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On the structure of the top homology group of the Johnson kernel

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The Johnson kernel is the subgroup $\mathcal{K}_g$ of the mapping class group $\mathrm{Mod}(\Sigma_g)$ of a genus-g oriented closed surface Σ_g generated by all Dehn twists about separating curves. We study the structure of the top homology group $\mathrm{H}_{2g-3}(\mathcal{K}_g,\mathbb{Z})$. For any collection of $2g-3$ disjoint separating curves on Σ_g, one can construct the corresponding abelian cycle in the group $\mathrm{H}_{2g-3}(\mathcal{K}_g,\mathbb{Z})$; such abelian cycles will be called simple. We describe the structure of a $\mathbb{Z}[\mathrm{Mod}(\Sigma_g)/\mathcal{K}_g]$–module on the subgroup of $\mathrm{H}_{2g-3}(\mathcal{K}_g,\mathbb{Z})$ generated by all simple abelian cycles and find all relations between them.

20F34; 20F36, 20J05, 57M07

1 Introduction

Let Σ_g be a compact oriented genus-g surface. Let $\mathrm{Mod}(\Sigma_g)=\pi_0(\mathrm{Homeo}^+(\Sigma_g))$ be the *mapping class group* of Σ_g, where $\mathrm{Homeo}^+(\Sigma_g)$ is the group of orientation-preserving homeomorphisms of Σ_g. The group $\mathrm{Mod}(\Sigma_g)$ acts on $H=H_1(\Sigma_g,\mathbb{Z})$. This action preserves the algebraic intersection form, so we have the representation $\mathrm{Mod}(\Sigma_g)\to\mathrm{Sp}(2g,\mathbb{Z})$, which is well known to be surjective. The kernel $\mathcal{I}_g$ of this representation is known as the *Torelli group*. This can be written as the short exact sequence

$$1\to\mathcal{I}_g\to\mathrm{Mod}(\Sigma_g)\to\mathrm{Sp}(2g,\mathbb{Z})\to 1. \tag{1}$$

The *Johnson kernel* $\mathcal{K}_g$ is the subgroup of $\mathcal{I}_g$ generated by all Dehn twists about separating curves. Johnson [15] proved that the group $\mathcal{K}_g$ can also be defined as the kernel of the surjective *Johnson homomorphism* $\tau:\mathcal{I}_g\to\bigwedge^3 H/H$, where the inclusion $H\hookrightarrow\bigwedge^3 H$ is given by $x\mapsto x\wedge\Omega$ and $\Omega\in\bigwedge^2 H$ is the inverse tensor of the algebraic intersection form. Therefore we have the short exact sequence

$$1\to\mathcal{K}_g\to\mathcal{I}_g\to\textstyle\bigwedge^3 H/H\to 1. \tag{2}$$

Denote by $\mathcal{G}_g$ the quotient group $\mathrm{Mod}(\Sigma_g)/\mathcal{K}_g$. The exact sequences (1) and (2) imply that $\mathcal{G}_g$ can be presented as the extension

$$1\to\textstyle\bigwedge^3 H/H\to\mathcal{G}_g\to\mathrm{Sp}(2g,\mathbb{Z})\to 1$$

of the symplectic group by the free abelian group $\bigwedge^3 H/H$. The group $H_*(\mathcal{K}_g,\mathbb{Z})$ has the natural structure of a $\mathcal{G}_g$–module.

In the case $g=1$ the representation $\mathrm{Mod}(\Sigma_1)\to\mathrm{Sp}(2,\mathbb{Z})=\mathrm{SL}(2,\mathbb{Z})$ is an isomorphism, so the group $\mathcal{I}_1$ is trivial. Mess [16] proved that the group $\mathcal{I}_2=\mathcal{K}_2$ is free with a countable number of generators. Therefore below we assume that $g\geq 3$ unless explicitly stated otherwise.

A natural problem is to study the homology of the group $\mathcal{K}_g$ for $g \geq 3$. The rational homology group $H_1(\mathcal{K}_g, \mathbb{Q})$ was shown to be finitely generated for $g \geq 4$ by Dimca and Papadima [7]. This group was computed explicitly for $g \geq 6$ by Morita, Sakasai and Suzuki [17] using the description due to Dimca, Hain and Papadima [6]. Recently Ershov and Sue He [8] proved that $\mathcal{K}_g$ is finitely generated in the case $g \geq 12$. This result was extended to any genus $g \geq 4$ by Church, Ershov and Putman [5]. This implies that the group $H_1(\mathcal{K}_g, \mathbb{Z})$ is finitely generated, provided that $g \geq 4$. It is still unknown whether $\mathcal{K}_3$ and $H_1(\mathcal{K}_3, \mathbb{Z})$ are finitely generated.

Bestvina, Bux and Margalit [2] computed the cohomological dimension of the Johnson kernel $\operatorname{cd}(\mathcal{K}_g) = 2g-3$. Gaifullin [11] proved that the top homology group $H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$ contains a free $\mathbb{Z}[\bigwedge^3 H/H]$–module of infinite rank. In particular, $H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$ is not finitely generated.

Recall that for n pairwise commuting elements $h_1, \dots, h_n$ of the group G, one can construct the *abelian cycle* $\mathcal{A}(h_1, \dots, h_n) \in H_n(G, \mathbb{Z})$ defined as follows. Consider the homomorphism $\phi\colon \mathbb{Z}^n \to G$ that maps the generator of the i^{th} factor to h_i. Then $\mathcal{A}(h_1, \dots, h_n) = \phi_*(\mu_n)$, where μ_n is the standard generator of $H_n(\mathbb{Z}^n, \mathbb{Z})$.

By a *curve* we always mean an essential simple closed curve on Σ_g. By an (oriented) *multicurve* we mean a finite union of pairwise disjoint and nonisotopic (oriented) curves on Σ_g. An *ordered multicurve* is a multicurve with a fixed order on its components. Usually we will not distinguish between a curve or a multicurve and its isotopy class. We denote by T_γ the left Dehn twist about a curve γ.

Definition 1.1 An *S–multicurve* is an ordered multicurve consisting of $2g-3$ separating components.

For example, the multicurve $\delta_1 \cup \dots \cup \delta_g \cup \epsilon_2 \cup \dots \cup \epsilon_{g-2}$ in Figure 1 is an S–multicurve. To an S–multicurve $M = \gamma_1 \cup \dots \cup \gamma_{2g-3}$ we assign the abelian cycle $\mathcal{A}(M) = \mathcal{A}(T_{\gamma_1}, \dots, T_{\gamma_{2g-3}}) \in H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$. If an S–multicurve M' is obtained from M by a permutation π of its components, then $\mathcal{A}(M') = (\operatorname{sign} \pi)\mathcal{A}(M)$. An easy computation of the Euler characteristic implies that any S–multicurve separates Σ_g into g one-punctured tori and $g-3$ three-punctured spheres. Throughout the paper we assume that the components of any S–multicurve are ordered so that the curves with numbers $1, \dots, g$ bound one-punctured tori from Σ_g.

Abelian cycles of the form $\mathcal{A}(M) \in H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$ for some S–multicurve M will be called *simple abelian cycles*. Denote by $\mathcal{A}_g \subseteq H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$ the subgroup generated by all simple abelian cycles. The author does not know the answer to the following question, which is an interesting problem itself:

Question 1.2 Is the inclusion $\mathcal{A}_g \subseteq H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$ strict for some $g \geq 3$?

The natural problem is to study the structure of the group $\mathcal{A}_g$. Let us identify Σ_g with the surface shown in Figure 1 and fix the multicurve $\Delta = \delta_1 \cup \dots \cup \delta_g \cup \epsilon_2 \cup \dots \cup \epsilon_{g-2}$ on Σ_g. Denote by $\mathcal{P}_g \subseteq \mathcal{A}_g$ the subgroup generated by all simple abelian cycles $\mathcal{A}(M)$, where $M = \delta_1 \cup \dots \cup \delta_g \cup \epsilon'_2 \cup \dots \cup \epsilon'_{g-2}$ for some separating curves $\epsilon'_2, \dots, \epsilon'_{g-2}$.

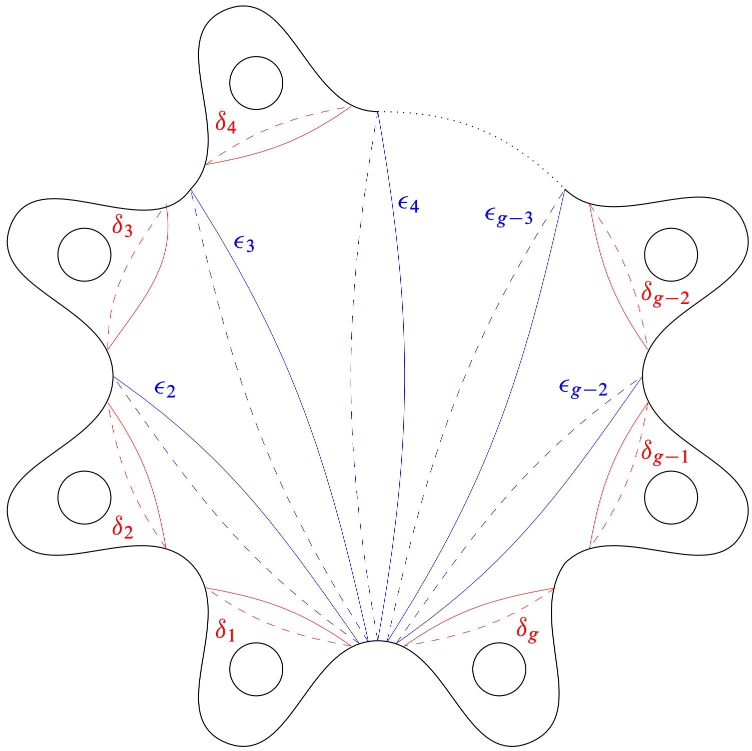

Figure 1: The surface Σ_g.

By an *unordered symplectic splitting* we mean an orthogonal (with respect to the intersection form) decomposition of H into a direct sum of g subgroups of rank 2. Write $N = \delta_1 \cup \cdots \cup \delta_g$, and for each i let X_i be the one-punctured torus bounded by δ_i. Set $V_i = H_1(X_i, \mathbb{Z}) \subset H$. Consider the corresponding unordered symplectic splitting $H = \bigoplus_i V_i$. The group $\mathrm{Sp}(2g, \mathbb{Z})$ acts on the set of all unordered symplectic splittings. Denote by $\mathcal{H}_g = \mathrm{SL}(2, \mathbb{Z})^{\times g} \rtimes S_g$ the stabilizer of the unordered splitting $\mathcal{V} = \{V_1, \dots, V_n\}$ in $\mathrm{Sp}(2g, \mathbb{Z})$, where S_g is the symmetric group.

The group $\mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(N)$ preserves the splitting $\mathcal{V}$; therefore the image of the natural homomorphism $\mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(N) \to \mathrm{Sp}(2g, \mathbb{Z})$ coincides with $\mathcal{H}_g$. Consider the corresponding mapping

$$\eta\colon \mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(N) \twoheadrightarrow \mathcal{H}_g.$$

We check in Proposition 2.1 that $\ker(\eta) \subseteq \mathcal{K}_g$, so we have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{K}_g & \longrightarrow & \mathrm{Mod}(\Sigma_g) & \longrightarrow & \mathcal{G}_g & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & \searrow & \\ & & & & & & & & \mathrm{Sp}(2g, \mathbb{Z}) \\ & & \cup & & \cup & & \cup & \nearrow & \\ 1 & \longrightarrow & \mathrm{Stab}_{\mathcal{K}_g}(N) & \longrightarrow & \mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(N) & \xrightarrow{\eta} & \mathcal{H}_g & \longrightarrow & 1 \end{array} \tag{3}$$

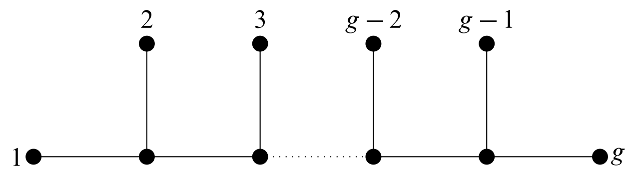

Figure 2: The dual tree $\mathcal{T}_0 = \mathcal{T}(\Delta)$ to the S–multicurve Δ.

Therefore we have the maps $\mathcal{H}_g \to \mathcal{G}_g$. Since the inclusion $\mathcal{H}_g \hookrightarrow \mathrm{Sp}(2g, \mathbb{Z})$ passes through $\mathcal{G}_g$, we have the inclusion $\mathcal{H}_g \hookrightarrow \mathcal{G}_g$. The second row of (3) implies that the group $\mathcal{H}_g = \mathrm{SL}(2,\mathbb{Z})^{\times g} \rtimes S_g$ acts on $\mathcal{P}_g$. The action of $\mathrm{SL}(2,\mathbb{Z})^{\times g}$ is trivial, and therefore $\mathcal{P}_g$ is an S_g–module. The first part of the main result is as follows:

Theorem 1.3 *There is an isomorphism of $\mathcal{G}_g$–modules*

$$\mathcal{A}_g \cong \mathrm{Ind}^{\mathcal{G}_g}_{\mathrm{SL}(2,\mathbb{Z})^{\times g} \rtimes S_g} \mathcal{P}_g.$$

In order to describe the S_g–module $\mathcal{P}_g$, we need to introduce some notation. Denote by $\boldsymbol{T}_g$ the set of trees $\mathcal{T}$ such that

- (I) $\mathcal{T}$ has g leaves (vertices of degree 1) marked by $1, \dots, g$, and
- (II) degrees of all other vertices of $\mathcal{T}$ equal 3.

We consider such trees up to an isomorphism preserving marking of the leaves. One can prove that $|\boldsymbol{T}_g| = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2g-5)$. For example, $\boldsymbol{T}_3$ consists of a single element.

For each S–multicurve M we consider the *dual tree* $\mathcal{T}(M)$, ie the graph that has a vertex for each connected component of $\Sigma_g \setminus M$ and where two vertices are adjacent if and only if the corresponding connected components are adjacent to each other. Since each component of M is separating, it follows that $\mathcal{T}(M)$ is a tree. The tree $\mathcal{T}(M)$ has g leaves corresponding to one-punctured tori; degrees of all other vertices equal 3. By definition components of M are ordered, so we also have an order on the set of curves that bound one-punctured tori on Σ_g. Each leaf of $\mathcal{T}(M)$ corresponds to a component of M with a number from 1 to g. Therefore the leaves of $\mathcal{T}(M)$ are numbered from 1 to g. Hence $\mathcal{T}(M)$ is an element of $\boldsymbol{T}_g$. For example, the dual tree $\mathcal{T}_0 = \mathcal{T}(\Delta)$ for the multicurve Δ in Figure 1 is shown in Figure 2.

Recall that we have the fixed curves $\delta_1, \dots, \delta_g$ on Σ_g as in Figure 1. For each $\mathcal{T} \in \boldsymbol{T}_g$ we can find a multicurve $\xi_2 \cup \dots \cup \xi_{g-2}$ disjoint from $\delta_1, \dots, \delta_g$ and consisting of separating components such that $\mathcal{T}$ is the dual tree to the multicurve $\Delta_{\mathcal{T}} = \delta_1 \cup \dots \cup \delta_g \cup \xi_2 \cup \dots \cup \xi_{g-2}$. Such a multicurve $\xi_2 \cup \dots \cup \xi_{g-2}$ is not unique, but we will prove that all such multicurves $\delta_1 \cup \dots \cup \delta_g \cup \xi_2 \cup \dots \cup \xi_{g-2}$ belong to the same $\mathcal{K}_g$–orbit; see Proposition 2.4. Therefore the simple abelian cycle $\mathcal{A}_{\mathcal{T}} = \mathcal{A}(\Delta_{\mathcal{T}}) \in H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$ is defined uniquely up to a sign. The sign of $\mathcal{A}_{\mathcal{T}}$ depends on the ordering of the curves $\xi_2, \dots, \xi_{g-2}$.

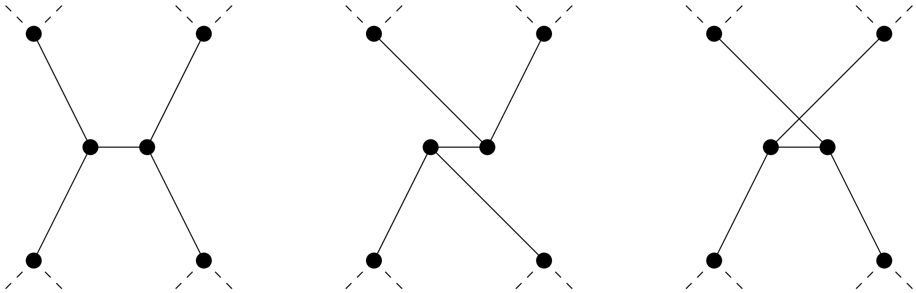

Figure 3: A cyclic triple of trees.

Let $h_1 = 1, h_2, h_3, \ldots \in \mathrm{Sp}(2g, \mathbb{Z})$ be representatives of all left cosets $\mathrm{Sp}(2g, \mathbb{Z})/\mathcal{H}_g$ and let $\hat{h}_1, \hat{h}_2, \hat{h}_3, \ldots \in \mathrm{Mod}(\Sigma_g)$ be mapping classes that go to $h_1, h_2, h_3, \ldots$ under the natural surjective homomorphism $\mathrm{Mod}(\Sigma_g) \twoheadrightarrow \mathrm{Sp}(2g, \mathbb{Z})$. Gaifullin [11, Theorem 1.3] proved that the abelian cycles

$$\hat{h}_s \cdot \mathcal{A}_{\mathcal{T}_0} \quad \text{for } s = 1, 2, 3, \ldots$$

form a basis of a free $\mathbb{Z}\big[\bigwedge^3 H/H\big]$–submodule of $H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$. In particular, these simple abelian cycles are nonzero and generate a free abelian group.

Definition 1.4 A triple of trees $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\} \subseteq \boldsymbol{T}_g$ is called *cyclic* if they differ only as shown in Figure 3 (upper and lower vertices in Figure 3 can be either leaves or not).

Theorem 1.5 *The abelian group $\mathcal{P}_g$ has a presentation where the generators are $\{\mathcal{A}_\mathcal{T} \mid \mathcal{T} \in \boldsymbol{T}_g\}$ and the relations are*

$$\{\mathcal{A}_{\mathcal{T}_1} + \mathcal{A}_{\mathcal{T}_2} + \mathcal{A}_{\mathcal{T}_3} = 0 \mid \{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\} \text{ is a cyclic triple}\}. \tag{4}$$

Remark 1.6 Recall that the signs of the simple abelian cycles $\mathcal{A}_\mathcal{T}$ depend on the order of the components of the corresponding S–multicurve. If $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$ is a cyclic triple, then the corresponding S–multicurves $\Delta_{\mathcal{T}_1}$, $\Delta_{\mathcal{T}_2}$ and $\Delta_{\mathcal{T}_3}$ differ by only one component. In (4) we mean that the components of these three S–multicurves are ordered so that the orderings coincide at $2g-4$ positions.

Remark 1.7 The "hard part" of Theorem 1.5 is the fact that any relation between simple abelian cycles follows from (4). However, the existence of such relations is not hard. For example, one can deduce (4) from the Lantern relation; see Farb and Margalit [9, Proposition 5.1]. Our proof is based on Arnold's relations in the cohomology of the pure braid group.

Also, we find an explicit basis of $\mathcal{P}_g$. For each tree $\mathcal{T} \in \boldsymbol{T}_g$ we say that the leaf with number g is the *root*, so $\mathcal{T}$ is a *rooted tree*. In this case, for each vertex the set of its *descendant leaves* is well defined.

Definition 1.8 Let $\mathcal{T} \in \boldsymbol{T}_g$. A vertex of $\mathcal{T}$ of degree 3 is called *balanced* if the paths from it to the two descendant leaves with the two smallest numbers have no common edges. The tree $\mathcal{T}$ is called *balanced* if all its vertices of degree 3 are balanced. The set of all balanced trees is denoted by $\boldsymbol{T}_g^b \subseteq \boldsymbol{T}_g$.

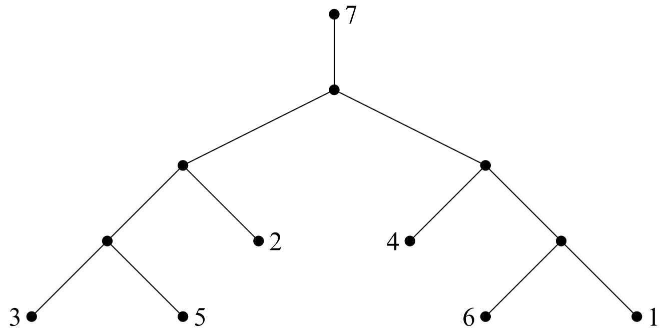

Figure 4: An example of a balanced tree in the case $g = 7$.

An example of a balanced tree for $g = 7$ is shown in Figure 4.

Theorem 1.9 *The simple abelian cycles* $\{\mathcal{A}_{\mathcal{T}} \mid \mathcal{T} \in \boldsymbol{T}_g^b\}$ *form a basis of* $\mathcal{P}_g$, *and* $\operatorname{rk} \mathcal{P}_g = |\boldsymbol{T}_g^b| = (g-2)!$.

Theorems 1.3, 1.5 and 1.9 provide a complete description of the group $\mathcal{A}_g$.

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2 Preliminaries and sketch of proof

2.1 Mapping class group of a surface with punctures and boundary components

Let Σ be an oriented surface, possibly with punctures and boundary components. We do not assume that Σ is connected. However, we require $H_*(\Sigma, \mathbb{Q})$ be a finite-dimensional vector space. The mapping class group of Σ is defined as $\operatorname{Mod}(\Sigma) = \pi_0(\operatorname{Homeo}^+(\Sigma, \partial\Sigma))$, where $\operatorname{Homeo}^+(\Sigma, \partial\Sigma)$ is the group of orientation-preserving homeomorphisms of Σ that restrict to the identity on $\partial\Sigma$. By $\operatorname{PMod}(\Sigma) \subseteq \operatorname{Mod}(\Sigma)$ we denote the *pure mapping class group* of Σ, ie the subgroup consisting of those elements fixing each of the punctures and each of the connected components. We have the exact sequence

$$1 \to \operatorname{PMod}(\Sigma_{g,n}^b) \to \operatorname{Mod}(\Sigma_{g,n}^b) \to S_n \to 1, \tag{5}$$

where $\Sigma_{g,n}^b$ denotes the connected genus-g surface with n punctures and b boundary components. For example, the pure mapping class group of the disk with n punctures is precisely the *pure braid group* $\mathrm{PB}_n = \operatorname{PMod}(\Sigma_{0,n}^1)$.

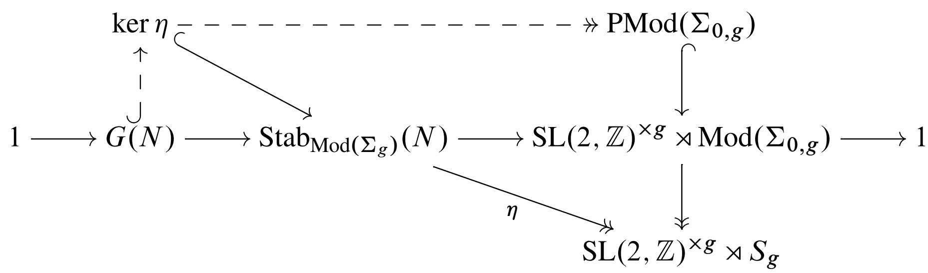
2.2 The Birman–Lubotzky–McCarthy exact sequence

Let M be a multicurve on Σ_g. Then, denoting by $G(M)$ the group generated by Dehn twists about the components of M, we have the Birman–Lubotzky–McCarthy exact sequence (see [3, Lemma 2.1])

$$1 \to G(M) \to \mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(M) \to \mathrm{Mod}(\Sigma_g \setminus M) \to 1. \tag{6}$$

Take $N = \delta_1 \cup \cdots \cup \delta_g$ as in Figure 1 and consider the group $\mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(N)$. We have $\Sigma_g \setminus N = \Sigma_{0,g} \sqcup X_1 \sqcup \cdots \sqcup X_g$, where X_i is the one-punctured torus bounded by δ_i. Since $\mathrm{Mod}(X_i) \cong \mathrm{SL}(2,\mathbb{Z})$, (5) and (6) imply the existence of the following commutative diagram:

(7)



This yields the exact sequence

$$1 \to G(N) \to \ker\eta \to \mathrm{PMod}(\Sigma_{0,g}) \to 1. \tag{8}$$

Proposition 2.1 *The following sequence is exact*:

$$1 \to \mathrm{Stab}_{\mathcal{K}_g}(N) \to \mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(N) \xrightarrow{\eta} \mathrm{SL}(2,\mathbb{Z})^{\times g} \rtimes S_g \to 1. \tag{9}$$

Proof First let us show that $\mathrm{Stab}_{\mathcal{K}_g}(N) \subseteq \ker\eta$. Indeed, any element $\phi \in \mathrm{Stab}_{\mathcal{K}_g}(N)$ stabilizes each component of N, so it also stabilizes each X_i. Since $\mathcal{K}_g$ is contained in the Torelli group, it follows that the restriction of ϕ to $\mathrm{Mod}(X_i) \cong \mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{Sp}(2g,\mathbb{Z})$ is trivial for all i.

Let us prove the opposite inclusion. The groups $G(M)$ and $\mathrm{PMod}(\Sigma_{g,n})$ are generated by Dehn twists about separating curves. The exact sequence (8) implies that the same is true for $\ker\eta$, and therefore $\ker\eta \subseteq \mathcal{K}_g$. Hence $\ker\eta \subseteq \mathrm{Stab}_{\mathcal{K}_g}(N)$. □

Lemma 2.2 *There is an isomorphism*

$$\mathrm{Stab}_{\mathcal{K}_g}(N) \cong \mathbb{Z}^{g-1} \times \mathrm{PB}_{g-1}. \tag{10}$$

Proof We need the following fact:

Fact 2.3 [9, Section 9.3] The center of the group PB_{g-1} is the infinite cyclic group, which is generated by the Dehn twist about the boundary curve. Moreover, we have the split exact sequence

$$1 \to \mathbb{Z} \xrightarrow{j_1} \mathrm{PB}_{g-1} \to \mathrm{PMod}(\Sigma_{0,g}) \to 1,$$

where j_1 is the inclusion of the center of PB_{g-1}.

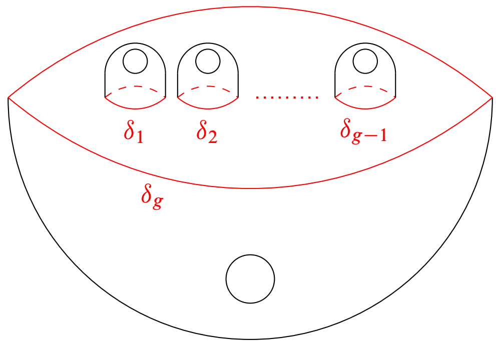


Figure 5

Consider the obvious map

$$j\colon \mathbb{Z}^g \cong \mathbb{Z}^{g-1} \times \mathbb{Z} \hookrightarrow \mathbb{Z}^{g-1} \times \mathrm{PB}_{g-1},$$

where the restriction of j on the first factor is the identity isomorphism and the restriction of j on the second factor is j_1. Fact 2.3 along with the exactness of (8) implies that in order to finish the proof of Lemma 2.2 we need to construct the map $\psi\colon \mathbb{Z}^{g-1} \times \mathrm{PB}_{g-1} \to \mathrm{Stab}_{\mathcal{K}_g}(N)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^g & \longrightarrow & \mathbb{Z}^{g-1} \times \mathrm{PB}_{g-1} & \longrightarrow & \mathrm{PMod}(\Sigma_{0,g}) & \longrightarrow & 1 \\ & & \| & & \downarrow \psi & & \| & & \\ 1 & \longrightarrow & G(N) & \longrightarrow & \mathrm{Stab}_{\mathcal{K}_g}(N) & \longrightarrow & \mathrm{PMod}(\Sigma_{0,g}) & \longrightarrow & 1 \end{array}$$

We define ψ as follows. The generator of the i^{th} factor of $\mathbb{Z}^{g-1}$ maps to T_{δ_i}. In order to define the restriction of ψ on the factor PB_{g-1}, let us identify Σ_g with the surface shown in Figure 5. We have the disk bounded by δ_g with $g-1$ handles bounded by $\delta_1, \dots, \delta_{g-1}$. We can replace all these handles by punctures and identify the group PB_{g-1} with the corresponding group $\mathrm{PMod}(\Sigma^1_{0,g-1})$. Then we extend the mapping classes in $\mathrm{PMod}(\Sigma^1_{0,g-1})$ to the handles so that the handles do not rotate.

Since the pure braid group is generated by Dehn twists about separating curves it follows that the image of ψ is contained in $\mathcal{K}_g$. The 5–lemma completes the proof of Lemma 2.2. □

2.3 Simple abelian cycles

Recall that for an S–multicurve $M = \gamma_1 \cup \dots \cup \gamma_{2g-3}$ on Σ_g there is the corresponding simple abelian cycle $\mathcal{A}(M) = \mathcal{A}(T_{\gamma_1}, \dots, T_{\gamma_{2g-3}}) \in H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$. We have already constructed the simple abelian cycles $\mathcal{A}_{\mathcal{T}} = \mathcal{A}(\Delta_{\mathcal{T}})$ for $\mathcal{T} \in \boldsymbol{T}_g$.

Proposition 2.4 *Let $\Delta_{\mathcal{T}} = \delta_1 \cup \dots \cup \delta_g \cup \xi_2 \cup \dots \cup \xi_{g-2}$ and $\Delta'_{\mathcal{T}} = \delta_1 \cup \dots \cup \delta_g \cup \xi'_2 \cup \dots \cup \xi'_{g-2}$ be two S–multicurves with the same dual tree $\mathcal{T}$. Then $\Delta_{\mathcal{T}}$ and $\Delta'_{\mathcal{T}}$ belong to the same $\mathcal{K}_g$–orbit (up to a permutation of the components). In particular, the simple abelian cycles $\{\mathcal{A}_{\mathcal{T}} \mid \mathcal{T} \in \boldsymbol{T}_g\}$ are well defined.*

Proof Since $\Delta_{\mathcal{T}}$ and $\Delta'_{\mathcal{T}}$ have the same dual tree, there is an element $\phi \in \mathrm{Mod}(\Sigma_g)$ such that after a permutation of $\xi_2, \dots, \xi_{g-2}$ we have $\phi(\xi_i) = \xi'_i$ and $\phi(\delta_j) = \delta_j$ for all i and j. Also, we can assume that $\phi|_{X_j} = \mathrm{id}$ for all $1 \le j \le g$. Then $\phi \in \ker \eta$ (see the exact sequence (9)), so $\phi \in \mathcal{K}_g$. Therefore all such S–multicurves $\Delta_{\mathcal{T}}$ belong to the same $\mathcal{K}_g$–orbit, so the simple abelian cycle $\mathcal{A}_{\mathcal{T}} = \mathcal{A}(\Delta_{\mathcal{T}}) \in H_{2g-3}(\mathcal{K}(\Sigma_g, \mathbb{Z}))$ is defined uniquely up to a sign. □

Proposition 2.5 *The simple abelian cycles $\{\mathcal{A}_{\mathcal{T}} \mid \mathcal{T} \in \boldsymbol{T}_g\}$ generate $\mathcal{A}_g$ as a $\mathbb{Z}[\mathcal{G}_g]$–module.*

Proof Consider an S–multicurve $M' = \gamma_1 \cup \dots \cup \gamma_{2g-3}$. We can assume that the curves $\gamma_1, \dots, \gamma_g$ bound one-punctured tori on Σ_g. There is an element $\phi \in \mathrm{Mod}(\Sigma_g)$ such that $\phi(\gamma_j) = \delta_j$ for all j. Then $\phi \cdot \mathcal{A}(M') = \pm \mathcal{A}_{\mathcal{T}(M')}$, so $\mathcal{A}(M') = \pm \phi^{-1} \cdot \mathcal{A}_{\mathcal{T}(M')}$. This implies the proposition. □

2.4 Sketches of the proofs of Theorems 1.3, 1.5 and 1.9

By Lemma 2.2 we can consider the simple abelian cycles $\{\mathcal{A}_{\mathcal{T}} \mid \mathcal{T} \in \boldsymbol{T}_g\}$ as elements of the group $H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(N), \mathbb{Z}) \cong H_{2g-3}(\mathbb{Z}^{g-1} \times \mathrm{PB}_{g-1})$. The following proposition will be proved in Section 3. Its proof is based on Arnold's relations in the cohomology of the pure braid group.

Proposition 2.6 *The abelian group $H_{2g-3}(\mathrm{PB}_{g-1} \times \mathbb{Z}^{g-1}, \mathbb{Z})$ is generated by the elements $\mathcal{A}_{\mathcal{T}}$, where $\mathcal{T} \in \boldsymbol{T}_g$. All relations among this generators follows from the relations*

$$\mathcal{A}_{\mathcal{T}_1} + \mathcal{A}_{\mathcal{T}_2} + \mathcal{A}_{\mathcal{T}_3} = 0$$

for each cyclic triple $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3\}$.

The next result will be proved in Sections 4 and 5. The proof is based on the spectral sequence for the action of $\mathcal{K}_g$ on the contractible complex of cycles, introduced by Bestvina, Bux and Margalit in [2], and certain new complexes which will be constructed below.

Proposition 2.7 *Let $f_1 = 1, f_2, f_3, \ldots \in \mathcal{G}_g$ be representatives of all left cosets $\mathcal{G}_g/\mathcal{H}_g$ and let $\hat{f}_1, \hat{f}_2, \hat{f}_3, \ldots \in \mathrm{Mod}(\Sigma_g)$ be their lifts in $\mathrm{Mod}(\Sigma_g)$. Then the inclusions*

$$i_s \colon \mathrm{Stab}_{\mathcal{K}_g}(\hat{f}_s \cdot N) \hookrightarrow \mathcal{K}_g \quad \text{for } s \in \mathbb{N}$$

induce an injective homomorphism

$$\bigoplus_{s \in \mathbb{N}} H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(\hat{f}_s \cdot N), \mathbb{Z}) \hookrightarrow H_{2g-3}(\mathcal{K}_g, \mathbb{Z}). \tag{11}$$

Proof of Theorem 1.5 By Proposition 2.7, the map

$$i_1 \colon H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(N), \mathbb{Z}) \hookrightarrow H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$$

is injective. Since $\mathcal{P}_g = i_1(H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(N), \mathbb{Z}))$, Proposition 2.6 implies the required result. □

Proof of Theorem 1.3 By Propositions 2.7 and 2.5, (11) induces an isomorphism

$$\bigoplus_{s \in \mathbb{N}} H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(\hat{f}_s \cdot N), \mathbb{Z}) \cong \mathcal{A}_g,$$

so

$$\mathcal{A}_g = \bigoplus_{s\in\mathbb{N}} \hat{f}_s \cdot \mathcal{P}_g.$$

Therefore, by the definition of an induced module, we have

$$\mathcal{A}_g \cong \mathcal{P}_g \otimes_{\mathcal{H}_g} \mathbb{Z}[\mathcal{G}_g] = \operatorname{Ind}_{\mathcal{H}_g}^{\mathcal{G}_g} \mathcal{P}_g. \qquad \square$$

Theorem 1.9 will be deduced from Proposition 2.6 in Section 3.

3 Proof of Proposition 2.6

3.1 Cohomology of the pure braid group

In order to prove Proposition 2.6 we conveniently consider the pure braid group on $g-1$ strands $q_1, \dots, q_{g-1}$. Let us recall Arnold's results on the structure of the ring $H^*(\mathrm{PB}_{g-1}, \mathbb{Z})$. The group PB_{g-1} has a standard set of generators $a_{i,j}$ for $1 \le i < j \le g-1$. These elements are the Dehn twists about curves enclosing the i^{th} and j^{th} strands (see Figure 6). Denote by $h_{i,j} \in H_1(\mathrm{PB}_{g-1}, \mathbb{Z})$ the corresponding homology classes. We denote by $\{w_{i,j}\}$ the dual basis of $H^1(\mathrm{PB}_{g-1}, \mathbb{Z})$. These cohomology classes can be interpreted as the homomorphisms

$$w_{i,j} : \mathrm{PB}_{g-1} \to \mathrm{PB}_2 \cong \mathbb{Z} \tag{12}$$

given by forgetting all strands besides q_i and q_j. It is convenient to put $w_{j,i} = w_{i,j}$.

Theorem 3.1 [1, Theorem 1] *The ring $H^*(\mathrm{PB}_{g-1}, \mathbb{Z})$ is the exterior graded algebra with $\binom{g-1}{2}$ generators $w_{i,j}$ of degree 1, satisfying $\binom{g-1}{3}$ relations*

$$w_{k,l}w_{l,m} + w_{l,m}w_{m,k} + w_{m,k}w_{k,l} = 0$$

for all $1 \le k < l < m \le n$.

Corollary 3.2 [1, Corollary 3] *The products*

$$w_{k_1,l_1} w_{k_2,l_2} \cdots w_{k_p,l_p} \quad \text{where } k_i < l_i \text{ and } l_1 < \cdots < l_p \tag{13}$$

form an additive basis of $H^*(\mathrm{PB}_{g-1}, \mathbb{Z})$.

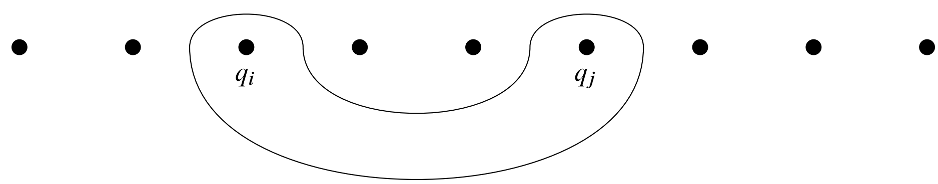


Figure 6: The element $a_{i,j}$ is the Dehn twist about the shown curve.

Corollary 3.2 implies that the products

$$w_{k_1,2}w_{k_2,3}\cdots w_{k_{g-2},g-1} \quad \text{where } k_i \le i \tag{14}$$

form an additive basis of $H^{g-2}(\mathrm{PB}_{g-1},\mathbb{Z})$. We denote the cohomology class (14) by $W_k = W_{k_1,\dots,k_{g-2}}$, where $k=(k_1,\dots,k_{g-2})$. We denote by $\boldsymbol{K}_g$ the set of all sequences $k=(k_1,\dots,k_{g-2})$ satisfying $1\le k_i\le i$.

3.2 Abelian cycles in $H_{g-2}(\mathrm{PB}_{g-1},\mathbb{Z})$

Corollary 3.2 implies that $\mathrm{cd}(\mathrm{PB}_n,\mathbb{Z})\ge n-1$. In fact $\mathrm{cd}(\mathrm{PB}_n)=n-1$. Indeed, let M_n be the ordered configuration space of n points on the disk. This space is aspherical, and $M_n\simeq K(\mathrm{PB}_n,1)$. We have the fiber bundle $M_n\to M_{n-1}$, where the fiber is homotopy equivalent to the wedge of $n-1$ circles. Hence, by induction, we obtain that M_n is homotopy equivalent to an $(n-1)$–dimensional CW–complex. Therefore $\mathrm{cd}(\mathrm{PB}_n,\mathbb{Z})\le n-1$.

So the isomorphism (10) implies

$$H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(N),\mathbb{Z})\cong H_{2g-3}(\mathbb{Z}^{g-1}\times \mathrm{PB}_{g-1},\mathbb{Z})\cong H_{g-2}(\mathrm{PB}_{g-1},\mathbb{Z}). \tag{15}$$

Let us recall the construction of the isomorphism $\mathrm{Stab}_{\mathcal{K}_g}(N)\cong\mathbb{Z}^{g-1}\times\mathrm{PB}_{g-1}$. We consider the surface $\Sigma^1_{0,g-1}$, given by replacing the boundary components corresponding to the curves $\delta_1,\dots,\delta_{g-1}$ on $\Sigma_{0,g}\subset\Sigma_g$ by the punctures $q_1,\dots,q_{g-1}$. Hence we obtain the pure braid group $\mathrm{PB}_{g-1}=\mathrm{PMod}(\Sigma^1_{0,g-1})$. The i^{th} factor in $\mathbb{Z}^{g-1}$ is generated by T_{δ_i}.

Consider a simple abelian cycle

$$\mathcal{A}_{\mathcal{T}}=\mathcal{A}(T_{\delta_1},\dots,T_{\delta_g},T_{\xi_2},\dots,T_{\xi_{g-2}})\in H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(N),\mathbb{Z})$$

for some $\mathcal{T}\in\boldsymbol{T}_g$. Isomorphism (15) sends $\mathcal{A}_{\mathcal{T}}$ to the abelian cycle

$$\mathcal{A}(T_{\delta_g},T_{\xi_2},\dots,T_{\xi_{g-2}})\in H_{g-2}(\mathrm{PB}_{g-1},\mathbb{Z}),$$

Let us set $\xi_1=\delta_g$ and

$$\widehat{\mathcal{A}}_{\mathcal{T}}=\mathcal{A}(T_{\xi_1},T_{\xi_2},\dots,T_{\xi_{g-2}})=\mathcal{A}(T_{\delta_g},T_{\xi_2},\dots,T_{\xi_{g-2}})\in H_{g-2}(\mathrm{PB}_{g-1},\mathbb{Z}).$$

Any simple closed curve on $\Sigma^1_{0,g-1}$ divides it into two parts. We say that a puncture q is *enclosed* by a curve γ on $\Sigma^1_{0,g-1}$ if q is contained in the part which does not contain the boundary component. For $k\in\boldsymbol{K}_g$, define the matrix $X_{k,\mathcal{T}}\in\mathrm{Mat}_{(g-2)\times(g-2)}(\mathbb{Z})$ by

$$(X_{k,\mathcal{T}})_{i,j}=\begin{cases}1 & \text{if the punctures } q_{k_i} \text{ and } q_{i+1} \text{ are enclosed by } \xi_j,\\ 0 & \text{otherwise.}\end{cases} \tag{16}$$

Lemma 3.3 *Let $k\in\boldsymbol{K}_g$ and $\mathcal{T}\in\boldsymbol{T}_g$. Then $\langle W_k,\widehat{\mathcal{A}}_{\mathcal{T}}\rangle=(-1)^{\binom{g-2}{2}}\det(X_{k,\mathcal{T}})$.*

Proof Consider a free abelian group $\mathbb{Z}^{g-2} = \langle c_1, \dots, c_{g-2} \rangle$ and the homomorphism $f : \mathbb{Z}^{g-2} \to \mathrm{PB}_{g-1}$ given by $c_i \mapsto T_{\xi_i}$. Denote by μ_{g-2} the standard generator of the group $H_{g-2}(\mathbb{Z}^{g-2}, \mathbb{Z})$. We have

$$\begin{aligned}\langle W_k, \widehat{\mathcal{A}}_{\mathcal{T}} \rangle &= \langle W_k, f_*(\mu_{g-2}) \rangle = \langle f^* W_k, \mu_{g-2} \rangle = \langle (f^* w_{k_1,2}) \cdots (f^* w_{k_{g-2},g-1}), \mu_{g-2} \rangle \\ &= (-1)^{\binom{g-2}{2}} \det(\langle f^* w_{k_i,i+1}, c_j \rangle)_{i,j=1}^{g-2} = (-1)^{\binom{g-2}{2}} \det(\langle w_{k_i,i+1}, f_* c_j \rangle)_{i,j=1}^{g-2} \\ &= (-1)^{\binom{g-2}{2}} \det(\langle w_{k_i,i+1}, T_{\xi_j} \rangle)_{i,j=1}^{g-2} = (-1)^{\binom{g-2}{2}} \det(X_{k,\mathcal{T}}).\end{aligned}$$

The last equality comes from the following corollary of (12):

$$\langle w_{k,l}, T_{\xi_j} \rangle = \begin{cases} 1 & \text{if the punctures } q_k \text{ and } q_l \text{ are enclosed by } \xi_j, \\ 0 & \text{otherwise.} \end{cases}$$ □

Let us denote by $\{D_k \in H_{g-2}(\mathrm{PB}_{g-1}, \mathbb{Z}) \mid k \in \boldsymbol{K}_g\}$ the dual basis to $\{W_k \mid k \in \boldsymbol{K}_g\}$.

Corollary 3.4 *Let $\mathcal{T} \in \boldsymbol{T}_g$. Then $\widehat{\mathcal{A}}_{\mathcal{T}} = \sum_{k \in \boldsymbol{K}_g} (-1)^{\binom{g-2}{2}} \det(X_{k,\mathcal{T}}) D_k$.*

3.3 Balanced trees

Recall that we consider the elements of $\boldsymbol{T}_g$ as marked trees such that the leaf with number g is the root. Also, we have already defined the subset $\boldsymbol{T}_g^b \subseteq \boldsymbol{T}_g$ of balanced trees. Take any $k \in \boldsymbol{K}_g$. Our goal is to construct a balanced tree $\mathcal{T}_k \in \boldsymbol{T}_g$ such that $\widehat{\mathcal{A}}_{\mathcal{T}_k} = (-1)^{\binom{g-2}{2}} D_k$ and the map $k \mapsto \mathcal{T}_k$ is a bijection between the sets $\boldsymbol{K}_g$ and $\boldsymbol{T}_g^b$. First let us construct the map $k \mapsto \mathcal{T}_k$ (then we will check that $\widehat{\mathcal{A}}_{\mathcal{T}_k} = (-1)^{\binom{g-2}{2}} D_k$; see Theorem 3.6).

Construction 3.5 We construct curves $\xi_1, \dots, \xi_{g-2}$ such that $\widehat{\mathcal{A}}_{\mathcal{T}_k} = \mathcal{A}(T_{\xi_1}, T_{\xi_2}, \dots, T_{\xi_{g-2}})$ by induction on g. The case $g = 3$ is trivial since $|\boldsymbol{T}_3^b| = |\boldsymbol{T}_3| = |\boldsymbol{K}_3| = 1$. Let us prove the induction step from $g-1$ to g. Consider any $k = (k_1, \dots, k_{g-2}) \in \boldsymbol{K}_g$ with $g > 3$. Let ξ_{g-2} be a curve enclosing exactly two points $q_{k_{g-2}}$ and q_{g-1}. Let us remove the curve ξ_{g-2} with its interior and denote the corresponding puncture by $q'_{k_{g-2}}$. Also take $q'_i = q_i$ for $i \le g-2$ and $i \ne k_{g-2}$. We obtain a disk with $g-2$ punctures $q'_1, \dots, q'_{g-2}$ and $k' = (k_1, \dots, k_{g-3}) \in \boldsymbol{K}_{g-1}$. The induction hypothesis implies that there is a balanced tree $\mathcal{T}_{k'} \in \boldsymbol{T}_{g-1}^b$ corresponding to k' given by some curves $\xi_1, \dots, \xi_{g-3}$. Now consider the curves $\xi_1, \dots, \xi_{g-3}, \xi_{g-2}$ and denote the dual tree by $\mathcal{T}_k \in \boldsymbol{T}_g$. It remains to show that $\mathcal{T}_k$ is balanced. Indeed, since the vertex q_{g-1} has the greatest number, all common vertices of $\mathcal{T}_{k'}$ and $\mathcal{T}_k$ are balanced. Also, this property holds for the vertex of $\mathcal{T}_k$ corresponding to the curve ξ_{g-2}, because it has only two descendant leaves. This implies the induction step.

Since for different $k, k' \in \boldsymbol{K}_g$ the corresponding trees $\mathcal{T}_k$ and $\mathcal{T}_{k'}$ are also different, it follows that the map $k \mapsto \mathcal{T}_k$ given by Construction 3.5 is injective. Moreover, direct computation shows that $|\boldsymbol{K}_g| = (g-2)!$. Therefore in order to prove that this map is a surjection to $\boldsymbol{T}_g^b$, it suffices to show that $|\boldsymbol{T}_g^b| = (g-2)!$. We use induction on g; the base case $g = 3$ is trivial. Consider a balanced tree $\mathcal{T} \in \boldsymbol{T}_g^b$ with $g \ge 4$. Let

$q_1, \dots, q_{g-1}$ be its leaves (besides the root). Let p be the vertex adjacent to q_{g-1}. Since $\mathcal{T}$ is balanced, another descendant vertex of p is a leaf q_i for some $1 \le i \le g-2$. Let us remove the vertices q_i and q_{g-1} (with the incident edges) and set $q_i = p$; denote the obtained tree by $\mathcal{T}'$. Then $\mathcal{T}' \in \boldsymbol{T}^b_{g-1}$. Since $|\boldsymbol{T}^b_{g-1}| = (g-3)!$ and there are $g-2$ ways to choose i, we have $|\boldsymbol{T}^b_g| = (g-2)!$. This implies the induction step.

Theorem 3.6 *Suppose that* $k \in \boldsymbol{K}_g$. *Then* $\widehat{\mathcal{A}}_{\mathcal{T}_k} = (-1)^{\binom{g-2}{2}} D_k$.

Proof By Corollary 3.4 it suffices to show that for any $k' \in \boldsymbol{K}_g$, we have $\det(X_{k',\mathcal{T}_k}) = 1$ if $k = k'$ and $\det(X_{k',\mathcal{T}_k}) = 0$ otherwise.

Lemma 3.7 *Let* $k \in \boldsymbol{K}_g$. *Then* $\det(X_{k,\mathcal{T}_k}) = 1$.

Proof Let $\xi_1, \dots, \xi_{g-2}$ be a multicurve with dual tree $\mathcal{T}_k$. By Construction 3.5 the punctures q_{k_i} and q_{i+1} are enclosed by the curve ξ_i for all $1 \le i \le g-2$. Indeed, for $i = g-2$ this follows from the construction of the curve ξ_{g-2}, and for $i < g-2$ this follows by the induction on g. Therefore $(X_{k,\mathcal{T}_k})_{i,i} = 1$ for all i.

Now let us check that $(X_{k,\mathcal{T}_k})_{i,j} = 0$ whenever $i < j$. Indeed, for $j = g-2$ this follows from the construction of the curve ξ_{g-2}, and for $j < g-2$ this follows by the induction on g. Therefore $X_{k,\mathcal{T}_k}$ is lower unitriangular, so $\det(X_{k,\mathcal{T}_k}) = 1$. □

Lemma 3.8 *Let* $k, k' \in \boldsymbol{K}_g$ *and* $k \ne k'$. *Then* $\det(X_{k',\mathcal{T}_k}) = 0$.

Proof Define $s = \max\{i \mid k_i \ne k'_i\}$. Let us check that the matrix $X_{k',\mathcal{T}_k}$ has the following form, where the s^{th} column is highlighted:

$$X_{k',\mathcal{T}_k} = \begin{pmatrix} * & * & \cdots & * & 0 & 0 & 0 & \cdots & 0 \\ * & * & \cdots & * & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * & 0 & 0 & 0 & \cdots & 0 \\ * & * & \cdots & * & 0 & 0 & 0 & \cdots & 0 \\ * & * & \cdots & * & * & 1 & 0 & \cdots & 0 \\ * & * & \cdots & * & * & * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * & * & * & * & \cdots & 1 \end{pmatrix}. \tag{17}$$

This will immediately imply $\det(X_{k',\mathcal{T}_k}) = 0$. We have $k_i = k'_i$ for all $i > s$, so similar arguments as in the proof of Lemma 3.7 show that the last $g-2-s$ columns have the required form.

Let $\mathcal{T}_k$ be given by curves $\xi_1, \dots, \xi_{g-2}$ as in Construction 3.5. Recall the construction of the curve ξ_s. At this step we have the punctures $q'_1, \dots, q'_{s+1}$. Some of them coincide with the q_i, others are identified with the interiors of some curves ξ_i, with $i \ge s+1$. Nevertheless, q_i is enclosed by ξ_s if and only if q'_i is enclosed by ξ_s for all $1 \le i \le s+1$.

By Construction 3.5, ξ_s is a curve enclosing exactly two punctures q'_{s+1} and q'_{k_s} from the set $\{q'_1,\dots,q'_{s+1}\}$. Therefore it does not enclose $q_{k'_s}$ as well as $q'_{k'_s}$, which implies $(X_{k',\mathcal{T}_k})_{s,s}=0$. Take any j with $1\le j<s$. The curve ξ_s encloses precisely one puncture among $q'_1,\dots,q'_s$, and so it also encloses precisely one puncture among $q_1,\dots,q_s$. Consequently, the curve ξ_s cannot enclose the punctures q_{j+1} and $q_{k'_j}$ simultaneously since $j+1\le s$ and $k'_j\le s$. Hence, by formula (16), we have $(X_{k',\mathcal{T}_k})_{j,s}=0$.

Therefore $(X_{k',\mathcal{T}_k})_{j,s}=0$ for $1\le j\le s$. □

Theorem 3.6 immediately follows form Lemmas 3.7 and 3.8. □

Corollary 3.9 *The abelian cycles $\{\widehat{\mathcal{A}}_{\mathcal{T}}\mid \mathcal{T}\in \boldsymbol{T}_g^b\}$ form a basis of the group $H_{g-2}(\mathrm{PB}_{g-1},\mathbb{Z})$. For any $\mathcal{T}\in\boldsymbol{T}_g$ we have $\widehat{\mathcal{A}}_{\mathcal{T}}=\sum_{k\in\boldsymbol{K}_g}(-1)^{\binom{g-2}{2}}\det(X_{k,\mathcal{T}})\widehat{\mathcal{A}}_{\mathcal{T}_k}$.*

Proof The result follows from Corollary 3.4 and Theorem 3.6. □

Proof of Theorem 1.9 By Proposition 2.7 and the first part of Corollary 3.9 there is an isomorphism

$$\mathcal{P}_g\cong H_{2g-3}(\mathbb{Z}^{g-1}\times\mathrm{PB}_{g-1},\mathbb{Z})\cong H_{g-2}(\mathrm{PB}_{g-1},\mathbb{Z}),$$

which maps $\mathcal{A}_{\mathcal{T}}$ to $\widehat{\mathcal{A}}_{\mathcal{T}}$ for all $\mathcal{T}\in\boldsymbol{T}_g$. The theorem follows from the second part of Corollary 3.9. □

3.4 Relations

Let $\{\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3\}\subseteq\boldsymbol{T}_g$ be a triple of trees. For $l=1,2,3$ denote by $\xi^l_1,\dots,\xi^l_{g-2}$ the corresponding sets of curves given by Construction 3.5. As before, the leaves of $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ (besides the root) are identified with the corresponding punctures and marked by $q_1,\dots,q_{g-1}$. One can check that the trees $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ form a cyclic triple if and only if, after some permutations of the corresponding sets of curves, the following conditions hold:

(a) There exists s with $1\le s\le g-2$ such that $\xi^1_i=\xi^2_i=\xi^3_i$ for $i\ne s$ and $1\le i\le g-2$.

(b) There exists $t\ne s$ with $1\le t\le g-2$ and pairwise disjoint nonempty subsets $B_1,B_2,B_3\subset\{q_1,\dots,q_{g-1}\}$ such that the set of punctures enclosed by the curve $\xi^1_t=\xi^2_t=\xi^3_t$ coincides with $B_1\cup B_2\cup B_3$.

(c) The set of punctures enclosed by ξ^1_s coincides with $B_2\cup B_3$.

(d) The set of punctures enclosed by ξ^2_s coincides with $B_3\cup B_1$.

(e) The set of punctures enclosed by ξ^3_s coincides with $B_1\cup B_2$.

Lemma 3.10 *Let $\{\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3\}\subseteq\boldsymbol{T}_g$ be a cyclic triple of trees. Then*

$$\widehat{\mathcal{A}}_{\mathcal{T}_1}+\widehat{\mathcal{A}}_{\mathcal{T}_2}+\widehat{\mathcal{A}}_{\mathcal{T}_3}=0. \tag{18}$$

Proof It suffices to prove that

$$\langle W_k,\widehat{\mathcal{A}}_{\mathcal{T}_1}+\widehat{\mathcal{A}}_{\mathcal{T}_2}+\widehat{\mathcal{A}}_{\mathcal{T}_3}\rangle=0 \tag{19}$$

for all $k \in \boldsymbol{K}_g$. The formula (19) is equivalent to

$$\det(X_{k,\mathcal{T}_1}) + \det(X_{k,\mathcal{T}_2}) + \det(X_{k,\mathcal{T}_3}) = 0. \tag{20}$$

We can assume that (a)–(e) hold. The matrices $X_{k,\mathcal{T}_1}$, $X_{k,\mathcal{T}_2}$ and $X_{k,\mathcal{T}_3}$ coincide everywhere besides the s^{th} column. Therefore the left-hand side of (20) equals the determinant of the matrix Y defined as follows. The s^{th} column of Y is the s^{th} column of the matrix $X_{k,\mathcal{T}_1} + X_{k,\mathcal{T}_2} + X_{k,\mathcal{T}_3}$ and all other columns are the corresponding columns of $X_{k,\mathcal{T}_1}$ (or, equivalently, $X_{k,\mathcal{T}_2}$ or $X_{k,\mathcal{T}_3}$). By (c)–(e) we have

$$(X_{k,\mathcal{T}_1})_{i,s} = \begin{cases} 1 & \text{if } i \in B_2 \cup B_3, \\ 0 & \text{otherwise,} \end{cases} \quad (X_{k,\mathcal{T}_2})_{i,s} = \begin{cases} 1 & \text{if } i \in B_3 \cup B_1, \\ 0 & \text{otherwise,} \end{cases} \quad (X_{k,\mathcal{T}_3})_{i,s} = \begin{cases} 1 & \text{if } i \in B_1 \cup B_2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$Y_{i,s} = (X_{k,\mathcal{T}_1} + X_{k,\mathcal{T}_2} + X_{k,\mathcal{T}_3})_{i,s} = \begin{cases} 2 & \text{if } i \in B_1 \cup B_2 \cup B_3, \\ 0 & \text{otherwise.} \end{cases}$$

By (b) we have

$$Y_{i,t} = (X_{k,\mathcal{T}_1})_{i,t} = \begin{cases} 1 & \text{if } i \in B_1 \cup B_2 \cup B_3, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the matrix Y has two proportional columns, so $\det(Y) = 0$. This implies (20). □

Lemma 3.11 *All relations between the abelian cycles* $\{\widehat{\mathcal{A}}_\mathcal{T} \mid \mathcal{T} \in \boldsymbol{T}_g\}$ *follow from* (18).

Proof Consider the abelian cycle $\widehat{\mathcal{A}}_\mathcal{T}$ for some $\mathcal{T} \in \boldsymbol{T}_g$. Corollary 3.9 implies that it suffices to decompose $\widehat{\mathcal{A}}_\mathcal{T}$ into a linear combination of abelian cycles $\{\widehat{\mathcal{A}}_\mathcal{T} \mid \mathcal{T} \in \boldsymbol{T}_g^b\}$ using (18).

Recall that a vertex of $\mathcal{T}$ of degree 3 is called balanced if the paths from it to the descendant leaves with the two smallest numbers have no common edges. If all vertices of $\mathcal{T}$ are balanced there is nothing to prove. Otherwise take any nonbalanced vertex v with the largest height (distance to the root) $h(v)$. Let v_1 and v_2 be its closest descendants and let w be its closest ancestor. Without loss of generality we may assume that the paths from v to the two descendant leaves with the smallest numbers start with the edge (v, v_1). Let u_1 and u_2 be the closest descendants of v_1.

Consider the trees $\mathcal{T}'$ and $\mathcal{T}''$ that differ from $\mathcal{T}$ as shown in Figure 7. The triple $\{\mathcal{T}, \mathcal{T}', \mathcal{T}''\}$ is cyclic, so $\widehat{\mathcal{A}}_\mathcal{T} = -\widehat{\mathcal{A}}_{\mathcal{T}'} - \widehat{\mathcal{A}}_{\mathcal{T}''}$. Note that the vertex v_1 is balanced in $\mathcal{T}$, and therefore the vertex v is balanced in $\mathcal{T}'$

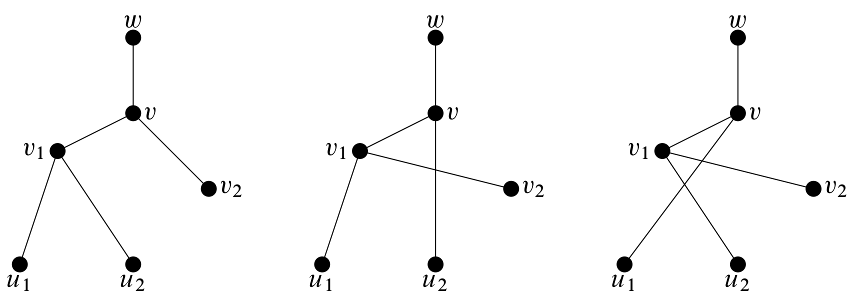


Figure 7: The trees $\mathcal{T}$, $\mathcal{T}'$ and $\mathcal{T}''$.

and $\mathcal{T}''$. Consequently, $\mathcal{T}'$ and $\mathcal{T}''$ have fewer nonbalanced vertices of height $h(v)$ and no nonbalanced vertices of greater height. Repeating this operation, we decompose $\widehat{\mathcal{A}}_{\mathcal{T}}$ into a linear combination of abelian cycles $\{\widehat{\mathcal{A}}_{\mathcal{T}} \mid \mathcal{T} \in \boldsymbol{T}_g^b\}$ using (18). □

Proof of Proposition 2.6 By Proposition 2.7 and the first part of Corollary 3.9 there is an isomorphism

$$\mathcal{P}_g \cong H_{2g-3}(\mathbb{Z}^{g-1} \times \mathrm{PB}_{g-1}, \mathbb{Z}) \cong H_{g-2}(\mathrm{PB}_{g-1}, \mathbb{Z}),$$

which maps $\mathcal{A}_{\mathcal{T}}$ to $\widehat{\mathcal{A}}_{\mathcal{T}}$ for all $\mathcal{T} \in \boldsymbol{T}_g$. The abelian cycles $\{\widehat{\mathcal{A}}_{\mathcal{T}} \mid \mathcal{T} \in \boldsymbol{T}_g\}$ generate $H_{g-2}(\mathrm{PB}_{g-1}, \mathbb{Z})$. Therefore the required assertion follows from Lemmas 3.10 and 3.11. □

4 The complex of cycles and the spectral sequence

Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \bigwedge^3 H/H & \longrightarrow & \mathcal{G}_g & \xrightarrow{\ p\ } & \mathrm{Sp}(2g,\mathbb{Z}) & \longrightarrow & 1 \\ & & & & \uparrow & \nearrow & & & \\ & & & & \mathcal{H}_g = \mathrm{SL}(2,\mathbb{Z})^{\times g} \rtimes S_g & & & & \end{array}$$

Let us choose elements $h_1 = 1, h_2, h_3, \ldots \in \mathcal{G}_g$ such that $1 = p(h_1), p(h_2), p(h_3), \ldots \in \mathrm{Sp}(2g, \mathbb{Z})$ are representatives of all left cosets $\mathrm{Sp}(2g, \mathbb{Z})/\mathcal{H}_g$. Let $\hat{h}_1, \hat{h}_2, \hat{h}_3, \ldots \in \mathrm{Mod}(\Sigma_g)$ be mapping classes that go to $h_1, h_2, h_3, \ldots$ under the natural surjective homomorphism $\mathrm{Mod}(\Sigma_g) \twoheadrightarrow \mathcal{G}_g$.

It is convenient to denote by U_g the abelian group $\bigwedge^3 H/H$ with multiplicative notation. For each $u \in U_g$ let $\hat{u} \in \mathcal{I}_g$ be the mapping class that goes to u under the Johnson homomorphism $\tau\colon \mathcal{I}_g \to U_g$. Let $f_1 = 1, f_2, f_3, \ldots \in \mathcal{G}_g$ be representatives of all left cosets $\mathcal{G}_g/\mathcal{H}_g$. Let $\hat{f}_1, \hat{f}_2, \hat{f}_3, \ldots \in \mathrm{Mod}(\Sigma_g)$ be mapping classes that go to $f_1, f_2, f_3, \ldots$ under the homomorphism $\mathrm{Mod}(\Sigma_g) \twoheadrightarrow \mathcal{G}_g$. For any $s \in \mathbb{N}$ the element f_s can be uniquely decomposed as $f_s = u \cdot h_r$ for some $u \in U_g$ and $r \in \mathbb{N}$. We can choose $\hat{f}_s$ such that $\hat{f}_s = \hat{u} \cdot \hat{h}_r$.

For each $r \in \mathbb{N}$ denote by G_r the subgroup of $H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$ generated by the images of homomorphisms

$$H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(\hat{u} \cdot \hat{h}_r \cdot N), \mathbb{Z}) \to H_{2g-3}(\mathcal{K}_g, \mathbb{Z}) \quad \text{for } u \in U_g. \tag{21}$$

In this section we prove the following result:

Lemma 4.1 *The inclusions*

$$G_r \hookrightarrow H_{2g-3}(\mathcal{K}_g, \mathbb{Z}) \quad \text{for } r \in \mathbb{N}$$

induce an injective homomorphism

$$\bigoplus_{r \in \mathbb{N}} G_r \hookrightarrow H_{2g-3}(\mathcal{K}_g, \mathbb{Z}).$$

In our proof we follow ideas of [11].

4.1 The complex of cycles

Bestvina, Bux and Margalit [2] constructed a contractible CW–complex $\mathcal{B}_g$ called the *complex of cycles* on which the Johnson kernel acts without rotations. "Without rotations" means that if an element $h \in \mathcal{K}_g$ stabilizes a cell σ setwise, then h stabilizes σ pointwise. Let us recall the construction of $\mathcal{B}_g$. More details can be found in [2; 13; 12; 10].

We denote by $\mathcal{C}$ the set of all isotopy classes of oriented nonseparating simple closed curves on Σ_g. Fix any nonzero element $x \in H$. The construction of $\mathcal{B}_g = \mathcal{B}_g(x)$ depends on the choice of the homology class x, however the CW–complexes $\mathcal{B}_g(x)$ are pairwise homeomorphic for different x.

A *basic* 1*–cycle* for a homology class x is a formal linear combination $\gamma = \sum_{i=1}^{n} k_i \gamma_i$, where $\gamma_i \in \mathcal{C}$ and $k_i \in \mathbb{N}$, such that

(I) the homology classes $[\gamma_1], \dots, [\gamma_n]$ are linearly independent,

(II) $\sum_{i=1}^{n} k_i [\gamma_i] = x$, and

(III) the isotopy classes $\gamma_1, \dots, \gamma_g$ contain pairwise disjoint representatives.

The oriented multicurve $\gamma_1 \cup \dots \cup \gamma_g$ is called the *support* of γ.

Denote by $\mathcal{M}(x)$ the set of oriented multicurves $M = \gamma_1 \cup \dots \cup \gamma_s$ such that

(i) no nontrivial linear combination of the homology classes $[\gamma_1], \dots, [\gamma_s]$ with nonnegative coefficients equals zero, and

(ii) for each $1 \le i \le s$ there exists a basic 1–cycle for x whose support is contained in M and contains γ_i.

For each $M \in \mathcal{M}(x)$ let us denote by $P_M \subset \mathbb{R}^{\mathcal{C}}_{\ge 0}$ the convex hull of the basic 1–cycles supported in M. We have that P_M is a convex polytope. By definition, the complex of cycles is the regular CW–complex given by $\mathcal{B}_g(x) = \bigcup_{M \in \mathcal{M}(x)} P_M$. Denote by $\mathcal{M}_0(x) \subseteq \mathcal{M}(x)$ the set of supports of basic 1–cycles for x. Then $\{P_M \mid M \in \mathcal{M}_0(x)\}$ is the set of 0–cells of $\mathcal{B}_g(x)$.

Theorem 4.2 [2, Theorem E] *Let $g \ge 1$ and $0 \ne x \in H_1(\Sigma_g, \mathbb{Z})$. Then $\mathcal{B}_g(x)$ is contractible.*

4.2 The spectral sequence

Suppose that a group G acts cellularly and without rotations on a contractible CW–complex X, let $C_*(X, \mathbb{Z})$ be the cellular chain complex of X and let $\mathcal{R}_*$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Consider the double complex $B_{p,q} = C_p(X, \mathbb{Z}) \otimes_G \mathcal{R}_q$ with the filtration by columns. The corresponding spectral sequence (see (7.7) in [4, Section VII.7]) has the form

$$E^1_{p,q} \cong \bigoplus_{\sigma \in \mathcal{X}_p} H_q(\mathrm{Stab}_G(\sigma), \mathbb{Z}) \Rightarrow H_{p+q}(G, \mathbb{Z}), \tag{22}$$

where $\mathcal{X}_p$ is a set containing exactly one representative in each G–orbit of p–cells of X. Let us remark that for an arbitrary CW–complex X, the spectral sequence (22) converges to the equivariant homology $H^G_{p+q}(X, \mathbb{Z})$. So for a contractible CW–complex X it converges to $H^G_{p+q}(X, \mathbb{Z}) \cong H_{p+q}(G, \mathbb{Z})$.

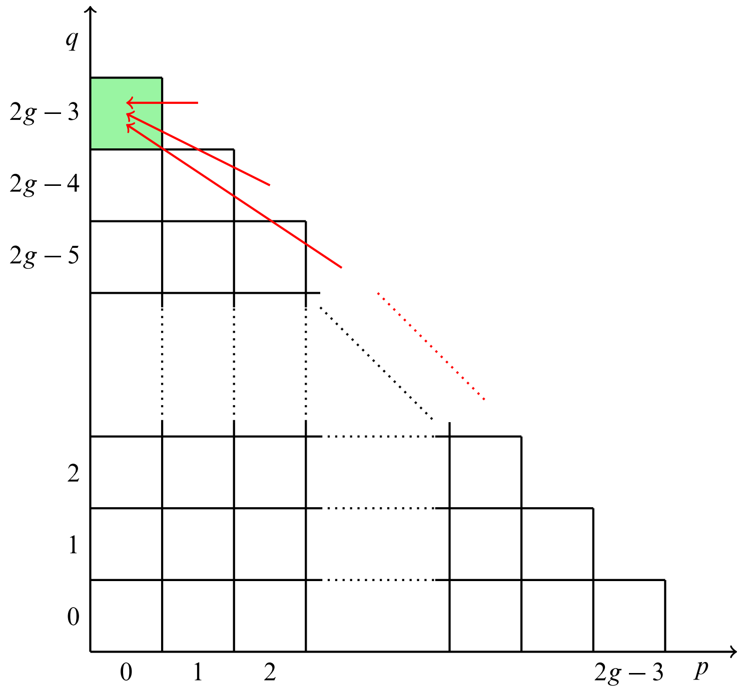


Figure 8

Now let $E^*_{*,*}$ be the spectral sequence (22) for the action of $\mathcal{K}_g$ on $\mathcal{B}_g(x)$ for some $0 \neq x \in H_1(\Sigma_g, \mathbb{Z})$. The fact that $\mathcal{K}_g$ acts on $\mathcal{B}_g$ without rotations follows from a result of Ivanov [14, Theorem 1.2]: if an element $h \in \mathcal{I}_g$ stabilizes a multicurve M then h stabilizes each component of M. Bestvina, Bux and Margalit proved [2, Proposition 6.2] that for each cell $\sigma \in \mathcal{B}_g(x)$ we have

$$\dim(\sigma) + \operatorname{cd}(\operatorname{Stab}_{\mathcal{K}_g}(\sigma)) \leq 2g - 3.$$

This immediately implies $E^1_{p,q} = 0$ for $p + q > 2g - 3$. Hence all differentials $d^1, d^2, \dots$ to the group $E^1_{0,2g-3}$ are trivial (see Figure 8, where the group $E^1_{0,2g-3}$ is shaded), so $E^1_{0,2g-3} = E^\infty_{0,2g-3}$. Therefore we have the following result:

Proposition 4.3 [11, Proposition 3.2] *Let $\mathfrak{M} \subseteq \mathcal{M}_0(x)$ be a subset consisting of oriented multicurves from pairwise different $\mathcal{K}_g$–orbits. Then the inclusions $\operatorname{Stab}_{\mathcal{K}_g}(M) \subseteq \mathcal{K}_g$, where $M \in \mathfrak{M}$, induce an injective homomorphism*

$$\bigoplus_{M \in \mathfrak{M}} H_{2g-3}(\operatorname{Stab}_{\mathcal{K}_g}(M), \mathbb{Z}) \hookrightarrow H_{2g-3}(\mathcal{K}_g), \mathbb{Z}).$$

Proof of Lemma 4.1 Denote by $X_{r,i} \subset \Sigma_g$ the one-punctured torus bounded by $\hat{h}_r \delta_i$ and $V_{r,i} = H_1(X_{r,i}, \mathbb{Z}) \subset H$. Then for each r we have the symplectic splittings $H = \bigoplus_i V_{r,i}$. Denote this unordered splitting by $\mathcal{V}_r = \{V_{r,1}, \dots, V_{r,g}\}$. Since $\mathcal{H}_g$ is the stabilizer of $\mathcal{V}_1$ in $\operatorname{Sp}(2g, \mathbb{Z})$, it follows that the $\mathcal{V}_r$ are pairwise distinct.

Assume the converse to the statement of Lemma 4.1 and consider a nontrivial linear relation

$$\sum_{r=1}^{k} \lambda_r \theta_r = 0 \quad \text{for } \lambda_r \in \mathbb{Z} \text{ and } \theta_r \in G_r. \tag{23}$$

For any homology class $x \in H_1(\Sigma_g, \mathbb{Z})$ and for any $1 \le r \le k$ we have a unique decomposition

$$x = \sum_{i=1}^{g} x_{r,i} \quad \text{for } x_{r,i} \in V_{r,i}.$$

The following result is proved in [11]:

Proposition 4.4 [11, Lemma 4.5] *There is a homology class $x \in H$ such that*

(I) *all homology classes $x_{r,i}$ are nonzero for $1 \le r \le k$ and $1 \le i \le g$, and*

(II) *for all $1 \le p \ne q \le k$ we have $\{x_{p,1}, \dots, x_{p,g}\} \ne \{x_{q,1}, \dots, x_{q,g}\}$ as unordered sets.*

Take any $x \in H$ satisfying the conditions of Proposition 4.4. For any $1 \le r \le k$ and $1 \le i \le g$ we have $x_{r,i} = n_{r,i} a_{r,i}$ where $a_{r,i} \in H$ is primitive and $n_{r,i} \in \mathbb{N}$. Let us check that for all $1 \le p \ne q \le k$ we have $\{a_{p,1}, \dots, a_{p,g}\} \ne \{a_{q,1}, \dots, a_{q,g}\}$ as unordered sets. Indeed, assume that there is a permutation $\pi \in S_g$ with $a_{p,i} = a_{q,\pi(i)}$. Therefore we have

$$\sum_{i=1}^{g} (n_{p,i} - n_{p,\pi(i)}) a_{p,i} = 0. \tag{24}$$

Since $a_{p,1}, \dots, a_{p_g}$ are linearly independent, (24) implies $n_{p,i} = n_{p,\pi(i)}$ for all $1 \le i \le g$. Hence $x_{p,i} = x_{p,\pi(i)}$ for all $1 \le i \le g$, which contradicts Proposition 4.4(II).

For any $1 \le r \le k$ and $1 \le i \le g$ let $\alpha_{r,i}$ be a simple curve on $X_{r,i}$ with $[\alpha_{r,i}] = a_{r,i} \in H$. Consider the oriented multicurve $A_r = \alpha_{r,1} \cup \dots \cup \alpha_{r,g}$. By construction $A_r \in \mathcal{M}_0(x)$.

Proposition 2.6 implies that the group $H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(\hat{u} \cdot \hat{h}_r \cdot N), \mathbb{Z})$ is generated by the primitive abelian cycles $\{\hat{u} \cdot \hat{h}_r \cdot \mathcal{A}_{\mathcal{T}} \mid \mathcal{T} \in \boldsymbol{T}_g\}$. Therefore for each $u \in U_g$ the homomorphisms (21) can be decomposed as

$$H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(\hat{u} \cdot \hat{h}_r \cdot N), \mathbb{Z}) \to H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(A_r), \mathbb{Z}) \to H_{2g-3}(\mathcal{K}_g, \mathbb{Z}). \tag{25}$$

Consequently there exists $\theta_r' \in H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(A_r), \mathbb{Z})$ which maps to θ_r under the second homomorphism in (25).

Proposition 4.3 implies that the inclusions $\mathrm{Stab}_{\mathcal{K}_g}(A_r) \subseteq \mathcal{K}_g$ for $r \in \mathbb{N}$ induce the injective homomorphism

$$\bigoplus_{r \in \mathbb{N}} H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(A_r), \mathbb{Z}) \hookrightarrow H_{2g-3}(\mathcal{K}_g, \mathbb{Z}).$$

So (23) implies that $\sum_{r=1}^{k} \lambda_r \theta_r' = 0$ as an element of the direct sum $\bigoplus_{r \in \mathbb{N}} H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(A_r), \mathbb{Z})$. Therefore $\lambda_r = 0$ for all r, which gives a contradiction. □

5 Proof of Proposition 2.7

In this section we prove the following lemma, which implies Proposition 2.7. Recall that G_r is the subgroup of $H_{2g-3}(\mathcal{K}_g, \mathbb{Z})$ generated by the images of homomorphisms

$$H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(\hat{u}\cdot\hat{h}_r\cdot N), \mathbb{Z}) \to H_{2g-3}(\mathcal{K}_g, \mathbb{Z}) \quad \text{for } u \in U_g.$$

Lemma 5.1 *Let $r \in \mathbb{N}$. Then the inclusions*

$$\mathrm{Stab}_{\mathcal{K}_g}(\hat{u}\cdot\hat{h}_r\cdot N) \hookrightarrow \mathcal{K}_g \quad \text{for } u \in U_g$$

induce an injective homomorphism

$$\bigoplus_{u\in U_g} H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(\hat{u}\cdot\hat{h}_r\cdot N), \mathbb{Z}) \hookrightarrow G_r.$$

Proof of Proposition 2.7 We can prove Proposition 2.7 for an arbitrary choice of $\hat{f}_s$, so we can assume that $\hat{f}_s = \hat{u}\cdot\hat{h}_r$ for some $u \in U_g$ and $r \in \mathbb{N}$. Combining Lemmas 4.1 and 5.1, we obtain

$$\bigoplus_{r\in\mathbb{N}}\bigoplus_{u\in U_g} H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(\hat{u}\cdot\hat{h}_r\cdot N), \mathbb{Z}) \hookrightarrow \bigoplus_{r\in\mathbb{N}} G_r \hookrightarrow H_{2g-3}(\mathcal{K}_g, \mathbb{Z}). \tag{26}$$

Then the sets $\{s \in \mathbb{N}\}$ and $\{\hat{u}\cdot\hat{h}_r \mid u \in U_g,\ r \in \mathbb{N}\}$ coincide, so (26) implies (11). □

To prove Lemma 5.1 we need to construct a new CW–complex, which will be called the *complex of relative cycles*. The idea is to introduce an analogue of $\mathcal{B}_g$ that makes sense for a sphere (ie the $g = 0$ case) with punctures.

5.1 The complex of relative cycles

Recall that $\Sigma_{0,2g}$ denotes a sphere with $2g$ punctures. In order to construct the complex of relative cycles $\mathcal{B}_{0,2g}$ we need to split the punctures into two disjoint sets: $P = \{p_1, \dots, p_g\}$ and $Q = \{q_1, \dots, q_g\}$.

By an *arc* on $\Sigma_{0,2g}$ we mean an embedded oriented curve with endpoints at punctures. By a *multiarc* we mean a disjoint union of arcs (common endpoints are allowed). We always consider arcs and multiarcs up to isotopy.

Denote by $\mathcal{D}$ the set of isotopy classes of arcs starting at a point in P and finishing at a point in Q. A *relative basic* 1*–cycle* is a formal sum $\gamma = \gamma_1 + \dots + \gamma_g$ where $\gamma_i \in \mathcal{D}$ such that

(I) $\partial\big(\sum_{i=1}^g \gamma_i\big) = \sum_{i=1}^g (q_i - p_i)$, and

(II) the isotopy classes $\gamma_1, \dots, \gamma_g$ contain pairwise disjoint representatives.

The multiarc $\gamma_1 \cup \dots \cup \gamma_g$ is called the *support* of γ.

Denote by $\mathcal{L}$ the set of multiarcs $L = \gamma_1 \cup \dots \cup \gamma_n$ (for arbitrary n) such that

(i) for each $1 \le i \le s$ there exists a relative basic 1–cycle, whose support is contained in L and contains γ_i.

For each $L \in \mathcal{L}$ we denote by $P_L \subset \mathbb{R}^{\mathcal{D}}_{\geq 0}$ the convex hull of all relative basic 1–cycles supported in L. We have that P_L is a convex polytope. By definition, the complex of relative cycles is the regular CW–complex given by $\mathcal{B}_{0,2g} = \bigcup_{L \in \mathcal{L}} P_L$. Denote by $\mathcal{L}_0 \subseteq \mathcal{L}$ the set of supports of all relative basic 1–cycles. Then $\{P_L \mid L \in \mathcal{L}_0\}$ is the set of 0–cells of $\mathcal{B}_{0,2g}$.

Remark 5.2 By construction, $\mathcal{B}_{0,2g}$ is the subset of $\mathbb{R}^{\mathcal{D}}_{\geq}$ consisting of the points (formal sums) $\sum_{i=1}^n k_i \gamma_i$ where $\gamma_i \in \mathcal{D}$ and $k_i \in \mathbb{R}_{\geq 0}$ satisfying the following conditions:

(I) $\partial(\sum_{i=1}^n k_i \gamma_i) = \sum_{i=1}^g (q_i - p_i)$.

(II) The isotopy classes $\gamma_1, \dots, \gamma_n$ contain pairwise disjoint representatives.

5.2 Contractability

Theorem 5.3 *Let $g \geq 1$. Then $\mathcal{B}_{0,2g}$ is contractible.*

In our proof we follow ideas of [2, Section 5]. Let us define an auxiliary complex $\widetilde{\mathcal{B}}_{0,2g}$. Denote by $\widetilde{\mathcal{D}}$ the union of $\mathcal{D}$ and the set consisting of the isotopy classes of all oriented simple closed curves on $\Sigma_{0,2g}$ (including contractible curves). Let us define $\widetilde{\mathcal{B}}_{0,2g}$ as the subset of $\mathbb{R}^{\widetilde{\mathcal{D}}}_{\geq 0}$ consisting of all points (formal sums) $\sum_{i=1}^n k_i \gamma_i$ where $\gamma_i \in \widetilde{D}$ and $k_i \in \mathbb{R}_{\geq 0}$ satisfying the following conditions:

(I) $\partial(\sum_{i=1}^n k_i \gamma_i) = \sum_{i=1}^g (q_i - p_i)$.

(II) The isotopy classes $\gamma_1, \dots, \gamma_n$ contain pairwise disjoint representatives.

Remark 5.2 implies that $\mathcal{B}_{0,2g} \subseteq \widetilde{\mathcal{B}}_{0,2g}$. Denote by $\mathrm{Drain}\colon \widetilde{\mathcal{B}}_{0,2g} \to \mathcal{B}_{0,2g}$ the retraction induced by the natural projection $\mathbb{R}^{\widetilde{\mathcal{D}}}_{\geq 0} \to \mathbb{R}^{\mathcal{D}}_{\geq 0}$.

Let d and d' be two points of $\mathcal{B}_{0,2g} \subseteq \mathbb{R}^{\mathcal{D}}_{\geq 0}$ and $t \in [0,1]$. The point $c = td + (1-t)d' \in \mathbb{R}^{\mathcal{D}}_{\geq 0}$ may not belong to $\mathcal{B}_{0,2g}$, because the arcs can have intersection points. We now explain how to do surgery to convert c into a point $\mathrm{Surger}(c) \in \widetilde{\mathcal{B}}_{0,2g} \subseteq \mathbb{R}^{\widetilde{\mathcal{D}}}_{\geq 0}$ which is canonical up to isotopy.

Let $c = \sum_{i=1}^n k_i c_i$ where the $c_i \in \mathcal{D}$ are in minimal position and $k_i \in \mathbb{R}_{\geq 0}$. We have $\partial\bigl(\sum_{i=1}^n k_i c_i\bigr) = \sum_{i=1}^g (q_i - p_i)$. Now it is convenient to replace the punctures $p_1, \dots, p_g, q_1, \dots, q_g$ by closed disks $P_1, \dots, P_g, Q_1, \dots, Q_g$. We thicken each c_i to a rectangle $R_i = [0,1] \times [0, k_i]$ of width k_i with coordinates $x_i \in [0,1]$ and $t_i \in [0, k_i]$ such that the curves $t_i = \mathrm{const}$ for different i are transverse to each other. We assume that the sides of R_i given by $x = 0$ and $x = 1$ are subsets of ∂P_a and ∂Q_b, respectively, where $\partial c_i = q_b - p_a$.

For a path $\alpha\colon [0,1] \to \Sigma_{0,2g}$, define $\mu_i(\alpha) = \int_\alpha dt_i$ and $\mu(\alpha) = \sum_{i=1}^n \mu_i(\alpha)$. Here we assume that $dt_i = 0$ outside R_i. Let us fix an arbitrary point $y_0 \in \Sigma_{0,2g}$. For each point $y \in \Sigma_{0,2g}$ choose a path α_y from y_0 to y and consider the map $\phi\colon \Sigma_{0,2g} \to S^1 = \mathbb{R}/\mathbb{Z}$ given by $\phi(y) = \mu(\alpha_y) \bmod 1$.

Let us check that the map ϕ is well defined. We have that $\phi(x)$ depends only on the homotopy class of α_x. Therefore it suffices to check that $\mu(\partial P_i) \in \mathbb{Z}$ and $\mu(\partial Q_i) \in \mathbb{Z}$ for all i. This follows from the fact that $\partial\bigl(\sum_{i=1}^n k_i c_i\bigr) = \sum_{i=1}^g (q_i - p_i)$.

The set of zeros of $d\phi$ is precisely $\Sigma_{0,2g} \setminus \bigcup_{i=1}^{g} R_i$, that is, a finite disjoint union of connected open sets. Therefore the map ϕ has a finite number of critical values separating S^1 into a finite number of intervals $w_1, \dots, w_l$. For any $1 \le j \le l$ take any point $y_j \in w_j$. The preimage $\eta_j = \phi^{-1}(y_j) \subset \Sigma_{0,2g}$ is a smooth 1–dimensional oriented submanifold, where the orientation on η_j is defined so that at each point of η_j the vector $\partial/\partial t_i$ and the positive tangent vector to η_s form a positive basis of the tangent space to the sphere. Moreover, $\eta_1, \dots, \eta_l$ are pairwise disjoint. Define $\mathrm{Surger}(c)$ as the formal sum $\sum_{j=1}^{l} |w_j| \eta_j$.

We claim that $\mathrm{Surger}(c) \in \mathbb{R}_{\ge 0}^{\widetilde{\mathcal{D}}}$. It suffices to show that each connected component of η_j is either closed or its initial point belongs to ∂P_a for some a and its terminal point belongs to ∂Q_b for some b. This follows from the orientation argument. Indeed, for all i the restrictions $\phi|_{\partial P_i}$ and $\phi|_{\partial Q_i}$ have degrees -1 and 1, respectively. Hence $\phi|_{\partial P_i}$ can only contain initial points of components of η_j, while $\phi|_{\partial Q_i}$ can only contain terminal points of components of η_j. Consequently, no component of $\mathrm{Surger}(c)$ connects ∂P_a with ∂P_b or ∂Q_a with ∂Q_b. Moreover, since the restrictions $\phi|_{\partial P_i}$ and $\phi|_{\partial Q_i}$ have degrees -1 and 1, respectively, we obtain $\partial(\mathrm{Surger}(c)) = \sum_{i=1}^{g} (q_i - p_i)$, so $\mathrm{Surger}(c) \in \widetilde{\mathcal{B}}_{0,2g}$.

Proof of Theorem 5.3 Take a point $c \in \mathcal{B}_{0,2g}$. Then the map

$$d \mapsto \mathrm{Drain}(\mathrm{Surger}(tc + (1-t)d))$$

is a deformation retraction from $\mathcal{B}_{0,2g}$ to the point c. □

5.3 Stabilizer dimensions

Proposition 5.4 *The group* $\mathrm{PMod}(\Sigma_{0,2g})$ *acts on* $\mathcal{B}_{0,2g}$ *without rotations.*

Proof Assume the converse and consider an element $\phi \in \mathrm{PMod}(\Sigma_{0,2g})$ and a cell corresponding to a multiarc $\gamma = \gamma_1 \cup \dots \cup \gamma_s$ such that $\phi(\gamma_i) = \gamma_{\pi(i)}$ for a nontrivial permutation π. Without loss of generality can assume that there exist arcs γ_1, γ_2 and γ_3 from $p \in P$ to $q \in Q$ satisfying $\gamma_1 \ne \gamma_2$ and $\gamma_2 \ne \gamma_3$ (and possibly $\gamma_1 = \gamma_3$), such that $\phi(\gamma_1) = \gamma_2$ and $\phi(\gamma_2) = \gamma_3$. Denote by $W_1 \subset \Sigma_{0,2g}$ and $W_2 \subset \Sigma_{0,2g}$ the subsurfaces bounded by the loops $\gamma_1\bar{\gamma}_2$ and $\gamma_2\bar{\gamma}_3$, respectively ($\bar{\gamma}_i$ denotes the arc γ_i with opposite direction). We assume that W_1 and W_2 are located on the left sides of $\gamma_1\bar{\gamma}_2$ and $\gamma_2\bar{\gamma}_3$, respectively.

By construction of $\mathcal{B}_{0,2g}$ we see that γ_1 is not isotopic to γ_2, so W_1 contains a nonempty set of punctures $\varnothing \ne Z_1 \subset P \sqcup Q$. Define $\varnothing \ne Z_2 \subset P \sqcup Q$ in a similar way. Since γ_2 separates W_1 from W_2 we have $Z_1 \ne Z_2$. The map f preserves the orientation, therefore $f(W_1) = W_2$ and so $f(Z_1) = Z_2$. However, $f \in \mathrm{PMod}(\Sigma_{0,2g})$ preserves the punctures, so we come to a contradiction. □

Theorem 5.5 *Let* σ *be a cell of* $\mathcal{B}_{0,2g}$. *Then*

$$\dim(\sigma) + \mathrm{cd}(\mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\sigma)) \le 2g - 3.$$

Proof The cell σ is given by a multiarc $\gamma_1 \cup \dots \cup \gamma_E$. Consider the planar graph Υ on the sphere with the vertices $p_1, \dots, p_g, q_1, \dots, q_g$ and the edges $\gamma_1, \dots, \gamma_E$. It is convenient for us to denote the number of vertices by $V = 2g$. Also let us denote by C the number of connected components of Υ and by F the number of its faces (ie the number of connected components of $\Sigma_{0,2g} \setminus \Upsilon$).

Lemma 5.6 *We have*

$$\dim(\sigma) = \dim(H_1(\Upsilon, \mathbb{R})) = E - V + C. \tag{27}$$

Proof The condition $\partial\big(\sum_{i=1}^{E} k_i\gamma_i\big) = \sum_{i=1}^{g}(q_i - p_i)$ is a nonhomogeneous system of linear equation in $\mathbb{R}^E$. The affine space of its solutions has the same dimension as the space of solutions of the homogeneous system $\partial\big(\sum_{i=1}^{E} k_i\gamma_i\big) = 0$. This space is precisely $H_1(\Upsilon, \mathbb{R})$. The cell σ is given by the intersection of this affine space with $\mathbb{R}^E_{\geq 0}$. Condition (i) in the construction of $\mathcal{B}_{0,2g}$ implies that σ contains a point in the interior of $\mathbb{R}^E_{\geq 0}$. Therefore $\dim(\sigma) = \dim(H_1(\Upsilon, \mathbb{R}))$. The second equality in (27) is trivial. □

Denote by $Y_1, \dots, Y_F$ the connected components of $\Sigma_{0,2g} \setminus \Upsilon$. We have $Y_i \cong \Sigma_{0,k_i}$ for some k_i. Recall that Σ_0^k denotes the sphere with k boundary components.

Proposition 5.7 $\mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\sigma) \cong \mathrm{Mod}(\Sigma_0^{k_1}) \times \cdots \times \mathrm{Mod}(\Sigma_0^{k_F})$.

Proof By construction we have $\mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\sigma) = \mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\Upsilon)$. Denote by $\overline{Y}_i$ the closure of $Y_i = \Sigma_{0,k_i}$ in the sphere. Let $\widetilde{Y}_i \cong \Sigma_0^{k_i}$ be the compactification of Y_i given by replacing each puncture by a boundary component. Let $p_i \colon \widetilde{Y}_i \to \overline{Y}_i$ be the natural projection. Then we have the corresponding mapping $\Phi_i \colon \mathrm{Mod}(\widetilde{Y}_i) \to \mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\Upsilon)$. It suffices to prove that the obvious mapping

$$\Phi \colon \mathrm{Mod}(\widetilde{Y}_1) \times \cdots \times \mathrm{Mod}(\widetilde{Y}_F) \to \mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\Upsilon)$$

is an isomorphism. We use the Alexander method; see [9, Proposition 2.8]. In the proof we need to distinguish between mapping classes and their representatives; the mapping class of a homeomorphism ψ is denoted by $[\psi]$.

First we prove the surjectivity of Φ. Let $[\psi] \in \mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\Upsilon)$. Then $\psi(\delta)$ is isotopic to δ for each arc δ of Υ. All such arcs are disjoint, so the Alexander method implies that there is a representative $\psi' \in [\psi]$ such that $\psi'(\delta) = \delta$ for each arc δ of Υ. Set $\phi'_i = \psi'|_{\overline{Y}_i}$. Since ϕ'_i is identical on $\partial\overline{Y}_i$, there exist $\phi_i \in \mathrm{Homeo}^+(\widetilde{Y}_i)$ such that $p_i \circ \phi_i = \phi'_i \circ p_i$. Hence $\Phi([\phi_1], \dots, [\phi_F]) = [\psi]$.

Now we prove that Φ is injective. Let $\Phi([\psi_1], \dots, [\psi_F]) = [\mathrm{id}]$. Since for each i the mapping $\psi_i|_{\partial\widetilde{Y}_i}$ is identical, there exists $\psi'_i \in \mathrm{Homeo}^+(\overline{Y}_i)$ such that $p_i \circ \psi_i = \psi'_i \circ p_i$. Consider the mapping $\psi' \in \mathrm{Homeo}^+(\Sigma_{0,2g})$ such that $\psi'|_{\overline{Y}_i} = \psi'_i$ for all i. By assumption ψ' is isotopic to the identity map.

Let $\widetilde{\Upsilon}$ be a planar graph on the sphere obtained from Υ by adding several arcs such that each face of $\widetilde{\Upsilon}$ is a disk. Let us show that there is an isotopy $\Psi_t \colon \Sigma_{0,2g} \to \Sigma_{0,2g}$ with $\Psi_0 = \psi'$ such that Ψ restricts to the identity on Υ and $\Psi_1(\psi'(\delta)) = \delta$ for each arc δ of $\widetilde{\Upsilon}$. It suffices to prove the existence of this isotopy in the case when we add only one arc γ to Υ. Let $\Upsilon' = \Upsilon \cup \{\gamma\}$. We can assume that $\psi'(\gamma)$ is transverse to γ. If $\psi'(\gamma)$ is disjoint from γ then these two arcs bound a disk on $\Sigma_{0,2g}$. This disk is contains no punctures, so it is disjoint from Υ. Hence in this case such an isotopy exists. If $\psi'(\gamma)$ and γ intersect, they form a bigon (see [9, Proposition 1.7]) that is disjoint from Υ for the same reason. So we can decrease the number of intersection points of γ and $\psi'(\gamma)$.

Set $\phi' = \Psi_1$, $\phi'_i = \phi'|_{\overline{Y}_i}$ and $\Psi'_i = \Psi|_{\overline{Y}_i}$. There exist homeomorphisms $\phi_i \in \mathrm{Homeo}(\widetilde{Y}_i)$ and isotopies Ψ_i of $\widetilde{Y}_i$ such that $p_i \circ \phi_i = \phi'_i \circ p_i$ and $p_i \circ \Psi_i = \Psi'_i \circ p_i$. Therefore Ψ_i is an isotopy between ψ_i and ϕ_i. By construction ϕ_i is identical on a collection of arcs that fill $\widetilde{Y}_i$ (*fill* means that each connected component of the complement to this collection is a disk). Hence the Alexander method implies that ϕ_i is isotopic to the identity for each i. Therefore ψ_i is also isotopic to the identity. □

For $k \geq 2$ we have $\mathrm{Mod}(\Sigma^1_{0,k-1}) \cong \mathrm{PB}_{k-1}$. If we replace the punctures on the disk S^1_0 by boundary components, the corresponding mapping class groups will by related to each other via the following exact sequence (see [9, Proposition 3.19]):

$$1 \to \mathbb{Z}^{k-1} \to \mathrm{Mod}(\Sigma^k_0) \to \mathrm{Mod}(\Sigma^1_{0,k-1}) \to 1.$$

Since the tangent bundle to the disk is trivial, this sequence splits. Therefore $\mathrm{Mod}(\Sigma^k_0) \cong \mathbb{Z}^{k-1} \times \mathrm{PB}_{k-1}$. Since $\mathrm{cd}(\mathrm{PB}_{k-1}) = k-2$ we have $\mathrm{cd}(\mathbb{Z}^{k-1} \times \mathrm{PB}_{k-1}) = 2k-3$. When $k=1$ we have $\mathrm{cd}(\mathrm{Mod}(\Sigma^1_0)) = 0$. Denote by D the number of Y_i that are homeomorphic to the disk. Proposition 5.7 immediately implies the following result:

Corollary 5.8
$$\mathrm{cd}(\mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\sigma)) = \sum_{i=1}^{F}(2k_i - 3) + D.$$

Let us finish the proof of Theorem 5.5. By Lemma 5.6 and Corollary 5.8 we have

$$\dim(\sigma) + \mathrm{cd}(\mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\sigma)) = E - V + C + \sum_{i=1}^{F}(2k_i - 3) + D = E - V + C + D - 3F + 2\sum_{i=1}^{F} k_i.$$

Let $\Theta_1, \dots, \Theta_C$ be the connected components of Υ. Note that

$$\begin{aligned}\sum_{i=1}^{F} k_i &= |\{(Y_i, \Theta_j) \mid Y_i \text{ is adjacent to } \Theta_j\}| = \sum_{j=1}^{C}(\dim(H_1(\Theta_j, \mathbb{R})) + 1) \\ &= \dim(H_1(\Upsilon, \mathbb{R})) + C = E - V + 2C.\end{aligned} \tag{28}$$

Therefore

$$\begin{aligned}E - V + C + D - 3F + 2\sum_{i=1}^{F} k_i &= E - V + C + D - 3F + 2(E - V + 2C) \\ &= 3E - 3V + 5C - 3F + D = 2C + D - 3(V - E + F - C).\end{aligned}$$

By Euler's formula we have

$$V - E + F - C = 1. \tag{29}$$

Therefore

$$\dim(\sigma) + \mathrm{cd}(\mathrm{Stab}_{\mathrm{PMod}(\Sigma_{0,2g})}(\sigma)) \leq 2C + D - 3. \tag{30}$$

In order to finish the proof of Theorem 5.5 we need the following result:

Lemma 5.9 *Let a planar graph Υ represent a cell of $\mathcal{B}_{0,2g}$ and $g \geq 2$. Then $2C + D \leq 2g$.*

Proof We proceed by induction on the number of connected components of Υ with only one edge.

Base case: Υ does not have a connected component with only one edge Since $D \le F$ and $V = 2g$, it suffices to check that

$$2C + F \le V. \tag{31}$$

Note that Υ is a bipartite graph and does not contain isotopic edges. Since Υ does not have a connected component with only one edge, if Y_i is adjacent to Θ_j for some i and j, then Y_i is adjacent to at least four edges of Θ_j. Then by (28) we have

$$E \ge 2\sum_{i=1}^{F} k_i = 2E - 2V + 4C = 2C + 2F - 2.$$

The last equality follows from (29). Since $C \ge 1$ we have

$$E \ge 2C + 2F - 2 \ge 2F + C - 1.$$

We can rewrite this as

$$2C + F \le 1 + C - F + E. \tag{32}$$

Equation (29) implies that the right-hand side of (32) equals V. Therefore (31) holds.

Induction step: Υ has a connected component with only one edge In the case $g = 2$ the graph Υ is a disjoint union of two closed intervals, so $C = 2$ and $D = 0$; in this case the required inequality $2C + D \le 2g$ is obvious. Hence we can assume that $g \ge 3$. Let p_i and q_j form such a component, that is, p_i and q_j are vertices of Υ of degree 1 connected by an edge α. Assume that after removing this component the remaining graph Υ_1 will not contain isotopic edges (and, consequently, will represent some cell of $\mathcal{B}_{0,2g-2}$). Then $C_1 = C - 1$ is the number of connected components of Υ_1. Denote by D_1 the number of faces of Υ_1 homeomorphic to the disk. We have $D_1 \le D + 1$, since at most one disk can appear. The graph Υ_1 has fewer connected components with only one edge than Υ. Since $g - 1 \ge 2$, by the induction assumption we have

$$2C + D \le 2C_1 + D_1 + 1 \le 2g - 2 + 1 < 2g.$$

Now assume that our previous assumption does not hold, that is, after removing the component consisting of one edge, the remaining graph will contain isotopic edges. This means that there exist punctures p_r and q_s and edges β_1 and β_2 between them such that p_i and q_j are the only vertices of Υ located inside of the disks bounded by β_1 and β_2. There exists an arc γ_1 from p_i to q_s and an arc γ_2 from p_r to q_j such that γ_1 and γ_2 are disjoint from Υ and from each other. Consider the graph Υ' obtained from Υ by adding the edges γ_1 and γ_2. Note that Υ' has fewer connected components with exactly one edge than Υ and also represents a cell of $\mathcal{B}_{0,2g}$. Then $C' = C - 1$ is the number of connected components of Υ' and $D' = D + 2$ is the number of faces of Υ' homeomorphic to the disk. Therefore $2C + D \le 2g$ if and only if $2C' + D' \le 2g$. The induction assumption concludes the proof. □

Lemma 5.9 and (30) imply that

$$\dim(\sigma) + \operatorname{cd}(\operatorname{Stab}_{\operatorname{PMod}(\Sigma_{0,2g})}(\sigma)) \le 2g - 3.$$ □

5.4 The spectral sequence

Let $K \subseteq \mathrm{PMod}(\Sigma_{0,2g})$ be a subgroup. Denote by $\widehat{E}^*_{*,*}$ the spectral sequence (22) for the action of K on $\mathcal{B}_{0,2g}$. Since cohomological dimension is monotonic, Theorem 5.5 implies that for any cell σ of $\mathcal{B}_{0,2g}$ we have

$$\dim(\sigma) + \mathrm{cd}(\mathrm{Stab}_K(\sigma)) \leq 2g-3.$$

This immediately implies $\widehat{E}^1_{p,q} = 0$ for $p+q > 2g-3$. Hence all differentials $d^1, d^2, \dots$ to the group $\widehat{E}^1_{0,2g-3}$ are trivial (Figure 8 is also applicable here, where the group $\widehat{E}^1_{0,2g-3}$ is shaded), so $\widehat{E}^1_{0,2g-3} = \widehat{E}^\infty_{0,2g-3}$. Therefore we have the following result:

Proposition 5.10 *Let $\mathfrak{L} \subseteq \mathcal{L}_0$ be a subset consisting of multiarcs from pairwise different K–orbits. Then the inclusions $\mathrm{Stab}_K(L) \subseteq K$, $L \in \mathfrak{L}$ induce the injective homomorphism*

$$\bigoplus_{L \in \mathfrak{L}} H_{2g-3}(\mathrm{Stab}_K(L), \mathbb{Z}) \hookrightarrow H_{2g-3}(K), \mathbb{Z}).$$

Proof of Lemma 5.1 It suffices to prove that the inclusions

$$j_u \colon \mathrm{Stab}_{\mathcal{K}_g}(\hat{u} \cdot N) \hookrightarrow \mathcal{K}_g \quad \text{for } u \in U_g$$

induce the injective homomorphism

$$\bigoplus_{u \in U_g} H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(\hat{u} \cdot N), \mathbb{Z}) \hookrightarrow G_1 \subseteq H_{2g-3}(\mathcal{K}_g, \mathbb{Z}).$$

Assume the converse and consider a nontrivial linear relation

$$\sum_{s=1}^{k} \lambda_s (j_{u_s})_*(\theta_s) = 0 \quad \text{for } \lambda_s \in \mathbb{Z} \text{ and } \theta_s \in H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(\hat{u}_s \cdot N), \mathbb{Z}) \tag{33}$$

for some pairwise different $u_1, \dots, u_s \in U_g$. For each $i = 1, \dots, g$ take an essential simple closed curve $\beta_i = \beta_{1,i}$ on the one-punctured torus X_i. Denote by $b_i = [\beta_{1,i}] \in H_1(\Sigma_g, \mathbb{Z})$ the corresponding homology class. For each $s \in \mathcal{N}$ denote by $\widehat{X}_{s,i} \subset \Sigma_g$ the one-punctured torus bounded by $\hat{u}_s \cdot \delta_i$. Since $\hat{u}_s$ belongs to the Torelli group $\mathcal{I}_g$, we have $H_1(\widehat{X}_{s,i}, \mathbb{Z}) = H_1(\widehat{X}_{t,i}, \mathbb{Z})$ for all $1 \leq s, t \leq k$. Denote by $\beta_{s,i}$ a unique curve on $\widehat{X}_{s,i}$ representing the homology class b_i.

Let $B_s = \beta_{s,1} \cup \dots \cup \beta_{s,g}$. Let $\{B_{d_1}, \dots, B_{d_l}\} \subseteq \{B_1, \dots, B_k\}$ be the maximal subset consisting of the multicurves from pairwise distinct $\mathcal{K}_g$–orbits. Take the homology class $x = \sum_{i=1}^{g} b_i$ and consider the complex of cycles $\mathcal{B}_g(x)$. Proposition 4.3 implies that the inclusions

$$\iota_i \colon \mathrm{Stab}_{\mathcal{K}_g}(B_{d_i}) \hookrightarrow \mathcal{K}_g$$

induce the injective homomorphism

$$\bigoplus_{i=1}^{l} H_{2g-3}(\mathrm{Stab}_{\mathcal{K}_g}(B_{d_i}), \mathbb{Z}) \hookrightarrow H_{2g-3}(\mathcal{K}_g, \mathbb{Z}). \tag{34}$$

Since the curves $\beta_{s,i}$ can be chosen in a unique way, we have the inclusions

$$\hat{j}_{u_s}\colon \mathrm{Stab}_{\mathcal{K}_g}(\hat{u}_s\cdot N)\hookrightarrow \mathrm{Stab}_{\mathcal{K}_g}(B_s).$$

Since $j_{u_i}=\iota_i\circ\hat{j}_{u_i}$, (34) and (33) imply that for each $i=1,\dots,l$ we have

$$\sum_{\{z\mid B_z\in\mathrm{Orb}_{\mathcal{K}_g}(B_{d_i})\}}\lambda_z(j_{u_z})_*(\theta_z)=0. \tag{35}$$

Equality (35) implies that it is sufficient to prove the statement of the lemma in the case where the multicurves $B_1,\dots,B_k$ belong to the same $\mathcal{K}_g$–orbit. Since we can prove Lemma 5.1 for an arbitrary choice of $\widehat{U}$, then by choosing the lifts $\hat{u}$ we can assume that $B_1=\dots=B_k=B$. Let $\zeta_{s,i}$ be a curve on $\widehat{X}_{s,i}$ intersecting β_i once and let $L_s=\zeta_{s,1}\cup\dots\cup\zeta_{s,g}$. Consider the surface $\Sigma_g\setminus B\cong\Sigma_{0,2g}$. Denote by p_i and q_i the punctures on $\Sigma_{0,2g}$ corresponding to the two sides of the curve β_i.

Consider the exacts sequence (6) in the case $M=B$. We have

$$1\to\langle T_{\beta_1},\dots,T_{\beta_g}\rangle\to\mathrm{Stab}_{\mathrm{Mod}(\Sigma_g)}(B)\to\mathrm{Mod}(\Sigma_{0,2g})\to 1.$$

Since the intersection $\langle T_{\beta_1},\dots,T_{\beta_g}\rangle\cap\mathcal{K}_g$ is trivial, we have the inclusion $K=\mathrm{Stab}_{\mathcal{K}_g}(B)\hookrightarrow\mathrm{Mod}(\Sigma_{0,2g})$. The action of $\mathcal{K}_g$ on the homology of Σ_g is trivial, so the image of this inclusion is contained in $\mathrm{PMod}(\Sigma_{0,2g})$. We have $K\hookrightarrow\mathrm{PMod}(\Sigma_{0,2g})$. Denote by $\zeta'_{s,i}$ the arc on $\Sigma_{0,2g}$ from p_i to q_i corresponding to the curve $\zeta_{s,i}$ and let $L'_s=\zeta'_{s,1}\cup\dots\cup\zeta'_{s,g}$. Let us show that $L'_1,\dots,L'_k$ belong to pairwise distinct K–orbits.

Assuming the converse, $f(L'_1)=L'_2$ for some $f\in K$. Then $f(L_1\cup B)=L_2\cup B$. Note that the surface $\Sigma_g\setminus(L_s\cup B)$ has g punctures, and each component of $\hat{u}_s\cdot N$ is homotopic into a neighborhood of its own puncture. Therefore the corresponding components of the multicurves $f(\hat{u}_1\cdot N)$ and $\hat{u}_2\cdot N$ are homotopic into a neighborhood of the same puncture. Consequently, the multicurves $f(\hat{u}_1\cdot N)$ and $\hat{u}_2\cdot N$ are isotopic. Since $\hat{u}_1,\hat{u}_2\in\mathcal{I}_g$, we obtain $\hat{u}_2^{-1}f\hat{u}_1\in\mathrm{Stab}_{\mathcal{I}_g}(N)$. It follows from the exactness of (9) that $\mathrm{Stab}_{\mathcal{I}_g}(N)\subseteq\mathcal{K}_g$. Hence $\hat{u}_2^{-1}f\hat{u}_1\in\mathcal{K}_g$ and we obtain

$$0=\tau(\hat{u}_2^{-1}f\hat{u}_1)=\tau(\hat{u}_1)-\tau(\hat{u}_2),$$

where τ is the Johnson homomorphism. This implies $u_1=u_2$, giving a contradiction.

Therefore $L'_1,\dots,L'_k$ belong to pairwise distinct K–orbits. Proposition 5.10 implies that the inclusions $\mathrm{Stab}_K(L'_s)\subseteq K$, $L'\in\mathfrak{L}$ induce the injective homomorphism

$$\bigoplus_s H_{2g-3}(\mathrm{Stab}_K(L'_s),\mathbb{Z})\hookrightarrow H_{2g-3}(K,\mathbb{Z}).$$

By Proposition 4.3 we also have the inclusion $H_{2g-3}(K,\mathbb{Z})\hookrightarrow H_{2g-3}(\mathcal{K}_g,\mathbb{Z})$ and so $\mathrm{Stab}_K(L'_s)=\mathrm{Stab}_{\mathcal{K}_g}(\hat{u}_s\cdot N)$. Therefore (33) implies $\lambda_s=0$ for all s. This contradiction proves Lemma 5.1. □

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