall the definition of the distance of a Heegaard splitting. To prove Theorem 1.2, we will need to deal with the graphic determined by a 4–parameter family of Heegaard surfaces. In Section 3, we give a quick review of the theory of graphics, and then we see that some ideas in [Johnson 2010] can be adapted to our setting. In Section 4, we prove Theorem 1.2. Finally, we give the proof of Theorem 1.3 in Section 5.

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2 Preliminaries

Throughout the paper, we will use the following notation. For a topological space X, we denote by $|X|$ the number of path-connected components of X. For a subspace Y of X, $\mathrm{Int}(Y)$ and $\mathrm{Cl}(Y)$ denote the interior and the closure of Y in X, respectively. We will denote by J the closed interval $[-1,1]$.

2.1 The space of Heegaard splittings

Let M be a compact orientable 3–manifold (possibly with boundary). Let (M,Σ) be a Heegaard splitting of M. This means that $\Sigma \subset M$ is a closed orientable embedded surface cutting M into the two compression bodies. Here, a *compression body* is a 3–manifold with nonempty boundary admitting a Morse function without critical points of index 2 and 3. A handlebody is a typical example of a compression body. The space $\mathcal{H}(M,\Sigma) = \mathrm{Diff}(M)/\mathrm{Diff}(M,\Sigma)$ is called the *space of Heegaard splittings* equivalent to (M,Σ). Note that there is a one-to-one correspondence between $\mathcal{H}(M,\Sigma)$ and the set of images of Σ under diffeomorphisms of M. We often identify an element of $\mathcal{H}(M,\Sigma)$ with the corresponding Heegaard surface. We always take the surface Σ as the basepoint of $\mathcal{H}(M,\Sigma)$, which corresponds to the left coset $\mathrm{id}_M \cdot \mathrm{Diff}(M,\Sigma)$. The space $\mathcal{H}(M,\Sigma)$ admits a structure of a Fréchet manifold, and this implies that $\mathcal{H}(M,\Sigma)$ has the homotopy type of a CW complex.

Theorem 2.1 [Johnson and McCullough 2013, Corollary 1] *Suppose that M is closed, orientable, irreducible and $\pi_1(M)$ is infinite, and that M is not a non-Haken infranilmanifold. Then $\pi_k(\mathcal{H}(M,\Sigma)) = 0$ for $k \geq 2$, and there is an exact sequence*

$$1 \to Z(\pi_1(M)) \to \pi_1(\mathcal{H}(M,\Sigma)) \to \mathrm{Isot}(M,\Sigma) \to 1.$$

A similar statement as above holds for handlebodies and the space $F_g \times J$:

Theorem 2.2 *Let $g' \geq g \geq 2$. Suppose that M is a genus g handlebody or the space $F_g \times J$, where F_g denotes a closed orientable surface of genus g. Suppose that (M,Σ) is a genus g' Heegaard splitting of M. Then $\pi_1(\mathcal{H}(M,\Sigma)) \cong \mathrm{Isot}(M,\Sigma)$ and $\pi_k(\mathcal{H}(M,\Sigma)) = 0$ for $k \geq 2$.*

Proof By [Johnson and McCullough 2013, Theorem 1], $\pi_k(\mathcal{H}(M,\Sigma)) = \pi_k(\mathrm{Diff}(M))$ for $k \geq 2$, and there is an exact sequence

$$1 \to \pi_1(\mathrm{Diff}(M)) \to \pi_1(\mathcal{H}(M,\Sigma)) \to \mathrm{Isot}(M,\Sigma) \to 1.$$

By Earle and Eells [1969] and Hatcher [1976], $\pi_k(\mathrm{Diff}(M)) = 0$ for $k \geq 1$. □

2.2 The distance of a Heegaard splitting

Let (M,Σ') be a genus $g(\Sigma') \geq 2$ Heegaard splitting of a closed orientable 3–manifold M. Denote by $V^-_{\Sigma'}$ and $V^+_{\Sigma'}$ the handlebodies in M with $V^-_{\Sigma'} \cap V^+_{\Sigma'} = \partial V^-_{\Sigma'} = \partial V^+_{\Sigma'} = \Sigma'$. The *curve graph* $\mathcal{C}(\Sigma')$ is the graph defined as follows. The vertices of $\mathcal{C}(\Sigma')$ are isotopy classes of nontrivial simple closed curves in Σ', and the edges are pairs of vertices that admit disjoint representatives. We denote by $d_{\mathcal{C}(\Sigma')}$ the simplicial metric on $\mathcal{C}(\Sigma')$.

Let $\mathcal{D}^-$ (resp. $\mathcal{D}^+$) denote the set of vertices in $\mathcal{C}(\Sigma')$ that are represented by simple closed curves bounding disks in $V^-_{\Sigma'}$ (resp. $V^+_{\Sigma'}$). Then the (Hempel) *distance* $d(\Sigma')$ of the Heegaard splitting (M,Σ') is defined to be

$$d(\Sigma') := d_{\mathcal{C}(\Sigma')}(\mathcal{D}^-, \mathcal{D}^+).$$

For example, if M contains an essential sphere, then any Heegaard splitting of M has distance zero (see Haken [1968]). If M contains an essential torus, then any Heegaard splitting of M has distance at most two. Furthermore, any Heegaard splitting of a Seifert manifold has distance at most two. See Hempel [2001] for these two facts. As a consequence of the geometrization theorem and these facts, we have:

Theorem 2.3 *Suppose that (M,Σ') is a Heegaard splitting of a closed orientable 3–manifold M. If $d(\Sigma') > 2$, then M admits a hyperbolic structure.*

3 Sweep-outs and graphics

In this section, we recall the definition of graphics and summarize their properties. In what follows, let M denote a closed orientable 3–manifold.

3.1 Graphics

Let (M, Σ) be a Heegaard splitting of M. A *sweep-out* associated with (M, Σ) is a function

$$h\colon M \to J = [-1, 1]$$

such that the level set $h^{-1}(t)$ is a Heegaard surface isotopic to Σ if $t \in \operatorname{Int}(J)$, and $h^{-1}(t)$ is a finite graph in M if $t \in \partial J$. The preimage $h^{-1}(\partial J)$ is called the *spine* of h.

Lemma 3.1 *Let $n > 0$ and (M, Σ) be a Heegaard splitting of a closed orientable 3–manifold M. Let $\varphi\colon D^n \to \mathcal{H}(M, \Sigma)$. Then there exists a family $\{h_u\colon M \to J \mid u \in D^n\}$ of sweep-outs such that $h_u^{-1}(0) = \varphi(u)$ for $u \in D^n$.*

Proof Take a sweep-out $h\colon M \to J$ with $h^{-1}(0) = \Sigma$. We note that

$$\operatorname{Diff}(M) \to \operatorname{Diff}(M)/\operatorname{Diff}(M, \Sigma) = \mathcal{H}(M, \Sigma)$$

is a fibration [Johnson and McCullough 2013]. So, the map φ lifts to a map $\tilde{\varphi}\colon D^n \to \operatorname{Diff}(M)$. Now define $h_u := h \circ \tilde{\varphi}(u)^{-1}$ for $u \in D^n$. □

Let (M, Σ) and (M, Σ') be Heegaard splittings of M. Let $f\colon M \to J$ be a sweep-out with $f^{-1}(0) = \Sigma'$. Furthermore, let $\{h_u\colon M \to J \mid u \in D^2\}$ be a family of sweep-outs associated with (M, Σ). We define the map $\Phi\colon M \times D^2 \to J^2 \times D^2$ by $\Phi(x, u) = (f(x), h_u(x), u)$.

Set $L := \Phi^{-1}(\partial J^2 \times D^2)$, and $W := (M \times D^2) \setminus L$. Define $S = S(\Phi|_W)$ to be the set of all points $w \in W$ such that $\operatorname{rank} d(\Phi|_W)_w < 4$. The image Γ of S in $J^2 \times D^2$ is called the *graphic* defined by f and $\{h_u\}$.

After a small perturbation, we may assume that the map Φ is generic in the following sense. First, for $u \in D^2$, the spine $h_u^{-1}(\partial J)$ intersects each level set of f at finitely many points. Similarly, for $u \in D^2$, the spine $f^{-1}(\partial J)$ intersects each level set of h_u at finitely many points. Furthermore, Φ is "excellent" on W. This means that the set S of singular points of $\Phi|_W$ is a 3–dimensional submanifold in W, and S is divided into four parts, S_2, S_3, S_4 and S_5, where S_k consists of singular points of codimension k. (In the notation of [Boardman 1967], we can write $S_2 = \Sigma^{2,0}$, $S_3 = \Sigma^{2,1,0}$, $S_4 = \Sigma^{2,1,1,0}$ and $S_5 = \Sigma^{2,1,1,1,0} \cup \Sigma^{2,2,0}$.) For $k \neq 5$, Φ has one of the following canonical forms around a point $w \in S_k$:[1] there exist local coordinates (a, b, c, x, y) centered at w and (A, B, X, Y) centered at $\Phi(w)$ such that

$$(A \circ \Phi, B \circ \Phi, X \circ \Phi, Y \circ \Phi) = \begin{cases} (a, b, c, x^2 + y^2) & \text{definite fold } (w \in S_2), \\ (a, b, c, x^2 - y^2) & \text{indefinite fold } (w \in S_2), \\ (a, b, c, x^3 + ax - y^2) & \text{cusp } (w \in S_3), \\ (a, b, c, x^4 + ax^2 + bx + y^2) & \text{definite swallowtail } (w \in S_4), \\ (a, b, c, x^4 + ax^2 + bx - y^2) & \text{indefinite swallowtail } (w \in S_4). \end{cases}$$

[1]We do not know if there exist canonical forms for the singularities of type $\Sigma^{2,2,0}$. However, the singularities in S_5 are not important for our present purpose.

Furthermore, for $2 \le k \le 5$, $\Phi|_{S_k}$ is an immersion with normal crossings, and the images of the S_k are in general position. The main reference about these materials is the book by Golubitsky and Guillemin [1973]. Hatcher and Wagoner [1973] also contains a helpful review for our present purpose.

In the remaining part of the paper, we always assume that the map Φ has the property described above. Under this assumption, Γ has the natural stratification: we can write $\Gamma = F_3 \cup F_2 \cup F_1 \cup F_0$, where $\dim F_k = k$ for $0 \le k \le 3$ and each F_k has the following description.

- F_3 This consists of those points $y \in \Gamma$ such that $(\Phi|_S)^{-1}(y) \subset S_2$ and $|(\Phi|_S)^{-1}(y)| = 1$.
- F_2 This consists of those points $y \in \Gamma$ such that
  - $(\Phi|_S)^{-1}(y) \subset S_2$ and $|(\Phi|_S)^{-1}(y)| = 2$, or
  - $(\Phi|_S)^{-1}(y) \subset S_3$ and $|(\Phi|_S)^{-1}(y)| = 1$.
- F_1 This consists of those points $y \in \Gamma$ such that
  - $(\Phi|_S)^{-1}(y) \subset S_2$ and $|(\Phi|_S)^{-1}(y)| = 3$,
  - $(\Phi|_S)^{-1}(y) \subset S_2 \cup S_3$ and $|(\Phi|_S)^{-1}(y)| = 2$, or
  - $(\Phi|_S)^{-1}(y) \subset S_4$ and $|(\Phi|_S)^{-1}(y)| = 1$.
- F_0 This consists of those points $y \in \Gamma$ such that
  - $(\Phi|_S)^{-1}(y) \subset S_2$ and $|(\Phi|_S)^{-1}(y)| = 4$,
  - $(\Phi|_S)^{-1}(y) \subset S_2 \cup S_3$ and $|(\Phi|_S)^{-1}(y)| = 3$,
  - $(\Phi|_S)^{-1}(y) \subset S_2 \cup S_4$ and $|(\Phi|_S)^{-1}(y)| = 2$,
  - $(\Phi|_S)^{-1}(y) \subset S_3$ and $|(\Phi|_S)^{-1}(y)| = 2$, or
  - $(\Phi|_S)^{-1}(y) \subset S_5$ and $|(\Phi|_S)^{-1}(y)| = 1$.

3.2 Labeling the regions of $J^2 \times D^2$

In this subsection, we will see that some definitions in [Johnson 2010] can be modified slightly and adapted to our setting.

Let (M, Σ) be a Heegaard splitting. We assume that one component of $M \setminus \Sigma$ is assigned the label $-$ and the other is assigned the label $+$ in some way. We denote by U_Σ^- and U_Σ^+ the components of $M \setminus \Sigma$ labeled by $-$ and $+$ respectively. (Typically, such a labeling is determined by a given sweep-out h with $h^{-1}(0) = \Sigma$. In this case, we can define $U_\Sigma^- = h^{-1}([-1, 0])$ and $U_\Sigma^+ = h^{-1}([0, 1])$.) Such an assignment of the labels $-$ or $+$ to the components of $M \setminus \Sigma$ is called a *transverse orientation* of Σ.

Definition Let (M, Σ) and $U_\Sigma^\pm$ be as above. Suppose $\Sigma' \subset M$ is a closed embedded surface. Then we say that Σ' is *mostly above* Σ if Σ' is transverse to Σ, and if every component of $\Sigma' \cap U_\Sigma^-$ is contained in a disk subset of Σ'. Similarly, we say that Σ' is *mostly below* Σ if Σ' is transverse to Σ, and if every component of $\Sigma' \cap U_\Sigma^+$ is contained in a disk subset of Σ'.

Suppose that $f : M \to J$ is a sweep-out, and that Σ is a transversely oriented Heegaard surface of M. We say that Σ is a *spanning surface* for f if there exist values $a, b \in \mathrm{Int}(J)$ such that $f^{-1}(a)$ is mostly above Σ and $f^{-1}(b)$ is mostly below Σ. We say that Σ is a *splitting surface* for f if it satisfies the following.

First, there does not exist value $s \in \mathrm{Int}(J)$ such that $f^{-1}(s)$ is mostly above or mostly below Σ. Second, $f|_\Sigma$ is *almost Morse*, that is, $f|_\Sigma$ has only nondegenerate critical points and $f|_\Sigma$ is Morse away from -1 and 1, but there may be more than one minima and maxima at the levels -1 and 1 respectively. We note that these definitions are coming from that in [Johnson 2010, Definitions 11 and 12].

Proposition 27 in [Johnson 2010], which will be used in the proof of Theorem 1.2, can be stated in our term as follows:

Lemma 3.2 [Johnson 2010, Proposition 27] *Let $f: M \to J$ be a sweep-out associated with a Heegaard splitting (M, Σ'). If f admits a splitting surface Σ, then $d(\Sigma') \le 2g(\Sigma)$.*

Let (M, Σ) and (M, Σ') be Heegaard splittings of M. Assume that $f: M \to J$ is a sweep-out with $f^{-1}(0) = \Sigma'$, and that $\{h_u: M \to J \mid u \in D^2\}$ is a family of sweep-outs associated with (M, Σ). Let $\Phi: M \times D^2 \to J^2 \times D^2$ be as in the previous subsection. Following [Johnson 2010], let us consider the two subsets $\mathcal{R}_a$ and $\mathcal{R}_b$ of $J^2 \times D^2$ defined as

$$\mathcal{R}_a := \{(s,t,u) \in J^2 \times D^2 \mid f^{-1}(s) \text{ is mostly above } h_u^{-1}(t)\},$$
$$\mathcal{R}_b := \{(s,t,u) \in J^2 \times D^2 \mid f^{-1}(s) \text{ is mostly below } h_u^{-1}(t)\}.$$

Here, for each $u \in D^2$ and $t \in J$, the transverse orientation of $h_u^{-1}(t)$ is determined by the sweep-out h_u. For example, if t is sufficiently close to -1, then the point (s,t,u) is in $\mathcal{R}_a$ because $f^{-1}(s) \cap h_u^{-1}([-1,t])$ consists of finitely many properly embedded disks in the handlebody $h_u^{-1}([-1,t])$. Similarly, if t is sufficiently close to 1, then the point (s,t,u) is in $\mathcal{R}_b$. The regions $\mathcal{R}_a$ and $\mathcal{R}_b$ are nonempty open subsets in $J^2 \times D^2$. The next proposition follows directly from the definition.

Proposition 3.3 *The following hold*:

1. (1) *$\mathcal{R}_a$ and $\mathcal{R}_b$ are disjoint as long as $g(\Sigma') \neq 0$.*
2. (2) *$\mathcal{R}_a$ and $\mathcal{R}_b$ are bounded by Γ.*
3. (3) *The regions $\mathcal{R}_a$ and $\mathcal{R}_b$ are convex in the t–direction, that is, if (s,t,u) is in $\mathcal{R}_a$ (resp. $\mathcal{R}_b$), then so is (s,t',u) for any $t' \le t$ (resp. $t' \ge t$).*

Set $J_u^2 := J^2 \times \{u\} \subset J^2 \times D^2$ for $u \in D^2$. Then, for $u \in D^2$, the intersection $\Gamma \cap J_u^2 \subset J_u^2$ can be viewed as the (2D) graphic defined by sweep-outs f and h_u.

Definition Let f and h_u be as above.

- (i) We say that h_u *spans* f if there exists $t \in J$ such that $h_u^{-1}(t)$ is a spanning surface for f.
- (ii) We say that h_u *splits* f if there exists $t \in J$ such that $h_u^{-1}(t)$ is a splitting surface for f.

We also say that the graphic defined by f and h_u is *spanned* if h_u spans f. Similarly, we say that the graphic defined by f and h_u is *split* if h_u splits f.

Remark 3.4 By Lemma 3.2, the graphic defined by f and h_u cannot be split if $d(\Sigma') > 2g(\Sigma)$.

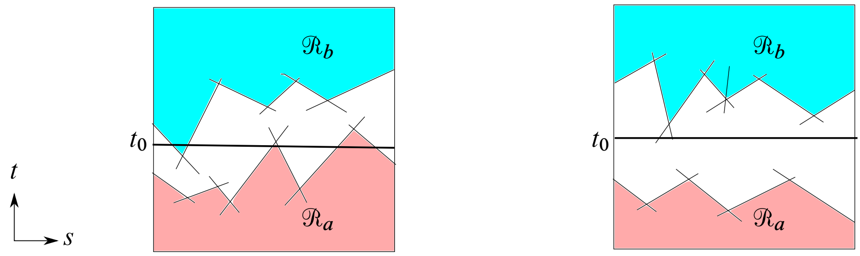

Figure 1: The graphic defined by f and h_u is spanned if there exists a horizontal segment in J_u^2 intersecting both $\mathcal{R}_a$ and $\mathcal{R}_b$ (left). On the other hand, the graphic is split if there exists a horizontal segment disjoint from both $\mathrm{Cl}(\mathcal{R}_a)$ and $\mathrm{Cl}(\mathcal{R}_b)$ (right).

Here are further remarks on the above definition. First, we remark that the condition (i) is equivalent to the following: there exists $t_0 \in J$ such that the horizontal segment $\{t = t_0\}$ in J_u^2 intersects both $\mathcal{R}_a$ and $\mathcal{R}_b$ (the left in Figure 1). We also note that J_u^2 intersects F_3 transversely for $u \in D^2$, and hence $J_u^2 \cap F_3$ consists of finitely many open arcs. This is because $d(p_3 \circ \Phi)_w$ has maximal rank for $w \in W$, where $p_3 \colon J^2 \times D^2 \to D^2$ denotes the projection onto the third coordinate. Furthermore, after perturbing Φ if necessary, $J_u^2 \cap F_k$ consists of finitely many points for $0 \le k \le 2$ and $u \in D^2$. Under this assumption, condition (ii) is equivalent to the following: there exists $t_0 \in J$ such that the horizontal segment $\{t = t_0\}$ in J_u^2 is disjoint from both $\mathrm{Cl}(\mathcal{R}_a)$ and $\mathrm{Cl}(\mathcal{R}_b)$ (the right in Figure 1).

Proposition 3.5 *If $g(\Sigma') \ge 2$, then $\mathrm{Cl}(\mathcal{R}_a)$ and $\mathrm{Cl}(\mathcal{R}_b)$ intersect only at points of F_0.*

Proof We first note that by Proposition 3.3(1) and (2), the intersection between $\mathrm{Cl}(\mathcal{R}_a)$ and $\mathrm{Cl}(\mathcal{R}_b)$ is contained in Γ. Suppose that $\mathrm{Cl}(\mathcal{R}_a) \cap \mathrm{Cl}(\mathcal{R}_b) \neq \varnothing$ and $y \in \mathrm{Cl}(\mathcal{R}_a) \cap \mathrm{Cl}(\mathcal{R}_b) \subset \Gamma$. Let (s_0, t_0, u_0) be the coordinate of y. Let l be the segment in $J^2 \times D^2$ defined by $l := \{s = s_0\} \cap \{u = u_0\}$. Note that $y \in l$ by definition. Furthermore, it follows from Proposition 3.3(3) that $l \subset \mathrm{Cl}(\mathcal{R}_a) \cup \mathrm{Cl}(\mathcal{R}_b)$. Consider a point (s_0', t_0, u_0') obtained by perturbing the point (s_0, t_0, u_0) in the s– and u–directions. We may assume that the segment $\tilde{l} := \{s = s_0'\} \cap \{u = u_0'\}$ is transverse to each stratum of Γ. The preimage $\Phi^{-1}(\tilde{l})$ of $\tilde{l}$ is the genus $g(\Sigma')$ Heegaard surface in $M \times \{u_0'\} \subset M \times D^2$, which can be naturally identified with Σ'. Let $h \colon \Sigma' \to J$ denote the function defined to be the restriction of $h_{u_0'}$ on $\Phi^{-1}(\tilde{l}) \cong \Sigma'$. Then h is almost Morse.

Now suppose, for the sake of contradiction, that y is in F_k with $k \ge 1$. Let (s_0', t_-, u_0') denote the coordinate of the intersection point between $\tilde{l}$ and the boundary of $\mathrm{Cl}(\mathcal{R}_a)$. Note that such a point is unique by Proposition 3.3(3). Similarly, let (s_0', t_+, u_0') denote the coordinate of the intersection point between $\tilde{l}$ and the boundary of $\mathrm{Cl}(\mathcal{R}_b)$. Then, h satisfies the following.

- For any regular value $t \in J \setminus [t_-, t_+]$, every loop of $h^{-1}(t)$ is trivial in Σ'.
- The interval $[t_-, t_+]$ contains at most three critical values of h.

It is easily seen that the Euler characteristic of such Σ' must be at least -1, but this is impossible because $g(\Sigma') \ge 2$ by assumption. □

4 Proof of Theorem 1.2

Suppose that M is a closed orientable 3–manifold, and that (M, Σ') is a genus $g(\Sigma') \geq 2$ Heegaard splitting with $d(\Sigma') > 2g(\Sigma') + 2$. Suppose that (M, Σ) is a once-stabilization of (M, Σ'). Let $\mathcal{H}$ denote the path-connected component of $\mathcal{H}(M, \Sigma)$ containing Σ. Let $K^{\pm}$ and $\mathcal{H}^{\pm} \subset \mathcal{H}$ be as in Section 1. Set $\mathcal{H}^{\cup} := \mathcal{H}^{-} \cup \mathcal{H}^{+}$ and $\mathcal{H}^{\cap} := \mathcal{H}^{-} \cap \mathcal{H}^{+}$.

By Theorem 2.2, $\pi_k(\mathcal{H}^{-}) = \pi_k(\mathcal{H}^{+}) = \pi_k(\mathcal{H}^{\cap}) = 0$ for $k \geq 2$. By the Mayer–Vietoris exact sequence, it follows that $H_k(\mathcal{H}^{\cup}; \mathbb{Z}) = 0$ for $k \geq 2$. Applying Hurewicz's theorem, we have $\pi_k(\mathcal{H}^{\cup}) = 0$ for $k \geq 2$. On the other hand, by Theorems 2.1 and 2.3, $\pi_k(\mathcal{H}) = 0$ for $k \geq 2$. So, to prove Theorem 1.2, it is enough to show the following.

Lemma 4.1 *The inclusion $\mathcal{H}^{\cup} \to \mathcal{H}$ induces the isomorphism $\pi_1(\mathcal{H}^{\cup}) \to \pi_1(\mathcal{H})$.*

Johnson [2011] proved that $\pi_1(\mathcal{H})$ is generated by $\pi_1(\mathcal{H}^{-})$ and $\pi_1(\mathcal{H}^{+})$, and hence the induced map is a surjection. (In fact, using the notations in this paper, what he proved in [Johnson 2011] can be written as

$$\pi_1(\mathcal{H}, \mathcal{H}^{\cup}) = 1.$$

See [Johnson 2011, Lemmas 2 and 3]. The following argument is motivated by this observation.) So in this paper, we focus on the proof of the injectivity of the induced map. In other words, we will show the following:

Lemma 4.2 *The second homotopy group $\pi_2(\mathcal{H}, \mathcal{H}^{\cup})$ of the pair $(\mathcal{H}, \mathcal{H}^{\cup})$ vanishes.*

Let $e_0 \in \partial D^2$ be the basepoint. Let $\varphi\colon (D^2, \partial D^2, e_0) \to (\mathcal{H}, \mathcal{H}^{\cup}, \Sigma)$. We will show that $[\varphi] = 0 \in \pi_2(\mathcal{H}, \mathcal{H}^{\cup})$. Let $f\colon M \to J$ be a sweep-out with $f^{-1}(0) = \Sigma'$ and $f^{-1}(\pm 1) = K^{\pm}$. By Lemma 3.1, there exists a family $\{h_u\colon M \to J \mid u \in D^2\}$ of sweep-outs such that $h_u^{-1}(0) = \varphi(u)$ for $u \in D^2$. The key of the proof is the following.

Lemma 4.3 *For any $u \in D^2$, the graphic defined by f and h_u is spanned.*

Proof Suppose, contrary to our claim, there exists $u_0 \in D^2$ such that the graphic defined by f and h_{u_0} is not spanned. Put $J^2_{u_0} := \{(s, t, u) \in J^2 \times D^2 \mid u = u_0\}$. For brevity, we denote the restriction of Φ on $W = (M \times D^2) \setminus L$ by the same symbol Φ in the following. Set $\Gamma := \Phi(S(\Phi))$. The intersection $\Gamma \cap J^2_{u_0} \subset J^2_{u_0}$ is precisely the graphic defined by f and h_{u_0}.

As noted in Remark 3.4, this graphic cannot be split. Then, there exists $t_0 \in J$ such that the horizontal segment $l := \{t = t_0\} \subset J^2_{u_0}$ intersects both $\mathrm{Cl}(\mathcal{R}_a)$ and $\mathrm{Cl}(\mathcal{R}_b)$ at their boundaries (Figure 2). By Proposition 3.5, $\mathrm{Cl}(\mathcal{R}_a)$ and $\mathrm{Cl}(\mathcal{R}_b)$ intersect only at points of F_0. So, pushing l out of $J^2_{u_0}$ slightly, we get an arc $\tilde{l} \subset J^2 \times D^2$ such that

- $\tilde{l}$ is disjoint from both $\mathcal{R}_a$ and $\mathcal{R}_b$, and
- $\tilde{l}$ is transverse to each stratum of Γ.

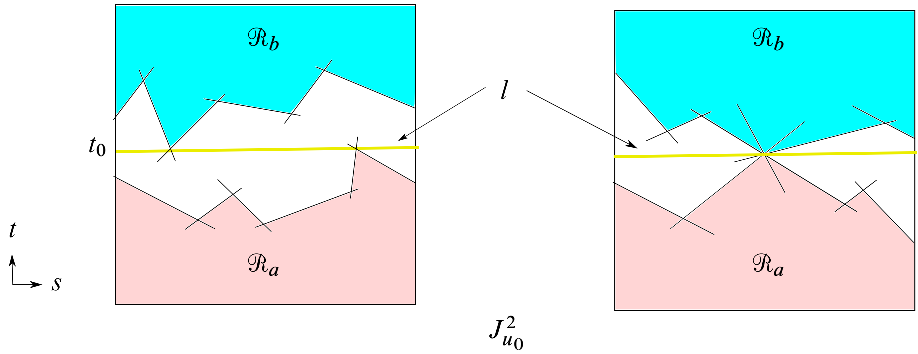

Figure 2: If the graphic defined by f and h_{u_0} is not spanned, then there exists a horizontal segment $l \subset J^2_{u_0}$ intersecting $\mathrm{Cl}(\mathcal{R}_a)$ and $\mathrm{Cl}(\mathcal{R}_b)$ at their boundaries; the intersection is either separate vertices (left) or a single common vertex (right). In either case l can be perturbed in $J^2 \times D^2$ so that l does not meet $\mathrm{Cl}(\mathcal{R}_a) \cup \mathrm{Cl}(\mathcal{R}_b)$.

Furthermore, there is a family $\{l_t \mid t \in I\}$ of arcs with $l_0 = l$ and $l_1 = \tilde{l}$ such that for any $t \in (0,1]$, l_t is transverse to each stratum of Γ. Note that $d(p_2 \circ \Phi)$ and $d(p_3 \circ \Phi)$ have maximal ranks, where $p_2 \colon J^2 \times D^2 \to J$ is the projection onto the second coordinate and $p_3 \colon J^2 \times D^2 \to D^2$ is the projection onto the third coordinate. This means that Φ is transverse to $l = l_0$. As a consequence, Φ is transverse to l_t for all $t \in [0,1]$, and hence $\tilde{\Sigma} := \Phi^{-1}(\tilde{l})$ is a closed embedded surface in $M \times D^2$ isotopic to $\Sigma_{t_0,u_0} := \Phi^{-1}(l)\ (= h^{-1}_{u_0}(t_0) \times \{u_0\})$. In particular, $g(\tilde{\Sigma}) = g(\Sigma') + 1$.

Let $q_1 \colon M \times D^2 \to M$ denote the projection onto the first coordinate. Since the restriction

$$q_1|_{\Sigma_{t_0,u_0}} \colon \Sigma_{t_0,u_0} \to M$$

is an embedding, so is $q_1|_{\tilde{\Sigma}} \colon \tilde{\Sigma} \to M$. We see that $q_1(\tilde{\Sigma})$ is a splitting surface for f. Consider the restriction $f \circ q_1|_{\tilde{\Sigma}}$ on $\tilde{\Sigma}$. The arc $\tilde{l}$ intersects Γ only at points in F_3, which correspond to fold points of Φ. Thus, $f \circ q_1|_{\tilde{\Sigma}}$ is almost Morse. Let $s \in J$ be any regular value of $f \circ q_1|_{\tilde{\Sigma}}$. By definition, we can write

$$(f \circ q_1|_{\tilde{\Sigma}})^{-1}(s) = (h^{-1}_u(t) \times \{u\}) \cap (f^{-1}(s) \times \{u\}) \subset M \times D^2$$

for some $t \in J$ and $u \in D^2$. Since $\tilde{l}$ is disjoint from both $\mathcal{R}_a$ and $\mathcal{R}_b$, the preimage $(f \circ q_1|_{\tilde{\Sigma}})^{-1}(s)$ contains at least one loop that is nontrivial in the surface $f^{-1}(s) \times \{u\}$. This implies that $q_1(\tilde{\Sigma})$ is a splitting surface for f. But it follows from Lemma 3.2 that $d(\Sigma') \le 2g(\tilde{\Sigma}) = 2g(\Sigma') + 2$, and this contradicts the assumption. □

We now return to the proof of Lemma 4.2.

Proof of Lemma 4.2 Let $p_2 \colon J^2 \times D^2 \to J$ denote the projection onto the second coordinate. For $u \in D^2$, set $I_u := p_2(\mathrm{Cl}(\mathcal{R}_a)) \cap p_2(\mathrm{Cl}(\mathcal{R}_b))$. Then $t \in J$ is in $\mathrm{Int}(I_u)$ if and only if $h^{-1}_u(t)$ is a spanning surface for f. By Lemma 4.3, each I_u is a nonempty subset of J. Furthermore, it follows from Proposition 3.3(3) that each I_u is a closed interval in J. So $\bigsqcup_{u \in D^2} I_u$ is an (trivial) I–bundle over D^2.

Let $\sigma\colon D^2 \to \bigsqcup_{u\in D^2} I_u$ be a section of this I–bundle. Define $\tilde{\sigma}\colon D^2 \to \mathcal{H}$ by $\tilde{\sigma}(u) := h_u^{-1}(\sigma(u))$. Recall that $\varphi(u) = h_u^{-1}(0)$ for $u \in D^2$. The straight line homotopy connecting the 0–section of $J \times D^2 \to D^2$ to σ induces the homotopy $\{\varphi_r\colon D^2 \to \mathcal{H} \mid r \in [0,1]\}$ with $\varphi_0 = \varphi$ and $\varphi_1 = \tilde{\sigma}$. By Proposition 3.3(3), we may choose σ such that for $u \in \partial D^2$, $\varphi(u)$ is isotopic to $\tilde{\sigma}(u)$ through surfaces disjoint from K^- or K^+, depending on if $\varphi(u) \in \mathcal{H}^+$ or $\varphi(u) \in \mathcal{H}^-$ holds. This means that $\varphi_r(u) \in \mathcal{H}^\cup$ for $u \in \partial D^2$ and $r \in [0,1]$. Clearly, $\{\varphi_r\}$ can be chosen so that it preserves the basepoint. Thus, φ and $\tilde{\sigma}$ represent the same element of $\pi_2(\mathcal{H}, \mathcal{H}^\cup)$. Applying the homotopy described above, from now on, we may assume that the map φ satisfies the following: for any $u \in D^2$, $\Sigma_u := \varphi(u)$ is a spanning surface for f.

We think about a fixed $u \in D^2$ for a moment. By assumption, there exist values $a, b \in J$ such that $\Sigma'^+ := f^{-1}(a)$ is mostly above Σ_u, and $\Sigma'^- := f^{-1}(b)$ is mostly below Σ_u. By definition, every loop of $\Sigma_u \cap (\Sigma'^+ \cup \Sigma'^-)$ bounds a disk in $\Sigma'^+ \cup \Sigma'^-$. The following observation is due to Johnson [2011].

Claim *One of the two (possibly both) holds:*

(1) *Every loop in $\Sigma_u \cap \Sigma'^+$ bounds a disk in Σ_u.*

(2) *Every loop in $\Sigma_u \cap \Sigma'^-$ bounds a disk in Σ_u.*

Proof If we compress the surface Σ_u along innermost loops in $\Sigma'^+ \cup \Sigma'^-$ repeatedly, we have a collection of surfaces disjoint from both Σ'^+ and Σ'^-. The point is that there is a surface S in the collection that separates Σ'^+ from Σ'^-, and so S is in the product region between Σ'^+ and Σ'^-. Note that such S must have genus at least $g(\Sigma')$. This means that at most one of the two surfaces Σ'^+ and Σ'^- contains an actual compression for Σ_u (ie a loop in $\Sigma_u \cap \Sigma'^+$ or $\Sigma_u \cap \Sigma'^-$ that is nontrivial in Σ_u) because $g(\Sigma_u) = g(\Sigma') + 1$. Therefore, either (1) or (2) holds. □

Put $T' = \Sigma'^+ \cup \Sigma'^-$. Take a loop ℓ in $\Sigma_u \cap T'$ satisfying the following condition:

$$(*) \qquad \ell \text{ is trivial in } \Sigma_u \text{ and } \ell \text{ is innermost in } T' \text{ among all the loops of } \Sigma_u \cap T'.$$

If we compress Σ_u along ℓ and discard the sphere component, then the loop ℓ (and possibly some other loops in $\Sigma_u \cap T'$) is removed. Since M is irreducible, this process can actually be achieved by an isotopy. Repeating this process as long as possible, all the loops in $\Sigma_u \cap T'$ satisfying the condition $(*)$ are finally removed. In particular, the resulting surface is disjoint from Σ'^+ or Σ'^- depending on if (1) or (2) holds.

We wish to do the above process simultaneously for $u \in D^2$. In fact, it is always possible using an argument of Hatcher [1976]. The following is a sketch of the argument in [Hatcher 1976].

We will construct a smooth family $\{\Theta_{u,r}\colon \Sigma_u \to M \mid u \in D^2, r \in [0,1]\}$ of isotopies such that for any $u \in D^2$, $\Theta_{u,0}(\Sigma_u) = \Sigma_u$ and $\Theta_{u,1}(\Sigma_u)$ is disjoint from either K^- or K^+. By the above argument, we can see that there exist a finite cover $\{B_i\}$ of D^2 with $B_i \cong D^2$ and a family $\{T_i'\}$ of (disconnected) surfaces with the following properties:

- T_i' is the union of the two level surfaces $\Sigma_i'^+$ and $\Sigma_i'^-$ of f.
- If $u \in B_i$, then $\Sigma_i'^+$ is mostly above Σ_u, and $\Sigma_i'^-$ is mostly below Σ_u.

For $u \in D^2$, let $\widetilde{\mathscr{C}}_u$ be the set of intersection loops between Σ_u and $\bigcup_i T_i'$, where the union is taken over all i such that $u \in B_i$. We denote by $D'(\ell)$ the disk in T_i' bounded by ℓ for $\ell \in \widetilde{\mathscr{C}}_u$. (Note that such a disk is unique because each component of T_i' is not homeomorphic to S^2.) For $u \in D^2$, let $\mathscr{C}_u$ be the subset of $\widetilde{\mathscr{C}}_u$ consisting of those loops ℓ such that ℓ is trivial in Σ_u, and that $D'(\ell)$ contains no other intersection loop that is nontrivial in Σ_u. Furthermore, for $\ell \in \mathscr{C}_u$, we denote by $D(\ell)$ the disk in Σ_u bounded by ℓ. (Again, note that such a disk is unique because $\Sigma_u \neq S^2$.) For $u \in D^2$, we define the partial order $<'$ on $\mathscr{C}_u$ by

$$\ell <' m \iff D'(\ell) \subset D'(m).$$

Let $\{B_i'\}$ be a finite cover of D^2 obtained by shrinking each B_i slightly so that $B_i' \subset \mathrm{Int}(B_i)$ for i. Take a family $\{\alpha_u \colon \mathscr{C}_u \to (0,2) \mid u \in D^2\}$ of functions with the following properties:

- If $\ell, m \in \mathscr{C}_u$ and $\ell <' m$, then $\alpha_u(\ell) < \alpha_u(m)$.
- If $u \in B_i'$ and $\ell \subset \Sigma_u \cap T_i'$, then $\alpha_u(\ell) < 1$.
- If $u \in \partial B_i$ and $\ell \subset \Sigma_u \cap T_i'$, then $\alpha_u(\ell) > 1$.

The function α_u shows the times when intersection loops that belong to $\mathscr{C}_u$ are eliminated by compressing. Let G denote the union of the images of the α_u in $D^2 \times [0,2]$. Note that for each intersection loop ℓ, the images of the loops corresponding to ℓ form a 2D sheet over some B_i, and so G can be written as the union of these sheets. We can view G as a "chart" to compress the surface Σ_u; if we compress Σ_u following this chart upward from $r = 0$ to $r = 2$, then we get the sequence of surfaces. Note that the following subtle case may occur: if ℓ and m are loops of $\Sigma_u \cap T_i'$ with $D(\ell) \subset D(m)$ and $\alpha_u(m) < \alpha_u(\ell)$, then the loop ℓ is eliminated automatically before the time $\alpha_u(\ell)$. This example shows that we should use the "reduced" chart $\widehat{G}$ rather than G, which is obtained from G by removing the parts of the sheets corresponding to any such ℓ.

For every u, we will define the isotopy $\Theta_{u,r}$ as follows. Let $N(\widehat{G})$ denote a small fibered neighborhood of $\widehat{G}$. The interval $\{u\} \times [0,2]$ intersects $N(\widehat{G})$ at its subintervals $J_u^{(k)}$, where $1 \leq k \leq n = n(u)$. Define $\widetilde{\Theta}_{u,r}$ to be the isotopy obtained by piecing together the isotopies $\theta_{u,r}^{(1)}, \dots, \theta_{u,r}^{(n)}$ in the way suggested by $\widehat{G}$. Here each $\theta_{u,r}^{(k)}$ is an isotopy with its r–support in $J_u^{(k)}$, and corresponds to the compression along a loop in $\mathscr{C}_u$. See Figure 3. Now we define $\Theta_{u,r}$ as the restriction of $\widetilde{\Theta}_{u,r}$ on $[0,1]$.

It remains to see that we can modify the above construction to get the isotopy $\{\Theta_{u,r}\}$ to be smooth for $u \in D^2$. It is enough to show that each factor $\theta_{u,r}^{(k)}$ of $\Theta_{u,r}$ can be chosen so that it varies smoothly for u. For simplicity, we will think about the isotopy $\theta_{u,r}^{(1)}$ in the following although the same argument applies to any $\theta_{u,r}^{(k)}$. The isotopy $\theta_{u,r}^{(1)}$ corresponds to the compression along a loop $\ell_u \in \mathscr{C}_u$ for each u. Assume that $\ell_u \subset T_i'$ for any u. Denote by $D^3(\ell_u)$ the 3–ball in M bounded by the 2–sphere $D(\ell_u) \cup D'(\ell_u)$. (Note that such a 3–ball is unique because $M \neq S^3$.) Let (D^3, D, D') be the standard triple of disks, that is, D and D' are the upper and lower hemispheres in the boundary ∂D^3 of the standard 3–ball D^3, respectively. There is an identification $\phi_u \colon (D^3(\ell_u), D(\ell_u), D'(\ell_u)) \to (D^3, D, D')$ for every u. Then the arguments

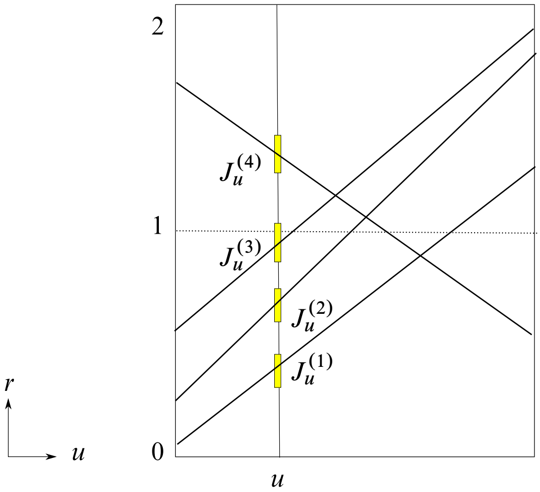

Figure 3: The isotopy $\widetilde{\Theta}_{u,r}$ is obtained by piecing together the isotopies $\theta_{u,r}^{(1)}$, $\theta_{u,r}^{(2)}$ $\theta_{u,r}^{(3)}$ and $\theta_{u,r}^{(4)}$. (In this example, $\widetilde{\Theta}_{u,r}$ can be written as the concatenation $\theta_{u,r}^{(1)} * \theta_{u,r}^{(2)} * \theta_{u,r}^{(3)} * \theta_{u,r}^{(4)}$ of the small isotopies.) The r–support of $\theta_{u,r}^{(k)}$ is contained in $J_u^{(k)}$.

in [Hatcher 1976] together with the Smale conjecture (the space $\mathrm{Diff}(D^3 \text{ rel } \partial D^3)$ is contractible), which is proved in [Hatcher 1983], show that ϕ_u can be chosen such that it varies smoothly for u.[2] Now we can define $\theta_{u,r}^{(1)} \equiv \phi_u^{-1} \circ F_r \circ \phi_u$ on $D(\ell_u) \subset \Sigma_u$ and $\theta_{u,r}^{(1)} \equiv \Theta_{u,0}$ on the complement of a small neighborhood of $D(\ell_u)$ in Σ_u, where $\{F_r : D^3 \to D^3 \mid r \in [0,1]\}$ is an isotopy that carries D to D' across D^3. Therefore, it follows that $\{\Theta_{u,r}\}$ is smooth for $u \in D^2$.

Finally, we see that $\Theta_{u,1}(\Sigma_u) \in \mathcal{H}^{\cup}$ for $u \in D^2$. Let u be any point in D^2. Take a path $\rho: [0,1] \to D^2$ with $\rho(0) = e_0$ and $\rho(1) = u$. It suffices to show that the path $\tilde{\rho}: [0,1] \to \mathcal{H}$ defined by $\tilde{\rho}(t) := \Theta_{\rho(t),1}(\Sigma_{\rho(t)})$ is wholly contained in $\mathcal{H}^{\cup}$.

For brevity, we denote by Σ_t the surface $\Theta_{\rho(t),1}(\Sigma_{\rho(t)})$ for $t \in [0,1]$ in the following. The cover $\{B_i\}$ of D^2 induces the cover $\{I_k \mid 0 \le k \le n\}$ of $[0,1]$ by finitely many closed intervals. By passing to a subcover if necessary, we may assume that $I_k \cap I_j = \varnothing$ if $|k-j| > 1$. As we have seen above, there exists a family $\{\Sigma_k'^{+} \cup \Sigma_k'^{-}\}$ $(= \{T_k'\})$ of level surfaces of f and the following hold:

- $\Sigma_k'^{+}$ is mostly above Σ_t if $t \in I_k$. Similarly, $\Sigma_k'^{-}$ is mostly below Σ_t if $t \in I_k$.
- For each k, one of the two surfaces $\Sigma_k'^{+}$ and $\Sigma_k'^{-}$ is disjoint from Σ_t if $t \in I_k$.

As is naturally expected, the following holds:

Claim *Suppose that $t \in I_k$. If $\Sigma_t \cap \Sigma_k'^{+} = \varnothing$ and $\Sigma_t \cap \Sigma_k'^{-} \neq \varnothing$, then $\Sigma_t \in \mathcal{H}^{-}$. Similarly, if $\Sigma_t \cap \Sigma_k'^{-} = \varnothing$ and $\Sigma_t \cap \Sigma_k'^{+} \neq \varnothing$, then $\Sigma_t \in \mathcal{H}^{+}$.*

[2] More specifically, we need the arguments at the end of Section 1 in [Hatcher 1976], where the sought isotopy, denoted by h_{tu} in that paper, is constructed. It starts by taking a suitable triangulation of D^n and then proceeds by extending the isotopy over the k–skeleton inductively. The homotopy group $\pi_k(\mathrm{Diff}(D^3 \text{ rel } D))$ appears as an obstruction to extending a map. (As we work in the smooth category, we use the Smale conjecture instead of the Alexander trick.)

Proof The proof is by induction on k. The following argument is based on the idea in [Johnson 2011]. By definition, $\Sigma_t \in \mathcal{H}^\cap$ for $t \in I_0$. Thus our claim holds on I_0. So, in what follows, we assume that $k > 0$ and that our claim holds on any interval I_j with $0 \le j < k$.

Let $t \in I_k$. Without loss of generality, we may assume that $\Sigma_t \cap \Sigma_k'^+ = \varnothing$ and $\Sigma_t \cap \Sigma_k'^- \neq \varnothing$. Fix $t_0 \in I_k \cap I_{k-1}$. Note that Σ_{t_0} and Σ_t are isotopic through surfaces disjoint from K^+. Thus it is enough to show that $\Sigma_{t_0} \in \mathcal{H}^-$. There are three cases to consider.

Case 1 $\Sigma_{t_0} \cap \Sigma_{k-1}'^+ = \varnothing$ and $\Sigma_{t_0} \cap \Sigma_{k-1}'^- \neq \varnothing$.

By the assumption of induction, this implies that $\Sigma_{t_0} \in \mathcal{H}^-$ and our claim holds in this case.

Case 2 $\Sigma_{t_0} \cap \Sigma_{k-1}'^+ \neq \varnothing$ and $\Sigma_{t_0} \cap \Sigma_{k-1}'^- = \varnothing$.

We will see that $\Sigma_{t_0} \in \mathcal{H}^\cap$. This is same as saying that Σ_{t_0} is a Heegaard surface of

$$M \setminus \operatorname{Int}(N(K^+ \cup K^-)) \cong \Sigma' \times J,$$

where $N(K^+ \cup K^-)$ is a sufficiently small neighborhood of $K^+ \cup K^-$. Since $\Sigma_{t_0} \cap \Sigma_{k-1}'^- = \Sigma_{t_0} \cap \Sigma_k'^+ = \varnothing$, Σ_{t_0} separates K^+ from K^-. First, we see that Σ_{t_0} is bicompressible in $M \setminus (K^+ \cup K^-)$. By assumption, there exists a loop $\ell \subset \Sigma_{t_0} \cap \Sigma_{k-1}'^+$ bounding a disk $D^- \subset \Sigma_{k-1}'^+$ such that ℓ is nontrivial in Σ_{t_0}. Similarly, there exists a loop $m \subset \Sigma_{t_0} \cap \Sigma_k'^-$ bounding a disk $D^+ \subset \Sigma_k'^-$ such that m is nontrivial in Σ_{t_0}. Since D^- and D^+ are in the opposite side of Σ_{t_0} to each other, Σ_{t_0} is bicompressible in $M \setminus (K^+ \cup K^-)$.

It is known that any genus $g(\Sigma') + 1$ bicompressible surface in $\Sigma' \times J$ separating $\Sigma' \times \{1\}$ from $\Sigma' \times \{-1\}$ must be reducible (see [Johnson 2011]). This means that there exists a 2–sphere $P \subset M \setminus (K^+ \cup K^-)$ intersecting Σ_{t_0} at a single nontrivial loop in Σ_{t_0}. Since M is irreducible, P cuts (M, Σ_{t_0}) into the two Heegaard splittings: one is a genus $g(\Sigma')$ Heegaard splitting of M and the other is a genus 1 Heegaard splitting of S^3. If we denote by S the genus $g(\Sigma')$ surface obtained by cutting Σ_{t_0} along P, then S still separates K^+ from K^-. Thus, S is isotopic to Σ' in the complement of $K^+ \cup K^-$. This shows that Σ_{t_0} is a genus $g(\Sigma') + 1$ Heegaard surface in $M \setminus \operatorname{Int}(N(K^+ \cup K^-))$. Therefore, we conclude that $\Sigma_{t_0} \in \mathcal{H}^\cap$ in this case.

Case 3 $\Sigma_{t_0} \cap \Sigma_{k-1}'^+ = \Sigma_{t_0} \cap \Sigma_{k-1}'^- = \varnothing$.

Let j denote the minimal integer such that for any $j < j' \le k$ and $t \in I_{j'}$, $\Sigma_t \cap \Sigma_{j'}'^+ = \Sigma_t \cap \Sigma_{j'}'^- = \varnothing$. If $j = 0$, then Σ_{t_0} is isotopic to Σ_0 through surfaces disjoint from $K^+ \cup K^-$. This shows that $\Sigma_{t_0} \in \mathcal{H}^\cap$. So we may assume that $j > 0$ in the following. Let $t_1 \in I_j \cap I_{j+1}$.

First, we assume that $\Sigma_{t_1} \cap \Sigma_j'^+ = \varnothing$ and that $\Sigma_{t_1} \cap \Sigma_j'^- \neq \varnothing$. By the assumption of induction, it follows that $\Sigma_{t_1} \in \mathcal{H}^-$. Since Σ_{t_1} and Σ_{t_0} are isotopic in $M \setminus (K^+ \cup K^-)$, we have $\Sigma_{t_0} \in \mathcal{H}^-$ in this case.

Next, we assume that $\Sigma_{t_1} \cap \Sigma_j'^+ \neq \varnothing$ and that $\Sigma_{t_1} \cap \Sigma_j'^- = \varnothing$. Then, there exists a compression disk $D^- \subset \Sigma_j'^+$ for Σ_{t_1}. Since Σ_{t_1} and Σ_{t_0} are isotopic in $M \setminus (K^+ \cup K^-)$, Σ_{t_0} has a compression disk disjoint from K^- as well. On the other hand, as we have seen above, $\Sigma_k'^-$ contains a compression disk

D^+ for Σ_{t_0} lying in the opposite side of Σ_{t_0} to D^-. Thus, Σ_{t_0} is bicompressible in $M \setminus (K^+ \cup K^-)$. Now applying the same argument as in Case 2, we have $\Sigma_{t_0} \in \mathcal{H}^\cap$ and this completes the proof. □

The above claim implies that the image of $\tilde{\rho}\colon [0,1] \to \mathcal{H}$ is contained in $\mathcal{H}^\cup$. In particular, $\Theta_{u,1}(\Sigma_u) \in \mathcal{H}^\cup$. Therefore, we conclude that $[\varphi] = 0 \in \pi_2(\mathcal{H}, \mathcal{H}^\cup)$ and this finishes the proof of Lemma 4.2. □

5 The isotopy subgroup of a Heegaard splitting of a handlebody

5.1 Proof of Theorem 1.3

We now give a proof of Theorem 1.3. Let V be a genus $g(V) \geq 2$ handlebody, and let (V, Σ) be a genus $g(V)+1$ Heegaard splitting of V. Fix a complete system $E_1, \ldots, E_{g(V)}$ of meridian disks for V. Consider a properly embedded, boundary parallel arc I in V that is disjoint from $\bigcup_{i=1}^{g(V)} E_i$. The surface Σ can be viewed as the boundary of a small neighborhood $N(\partial V \cup I)$ of $\partial V \cup I$. In the same spirit in [Johnson and McCullough 2013], we define the space $\mathrm{Unk}(V, I)$ of unknotted arcs to be $\mathrm{Diff}(V)/\mathrm{Diff}(V, I)$. Then, the following holds:

Theorem 5.1 [Scharlemann 2013, Theorem 5.1] *The group* $\mathrm{Isot}(V, \Sigma)$ *is isomorphic to* $\pi_1(\mathrm{Unk}(V, I))$.

Thus, it suffices to show that $\pi_1(\mathrm{Unk}(V, I))$ is finitely presented.

Fix a parallelism disk E for I disjoint from $\bigcup_{i=1}^{g(V)} E_i$. Furthermore, fix a spine K of V such that $K \cap E = \varnothing$ and K intersects each E_i at a single point. We now consider the two subspaces of $\mathrm{Unk}(V, I)$:

$$U_1 := \{I' \in \mathrm{Unk}(V, I) \mid I' \text{ admits a parallelism disk } E' \text{ with } E' \cap K = \varnothing\},$$

$$U_2 := \left\{ I' \in \mathrm{Unk}(V, I) \;\middle|\; I' \cap \bigcup_{i=1}^{g(V)} E_i = \varnothing \right\}.$$

Note that U_1, U_2 and $U_1 \cap U_2$ are all connected.

The group $\pi_1(U_1)$ is identical to the group $\mathfrak{F}_E$ in [Scharlemann 2013], which is called the *freewheeling* subgroup in that paper. This group is an extension of $\pi_1(\partial V)$ by $\mathbb{Z}$, and generated by λ_i, μ_i $(1 \leq i \leq g(V))$ and ρ shown in Figure 4. For each i, λ_i is represented by an isotopy of parallelism disk E along a longitudinal loop that intersects ∂E_i at a single point. Similarly, μ_i is represented by an isotopy of the parallelism disk E along a meridional loop corresponding to ∂E_i. The set $\{\lambda_i, \mu_i \mid 1 \leq i \leq g(V)\}$ corresponds to a generating set of $\pi_1(\partial V)$, and ρ is defined to be the half rotation of the parallelism disk E. Let P denote the planar surface obtained by cutting ∂V along simple closed curves $\partial E_1, \ldots, \partial E_{g(V)}$. Then, the group $\pi_1(U_2)$ is isomorphic to the 2–braid group $B_2(P)$ of P. Following [Scharlemann 2013], we define the *anchored* subgroup $\mathfrak{A}_{E_1,\ldots,E_{g(V)}}$ of $\pi_1(U_2)$ as follows. This is generated by $2g(V)$ elements α_i and α_i' $(1 \leq i \leq g(V))$ shown in Figure 5. Here each of α_i and α_i' is represented by an isotopy of I that moves the one endpoint p_1 of I along a meridional loop and fixes the other endpoint p_0. Note that we can write $\alpha_i' = \lambda_i^{-1} \alpha_i \lambda_i$ as elements of $\pi_1(\mathrm{Unk}(V, I))$. The group $\pi_1(U_2)$ is generated by $\mathfrak{A}_{E_1,\ldots,E_{g(V)}}$ and ρ.

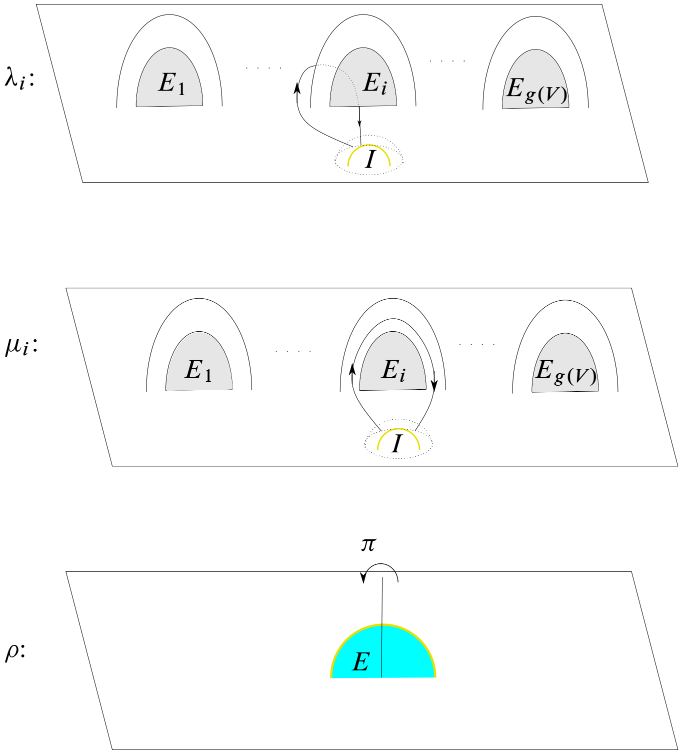

Figure 4: The group $\pi_1(U_1)$ is generated by $2g(V)+1$ elements.

The groups $\pi_1(U_1)$, $\pi_1(U_2)$ and $\pi_1(U_1 \cap U_2)$ are all finitely presented. By van Kampen's theorem, the proof is finished if the following is shown:

Lemma 5.2 *The inclusion* $U_1 \cup U_2 \to \mathrm{Unk}(V, I)$ *is a homotopy equivalence.*

In fact, by the same argument as in Section 4, it is easily seen that $\pi_k(U_1 \cup U_2) = \pi_k(\mathrm{Unk}(V, I)) = 0$ for $k \geq 2$. (And of course, this fact is unnecessary for our present purpose.) So we will see that the natural map $\pi_1(U_1 \cup U_2) \to \pi_1(\mathrm{Unk}(V, I))$ is an isomorphism.

Proof For brevity, set $U := \mathrm{Unk}(V, I)$. In [Scharlemann 2013], it was shown that $\pi_1(U)$ is generated by the two subgroups $\pi_1(U_1)$ $(= \mathfrak{F}_E)$ and $\mathfrak{A}_{E_1,\dots,E_{g(V)}}$ $(\subset \pi_1(U_2))$. It follows from this fact that the map

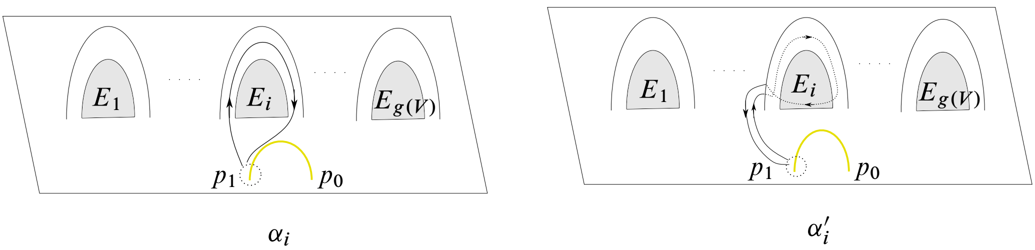

Figure 5: The group $\mathfrak{A}_{E_1,\dots,E_{g(V)}} \subset \pi_1(U_2)$ is generated by $2g(V)$ elements.

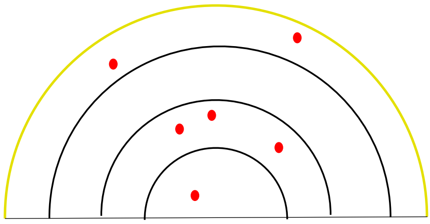

Figure 6: For $u \in \partial D^2$, the parallelism disk E_u intersects $\bigcup_{i=1}^{g(V)} E_i$ only at arcs parallel to I_u and intersects K at finitely many points.

$\pi_1(U_1 \cup U_2) \to \pi_1(U)$ is a surjection. We will see that the map $\pi_1(U_1 \cup U_2) \to \pi_1(U)$ is an injection. In other words, we will see that $\pi_2(U, U_1 \cup U_2) = 0$. Let $\varphi\colon (D^2, \partial D^2) \to (U, U_1 \cup U_2)$. Put $I_u := \varphi(u)$ for $u \in D^2$. In the same spirit of the proof of Lemma 3.1, we can show the following.

Claim 1 *There exists a (smooth) family of disks $\{E_u \mid u \in D^2\}$ in V such that E_u is a parallelism disk for I_u.*

Proof By [Scharlemann 2013], the map $\mathrm{Diff}(V) \to \mathrm{Diff}(V)/\mathrm{Diff}(V, \Sigma)$ is homotopy equivalent to $\mathrm{Diff}(V) \to \mathrm{Diff}(V)/\mathrm{Diff}(V, I)$. The former is a fibration [Johnson and McCullough 2013], and so is the latter. Thus, the map $\varphi\colon D^2 \to U$ lifts to a map $\tilde{\varphi}\colon D^2 \to \mathrm{Diff}(V)$. Now define $E_u := \tilde{\varphi}(u)(E)$. □

Since $\varphi(\partial D^2) \subset U_1 \cup U_2$, the isotopy $\{I_u \mid u \in \partial D^2\}$ represents an element of $\pi_1(U_1 \cup U_2)$. So we can write this isotopy as a product $\omega_1 \omega_2 \cdots \omega_n$ of the ω_k's, where each ω_k is either λ_i, μ_i, ρ, α_i, α_i' or their inverses. Corresponding to this factorization, there is a division of ∂D^2 into the intervals $J_1 = [u_0, u_1], \ldots, J_n = [u_{n-1}, u_n]$ with $u_0 = u_n$.

Claim 2 *After a deformation of $\{E_u \mid u \in D^2\}$ near ∂D^2, the following hold for any $u \in \partial D^2$:*

(i) *E_u intersects $\bigcup_{i=1}^{g(V)} E_i$ at finitely many arcs, and E_u intersects K at finitely many points.*

(ii) *Each arc of $E_u \cap \bigcup_{i=1}^{g(V)} E_i$ is parallel to I_u in E_u.*

(iii) *If a and a' are arcs of $E_u \cap \bigcup_{i=1}^{g(V)} E_i$, then a and a' are nested in the following sense: if Δ and Δ' are bigons in E_u cut by a and a' respectively, then either $\Delta \subset \Delta'$ or $\Delta' \subset \Delta$ holds.*

See Figure 6.

Proof The key is the following simple observation. For each interval J_k, there are the three possibilities:

- $\omega_k = \lambda_i^\epsilon$ for some $1 \le i \le g(V)$ and $\epsilon = \pm 1$. Then, during the move ω_k, some intersection arcs of $E_u \cap \bigcup_{i=1}^{g(V)} E_i$ are introduced or removed (possibly both may occur). All such arcs are parallel to I_u in E_u. The intersection pattern of $E_u \cap K$ is not changed by ω_k. See Figure 7, top.
- $\omega_k = \alpha_i^\epsilon$ or $\omega_k = \alpha_i'^\epsilon$ for some $1 \le i \le g(V)$ and $\epsilon = \pm 1$. Then, during the move ω_k, a single intersection point of $E_u \cap K$ is introduced or removed. The intersection pattern of $E_u \cap \bigcup_{i=1}^{g(V)} E_i$ is not changed by ω_k. See Figure 7, bottom.

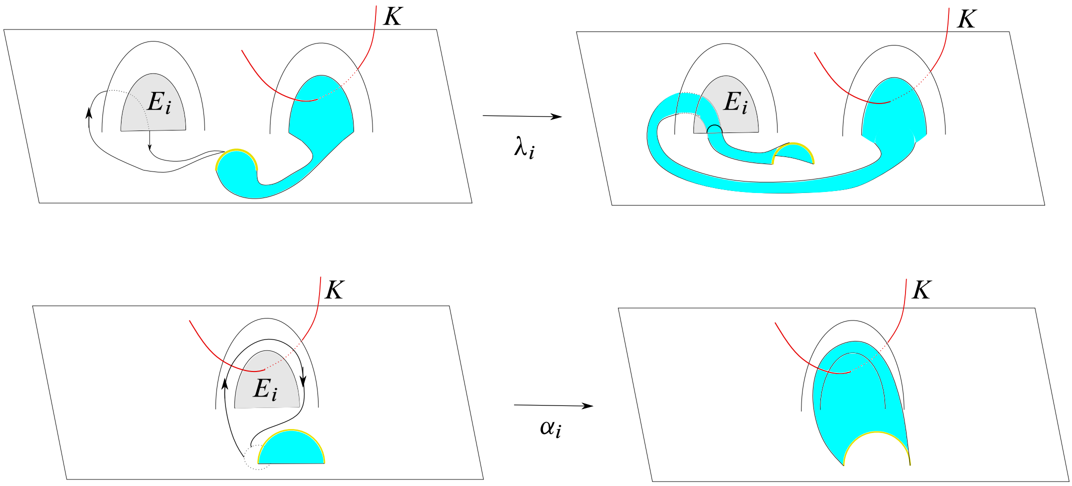

Figure 7: The move λ_i introduces or removes arcs parallel to I_u (top), and the move α_i introduces or removes a single point (bottom).

- $\omega_k = \mu_i^\epsilon$ or $\omega_k = \rho^\epsilon$ for some $1 \le i \le g(V)$ and $\epsilon = \pm 1$. Then, during the move ω_k, the intersection pattern of $E_u \cap \left(\bigcup_{i=1}^{g(V)} E_i \cup K\right)$ does not change.

Recall that $E_{u_0} = E_{u_n} = E$ by definition. In particular, $E_{u_0} \cap \left(\bigcup_{i=1}^{g(V)} E_i \cup K\right) = \varnothing$. By the above observation, it follows that the conditions (i), (ii) and (iii) are satisfied on the interval J_1. By an inductive argument, we can see that these three conditions are satisfied on any interval J_k as well. □

Put $B := \{re^{\sqrt{-1}\theta} \in \mathbb{C} \mid 0 \le r \le 1,\ 0 \le \theta \le \pi\}$. There is a smooth family $\{f_u \colon E_u \to B \mid u \in D^2\}$ of diffeomorphisms between E_u and B. (More rigorously, this is a consequence of the fact that the space $\mathrm{Diff}(D^2 \operatorname{rel} \partial D^2)$ is contractible [Smale 1959].) Furthermore, by Claim 2, we may choose $\{f_u\}$ such that for any $u \in \partial D^2$, each arc of $E_u \cap \bigcup_{i=1}^{g(V)} E_i$ is mapped to an arc $\{re^{\sqrt{-1}\theta} \in B \mid r = r_0, 0 \le \theta \le \pi\}$ for some $0 < r_0 \le 1$. For $t \in [0, 1)$, define $\sigma_t \colon B \to B$ by $\sigma_t(re^{\sqrt{-1}\theta}) := (1-t)re^{\sqrt{-1}\theta}$. Set $\Theta_{u,t} := f_u^{-1} \circ \sigma_t \circ f_u$ for $u \in D^2$ and $t \in [0, 1)$. Then, the isotopy $\Theta_{u,t}$ shrinks I_u along E_u into a small neighborhood of a point in $E_u \cap \partial V$ as $t \to 1$. If t is sufficiently close to 1, then $\Theta_{u,t}(I_u) \in U_1$. Furthermore, by definition, for $u \in \partial D^2$ and $t \in [0, 1)$, $\Theta_{u,t}(I_u)$ is disjoint from either K or $\bigcup_{i=1}^{g(V)} E_i$. Let $u \in \partial D^2$ and $t \in [0, 1)$. If $\Theta_{u,t}(I_u) \cap K = \varnothing$, then $\Theta_{u,t}(I_u) \in U_1$. On the other hand, if $\Theta_{u,t}(I_u) \cap \bigcup_{i=1}^{g(V)} E_i = \varnothing$, then $\Theta_{u,t}(I_u) \in U_2$. This means that $\Theta_{u,t}(I_u) \in U_1 \cup U_2$ for $u \in \partial D^2$ and $t \in [0, 1)$. Therefore, we conclude that $[\varphi] = 0 \in \pi_2(U, U_1 \cup U_2)$. □

5.2 Proof of Corollary 1.4

Proof of Corollary 1.4 By Theorems 1.1 and 1.3, $\mathrm{Isot}(M, \Sigma)$ is finitely presented. It remains to show that $\mathrm{MCG}(M, \Sigma)$ is finitely presented. By definition, there exists an exact sequence

$$1 \to \mathrm{Isot}(M, \Sigma) \to \mathrm{MCG}(M, \Sigma) \to \mathrm{MCG}(M).$$

By Theorem 2.3, M is hyperbolic, and hence $\mathrm{MCG}(M)$ is finite. Therefore, $\mathrm{MCG}(M,\Sigma)$ is finitely presented. □

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