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**The Heisenberg double of involutory Hopf algebras
and invariants of closed 3-manifolds**

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We construct an invariant of closed oriented 3–manifolds using a finite-dimensional involutory unimodular and counimodular Hopf algebra H . We use the framework of normal \mathfrak{o} –graphs introduced by R Benedetti and C Petronio, in which one can represent a branched ideal triangulation via an oriented virtual knot diagram. We assign a copy of the canonical element of the Heisenberg double $\mathcal{H}(H)$ of H to each real crossing, which represents a branched ideal tetrahedron. The invariant takes values in the cyclic quotient $\mathcal{H}(H)/[\mathcal{H}(H), \mathcal{H}(H)]$, which is isomorphic to the base field. In the construction we use only the canonical element and structure constants of H , and not any representations of H . This, together with the finiteness and locality conditions of the moves for normal \mathfrak{o} –graphs, makes the calculation of our invariant rather simple and easy to understand. When H is the group algebra of a finite group, the invariant counts the number of group homomorphisms from the fundamental group of the 3–manifold to the group.

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1 Introduction

S Baaj and G Skandalis [1] and R M Kashaev [6] found a striking fact: for the Heisenberg double $\mathcal{H}(H)$ of any finite-dimensional Hopf algebra H , there exists a canonical element T in $\mathcal{H}(H)^{\otimes 2}$ satisfying the pentagon equation

$$T_{12}T_{13}T_{23} = T_{23}T_{12}.$$

In [6], Kashaev also showed that the Drinfeld double $\mathcal{D}(H)$ of H can be realized as a subalgebra of $\mathcal{H}(H)^{\otimes 2}$, and observed that the universal R –matrix of $\mathcal{D}(H)$ can be represented as a combination of four copies of T , where the quantum Yang–Baxter equation of the universal R –matrix follows from a sequence of the pentagon equation of T . Using his results, the second author [16] reconstructed the universal quantum $\mathcal{D}(H)$ invariant of framed tangles by assigning a copy of the canonical element T to each branched ideal tetrahedron of the tangle complements, and expected that this construction leads to invariants of pairs of a 3–manifold and geometrical input. We show that this construction defines an invariant of closed oriented 3–manifolds when the Hopf algebra H is involutory unimodular and counimodular.

In the formulation of our invariant, we use a diagrammatic representation of closed oriented 3–manifolds introduced by R Benedetti and C Petronio [3]. Their diagrams, which are called closed normal \mathfrak{o} –graphs,

are oriented virtual knot diagrams satisfying certain conditions. They showed that homeomorphism classes of closed oriented 3-manifolds are identified with equivalence classes of closed normal o-graphs up to certain moves. A crossing of a closed normal o-graph represents a branched ideal tetrahedron in the corresponding 3-manifold, and the orientation of the strand specifies a way to extend these local branching structures to a global one. Our invariant is obtained by assigning a copy of the canonical element T (or its inverse) to each crossing of closed normal o-graphs, and by reading them along the strands. The invariant takes values in the cyclic quotient $\mathcal{H}(H)/[\mathcal{H}(H), \mathcal{H}(H)]$, which is isomorphic to the base field through the character of the Fock space representation.

The proof of the invariance will be performed by checking the invariance under each move for normal o-graphs. There are two important types: the MP-moves and the CP-move. An MP-move represents a Pachner move equipped with a branching structure, which corresponds to a modified pentagon equation in the level of the invariant. Here, the modification is related to the antipode of H , and we need to assume that the antipode is involutive. Up to the MP-moves (and the 0–2 move), a closed normal o-graph represents a closed oriented 3-manifold with a combing. The CP-move is a special move for branched triangulations, which changes the combing. The invariance under the CP-move will be shown using tensor networks, where we need to assume that H is in addition both unimodular and counimodular. Using tensor networks, we will also show a connected sum formula of the invariant.

The main example of finite-dimensional involutory unimodular and counimodular Hopf algebras is the group algebra $\mathbb{C}[G]$ of a finite group G . We show that, in this case, the invariant is same as the number of homomorphisms from the fundamental group of the 3-manifold to G . There are several other examples of Hopf algebras which would be interesting to consider. The restricted enveloping algebras of restricted Lie algebras (see Jacobson [5]) are finite-dimensional involutory Hopf algebras. In [14], S Majid and A Pachol classified Hopf algebras of dimension ≤ 4 over the field of characteristic 2. M Kim [8] also gave some examples of finite-dimensional involutory unimodular and counimodular (commutative and cocommutative) Hopf algebras which are not group algebras.

There are several invariants based on triangulations; see Barrett and Westbury [2] and Turaev and Viro [17]. Ours uses only the structure constants of Hopf algebras and not representation categories, and thus the construction is rather simple. It would be interesting to compare our invariant and the Kuperberg invariant [11; 12], which also uses only the structure constants, and agrees with our invariant on group algebras. We remark that the Kuperberg invariant is constructed based on Heegaard diagrams, and the handle slide moves in Heegaard diagrams are global and infinite. The moves for closed normal o-graphs are local and finite, and each of these finite moves corresponds nicely to an algebraic equation, like the MP-moves and the pentagon equation. Thus the proof of the invariance is rather easy to understand.

The rest of the paper is organized as follows. In Section 2, we discuss Hopf algebras and their Heisenberg doubles. In Section 3, following Benedetti and Petronio [3], we explain how to represent closed oriented 3-manifolds in a combinatorial manner using closed normal o-graphs. In Section 4, we explain the construction of our invariant $Z(M; \mathcal{H}(H)) = Z(\Gamma; \mathcal{H}(H))$ using the Heisenberg double $\mathcal{H}(H)$ and a

closed normal o-graph Γ which represents a 3-manifold M . The proof of the invariance will be given in two sections. In Section 5, we prove the invariance under all moves except for the CP-move. In Section 6, we reformulate our invariant using tensor networks, and use them to prove the invariance under the CP-move. In Section 7, we show the connected sum formula, and study the case for the group algebra $\mathbb{C}[G]$ of a finite group G .

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2 Hopf algebra and Heisenberg double

In this section, we quickly review the definition and some properties of the Heisenberg double of Hopf algebras.

2.1 Hopf algebra

A Hopf algebra H over a field \mathbb{K} is a vector space equipped with five linear maps,

$$M : H \otimes H \rightarrow H, \quad 1 : \mathbb{K} \rightarrow H, \quad \Delta : H \rightarrow H \otimes H, \quad \epsilon : H \rightarrow \mathbb{K} \quad \text{and} \quad S : H \rightarrow H,$$

called multiplication, unit, comultiplication, counit and antipode, respectively, satisfying the standard axioms of Hopf algebras. When the antipode is involutive, ie $S^2 = \text{id}_H$, we call H involutory. Throughout the paper, H will denote a finite-dimensional Hopf algebra and H^* will denote the dual Hopf algebra of H . We will also use the Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for $x \in H$.

Recall that a right integral of a Hopf algebra H is an element $\mu_R \in H^*$ satisfying $\mu_R \cdot f = \mu_R f(1)$ for every $f \in H^*$. A left integral is defined similarly. Since H is finite-dimensional, a left (resp. right) integral of the dual Hopf algebra H^* is an element of H and is called a left (resp. right) cointegral of H . It is well known that for a finite-dimensional Hopf algebra, an integral always exists and is unique up to scalar multiplication. We say H is unimodular when the left cointegrals are also the right cointegrals, and counimodular when the left integrals are also the right integrals. For more details on Hopf algebras and their integrals, see [15; 11].

2.2 Heisenberg double

We use the left action of H on H^* defined by $(a \rightharpoonup f)(x) := f(xa)$, for $a, x \in H$ and $f \in H^*$. The Heisenberg double

$$\mathcal{H}(H) = H^* \otimes H$$

of a Hopf algebra H is a \mathbb{K} -algebra with unit $\epsilon \otimes 1$ and multiplication given by

$$(f \otimes a) \cdot (g \otimes b) = f \cdot (a_{(1)} \rightharpoonup g) \otimes a_{(2)} b,$$

for $a, b \in H$ and $f, g \in H^*$.

Let $\{e_i\}$ be the basis of H and $\{e^i\}$ its dual basis. Then the canonical element is given by

$$(2-1) \quad T = \sum_i (\epsilon \otimes e_i) \otimes (e^i \otimes 1) \in \mathcal{H}(H)^{\otimes 2}$$

and its inverse by

$$(2-2) \quad \bar{T} = \sum_i (\epsilon \otimes S(e_i)) \otimes (e^i \otimes 1) \in \mathcal{H}(H)^{\otimes 2}.$$

In the case of the Drinfeld double $\mathcal{D}(H)$ of H , the canonical element satisfies the quantum Yang–Baxter equation, which in turn produces invariants of links and 3–manifolds [4; 7]. One important feature of the Heisenberg double, which plays a central role in our construction of invariants, is that the canonical element satisfies the pentagon equation.

Proposition 2.1 [1; 6] *The pentagon equation*

$$(2-3) \quad T_{12}T_{13}T_{23} = T_{23}T_{12}$$

holds in $\mathcal{H}(H)^{\otimes 3}$.

Proof Note that

$$\begin{aligned} T_{12}T_{13}T_{23} &= \sum_{i,j,k} (\epsilon \otimes e_i)(\epsilon \otimes e_j) \otimes (e^i \otimes 1)(\epsilon \otimes e_k) \otimes (e^j \otimes 1)(e^k \otimes 1) \\ &= \sum_{i,j,k} (\epsilon \otimes e_i e_j) \otimes (e^i \otimes e_k) \otimes (e^j e^k \otimes 1) \in \epsilon \otimes (H \otimes H^*)^{\otimes 2} \otimes 1, \\ T_{23}T_{12} &= \sum_{i,j} (\epsilon \otimes e_j) \otimes (\epsilon \otimes e_i)(e^j \otimes 1) \otimes (e^i \otimes 1) \\ &= \sum_{i,j} (\epsilon \otimes e_j) \otimes (e_{i(1)} \rightarrow e^j \otimes e_{i(2)}) \otimes (e^i \otimes 1) \in \epsilon \otimes (H \otimes H^*)^{\otimes 2} \otimes 1. \end{aligned}$$

Let us identify $(H \otimes H^*)^{\otimes 2}$ with $\text{End}(H \otimes H)$ through the map

$$\iota: x \otimes f \otimes y \otimes g \mapsto (a \otimes b \mapsto f(a)x \otimes g(b)y),$$

for $x, y, a, b \in H$ and $f, g \in H^*$. Then, after identifying $\epsilon \otimes (H \otimes H^*)^{\otimes 2} \otimes 1$ with $(H \otimes H^*)^{\otimes 2}$, we can see that the both elements $T_{12}T_{13}T_{23}$ and $T_{23}T_{12}$ are sent by ι to the same element as follows:

$$\begin{aligned} \iota(T_{12}T_{13}T_{23})(a \otimes b) &= e^i(a)e_i e_j \otimes e^j e^k(b)e_k = a e_j \otimes e^j(b_{(1)})e^k(b_{(2)})e_k = ab_{(1)} \otimes b_{(2)}, \\ \iota(T_{23}T_{12})(a \otimes b) &= e_j(e_{i(1)} \rightarrow e^j)(a) \otimes e^i(b)e_{i(2)} = e_j e^j(ae_{i(1)}) \otimes e^i(b)e_{i(2)} \\ &= a e_{i(1)} \otimes e^i(b)e_{i(2)} = ab_{(1)} \otimes b_{(2)}. \end{aligned} \quad \square$$

The Heisenberg double $\mathcal{H}(H)$ has a canonical left module $F(H^*) = H^*$, which we call the *Fock space*, with the action $\phi: \mathcal{H}(H) \rightarrow \text{End}(H^*)$ given by

$$(2-4) \quad \phi(f \otimes a)(g) = (f \otimes a) \triangleright g := f \cdot (a \rightarrow g),$$

for $(f \otimes a) \in \mathcal{H}(H)$ and $g \in H^*$. Let χ_{Fock} be the character associated to the Fock space. For a \mathbb{K} -algebra A , let $[A, A]$ be the subspace spanned by $\{xy - yx \mid x, y \in A\}$ over \mathbb{K} . We are interested in the quotient space $\mathcal{H}(H)/[\mathcal{H}(H), \mathcal{H}(H)]$ since this is the space in which our invariant takes values.

Proposition 2.2 *The character of the Fock space*

$$\chi_{\text{Fock}}: \mathcal{H}(H)/[\mathcal{H}(H), \mathcal{H}(H)] \rightarrow \mathbb{K}$$

is an isomorphism between vector spaces.

Proof In [13, Proposition 6.1] it was shown that for $F \in \text{End}(H^*)$, the element

$$\sum_{i,j} F(e^i)e^j \otimes S^{-1}(e_j)e_i \in \mathcal{H}(H)$$

is in the preimage of F by ϕ , ie for any $g \in H^*$ we have

$$\begin{aligned} \sum_{i,j} \phi(F(e^i)e^j \otimes S^{-1}(e_j)e_i)(g) &= \sum_{i,j} F(e^i)e^j \cdot (S^{-1}(e_j)e_i \rightarrow g) \\ &= \sum_{i,j} g_{(3)}(e_i) \cdot F(e^i) \cdot g_{(2)}(S^{-1}(e_j))e^j \cdot g_{(1)} \\ &= F(g_{(3)}) \cdot S^{-1}(g_{(2)}) \cdot g_{(1)} = F(g_{(2)})\epsilon(g_{(1)}) = F(g). \end{aligned}$$

Since $\dim \mathcal{H}(H) = \dim \text{End}(H^*)$, it follows that ϕ is bijective, and hence ϕ is an algebra isomorphism.

Note that $\text{End}(H^*)$ is a matrix algebra, and thus the canonical trace

$$\text{tr}: \text{End}(H^*)/[\text{End}(H^*), \text{End}(H^*)] \rightarrow \mathbb{K}$$

is also an isomorphism. □

When the antipode S is involutive, χ_{Fock} can be given in terms of integrals. Let $\mu_R \in H^*$ and $e_L \in H$ be a right integral and a left cointegral satisfying $\mu_R(e_L) = 1$.

Proposition 2.3 *For an involutory Hopf algebra H , we have*

$$\chi_{\text{Fock}}(f \otimes a) = f(e_L)\mu_R(a)$$

for $f \otimes a \in \mathcal{H}(H)$.

Proof For $F \in \text{End}(H^*)$, the trace map is given (see [15, Chaper 10]) by

$$\text{tr}(F) = \langle e_L, F(\mu_{R(2)})S(\mu_{R(1)}) \rangle.$$

Thus

$$\chi_{\text{Fock}}(f \otimes a) = \langle e_L, f(a \rightarrow \mu_{R(2)})S(\mu_{R(1)}) \rangle = \langle e_L, \mu_{R(3)}(a)f \cdot \mu_{R(2)} \cdot S(\mu_{R(1)}) \rangle = f(e_L)\mu_R(a).$$

The third equality follows from the fact that $x_{(2)}S(x_{(1)}) = S(x_{(1)}S(x_{(2)})) = \epsilon(x)1$ for an involutory Hopf algebra. □



Figure 1: Vertices of type + (left) and - (right).

3 Closed normal o-graph

In order to define an invariant, we first recall the method introduced in [3] by Benedetti and Petronio to represent closed oriented 3-manifolds in a combinatorial manner. This method is based on the theory of branched spines, which are the dual of branched ideal triangulations.

Definition 3.1 [3] A closed normal o-graph is an oriented virtual knot diagram, ie a finite connected 4-valent graph Γ immersed in \mathbb{R}^2 with the following data and conditions:

- N1 Only two types (+ or -) of 4-valent vertices are considered, which are represented by the over-under notation as in Figure 1.
- N2 Each edge has an orientation such that it matches among two edges which are opposite to each other at a vertex.
- C1 If one removes the vertices and joins the edges which are opposite to each other, the result is a unique oriented circuit.

This diagram satisfies the following additional conditions:

- C2 The trivalent graph obtained from Γ by the rule defined in Figure 2 is connected.
- C3 Consider the disjoint union of oriented circuits obtained from Γ by the rule defined in Figure 3. The number of these circuits is exactly one more than the number of vertices of Γ .

Let \mathcal{G} be the set of closed normal o-graphs and \mathcal{M} the set of oriented closed 3-manifolds up to orientation-preserving homeomorphisms. Given a closed normal o-graph $\Gamma \in \mathcal{G}$, one can canonically construct a 3-manifold $\Phi(\Gamma) \in \mathcal{M}$ as follows. We fix an orientation of \mathbb{R}^3 and place a closed normal o-graph on $\mathbb{R}^2 \subset \mathbb{R}^3$. Then we replace each of its vertices with a tetrahedron (with the orientation given by \mathbb{R}^3), and

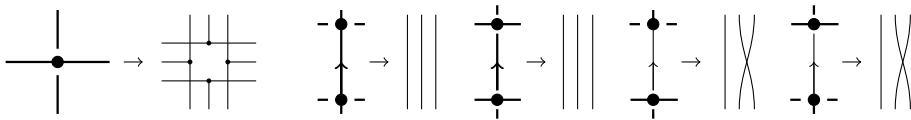


Figure 2

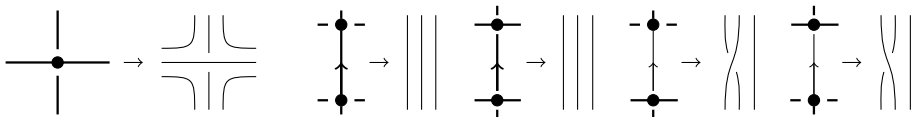


Figure 3

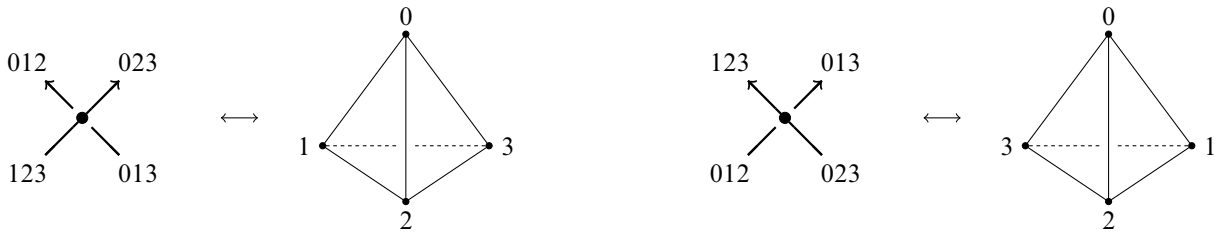


Figure 4

glue the faces of the ideal tetrahedra. This gluing is specified by the order of vertices of ideal tetrahedra defined as in Figure 4: we glue faces by the orientation-reversing map which preserves the order of vertices. Conditions on closed normal o-graphs ensure that the result is an ideally triangulated 3-manifold with S^2 boundary. Then, after we cap the boundary, the result defines an element $\Phi(\Gamma) \in \mathcal{M}$. Here, for the geometrical meaning of the order of the vertices of ideal tetrahedra, see Remark 3.4, where the meaning of the technical conditions C1, C2 and C3 are also explained. We denote the construction map obtained in the above way by $\Phi: \mathcal{G} \rightarrow \mathcal{M}$.

The map Φ is not one-to-one, and in order to make it so we need the following local moves of the diagrams:

- (1) planar isotopy of the diagram and the Reidemeister-type moves described in Figure 5, left,
- (2) the 0–2 move and the Matveev–Piergallini moves (MP–moves), described in Figure 5, right, and Figure 6, respectively,
- (3) the combinatorial Pontryagin move (CP–move) in Figure 7.

All of the above local moves preserve the axioms of closed normal o-graphs. We say that two closed normal o-graphs are equivalent if one can be obtained from the other by planar isotopy and a finite sequence of moves defined above. Let us denote this equivalence relation by \sim . The following was proved in [3]:

Proposition 3.2 *The map*

$$(3-1) \quad \Phi: \mathcal{G}/\sim \rightarrow \mathcal{M}$$

is well defined and bijective.

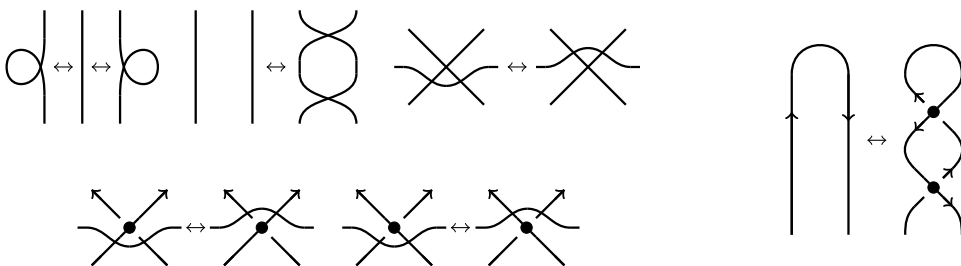


Figure 5: Left: the Reidemeister-type moves. Right: the 0–2 move.

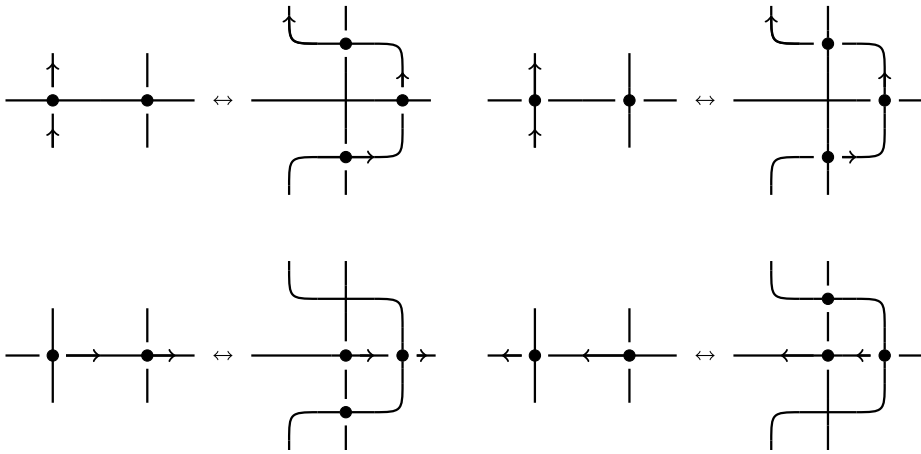


Figure 6: The MP-moves. The orientation of each nonoriented edge in the figure is arbitrary if it matches before and after the move.

Example 3.3 The closed normal o-graph of the lens space $L(p, 1)$, for $p \geq 1$, is given by the following graph with p vertices:



Remark 3.4 We briefly remark on the geometrical meaning of the representation of 3-manifolds by closed normal o-graphs; see [3] for more details. The order of vertices of ideal tetrahedra as in Figure 4 specifies a branching structure for the ideal triangulation, which gives a combing, ie a nonvanishing vector field up to homotopy, to the underlying 3-manifold. The technical conditions C1, C2 and C3 ensure that the 3-manifold corresponding to a closed normal o-graph has an S^2 boundary with a nice branching structure, where the associated combing can be extended canonically to the closed 3-manifold after we can cap off the boundary by B^3 . In this case, a closed normal o-graph up to the 0-2 move and the MP-moves represents a 3-manifolds with a combing. The CP-move in Figure 7 changes the combing while preserving the underlying 3-manifold; thus one gets the complete representation of \mathcal{M} as in Proposition 3.2. Here the CP-move is an interpretation of the Pontryagin surgery in terms of branched standard spines; see [3, Chaper 6] for details.

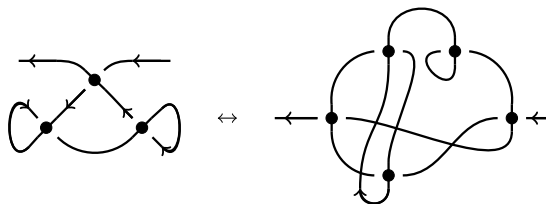


Figure 7: The CP-move.

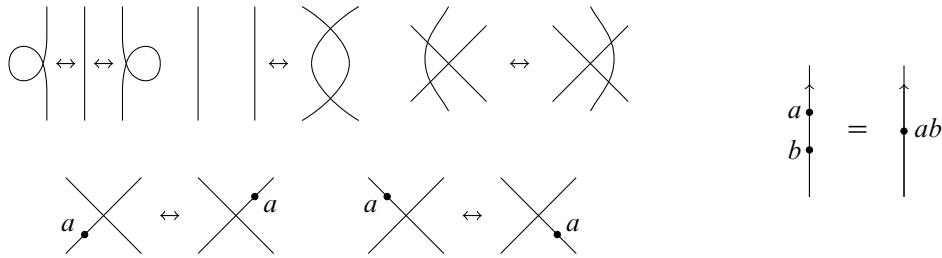


Figure 8

4 Invariant

In this section, we define a scalar $Z(\Gamma; \mathcal{H}(H))$ for a closed normal o-graph Γ . In the following section, we show that this scalar is an invariant of closed oriented 3-manifolds when the Hopf algebra is involutory, unimodular and counimodular.

Let A be a \mathbb{K} -algebra. An A -decorated diagram is an oriented closed curve immersed in \mathbb{R}^2 , where the self-intersections are transverse double points, with a finite number of dots, each of which is labeled with an element of A . These dots are called beads. We shall consider the A -decorated diagram up to planar isotopy and moves in Figure 8. We also allow the beads to slide along the curve.

We define the scalar $Z(\Gamma; \mathcal{H}(H))$ as follows. Recall the definition of the canonical element T and its inverse \bar{T} given in (2-1)–(2-2). Using Sweedler notation, we write the canonical element as $T = T_1 \otimes T_2 \in \mathcal{H}(H)^{\otimes 2}$ and its inverse as $\bar{T} = \bar{T}_1 \otimes \bar{T}_2 \in \mathcal{H}(H)^{\otimes 2}$. Given a closed normal o-graph, we replace its vertices with the diagram in Figure 10 to get an $\mathcal{H}(H)$ -decorated diagram C_Γ .

Since a closed normal o-graph satisfies axiom C1 in Definition 3.1, we can perform the moves in Figure 8 and slide beads on C_Γ to get a closed circle with a single bead labeled by J_Γ in $\mathcal{H}(H)$. Because one can permute the beads as in Figure 9, J_Γ is well defined in the quotient space $\mathcal{H}(H)/[\mathcal{H}(H), \mathcal{H}(H)]$, which can be identified with \mathbb{K} by Proposition 2.2.

Definition 4.1 $Z(\Gamma; \mathcal{H}(H)) := \chi_{\text{Fock}}(J_\Gamma)$.

Recall from Section 2.1 that a Hopf algebra H is called unimodular if the left cointegrals are also the right cointegrals, and counimodular if the left integrals are also the right integrals.

Theorem 4.2 *Let H be a finite-dimensional involutory unimodular counimodular Hopf algebra over \mathbb{K} , and Γ a closed normal o-graph of a closed oriented 3-manifold M . Then $Z(\Gamma, \mathcal{H}(H))$ is an invariant of M .*

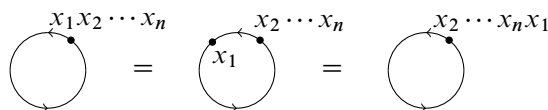


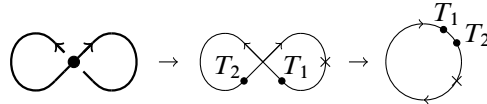
Figure 9



Figure 10: How to associate beads to vertices.

We end this section with some calculations, and prove [Theorem 4.2](#) in [Section 5](#).

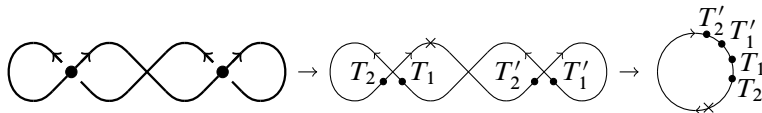
Example 4.3 The invariant of $S^3 = L(1, 1)$ is given by



Thus $Z(S^3; \mathcal{H}(H)) = \chi_{\text{Fock}}(T_2 T_1) = \sum_i \chi_{\text{Fock}}(e^i \otimes e_i)$. By [Proposition 2.3](#),

$$Z(S^3; \mathcal{H}(H)) = e^i(e_L)\mu_R(e_i) = \mu_R(e^i(e_L)e_i) = \mu_R(e_L) = 1.$$

Example 4.4 The invariant of $\mathbb{R}P^3 = L(2, 1)$ is given by



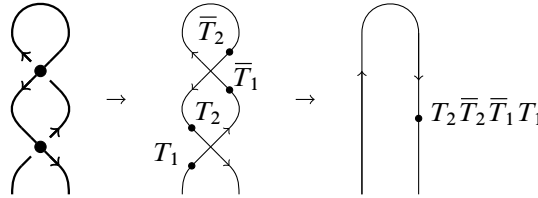
Thus $Z(\mathbb{R}P^3; \mathcal{H}(H)) = \chi_{\text{Fock}}(T_2 T_1 T_1' T_2') = \sum_{i,j} \chi_{\text{Fock}}(e^i \cdot (e_{i(1)} e_{j(1)} \rightarrow e^j) \otimes e_{i(2)} e_{j(2)}) = \text{tr}(S)$, where S is the antipode.

5 Main theorem

In this section we prove [Theorem 4.2](#). According to [Proposition 3.2](#), in order to prove that $Z(\Gamma, \mathcal{H}(H))$ is an invariant, we need to show that $Z(\Gamma, \mathcal{H}(H))$ is an invariant under planar isotopy and the local moves (the Reidemeister-type moves, the 0–2 move, the MP–moves and the CP–move) of closed normal o–graphs described in [Figures 5–7](#). In [\[16\]](#), it was essentially proved that the value $Z(\Gamma, \mathcal{H}(H))$ is an invariant under planar isotopy, the Reidemeister-type moves, the 0–2 move and the MP–moves. Here the Reidemeister-type moves are nothing but the *symmetry moves* in [\[16\]](#), the 0–2 move is a special case of the *colored (0, 2) moves* and the MP–moves are obtained by the *colored Pachner (2, 3) moves* and the colored (0, 2) moves. Even so, since the frameworks are slightly different, we give another proof in this section. The invariance under the CP–move was not observed in [\[16\]](#), and we give the proof in the following section, where we use a framework of tensor networks.

Proof of [Theorem 4.2](#) Invariance under planar isotopy and the Reidemeister-type moves It is obvious from the construction that $Z(\Gamma, \mathcal{H}(H))$ is an invariant under planar isotopy and the Reidemeister-type moves described in the [Figure 5](#), left.

Invariance under the 0–2 move Let us calculate the local tensor associated to the right-hand side of Figure 5, right:



So we need to show $T_2\bar{T}_2\bar{T}_1T_1 = \epsilon \otimes 1$. Let $T = \sum_i (\epsilon \otimes e_i) \otimes (e^i \otimes 1)$ and $\bar{T} = \sum_j (\epsilon \otimes S(e_j)) \otimes (e^j \otimes 1)$. Then

$$(5-1) \quad T_2\bar{T}_2\bar{T}_1T_1 = \sum_{i,j} (e^i \otimes 1)(e^j \otimes 1)(\epsilon \otimes S(e_j))(\epsilon \otimes e_i) = \sum_{i,j} e^i e^j \otimes S(e_j) e_i.$$

Identifying $H^* \otimes H$ with $\text{End}(H)$, the right side of (5-1) becomes $x \mapsto S(x_{(2)})x_{(1)}$. Since S is assumed to be involutive, $S(x_{(2)})x_{(1)} = \epsilon(x)1$. Thus $T_2\bar{T}_2\bar{T}_1T_1 = \epsilon \otimes 1$.

Invariance under the MP moves There are 16 MP moves. We write the calculations in Figures 11–12. Namely, we need to check the following 16 equalities:

- | | |
|--|--|
| MP 1.1 $T_{23}T_{13} = \bar{T}_{21}T_{13}T_{21}$, | MP 2.1 $\bar{T}_{32}\bar{T}_{31} = T_{12}\bar{T}_{31}\bar{T}_{12}$, |
| MP 1.2 $\bar{T}_{13}\bar{T}_{23} = \bar{T}_{21}\bar{T}_{13}T_{21}$, | MP 2.2 $T_{31}T_{32} = T_{12}T_{31}\bar{T}_{12}$, |
| MP 1.3 $\bar{T}_{23}T_{13} = T'_2T_1\bar{T}_2 \otimes \bar{T}_1T'_1 \otimes T_2$, | MP 2.3 $T_{32}\bar{T}_{31} = \bar{T}'_1\bar{T}_2T_1 \otimes T_2\bar{T}'_2 \otimes \bar{T}_1$, |
| MP 1.4 $\bar{T}_{13}T_{23} = T_2\bar{T}'_1\bar{T}_2 \otimes \bar{T}_1T_1 \otimes \bar{T}'_2$, | MP 2.4 $T_{31}\bar{T}_{32} = \bar{T}_1T'_2T_1 \otimes T_2\bar{T}_2 \otimes T'_1$, |
| MP 3.1 $T_{23}\bar{T}_{31} = \bar{T}_{31}T_{23}T_{21}$, | MP 4.1 $T_{31}\bar{T}_{23} = \bar{T}_{21}\bar{T}_{23}T_{31}$, |
| MP 3.2 $\bar{T}_{23}\bar{T}_{31} = \bar{T}_{31}\bar{T}_{21}\bar{T}_{23}$, | MP 4.2 $T_{31}T_{23} = T_{23}T_{21}T_{31}$, |
| MP 3.3 $T_{23}T_{31} = \bar{T}_2T'_2 \otimes T_1\bar{T}_1 \otimes T'_1T_2$, | MP 4.3 $\bar{T}_{31}\bar{T}_{23} = \bar{T}_2T_2 \otimes T_1\bar{T}'_1 \otimes \bar{T}'_2\bar{T}_1$, |
| MP 3.4 $\bar{T}_{23}\bar{T}_{31} = T_{21}T_{31}\bar{T}_{23}$, | MP 4.4 $\bar{T}_{31}T_{23} = T_{23}\bar{T}_{31}\bar{T}_{21}$. |

We verify that each of these is equivalent to the pentagon equation (2-3). Define $\tau_{\mathcal{H}}: \mathcal{H}(H)^{\otimes 2} \rightarrow \mathcal{H}(H)^{\otimes 2}$ by $\tau_{\mathcal{H}}(x \otimes y) = y \otimes x$ for $x, y \in \mathcal{H}(H)$. Then, for example, if we multiply MP 1.1 by T_{21} from the left and apply $\tau_{\mathcal{H}} \otimes \text{id}$, the result is exactly the pentagon equation. Similarly, we can reduce MP 1.2, MP 2.1, MP 2.2, MP 3.1, MP 3.2, MP 3.4, MP 4.1, MP 4.2, and MP 4.4 to the pentagon equation.

For the other six equalities, define the map $\mathcal{S}: \mathcal{H}(H) \rightarrow \mathcal{H}(H)$ by $\mathcal{S}(f \otimes a) = S(f) \otimes S(a)$, where S is the antipode of the Hopf algebra H . Then, for example, we can transform MP 1.3 into MP 1.1 by applying $\text{id} \otimes \mathcal{S} \otimes \text{id}$ to both sides as follows:

$$\begin{aligned} (\text{id} \otimes \mathcal{S} \otimes \text{id})(\bar{T}_{23}T_{13}) &= (\text{id} \otimes \mathcal{S} \otimes \text{id})(T'_2T_1\bar{T}_2 \otimes \bar{T}_1T'_1 \otimes T_2) \\ &\iff \sum_{i,j} (\epsilon \otimes e_j) \otimes (\epsilon \otimes S^2(e_i)) \otimes (e^i e^j \otimes 1) = \sum_{i,j,k} (e^j (e_{i(1)} \rightarrow e^k) \otimes e_{i(2)}) \otimes (\epsilon \otimes S(e_j) S^2(e_k)) \otimes (e^i \otimes 1) \\ &\iff T_{23}T_{13} = \bar{T}_{21}T_{13}T_{21}. \end{aligned}$$

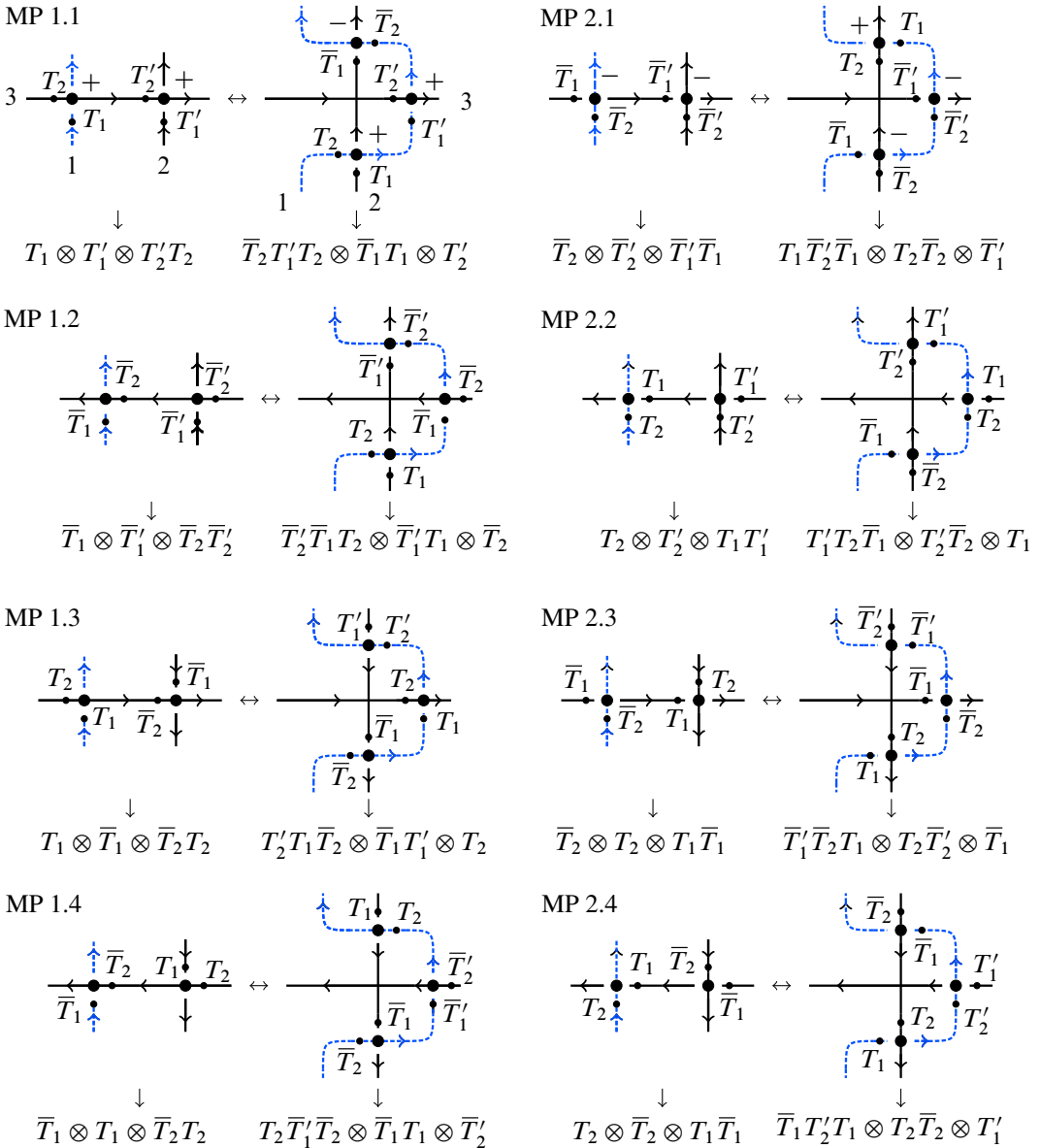


Figure 11

Here we used the involutivity of S for the last equivalence. Thus MP 1.3 is also equivalent to the pentagon equation. We can show that MP 1.4, MP 2.3, MP 2.4, MP 3.3 and MP 4.3 are also equivalent to the pentagon equation in a similar manner.

Since the pentagon equation holds in the Heisenberg double $\mathcal{H}(H)$, we conclude that Z is invariant under the MP-moves.

Invariance under the CP-move This will be proved in Proposition 6.6 using tensor networks. □

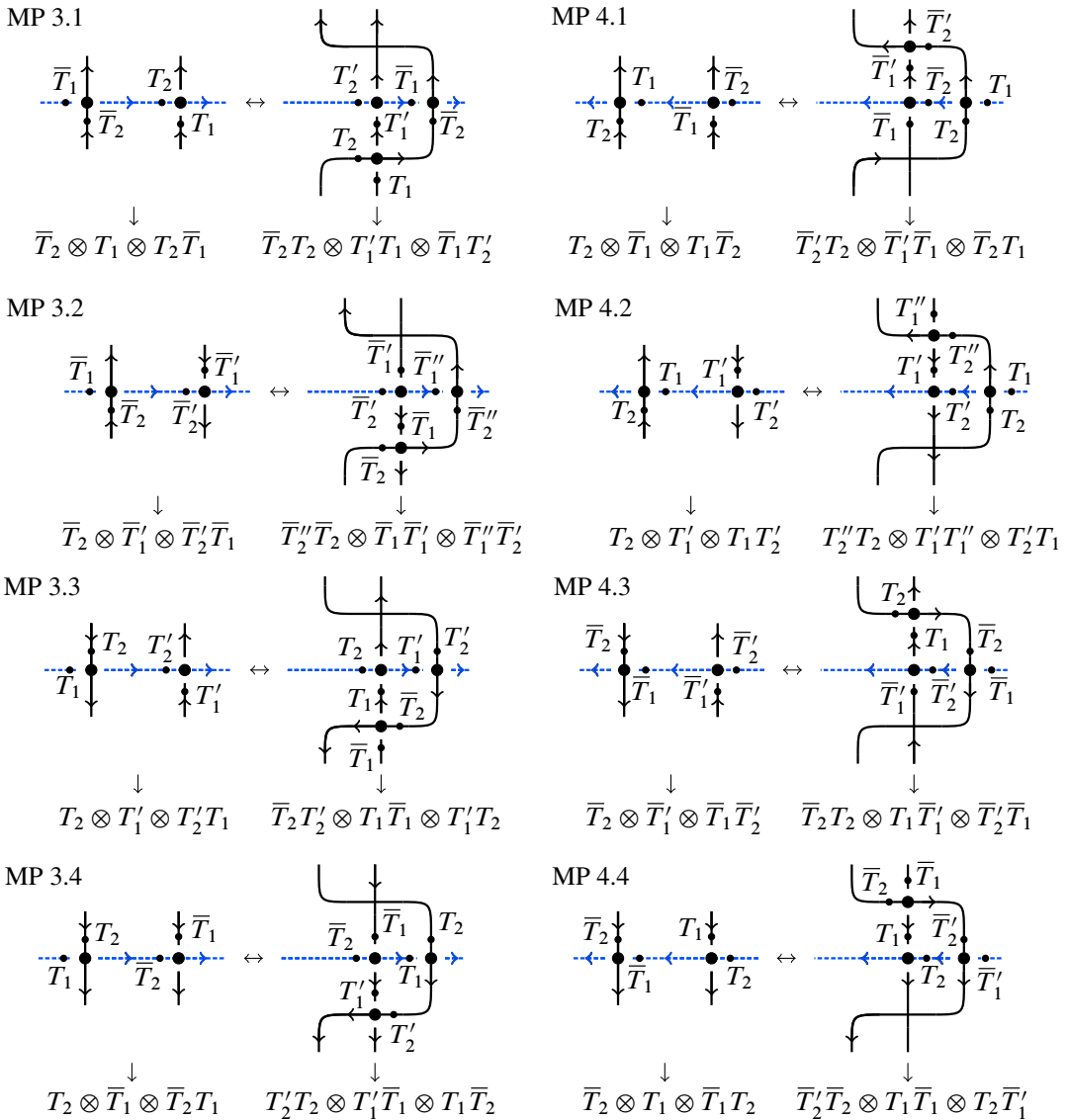


Figure 12

Remark 5.1 The assumption of unimodularity and counimodularity of H will be used only for invariance of the CP-move. Involutivity was used for the 0–2 move and the MP-moves MP 1.3, MP 1.4, MP 2.3, MP 2.4, MP 3.3 and MP 4.3. The other 10 equalities for the MP-moves do not need any restriction on Hopf algebras to hold.

Remark 5.2 In [16, Theorem 5.1], there is an error; even for an involutory Hopf algebra, J is not an invariant under the colored Pachner (2, 3) move in [16, Figure 16], which cannot be obtained by rotating the allowed one. This excluded move corresponds to the MP-moves with some strands reversed, which are not actually the MP-moves.

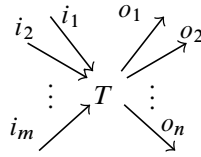


Figure 13

6 Tensor network approach

In this section, we give a quick review of tensor networks, which enable graphical calculus of tensors and linear maps. Then we reformulate the invariant using tensor networks, and prove the invariance under the CP-move.

6.1 Tensor network

A *tensor network* over a vector space V is an oriented graph which represents a tensor labeled by the set of open edges, where each incoming (resp. outgoing) edge labels V^* (resp. V). For example, the diagram in Figure 13 presents an (m, n) tensor $T \in (V^*)^{\otimes \mathcal{I}} \otimes V^{\otimes \mathcal{O}} = \text{Hom}(V^{\otimes \mathcal{I}}, V^{\otimes \mathcal{O}})$, where $\mathcal{I} = \{i_1, \dots, i_m\}$ is the set of incoming edges and $\mathcal{O} = \{o_1, \dots, o_n\}$ is the set of outgoing edges.

One important feature of tensor networks, which makes this notion practical, is the contraction of tensors. Given two tensor networks T and S , one gets a new tensor network by connecting an outgoing edge o of T and an incoming edge i of S (see Figure 14), which represents the tensor obtained from $T \otimes S$ by contracting V_o and $(V^*)_i$.

For example, the left diagram in Figure 15 represents the composition $g \circ f$ of two maps $f, g: V \rightarrow V$ and the right diagram represents the trace $\sum_i f_i^i = \text{tr}(f) \in \mathbb{K}$ of a map $f: V \rightarrow V$.

Note that a *tensor labeled by a set \mathcal{I}* does not fix the order of the tensorands. More precisely, $V^{\otimes \mathcal{I}}$ is the tensor product constructed from the product $V^{\mathcal{I}} = \{v: \mathcal{I} \rightarrow V\}$ labeled by \mathcal{I} . For example, for $\mathcal{I} = \{a, b, c\}$, the linear space $V^{\otimes \mathcal{I}}$ is isomorphic to $V^{\otimes 3}$, but such an isomorphism is not canonical, ie there is not a canonical order of labeled V to be written. On the other hand, for $\mathcal{I} = \{1, 2, \dots, n\}$, we have the canonical isomorphism $V^{\otimes \mathcal{I}} \rightarrow V^{\otimes n}$, $v \mapsto v(1) \otimes \dots \otimes v(n)$. In order to present calculations of a Hopf algebra $(H, M, 1, \Delta, \epsilon, S)$ using tensor networks over H , we fix an order of the incoming

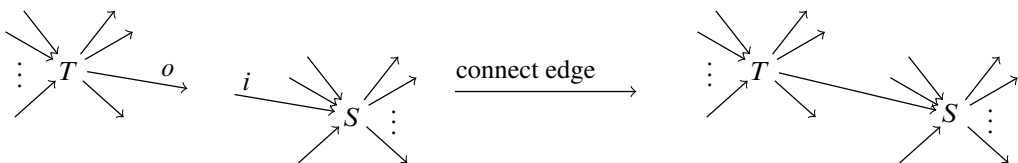


Figure 14



Figure 15

and the outgoing edges of the graphs representing multiplication $M: H^{\otimes 2} \rightarrow H$ and comultiplication $\Delta: H \rightarrow H^{\otimes 2}$, respectively, as below:



Note that the incoming edges of multiplication are counted in counterclockwise order, and the outgoing edges of comultiplication are counted in clockwise order. Using this notation, for example, the axioms of Hopf algebras are represented in Figure 16, where we use the subscript 1 to make the order clear. This subscript will be omitted sometimes when it is obvious.

6.2 Reformulation of invariant

Assume $H = (H, M, 1, \Delta, \epsilon, S)$ is a finite-dimensional involutory unimodular counimodular Hopf algebra.

An *o-tangle* is an oriented virtual tangle diagram in $[0, 1]^2$ such that each boundary point is on the bottom, $[0, 1] \times \{0\}$, or on the top, $[0, 1] \times \{1\}$. For finite sequences ε and ε' of \pm , an $(\varepsilon, \varepsilon')$ o-tangle is an o-tangle having boundary points on the bottom and top compatible to ε and ε' , respectively, where compatible means that if an edge is oriented upwards (resp. downwards) then it is connected to $+$ (resp. $-$).

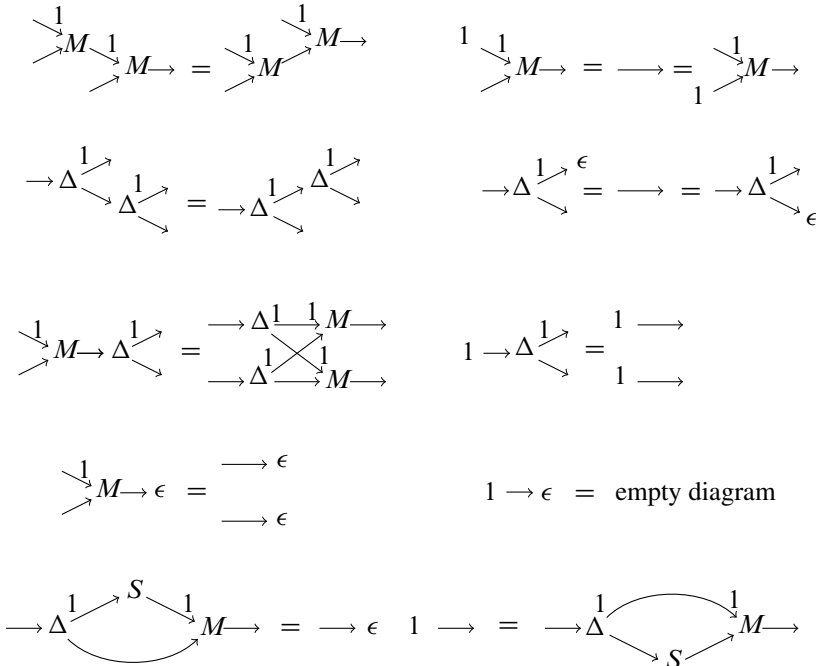
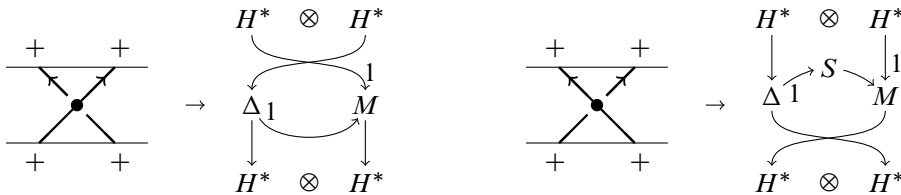


Figure 16

Let \mathcal{T}_o be the *category of o-tangles*, where objects are finite sequences of \pm including the empty sequence \emptyset , and morphisms $\mathcal{T}_o(\varepsilon, \varepsilon')$ from ε to ε' are isotopy classes of $(\varepsilon, \varepsilon')$ o-tangles. As usual, \mathcal{T}_o is a strict monoidal category with the unit object \emptyset , and the composition $\Gamma' \circ \Gamma$ of $(\varepsilon, \varepsilon')$ o-tangle Γ and $(\varepsilon', \varepsilon'')$ o-tangle Γ' is obtained by connecting the ε' -type boundary points on the top of Γ to these on the bottom of Γ' . We can construct a monoidal functor $Z(*; H)$ from the category of o-tangles \mathcal{T}_o to the category of finite-dimensional vector spaces $\text{Vect}_{\mathbb{K}}$ as follows.

For the object $+$ (resp. $-$), we set $Z(+; H) := H^*$ (resp. $Z(-; H) := H$). For a sequence $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ in \pm , let H^ε denote $H^{\varepsilon_1} \otimes \dots \otimes H^{\varepsilon_n}$ with $H^+ = H^*$ and $H^- = H$. To a given $(\varepsilon, \varepsilon')$ o-tangle Γ , we associate a tensor network over H , which represents a linear map $Z(\Gamma; H) \in \text{Hom}(H^\varepsilon, H^{\varepsilon'})$ as follows; we replace each positive (resp. negative) crossing of Γ with the tensor network as in the left (resp. right) picture below, and then connect the boundary points of these tensor networks following the strands of Γ . The boundary edges on the bottom and top of the resulting tensor network are counted from the left, and an input element to $Z(\Gamma; H) \in \text{Hom}(H^\varepsilon, H^{\varepsilon'})$ is a tensor $T \in H^\varepsilon$ labeled by the bottom edges. Then $Z(\Gamma; H)$ sends T to the tensor $Z(\Gamma; H)(T) \in H^{\varepsilon'}$ (labeled by the top edges) which is obtained by concatenating T to the bottom edges of the tensor network. Note that the corresponding strands of the tensor network are oriented in the opposite direction to these of Γ :

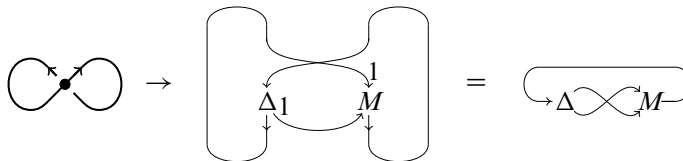


Note that a maximum point plays the role of evaluation map, and a minimum point plays the role of coevaluation map:

$$\begin{aligned} \text{cap} &: H^* \otimes H \rightarrow \mathbb{K}, & f \otimes a &\mapsto f(a), & \text{cup} &: \mathbb{K} \rightarrow H \otimes H^*, & 1 &\mapsto \sum e_i \otimes e^i, \\ \text{cocap} &: H \otimes H^* \rightarrow \mathbb{K}, & a \otimes f &\mapsto f(a), & \text{cucup} &: \mathbb{K} \rightarrow H^* \otimes H, & 1 &\mapsto \sum e^i \otimes e_i. \end{aligned}$$

Since a closed normal o-graph Γ is an (\emptyset, \emptyset) o-tangle, it is sent to an endomorphism $Z(\Gamma; H)$ of \mathbb{K} , which is represented by a scalar in \mathbb{K} . By abusing notation we also denote the scalar by $Z(\Gamma; H) \in \mathbb{K}$.

Example 6.1 For a closed normal o-graph Γ for S^3 , which is an (\emptyset, \emptyset) o-tangle as in the left picture below, the resulting tensor network $Z(\Gamma; H)$ is the right picture below.



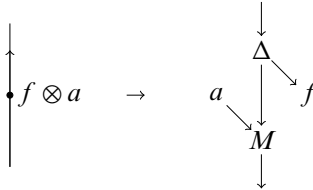
Thus $Z(\Gamma; H) = \text{tr}(M \circ \tau \circ \Delta) \in \mathbb{K}$, where $\tau(x \otimes y) = y \otimes x$.

Let Γ be a closed normal o-graph and $Z(\Gamma; \mathcal{H}(H))$ the invariant defined in Section 4.

Proposition 6.2

$$Z(\Gamma; \mathcal{H}(H)) = Z(\Gamma; H).$$

Proof First, let us consider an oriented strand with a bead $f \otimes a \in \mathcal{H}(H)$ as in the left picture below. We will think of this strand with a bead as the action of $f \otimes a$ on the Fock space $F(H^*)$, which was given by $\phi(f \otimes a): g \mapsto f(a \rightarrow g)$ for $g \in H^*$, where $f(a \rightarrow g): x \mapsto f(x_{(1)})g(x_{(2)}a)$ for $x \in H$. Graphically this map $\phi(f \otimes a) \in \text{Hom}(H^*, H^*)$ can be represented by the tensor network in the right picture below.

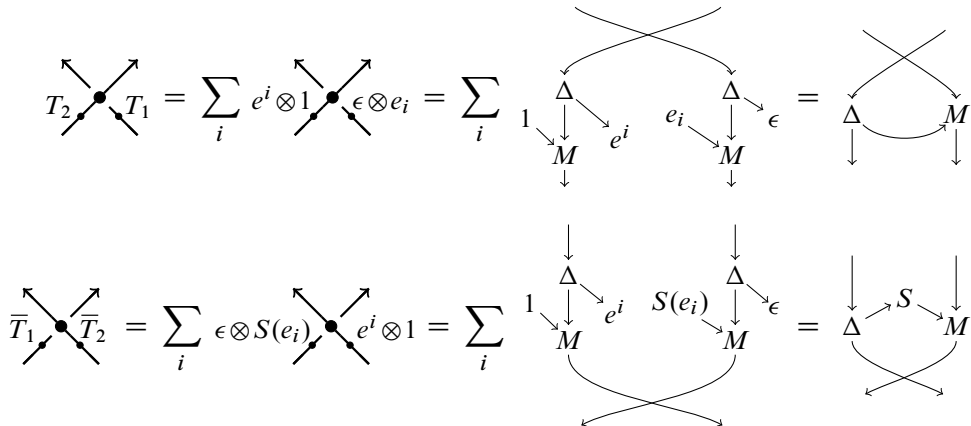


If there are multiple beads, we replace each one with the above tensor network. Since each map is an action, the result is well defined under the move of Figure 8, right.

Then, let us consider the oriented closed curve with a bead $f \otimes a \in \mathcal{H}(H)$. Recall that $\chi_{\text{Fock}}(f \otimes a)$ is the trace of the linear map defined by the action of $f \otimes a$. As remarked in Section 6.1, in terms of tensor networks, taking a trace is just connecting the incoming edge with the outgoing edge. Thus

$$\chi_{\text{Fock}}(f \otimes a) = \text{Diagram of a closed loop with the tensor network from the previous block inside it. The loop is formed by two vertical strands, one on the left and one on the right, connected at the top and bottom by curved arrows pointing downwards.$$

Finally, let Γ be a closed normal o-graph. Replacing its vertices as in Figure 10 and sliding beads, we get a single bead $J_\Gamma \in \mathcal{H}(H)$, and the invariant $Z(\Gamma, \mathcal{H}(H))$ was defined as $\chi_{\text{Fock}}(J_\Gamma)$. Here, before the sliding process, we replace each bead with a corresponding tensor network as above, and compare the result to $Z(\Gamma; H)$. Since the beads only appear at the vertices of Γ , we just need to look at the associated tensor network for these beads:



These are the same tensor networks associated with vertices in the definition of $Z(\Gamma; H)$. □

6.3 Invariance under CP-move

We prove the invariance of $Z(\Gamma; H)$ under the CP-move. Refer to [Section 2](#) for the definition of integrals. Let $e_L \in H$ and $\mu_L \in H^*$ be a left cointegral and a left integral satisfying $\mu_L(e_L) = 1$. In terms of tensor networks, we have

$$\begin{array}{ccc}
 \begin{array}{c} \nearrow \\ \searrow \\ \rightarrow M \end{array} & \xrightarrow{\quad} & \epsilon \\
 e_L \rightarrow & = & e_L \rightarrow
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} \rightarrow \Delta \\ \nearrow \\ \searrow \\ \mu_L \end{array} & \xrightarrow{\quad} & 1 \\
 & = & \mu_L
 \end{array}
 \end{array}$$

Lemma 6.3 [12, Lemma 3.3] *The following equality holds in any Hopf algebra:*

$$\begin{array}{ccc}
 \begin{array}{c} \rightarrow \mu_L \\ e_L \rightarrow \end{array} & = & \begin{array}{c} \rightarrow M \\ \nearrow S \\ \searrow S \\ \Delta \end{array} \xrightarrow{1}
 \end{array}$$

Set $e_R := S(e_L)$. Since S is an antialgebra map, e_R is a (nonzero) right cointegral.

Lemma 6.4 [15, Theorem 10.5.4] *For a finite-dimensional involutory counimodular Hopf algebra,*

$$\Delta^{\text{op}}(e_R) = \Delta(e_R),$$

where $\Delta^{\text{op}}(x) := x_{(2)} \otimes x_{(1)}$.

Lemma 6.5 *Let H be an involutory Hopf algebra. Then*

$$\begin{array}{ccc}
 \begin{array}{c} \curvearrowright \\ \bullet \\ \curvearrowleft \end{array} & \xrightarrow{Z(*; H)} & \begin{array}{c} \downarrow \mu_L \\ e_R \\ \downarrow \end{array}
 \end{array}$$

Proof Replacing the vertex of the o-tangle with the corresponding tensor network, we have

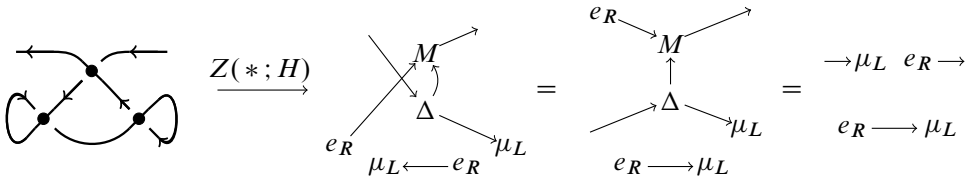
$$\begin{array}{ccc}
 \begin{array}{c} \curvearrowright \\ \bullet \\ \curvearrowleft \end{array} & \xrightarrow{z(*; H)} & \begin{array}{c} \downarrow 1 \\ M \\ \Delta \end{array} \\
 & & \begin{array}{c} \uparrow 1 \\ \Delta^{\text{op}} \\ M \end{array}
 \end{array}$$

Using [Lemma 6.3](#) we get

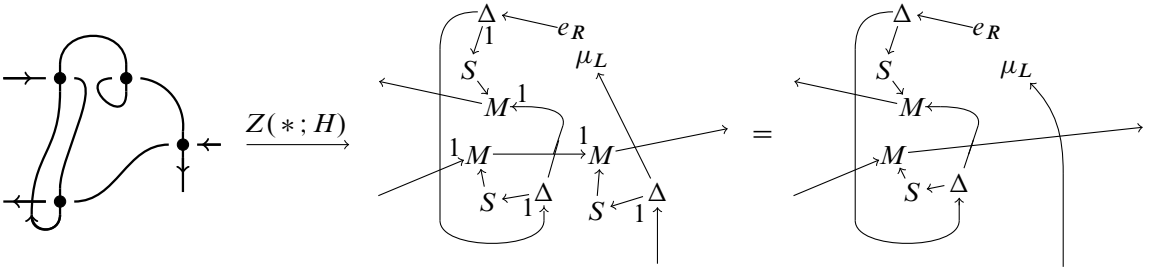
$$\begin{array}{ccccccc}
 \begin{array}{c} \rightarrow M \\ \curvearrowright \\ \Delta^{\text{op}} \end{array} & \rightarrow & \begin{array}{c} \rightarrow M \\ \nearrow S^2 \\ \searrow S^2 \\ \Delta^{\text{op}} \end{array} & \rightarrow & \begin{array}{c} \rightarrow M \\ \nearrow S \\ \searrow S \\ \Delta \end{array} \rightarrow S & \rightarrow & \begin{array}{c} \rightarrow \mu_L \\ e_R \end{array}
 \end{array}
 \quad \square$$

Proposition 6.6 Let H be a finite-dimensional involutory unimodular counimodular Hopf algebra. Then $Z(\Gamma; H)$ is an invariant under the CP-move in Figure 7.

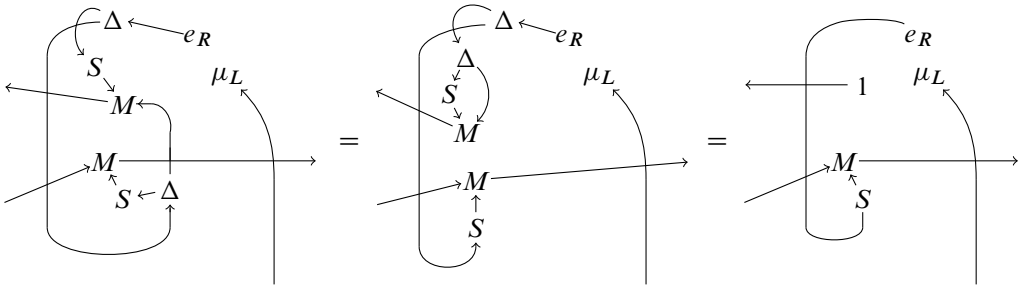
Proof First, we evaluate the left-hand side of the CP-move. Using Lemma 6.5, the twists in the closed normal o-graph can be replaced by integrals, and thus



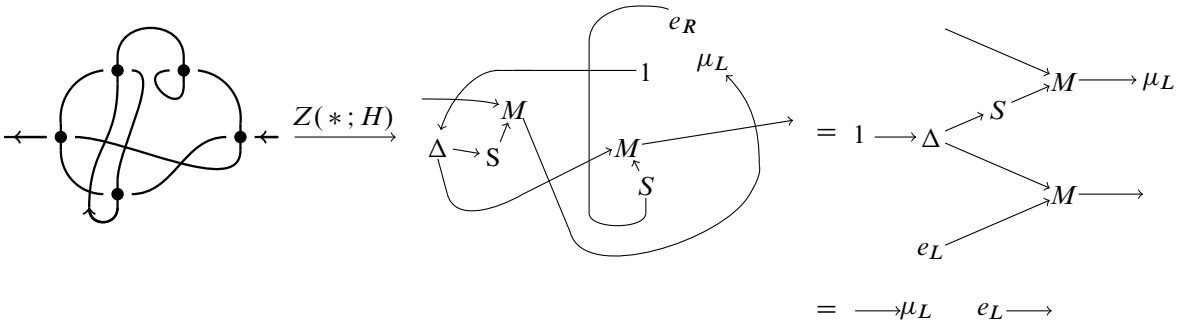
Next, we evaluate a part of the right-hand side of the CP-move:



We used the definition of left integral μ_L for the equality above. From Lemma 6.4, e_R is cyclic. Thus this equals



The first equality follows from coassociativity of the comultiplication, and the second follows from the property of the antipode. Thus right-hand side of the CP-move becomes



Finally, we show that the following equality holds for involutory unimodular counimodular Hopf algebra:

$$\begin{array}{c} \longrightarrow \mu_L \quad e_R \longrightarrow \\ e_R \longrightarrow \mu_L \end{array} = \begin{array}{c} \longrightarrow \mu_L \quad e_L \longrightarrow \\ e_L \longrightarrow \mu_L \end{array}$$

Since H is unimodular, $e_R = ke_L$ for some nonzero $k \in \mathbb{K}$. Since $\mu_L(e_L)$ is 1, $\mu_L(e_R) = k$ and the left-hand side of the above tensor network is the same as right-hand side times k^2 . Applying S^2 to e_L and using the fact that S is involutive, we see that $k^2 = 1$, and the above equality follows. \square

7 Properties

We continue to assume that H is a finite-dimensional involutory unimodular counimodular Hopf algebra over \mathbb{K} . For a closed oriented 3-manifold M and a closed normal o-graph Γ representing M , set $Z(M; H) := Z(\Gamma; H)$.

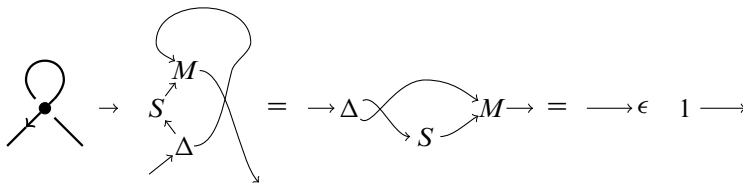
7.1 Connected sum formula

For closed oriented 3-manifolds M and N , let $M \# N$ be their connected sum.

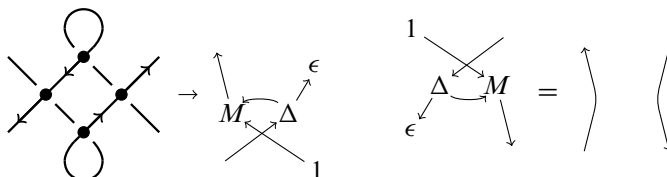
Proposition 7.1 $Z(M \# N; H) = Z(M; H)Z(N; H)$.

Proof Let Γ_M and Γ_N be closed normal o-graphs representing M and N , respectively. Let $\Gamma_M \# \Gamma_N$ be the connected sum of closed normal o-graphs defined in Figure 17.

In [10], Y Koda showed that for two closed normal o-graphs Γ_M and Γ_N , the 3-manifold represented by $\Gamma_M \# \Gamma_N$ is $M \# N$. We show the assertion by comparing the tensor networks for the closed normal o-graphs in the left and right-hand sides of the Figure 17. Note that



Thus



which implies the assertion. \square

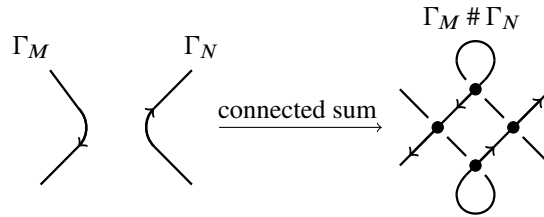


Figure 17

7.2 Group algebra

We show that the invariant $Z(M; \mathbb{C}[G])$ with the group algebra $\mathbb{C}[G]$ of a finite group G counts the number $|\text{Hom}(\pi_1 M, G)|$ of group homomorphisms from the fundamental group $\pi_1 M$ of M to G . The proof essentially follows the line of [7].

The algebra $\mathbb{C}[G]$ has a canonical basis given by $\{g\}_{g \in G}$ and the Hopf algebra structure is given by $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1}$. Note that $\mathbb{C}[G]$ is involutory, unimodular and counimodular. The dual group algebra $\mathbb{C}(G) := \mathbb{C}[G]^*$ has the dual basis $\{\delta_g\}_{g \in G}$, and the dual Hopf algebra structure is given by

$$\delta_g \cdot \delta_h = \delta_{g,h} \delta_g, \quad 1_{\mathbb{C}(G)} = \sum_{g \in G} \delta_g, \quad \Delta(\delta_g) = \sum_{hk=g} \delta_h \otimes \delta_k, \quad \epsilon(\delta_g) = \delta_g(e), \quad S(\delta_g) = \delta_{g^{-1}},$$

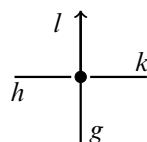
where $\delta_{g,h} \in \{0, 1\}$ is 1 if $g = h$ and 0 otherwise, and e is the unit of G .

The left action of $x \in G \subset \mathbb{C}[G]$ on $\delta_g \in \mathbb{C}(G)$ is given by

$$x \cdot \delta_g = \delta_{gx^{-1}}.$$

Proposition 7.2 For a closed oriented 3-manifold M and a finite group G , we have $Z(M; \mathbb{C}[G]) = |\text{Hom}(\pi_1 M, G)|$.

Proof Let Γ be a closed normal o-graph representing M . In [10; 9], Koda gave an explicit formula for the fundamental group $\pi_1 M$ in terms of the closed normal o-graph Γ . Let E be the set of all edges of Γ . We consider the group generated by E and the relation set R consisting of $g = l$ and $hg = k$ for the edges g, l, h and k around each vertex, as below. Then the resulting group $\langle E \mid R \rangle$ is isomorphic to the fundamental group $\pi_1 M$ of M , and the number $|\text{Hom}(\pi_1 M, G)|$ is equal to the number of edge colorings $c: E \rightarrow G$ such that $c(R)$ holds in G .



We show that the invariant $Z(M; \mathbb{C}[G])$ indeed counts such edge colorings. As we explained in Section 6.2, each vertex of a closed normal o-graph can be treated as a linear map between $\mathbb{C}(G)^{\otimes 2}$:

$$\begin{array}{l}
 \begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array} : \mathbb{C}(G) \otimes \mathbb{C}(G) \rightarrow \mathbb{C}(G) \otimes \mathbb{C}(G), \quad \delta_g \otimes \delta_h \mapsto T_1 \triangleright \delta_h \otimes T_2 \triangleright \delta_g \\
 \begin{array}{c} \nwarrow \\ \bullet \\ \nearrow \end{array} : \mathbb{C}(G) \otimes \mathbb{C}(G) \rightarrow \mathbb{C}(G) \otimes \mathbb{C}(G), \quad \delta_h \otimes \delta_g \mapsto \bar{T}_2 \triangleright \delta_g \otimes \bar{T}_1 \triangleright \delta_h
 \end{array}$$

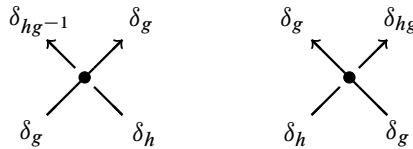
Here

$$T_1 \triangleright \delta_h \otimes T_2 \triangleright \delta_g = \sum_{x \in G} (\epsilon \otimes x) \triangleright \delta_h \otimes (\delta_x \otimes e) \triangleright \delta_g = \sum_{x \in G} x \mapsto \delta_h \otimes \delta_x \cdot \delta_g = \sum_{x \in G} \delta_{hx^{-1}} \otimes \delta_{x,g} \delta_g = \delta_{hg^{-1}} \otimes \delta_g$$

and

$$\begin{aligned}
 \bar{T}_2 \triangleright \delta_g \otimes \bar{T}_1 \triangleright \delta_h &= \sum_{x \in G} (\delta_x \otimes e) \triangleright \delta_g \otimes (\epsilon \otimes x^{-1}) \triangleright \delta_h = \sum_{x \in G} \delta_x \cdot \delta_g \otimes x^{-1} \mapsto \delta_h \\
 &= \sum_{x \in G} \delta_{x,g} \delta_g \otimes \delta_{hx} = \delta_g \otimes \delta_{hg}.
 \end{aligned}$$

We draw these maps as follows:



Note that the subscripts of δ give nothing but an edge coloring (around the vertices) as desired. Furthermore, to connect those vertices by strands means that we insert maximum and minimum points among them. Recall from Section 6.2 that maximum and minimum points correspond to evaluations and coevaluations, respectively. In the present case they are the maps shown below, which means after all we sum up all edge colorings:

$$\begin{array}{l}
 \begin{array}{c} \curvearrowright \end{array} : \mathbb{C}(G) \otimes \mathbb{C}[G] \rightarrow \mathbb{C}, \quad \delta_g \otimes h \mapsto \delta_{g,h}, \quad \begin{array}{c} \curvearrowleft \end{array} : \mathbb{C} \rightarrow \mathbb{C}[G] \otimes \mathbb{C}(G), \quad 1 \mapsto \sum g \otimes \delta_g, \\
 \begin{array}{c} \curvearrowleft \end{array} : \mathbb{C}[G] \otimes \mathbb{C}(G) \rightarrow \mathbb{C}, \quad h \otimes \delta_g \mapsto \delta_{g,h}, \quad \begin{array}{c} \curvearrowright \end{array} : \mathbb{C} \rightarrow \mathbb{C}(G) \otimes \mathbb{C}[G], \quad 1 \mapsto \sum \delta_g \otimes g. \quad \square
 \end{array}$$

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